ON THE QUASI-ISOMETRIC RIGIDITY OF A CLASS OF RIGHT-ANGLED COXETER
GROUPS

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ABSTRACT

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To each finite simplicial graph Γ there is an associated right-angled Coxeter group given by the presentation

$$W_\Gamma = \langle v \in V(\Gamma) | v^2 = 1 \text{ for all } v \in V(\Gamma); v_1 v_2 = v_2 v_1 \text{ if and only if } (v_1, v_2) \in E(\Gamma) \rangle,$$

where $V(\Gamma), E(\Gamma)$ denote the vertex set and edge set of Γ respectively. In this dissertation, we discuss the quasi-isometric rigidity of the class of right-angled Coxeter groups whose defining graphs are given by generalized polygons. We begin with a review of some helpful preliminary concepts, including a discussion on the current state of the art of the quasi-isometric classification of right-angled Coxeter groups. We then prove in detail that for any given joins of finite generalized thick m -gons Γ_1, Γ_2 with $m \in \{3, 4, 6, 8\}$, the corresponding right-angled Coxeter groups are quasi-isometric if and only if Γ_1 and Γ_2 are isomorphic.

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PREFACE

Much of the work presented in this dissertation was completed jointly by the author and Xiangdong Xie and is featured in the paper “Quasi-isometric rigidity of a class of right-angled Coxeter groups” to appear in the Proceedings of the AMS. A pre-publication version can be found here: <https://arxiv.org/abs/1804.03123>.

CHAPTER 1 PRELIMINARIES

The primary focus of this dissertation is to discuss key elements of the theory involved in the quasi-isometric classification of the family of finitely generated groups known as *right-angled Coxeter groups*.

Given a finite simplicial graph Γ there is an associated right-angled Coxeter group (RACG for short) given by the presentation

$$W_\Gamma = \langle v \in V(\Gamma) \mid v^2 = 1 \text{ for all } v \in V(\Gamma); v_1 v_2 = v_2 v_1 \text{ if and only if } (v_1, v_2) \in E(\Gamma) \rangle$$

where $V(\Gamma), E(\Gamma)$ denote the vertex set and edge set of Γ respectively. To each RACG W_Γ there is an associated CAT(0) cube complex, the *Davis complex*, upon which W_Γ acts geometrically (properly and cocompactly by isometries). The Davis complex should be thought of as a geometric model for the associated RACG.

The quasi-isometric classification of RACGs is a wide open area of study. The current state of the art is briefly discussed here in Chapter 3, however, the interested reader should consult the survey paper by Dani [11] for a more detailed account of these topics. A common theme in this field of study is to associate properties of RACGs to the structure of their defining graphs. The main result of particular interest in this dissertation is to classify (up to quasi-isometry) the family of RACGs whose defining graphs are generalized polygons.

Let L be a connected bipartite graph whose vertices are colored red and blue such that no two adjacent vertices share the same color. L is called a *generalized m -gon* ($m \in \mathbb{N}, m \geq 2$) if it has the following two properties:

1. Given any pair of edges there is a circuit with combinatorial length $2m$ containing both.
2. For two circuits A_1, A_2 of combinatorial length $2m$ that share an edge there is an isomorphism $f : A_1 \rightarrow A_2$ that pointwise fixes $A_1 \cap A_2$.

A generalized m -gon is called *thick* if each vertex has valence at least 3.

The main focus of this dissertation is proving the following result.

Theorem ([8], Corollary 1.3). *For each $m \in \{3, 4, 6, 8\}$ let $\mathcal{S}_m$ denote the set of finite thick generalized m -gons and $\mathcal{S} = \cup_m \mathcal{S}_m$. If Γ_1, Γ_2 are finite joins of elements in $\mathcal{S}$, then any quasi-isometry $f : \Sigma_{\Gamma_1} \rightarrow \Sigma_{\Gamma_2}$ is at a finite distance from an isometry. In particular, W_{Γ_1} and W_{Γ_2} are quasi-isometric if and only if Γ_1 and Γ_2 are isomorphic.*

A key element in establishing the above theorem is proving that the Davis complex associated to the graph of a finite thick generalized m -gon where $m \in \{3, 4, 6, 8\}$ admits a metric structure that makes it a Fuchsian building. The desired result then follows from applying the quasi-isometric rigidity of Fuchsian buildings due to Xie [40] in conjunction with a result of Kapovich-Kleiner-Leeb [24] on the quasi-isometric rigidity of product spaces.

Throughout this work it is assumed that the reader has a basic knowledge of group theory, analysis, and topology. For any material beyond this basic level we provide the necessary details. The remainder of this chapter is devoted to reviewing several elements of geometric group theory that will aid in comprehending the theory presented in later chapters.

1.1 Finitely generated groups

A subset S of a group G is called a *generating set* of G if every element of G can be expressed as a finite product of elements in $S \cup S^{-1}$ where $S^{-1} = \{x^{-1} : x \in S\}$. G is called *finitely generated* if it has a finite generating set.

Example 1.1.1. (1) The set of integers $\mathbb{Z}$ paired with addition is a finitely generated group. Clearly, the sets $\{1\}$ and $\{-1\}$ each generate $\mathbb{Z}$. Moreover, given coprime integers p and q there exist integers n and m such that $np + mq = 1$. Hence any subgroup of $\mathbb{Z}$ containing p and q must also contain 1 and therefore all of $\mathbb{Z}$. Thus any set of the form $\{p, q\}$ where p and q are coprime integers also generates $\mathbb{Z}$.

(2) While the set of rational numbers $\mathbb{Q}$ under addition is a group, it is not finitely generated.

Suppose that $\mathbb{Q}$ is generated by $\{\frac{n_i}{m_i} : 1 \leq i \leq k, n_i, m_i \in \mathbb{Z}\}$ and set $m = \prod_i m_i$. Then

$\frac{n_i}{m_i} = (m_1 \cdots m_{i-1} m_{i+1} \cdots m_k \cdot n_i) \cdot \frac{1}{m}$, implying that $\frac{n_i}{m_i} \in \langle \frac{1}{m} \rangle$, the subgroup generated by $\frac{1}{m}$. Hence $\mathbb{Q} = \langle \frac{1}{m} \rangle$ where m is an integer, forcing every element of $\mathbb{Q}$ to be of the form $\frac{n}{m}$ where n is an integer.. This is clearly not possible since $\frac{1}{2m} \in \mathbb{Q}$ and there is no integer n such that $\frac{1}{2m} = \frac{n}{m}$.

Given a set X one can always define a group generated by X in the following manner. A *word* in $X \cup X^{-1}$ is any expression of the form $x_1^{\epsilon_1} x_2^{\epsilon_2} \cdots x_k^{\epsilon_k}$, $k \geq 0$, where $x_i \in X$ and $\epsilon_i \in \{\pm 1\}$ for all i . Let 1 denote the *empty word* of $X \cup X^{-1}$, the word of length 0. For convenience we let X^* denote the set of all words in $X \cup X^{-1}$. A word $w \in X^*$ is called *reduced* if it contains no pair of consecutive letters of the form xx^{-1} or $x^{-1}x$. An equivalence relation on X^* is defined by saying $w_1 \sim w_2$ if w_2 can be obtained from w_1 through inserting and deleting finitely many pairs of consecutive elements of the form xx^{-1} or $x^{-1}x$.

Proposition 1.1.2 ([18], Proposition 7.17 p.204). *Any word $w \in X^*$ is equivalent to a unique reduced word.*

Proof. We first show existence by induction on the length of words in X^* . The statement clearly holds for the empty word and any word of length one. So assume it is true for any word of length n and consider $w = x_1 \cdots x_n x_{n+1}$, $x_i \in X \cup X^{-1}$ for $1 \leq i \leq n+1$. Applying the inductive hypothesis to the word $x_2 \cdots x_n x_{n+1}$ yields a reduced word $u = y_1 \cdots y_k$ such that $u \sim x_2 \cdots x_n x_{n+1}$. Then $w \sim x_1 u$. If $x_1 \neq y_1^{-1}$, then the word $x_1 u$ is reduced. If $x_1 = y_1^{-1}$, then $x_1 u \sim y_2 \cdots y_k$ and $y_2 \cdots y_k$ is reduced.

Proving uniqueness is equivalent to showing that if u, v are reduced words satisfying $u \sim v$, then $u = v$. First let $F(X)$ denote the set of reduced words in $X \cup X^{-1}$. For every $a \in X \cup X^{-1}$ define the map $L_a : F(X) \rightarrow F(X)$ by

$$L_a(x_1 \cdots x_k) = \begin{cases} ax_1 \cdots x_k & \text{if } a \neq x_1^{-1} \\ x_2 \cdots x_k & \text{if } a = x_1^{-1}. \end{cases}$$

For every word $w = a_1 \cdots a_n \in X^*$ define $L_w = L_{a_1} \circ \cdots \circ L_{a_n}$ with $L_1 = \text{Id}_{F(X)}$. It is immediate

that $L_w \circ L_{w^{-1}} = \text{Id}_{F(X)}$ for every $w \in X^*$ and $u \sim v$ implies $L_u = L_v$. We now show that $L_w(1) = w$ for every reduced word $w \in F(X)$ by induction on length.

The statement clearly holds for the empty word and words of length one, so assume it is true for all reduced words of length n and let w be a reduced word of length $n + 1$. Then we can write $w = au$ where $a \in X \cup X^{-1}$ and u is a reduced word such that $L_a(u) = au$. It follows from the definition and the inductive hypothesis that $L_w(1) = L_a \circ L_u(1) = L_a(u) = au = w$.

Finally, let u, v be reduced with $u \sim v$. Then $L_u = L_v$. In particular, $L_u(1) = L_v(1)$, hence $u = v$. □

Definition 1.1.3. *The free group with basis X is the set $F(X)$ of reduced words in X^* with the group operation $*$ given by the following definition: $w_1 * w_2$ is the unique reduced word in $X \cup X^{-1}$ that is equivalent to the word $w_1 w_2$.*

Proposition 1.1.4 (Universal Property of Free Groups, [18], Proposition 7.21 p.205). *A map $\phi : X \rightarrow G$ from the set X to a group G can be extended to a unique homomorphism $\Phi : F(X) \rightarrow G$.*

Proof. The map ϕ can be extended to a map $\tilde{\phi}$ on $X \cup X^{-1}$ by defining $\tilde{\phi}(x^{-1}) = \phi(x)^{-1}$. For every word $w = x_1 \cdots x_n$ in $F(X)$ define $\Phi(x_1 \cdots x_n) = \tilde{\phi}(x_1) \cdots \tilde{\phi}(x_n)$. Set $\Phi(1_{F(X)}) = 1_G$ where $1_{F(X)}$ denotes the empty word in $F(X)$ and 1_G the identity element of G . Then given elements $\alpha = a_1 \cdots a_l, \beta = b_1 \cdots b_p \in F(X)$ it follows that

$$\Phi(\alpha\beta) = \Phi(a_1 \cdots a_l b_1 \cdots b_p) = \tilde{\phi}(a_1) \cdots \tilde{\phi}(a_l) \tilde{\phi}(b_1) \cdots \tilde{\phi}(b_p) = \Phi(\alpha)\Phi(\beta),$$

making Φ a homomorphism.

Let $\Psi : F(X) \rightarrow G$ be a homomorphism such that Ψ and ϕ agree on X . Then given $w = x_1 \cdots x_n$ in $F(X)$,

$$\Psi(w) = \Psi(x_1) \cdots \Psi(x_n) = \phi(x_1) \cdots \phi(x_n) = \Phi(w).$$

□

It is often helpful to describe a group in terms of a generating set and a list of “rules” that describe how the generators behave. Let G be a group and H a normal subgroup of G . A subset $K \subset H$ is said to *normally generate* H if H is the smallest normal subgroup of G containing K . If K normally generates H , then we write $\langle\langle K \rangle\rangle = H$.

Let S be a generating set of G . The inclusion map $\iota : S \rightarrow G$ may be extended to a unique homomorphism $\pi_S : F(S) \rightarrow G$ via Proposition 1.1.4. Note that π_S is surjective since S is a generating set of G . Thus, $F(S)/\ker \pi_S \cong G$ by the first isomorphism theorem of group theory. The elements of $\ker \pi_S$ are called *relators* of G . Let $R = \{r_i | i \in I\}$ be a subset of $F(S)$ such that $\langle\langle R \rangle\rangle = \ker \pi_S$. Then the pair (S, R) is called a *presentation for the group* G and we write $G = \langle S | R \rangle$. G is said to be finitely presented if there exists a presentation (S, R) for G such that S and R are both finite.

Example 1.1.5. (1) $\mathbb{Z}^n$ is given by the presentation $\langle a_1, \dots, a_n \mid [a_i, a_j], 1 \leq i \leq n, 1 \leq j \leq n \rangle$.

(2) The finite dihedral group D_{2n} is given by the presentation $\langle x, y \mid x^n, y^2, yxyx \rangle$.

Suppose $G = \langle S | R \rangle$. Then every element of G can be expressed as a word in $S \cup S^{-1}$. Given a word $w = x_1 \cdots x_n$ in $S \cup S^{-1}$, it is the case that w is equal to the identity in G if and only if w , when considered as an element of $F(S)$, can be written as a product of finitely many conjugates of words in R . The following lemma provides a useful method for constructing homomorphisms that are needed in later sections.

Lemma 1.1.6 ([18], Lemma 7.26 p.207). *Let G have presentation $G = \langle S | R \rangle$ and let H be a group. Suppose $\phi : S \rightarrow H$ is a map such that $\phi(r) = 1$ for every $r \in R$. Then ϕ extends to a homomorphism $\Psi : G \rightarrow H$.*

Proof. By Proposition 1.1.4 the map ϕ extends to a homomorphism $\Psi : F(S) \rightarrow H$. The elements of $\langle\langle R \rangle\rangle$ are finite products of elements of the form grg^{-1} where $g \in F(S), r \in R$. Thus,

$$\Psi(grg^{-1}) = \Psi(g)\Psi(r)\Psi(g)^{-1} = \Psi(g)\phi(r)\Psi(g)^{-1} = \Psi(g)\Psi(g)^{-1} = 1_H.$$

Therefore $\langle\langle R \rangle\rangle \subset \ker \Psi$ and the desired result follows. $\square$

1.2 Cayley graphs

Let G be a finitely generated group and S a finite generating set for G . One can endow G with a metric space structure through defining the *Cayley graph of G with respect to S* , which is denoted here as $\text{Cay}(G, S)$. $\text{Cay}(G, S)$ is the graph whose vertex set is G and edge set consists of all pairs of elements (g, h) such that $h = gs$ for some $s \in S \cup S^{-1}$. That is, there is an edge joining the vertices labeled g and h if and only if there exists $s \in S \cup S^{-1}$ such that $h = gs$ in G . $\text{Cay}(G, S)$ is equipped with the standard length metric, assigning each edge unit length. The *word metric associated to S* , denoted by d_S , is the restriction of the standard metric on $\text{Cay}(S, G)$ to G . We denote $d_S(1, g)$ by $|g|_S$. Note that $|g|_S$ is the minimal length of a word in $S \cup S^{-1}$ representing g .

Suppose $g, h \in G$. g can be expressed as a reduced word $g = x_1 \cdots x_n$ with $x_i \in S \cup S^{-1}$ for all i . The sequence $h, hx_1, hx_1x_2, \dots, hx_1 \cdots x_{n-1}, hg$ is an edge path in $\text{Cay}(G, S)$ joining the elements h and hg . It follows that $|g|_S \geq d_S(h, hg)$. On the other hand, suppose $h = z_0, z_1, \dots, z_{t-1}, z_t = hg$ is an edge path joining h and hg . Then each z_j and z_{j+1} differ only by multiplication on the right by some $s \in S \cup S^{-1}$. As a result, the sequence $1, h^{-1}z_1, h^{-1}z_2, \dots, h^{-1}z_t, g$ is an edge path in $\text{Cay}(G, S)$ joining 1 and g , demonstrating that $d_S(h, hg) \geq |g|_S$. Therefore, $|g|_S = d_S(h, hg)$, showing that d_S is left-invariant.

Example 1.2.1. (1) Consider the free group on two generators, denoted by F_2 , with basis $S = \{a, b\}$. The Cayley graph $\text{Cay}(F, X)$ is the infinite 4-valent tree depicted in Figure 1.1.

