QUASICONFORMAL MAPS ON A 2-STEP CARNOT GROUP

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ABSTRACT

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In this paper, we find all the quasiconformal maps on a particular non-rigid 2-step Carnot group. In particular, all quasiconformal maps on this Carnot group preserve the vertical direction. Given that a Carnot group is a Lie algebra with a group structure, we employ concepts from linear algebra and abstract algebra to gain information about the group. Utilizing the theory of Pansu differentiability along with the biLipschitz nature of quasisymmetric maps, we use an analytical approach to help determine the form of any quasiconformal map on the Carnot group. The main result has consequences for the rigidity of quasiisometries of negatively curved solvable Lie groups.

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CHAPTER 1 INTRODUCTION TO CARNOT GROUPS

1.1 Lie algebras

Lie groups are inherently tied to Lie algebras. For every Lie group, there is a corresponding Lie algebra, and conversely, for every finite-dimensional Lie algebra over the real numbers, $\mathbb{R}$, there is a corresponding Lie group. For our purposes, we begin with the notion of a Lie algebra and show how the particular Lie algebra we work with in this paper can be considered as a Lie group. We begin by defining an integral piece of any Lie algebra, the Lie bracket.

Definition 1.1.1. *Let V be a vector space over a field F . A Lie bracket is a map from $V \times V$ to V , where the image of $(\vec{x}, \vec{y}) \in V \times V$ is denoted by $[\vec{x}, \vec{y}]$, and which satisfies the following conditions:*

1. $[a\vec{x} + b\vec{y}, \vec{z}] = a[\vec{x}, \vec{z}] + b[\vec{y}, \vec{z}]$ for $a, b \in F$ and $\vec{x}, \vec{y}, \vec{z} \in V$.
2. $[\vec{x}, \vec{y}] = -[\vec{y}, \vec{x}]$ for $\vec{x}, \vec{y} \in V$.
3. $[[\vec{x}, \vec{y}], \vec{z}] + [[\vec{y}, \vec{z}], \vec{x}] + [[\vec{z}, \vec{x}], \vec{y}] = 0$ for $\vec{x}, \vec{y}, \vec{z} \in V$.

That is, a Lie bracket is a binary operation on the vector space V , that is bilinear, anticommutative, and satisfies the Jacobi identity. The only case that we consider is when $F = \mathbb{R}$, and anticommutativity implies that $[\vec{x}, \vec{x}] = 0$ for any $\vec{x} \in V$ in this case. (This property is referred to as alternativity.)

Definition 1.1.2. *A Lie algebra is a vector space V together with a Lie bracket.*

Definition 1.1.3. *A Lie algebra, $\mathfrak{n}$, is called an r -step Carnot Lie algebra if it admits a direct sum decomposition $\mathfrak{n} = V_1 \oplus V_2 \oplus \dots \oplus V_r$ such that $V_r \neq 0$, $[V_1, V_j] = V_{j+1}$ for $j = 1, 2, \dots, r-1$, and $[V_1, V_r] = 0$.*

It should be noted that $[V_1, V_j]$ is the linear subspace spanned by $[\vec{x}, \vec{y}]$ where $\vec{x} \in V_1$ and $\vec{y} \in V_j$. Each V_i in the definition of a Carnot Lie algebra is referred to as the i th layer. Furthermore, V_1 is

considered to be the horizontal direction, while the remaining layers, $V_2, \dots, V_r$, are considered to be the vertical directions.

The specific Lie algebra we consider is $\mathfrak{n} = \mathbb{R}X \oplus \mathbb{R}Y_1 \oplus \mathbb{R}Y_2 \oplus \mathbb{R}Z_1 \oplus \mathbb{R}Z_2$, where X, Y_1, Y_2, Z_1 , and Z_2 are orthonormal vectors. (For ease of notation, these vectors are written without arrows.) We define the brackets $[X, Y_1] = Z_1$ and $[X, Y_2] = Z_2$. All the other brackets of the basis vectors are 0. Setting $V_1 = \mathbb{R}X \oplus \mathbb{R}Y_1 \oplus \mathbb{R}Y_2$ and $V_2 = \mathbb{R}Z_1 \oplus \mathbb{R}Z_2$, we get a 2-step Carnot Lie algebra, $\mathfrak{n} = V_1 \oplus V_2$. From now on, we reserve the use of $\mathfrak{n}$ to refer specifically to this Carnot Lie algebra.

1.2 Group operation, norm, and metric

Given the basic framework of a Carnot Lie algebra, as well as the particular example we are working with, we discuss the corresponding Carnot group by first defining the group operation. The operation, $*$, is given by the Baker-Campbell-Hausdorff formula (BCH formula).

Definition 1.2.1. *Given $p, q \in \mathfrak{n}$, we define $p * q = p + q + \frac{1}{2} [p, q]$.*

It's easy to check that $(\mathfrak{n}, *)$ does indeed define a group. (The identity and inverses match those of vector addition.) So, $\mathfrak{n}$ is a Lie algebra with corresponding Lie group $(\mathfrak{n}, *)$. We refer to the Lie algebra and the associated Lie group interchangeably as simply $\mathfrak{n}$, with the understanding that $\mathfrak{n}$ has its group structure provided by the BCH formula.

Our eventual goal is to discuss quasiconformal maps on $\mathfrak{n}$. In order to do so, a metric on $\mathfrak{n}$ is needed. The metric associated with Carnot groups is defined to be the infimum of the length of horizontal curves joining two points. This is referred to as the Carnot metric, d_c . Rather than working with the Carnot metric, we define a distance function that can be used in its place. We proceed by first defining a 'norm' on $\mathfrak{n}$.

Definition 1.2.2. *Let $|\cdot|_1$ and $|\cdot|_2$ be norms on V_1 and V_2 , respectively. The 'norm,' $\| \cdot \|$, is given by $\|p\| = \|v_1 + v_2\| = |v_1|_1 + (|v_2|_2)^{\frac{1}{2}}$ for $p \in \mathfrak{n}$, where $p = v_1 + v_2$ for some $v_1 \in V_1$ and $v_2 \in V_2$.*

We can say the specific norms used on V_1 and V_2 are the Euclidean norm and write $\|p\| = \|v_1 + v_2\| = |v_1| + |v_2|^{\frac{1}{2}}$. For clarification, the 'norm' defined is not a norm in the same sense

that the Euclidean norm is a norm. (Hence, the use of quotes.) From now on, we drop the quotes and simply say norm to refer to the ‘norm,’ as is standard practice for Lie groups. The following proposition states precisely how we use the norm to define the distance function on $\mathfrak{n}$.

Definition 1.2.3. *The distance function $d : \mathfrak{n} \times \mathfrak{n} \longrightarrow [0, \infty)$ is given by $d(p, q) = \|(-p) * q\|$ for $p, q \in \mathfrak{n}$.*

We’re not concerned with whether or not d defines a metric in the analytical sense because it’s biLipschitz equivalent with the Carnot metric, d_c .

Proposition 1.2.4. *There is a constant $C \geq 1$ such that*

$$\frac{1}{C} \cdot d(p, q) \leq d_c(p, q) \leq C \cdot d(p, q)$$

for all $p, q \in \mathfrak{n}$.

This property is what allows us to make use of d , rather than d_c , as we do our calculations. Another benefit of defining d in this manner is that it’s left-invariant.

Proposition 1.2.5. *$d(p, q) = d(s * p, s * q)$ for $p, q, s \in \mathfrak{n}$.*

Proof. $d(p, q) = \|(-p) * q\| = \|(-p) * (-s) * s * q\| = \|(s * p) * (s * q)\| = d(s * p, s * q).$

□

When we begin to discuss quasiconformal maps on metric spaces, the metric associated with the Lie group is d_c , however, we always make use of d to do our calculations.

1.3 Subalgebras, subgroups, and left cosets

A subalgebra is simply a subspace that is closed under the algebra’s associated product. In the case of a Lie algebra, the product is the Lie bracket. So a subalgebra, in the context we’re using it, is a subspace of $\mathfrak{n}$ that’s also closed under the Lie bracket. It’s beneficial for us to consider two particular subalgebras, which can be considered as subgroups of $\mathfrak{n}$.

Definition 1.3.1. *A subspace, H , of $\mathfrak{n}$ is a subalgebra if $[p, q] \in H$ for $p, q \in H$.*

Given a fixed $p \in \mathfrak{n}$, we consider the map $ad(p) : \mathfrak{n} \longrightarrow \mathfrak{n}$, where $ad(p)(q) = [p, q]$ for $q \in \mathfrak{n}$. By bilinearity of the Lie bracket, $ad(p)$ is a linear transformation. We refer to the rank of this linear transformation as $r(p)$, and we define $r(\mathfrak{n}) = \min \{r(p) \mid p \in V_1 \setminus \{0\}\}$. Recall, the rank of a linear transformation is the dimension of the image of the map. Let W be the subspace of $\mathfrak{n}$ spanned by $p \in V_1$ with $r(p) = r(\mathfrak{n})$.

Proposition 1.3.2. $W = \mathbb{R}Y_1 \oplus \mathbb{R}Y_2$.

Proof. We need only consider $ad(p)$ for $p = xX + y_1Y_1 + y_2Y_2$, where $x \neq 0$, and $p = y_1Y_1 + y_2Y_2$, where y_1, y_2 aren't both 0. Let $q \in \mathfrak{n}$. So, $q = \tilde{x}X + \tilde{y}_1Y_1 + \tilde{y}_2Y_2 + \tilde{z}_1Z_1 + \tilde{z}_2Z_2$ for some $\tilde{x}, \tilde{y}_1, \tilde{y}_2, \tilde{z}_1, \tilde{z}_2 \in \mathbb{R}$.