(2) As stated in Example 1.1.5, $\mathbb{Z}^2$ is given by the presentation $\mathbb{Z}^2 = \langle a, b \mid [a, b] \rangle$. The resulting Cayley graph, $\text{Cay}(G, \{a, b\})$ is a square grid in the Euclidean plane. See Figure 1.1.

Proposition 1.2.2. *Let S be a finite generating set for the group G . Then $\text{Cay}(G, S)$ is connected.*

Proof. It suffices to show that there exists an edge path in $\text{Cay}(G, S)$ from 1 to g for every $g \in G$. Let $g \in G$ and express g as a reduced word $g = x_1 \cdots x_n$, $x_i \in S \cup S^{-1}$. Then $1, x_1, x_1x_2, \dots, x_1 \cdots x_{n-1}, g$ is an edge path in $\text{Cay}(G, S)$ joining g and 1. $\square$

Figure 1.1: Cayley graph of $\mathbb{Z}^2$ (left) and Cayley graph of F_2 (right).

Remark 1.2.3. The Cayley graph has several useful properties when considered as a metric space.

Let S be a finite generating set for the group G .

- (1) $\text{Cay}(G, S)$ is a *geodesic metric space*. A *geodesic metric space* is a metric space X in which any pair of points x, y can be joined by an isometric embedding $\alpha : [0, d(x, y)] \rightarrow X$ such that $\alpha(0) = x$ and $\alpha(d(x, y)) = y$. Some additional details regarding geodesics can be found in Section 2.1.
- (2) Let $x \in G$, $r \geq 0$. Then the closed ball $\bar{B}(x, r) = \{g \in G : d_S(g, x) \leq r\}$ is finite (due to the construction of the Cayley graph and the finiteness of S) and therefore compact. Thus, $\text{Cay}(G, S)$ is a proper metric space.

It should be emphasized that the definition of d_S depends on the choice of generating set. In general, different generating sets will produce non-isomorphic Cayley graphs and thus induce different word metrics on G . As a result d_S fails to serve as a well-defined metric associated to G independent of the choice of S . However, by exploring the coarse geometry of G one can hope to at least partially address this issue.

Let S and T be different finite generating sets for the group G . Set $\lambda = \max_{s \in S} \{d_T(1_G, s)\}$ and $\lambda' = \max_{t \in T} \{d_S(1_G, t)\}$. Let $g, h \in G$. Then $d_S(g, h) = n$ and $d_T(g, h) = m$ for some n and m . As $d_S(g, h) = n$, there exist $s_1, \dots, s_n \in S \cup S^{-1}$ such that $h = gs_1 \cdots s_n$. Thus,

$$\begin{aligned}
d_T(g, h) &= d_T(g, gs_1 \cdots s_n) \\
&= d_T(1_G, s_1 \cdots s_n) \quad \text{by left invariance of } d_T \\
&\leq \sum_{i=1}^n d_T(1_G, s_i) \quad \text{by triangle inequality and left invariance} \\
&\leq \lambda n = \lambda d_S(g, h).
\end{aligned}$$

As $d_T(g, h) = m$, there exist $t_1, \dots, t_m \in T \cup T^{-1}$ such that $h = gt_1 \cdots t_m$. Through a similar calculation it follows that $d_S(g, h) \leq \lambda' d_T(g, h)$. Therefore, $\Lambda^{-1} d_S(g, h) \leq d_T(g, h) \leq \Lambda d_S(g, h)$ for every $g, h \in G$ where $\Lambda = \max\{\lambda, \lambda'\}$. The metric spaces (G, d_S) and (G, d_T) are then bi-Lipschitz equivalent under the identity map. As a slight abuse of notation, we will often say that all word metrics on a finitely generated group G are bi-Lipschitz equivalent.

1.3 Quasi-isometries

An *isometry* between metric spaces is a surjective, distance preserving map. Given different finite generating sets S and T for a group G there is no reason to expect the metric spaces (G, d_S) and (G, d_T) to be isometric. However, the two are coarsely related in the following sense.

Definition 1.3.1. Let (X, d_X) and (Y, d_Y) be metric spaces and $f : X \rightarrow Y$ a map.

(1) f is called a (λ, ϵ) -**quasi-isometric embedding**, $\epsilon \geq 0, \lambda \geq 1$, if

$$\lambda^{-1} d_X(x, x') - \epsilon \leq d_Y(f(x), f(x')) \leq \lambda d_X(x, x') + \epsilon$$

for all $x, x' \in X$.

(2) f is called **coarsely surjective** if there exists $\epsilon \geq 0$ so that for any $y \in Y$ there exists $x_y \in X$ satisfying $d_Y(f(x_y), y) \leq \epsilon$.

(3) A (λ, ϵ) -**quasi-isometry** from X to Y is a coarsely surjective (λ, ϵ) -quasi-isometric embedding.

Two spaces are said to be **quasi-isometric** if there exists a (λ, ϵ) -quasi-isometry between them for some $\lambda \geq 1, \epsilon \geq 0$.

Example 1.3.2. (1) Given $u, v \in \mathbb{R}^2$, $u \neq 0$, the map $t : \mathbb{R} \rightarrow \mathbb{R}^2$ defined by $t \mapsto tu + v$ is a $(\|u\|, 0)$ -quasi-isometric embedding, however it is not coarsely surjective.

(2) The natural inclusion map $\iota : \mathbb{Z} \hookrightarrow \mathbb{R}$ is a $(1, 1/2)$ -quasi-isometry.

Quasi-isometries are, in general, poorly behaved maps that are often not even continuous. In spite of this, their defining properties do allow them to preserve some structure between metric spaces.

Proposition 1.3.3. *The relation $\sim$ given by $X \sim Y$ if and only if X and Y are quasi-isometric defines an equivalence relation on metric spaces.*

Proof. Reflexive: The identity map is clearly a quasi-isometry from X into itself, hence $X \sim X$.

Transitive: Given X, Y, Z such that $X \sim Y$ and $Y \sim Z$, there is a (λ, ϵ) -quasi-isometry $f : X \rightarrow Y$ and a (λ', ϵ') -quasi-isometry $g : Y \rightarrow Z$. Then

$$\lambda^{-1}d_X(a, b) - \epsilon \leq d_Y(f(a), f(b)) \leq \lambda'(d_Z(g \circ f(a), g \circ f(b)) + \epsilon')$$

and

$$\lambda'^{-1}(d_Z(g \circ f(a), g \circ f(b)) - \epsilon') \leq d_Y(f(a), f(b)) \leq \lambda d_X(a, b) + \epsilon$$

for all $a, b \in X$. It follows that the composition $g \circ f : X \rightarrow Z$ is a quasi-isometric embedding.

Given $z \in Z$ there exists $y \in Y$ such that $d_Z(g(y), z) \leq \epsilon'$ and there exists $x \in X$ such that $d_Y(f(x), y) \leq \epsilon$. Thus,

$$\begin{aligned} d_Z(g \circ f(x), z) &\leq d_Z(g \circ f(x), g(y)) + d_Z(g(y), z) && \text{by triangle inequality} \\ &\leq \lambda' d_Y(f(x), y) + \epsilon' + \epsilon' && \text{since } g \text{ is a } (\lambda', \epsilon')\text{-quasi-isometry} \\ &\leq \lambda' \epsilon + 2\epsilon' && \text{since } f \text{ is a } (\lambda, \epsilon)\text{-quasi-isometry.} \end{aligned}$$

Hence $g \circ f$ is a quasi-isometry and $X \sim Z$.

Symmetric: Suppose $X \sim Y$. Then there is a (λ, ϵ) -quasi-isometry $f : X \rightarrow Y$. By definition, for each $y \in Y$ there exists $x_y \in X$ such that $d_Y(f(x_y), y) \leq \epsilon$. Define $g : Y \rightarrow X$ by $g(y) = x_y$.

For each $x \in X$ we have

$$d_X(x, g \circ f(x)) = d_X(x, x_{f(x)}) \leq \lambda d_Y(f(x), f(x_{f(x)})) + \lambda \epsilon \leq 2\lambda \epsilon$$

because f is (λ, ϵ) -quasi-isometry, hence g is coarsely surjective.

Now let $a, b \in Y$. Then

$$\begin{aligned} d_X(g(a), g(b)) &= d_X(x_a, x_b) \\ &\leq \lambda(d_Y(f(x_a), f(x_b)) + \epsilon) && \text{since } f \text{ is a } (\lambda, \epsilon)\text{-quasi-isometry} \\ &\leq \lambda(d_Y(f(x_a), a) + d_Y(a, b) + d_Y(b, f(x_b)) + \epsilon) && \text{by triangle inequality} \\ &\leq \lambda(3\epsilon + d_Y(a, b)) = \lambda d_Y(a, b) + 3\lambda \epsilon && \text{because } d_Y(f(x_a), a) \leq \epsilon. \end{aligned}$$

Similarly,

$$\begin{aligned} d_X(g(a), g(b)) &= d_X(x_a, x_b) \\ &\geq \lambda^{-1}(d_Y(f(x_a), f(x_b)) - \epsilon) \\ &\geq \lambda^{-1}(d_Y(f(x_a), b) - d_Y(b, f(x_b)) - \epsilon) \\ &\geq \lambda^{-1}(d_Y(a, b) - d_Y(f(x_a), a) - d_Y(b, f(x_b)) - \epsilon) \\ &\geq \lambda^{-1}d_Y(a, b) - 3\lambda^{-1}\epsilon. \end{aligned}$$

Thus g defines a quasi-isometry from Y to X and $Y \sim X$. □

Remark 1.3.4. The map g defined when verifying that the above relation is symmetric is often referred to as a *quasi-inverse* for the quasi-isometry f .

A common theme of geometric group theory is the study of group actions on metric spaces. The properties of the action together with the choice of appropriate metric spaces can often reveal useful information about the group. One type of action in particular, namely *geometric actions*, plays a central role in later chapters.

Definition 1.3.5. Let X be a metric space and G a group. Suppose $\Phi : G \rightarrow \text{Sym } X$ defines a group action of G on X by isometries, meaning $\Phi(G) \subset \text{Isom } X$. Φ is called **properly discontinuous** if for any compact set $C \subset X$ the set $\{g \in G : g(C) \cap C \neq \emptyset\}$ is finite. Φ is called **cocompact** if there is some compact set $C \subset X$ such that for any $x \in X$ there exists $g_x \in G$ satisfying $x \in g_x(C)$. G acts **geometrically** on X if the action is both properly discontinuous and cocompact.

Theorem 1.3.6 ([18], Theorem 8.37 p.262). (*The Milnor-Schwarz Theorem*) Let (X, d) be a proper geodesic metric space and G a group acting geometrically on X . Then G is finitely generated and for any word metric d_S on G and any point $x \in X$ the orbit map $\phi : G \rightarrow X$ given by $g \rightarrow gx$ is a quasi-isometry.

Proof. For convenience let G_y denote the orbit of the point $y \in X$ and $GA = \bigcup_{a \in A} G_a$. As the action $G \curvearrowright X$ is geometric, there exists a compact set C such that $GC = X$. Hence, there exists a closed ball B with radius D satisfying $GB = X$. B is compact since X is proper and so the set $S = \{s \in G : sB \cap B \neq \emptyset\}$ is finite. Moreover, $1 \in S$ by definition of S , showing that S is nonempty. We show that S generates G .

If $S = G$, then we're done. So suppose $S \neq G$. Define $c := 2^{-1} \inf\{d(B, gB) : g \in G \setminus S\}$ and fix $g \in G \setminus S$. By the definition of S the distance $d(B, gB)$ is a positive constant, say R . The set $H = \{h \in G : d(B, hB) \leq R\}$ is finite as it is contained in the finite set $\{g \in G : g\bar{B}(x, D+R) \cap \bar{B}(x, D+R) \neq \emptyset\}$. Now, $\inf\{d(B, gB) : g \in G \setminus S\} = \inf\{d(B, gB) : g \in H \setminus S\}$ where the latter infimum is taken over finitely many positive numbers. Thus there exists $h \in H \setminus S$ such that $d(B, hB)$ realizes this infimum, forcing it to be a positive value. Then $d(B, g'B) < 2c$ implies that $g' \in S$ by definition.

Let γ be a geodesic in X joining x and gx and define k to be the floor of the value $c^{-1}d(x, gx)$. Then there exists a finite sequence of points on γ , say $y_0 = x, y_1, \dots, y_{k+1} = gx$, such that $d(y_i, y_{i+1}) \leq c$ for every $i \in \{0, \dots, k\}$. For each i choose $h_i \in G$ such that $y_i \in h_i B$. Set $h_0 = 1_G$ and $h_{k+1} = g$. Since $d(B, h_i^{-1}h_{i+1}B) = d(h_i B, h_{i+1}B) \leq d(y_i, y_{i+1}) \leq c$, it follows that $h_i^{-1}h_{i+1} = s_i \in S$, i.e., $h_{i+1} = h_i s_i$. Then $g = h_{k+1} = s_0 s_1 \cdots s_k$ and it follows that G is

generated by S .

Fix $x_0 \in X$. We now prove that when G is equipped with any word metric the orbit map $\phi : G \rightarrow X$ defined by $\phi(g) = gx_0$ is a quasi-isometry. As all word metrics on G are bi-Lipschitz equivalent, it suffices to prove this statement when G is equipped with d_S where S is the finite generating set constructed above. The map ϕ is clearly coarsely surjective since $GB = X$, so all that remains is to verify that ϕ is a quasi-isometric embedding.

Let $g \in G$ and let k be the floor of the value $c^{-1}d(x_0, gx_0)$. As already seen, $|g|_S \leq k + 1$, hence $|g|_S \leq c^{-1}d(x_0, gx_0) + 1$, implying that $c|g|_S - c \leq d(x_0, gx_0)$. Let $L = \max\{d(x_0, sx_0) : s \in S\}$. By applying the triangle inequality it follows that $d(x_0, gx_0) \leq L|g|_S$. Hence $c|g|_S - c \leq d(x_0, gx_0) \leq L|g|_S$ for all $g \in G$. This is equivalent to saying that

$$cd_S(1_G, g) - c \leq d(x_0, gx_0) \leq Ld_S(1_G, g) \text{ for all } g \in G. \quad (1.3.7)$$

Since G acts on X by isometries, we have $d(gx_0, hx_0) = d(x_0, g^{-1}hx_0)$ for all $g, h \in G$. Combining this fact with the left invariance of d_S allows us to replace 1_G in equation 1.3.7 with any element $h \in G$, concluding that ϕ is a (λ, ϵ) -quasi-isometric embedding where $\epsilon = c$ and $\lambda = \max\{c, L\}$.

□

Corollary 1.3.8. *Let G be a group and X_i , $i = 1, 2$, proper geodesic metric spaces such that the action of G on each X_i is geometric. Then X_1, X_2 are quasi-isometric.*

Proof. Fix $x_i \in X_i$ for $i = 1, 2$. G is finitely generated by Theorem 1.3.6, so fix a word metric d_S on G with respect to some finite generating set S of G . The orbit maps $f_i : G \rightarrow X_i$ defined by $f_i(g) = gx_i$ are then quasi-isometries, also by Theorem 1.3.6. Let $\bar{f}_i$, $i = 1, 2$, denote their quasi-inverses. Then the map $f_2 \circ \bar{f}_1$ is a quasi-isometry from X_1 to X_2 .

□

Proposition 1.3.3 allows one to classify metric spaces *up to quasi-isometry*. While this may not seem to be a natural approach for general metric spaces (Example 1.3.2 says that $\mathbb{Z}$ and $\mathbb{R}$ as metric spaces lie in the same quasi-isometry class), it does help to address the previously mentioned concerns associated to the word metric of finitely generated groups.

Definition 1.3.9. Two finitely generated groups G, H are said to be **quasi-isometric** if there exist finite generating sets S and T of G and H respectively such that $\text{Cay}(G, S)$ and $\text{Cay}(H, T)$ are quasi-isometric as metric spaces.

Example 1.3.10. (1) Any isomorphism of groups induces a quasi-isometry of Cayley graphs. Hence, any two isomorphic groups are quasi-isometric.

(2) Let H be a finite index subgroup of a finitely generated group G . Then G and H are quasi-isometric. This is a direct application of Theorem 1.3.6 where X is a Cayley graph of G .

(3) Let N be a finite normal subgroup of G . Then applying Theorem 1.3.6 to the action of G on a Cayley graph of G/N implies that G and G/N are quasi-isometric.

Proposition 1.3.11. Let S be a finite generating set for the group G and $X = \text{Cay}(G, S)$. Then G acts geometrically on X .