Case 1: $p = xX + y_1Y_1 + y_2Y_2$, with $x \neq 0$.

$$\begin{aligned} ad(p)(q) &= [xX + y_1Y_1 + y_2Y_2, \tilde{x}X + \tilde{y}_1Y_1 + \tilde{y}_2Y_2 + \tilde{z}_1Z_1 + \tilde{z}_2Z_2] \\ &= [xX + y_1Y_1 + y_2Y_2, \tilde{x}X + \tilde{y}_1Y_1 + \tilde{y}_2Y_2] \\ &= x\tilde{y}_1Z_1 + x\tilde{y}_2Z_2 - \tilde{x}y_1Z_1 - \tilde{x}y_2Z_2 \end{aligned}$$

So $r(p) = 2$.

Case 2: $p = y_1Y_1 + y_2Y_2$, with y_1, y_2 not both 0.

$$\begin{aligned} ad(p)(q) &= [y_1Y_1 + y_2Y_2, \tilde{x}X + \tilde{y}_1Y_1 + \tilde{y}_2Y_2 + \tilde{z}_1Z_1 + \tilde{z}_2Z_2] \\ &= [y_1Y_1 + y_2Y_2, \tilde{x}X] \\ &= -\tilde{x}y_1Z_1 - \tilde{x}y_2Z_2 \end{aligned}$$

So $r(p) = 1$.

Every nonzero vector in V_1 falls into one of these two cases, hence $r(\mathfrak{n}) = 1$.

Therefore $r(p) = r(\mathfrak{n})$ when $0 \neq p \in \mathbb{R}Y_1 \oplus \mathbb{R}Y_2$. □

W is clearly a subspace of $\mathfrak{n}$. It's trivial to show that this subspace is closed under the Lie bracket, since any bracket is 0. So W is a subalgebra of $\mathfrak{n}$. The next subalgebra we consider is defined off of W as $N(W) = \{p \in \mathfrak{n} \mid [p, W] \subseteq W\}$.

Proposition 1.3.3. $N(W) = \mathbb{R}Y_1 \oplus \mathbb{R}Y_2 \oplus \mathbb{R}Z_1 \oplus \mathbb{R}Z_2$.

Proof. It's easy to see that if $p = xX + y_1Y_1 + y_2Y_2 + z_1Z_1 + z_2Z_2$ for some $x, y_1, y_2, z_1, z_2 \in \mathbb{R}$ with $x \neq 0$, then $[p, w] \notin W$ for $0 \neq w \in W$, as the bracket exists in the second layer, V_2 . On the other hand, if $x = 0$, then $[p, w] = 0 \in W$ for $w \in W$. $\square$

In other words, $N(W) = W \oplus V_2$. Similar to W , $N(W)$ is a subspace of $\mathfrak{n}$ that is trivially closed under the Lie bracket. So, $N(W)$ is indeed another subalgebra of $\mathfrak{n}$.

It was mentioned before that the subalgebras, W and $N(W)$, can actually be considered as subgroups of $\mathfrak{n}$. All that needs to be shown is that they're each closed under the group operation, $*$, given by the BCH formula. (This is a simple task as each is closed under vector addition, as a subspace, and closed under the Lie bracket, as a subalgebra.) Given that these subalgebras are subgroups, it's useful to consider the form of left cosets.

First, we can write $p \in \mathfrak{n}$ as $p = xX * (y_1Y_1 + y_2Y_2 + z_1Z_1 + z_2Z_2)$ for some $x, y_1, y_2, z_1, z_2 \in \mathbb{R}$. Since $y_1Y_1 + y_2Y_2 + z_1Z_1 + z_2Z_2 \in N(W)$, left cosets of $N(W)$ have the following form:

$$\begin{aligned} p * N(W) &= xX * (y_1Y_1 + y_2Y_2 + z_1Z_1 + z_2Z_2) * N(W) \\ &= xX * N(W). \end{aligned} \tag{1.3.4}$$

All elements in the same left coset of $N(W)$ have the same X component, matching that of a representative element. We can also write

$$\begin{aligned} p &= xX * (y_1Y_1 + y_2Y_2 + z_1Z_1 + z_2Z_2) \\ &= xX * (y_1Y_1 + y_2Y_2) * (z_1Z_1 + z_2Z_2) \\ &= xX * (z_1Z_1 + z_2Z_2) * (y_1Y_1 + y_2Y_2). \end{aligned}$$

While it's easy enough to see we can write p in this form through direct computation, it's easiest to just notice that $[y_1Y_1 + y_2Y_2, z_1Z_1 + z_2Z_2] = 0$. (It's worth mentioning that the operation, $*$, is not, in general, commutative.) Given that $y_1Y_1 + y_2Y_2 \in W$, left cosets of W have the following form:

$$\begin{aligned} p * W &= xX * (z_1Z_1 + z_2Z_2) * (y_1Y_1 + y_2Y_2) * W \\ &= xX * (z_1Z_1 + z_2Z_2) * W. \end{aligned} \tag{1.3.5}$$

All elements in the same left coset of W have the same X , Z_1 , and Z_2 components, when written in the same form as we initially wrote p . They match that of a representative element.

In the next chapter, we see how the subalgebras W and $N(W)$ are affected by different mappings. In particular, we examine how W behaves under graded isomorphisms, and we use this behavior to help identify the form of any graded isomorphism on $\mathfrak{n}$. Considering these subalgebras as subgroups, we also look at what quasiconformal mappings do to left cosets. Ultimately, this helps us to identify the form of any quasiconformal mapping on $\mathfrak{n}$.

CHAPTER 2 MAPPINGS ON LIE ALGEBRAS

2.1 Lie algebra isomorphisms

A linear isomorphism is a bijective linear transformation between vector spaces. In other words, it's a one-to-one and onto mapping that preserves the operations of scalar multiplication and vector addition. Lie algebra isomorphisms retain these properties, as well as preserving the Lie bracket operation.

Definition 2.1.1. *Let $\mathfrak{n}_1$ and $\mathfrak{n}_2$ be Lie algebras. A linear map $A : \mathfrak{n}_1 \longrightarrow \mathfrak{n}_2$ is a Lie algebra homomorphism if $A([\vec{x}, \vec{y}]) = [A(\vec{x}), A(\vec{y})]$ for $\vec{x}, \vec{y} \in \mathfrak{n}_1$. If A is a linear isomorphism as well, then we refer to A as a Lie algebra isomorphism.*

From now on, the term isomorphism implies a Lie algebra isomorphism. In any context, an isomorphism tends to preserve certain properties of the isomorphic objects. Recall the definition of the map $ad(p)$ from the previous chapter. If we generalize the notion of this linear transformation to any Lie algebra, we can still discuss the rank of such a map. So, for some Lie algebra $\mathfrak{n}_1$ and a fixed $\vec{x} \in \mathfrak{n}_1$, we refer to the rank of $ad(\vec{x})$ as $r(\vec{x})$, just as before. What we see in the next theorem is that the rank of this map is one such property that's preserved by an isomorphism.

Theorem 2.1.2. *If $A : \mathfrak{n}_1 \longrightarrow \mathfrak{n}_2$ is an isomorphism, then $r(\vec{x}) = r(A(\vec{x}))$ for $\vec{x} \in \mathfrak{n}_1$.*

Proof. Fix $\vec{x} \in \mathfrak{n}_1$. Consider the maps $ad(\vec{x}) : \mathfrak{n}_1 \longrightarrow \mathfrak{n}_1$ and $ad(A(\vec{x})) : \mathfrak{n}_2 \longrightarrow \mathfrak{n}_2$. It suffices to show the dimensions of the images of the maps, $ad(\vec{x})(\mathfrak{n}_1)$ and $ad(A(\vec{x}))(\mathfrak{n}_2)$, are equal. So we need only show there is a linear isomorphism from $ad(\vec{x})(\mathfrak{n}_1)$ to $ad(A(\vec{x}))(\mathfrak{n}_2)$. We can simply use the isomorphism A restricted to $ad(\vec{x})(\mathfrak{n}_1)$. Then, all that needs to be shown is that the restriction map is onto $ad(A(\vec{x}))(\mathfrak{n}_2)$.

Let $\vec{y} \in ad(A(\vec{x}))(\mathfrak{n}_2)$. So $\vec{y} = [A(\vec{x}), \vec{y}_1]$ for some $\vec{y}_1 \in \mathfrak{n}_2$. There exists $\vec{x}_1 \in \mathfrak{n}_1$ such that $A(\vec{x}_1) = \vec{y}_1$. Now, $\vec{y} = [A(\vec{x}), A(\vec{x}_1)] = A([\vec{x}, \vec{x}_1])$ where $[\vec{x}, \vec{x}_1] \in ad(\vec{x})(\mathfrak{n}_1)$. □

2.2 Graded isomorphisms

As alluded to in the last chapter, our interest in isomorphisms is with graded isomorphisms, which are Lie algebra isomorphisms that preserve the layers of a Carnot Lie algebra.

Definition 2.2.1. Let $\mathfrak{n}_1 = V_1 \oplus V_2 \oplus \dots \oplus V_r$ and $\mathfrak{n}_2 = V'_1 \oplus V'_2 \oplus \dots \oplus V'_s$ be Carnot Lie algebras. A Lie algebra homomorphism, $A : \mathfrak{n}_1 \longrightarrow \mathfrak{n}_2$, is called a *graded homomorphism* if $A(V_i) \subset V'_i$ for each $i = 1, 2, \dots, r$. If A is an isomorphism, then we refer to A as a *graded isomorphism*.

If A happens to be an isomorphism, then each layer is bijective. So $A(V_i) = V'_i$ for each $i = 1, 2, \dots, r$, and $r = s$. Graded isomorphisms are important to the discussion of the Pansu differential of a quasiconformal map, but first we need more information about the form of such a map on $\mathfrak{n}$. So, we consider how a graded isomorphism affects W .

Proposition 2.2.2. If $A : \mathfrak{n} \longrightarrow \mathfrak{n}$ is a graded isomorphism, then $A(W) = W$.

Proof. Let $0 \neq w \in W$. By Theorem 2.1.2, $r(A(w)) = r(w) = r(\mathfrak{n})$. Also, $A(w) \in V_1$. Hence, $A(w) \in W$. Now, $A(W) \subseteq W$, but since A is onto $\mathfrak{n}$, and for any $p \notin W$, $A(p) \notin W$, it must be that $A(W) = W$. $\square$

Let A be a graded isomorphism on $\mathfrak{n}$. From Proposition 2.2.2,

$$A(X) = eX + e_1Y_1 + e_2Y_2,$$

$$A(Y_1) = aY_1 + bY_2,$$

$$A(Y_2) = cY_1 + dY_2,$$

for some $a, b, c, d, e, e_1, e_2 \in \mathbb{R}$. We can use the Lie bracket to determine how A acts on the second layer of $\mathfrak{n}$, and we get

$$A(Z_1) = A([X, Y_1]) = [A(X), A(Y_1)] = [eX + e_1Y_1 + e_2Y_2, aY_1 + bY_2] = eaZ_1 + ebZ_2,$$

$$A(Z_2) = A([X, Y_2]) = [A(X), A(Y_2)] = [eX + e_1Y_1 + e_2Y_2, cY_1 + dY_2] = ecZ_1 + edZ_2.$$

What we see is that any graded isomorphism on $\mathfrak{n}$ is determined purely by how it acts on the first layer, V_1 . Also, since $A : W \longrightarrow W$ represents an isomorphism, $ad - bc \neq 0$. (Otherwise, A isn't a linear isomorphism.)

2.3 Quasiconformal maps

Our overarching goal is to give the form of any quasiconformal map on $\mathfrak{n}$. To do so, we need to make use of some established results. The following definitions give context to the results we need.

Definition 2.3.1. Let $K \geq 1$ and $C > 0$. A bijection $F : M \longrightarrow N$ between two metric spaces is called a (K, C) -quasi-similarity if

$$\frac{C}{K} \cdot d(x, y) \leq d(F(x), F(y)) \leq C \cdot K \cdot d(x, y)$$

for all $x, y \in M$.

It's clear that a quasi-similarity is the same as a biLipschitz map.

Definition 2.3.2. Let $F : M \longrightarrow N$ be a homeomorphism between two metric spaces. For any $x \in M$ and any $r > 0$, set

$$H_F(x, r) = \frac{\sup \{d(F(y), F(x)) \mid d(y, x) \leq r\}}{\inf \{d(F(y), F(x)) \mid d(y, x) \geq r\}}.$$

The map F is called quasiconformal if there is some $H < \infty$ such that

$$\limsup_{r \rightarrow 0} H_F(x, r) \leq H$$

for all $x \in M$.

Notice this definition doesn't indicate the form of F , but rather what condition must be satisfied for a homeomorphism to be considered quasiconformal.

Definition 2.3.3. Let $\eta : [0, \infty) \longrightarrow [0, \infty)$ be a homeomorphism. A homeomorphism $F : M \longrightarrow N$ between two metric spaces is η -quasisymmetric if for all distinct triples $x, y, z \in M$, we have

$$\frac{d(F(x), F(y))}{d(F(x), F(z))} \leq \eta \left(\frac{d(x, y)}{d(x, z)} \right).$$

For Carnot groups, a quasiconformal map is equivalent to a quasisymmetric map (see [1]). Bearing this in mind, the results we cite on quasisymmetric maps are immediately useful to us.

Theorem 2.3.4 ([2] Theorem 1.2). Let G be a Carnot group. If there is a non-trivial proper linear subspace, $W_1 \subset V_1$, such that $A(W_1) = W_1$ for every graded isomorphism A , then every quasisymmetric map $F : G \longrightarrow G$ is a quasi-similarity.

A Carnot group that satisfies the hypothesis of the above theorem is said to have reducible first stratum. Given the result of Proposition 2.2.2, $\mathfrak{n}$ has reducible first stratum. So, we can rephrase Theorem 2.3.4 for our situation.

Corollary 2.3.5. *If $F : \mathfrak{n} \longrightarrow \mathfrak{n}$ is a quasiconformal map, then F is biLipschitz.*

The fact that quasiconformal maps on $\mathfrak{n}$ are biLipschitz is very important, as we make use of this frequently in the last chapter. A result that is useful much sooner is given in the following theorem.

Theorem 2.3.6 ([3] Proposition 3.4). *Let G and G' be two Carnot groups, $W_1 \subset V_1, W'_1 \subset V'_1$ be subspaces. Denote by $\mathfrak{g}_{W_1} \subset \mathfrak{g}$ and $\mathfrak{g}'_{W'_1} \subset \mathfrak{g}'$ respectively the Lie subalgebras generated by W_1 and W'_1 . Let $H \subset G$ and $H' \subset G'$ respectively be the connected Lie subgroups of G and G' corresponding to $\mathfrak{g}_{W_1}$ and $\mathfrak{g}'_{W'_1}$. Let $F : G \longrightarrow G'$ be a quasisymmetric homeomorphism. If $dF(x)(W_1) \subset W'_1$ for a.e. $x \in G$, then F sends left cosets of H into left cosets of H' .*

Here, $\mathfrak{g}$ and $\mathfrak{g}'$ are used to distinguish the Lie algebras from their respective Lie groups G and G' , and $dF(x)$ is a particular graded isomorphism. (We discuss what this notation means for a graded isomorphism in the next chapter.) A property holds almost everywhere (a.e.) if it's true except on a set of Lebesgue measure 0. Making use of Proposition 2.2.2 again, we know W satisfies the conditions of Theorem 2.3.6.

Corollary 2.3.7. *If $F : \mathfrak{n} \longrightarrow \mathfrak{n}$ is a quasiconformal map, then F permutes left cosets of W .*

To say F permutes left cosets of W means that F maps a left coset of W to another left coset of W . The following lemma gives the form of the image of a left coset under F .

Lemma 2.3.8. *If $F : \mathfrak{n} \longrightarrow \mathfrak{n}$ is a quasiconformal map and $p \in \mathfrak{n}$ so that $p * W$ is a left coset of W , then*

$$F(p * W) = F(p) * W.$$

Proof. Let $p * W$ be a left coset of W . So, $p \in p * W$, which implies $F(p) \in F(p * W)$. By Proposition 2.3.7, $F(p * W)$ is a left coset of W with representative $F(p)$. Therefore, $F(p * W) = F(p) * W$. $\square$

It's also the case that F permutes left cosets of $N(W)$, but we require the following lemma and a definition of Hausdorff distance.

Definition 2.3.9. Let $M_1, M_2 \subseteq M$ be non-empty subsets of a metric space M . The Hausdorff distance between M_1 and M_2 is

$$d_H(M_1, M_2) = \max \left\{ \sup_{x \in M_1} \inf_{y \in M_2} d(x, y), \sup_{y \in M_2} \inf_{x \in M_1} d(x, y) \right\}.$$

Lemma 2.3.10. $d_H(p * W, q * W) < \infty$ if and only if $p * W, q * W$ lie in the same left coset of $N(W)$.

Proof. Suppose $p * W, q * W$ lie in the same left coset of $N(W)$. Hence, p and q have the same X component (see 1.3.4). We can write $p = xX * (z_1Z_1 + z_2Z_2)$, $q = xX * (\tilde{z}_1Z_1 + \tilde{z}_2Z_2)$ (see 1.3.5).

Fix $r = xX * (y_1Y_1 + y_2Y_2 + z_1Z_1 + z_2Z_2) \in p * W$. Then,

$$\inf_{s \in q * W} d(r, s) = |(\tilde{z}_1 - z_1)Z_1 + (\tilde{z}_2 - z_2)Z_2|^{\frac{1}{2}}$$

occurs if we choose s with identical Y_1, Y_2 components to r . We can make a similar choice for s regardless of r , so,

$$\sup_{r \in p * W} \inf_{s \in q * W} d(r, s) = |(\tilde{z}_1 - z_1)Z_1 + (\tilde{z}_2 - z_2)Z_2|^{\frac{1}{2}}.$$

We get the same finite expression for $\sup_{s \in q * W} \inf_{r \in p * W} d(p, q)$. Hence, $d_H(p * W, q * W) < \infty$.