Proof. Throughout this proof let d denote the word metric on G with respect to S . Denote the open (respectively closed) ball with center x and radius r by $B(x, r)$ ($\bar{B}(x, r)$). As previously noted, for each $x \in X$ and $r \geq 0$ any ball centered at x with radius r contains finitely many elements due to the construction of the $\text{Cay}(G, S)$.

Since d is left invariant, G acts on X isometrically by left multiplication. This action is clearly cocompact as the closed unit ball $\bar{B}(1_G, 1)$ is a compact subset of X satisfying $G \cdot \bar{B}(1_G, 1) = X$. All that remains is to verify that this action is properly discontinuous. Let $C \subset X$ be compact. Hence there is $r > 0$ such that $C \subset B(1_G, r)$.

Suppose $g \in G$ satisfies $gC \cap C \neq \emptyset$. Then there exists $x, y \in C$ such that $gx = y$. Thus, $d(1_G, g) \leq d(1_G, y) + d(y, g) = d(1_G, y) + d(gx, g) = d(1_G, y) + d(1_G, x) \leq r + r = 2r$. Therefore $g \in B(1, 2r)$, i.e., $\{g \in G : gC \cap C \neq \emptyset\} \subset B(1, 2r)$ where the latter set is finite.

□

Corollary 1.3.12. Let S and T be different finite generating sets for the group G . Then $\text{Cay}(G, S)$ and $\text{Cay}(G, T)$ are quasi-isometric.

Proof. By the previous proposition G acts geometrically on each of $\text{Cay}(G, S)$ and $\text{Cay}(G, T)$.

The desired result follows from Corollary 1.3.8. $\square$

CHAPTER 2 SPACES OF NONPOSITIVE CURVATURE

2.1 Geodesics

Definition 2.1.1. Let (X, d) be a metric space.

- (1) A **geodesic** from x to y is a map c from a (possibly infinite) interval $[0, r] \subset \mathbb{R}$ to X such that $c(0) = x$, $c(r) = y$, and $d(c(t), c(t')) = |t - t'|$ for all t, t' in $[0, r]$. The image of c is called a **geodesic segment** (or **geodesic path**) with endpoints x and y .
- (2) X is called a **geodesic metric space** if every two points in X are joined by a geodesic segment in X .
- (3) Let $r > 0$. X is said to be **r -geodesic** if for every pair of points $x, y \in X$ with $d(x, y) < r$ there exists a geodesic joining x to y .

Let $[xy]$ denote the geodesic segment with endpoints x and y , though this is a slight abuse of notation as there may be multiple such geodesic paths.

Example 2.1.2. (1) Euclidean space, denoted by $\mathbb{E}^n$ and formed from equipping the real vector space $\mathbb{R}^n$ with the standard Euclidean norm, is a geodesic metric space where the geodesic segments are subsets of the form $[xy] = \{ty + (1 - t)x : 0 \leq t \leq 1\}$.

- (2) Any normed space V equipped with the metric $d(u, v) = \|u - v\|$ is a geodesic metric space as the map $t \rightarrow (1 - t)u + tv$ defines a path from $[0, 1] \rightarrow V$.

2.2 The model space $\mathbb{X}_\kappa^n$

For each positive integer n there are three standard classes of metric spaces which serve as comparative models for more general geodesic spaces. They are Euclidean n -space $\mathbb{E}^n$, the n -sphere $\mathbb{S}^n$, and hyperbolic n -space $\mathbb{H}^n$. There is an abundance of source material regarding the construction of each of these spaces ([9] and [18] for example), hence only a brief description of each is provided here.

Euclidean n -space is obtained from equipping the real vector space $\mathbb{R}^n$ with the metric induced by the inner product $\langle x, y \rangle = \sum_{i=1}^n x_i y_i$. The resulting norm, given by $\|x\| = \sqrt{\langle x, x \rangle}$, is called the Euclidean norm and in turn induces the Euclidean metric $d(x, y) = \|x - y\|$. As previously mentioned, the geodesic segments in $\mathbb{E}^n$ are precisely the straight line paths, i.e., sets of the form $\{ty + (1 - t)x : 0 \leq t \leq 1\}$.

The n -sphere is the subset of $\mathbb{E}^{n+1}$ consisting of all vectors with Euclidean norm 1. That is, $\mathbb{S}^n = \{x \in \mathbb{E}^{n+1} : \|x\| = 1\}$. $\mathbb{S}^n$ inherits the inner product from $\mathbb{E}^{n+1}$ but is equipped with the metric $d_{\mathbb{S}}$ which assigns to each pair $(a, b) \in \mathbb{S}^n \times \mathbb{S}^n$ the unique real number $d_{\mathbb{S}}(a, b) \in [0, \pi]$ satisfying $\cos(d_{\mathbb{S}}(a, b)) = \langle a, b \rangle$. A *great circle* in $\mathbb{S}^n$ is defined to be the intersection of $\mathbb{S}^n$ with a 2-dimensional vector subspace of $\mathbb{E}^{n+1}$. The geodesics in $\mathbb{S}^n$ are the *minimal great arcs*, which are defined to be subarcs of great circles of $\mathbb{S}^n$ with length at most π .

There are three main models of hyperbolic space with which the reader may be familiar: the hyperboloid, the upper half plane, and the Poincaré disk. The latter two models are typically defined in the plane and are then generalized to higher dimensions. The hyperboloid model is sufficient for our needs and is constructed as follows.

Let $\mathbb{E}^{n,1}$ denote $\mathbb{R}^{n+1}$ paired with the symmetric bilinear form defined by $\langle x, y \rangle_{\mathbb{H}} = -x_{n+1}y_{n+1} + \sum_{i=1}^n x_i y_i$. The space $\mathbb{H}^n$ is defined to be $\{x \in \mathbb{E}^{n,1} : \langle x, x \rangle_{\mathbb{H}} = -1, x_{n+1} > 0\}$. That is, $\mathbb{H}^n$ is the upper sheet of the hyperboloid $\{x \in \mathbb{E}^{n,1} : \langle x, x \rangle_{\mathbb{H}} = -1\}$. The metric associated to $\mathbb{H}^n$ is the function $d_{\mathbb{H}}$ which assigns to each pair $(a, b) \in \mathbb{H}^n \times \mathbb{H}^n$ the unique non-negative number $d_{\mathbb{H}}(a, b)$ satisfying $\cosh(d_{\mathbb{H}}(a, b)) = -\langle a, b \rangle_{\mathbb{H}}$. The geodesics in $\mathbb{H}^n$ are the *hyperbolic segments*, compact subarcs of the intersections of $\mathbb{H}^n$ with 2-dimensional subspaces of $\mathbb{E}^{n,1}$.

Definition 2.2.1. Let $\kappa \in \mathbb{R}$. $\mathbb{X}_{\kappa}^n$ denotes the following metric spaces:

- (1) $\mathbb{X}_0^n = \mathbb{E}^n$;
- (2) if $\kappa > 0$, then $\mathbb{X}_{\kappa}^n$ is obtained from $\mathbb{S}^n$ by scaling the metric by the constant $1/\sqrt{\kappa}$;
- (3) if $\kappa < 0$, then $\mathbb{X}_{\kappa}^n$ is obtained from $\mathbb{H}^n$ by scaling the metric by the constant $1/\sqrt{-\kappa}$.

Let D_{κ} denote the **diameter** of $\mathbb{X}_{\kappa}^2$. That is, $D_{\kappa} := \pi/\sqrt{\kappa}$ for $\kappa > 0$ and $D_{\kappa} := \infty$ for $\kappa \leq 0$.

2.3 CAT(κ) spaces

Let X be a metric space. A *geodesic triangle* $\Delta(p, q, r)$ in X is the union of the geodesics $[pq]$, $[qr]$, and $[rp]$ together with the vertices p, q , and r . A *comparison triangle* $\Delta^* = \Delta^*(p, q, r)$ for the geodesic triangle $\Delta(p, q, r)$ is a triangle in $\mathbb{X}_\kappa^2$ with the same edge lengths as $\Delta(p, q, r)$. The vertices of Δ^* are denoted by p^*, q^*, r^* . A point $x^* \in [p^*q^*]$ is called a *comparison point* for $x \in [pq]$ if $d(q, x) = d(q^*, x^*)$.

Lemma 2.14 p.24 of [9] says that a comparison triangle for a geodesic triangle Δ always exists when $\kappa \leq 0$. When $\kappa > 0$ such a triangle exists if and only if $l(\Delta) \leq 2D_\kappa$ where $l(\Delta)$ denotes the perimeter of Δ . If Δ^* is a comparison triangle for Δ , then for each edge of Δ there exists a well-defined isometry which maps the given edge of Δ onto the corresponding edge of Δ^* of the same length.

Definition 2.3.1. Let X be a metric space and Δ a geodesic triangle in X with $l(\Delta) < 2D_\kappa$. Let $\Delta^* \subset \mathbb{X}_\kappa^2$ be a comparison triangle for Δ . Then Δ satisfies the **CAT(κ) inequality** if for all $x, y \in \Delta$ and all comparison points $x^*, y^* \in \Delta^*$ the inequality $d(x, y) \leq d(x^*, y^*)$ holds.

If $\kappa \leq 0$, then X is called a **CAT(κ) space** if X is a geodesic space where each geodesic triangle satisfies the CAT(κ) inequality.

If $\kappa > 0$, then X is called a **CAT(κ) space** if X is D_κ -geodesic and every geodesic triangle in X with perimeter less than $2D_\kappa$ satisfies the CAT(κ) inequality.

The terminology “CAT(κ)” was introduced by Gromov [21] and the initials are chosen to represent E. Cartan, A.D. Alexandrov, and V.A. Toponogov as each of them considered similar conditions in varying degrees of generality.

Proposition 2.3.2 ([15], Theorem I.2.5 p.501). Let X be a CAT(κ) space. If $\kappa \leq 0$, then there is a unique geodesic between any two points of X . If $\kappa > 0$, the geodesic segment between any two points of distance less than D_κ is unique.

Proof. Let $x, y \in X$ (with $d(x, y) < D_\kappa$ if $\kappa > 0$). As X is CAT(κ) there exists some geodesic path from x to y . So suppose there exist two distinct geodesic segments γ_1, γ_2 between x and y .

Then there exists some $[a, b] \subset \mathbb{R}$ such that $\gamma_1(t) \neq \gamma_2(t)$ for all $t \in [a, b]$. Choose $c \in [a, b]$ and set $z = \gamma_1(c)$. Then the geodesic triangle in X with sides γ_2 , $[xz]$, and $[zy]$ has comparison triangle in $\mathbb{X}_\kappa^2$ which degenerates to a single geodesic segment. The $\text{CAT}(\kappa)$ inequality then says that γ_1 and γ_2 must be equal. $\square$

Example 2.3.3. (1) $\mathbb{X}_\kappa^n$ is $\text{CAT}(\kappa)$ for each real number κ .

(2) A circle is $\text{CAT}(\kappa)$, $\kappa > 0$, if and only if its circumference is at least $2D_\kappa$.

(3) If X is a $\text{CAT}(\kappa)$ space, then it is $\text{CAT}(\kappa')$ for every $\kappa' \geq \kappa$. ([9], Theorem 1.12 p.165)

Definition 2.3.4. A metric space X is said to have **curvature** $\leq \kappa$ if X satisfies the $\text{CAT}(\kappa)$ inequality locally. That is, for every $x \in X$ there exists $r_x > 0$ such that the ball $B(x, r_x)$ under the induced metric is a $\text{CAT}(\kappa)$ space.

Properties of spaces of nonpositive curvature play important roles throughout the pages to come. In particular, it is helpful to describe some criteria for recognizing when a given cell complex is a $\text{CAT}(\kappa)$ space.

2.4 Gromov hyperbolic spaces

There are several equivalent ways of formulating hyperbolic groups. The following definition is attributed to Rips.

Definition 2.4.1. Let X be a metric space and $\delta > 0$. A geodesic triangle $\Delta \subset X$ is called **δ -slim** if each of its sides is contained in the δ -neighborhood of the union of the other two sides. A geodesic metric space X is called **δ -hyperbolic** if every geodesic triangle in X is δ -slim.

Example 2.4.2. (1) Every geodesic metric space of diameter $D \leq \delta < \infty$ is δ -hyperbolic.

(2) If $\kappa < 0$, then every $\text{CAT}(\kappa)$ space is δ_κ -hyperbolic where δ depends only on κ ([9] Proposition 1.2 p.399). In particular, $\mathbb{H}^n$ is $\text{arccosh}(\sqrt{2})$ -hyperbolic ([18] Example 11.7).

Remark 2.4.3. The above definition of hyperbolicity relies on geodesics while it is well-known that not every metric space is geodesic (simply consider $\mathbb{R}^2$ with the origin removed). A less

intuitive, though more general, definition due to Gromov applies to any metric space. It relies on the Gromov product which, roughly speaking, is a measure of how far the triangle inequality for a triple $x, y, p \in X$ is from being an equality. In general, however, hyperbolicity in the sense of Gromov is not invariant under quasi-isometries (see Example 11.36 in [18]).

Theorem 2.4.4 ([9], Theorem 1.9 p.402). *Let X, X' be geodesic metric spaces and $f : X \rightarrow X'$ a (λ, ϵ) -quasi-isometry. If X is δ -hyperbolic, then X' is δ' -hyperbolic where δ' depends only on δ, λ , and ϵ .*

The proof of this theorem requires a few results regarding the stability of *quasi-geodesics* in δ -hyperbolic spaces that are not necessary for our discussion and is thus omitted here.

Definition 2.4.5. *Let G be a finitely generated group. G is called **hyperbolic** if one of its Cayley graphs is hyperbolic.*

Let G be a finitely generated group. The concept of a hyperbolic group is well-defined as hyperbolicity is invariant under quasi-isometries and any two Cayley graphs of G are quasi-isometric. Thus, G is hyperbolic if and only if all of its Cayley graphs are hyperbolic. In particular, if G' is a finitely generated group quasi-isometric to G , then G is hyperbolic if and only if G' is hyperbolic. In view of the Milnor-Schwarz Theorem (Theorem 1.3.6), if G acts geometrically on a hyperbolic metric space, then G is hyperbolic.

2.5 Gromov's link condition

Let A be an index set. A *CW complex* is a Hausdorff space X constructed via successively attaching cells to a union of lower-dimensional cells. Let X^0 be a discrete set. The points of X^0 are regarded as *0-cells*. Inductively, the *n-skeleton* X^n is formed from X^{n-1} by attaching n -cells e_α^n via maps $\phi_\alpha : S^{n-1} \rightarrow X^{n-1}$ which map the interior of D^n homeomorphically onto e_α^n . The restriction of ϕ_α to S^{n-1} is the *attaching map* for e_α^n . This inductive process can either be stopped at a finite stage, setting $X = X^n$, or continued indefinitely with $X = \bigcup_n X^n$. X is equipped with the weak topology, meaning a set $U \subset X$ is open (closed) in X if and only if $U \cap X^n$ is open (closed) for all n . A *subcomplex* of a CW-complex X is a subspace $A \subset X$ such that the closure

of each cell in A is also contained in A . A subcomplex A is itself a CW complex since the image of the attaching map of each cell in A is contained in A .

Definition 2.5.1. A **convex polytope** in $\mathbb{X}_\kappa^n$ is the convex hull of a finite set of points in $\mathbb{X}_\kappa^n$. Equivalently, a convex polytope is a compact intersection of a finite number of half spaces. The **dimension** of a convex polytope is the dimension of the affine subspace it spans.

A 0, 1, 2-dimensional convex polytope is a point, interval, polygon respectively. Let P be a convex polytope and H a hyperplane in $\mathbb{X}_\kappa^n$ such that $P \cap H \neq \emptyset$ and P is contained in one of the two closed half-spaces bounded by H . Then $P \cap H$ is itself a convex polytope called a *face* of P . A 0-dimensional face of P is a *vertex* and a 1-dimensional face is an *edge* of P .

Example 2.5.2. (1) A 1-dimensional cell complex, i.e., $X = X^1$, is simply a graph consisting of vertices and edges where two ends of an edge can be attached to the same vertex.

(2) The sphere is a CW complex with two cells, e^0 and e^n . The n -cell e^n is attached by the constant map $S^{n-1} \rightarrow e^0$, regarding S^n as the quotient space $D^n/\partial D^n$.