The reverse implication follows by contrapositive. So, suppose $p * W, q * W$ don't lie in the same left coset of $N(W)$. It suffices to consider the case when $p * W = xX * (z_1Z_1 + z_2Z_2) * W$, $x \neq 0$, and $q * W = W$. So, $p(t) = xX * (tY_1 + z_1Z_1 + z_2Z_2) \in xX * (z_1Z_1 + z_2Z_2) * W$. For $w = y_1Y_1 + y_2Y_2 \in W$,

$$d(p(t), w) = |-xX + (y_1 - t)Y_1 + y_2Y_2| + \left| \left(\frac{1}{2}x(y_1 + t) + z_1 \right) Z_1 + \left(\frac{1}{2}xy_2 + z_2 \right) Z_2 \right|^{\frac{1}{2}}.$$

We claim: for $t > \frac{4|z_1|}{|x|}$, we have $d(p(t), w) \geq \min \left\{ t, \frac{1}{2} |xt|^{\frac{1}{2}} \right\}$. Since the right hand side goes to infinity as $t \rightarrow \infty$, $d_H(xX * (z_1Z_1 + z_2Z_2) * W, W) = \infty$.

Case 1: $|y_1 - t| \geq t$.

$$d(p(t), w) > |y_1 - t| \geq t$$

Case 2: $|y_1 - t| < t$.

$$\begin{aligned}
 d(p(t), w) &> \left| \frac{1}{2} x(y_1 + t) + z_1 \right|^{\frac{1}{2}} \\
 &\geq \left(\frac{1}{2} |x(y_1 + t)| - |z_1| \right)^{\frac{1}{2}} \\
 &= \left(\frac{1}{2} |x(y_1 - t + 2t)| - |z_1| \right)^{\frac{1}{2}} \\
 &\geq \left(\frac{1}{2} |x(-|y_1 - t| + 2t)| - |z_1| \right)^{\frac{1}{2}} \\
 &> \left(\frac{1}{2} |x(-t + 2t)| - \frac{1}{4} |xt| \right)^{\frac{1}{2}} \\
 &= \frac{1}{2} |xt|^{\frac{1}{2}}
 \end{aligned}$$

□

Proposition 2.3.11. *If $F : \mathfrak{n} \longrightarrow \mathfrak{n}$ is a quasiconformal map, then F permutes left cosets of $N(W)$.*

Proof. Suppose p, q are in the same left coset of $N(W)$. Then, $p * W, q * W$ lie in the same left coset of $N(W)$. By Lemma 2.3.10, $d_H(p * W, q * W) < \infty$. Since F is biLipschitz, $d_H(F(p * W), F(q * W)) < \infty$. Applying Lemma 2.3.8, $d_H(F(p) * W, F(q) * W) < \infty$. By Lemma 2.3.10 $F(p) * W, F(q) * W$ lie in the same left coset of $N(W)$. Hence, $F(p), F(q)$ lie in the same left coset of $N(W)$. □

In a similar manner to Lemma 2.3.8, we can give the form of the image of left cosets of $N(W)$ under a quasiconformal map.

Lemma 2.3.12. *If $F : \mathfrak{n} \longrightarrow \mathfrak{n}$ is a quasiconformal map and $p \in \mathfrak{n}$ so that $p * N(W)$ is a left coset of $N(W)$, then*

$$F(p * N(W)) = F(p) * N(W).$$

Proof. See proof of Lemma 2.3.8. □

The fact that any quasiconformal map permutes left cosets of W and $N(W)$ is important in terms of determining the form of any quasiconformal map on $\mathfrak{n}$.

Theorem 2.3.13. *Let $F : \mathfrak{n} \longrightarrow \mathfrak{n}$ be a quasiconformal map. There are real valued functions $f = f(x)$, $g_1 = g_1(x, y_1, y_2, z_1, z_2)$, $g_2 = g_2(x, y_1, y_2, z_1, z_2)$, $h_1 = h_1(x, z_1, z_2)$, $h_2 = h_2(x, z_1, z_2)$ such that for any $p = xX * (y_1Y_1 + y_2Y_2 + z_1Z_1 + z_2Z_2)$,*

$$F(p) = fX * (g_1Y_1 + g_2Y_2 + h_1Z_1 + h_2Z_2).$$

Proof. By 1.3.4, $p \in xX * N(W)$, so $F(p) \in F(xX * N(W))$.

By Lemma 2.3.12, $F(xX * N(W)) = F(xX) * N(W)$. The X component of every element of $F(xX) * N(W)$ matches the X component of $F(xX)$, which is determined by where F maps xX . Hence, the X component of $F(p)$ is determined by x alone.

Similarly, by 1.3.5, $p \in xX * (z_1Z_1 + z_2Z_2) * W$, so $F(p) \in F(xX * (z_1Z_1 + z_2Z_2) * W)$.

By Lemma 2.3.8, $F(xX * (z_1Z_1 + z_2Z_2) * W) = F(xX * (z_1Z_1 + z_2Z_2)) * W$. Every element of $F(xX * (z_1Z_1 + z_2Z_2)) * W$ has the same Z_1 and Z_2 components, matching that of $F(xX * (z_1Z_1 + z_2Z_2))$. So, the Z_1, Z_2 components are determined independently of y_1 and y_2 . □

In regards to finding the form of any quasiconformal map on $\mathfrak{n}$, we now have a starting point to work from. We work towards further refining the component functions, f, g_1, g_2, h_1, h_2 , in the following chapters.

CHAPTER 3 DIFFERENTIABILITY OF QUASICONFORMAL MAPS

3.1 Pansu differentiability

By having a notion of differentiability of a quasiconformal map, we're able to gain more information about the component functions of the map. This, in turn, refines our knowledge of the map itself. We borrow the theory of Pansu differentiability from Pansu [4].

Definition 3.1.1. *Let G and G' be two Carnot groups endowed with Carnot metrics, and $U \subset G, U' \subset G'$ open subsets. A map $F : U \rightarrow U'$ is Pansu differentiable at $p \in U$ if there exists a graded homomorphism $A : G \rightarrow G'$ such that*

$$\frac{d((-F(p)) * F(p * s), A(s))}{d(s, 0)} \rightarrow 0 \text{ as } s \rightarrow 0.$$

In the definition of Pansu differentiability, we refer to the graded homomorphism, A , as the Pansu differential of F at p , and write $A = dF(p)$. This notation helps to put emphasis on the fact that the graded homomorphism is dependent on the point p .

Since the metric d is left invariant,

$$d((-F(p)) * F(p * s), dF(p)(s)) = d(-dF(p)(s) * (-F(p)) * F(p * s), 0).$$

So we can rephrase the definition of Pansu differentiability to say that

$$\frac{d(-dF(p)(s) * (-F(p)) * F(p * s), 0)}{d(s, 0)} \rightarrow 0 \text{ as } s \rightarrow 0.$$

We use this version of the limit to do our calculations.

Theorem 3.1.2 (Pansu Differentiability Theorem). *Let G, G' be Carnot lie groups and $U_1 \subseteq G, U_2 \subseteq G'$ open subsets. Let $F : U_1 \rightarrow U_2$ be a quasiconformal map. Then, F is a.e. Pansu differentiable. Furthermore, at a.e. $p \in U$, the Pansu differential, $dF(p)$, is a graded isomorphism.*

From now on, the term differentiability is used to refer specifically to Pansu differentiability. If we apply this theorem in conjunction with Fubini's Theorem, we get that for a.e. left coset $p * W$, F is differentiable at a.e. point in $p * W$.

3.2 Partial derivatives

With the notion of differentiability of a quasiconformal map on $\mathfrak{n}$ established, we can now undergo the task of calculating some of the partial derivatives of the component functions, f , g_1 , g_2 , h_1 , h_2 .

Let

$$-dF(p)(s) * (-F(p)) * F(p * s) = \hat{x}X + \hat{y}_1Y_1 + \hat{y}_2Y_2 + \hat{z}_1Z_1 + \hat{z}_2Z_2$$

for some $\hat{x}, \hat{y}_1, \hat{y}_2, \hat{z}_1, \hat{z}_2 \in \mathbb{R}$. So,

$$\begin{aligned} d(-dF(p)(s) * (-F(p)) * F(p * s), 0) &= \| -(\hat{x}X + \hat{y}_1Y_1 + \hat{y}_2Y_2 + \hat{z}_1Z_1 + \hat{z}_2Z_2) * 0 \| \\ &= \| -(\hat{x}X + \hat{y}_1Y_1 + \hat{y}_2Y_2 + \hat{z}_1Z_1 + \hat{z}_2Z_2) \| \\ &= |\hat{x}X + \hat{y}_1Y_1 + \hat{y}_2Y_2| + |\hat{z}_1Z_1 + \hat{z}_2Z_2|^{\frac{1}{2}}. \end{aligned}$$

Since we're working with the Euclidean norm on the first and second layers, we have the following inequalities:

$$\begin{aligned} |\hat{x}| &= |\hat{x}X| \leq |\hat{x}X + \hat{y}_1Y_1 + \hat{y}_2Y_2|, \\ |\hat{y}_1| &= |\hat{y}_1Y_1| \leq |\hat{x}X + \hat{y}_1Y_1 + \hat{y}_2Y_2|, \\ |\hat{y}_2| &= |\hat{y}_2Y_2| \leq |\hat{x}X + \hat{y}_1Y_1 + \hat{y}_2Y_2|, \\ |\hat{z}_1|^{\frac{1}{2}} &= |\hat{z}_1Z_1|^{\frac{1}{2}} \leq |\hat{z}_1Z_1 + \hat{z}_2Z_2|^{\frac{1}{2}}, \\ |\hat{z}_2|^{\frac{1}{2}} &= |\hat{z}_2Z_2|^{\frac{1}{2}} \leq |\hat{z}_1Z_1 + \hat{z}_2Z_2|^{\frac{1}{2}}. \end{aligned}$$

All of which are less than or equal to $|\hat{x}X + \hat{y}_1Y_1 + \hat{y}_2Y_2| + |\hat{z}_1Z_1 + \hat{z}_2Z_2|^{\frac{1}{2}}$. Hence, if

$$\frac{d(-dF(p)(s) * (-F(p)) * F(p * s), 0)}{d(s, 0)} \rightarrow 0 \text{ as } s \rightarrow 0,$$

then

$$\frac{|\hat{x}|}{d(s, 0)} \rightarrow 0, \frac{|\hat{y}_1|}{d(s, 0)} \rightarrow 0, \frac{|\hat{y}_2|}{d(s, 0)} \rightarrow 0, \frac{|\hat{z}_1|^{\frac{1}{2}}}{d(s, 0)} \rightarrow 0, \frac{|\hat{z}_2|^{\frac{1}{2}}}{d(s, 0)} \rightarrow 0 \text{ as } s \rightarrow 0. \quad (3.2.1)$$

We're able to find the partial derivatives we're concerned with by utilizing the limits of 3.2.1 and making careful choices for s .

Let $F : \mathfrak{n} \rightarrow \mathfrak{n}$ be a quasiconformal map such that F is differentiable at

$p = xX * (y_1Y_1 + y_2Y_2 + z_1Z_1 + z_2Z_2)$. Let $dF(p)$ be the Pansu differential of F at p . Let $s = \tilde{x}X * (\tilde{y}_1Y_1 + \tilde{y}_2Y_2 + \tilde{z}_1Z_1 + \tilde{z}_2Z_2)$. (We specify what $\tilde{x}, \tilde{y}_1, \tilde{y}_2, \tilde{z}_1, \tilde{z}_2$ are later.)

First, recall that a graded isomorphism acts on V_1 as follows:

$$dF(p)(X) = eX + e_1Y_1 + e_2Y_2,$$

$$dF(p)(Y_1) = aY_1 + bY_2,$$

$$dF(p)(Y_2) = cY_1 + dY_2.$$

For $s = \tilde{x}X + \tilde{y}_1Y_1 + \tilde{y}_2Y_2 + v_2$, with $v_2 \in V_2$,

$$\begin{aligned} dF(p)(s) &= \tilde{x}dF(p)(X) + \tilde{y}_1dF(p)(Y_1) + \tilde{y}_2dF(p)(Y_2) + dF(p)(v_2) \\ &= \tilde{x}(eX + e_1Y_1 + e_2Y_2) + \tilde{y}_1(aY_1 + bY_2) + \tilde{y}_2(cY_1 + dY_2) + dF(p)(v_2) \\ &= \tilde{x}eX + (\tilde{x}e_1 + \tilde{y}_1a + \tilde{y}_2c)Y_1 + (\tilde{x}e_2 + \tilde{y}_1b + \tilde{y}_2d)Y_2 + dF(p)(v_2). \end{aligned} \quad (3.2.2)$$

It's important to note that a, b, c, d, e, e_1, e_2 depend on the Pansu differential, $dF(p)$, which depends on p . So a, b, c, d, e, e_1, e_2 are functions of p . (We can further consider p as a function of x, y_1, y_2, z_1, z_2 .)

Next, by Theorem 2.3.13,

$$F(p) = fX * (g_1Y_1 + g_2Y_2 + h_1Z_1 + h_2Z_2) \quad (3.2.3)$$

for some real valued functions $f = f(x)$, $g_1 = g_1(x, y_1, y_2, z_1, z_2)$, $g_2 = g_2(x, y_1, y_2, z_1, z_2)$, $h_1 = h_1(x, z_1, z_2)$, $h_2 = h_2(x, z_1, z_2)$.

Last,

$$p * s = (x + \tilde{x})X * ((y_1 + \tilde{y}_1)Y_1 + (y_2 + \tilde{y}_2)Y_2 + (z_1 + \tilde{z}_1 - \tilde{x}y_1)Z_1 + (z_2 + \tilde{z}_2 - \tilde{x}y_2)Z_2).$$

So, again by Theorem 2.3.13,

$$F(p * s) = \tilde{f}X * (\tilde{g}_1Y_1 + \tilde{g}_2Y_2 + \tilde{h}_1Z_1 + \tilde{h}_2Z_2) \quad (3.2.4)$$

where

$$\begin{aligned} \tilde{f} &= f(x + \tilde{x}), \\ \tilde{g}_1 &= g_1(x + \tilde{x}, y_1 + \tilde{y}_1, y_2 + \tilde{y}_2, z_1 + \tilde{z}_1 - \tilde{x}y_1, z_2 + \tilde{z}_2 - \tilde{x}y_2), \\ \tilde{g}_2 &= g_2(x + \tilde{x}, y_1 + \tilde{y}_1, y_2 + \tilde{y}_2, z_1 + \tilde{z}_1 - \tilde{x}y_1, z_2 + \tilde{z}_2 - \tilde{x}y_2), \\ \tilde{h}_1 &= h_1(x + \tilde{x}, z_1 + \tilde{z}_1 - \tilde{x}y_1, z_2 + \tilde{z}_2 - \tilde{x}y_2), \\ \tilde{h}_2 &= h_2(x + \tilde{x}, z_1 + \tilde{z}_1 - \tilde{x}y_1, z_2 + \tilde{z}_2 - \tilde{x}y_2). \end{aligned}$$

Using equations 3.2.2, 3.2.3, and 3.2.4, we get that

$$\begin{aligned} & -dF(p)(s) * (-F(p)) * F(p * s) \\ & = (\tilde{f} - f - \tilde{x}e)X + (\tilde{g}_1 - g - \tilde{x}e_1 - \tilde{y}_1a - \tilde{y}_2c)Y_1 + (\tilde{g}_2 - g_2 - \tilde{x}e_2 - \tilde{y}_1b - \tilde{y}_2d)Y_2 + \tilde{v}_2 \end{aligned}$$

for some $\tilde{v}_2 \in V_2$. (We calculate $\tilde{v}_2$ explicitly after choosing a particular s .) We can now apply the first three limits of 3.2.1 to get

$$\frac{|\tilde{f} - f - \tilde{x}e|}{d(s, 0)} \longrightarrow 0, \quad (3.2.5)$$

$$\frac{|\tilde{g}_1 - g_1 - \tilde{x}e_1 - \tilde{y}_1a - \tilde{y}_2c|}{d(s, 0)} \longrightarrow 0, \quad (3.2.6)$$

$$\frac{|\tilde{g}_2 - g_2 - \tilde{x}e_2 - \tilde{y}_1b - \tilde{y}_2d|}{d(s, 0)} \longrightarrow 0, \quad (3.2.7)$$

as $s \longrightarrow 0$.

Let $s = \tilde{x}X$. (In other words, $\tilde{y}_1 = \tilde{y}_2 = \tilde{z}_1 = \tilde{z}_2 = 0$.) If we consider 3.2.5, we get

$$\frac{|\tilde{f} - f - \tilde{x}e|}{|\tilde{x}|} \longrightarrow 0 \text{ as } \tilde{x} \longrightarrow 0. \text{ So } \frac{f(x + \tilde{x}) - f(x)}{\tilde{x}} \longrightarrow e \text{ as } \tilde{x} \longrightarrow 0.$$

Hence, the derivative of $f(x)$ is $f'(x) = e = e(p)$. (This is simply the definition of the derivative of a function from Calculus.) So, we see that e is a function that only depends on the X component of p . (It doesn't depend on y_1, y_2, z_1, z_2 .) We can write $e = e(x)$.

Now, instead, let $s = \tilde{y}_1Y_1$. If we consider 3.2.6, we get

$$\frac{|\tilde{g}_1 - g_1 - \tilde{x}e_1 - \tilde{y}_1a - \tilde{y}_2c|}{d(s, 0)} = \frac{|\tilde{g}_1 - g_1 - \tilde{y}_1a|}{|\tilde{y}_1|} \longrightarrow 0 \text{ as } \tilde{y}_1 \longrightarrow 0,$$

which implies

$$\frac{g_1(x, y_1 + \tilde{y}_1, y_2, z_1, z_2) - g_1(x, y_1, y_2, z_1, z_2)}{\tilde{y}_1} \longrightarrow a \text{ as } \tilde{y}_1 \longrightarrow 0.$$

Hence, the partial derivative of g_1 with respect to y_1 is $\frac{\partial}{\partial y_1} g_1(x, y_1, y_2, z_1, z_2) = a = a(p)$.

If we consider 3.2.7, we get

$$\frac{|\tilde{g}_2 - g_2 - \tilde{x}e_2 - \tilde{y}_1b - \tilde{y}_2d|}{d(s, 0)} = \frac{|\tilde{g}_2 - g_2 - \tilde{y}_1b|}{|\tilde{y}_1|} \longrightarrow 0 \text{ as } \tilde{y}_1 \longrightarrow 0,$$

which implies

$$\frac{g_2(x, y_1 + \tilde{y}_1, y_2, z_1, z_2) - g_2(x, y_1, y_2, z_1, z_2)}{\tilde{y}_1} \longrightarrow b \text{ as } \tilde{y}_1 \longrightarrow 0.$$

Hence, the partial derivative of g_2 with respect to y_1 is $\frac{\partial}{\partial y_1} g_2(x, y_1, y_2, z_1, z_2) = b = b(p)$.