(3) A *simplicial complex* consists of a vertex set V and a collection X of finite subsets of V with the following properties:

- (i) $\{v\} \in X$ for all $v \in V$.
- (ii) If $\Delta \in X$ and $\Delta' \subset \Delta$, then $\Delta' \in X$.

An element of X is called a *simplex*. If Δ' is a simplex properly contained in Δ , then Δ' is called a *face* of Δ .

Definition 2.5.3. A **polyhedral complex** X is a CW complex where each n -cell is metrized as a convex, compact polytope in $\mathbb{X}_\kappa^n$ such that the metrics agree on intersections of closed cells. A polyhedral complex is called *spherical*, *Euclidean*, or *hyperbolic* if κ is positive, 0, or negative respectively.

The elements of a polyhedral complex X are typically referred to as cells instead of polytopes. Simple examples of polyhedral complexes are tessellations of $\mathbb{X}_\kappa^n$ by regular convex polygons.

Let X be a polyhedral complex and v a vertex of X . The *link of v in X* , denoted by $\text{Lk}(v) = \text{Lk}(v, X)$, is the spherical polyhedral complex obtained from intersecting X with a small sphere centered at v . In the 2-dimensional case $\text{Lk}(v)$ is a graph with a vertex corresponding to each 1-cell of X incident to v and an edge between a pair of vertices if both of their corresponding edges lie in some 2-cell of X .

Definition 2.5.4 (Gromov's link condition). *A polyhedral complex X is said to satisfy the **link condition** if $\text{Lk}(v)$ is a $\text{CAT}(1)$ space for every vertex v of X .*

Theorem 2.5.5 ([36], Theorem 11). *Let X be a simply connected polyhedral complex.*

1. *If X is Euclidean, then X is $\text{CAT}(0)$ if and only if X satisfies the link condition.*
2. *If X is hyperbolic, then X is $\text{CAT}(-1)$ if and only if X satisfies the link condition.*

Theorem 2.5.6 ([9], Lemma 5.6 p.207). *A 2-dimensional $\mathbb{X}^n$ -polyhedral complex X satisfies the link condition if and only if for each vertex $v \in X$ every cycle in $\text{Lk}(v)$ has length at least 2π .*

CHAPTER 3 RIGHT-ANGLED COXETER GROUPS

This chapter discusses some of the theory involved in examining RACGs in addition to providing an overview of recent results in the quasi-isometric classification of this particular family of groups. Those interested in a more detailed treatment of the theory of Coxeter groups should consult the book by Davis [15].

3.1 Background in right-angled Coxeter groups

An *edge loop* in a graph is an edge whose initial and terminal vertices are equal. A graph is called *simplicial* if it has no edge loops and there is at most one edge between every pair of vertices. An *n-clique* is the complete graph on n vertices. Given two graphs Γ_1 and Γ_2 , the graph join $\Lambda = \Gamma_1 \star \Gamma_2$ is formed by taking the disjoint union of the graphs Γ_1, Γ_2 and connecting every vertex of Γ_1 to every vertex of Γ_2 . If $X \subset Y$ where X and Y are graphs, then X is said to *separate* Y if $Y \setminus X$ has more than one component. For the remainder of this work the vertex and edge sets of a graph Γ are denoted by $V(\Gamma)$ and $E(\Gamma)$ respectively.

Given a graph Γ , the flag complex defined by Γ is the simplicial complex with Γ as its 1-skeleton and whose k -simplices are in bijective correspondence with the k -cliques of Γ . If X and Y are graphs (similarly complexes), then X is said to separate Y if $Y \setminus X$ has more than one component.

Definition 3.1.1. *Given a finite simplicial graph Γ the associated **right-angled Coxeter group** (RACG for short), denoted by W_Γ , is the group defined by the following presentation:*

$$W_\Gamma = \langle v \in V(\Gamma) \mid v^2 = 1 \text{ for all } v \in V(\Gamma); v_1 v_2 = v_2 v_1 \text{ if and only if } (v_1, v_2) \in E(\Gamma) \rangle.$$

Example 3.1.2. If Γ is the graph on n vertices with no edges, then W_Γ is the free product of n copies of $\mathbb{Z}_2$.

- Remark 3.1.3.** (1) Let Γ be an n -clique with vertices $v_1, \dots, v_n$. Then any pair of generators commute in W_Γ , hence every element of W_Γ can be written in the form $v_1^{\epsilon_1} \cdots v_n^{\epsilon_n}$ where $\epsilon_i \in \{0, 1\}$ for each i . It follows that W_Γ is finite. In fact W_Γ is finite if and only if Γ is a clique.
- (2) The graph Γ decomposes as a graph join $\Gamma = \Gamma_1 \star \Gamma_2$ if and only if the associated RACG W_Γ is the direct product $W_\Gamma = W_{\Gamma_1} \times W_{\Gamma_2}$.
- (3) A theorem by Moussong [30] proves that W_Γ is hyperbolic if and only if Γ contains no induced four cycles.

A subgraph Λ of Γ is called *induced* if whenever two vertices of Λ share an edge in Γ , they also share an edge in Λ . A *special subgroup* of W_Γ is the subgroup generated by the vertices of an induced subgraph Λ of Γ and is isomorphic to the RACG W_Λ .

To each RACG W_Γ there is an associated CAT(0) cube complex Σ_Γ upon which W_Γ acts geometrically (see Definition 1.3.5). This complex, originally constructed by Gromov, is typically referred to as the *Davis complex associated to W_Γ* due to its treatment in the book by Davis [15]. While the precise definition and construction can be found in Chapter 7 of [15], a description of Σ_Γ as it appears in Chapter 1 of [15] is more directly applicable in establishing our main results in the following chapter.

Definition 3.1.4. Let Γ be a finite simplicial graph with n vertices and L the flag complex on Γ . Let $C = [-1, 1]^n \subset \mathbb{R}^n$ denote the standard n -dimensional unit cube in $\mathbb{R}^n$. Fix an ordering on the vertex set $V(\Gamma)$ of Γ so that the i th generator of W_Γ corresponds to the i th factor of C . Then each face of C determines a unique subset of $V(\Gamma)$.

Define a subcomplex P_L of C to have the property that P_L contains a face $F \subset C$ if and only if the subset of $V(\Gamma)$ which corresponds to F forms a clique in Γ . As a result, P_L has the same vertex set of C and the link of each of its vertices is isometric to L with its all right structure. The **Davis complex Σ_Γ associated to W_Γ** is the universal cover of P_L .

Figure 3.1: Finite example of Davis complex. The graph Γ is a clique and defines a finite RACG. The corresponding Davis complex is the 2-cell depicted on the right.

Remark 3.1.5. The *all right structure* of L mentioned above simply means that each edge of L has length $\pi/2$. As L is a flag complex, it is CAT(1) due to Gromov's lemma ([15], Lemma I.6.1 p.516).

Equivalently, one can construct the Davis complex from the Cayley graph $\text{Cay}(W_\Gamma, V(\Gamma))$. For each k -clique $T \subset \Gamma$, the special subgroup W_T of W_Γ is isomorphic to the direct product of k copies of $\mathbb{Z}_2$. Hence $\text{Cay}(W_T, V(T))$ is isomorphic to the 1-skeleton of a k -cube. The 1-skeleton of Σ_Γ is $\text{Cay}(W_\Gamma, V(\Gamma))$ where each edge is endowed with unit length. For each k -clique $T \subset \Gamma$ and each coset gW_T attach a unit k -cube to gW_T . In other words, the following theorem holds.

Theorem 3.1.6 ([37], Theorem 5.24). *The Davis complex Σ_Γ may be identified with the CW complex which has 1-skeleton $\text{Cay}(W_\Gamma, V(\Gamma))$ and a cell with vertex set $U \subset W_\Gamma$ whenever $U = wW_T$ for some $w \in W_\Gamma$ and $T \subset S$ with W_T finite.*

Example 3.1.7 ([37], Example 5.25). (1) If W_Γ is finite, then Σ_Γ has one $|V(\Gamma)|$ -dimensional cell P with vertex set W_Γ and all other cells are faces of P . The case where Γ is the clique consisting of two vertices and an edge joining them is given in Figure 3.1.

(2) Consider the infinite dihedral group $D_\infty = \langle s_1, s_2 \mid s_1^2 = s_2^2 = 1 \rangle$. This is the RACG whose defining graph consists of two vertices and no edges. D_∞ contains no spherical special sub-

Figure 3.2: The Davis complex associated to D_∞ . The graph Γ (left) generates the RACG isomorphic to D_∞ . As Γ contains no cliques with more than one vertex, Σ_Γ is simply the Cayley graph of D_∞ (right).

groups W_T with $|T| \geq 2$, hence Σ_Γ is simply the Cayley graph $\text{Cay}(D_\infty, \{s_1, s_2\})$. See Figure 3.2.

Remark 3.1.8. (1) The group $\mathbb{Z}_2^n$ acts on C by reflections in hyperplanes of $\mathbb{R}^n$ which pass through the origin. As a result, $\mathbb{Z}_2^n$ acts on P_L in the same manner. This action induces a finer cube complex structure on P_L , which then lifts to a cube complex structure on Σ_Γ . The $\mathbb{Z}_2^n$ action on P_L lifts to an action of W_Γ on Σ_Γ by cubical isometries which is geometric in nature (see Chapter 7 of [15] for more detail).

(2) Suppose the graph Γ is triangle free (contains no three cycles) and $T \subset \Gamma$ is a k -clique. Then k must be either 0, 1, or 2. As long as Γ contains at least one edge, Σ_Γ is a two dimensional cube complex. Clearly the converse is also true, meaning Σ_Γ is a two dimensional cube complex if and only if Γ is triangle free and contains at least one edge.

Theorem 3.1.9 ([15], Theorem 12.2.1 p.233). *The piecewise Euclidean cubical structure on Σ_Γ is $CAT(0)$.*

Proof. Each vertex link of Σ_Γ is $CAT(1)$ (see Remark 3.1.5). Hence Σ_Γ satisfies Gromov's link condition (see definition 2.5.4). As Σ_Γ is the universal cover of P_L , it is simply connected. Applying Theorem 2.5.5 it follows that Σ_Γ is $CAT(0)$. $\square$

An important graph theoretical property that has been useful in the quasi-isometric classification of RACGs is the $\mathcal{CFS}$ condition, initially defined in the case of triangle free graphs by Dani-Thomas in [13] and then generalized to all graphs by Behrstock-Falgas-Ravry-Hagen-Susse

in [5]. Generally speaking, a graph has the $\mathcal{CFS}$ property if every pair of four cycles are connected by a chain of four cycles where the intersection of successive four cycles consists of two nonadjacent vertices.

Definition 3.1.10 ($\mathcal{CFS}$ property). *The **four-cycle graph** Γ^4 associated to Γ is defined as follows. Γ^4 has a vertex for each induced four cycle in Γ . Two vertices of Γ^4 share an edge if the corresponding four cycles in Γ share a pair of non-adjacent vertices. The **support** of a subgraph K of Γ^4 is the collection of vertices of Γ that appear in the four cycles of Γ that correspond to the vertices of K . The graph Γ is $\mathcal{CFS}$ if $\Gamma = \Omega \star K$ where K is a (possibly empty) clique and Ω is a non-empty subgraph such that Ω^4 has a connected component whose support is $V(\Omega)$.*

Having established the necessary preliminaries associated to right-angled Coxeter groups we now turn our attention to some quasi-isometric results. Much has been accomplished in recent years regarding the quasi-isometric classification of RACGs. Behrstock-Caprace-Hagen-Sisto [6] proved that RACGs are divided into two classes of groups: those that are algebraically thick and those that are relatively hyperbolic. In [31], Nguyen-Tran classified up to quasi-isometry a class of algebraically thick RACGs whose graphs are planar $\mathcal{CFS}$ graphs. Dani-Thomas [14] classified up to quasi-isometry a class of 2 dimensional hyperbolic RACGs that split over 2-ended subgroups. Results from [14] and [23] together classified up to quasi-isometry all RACGs whose graphs are generalized theta graphs.

In the sections to follow, we briefly discuss results related to divergence, thickness, and the number of ends. As previously mentioned, the survey paper by Dani [11] serves as a great source of what is known and has been established in recent years. Many of these results are given here without proof, though the appropriate source material is provided as needed.

3.2 Ends

For a topological space U let $\pi_0(U)$ denote the set of path-connected components of U . Let X be a proper geodesic metric space. For each closed subset $B \subset X$ define the set $\pi_0^u(B) = \pi_0(U_B)$ where U_B is the union of the unbounded path-connected components of B^c .

Definition 3.2.1 (Ends of a space). *The number of ends of X is $\eta(X) = \sup_{\substack{K \subset X \\ K \text{ compact}}} |\pi_0^u(K)|$.*

It follows from the definition that X is 0-ended if and only if it is compact. Moreover, X is 1-ended if and only if it is not compact and for each compact $K \subset X$, K^c has exactly one unbounded component. It is well-known that the number of ends of a space is invariant under quasi-isometry (see, for example, Lemma 9.5 p. 289 of [18]).

Definition 3.2.2 (Ends of a group). *Let G be a finitely generated group and S a finite generating set. Then the number of ends of G , denoted by $\eta(G)$, is the number of ends of $\text{Cay}(G, S)$.*

This notion of the number of ends of a finitely generated group is well-defined given that the number of ends of a metric space is a quasi-isometric invariant and all Cayley graphs of G with respect to finite generating sets are quasi-isometric. Moreover, $\eta(G)$ is a quasi-isometric invariant of G . The following theorem, due to Hopf [22], summarizes some of the restrictions on the possible number of ends of a finitely generated group. See [9] pp.146-147 for a complete proof.

Theorem 3.2.3 ([9], Theorem 8.32 p.146). *Let G be a finitely generated group. Then*

1. $\eta(G) = 0, 1, 2$, or ∞ ;
2. $\eta(G) = 0$ if and only if G is finite;
3. $\eta(G) = 2$ if and only if G contains an infinite cyclic subgroup of finite index.

In terms of RACGs, the 0-ended case is easily characterized in terms of defining graphs. It is clear from the definition that a group is 0-ended if and only if it is finite. Hence, the following lemma is immediate.

Lemma 3.2.4. *The following are equivalent:*

1. W_Γ is 0-ended.
2. W_Γ is finite.
3. Γ is a clique.

Furthermore, 2-ended RACGs also prove to be easily characterized in terms of defining graphs.

Theorem 3.2.5 ([15], Theorem 8.7.3 p.160). *The following are equivalent:*

1. W_Γ is 2-ended.
2. W_Γ is the product of a finite group and the infinite dihedral group D_∞ .
3. Γ is the 1-skeleton of a suspension of a clique.

Based on these results it has been a simple task to classify 0- and 2-ended RACGs up to quasi-isometry. All 0-ended groups are quasi-isometric to the trivial group and all 2-ended groups are quasi-isometric to $\mathbb{Z}$. A famous theorem of Stallings [35] says that a group has infinitely many ends if and only if it splits over some finite group, hence some attention has also been devoted to RACGs that split over 2-ended groups. In this endeavor, Mihalik-Tschantz [29] characterized the defining graphs of RACGs which split over 2-ended groups while Dani-Thomas [14] classified up to quasi-isometry a class of 2 dimensional hyperbolic RACGs which split over 2-ended subgroups. Mihalik-Tschantz [29] also showed that the splitting of RACGs is visible in the defining graph.

Theorem 3.2.6 ([29], Corollary 16). *W_Γ has infinitely many ends if and only if Γ has a separating clique.*

In the case of infinitely many ends, it has been shown that a RACG with infinitely many ends splits as a tree of groups where each vertex group is either a 0- or 1-ended special subgroup and each edge group is a finite special subgroup (see [15] Theorem 8.8.2 or [29] Corollary 18). Theorems 0.3 and 0.4 of Papasoglu-Whyte [32] then provide necessary and sufficient conditions on the tree of groups decompositions for two RACGs with infinitely many ends to be quasi-isometric. This condition relies on the quasi-isometry types of the 1-ended vertex groups, implying that understanding and classifying the 1-ended case may be the key to understanding the quasi-isometric classification of RACGs. Many researchers then choose to restrict attention to this case.

By default, the 1-ended RACGs are precisely the following.

Corollary 3.2.7 ([11], Corollary 2.4). W_Γ is one-ended if and only if Γ is connected, has no separating cliques, and is not equal to a clique.

3.3 Divergence and thickness

Let X be a metric space. Consider the linear function $\rho(r) = \delta r - \lambda$ where $0 < \delta \leq 1$ and $\lambda \geq 0$ are fixed constants.