If we let $s = \tilde{y}_2Y_2$ and consider 3.2.6 and 3.2.7 in a similar manner to the above calculations,

then we find that the partial derivatives of g_1 and g_2 with respect to y_2 are $\frac{\partial}{\partial y_2} g_1(x, y_1, y_2, z_1, z_2) = c = c(p)$ and $\frac{\partial}{\partial y_2} g_2(x, y_1, y_2, z_1, z_2) = d = d(p)$.

Recall that a graded isomorphism acts on V_2 as follows:

$$dF(p)(Z_1) = eaZ_1 + ebZ_2,$$

$$dF(p)(Z_2) = ecZ_1 + edZ_2.$$

Now, let $s = \tilde{z}_1 Z_1 + \tilde{z}_2 Z_2$, and we get

$$\begin{aligned} dF(p)(s) &= \tilde{z}_1 dF(p)(Z_1) + \tilde{z}_2 dF(p)(Z_2) \\ &= \tilde{z}_1 (eaZ_1 + ebZ_2) + \tilde{z}_2 (ecZ_1 + edZ_2) \\ &= (\tilde{z}_1 ea + \tilde{z}_2 ec) Z_1 + (\tilde{z}_1 eb + \tilde{z}_2 ed) Z_2. \end{aligned} \tag{3.2.8}$$

Equations 3.2.8, 3.2.3, and 3.2.4 give us that

$$\begin{aligned} &-dF(p)(s) * (-F(p)) * F(p * s) \\ &= (\tilde{g}_1 - g) Y_1 + (\tilde{g}_2 - g_2) Y_2 + \left(\tilde{h}_1 - h_1 - \tilde{z}_1 ea - \tilde{z}_2 ec \right) Z_1 + \left(\tilde{h}_2 - h_2 - \tilde{z}_1 eb - \tilde{z}_2 ed \right) Z_2. \end{aligned}$$

We can now apply the last two limits of 3.2.1 to get

$$\frac{\left| \tilde{h}_1 - h_1 - \tilde{z}_1 ea - \tilde{z}_2 ec \right|^{\frac{1}{2}}}{d(s, 0)} \longrightarrow 0, \tag{3.2.9}$$

$$\frac{\left| \tilde{h}_2 - h_2 - \tilde{z}_1 eb - \tilde{z}_2 ed \right|^{\frac{1}{2}}}{d(s, 0)} \longrightarrow 0, \tag{3.2.10}$$

as $s \longrightarrow 0$.

Using 3.2.9 and setting $\tilde{z}_2 = 0$ we get

$$\frac{\left| \tilde{h}_1 - h_1 - \tilde{z}_1 ea - \tilde{z}_2 ec \right|^{\frac{1}{2}}}{d(s, 0)} = \frac{\left| \tilde{h}_1 - h_1 - \tilde{z}_1 ea \right|^{\frac{1}{2}}}{|\tilde{z}_1|^{\frac{1}{2}}} \longrightarrow 0 \text{ as } \tilde{z}_1 \longrightarrow 0,$$

which implies

$$\frac{h_1(x, z_1 + \tilde{z}_1, z_2) - h_1(x, z_1, z_2)}{\tilde{z}_1} \longrightarrow ea \text{ as } \tilde{z}_1 \longrightarrow 0.$$

Using 3.2.10 with $\tilde{z}_2 = 0$ we get

$$\frac{\left| \tilde{h}_2 - h_2 - \tilde{z}_1 eb - \tilde{z}_2 ed \right|^{\frac{1}{2}}}{d(s, 0)} = \frac{\left| \tilde{h}_2 - h_2 - \tilde{z}_1 eb \right|^{\frac{1}{2}}}{|\tilde{z}_1|^{\frac{1}{2}}} \longrightarrow 0 \text{ as } \tilde{z}_1 \longrightarrow 0,$$

which implies

$$\frac{h_2(x, z_1 + \tilde{z}_1, z_2) - h_2(x, z_1, z_2)}{\tilde{z}_1} \longrightarrow eb \text{ as } \tilde{z}_1 \longrightarrow 0.$$

Hence, the partial derivatives of h_1 and h_2 with respect to z_1 are $\frac{\partial}{\partial z_1} h_1(x, z_1, z_2) = ea = e(x) a(p)$ and $\frac{\partial}{\partial z_1} h_2(x, z_1, z_2) = eb = e(x) b(p)$ respectively. However, h_1, h_2 only depend on x, z_1, z_2 . So, we can write $ea = e(x) a(x, z_1, z_2)$ and $eb = e(x) b(x, z_1, z_2)$.

If we instead let $\tilde{z}_1 = 0$ and consider 3.2.9 and 3.2.10 in a similar manner, we get the partial derivatives of h_1 and h_2 with respect to z_2 are $\frac{\partial}{\partial z_2} h_1(x, z_1, z_2) = ec = e(x) c(x, z_1, z_2)$ and $\frac{\partial}{\partial z_2} h_2(x, z_1, z_2) = ed = e(x) d(x, z_1, z_2)$.

While it can also be shown that the partial derivatives of g_1 and g_2 with respect to x are $e_1(p)$ and $e_2(p)$, respectively, this isn't be important to us. After we've gained more information about $e(x)$ in the next chapter, we calculate the partial derivatives of h_1 and h_2 with respect to x , and this information is beneficial. (It's not recommended to attempt to calculate the partials of g_1, g_2 with respect to z_1, z_2 at this stage; the calculation is long and when we're finished they're trivial.)

CHAPTER 4 FORM OF A QUASICONFORMAL MAP

4.1 Component functions for Y_1, Y_2

Recall, every quasiconformal map, F , has the form

$$F(p) = fX * (g_1Y_1 + g_2Y_2 + h_1Z_1 + h_2Z_2),$$

where $f = f(x)$, $g_1 = g_1(x, y_1, y_2, z_1, z_2)$, $g_2 = g_2(x, y_1, y_2, z_1, z_2)$, $h_1 = h_1(x, z_1, z_2)$, and $h_2 = h_2(x, z_1, z_2)$. Also, there are functions $a(x, z_1, z_2)$, $b(x, z_1, z_2)$, $c(x, z_1, z_2)$, and $d(x, z_1, z_2)$ such that if F is differentiable at $p = xX * (y_1Y_1 + y_2Y_2 + z_1Z_1 + z_2Z_2)$, then the Pansu differential has the form

$$dF(p)(Y_1) = a(x, z_1, z_2)Y_1 + b(x, z_1, z_2)Y_2,$$

$$dF(p)(Y_2) = c(x, z_1, z_2)Y_1 + d(x, z_1, z_2)Y_2.$$

In the last chapter, we found the partial derivatives of g_1 and g_2 with respect to y_1 and y_2 to be

$$\begin{aligned} \frac{\partial}{\partial y_1} g_1(x, y_1, y_2, z_1, z_2) &= a(x, z_1, z_2), \\ \frac{\partial}{\partial y_1} g_2(x, y_1, y_2, z_1, z_2) &= b(x, z_1, z_2), \\ \frac{\partial}{\partial y_2} g_1(x, y_1, y_2, z_1, z_2) &= c(x, z_1, z_2), \\ \frac{\partial}{\partial y_2} g_2(x, y_1, y_2, z_1, z_2) &= d(x, z_1, z_2). \end{aligned}$$

Notice, these partial derivatives are independent of y_1 and y_2 . Since any biLipschitz map is absolutely continuous, we can recover g_1 and g_2 from their partial derivatives.

Proposition 4.1.1. *There are functions $t_1(x, z_1, z_2)$, $t_2(x, z_1, z_2)$ such that the component functions g_1, g_2 can be written as*

$$g_1(x, y_1, y_2, z_1, z_2) = a(x, z_1, z_2)y_1 + c(x, z_1, z_2)y_2 + t_1(x, z_1, z_2),$$

$$g_2(x, y_1, y_2, z_1, z_2) = b(x, z_1, z_2)y_1 + d(x, z_1, z_2)y_2 + t_2(x, z_1, z_2).$$

After finding the partial derivatives of h_1 and h_2 , we saw that a, b, c, d don't depend on all the components of p , but rather on just the X, Z_1, Z_2 components. Our goal now is to show that a, b, c, d actually don't depend on p at all, hence are constant. We begin by showing these values

don't depend on the Z_1, Z_2 components.

Proposition 4.1.2. *For a, b, c, d as in Proposition 4.1.1, we have*

$$a(x, z_1, z_2) = a(x), b(x, z_1, z_2) = b(x), c(x, z_1, z_2) = c(x), d(x, z_1, z_2) = d(x).$$

Proof. Let $q = xX * (y_1Y_1 + y_2Y_2 + \tilde{z}_1Z_1 + \tilde{z}_2Z_2)$, so that p and q only vary in the Z_1, Z_2 components. F is biLipschitz, so $d(F(p), F(q)) \leq L \cdot d(p, q)$ for some $L > 0$. Hence,

$$\begin{aligned} & |g_1(x, y_1, y_2, \tilde{z}_1, \tilde{z}_2) - g_1(x, y_1, y_2, z_1, z_2)| \\ &= |a(x, \tilde{z}_1, \tilde{z}_2)y_1 + c(x, \tilde{z}_1, \tilde{z}_2)y_2 + t_1(x, \tilde{z}_1, \tilde{z}_2) - a(x, z_1, z_2)y_1 - c(x, z_1, z_2)y_2 - t_1(x, z_1, z_2)| \\ &= |(a(x, \tilde{z}_1, \tilde{z}_2) - a(x, z_1, z_2))y_1 + (c(x, \tilde{z}_1, \tilde{z}_2) - c(x, z_1, z_2))y_2 + t_1(x, \tilde{z}_1, \tilde{z}_2) - t_1(x, z_1, z_2)| \\ &\leq d(F(p), F(q)) \leq L \cdot |(\tilde{z}_1 - z_1)Z_1 + (\tilde{z}_2 - z_2)Z_2|^{\frac{1}{2}} \end{aligned}$$

holds for any choice of y_1, y_2 .