Definition 3.3.1. Let $a, b, c \in X$ and set $k = d(c, \{a, b\})$. Define $\text{div}_\lambda(a, b, c, \delta)$ to be the length of the shortest path in X from a to b that lies outside the ball $B(c, \rho(k))$. If no such path exists set $\text{div}_\lambda(a, b, c, \delta) = \infty$. $\text{Div}_\lambda^X(r, \delta)$ is defined to be the supremum of $\text{div}_\lambda(a, b, c, \delta)$ over all a, b, c with $d(a, b) \leq r$. The **divergence of X** is the function $\text{Div}(X) = \text{Div}_\lambda^X(r, \delta)$.

We define an equivalence relation on the set of functions from $\mathbb{R}_+ \rightarrow \mathbb{R}_+$ by saying $f(x) \asymp g(x)$ if there exists a constant C such that

$$C^{-1}g(C^{-1}x) - Cx - C < f(x) < Cg(Cx) + Cx + C$$

for all x .

It is well-known that under this equivalence relation the divergence of the Cayley graph of a finitely generated group is a quasi-isometric invariant (see, for example, Proposition 3.5 of [3]). Thus, given a finitely generated group G one can define $\text{Div}(G)$ to be the divergence of $\text{Cay}(G, S)$ for some finite generating set S of G .

Thickness, as introduced in [4], is a quasi-isometric invariant closely related to divergence. Thick spaces are defined inductively. Spaces that are *thick of order 0* correspond to those with linear divergence. Roughly speaking, a space is *thick of order n* if it is formed from a network of subsets that are all thick of order $n - 1$ where any pair of points in the space can be connected via a chain of subsets that are thick of order $n - 1$ in such a way that the intersection of successive subsets in the chain have infinite diameter. By refining the definition one can obtain the notion of *strong algebraic thickness*, introduced in [3] where Behrstock-Drutu prove that any group which is strongly algebraically thick of order n has divergence at most x^{n+1} .

As previously mentioned, Behrstock-Caprace-Hagen-Sisto [6] proved that RACGs are divided into two classes of groups: those that are algebraically thick and those that are relatively hyperbolic. Relatively hyperbolic RACGs are known to have exponential divergence ([34], Theorem 1.3) while those RACGs that are thick have divergence bounded above by a polynomial ([6]). It is still an open question as to whether the divergence of any algebraically thick RACG is exactly polynomial.

Restricting focus to the algebraically thick case, Dani-Thomas [13] proved the existence of a RACG with divergence x^d for every positive integer d . A complete characterization of the RACGs which have divergence x^d , $d > 2$, is still open, however Levcovitz has made progress in this endeavor [27],[28]. In [27] he defines a *rank n pair* via an inductive criterion. A pair of non-adjacent vertices is rank 1 if it is not contained in any four cycle. A pair of non-adjacent vertices v_1, v_2 is rank n if every pair of nonadjacent vertices in $\text{Lk}(v_i)$ is rank $n - 1$ for some $i \in \{1, 2\}$. Levcovitz shows that the existence of a rank n pair implies that the divergence is bounded below by x^{n+1} . In [28], he defines the *hypergraph index*, a graph theoretical value that can be calculated from the defining graph through an algorithmic procedure in finite time. The hypergraph index is shown to be a quasi-isometric invariant for triangle free RACGs and that having hypergraph index n yields an upper bound of n on the order of thickness and an upper bound of x^{n+1} on the divergence. Combining these results yields the following.

Theorem 3.3.2 ([27], [28]). *If Γ contains a rank n pair and has hypergraph index n , then W_Γ has divergence x^{n+1} .*

Furthermore, Levcovitz conjectures that the notions of hypergraph index n , thickness of order n , and divergence x^{n+1} are all equivalent, proving that this is true in the cases where $n = 0$ and $n = 1$.

Dani-Thomas [13] provide a complete characterization of the RACGs with linear and quadratic divergence whose defining graphs are triangle free. Behrstock-Hagen-Sisto [6] then generalized the case of linear divergence to all RACGs. The results of Levcovitz [27] generalized the case of quadratic divergence to include all RACGs. Combining all of these findings we obtain the following theorems.

Theorem 3.3.3 ([13], [6], [28]). *The following are equivalent.*

1. Γ decomposes as a nontrivial join ($\Gamma = A \star B$ where A and B each contain a pair of non-adjacent vertices).
2. W_Γ is thick of order 0.
3. W_Γ is strongly algebraically thick of order 0.
4. W_Γ has linear divergence.
5. Γ has hypergraph index 0.

Theorem 3.3.4 ([13], [27], [28]). *The following are equivalent.*

- (1) Γ is $\mathcal{CFS}$ and Γ does not decompose as a nontrivial join.
- (2) W_Γ is strongly algebraically thick of order 1.
- (3) W_Γ is thick of order 1.
- (4) The divergence of W_Γ is quadratic.
- (5) Γ has hypergraph index 1.

CHAPTER 4 QUASI-ISOMETRIC RIGIDITY OF A CLASS OF RIGHT-ANGLED COXETER GROUPS

In this chapter we establish the quasi-isometric rigidity of a particular class of right-angled Coxeter groups (RACGs). The work presented here in section 4.3 was completed by the author jointly with Xiangdong Xie [8] at Bowling Green State University. We prove the following theorem.

Theorem 4.4.2. *For $i = 1, 2$, let Γ_i be a finite thick generalized m_i -gon with $m_i \in \{3, 4, 6, 8\}$. Then any quasi-isometry $f : \Sigma_{\Gamma_1} \rightarrow \Sigma_{\Gamma_2}$ is at a finite distance from an isometry. In particular, W_{Γ_1} and W_{Γ_2} are quasi-isometric if and only if Γ_1 and Γ_2 are isomorphic.*

For each $m \in \{3, 4, 6, 8\}$ let $\mathcal{S}_m$ be the collection of finite thick generalized m -gons. Let $\mathcal{S} = \bigcup_m \mathcal{S}_m$. By combining Theorem 4.4.2 with a result of Kapovich-Kleiner-Leeb [24] on the quasi-isometric rigidity of product spaces we also prove the following generalization.

Corollary 4.4.4. *Let Γ_1, Γ_2 be finite joins of graphs from $\mathcal{S}$. Then any quasi-isometry $f : \Sigma_{\Gamma_1} \rightarrow \Sigma_{\Gamma_2}$ is at a finite distance from an isometry. In particular, W_{Γ_1} and W_{Γ_2} are quasi-isometric if and only if Γ_1 and Γ_2 are isomorphic.*

Two maps $f : X \rightarrow Y$ and $g : X \rightarrow Y$ are said to lie at a *finite distance* if the value $d(f, g) = \sup\{d(f(x), g(x)) : x \in X\}$ is finite. The key point in the proof of Theorem 4.4.2 is that Σ_Γ admits a Fuchsian building structure when Γ is a finite thick generalized m -gon with $m \geq 3$. Theorem 4.4.2 then follows from applying the quasi-isometric rigidity for Fuchsian buildings established by Xie [40].

Under the assumptions of Theorem 4.4.2 the Davis complex is a 2-dimensional CAT(0) cube complex where each 2-cell is isometric to a Euclidean unit square. We replace each square with a regular 4-gon in the hyperbolic plane with angle π/m at each vertex. The Davis complex with this piecewise hyperbolic metric is then a Fuchsian building (see Theorem 4.4.1).

We first provide the necessary background regarding generalized polygons and Fuchsian buildings. We then construct a labeling of the Davis complex necessary for proving it admits a Fuchsian building structure. Finally, we prove Theorem 4.4.2 and Corollary 4.4.4.

4.1 Fuchsian buildings

Definition 4.1.1. A **geometric reflection group** (W, S) is the action of a group W on $\mathbb{X}_\kappa^n$ which is generated by a set S of reflections of a convex polytope with dihedral angles of the form π/m , $m \in \mathbb{Z}$. A geometric reflection group W acting on $\mathbb{X}_\kappa^n$ is *spherical*, *Euclidean*, or *hyperbolic* when $\mathbb{X}_\kappa^n$ is, respectively, $\mathbb{S}^n$, $\mathbb{E}^n$, or $\mathbb{H}^n$.

Example 4.1.2. Let P be a regular hyperbolic n -gon, $n \geq 5$, with angle $\pi/2$ at each vertex. Reflections about the edges of P generate a tiling of $\mathbb{H}^2$. Reflections in adjacent sides of P commute while reflections in nonadjacent sides of P generate an infinite dihedral group.

Let (W, S) be a geometric reflection group. Roughly speaking, a building Δ is a polyhedral complex that is a union of apartments where each apartment is a copy of $\mathbb{X}_\kappa^n$ tessellated by the action of W . Δ is called *spherical*, *Euclidean*, or *hyperbolic* if $\mathbb{X}_\kappa^n$ is $\mathbb{S}^n$, $\mathbb{E}^n$, or $\mathbb{H}^n$ respectively. The precise definition of a *two dimensional hyperbolic building* is given here.

Let R denote a compact convex polygon in $\mathbb{H}^2$ whose angles are of the form π/m , $m \in \mathbb{N}$, $m \geq 2$. Let W be the Coxeter group generated by the reflections about the edges of R . Label the edges and vertices of R cyclically by $\{1\}, \{2\}, \dots, \{k\}$ and $\{1, 2\}, \dots, \{k-1, k\}, \{k, 1\}$ respectively so that the edges i and $i+1$ intersect at the vertex $\{i, i+1\}$. It is well-known that the images of R under W form a tessellation of $\mathbb{H}^2$ and that the quotient $\mathbb{H}^2/W = R$. Thus, there is a labeling of edges and vertices of the tessellation of $\mathbb{H}^2$ that is W -invariant and compatible with the labeling of R . Let A_R denote the obtained labeled 2-complex.

Definition 4.1.3. Let Δ be a connected cellular 2-complex whose edges and vertices are labeled by $\{1\}, \{2\}, \dots, \{k\}$ and $\{1, 2\}, \{2, 3\}, \dots, \{k-1, k\}, \{k, 1\}$ respectively, such that each 2-cell (called a *chamber*) is isomorphic to R as labeled 2-complexes. We call Δ a **two dimensional hyperbolic building** if it has a family of subcomplexes (called *apartments*) isomorphic to A_R (as

labeled 2-complexes) satisfying the following properties:

1. Given any two chambers there is an apartment containing both.
2. For any two apartments A_1, A_2 that share a chamber there is an isomorphism of labeled 2-complexes $f : A_1 \rightarrow A_2$ which pointwise fixes $A_1 \cap A_2$.

If additionally there are integers $q_i \geq 2, i = 1, 2, \dots, k$ such that each edge of Δ labeled by i is contained in exactly $q_i + 1$ chambers, then Δ is called a **Fuchsian building**.

Remark 4.1.4. The condition requiring that each edge labeled i be contained in exactly $q_i + 1$ chambers is not always taken to be part of the definition of a Fuchsian building. Fuchsian buildings that satisfy this additional condition are often called *semi-regular*.

The following result, due to Xie [40], is a key element in the proof of Theorem 4.4.2 as it establishes the quasi-isometric rigidity of Fuchsian buildings.

Theorem 4.1.5 ([40], Theorem 1.1). *Let Δ_1, Δ_2 be two Fuchsian buildings and $g : \Delta_1 \rightarrow \Delta_2$ a quasi-isometry. If Δ_1, Δ_2 admit cocompact lattices, then g lies at a finite distance from an isomorphism.*

Let X be a connected polyhedral complex and (W, S) a geometric reflection group where S is the set of reflections of a convex polytope P . X is said to be of *type* (W, S) if there exists a morphism $\tau : X \rightarrow P$ of CW-complexes where the restriction of τ to each maximal cell is an isometry.

The work of Gaboriau-Paulin [20] provides a local-to-global result that is useful in constructing Fuchsian buildings. An exposition of their work can be found in Section 10.1 of [37]. Roughly speaking, Corollary 2.4 of [20] states that the universal cover of a connected polyhedral complex of type (W, S) is a building of type (W, S) if the link of each vertex is CAT(1) and each link satisfying certain conditions contains an isometrically embedded sphere passing through any pair of points. In particular, if X is a simply connected polygonal complex of type (W, S) such that each vertex link of X is a 1-dimensional spherical building, then X is a building of type (W, S) . More precisely, we have the following theorem.

Theorem 4.1.6. *Let L be a generalized polygon and let X be a simply connected polyhedral complex with all faces regular hyperbolic k -gons, for some $k \geq 3$, with vertex angles π/m and each vertex link isomorphic to L . Then X is a hyperbolic building.*

We note that X is of type (W, S) if and only if X admits a labeled 2-complex structure as in Definition 4.1.3.

4.2 Generalized polygons

There are several equivalent definitions of *generalized polygons*, all of which are detailed in the survey paper by Van Malgdeghem [39], two of which are of particular interest for our purposes here. Let L be a connected bipartite graph whose vertices are colored red and blue such that no two adjacent vertices share the same color. Let $m \in \mathbb{N}, m \geq 2$.

Definition 4.2.1 (Type i). *L is a **generalized m -gon of type (i)** if it has the following two properties:*

1. *Given any pair of edges there is a circuit with combinatorial length $2m$ containing both.*
2. *For two circuits A_1, A_2 of combinatorial length $2m$ that share an edge there is an isomorphism $f : A_1 \rightarrow A_2$ that pointwise fixes $A_1 \cap A_2$.*

Definition 4.2.2 (Type ii). *Suppose each edge of L has unit length. The diameter of L is the maximum distance between any two points and its girth is the length of the shortest circuit. L is a **generalized m -gon of type (ii)** if its diameter is m and its girth is $2m$.*

Here a *circuit* refers to a closed edge path which is topologically a circle.

Theorem 4.2.3 ([39], Theorem 1.1). *If each vertex of Γ has valence at least two, then Γ is a generalized m -gon of type i if and only if it is a generalized m -gon of type ii.*

A generalized m -gon is called *thick* if each vertex has valence at least 3. Let L be a thick generalized m -gon. An edge in L is called a *chamber* and a circuit with combinatorial length $2m$ in L is called an *apartment*. Two vertices of the same color are said to have the same type.

Figure 4.1: Thick generalized 3-gon.

A generalized 3-gon is simply a projective plane. For generalized m -gons with $m > 3$ many use terminology associated with familiar polygons, i.e., generalized octagon (8-gon) or hexagon (6-gon), with the exception being a generalized quadrangle (4-gon). Numerous examples can be found throughout the works of Van Maldeghem ([38], [39]).

A well known theorem of Feit-Higman [19] says that finite thick generalized m -gons exist only for $m \in \{2, 3, 4, 6, 8\}$. For each $m \in \{2, 3, 4, 6, 8\}$ there exist infinitely many finite thick generalized m -gons.

Example 4.2.4. (1) Any cycle of length $2m$, $m \geq 1$, is a generalized m -gon. An example of a *thick* generalized 3-gon is given in Figure 4.1.

(2) Given any prime q the incidence graph of the projective plane over the finite field of order q is a finite thick generalized 3-gon where all vertices have valence $q + 1$.

Let L be a thick generalized m -gon. Then there are integers $r, b \geq 2$ such that each red vertex is contained in exactly $r + 1$ chambers and each blue vertex is contained in exactly $b + 1$ chambers (see p.29 of [33]). It is known that $r = b$ in the case where m is odd and $r \neq b$ when $m = 8$.

Remark 4.2.5. (1) In view of Definition 4.2.1 there appears to be a clear connection between generalized polygons and Fuchsian buildings. In fact, generalized polygons are rank 2 (1-dimensional) spherical buildings.

- (2) For each chamber in L we equip a metric such that it is isometric to the closed interval of length π/m . We then equip L with the path metric, making L a CAT(1) space in view of Definition 4.2.2 and Theorem 2.5.6.
- (3) Every apartment A in L is convex in L , meaning that for any $x, y \in A$ with $d(x, y) < \pi$ the geodesic segment xy also lies in A . Thus, if two apartments A_1, A_2 share a chamber, then either $A_1 = A_2$ or $A_1 \cap A_2$ is a segment.

4.3 The Davis complex as a labeled 2-complex

The goal of this section is to show that the Davis complex Σ_Γ associated to the graph of a finite generalized m -gon Γ , $m \geq 3$, has the structure of a labeled 2-complex. More precisely, we prove the following proposition.