If we let $y_1 \rightarrow \infty$, we see that $a(x, \tilde{z}_1, \tilde{z}_2) - a(x, z_1, z_2) = 0$. If we let $y_2 \rightarrow \infty$, we see that $c(x, \tilde{z}_1, \tilde{z}_2) - c(x, z_1, z_2) = 0$. Therefore, $a(x, \tilde{z}_1, \tilde{z}_2) = a(x, z_1, z_2)$, and $c(x, \tilde{z}_1, \tilde{z}_2) = c(x, z_1, z_2)$.

A similar argument yields $b(x, \tilde{z}_1, \tilde{z}_2) = b(x, z_1, z_2)$, and $d(x, \tilde{z}_1, \tilde{z}_2) = d(x, z_1, z_2)$. $\square$

The next step is to show that a, b, c, d don't depend on the X component of p . To make things easier, we first consider the component functions, g_1, g_2 , in terms of a matrix equation.

$$\begin{pmatrix} g_1(x, y_1, y_2, z_1, z_2) \\ g_2(x, y_1, y_2, z_1, z_2) \end{pmatrix} = \begin{pmatrix} a(x) & c(x) \\ b(x) & d(x) \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} t_1(x, z_1, z_2) \\ t_2(x, z_1, z_2) \end{pmatrix} \quad (4.1.3)$$

The following proposition is equivalent to saying a, b, c, d are constant.

Proposition 4.1.4. *For the matrix given in 4.1.3,*

$$\begin{pmatrix} a(x) & c(x) \\ b(x) & d(x) \end{pmatrix} = \begin{pmatrix} a(\tilde{x}) & c(\tilde{x}) \\ b(\tilde{x}) & d(\tilde{x}) \end{pmatrix}$$

for any $x, \tilde{x} \in \mathbb{R}$.

Proof. We proceed by contradiction.

Assume the matrices aren't equal, therefore, there exists a $w \in W$, $w = w_1Y_1 + w_2Y_2 = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$,

such that

$$Mw = \begin{pmatrix} a(x) & c(x) \\ b(x) & d(x) \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \neq \begin{pmatrix} a(\tilde{x}) & c(\tilde{x}) \\ b(\tilde{x}) & d(\tilde{x}) \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \tilde{M}w$$

for some $x, \tilde{x} \in \mathbb{R}, x \neq \tilde{x}$.

Let $p = xX * (y_1Y_1 + y_2Y_2 + z_1Z_1 + z_2Z_2)$ and $q = \tilde{x}X * (\tilde{y}_1Y_1 + \tilde{y}_2Y_2 + \tilde{z}_1Z_1 + \tilde{z}_2Z_2)$. Since F is biLipschitz, we get

$$d(F(p * tw), F(q * tw)) \leq L \cdot d(p * tw, q * tw). \quad (4.1.5)$$

It's not difficult to show, by direct computation, that $F(p * tw) = F(p) * tMw$ and $F(q * tw) = F(q) * t\tilde{M}w$. So,

$$\begin{aligned} d(F(p * tw), F(q * tw)) &= d(F(p) * tMw, F(q) * t\tilde{M}w) \\ &= d\left(\left(-t\tilde{M}w\right) * (-F(q)) * F(p) * tMw, 0\right) \\ &= \left\| \left(-t\tilde{M}w\right) * (-F(q)) * F(p) * tMw \right\| \end{aligned}$$

If we set $F(p) = \hat{x}X + \hat{y}_1Y_1 + \hat{y}_2Y_2 + \hat{z}_1Z_1 + \hat{z}_2Z_2$ and $F(q) = x'X + y'_1Y_1 + y'_2Y_2 + z'_1Z_1 + z'_2Z_2$,

we get

$$\begin{aligned} &\left(-t\tilde{M}w\right) * (-F(q)) * F(p) * tMw \\ &= (\hat{x} - x')X \\ &\quad + (\hat{y}_1 - y'_1 - ta(\tilde{x})w_1 - tc(\tilde{x})w_2 + ta(x)w_1 + tc(x)w_2)Y_1 \\ &\quad + (\hat{y}_2 - y'_2 - tb(\tilde{x})w_1 - td(\tilde{x})w_2 + tb(x)w_1 + td(x)w_2)Y_2 \\ &\quad + \left(\hat{z}_1 - z'_1 - \frac{1}{2}x'\hat{y}_1 + \frac{1}{2}\hat{x}y'_1 - \frac{1}{2}tx'a(\tilde{x})w_1 - \frac{1}{2}tx'c(\tilde{x})w_2 - \frac{1}{2}tx'a(x)w_1 - \frac{1}{2}tx'c(x)w_2 \right. \\ &\quad \left. + \frac{1}{2}t\hat{x}a(\tilde{x})w_1 + \frac{1}{2}t\hat{x}c(\tilde{x})w_2 + \frac{1}{2}t\hat{x}a(x)w_1 + \frac{1}{2}t\hat{x}c(x)w_2\right)Z_1 \\ &\quad + \left(\hat{z}_2 - z'_2 - \frac{1}{2}x'\hat{y}_2 + \frac{1}{2}\hat{x}y'_2 - \frac{1}{2}tx'b(\tilde{x})w_1 - \frac{1}{2}tx'd(\tilde{x})w_2 - \frac{1}{2}tx'b(x)w_1 - \frac{1}{2}tx'd(x)w_2 \right. \\ &\quad \left. + \frac{1}{2}t\hat{x}b(\tilde{x})w_1 + \frac{1}{2}t\hat{x}d(\tilde{x})w_2 + \frac{1}{2}t\hat{x}b(x)w_1 + \frac{1}{2}t\hat{x}d(x)w_2\right)Z_2 \end{aligned}$$

Notice,

$$-t\tilde{M}w + tMw = \begin{pmatrix} -ta(\tilde{x})w_1 - tc(\tilde{x})w_2 + ta(x)w_1 + tc(x)w_2 \\ -tb(\tilde{x})w_1 - td(\tilde{x})w_2 + tb(x)w_1 + td(x)w_2 \end{pmatrix}.$$

Under the assumption that $tMw \neq t\tilde{M}w$, the t term can't vanish in both the Y_1 and Y_2 component

of $(-t\tilde{M}w) * (-F(q)) * F(p) * tMw$. So, $d(F(p * tw), F(q * tw))$ grows linearly as $t \rightarrow \infty$.
Now,

$$\begin{aligned} d(p * tw, q * tw) &= d((-tw) * (-q) * p * tw, 0) \\ &= \|(-tw) * (-q) * p * tw\|, \end{aligned}$$

and

$$\begin{aligned} (-tw) * (-q) * p * tw &= (x - \tilde{x})X + (y_1 - \tilde{y}_1)Y_1 + (y_2 - \tilde{y}_2)Y_2 \\ &\quad + \left(z_1 - \tilde{z}_1 + \frac{1}{2}xy_1 - \frac{1}{2}\tilde{x}\tilde{y}_1 + \frac{1}{2}x\tilde{y}_1 - \frac{1}{2}\tilde{x}y_1 + txw_1 - t\tilde{x}w_1 \right) Z_1 \\ &\quad + \left(z_2 - \tilde{z}_2 + \frac{1}{2}xy_2 - \frac{1}{2}\tilde{x}\tilde{y}_2 + \frac{1}{2}x\tilde{y}_2 - \frac{1}{2}\tilde{x}y_2 + txw_2 - t\tilde{x}w_2 \right) Z_2. \end{aligned}$$

Applying the norm to this expression, we see $d(p * tw, q * tw)$ grows like $t^{\frac{1}{2}}$ as $t \rightarrow \infty$. However, this contradicts 4.1.5. $\square$

Now that a, b, c, d are constant, we can rewrite 4.1.3.

$$\begin{pmatrix} g_1(x, y_1, y_2, z_1, z_2) \\ g_2(x, y_1, y_2, z_1, z_2) \end{pmatrix} = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} t_1(x, z_1, z_2) \\ t_2(x, z_1, z_2) \end{pmatrix}$$

This effectively simplifies Proposition 4.1.1.

4.2 Component functions for Z_1, Z_2

After discovering a, b, c, d to be constant, we can rephrase our knowledge of the Pansu differential stated in the last section. So, there are functions $e(x), e_1(p), e_2(p)$ and constants a, b, c, d such that if F is differentiable at $p = xX * (y_1Y_1 + y_2Y_2 + z_1Z_1 + z_2Z_2)$, then the Pansu differential has the form

$$dF(p)(X) = e(x)X + e_1(p)Y_1 + e_2(p)Y_2,$$

$$dF(p)(Y_1) = aY_1 + bY_2,$$

$$dF(p)(Y_2) = cY_1 + dY_2.$$

In the last chapter, we found the partial derivatives of h_1 and h_2 with respect to z_1 and z_2 to be

$$\begin{aligned}\frac{\partial}{\partial z_1} h_1(x, z_1, z_2) &= e(x) a, \\ \frac{\partial}{\partial z_1} h_2(x, z_1, z_2) &= e(x) b, \\ \frac{\partial}{\partial z_2} h_1(x, z_1, z_2) &= e(x) c, \\ \frac{\partial}{\partial z_2} h_2(x, z_1, z_2) &= e(x) d.\end{aligned}$$

Now that a, b, c, d are constant, these partial derivatives are independent of z_1 and z_2 . Again, the differential equations are easily solved in the following proposition.