Proposition 4.3.1. *Let Γ be the graph of a finite generalized m -gon. Then there is a labeling of the edges of Σ_Γ by $\{1\}, \{2\}, \{3\}, \{4\}$ and a labeling of the vertices by $\{1, 2\}, \{2, 3\}, \{3, 4\}, \{4, 1\}$, such that for every 2-cell S in Σ_Γ its edges are cyclically labeled by 1, 2, 3, 4 and the vertex of S incident to the edges labeled by $i, i + 1$ is labeled by $\{i, i + 1\}$.*

This is the first step in proving that Σ_Γ admits a Fuchsian building structure under the additional assumption that Γ is thick. Throughout, given $i \in \{1, 2, 3, 4\}$ set $i + 1 = 1$ if $i = 4$. As Σ_Γ is taken to be the universal cover of P_L (see Definition 3.1.4), it suffices to show that P_L admits the structure of a labeled 2-complex and then lift this structure to obtain a labeled 2-complex structure on Σ_Γ . Recall that a one-to-one correspondence was fixed between the set of coordinate axes of $\mathbb{R}^n$ and $V(\Gamma)$. The vertices of Γ are divided into red and blue vertices. A coordinate axis is red (blue) if it corresponds to a red (blue) vertex. An edge e of P_L is red (blue) if it is parallel to a red (blue) coordinate axis.

Definition 4.3.2. *Given a vertex v of P_L let E_v denote the set of all edges in P_L incident to v . A labeling of P_L is a map l from the edge set of P_L to the set $\{1, 2, 3, 4\}$ with the following two properties:*

1. For any 2-cell S in P_L , the edges of S are cyclically labeled 1,2,3,4.
2. For any vertex v , $l(E_v) = \{i, i + 1\}$ for some $i \in \{1, 2, 3, 4\}$.

A labeling l of P_L clearly induces a labeled 2-complex structure of P_L by labeling each vertex v of P_L with $l(E_v)$. A labeling of P_L is obtained by first selecting with a labeling of E_{v_0} for a fixed vertex v_0 and then obtaining a labeling of E_v for every vertex v in a compatible manner (see Definition 4.3.4).

Definition 4.3.3. For a vertex $v \in P_L$ a **labeling** of E_v is a map l from E_v to the set $\{1, 2, 3, 4\}$ with the following properties:

1. $l(E_v) = \{i, i + 1\}$ for some $i \in \{1, 2, 3, 4\}$.
2. For all $e_1, e_2 \in E_v$, $l(e_1) = l(e_2)$ if and only if e_1, e_2 have the same color.

Given a vertex v such a labeling always exists as defining $l(e) = i$ for all red edges in E_v and $l(e) = i + 1$ for all blue edges in E_v provides a clear example of a labeling on E_v .

Definition 4.3.4. Let v_1, v_2 be two adjacent vertices in P_L and l_1, l_2 labelings of E_{v_1}, E_{v_2} respectively. Let e be the edge with vertices v_1, v_2 . We say l_1 and l_2 are **compatible** if there is some $i \in \{1, 2, 3, 4\}$ such that $l_1(e) = l_2(e) = i$ and either $l_1(E_{v_1}) = \{i - 1, i\}$, $l_2(E_{v_2}) = \{i, i + 1\}$ or $l_2(E_{v_2}) = \{i - 1, i\}$, $l_1(E_{v_1}) = \{i, i + 1\}$.

Remark 4.3.5. Let v_1, v_2 be adjacent vertices in P_L and l_1 any labeling of E_{v_1} and e_0 the edge with vertices v_1, v_2 . Without loss of generality we may assume $l_1(e) = i$ for red edges and $l_1(e) = i + 1$ for blue edges in E_{v_1} . If e_0 is red, then the labeling l_2 of E_{v_2} defined by $l_2(e) = i$ if e is red and $l_2(e) = i - 1$ if e is blue is the unique labeling of E_{v_2} compatible with l_1 . Similarly, if e_0 is blue, then the labeling l_2 of E_{v_2} given by $l_2(e) = i + 1$ if e is blue and $l_2(e) = i + 2$ if e is red defines the unique labeling of E_{v_2} compatible with l_1 .

Fix a vertex v of P_L and a labeling l of E_v . Given adjacent vertices v_1, v_2 and a labeling l_1 of E_{v_1} there is a unique labeling l_2 of E_{v_2} compatible with l_1 by Remark 4.3.5. If $C = e_1 e_2 \dots e_m$ is

Figure 4.2: Labeling of 4-cycle in P_L .

an edge path from v to a vertex v' , then we can obtain a labeling l' of $E_{v'}$ via compatibility along C . Note that there may be many edge paths from v to v' , hence it is necessary to verify that l' is independent of the chosen path C . This can be done by showing that if C is an edge loop based at the vertex v , then the labeling l' of E_v obtained from l by compatibility along the loop C coincides with l . This is proven by first considering the case when C is a 4-cycle (Lemma 4.3.6) and then express a general edge loop as the concatenation of conjugates of 4-cycles (Lemma 4.3.7).

For any oriented edge e of P_L let e^{-1} denote the same edge given with the opposite orientation, $\tilde{e}$ any edge parallel to e with the same orientation, and $\bar{e}$ any edge parallel to e with the opposite orientation. In particular if $\tilde{e}, \bar{e}$ occur consecutively in an edge path of P_L , then this $\bar{e}$ is equal to $\tilde{e}^{-1}$.

Lemma 4.3.6. *Let $C = e_1 e_2 \bar{e}_1 \bar{e}_2$ be a 4-cycle in P_L based at v . Then for any labeling l of E_v we have $l' = l$ where l' is the labeling of E_v obtained from l by compatibility along C .*

Proof. Suppose v is the vertex in C incident to both e_1 and $\bar{e}_2$ and let l be a labeling of E_v . Denote the remaining vertices in C as in Figure 4.2. We have two cases to consider.

Case 1: Suppose $e_1, \bar{e}_2$ are of the same color. As $\bar{e}_1$ and e_2 are parallel to e_1 and $\bar{e}_2$ respectively, all of $e_1, e_2, \bar{e}_1, \bar{e}_2$ are of the same color. Thus there is some $i \in \{1, 2, 3, 4\}$ such that $l(e_1) = l(\bar{e}_2) = i$. Furthermore either $l(E_v) = \{i, i+1\}$ or $l(E_v) = \{i, i-1\}$. Without loss of generality we may assume $l(E_v) = \{i, i+1\}$. Let l_2 be the labeling of E_{v_2} obtained from l by compatibility along e_1 . Then $l_2(E_{v_2}) = \{i-1, i\}$ and $l_2(e_1) = l_2(e_2) = i$. Similarly, let l_3 be the labeling of E_{v_3} obtained from l_2 by compatibility along e_2 . Then $l_3(E_{v_3}) = \{i, i+1\}$ and $l_3(e_2) = l_3(\bar{e}_1) = i$. Next

let l_4 be the labeling of E_{v_4} obtained from l_3 by compatibility along $\bar{e}_1$. Then $l_4(E_{v_4}) = \{i - 1, i\}$ and $l_4(\bar{e}_1) = l_4(\bar{e}_2) = i$. Finally, we obtain the labeling l' of E_v from l_4 by compatibility along $\bar{e}_2$. It follows that $l'(E_v) = \{i, i + 1\}$ and $l'(\bar{e}_2) = l'(e_1) = i$. Thus $l' = l$.

Case 2: Suppose $e_1, \bar{e}_2$ are of different colors. Then there must be some $i \in \{1, 2, 3, 4\}$ such that either $l(e_1) = i, l(\bar{e}_2) = i + 1$ or $l(e_1) = i + 1, l(\bar{e}_2) = i$. Without loss of generality, suppose $l(e_1) = i, l(\bar{e}_2) = i + 1$. Hence $l(E_v) = \{i, i + 1\}$. Let l_2 be the labeling of E_{v_2} obtained from l by compatibility along e_1 . Then $l_2(e_1) = i$ and $l_2(E_{v_2}) = \{i - 1, i\}$. As $e_1, \bar{e}_2$ are of different colors, so are e_1, e_2 . Then $l_2(e_1) \neq l_2(e_2)$ must be the case. It then follows that $l_2(e_2) = i - 1$. Now let l_3 be the labeling of E_{v_3} obtained from l_2 by compatibility along e_2 . Then $l_3(e_2) = i - 1$ and so $l_3(E_{v_3}) = \{i - 2, i - 1\}$ by property 2 of Definition 4.3.3. As before, $e_2, \bar{e}_1$ are of different colors, implying that $l_3(\bar{e}_1) = i - 2$. Similarly, let l_4 be the labeling of E_{v_4} obtained from l_3 by compatibility along $\bar{e}_1$. Then $l_4(\bar{e}_1) = i - 2, l_4(E_{v_4}) = \{i - 3, i - 2\}$, and $l_4(\bar{e}_2) = i - 3$. Finally, let l' be the labeling of E_v obtained from l_4 by compatibility along $\bar{e}_2$. It follows that $l'(\bar{e}_2) = l_4(\bar{e}_2) = i - 3, l'(E_v) = \{i - 4, i - 3\}$, and $l'(e_1) = i - 4$. By our previously stated convention, $i - 4 = i$ and $i - 3 = i + 1$, hence $l' = l$. $\square$

We now define an equivalence relation $\sim$ on the edge paths of P_L . We say that two edge paths C, C' satisfy $C \sim C'$ if C' can be obtained from C by inserting and deleting pairs of the form ee^{-1} . In this case C and C' have the same initial and terminal vertices. Observe that if v_0 is the initial vertex, l_0 is a labeling of E_{v_0} and l, l' are the labelings at the terminal vertex obtained from l_0 by compatibility along C and C' respectively, then $l = l'$.

Lemma 4.3.7. *Let $C = e_1 \cdots e_{2m}$ be an edge loop in P_L . Then there exists an edge loop $C' = f_1 \cdots f_k$ in P_L with $C' \sim C$ such that each f_j is of the form $\alpha_j(\tilde{e}_s \tilde{e}_t \bar{e}_s \bar{e}_t) \alpha_j^{-1}$ with $1 \leq s < t \leq 2m$ and α_j an edge path of P_L .*

Proof. By deleting pairs of the form ee^{-1} , we may assume $e_{i+1} \neq e_i^{-1}$ for any i . We induct on the length $2m$ of the edge loop C . Taking C to have length 4 as our base case, $C = e_1 e_2 \bar{e}_1 \bar{e}_2$ already has the required form. Suppose the statement holds for edge loops with length $< 2m$. Now assume

the edge loop C has length $2m$.

As C is an edge loop in P_L there exists j such that $e_j = \bar{e}_1$. We then have that

$$\begin{aligned}
C &= e_1 e_2 e_3 \cdots e_j \cdots e_{2m} \\
&\sim (e_1 e_2 \bar{e}_1 \bar{e}_2) \bar{e}_2^{-1} \tilde{e}_1 e_3 \cdots e_j \cdots e_{2m} && \text{with } \tilde{e}_1 = \bar{e}_1^{-1} \\
&= f_1 \bar{e}_2^{-1} \tilde{e}_1 e_3 \cdots e_j \cdots e_{2m} && \text{with } f_1 = e_1 e_2 \bar{e}_1 \bar{e}_2 \\
&\sim f_1 \bar{e}_2^{-1} (\tilde{e}_1 e_3 \bar{e}_1 \bar{e}_3) \bar{e}_2 \bar{e}_2^{-1} \bar{e}_3^{-1} \tilde{e}_1 e_4 \cdots e_j \cdots e_{2m} \\
&= f_1 f_2 \bar{e}_2^{-1} \bar{e}_3^{-1} \tilde{e}_1 e_4 \cdots e_j \cdots e_{2m} && \text{with } f_2 = \bar{e}_2^{-1} (\tilde{e}_1 e_3 \bar{e}_1 \bar{e}_3) \bar{e}_2 \\
&\dots \\
&\sim f_1 f_2 \cdots f_j \bar{e}_2^{-1} \cdots \bar{e}_{j-1}^{-1} \tilde{e}_1 e_j \cdots e_{2m} \\
&\sim f_1 f_2 \cdots f_j \bar{e}_2^{-1} \cdots \bar{e}_{j-1}^{-1} e_{j+1} \cdots e_{2m} && \text{since } e_j = \bar{e}_1.
\end{aligned}$$

The length of the edge loop $\bar{e}_2^{-1} \cdots \bar{e}_{j-1}^{-1} e_{j+1} \cdots e_{2m}$ is less than $2m$. Thus it follows inductively that there exist $f_{j+1}, \dots, f_k$, each of the desired form, such that $C \sim f_1 \cdots f_j f_{j+1} \cdots f_k$. $\square$

We now prove Proposition 4.3.1.

Proof of Proposition 4.3.1. As already observed, it suffices to construct a labeling of P_L as this induces a labeled 2-complex structure on P_L which then lifts to a labeled 2-complex structure on Σ_Γ . Fix a vertex v_0 of P_L and a labeling l_0 of E_{v_0} . By Lemmas 4.3.6 and 4.3.7, we obtain a labeling l_v of E_v for every vertex v of P_L such that l_{v_1}, l_{v_2} are compatible whenever v_1, v_2 are adjacent. We define a labeling l of P_L as follows. For every edge e of P_L define $l(e) = l_{v_1}(e)$ where v_1 is a vertex of e . Note that if v_2 is the other vertex of e , then by compatibility we have $l_{v_2}(e) = l_{v_1}(e)$. Thus l is well-defined. Property 2 in Definition 4.3.2 is clearly satisfied. We show that property 1 holds as well.

Let S be a 2-cell in P_L . Denote its vertices by v_1, v_2, v_3, v_4 and edges by e_1, e_2, e_3, e_4 such that the edge e_i is incident to both v_i and v_{i+1} . Let l_{v_i} be obtained labelings of E_{v_i} , $i = 1, 2, 3, 4$. First note that each pair of adjacent edges e_i, e_{i+1} must have different colors. We proceed with an argument similar to that of Case 2 in the proof of Lemma 4.3.6. Given that the labelings obtained via compatibility are independent of chosen edge path, $l_{v_{i+1}}$ can be obtained from l_{v_i} by

compatibility along $e_i, i = 1, 2, 3, 4$. Without loss of generality we may assume $l_1(E_{v_1}) = \{i, i+1\}$ and $l_1(e_1) = i$. By compatibility along e_1 it follows that $l_2(E_{v_2}) = \{i-1, i\}$, $l_2(e_1) = i$, and $l_2(e_2) = i-1$. By compatibility along e_2 we obtain $l_3(E_{v_3}) = \{i-2, i-1\}$, $l_3(e_2) = i-1$, and $l_3(e_3) = i-2$. Finally, $l_4(E_{v_4}) = \{i-3, i-2\}$, $l_4(e_3) = i-2$, and $l_4(e_4) = i-3$. Thus, the boundary of S is labeled cyclically as $1, 2, 3, 4$. $\square$

For any edge e of P_L , the degree $d(e)$ of e is defined to be the number of squares in P_L that contain e . We next show that edges with the same label have the same degree (Lemma 4.3.8 (3)). This result is needed to show that Σ_Γ admits a Fuchsian building structure.

Lemma 4.3.8. *Let l be a labeling of P_L and e, e' be two edges of P_L .*

1. *If $l(e), l(e')$ have the same parity, then e, e' have the same color;*
2. *If $l(e), l(e')$ have different parity, then e, e' have different colors;*
3. *If $l(e), l(e')$ have the same parity, then $d(e) = d(e')$. In particular, edges with the same label have the same degree.*

Proof. (1) and (2): We prove (1) and (2) at the same time by inducting on the minimal length m of edge paths from e to e' . In the case where $m = 0$, e and e' share a vertex. If $l(e), l(e')$ have the same parity, then $l(e) = l(e')$ and e, e' are of the same color. If $l(e), l(e')$ have different parity, then there is some i such that $l(e) = i, l(e') = i+1$ or $l(e) = i+1, l(e') = i$ and so e, e' are of different colors. Assume (1) and (2) hold for m . Now suppose the minimal length of an edge path from e to e' is $m+1$. Let $ee_1 \cdots e_{m+1}e'$ be such an edge path. First suppose $l(e), l(e')$ have the same parity. If $l(e_1)$ has the same parity as $l(e)$, then $l(e_1), l(e')$ have the same parity. By the inductive assumption e_1, e' are of the same color, e, e_1 are of the same color, and so e, e' are of the same color. If $l(e_1), l(e)$ have different parity, then $l(e_1), l(e')$ have different parity. By the inductive assumption e_1, e' are of different colors, e, e_1 are of different colors, and so e, e' are of the same color. Next suppose $l(e), l(e')$ have different parity. If $l(e), l(e_1)$ have the same parity, then $l(e_1), l(e')$ have different parity. Then by the inductive assumption e_1, e' are of different colors, e, e_1 are of the same color,

and so e, e' are of different colors. If $l(e), l(e_1)$ have different parity, then $l(e_1), l(e')$ have the same parity. Again by the inductive assumption e_1, e' are of the same color, e, e_1 are of different color, and so e, e' are of different colors.

(3): Suppose $l(e), l(e')$ have the same parity. By (1) e, e' have the same color. Let u, u' be vertices of Γ that correspond to e and e' respectively. Then u, u' have the same color. Also observe $d(e) = d(u)$ (valence of u) and $d(e') = d(u')$. Since vertices of the same color in Γ have the same valence, we have $d(u) = d(u')$. It follows that $d(e) = d(e')$. $\square$

4.4 Proof of main results

In this section we prove Theorem 4.4.2 and Corollary 4.4.4 as in Section 4 of [8]. Throughout let $\Gamma \in \mathcal{S}$.

As previously indicated, the key to the proof of Theorem 4.4.2 is establishing that Σ_Γ admits a Fuchsian building structure and then applying the quasi-isometric rigidity theorem for Fuchsian buildings due to Xie [40] (stated here as Theorem 4.1.5). Σ_Γ admits the structure of a labeled 2-complex by Proposition 4.3.1 and all edges with the same label have the same degree by Lemma 4.3.8. Thus, to show that Σ_Γ admits a Fuchsian building structure, it remains to put a piecewise hyperbolic metric on Σ_Γ so that it satisfies conditions (1) and (2) in the definition of a Fuchsian building.

Theorem 4.4.1. *Let Γ be the graph of a finite generalized thick m -gon with $m \in \{3, 4, 6, 8\}$. Then the Davis complex Σ_Γ associated with the right-angled Coxeter group W_Γ admits a Fuchsian building structure.*

Proof. We equip the Davis complex Σ_Γ with a piecewise hyperbolic metric by replacing each Euclidean square with a regular four sided polygon in $\mathbb{H}^2$ that has angle π/m at each vertex. Observe that every vertex link in Σ_Γ is Γ where each edge has length π/m and so is CAT(1).

Let R be the regular 4-gon in the hyperbolic plane with all angles equal to π/m . Label the edges of R cyclically by 1, 2, 3, 4 and the vertex incident to edges labeled $\{i\}$ and $\{i + 1\}$ by $\{i, i + 1\}$. The labeling of Σ_Γ provided by Proposition 4.3.1 induces a type function $\tau : \Sigma_\Gamma \rightarrow R$

where τ preserves the labeling of the edges and vertices. By Theorem 4.1.6 Σ_Γ admits a hyperbolic building structure. It then follows from Lemma 4.3.8 that Σ_Γ is a Fuchsian building. $\square$

Now for the proof of our main result.

Theorem 4.4.2. *For $i = 1, 2$, let Γ_i be a finite generalized thick m_i -gon with $m_i \in \{3, 4, 6, 8\}$. Then any quasi-isometry $f : \Sigma_{\Gamma_1} \rightarrow \Sigma_{\Gamma_2}$ is at a finite distance from an isometry. In particular, W_{Γ_1} and W_{Γ_2} are quasi-isometric if and only if Γ_1 and Γ_2 are isomorphic.*

Proof. By Theorem 4.4.1 Σ_{Γ_i} , $i = 1, 2$, is a Fuchsian building. Applying the quasi-isometric rigidity of Fuchsian buildings, Theorem 4.1.5, yields that any quasi-isometry $f : \Sigma_{\Gamma_1} \rightarrow \Sigma_{\Gamma_2}$ is at a finite distance from an isomorphism. In particular, if W_{Γ_1} and W_{Γ_2} are quasi-isometric, then there exists a quasi-isometry $f : \Sigma_{\Gamma_1} \rightarrow \Sigma_{\Gamma_2}$. As previously stated, f lies at finite distance from an isomorphism implying that Σ_{Γ_1} and Σ_{Γ_2} are isomorphic. Recall that the vertex links of Σ_{Γ_i} are Γ_i . Since Σ_{Γ_1} and Σ_{Γ_2} are isomorphic, the vertex links Γ_1 and Γ_2 must also be isomorphic. Conversely, if Γ_1, Γ_2 are isomorphic, then clearly $W_{\Gamma_1}, W_{\Gamma_2}$ are isometric, and so are quasi-isometric. $\square$

The following theorem due to Kapovitch-Kleiner-Leeb allows us to prove Corollary 4.4.4.

Theorem 4.4.3 (Theorem B in [24]). *Suppose $M = Z \times \prod_{i=1}^k M_i$ and $N = W \times \prod_{i=1}^l N_i$ are geodesic metric spaces such that the asymptotic cones of Z and W are homeomorphic to $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively, and the components M_i, N_j are of coarse type I or II. Then for every $L \geq 1, A \geq 0$ there is a constant D so that for each (L, A) -quasi-isometry $\phi : M \rightarrow N$ we have $k = l, n = m$ and after reindexing the factors N_j there are quasi-isometries $\phi_i : M_i \rightarrow N_i$ such that for every i the following diagram commutes up to error at most D :*

$$\begin{array}{ccc} M & \xrightarrow{\phi} & N \\ \downarrow & & \downarrow \\ M_i & \xrightarrow{\phi_i} & N_i \end{array}$$

We refer the reader to [25] for precise definitions of asymptotic cones and coarse types. We note that the asymptotic cone of a Gromov hyperbolic space is an $\mathbb{R}$ -tree. Thus given a generalized m -gon with $m \in \{3, 4, 6, 8\}$ the corresponding Davis complex Σ_Γ is of coarse type I.

Corollary 4.4.4. *Let Γ_1, Γ_2 be finite joins of graphs from $\mathcal{S}$. Then any quasi-isometry $f : \Sigma_{\Gamma_1} \rightarrow \Sigma_{\Gamma_2}$ is at a finite distance from an isometry. In particular, W_{Γ_1} and W_{Γ_2} are quasi-isometric if and only if Γ_1 and Γ_2 are isomorphic.*

Proof. Say $\Gamma_1 = \mathcal{M}_1 \star \mathcal{M}_2 \star \cdots \star \mathcal{M}_k$ and $\Gamma_2 = \mathcal{N}_1 \star \mathcal{N}_2 \star \cdots \star \mathcal{N}_l$ with $\mathcal{M}_i, \mathcal{N}_j \in \mathcal{S}$. As seen in Remark 3.1.3, the corresponding RACGs are then the direct products $W_{\Gamma_1} = \prod_{i=1}^k W_{\mathcal{M}_i}$ and $W_{\Gamma_2} = \prod_{i=1}^l W_{\mathcal{N}_i}$. It follows that $\Sigma_{\Gamma_1} = \prod_{i=1}^k \Sigma_{\mathcal{M}_i}$ and $\Sigma_{\Gamma_2} = \prod_{i=1}^l \Sigma_{\mathcal{N}_i}$.

First suppose $f : \Sigma_{\Gamma_1} \rightarrow \Sigma_{\Gamma_2}$ is a quasi-isometry. All of the hypotheses of Theorem 4.4.3 are clearly satisfied with Z and W homeomorphic to $\mathbb{R}^0$, hence $k = l$ and, after reindexing, there are quasi-isometries $\phi_i : W_{\mathcal{M}_i} \rightarrow W_{\mathcal{N}_i}$ such that f is at finite distance from $(\phi_1, \phi_2, \dots, \phi_k)$. By Theorem 4.4.2, each ϕ_i is at a finite distance from an isomorphism, resulting in f itself being at a finite distance from an isomorphism.

In particular, if W_{Γ_1} and W_{Γ_2} are quasi-isometric, then Σ_{Γ_1} and Σ_{Γ_2} are quasi-isometric. Thus, after reindexing, $k = l$ and $W_{\mathcal{M}_i}$ is quasi-isometric to $W_{\mathcal{N}_i}$, for each $i = 1, \dots, k$. By Theorem 4.4.2, $\Sigma_{\mathcal{M}_i}$ is isomorphic to $\Sigma_{\mathcal{N}_i}$ and M_i is isomorphic to N_i for each i . It must then be the case that Γ_1 and Γ_2 are isomorphic. Conversely, if Γ_1 and Γ_2 are isomorphic, then clearly W_{Γ_1} and W_{Γ_2} are isometric, and therefore quasi-isometric. $\square$

CHAPTER 5 CONSTRUCTING COMMENSURABLE RACGS

5.1 Commensurability

Definition 5.1.1. *Two groups G and H are said to be **(abstractly) commensurable** if they have isomorphic finite index subgroups.*

Given finitely generated commensurable groups G and H it follows from Example 1.3.10 and Proposition 1.3.3 that G and H are quasi-isometric. As a result, the question of commensurability often arises in the study of the quasi-isometric classification of RACGs.

In general, there exist non-isomorphic graphs which induce commensurable, and hence quasi-isometric, RACGs. For example, consider the class of RACGs defined by n -cycles, Γ_n , with $n \geq 5$. Each W_{Γ_n} is generated by reflections about the sides of a right-angled n -gon in the hyperbolic plane. The resulting groups are, in fact, all commensurable and it is clear that Γ_n is isomorphic to Γ_m if and only if $n = m$.

Work has been done in establishing commensurability classes within the family of RACGs. Particular focus has been devoted to RACGs whose defining graphs are generalized theta graphs, graphs with two distinguished vertices of valence $k \geq 3$ with k branches connecting them and restrictions on the number of vertices that lie on each branch. Crisp-Paoluzzi [10] establish criteria for two RACGs with defining graphs given by generalized theta graphs of linear degree 1 to be commensurable. This result was then generalized to all RACGs defined by generalized theta graphs by Dani-Stark-Thomas [12]. Their work with generalized theta graphs shows that there are infinitely many commensurability classes within each quasi-isometry class that contains a RACG defined by a generalized theta graph.

Additionally, much has been discovered about the commensurability of RACGs with other classes of finitely generated groups. In particular, one can consider the *right-angled Artin group* (RAAG for short) A_Γ associated to a finite simplicial graph Γ . A_Γ is given by the following

presentation:

$$A_\Gamma = \langle v \in V(\Gamma) \mid v_1 v_2 = v_2 v_1 \text{ if and only if } (v_1, v_2) \in E(\Gamma) \rangle.$$

Example 5.1.2. (1) Let Γ be the graph with n vertices and no edges. Then A_Γ is the free group on n generators.

(2) If Γ is an n -clique, then $A_\Gamma = \mathbb{Z}^n$.

There exists a CAT(0) cube complex upon which A_Γ acts cocompactly by isometries called the *Salveti complex*. For a complete description see [16]. Davis-Januszkiewicz [17] establish that given a RAAG A , there exists a RACG W such that A and W are commensurable by showing the corresponding Salvetti and Davis complexes coincide.

Conversely, it is quite a simple task to find RACGs which are not commensurable to any RAAG. A 1-ended RAAG has at most quadratic divergence [2]. Thus, for a 1-ended RACG W_Γ to be commensurable with a RAAG, Γ must at least be $\mathcal{CFS}$ or a non-trivial join. However, these conditions are neither necessary nor sufficient as illustrated by examples due to Behrstock [1] and LaForge [26]. It is still an open question as to what conditions are necessary and sufficient on Γ so that W_Γ is commensurable with some RAAG.

5.2 A construction of commensurable RACGs

The ability to generate examples of commensurable and quasi-isometric RACGs can be helpful in examining the various classifications within this family of groups. This section describes a method for constructing examples of commensurable RACGs as detailed in [8]. This method is inspired by a construction of Bestvina-Kleiner-Sageev in the case of right-angled Artin groups (see Example 1.4 of [7]).

Let Γ be a finite simplicial graph and $K \subset \Gamma$ a clique. By definition of W_Γ a reduced word for an element $g \in W_\Gamma$ is a word in the free group with basis $V(\Gamma)$ that cannot be shortened via a sequence of operations of either deleting a consecutive pair of the form uu , $u \in V(\Gamma)$, or transposing two consecutive letters u, v such that $uv = vu$ in W_Γ (see Section 1.1).

Define $\rho : V(\Gamma) \rightarrow V(K) \cup \{1\}$ by

$$\rho(v) = \begin{cases} v & \text{if } v \in V(K) \\ 1 & \text{otherwise.} \end{cases}$$

Clearly $\rho(v)^2 = 1$ for every $v \in V(\Gamma)$. Moreover, given $v_1, v_2 \in V(\Gamma)$ such that $v_1 v_2 = v_2 v_1$ we also have $\rho(v_1) \rho(v_2) = \rho(v_1) \rho(v_2)$. Hence ρ can be extended to a homomorphism $\phi : W_\Gamma \rightarrow W_K$ by Lemma 1.1.6. As K is a clique, the corresponding RACG W_K is finite. It follows from the first isomorphism theorem of group theory that $\ker \phi$ is a finite index subgroup of W_Γ . We construct a graph Γ' such that $W_{\Gamma'}$ is isomorphic to $\ker \phi$. The commensurability of W_Γ and $W_{\Gamma'}$ is then immediate.

Roughly speaking, Γ' is constructed from Γ by removing K to obtain a graph Λ and then gluing together $2^{|K|}$ copies of Λ where each copy corresponds to conjugation by an element of W_K . When K is chosen to be a single vertex this method is the same as the doubling along a vertex construction (see Remark 1.9 of [12]). The resulting RACG $W_{\Gamma'}$ is isomorphic to a finite index subgroup of W_Γ . In particular, it follows immediately that W_Γ and $W_{\Gamma'}$ are quasi-isometric. An example is given in Figure 5.1.

Before constructing Γ' it is helpful to find a useful generating set for $\ker \phi$. As a finite index subgroup of a finitely generated group, $\ker \phi$ must itself be finitely generated (a consequence of Theorem 1.3.6). Thus we first focus attention on finding a suitable finite generating set for $\ker \phi$.

Enumerate the vertices of K as $V(K) = \{v_1, \dots, v_k\}$. As seen in Remark 3.1.3 every element of W_K can be written in the form $v_1^{\epsilon_1} v_2^{\epsilon_2} \dots v_k^{\epsilon_k}$ where $\epsilon_i \in \mathbb{Z}_2$ for all i . As a result, there is a clear bijection $g : \mathbb{Z}_2^k \rightarrow W_K$ defined by $g(\epsilon_1, \dots, \epsilon_k) = v_1^{\epsilon_1} \dots v_k^{\epsilon_k}$.

Consider the sets $T = \{g(\epsilon) v g(\epsilon) : v \in V(\Gamma) \setminus V(K), \epsilon \in \mathbb{Z}_2^k\}$ and $R = \{g(\epsilon) v g(\epsilon) \in T : g(\epsilon) v g(\epsilon) \text{ is reduced in } W_\Gamma\}$. Given $g(\epsilon) v g(\epsilon) \in T \setminus R$ there must be some j with $\epsilon_j = 1$ such that $(v, v_j) \in E(\Gamma)$. Then v, v_j commute in W_Γ and we have

$$v_1^{\epsilon_1} \dots v_j \dots v_k^{\epsilon_k} v v_1^{\epsilon_1} \dots v_j \dots v_k^{\epsilon_k} = v_1^{\epsilon_1} \dots v_{j-1}^{\epsilon_{j-1}} v_{j+1}^{\epsilon_{j+1}} \dots v_k^{\epsilon_k} v_j v v_1^{\epsilon_1} \dots v_{j-1}^{\epsilon_{j-1}} v_{j+1}^{\epsilon_{j+1}} \dots v_k^{\epsilon_k}$$

$$= v_1^{\epsilon_1} \cdots v_{j-1}^{\epsilon_{j-1}} v_{j+1}^{\epsilon_{j+1}} \cdots v_k^{\epsilon_k} v v_1^{\epsilon_1} \cdots v_{j-1}^{\epsilon_{j-1}} v_{j+1}^{\epsilon_{j+1}} \cdots v_k^{\epsilon_k} \text{ in } W_\Gamma.$$

It follows inductively that there exists $\delta = (\delta_1, \dots, \delta_k) \in \mathbb{Z}_2^k$ such that $g(\epsilon)vg(\epsilon) = g(\delta)vg(\delta)$ in W_Γ and $g(\delta)vg(\delta)$ is reduced, hence R generates $\langle T \rangle$.

Lemma 5.2.1. $\ker \phi = \langle R \rangle$.

Proof. Let $g(\epsilon_1)u_1g(\epsilon_1)g(\epsilon_2)u_2g(\epsilon_2) \cdots g(\epsilon_p)u_pg(\epsilon_p) \in \langle R \rangle$. Then

$$\begin{aligned} & \phi(g(\epsilon_1)u_1g(\epsilon_1)g(\epsilon_2)u_2g(\epsilon_2) \cdots g(\epsilon_p)u_pg(\epsilon_p)) \\ &= g(\epsilon_1)\phi(u_1)g(\epsilon_1)g(\epsilon_2)\phi(u_2)g(\epsilon_2) \cdots g(\epsilon_p)\phi(u_p)g(\epsilon_p) \\ &= g(\epsilon_1)g(\epsilon_1)g(\epsilon_2)g(\epsilon_2) \cdots g(\epsilon_p)g(\epsilon_p) = 1, \end{aligned}$$

hence $\langle T \rangle \subset \ker \phi$.

To verify the reverse containment let $w \in \ker \phi \setminus \{1\}$. Then there exist $u_1, u_2, \dots, u_n \in V(\Gamma)$, $n > 0$, such that $u_i \neq u_{i+1}$ for all i and $w = u_1u_2 \cdots u_n$. We show by induction on n that $w \in \langle R \rangle$. When $n = 1$ we have $1 = \phi(w) = \phi(u_1)$. Thus $u_1 \in V(\Gamma) \setminus V(K)$ and $w = g(0)u_1g(0) \in \langle T \rangle$ where $0 = (0, \dots, 0) \in \mathbb{Z}_2^k$. So suppose $n > 1$ and that every element of $\ker \phi$ that is a product of at most $n - 1$ elements of $V(\Gamma)$ lies in $\langle T \rangle$. If $u_1 \in V(\Gamma) \setminus V(K)$, then $1 = \phi(u_1)\phi(w) = \phi(u_1w) = \phi(u_2 \cdots u_n)$. Hence $u_2 \cdots u_n \in \ker \phi$, has length $n - 1$, and therefore $u_2 \cdots u_n \in \langle T \rangle$ by the inductive hypothesis. Since $u_1 \in V(\Gamma) \setminus V(K)$ it follows that $u_1 = g(0)u_1g(0) \in \langle T \rangle$ and $w \in \langle T \rangle$.

Now suppose $u_1 \in V(K)$. Let $i > 1$ be the smallest integer such that $u_{i+1} \notin V(K)$ and $j > i$ the smallest integer such that $u_j \in V(K)$. For each t with $1 \leq t \leq j - 1 - i$ set $w_t = (u_1 \cdots u_i)u_{i+t}(u_1 \cdots u_i)$. Note that $(u_1 \cdots u_i)^2 = u_1 \cdots u_i u_1 \cdots u_i = u_1^2 \cdots u_i^2 = 1$ since $u_1, \dots, u_i \in V(K)$. Thus

$$\begin{aligned} w &= u_1 \cdots u_i u_{i+1} \cdots u_j \cdots u_n \\ &= (u_1 \cdots u_i)u_{i+1}(u_1 \cdots u_i)(u_1 \cdots u_i)u_{i+2} \cdots u_j \cdots u_n \end{aligned}$$

$$\begin{aligned}
&= w_1(u_1 \cdots u_i)u_{i+2} \cdots u_j \cdots u_n \\
&= w_1(u_1 \cdots u_i)u_{i+2}(u_1 \cdots u_i)(u_1 \cdots u_i)u_{i+3} \cdots u_j \cdots u_n \\
&= w_1w_2(u_1 \cdots u_i)u_{i+3} \cdots u_j \cdots u_n \\
&\quad \vdots \\
&= w_1w_2 \cdots w_{j-1-i}u_1 \cdots u_iu_j \cdots u_n.
\end{aligned}$$

Each $u_{i+t} \in V(\Gamma) \setminus V(K)$ for $1 \leq t \leq j-1-i$, implying that $\phi(w_t) = (u_1 \cdots u_i) \cdot 1 \cdot (u_1 \cdots u_i) = (u_1 \cdots u_i)^2 = 1$ for all t . Therefore,

$$\begin{aligned}
1 = \phi(w) &= \phi(w_1w_2 \cdots w_{j-1-i}u_1 \cdots u_iu_j \cdots u_k) = \phi(w_1) \cdots \phi(w_{j-1-i})\phi(u_1 \cdots u_iu_j \cdots u_k) \\
&= \phi(u_1 \cdots u_iu_j \cdots u_k).
\end{aligned}$$

The element $u_1 \cdots u_iu_j \cdots u_k$ then lies in $\ker \phi$ and has length at most $n-1$. It follows from the inductive hypothesis that $u_1 \cdots u_iu_j \cdots u_k \in \langle T \rangle$. Thus $w \in \langle T \rangle$. As previously noted, R generates $\langle T \rangle$. Hence $w \in \langle R \rangle$ and $\ker \phi \subset \langle R \rangle$. $\square$

Construction 5.2.2 (Construction of Γ'). Set $V(\Gamma') = R$. Two vertices $g(\epsilon)vg(\epsilon)$ and $g(\delta)ug(\delta)$ are joined by an edge in Γ' if $(v, u) \in E(\Gamma)$ and there exists $\gamma \in \mathbb{Z}_2^k$ such that $g(\epsilon)vg(\epsilon) = g(\gamma)vg(\gamma)$, $g(\delta)ug(\delta) = g(\gamma)ug(\gamma)$ in W_Γ . Γ' is clearly a simplicial graph.

Visually, Γ' is formed by joining together multiple isomorphic copies of the graph Λ , the graph formed by removing the subgraph K along with all edges incident to a vertex in K from Γ . For each $\epsilon \in \mathbb{Z}_2^k$ define the graph Λ_ϵ to have vertex set $T_\epsilon = \{g(\epsilon)vg(\epsilon) : v \in V(\Gamma) \setminus V(K)\}$ and an edge joining vertices $g(\epsilon)vg(\epsilon), g(\epsilon)ug(\epsilon)$ if v, u are adjacent in Λ . Γ' is obtained by taking the union $\bigcup_{\epsilon \in \mathbb{Z}_2^k} \Lambda_\epsilon$ and identifying the vertices $g(\epsilon)vg(\epsilon)$ and $g(\delta)ug(\delta)$ if $g(\epsilon)vg(\epsilon) = g(\delta)ug(\delta)$ in W_Γ . See Figure 5.1 for an example.

Lemma 5.2.3. $\ker \phi \cong W_{\Gamma'}$.

Proof. We verify the isomorphic relationship by constructing an isomorphism from $W_{\Gamma'}$ into

Figure 5.1: Commensurable construction when Γ is a 5-cycle. Let Γ be the given five cycle and K the clique consisting of the vertices v_1, v_2 and their shared edge. We obtain Γ' by taking the union of the graphs $\Lambda, \Lambda_{(1,0)}, \Lambda_{(0,1)}, \Lambda_{(1,1)}$ and identifying the vertices $v_3 = v_2v_3v_2$, $v_2v_5v_2 = v_1v_2v_5v_1v_2v_5$, $v_1v_3v_1 = v_1v_2v_3v_1v_2$, and $v_5 = v_1v_5v_1$. The result is the 8-cycle pictured above. As previously noted, the RACGs generated from a 5-cycle and 8-cycle are commensurable.

$\ker \phi$. There is a clear inclusion map $\iota : R \rightarrow \ker \phi$ by Lemma 5.2.1. Moreover, ι satisfies $\iota((g(\epsilon)vg(\epsilon))^2) = (g(\epsilon)vg(\epsilon))^2 = g(\epsilon)vg(\epsilon)g(\epsilon)vg(\epsilon) = 1$ for all $g(\epsilon)vg(\epsilon) \in R$. Additionally, given $g(\epsilon)vg(\epsilon), g(\delta)ug(\delta) \in R$ with $(g(\epsilon)vg(\epsilon), g(\delta)ug(\delta)) \in E(\Gamma')$ it must be the case that $(u, v) \in E(\Gamma)$ and there is some $\gamma \in \mathbb{Z}_2^k$ such that $g(\epsilon)vg(\epsilon) = g(\gamma)vg(\gamma)$ and $g(\delta)ug(\delta) = g(\gamma)ug(\gamma)$ in W_Γ . Thus

$$\begin{aligned}
 & \iota(g(\epsilon)vg(\epsilon)g(\delta)ug(\delta)g(\epsilon)vg(\epsilon)g(\delta)ug(\delta)) \\
 &= g(\epsilon)vg(\epsilon)g(\delta)ug(\delta)g(\epsilon)vg(\epsilon)g(\delta)ug(\delta) \\
 &= g(\gamma)vg(\gamma)g(\gamma)ug(\gamma)g(\gamma)vg(\gamma)g(\gamma)ug(\gamma) \\
 &= g(\gamma)vuvug(\gamma) = 1.
 \end{aligned}$$

As ι preserves the relators of $W_{\Gamma'}$ it can be extended to a surjective homomorphism $f : W_{\Gamma'} \rightarrow \ker \phi$ by Lemmas 1.1.6 and 5.2.1. All that remains is to show that f is an injection, i.e., $\ker f$ is trivial.

Suppose by way of contradiction that $\ker f$ is nontrivial. Let $w = g(\epsilon_1)u_1g(\epsilon_1) \cdots g(\epsilon_p)u_pg(\epsilon_p) \in \ker f \setminus \{1\}$ be of shortest length. Note that $p \neq 1$ for if it did, then $1 = f(w) = f(g(\epsilon_1)u_1g(\epsilon_1)) = g(\epsilon_1)u_1g(\epsilon_1) \in R$ which is not possible by definition of R . Hence $p > 1$. Since

$$1 = f(w) = f(g(\epsilon_1)u_1g(\epsilon_1) \cdots g(\epsilon_p)u_pg(\epsilon_p)) = g(\epsilon_1)u_1g(\epsilon_1) \cdots g(\epsilon_p)u_pg(\epsilon_p)$$

in W_Γ , this expression for $f(w)$ can be reduced through a sequence of deletion and transposition operations in order to obtain the empty word in W_Γ .

Rewriting the above expression yields

$$f(w) = g(\epsilon_1)u_1g(\epsilon_1) \cdots g(\epsilon_p)u_pg(\epsilon_p) = g(\epsilon_1)u_1g(\epsilon_1 + \epsilon_2)u_2g(\epsilon_2 + \epsilon_3) \cdots g(\epsilon_{p-1} + \epsilon_p)u_pg(\epsilon_p).$$

Set $\epsilon_0 = \epsilon_{p+1} = 0$ and $\delta_i = \epsilon_{i-1} + \epsilon_i$ for $1 \leq i \leq p+1$ and let $v_1^{a_{i,1}}v_2^{a_{i,2}} \cdots v_k^{a_{i,k}} = g(\delta_i)$ for $1 \leq i \leq p+1$. In order for $f(w)$ to equal 1 in W_Γ there must be s, t , with $t > s$, such that $u_s = u_t$, $u_s u_j = u_j u_s$ for all $s < j < t$, and $u_s v_r^{a_{i,r}} = v_r^{a_{i,r}} u_s$ for all $s+1 \leq i \leq t$, $1 \leq r \leq k$. Then

$$\begin{aligned} & g(\epsilon_s)u_sg(\epsilon_s) \cdots g(\epsilon_t)u_tg(\epsilon_t) \\ &= g(\epsilon_s)u_sg(\epsilon_s)g(\epsilon_{s+1})u_{s+1}g(\epsilon_{s+1}) \cdots g(\epsilon_t)u_tg(\epsilon_t) \\ &= g(\epsilon_s)u_sg(\delta_{s+1})u_{s+1}g(\delta_{s+2}) \cdots g(\delta_t)u_tg(\epsilon_t) && \text{with } \delta_i = \epsilon_{i-1} + \epsilon_i \\ &= g(\epsilon_s)g(\delta_{s+1})u_{s+1}g(\delta_{s+2}) \cdots g(\delta_t)u_s u_tg(\epsilon_t) && \text{since } u_s u_j = u_j u_s, u_s v_r^{a_{i,r}} = v_r^{a_{i,r}} u_s \\ &= g(\epsilon_s)g(\delta_{s+1})u_{s+1}g(\delta_{s+2}) \cdots g(\delta_t)g(\epsilon_t) && \text{because } u_s = u_t \\ &= g(\epsilon_s)g(\epsilon_s)g(\epsilon_{s+1})u_{s+1}g(\epsilon_{s+1}) \cdots g(\epsilon_{t-1})g(\epsilon_t)g(\epsilon_t) && \text{with } \delta_i = \epsilon_{i-1} + \epsilon_i \\ &= g(\epsilon_{s+1})u_{s+1}g(\epsilon_{s+1}) \cdots g(\epsilon_{t-1})u_{t-1}g(\epsilon_{t-1}). \end{aligned}$$

Setting $g(\epsilon_{s+1})u_{s+1}g(\epsilon_{s+1}) \cdots g(\epsilon_{t-1})u_{t-1}g(\epsilon_{t-1}) = w'$ yields

$$\begin{aligned} f(w) &= g(\epsilon_1)u_1g(\epsilon_1) \cdots g(\epsilon_s)u_sg(\epsilon_s) \cdots g(\epsilon_t)u_tg(\epsilon_t) \cdots g(\epsilon_p)u_pg(\epsilon_p) \\ &= g(\epsilon_1)u_1g(\epsilon_1) \cdots g(\epsilon_{s-1})u_{s-1}g(\epsilon_{s-1})w'g(\epsilon_{t+1})u_{t+1}g(\epsilon_{t+1}) \cdots g(\epsilon_p)u_pg(\epsilon_p) \end{aligned}$$

in W_Γ . Let $\hat{w} = g(\epsilon_1)u_1g(\epsilon_1) \cdots g(\epsilon_{s-1})u_{s-1}g(\epsilon_{s-1})w'g(\epsilon_{t+1})u_{t+1}g(\epsilon_{t+1}) \cdots g(\epsilon_p)u_pg(\epsilon_p)$ for convenience. Certainly $\hat{w} \in W_{\Gamma'}$ and $f(\hat{w}) = \hat{w} = f(w) = 1$, hence $\hat{w} \in \ker f$ and has length $p-2$ in $W_{\Gamma'}$. As w was chosen to be an element of $\ker f \setminus \{1\}$ with shortest length it follows that $\hat{w}$ must be the empty word in $W_{\Gamma'}$, forcing $p = 2$. Then $w = g(\epsilon_1)u_1g(\epsilon_1)g(\epsilon_2)u_1g(\epsilon_2)$ with $u_1 v_j^{a_{2,j}} = v_j^{a_{2,j}} u_1$, $1 \leq j \leq k$, by our previous observations.

For $i = 1, 2$ let $\epsilon_i = (\epsilon_{i,1}, \epsilon_{i,2}, \dots, \epsilon_{i,k})$. Note that

$$v_j^{a_{2,j}} = v_j^{\epsilon_{1,j} + \epsilon_{2,j}} = \begin{cases} 1 & \text{if } \epsilon_{1,j} = \epsilon_{2,j} \\ v_j & \text{if } \epsilon_{1,j} \neq \epsilon_{2,j} \end{cases}.$$

Suppose there is some j such that $v_j^{a_{2,j}} = v_j$. This can only happen if exactly one of ϵ_1, ϵ_2 have j^{th} coordinate 1, say ϵ_1 . But this implies that the expression $g(\epsilon_1)u_1g(\epsilon_1)$ can be reduced, a contradiction since $g(\epsilon_1)u_1g(\epsilon_1) \in R$ and is reduced by definition of R . It must then be the case that $v_j^{a_{2,j}} = 1$ for all j . That is, $a_{1,j} = a_{3,j}$ for all j and $\epsilon_1 = \epsilon_2$. It follows that $w = g(\epsilon_1)u_1g(\epsilon_1)g(\epsilon_1)u_1g(\epsilon_1) = 1$ in $W_{\Gamma'}$, contradicting our assumption that w was nontrivial. $\square$

Proposition 5.2.4. W_{Γ} and $W_{\Gamma'}$ are commensurable.

Proof. $W_{\Gamma'} \cong \ker \phi$ by Lemma 5.2.3. As previously observed, $\ker \phi$ is a finite index subgroup of W_{Γ} because K is a clique. Therefore W_{Γ} and $W_{\Gamma'}$ are commensurable. $\square$

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