Proposition 4.2.1. *There are functions $r_1(x), r_2(x)$ such that the component functions h_1, h_2 can be written as*

$$\begin{aligned}h_1(x, z_1, z_2) &= e(x) a z_1 + e(x) c z_2 + r_1(x), \\ h_2(x, z_1, z_2) &= e(x) b z_1 + e(x) d z_2 + r_2(x).\end{aligned}$$

In a similar manner to how we approached the component functions, g_1, g_2 , in the last section, we use the fact that F is biLipschitz and compare the distance between elements of $\mathbf{n}$ to reduce the form of h_1, h_2 . In doing so, we show that $e(x)$ is actually a constant function as well.

Lemma 4.2.2. *The functions $t_1(x, z_1, z_2), t_2(x, z_1, z_2)$ given in 4.1.1 satisfy*

$$\frac{t_1(x, t^2 z_1, t^2 z_2)}{t^2} \longrightarrow 0, \frac{t_2(x, t^2 z_1, t^2 z_2)}{t^2} \longrightarrow 0, \text{ as } t \longrightarrow \infty.$$

Proof. Let $p = xX$ and $q = xX * (t^2 z_1 Z_1 + t^2 z_2 Z_2)$. So, $d(p, q) = |t| \cdot |z_1 Z_1 + z_2 Z_2|^{\frac{1}{2}}$.

Now,

$$\begin{aligned}F(p) &= f(x) X * (t_1(x, 0, 0) Y_1 + t_2(x, 0, 0) Y_2 + r_1(x) Z_1 + r_2(x) Z_2), \\ F(q) &= f(x) X * (t_1(x, t^2 z_1, t^2 z_2) Y_1 + t_2(x, t^2 z_1, t^2 z_2) Y_2 \\ &\quad + (e(x) a t^2 z_1 + e(x) c t^2 z_2 + r_1(x)) Z_1 + (e(x) b t^2 z_1 + e(x) d t^2 z_2 + r_2(x)) Z_2).\end{aligned}$$

Thus,

$$\begin{aligned}d(F(p), F(q)) &= |(t_1(x, t^2 z_1, t^2 z_2) - t_1(x, 0, 0)) Y_1 + (t_2(x, t^2 z_1, t^2 z_2) - t_2(x, 0, 0)) Y_2| \\ &\quad + |(e(x) a t^2 z_1 + e(x) c t^2 z_2) Z_1 + (e(x) b t^2 z_1 + e(x) d t^2 z_2) Z_2|^{\frac{1}{2}}.\end{aligned}$$

Since F is biLipschitz, we get

$$\left| \left(t_1(x, t^2 z_1, t^2 z_2) - t_1(x, 0, 0) \right) Y_1 + \left(t_2(x, t^2 z_1, t^2 z_2) - t_2(x, 0, 0) \right) Y_2 \right| \leq L \cdot |t| \cdot |z_1 Z_1 + z_2 Z_2|^{\frac{1}{2}}.$$

So,

$$\left| t_1(x, t^2 z_1, t^2 z_2) - t_1(x, 0, 0) \right| \leq L \cdot |t| \cdot |z_1 Z_1 + z_2 Z_2|^{\frac{1}{2}},$$

$$\left| t_2(x, t^2 z_1, t^2 z_2) - t_2(x, 0, 0) \right| \leq L \cdot |t| \cdot |z_1 Z_1 + z_2 Z_2|^{\frac{1}{2}}.$$

Dividing both sides of each inequality by t^2 and letting $t \rightarrow \infty$, we get the result. $\square$

Proposition 4.2.3. *For e as in Proposition 4.2.1, we have $e(x) = e$.*

Proof. Let $p = xX * (t^2 z_1 Z_1 + t^2 z_2 Z_2)$ and $q = \tilde{x}X * (t^2 z_1 Z_1 + t^2 z_2 Z_2)$. So, $d(p, q) = |\tilde{x} - x|$.

After some computation, we get

$$\begin{aligned} & (-F(p)) * F(q) \\ &= (f(\tilde{x}) - f(x)) X \\ &+ (t_1(\tilde{x}, t^2 z_1, t^2 z_2) - t_1(x, t^2 z_1, t^2 z_2)) Y_1 \\ &+ (t_2(\tilde{x}, t^2 z_1, t^2 z_2) - t_2(x, t^2 z_1, t^2 z_2)) Y_2 \\ &+ \left(t^2 (az_1 + cz_2) (e(\tilde{x}) - e(x)) + r_1(\tilde{x}) - r_1(x) \right. \\ &\quad \left. + \frac{1}{2} (f(\tilde{x}) - f(x)) (t_1(\tilde{x}, t^2 z_1, t^2 z_2) + t_1(x, t^2 z_1, t^2 z_2)) \right) Z_1 \\ &+ \left(t^2 (bz_1 + dz_2) (e(\tilde{x}) - e(x)) + r_2(\tilde{x}) - r_2(x) \right. \\ &\quad \left. + \frac{1}{2} (f(\tilde{x}) - f(x)) (t_2(\tilde{x}, t^2 z_1, t^2 z_2) + t_2(x, t^2 z_1, t^2 z_2)) \right) Z_2. \end{aligned}$$

If we refer to the Z_1 component of this expression as z'_1 , then

$$|z'_1|^{\frac{1}{2}} \leq d(F(p), F(q)) \leq L \cdot d(p, q) = L \cdot |\tilde{x} - x|$$

since F is biLipschitz.

Dividing both sides of the inequality by t , letting $t \rightarrow \infty$, and noting Lemma 4.2.2 gives

$$|(az_1 + cz_2) (e(\tilde{x}) - e(x))| \leq 0$$

for all $z_1, z_2 \in \mathbb{R}$. So, $e(\tilde{x}) - e(x) = 0$. $\square$

We now have that e is constant. If we write the component functions h_1, h_2 in a matrix equation,

as we did in 4.1.3, then we get the following:

$$\begin{pmatrix} h_1(x, z_1, z_2) \\ h_2(x, z_1, z_2) \end{pmatrix} = e \begin{pmatrix} a & c \\ b & d \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} + \begin{pmatrix} r_1(x) \\ r_2(x) \end{pmatrix}.$$

4.3 Component function for X

We can immediately make use of the fact that e is constant to give the form of the final component function, f . Recall that $f'(x) = e$. Since e is now constant, we get the following proposition.

Proposition 4.3.1. *The component function f can be written as*

$$f(x) = ex + C, \text{ for some } C \in \mathbb{R}.$$

Since F is a homeomorphism, $e \neq 0$. (Otherwise, F isn't onto n .)

This new knowledge about the function f allows us to find more information about the functions t_1 , t_2 , r_1 , and r_2 . To do so, we revisit the steps we took in the last chapter and calculate the derivatives of r_1 and r_2 .

To start, let $p = xX * (y_1Y_1 + y_2Y_2 + z_1Z_1 + z_2Z_2)$ and $s = \tilde{x}X$. We use the following notation:

$$\begin{aligned} f &= f(x), & \tilde{f} &= f(x + \tilde{x}), \\ g_1 &= g_1(x, y_1, y_2, z_1, z_2), & \tilde{g}_1 &= g_1(x + \tilde{x}, y_1, y_2, z_1 - \tilde{x}y_1, z_2 - \tilde{x}y_2), \\ h_1 &= h_1(x, z_1, z_2), & \tilde{h}_1 &= h_1(x + \tilde{x}, z_1 - \tilde{x}y_1, z_2 - \tilde{x}y_2). \end{aligned}$$

Then, the Z_1 component of $-dF(p)(s) * (-F(p)) * F(p * s)$ is

$$\begin{aligned} \tilde{h}_1 - h_1 + \frac{1}{2}\tilde{f}\tilde{g}_1 - \frac{1}{2}f\tilde{g}_1 + \frac{1}{2}\tilde{f}g_1 - \frac{1}{2}fg_1 + \frac{1}{2}e\tilde{x}g_1 - \frac{1}{2}e\tilde{x}\tilde{g}_1 + \frac{1}{2}\tilde{x}e_1\tilde{f} - \frac{1}{2}\tilde{x}e_1f \\ = \tilde{h}_1 - h_1 + \frac{1}{2}(\tilde{f} - f)(\tilde{g}_1 + g_1 + \tilde{x}e_1) + \frac{1}{2}e\tilde{x}(g_1 - \tilde{g}_1). \end{aligned}$$

Now, $\tilde{f} - f = e\tilde{x}$, so the above expression for the Z_1 component reduces to

$$\tilde{h}_1 - h_1 + \frac{1}{2}e\tilde{x}(2g_1 + \tilde{x}e_1).$$

Also, $\tilde{h}_1 - h_1 = -ea\tilde{x}y_1 - ec\tilde{x}y_2 + r_1(x + \tilde{x}) - r_1(x)$ and $g_1 = ay_1 + cy_2 + t_1(x, z_1, z_2)$, so the above expression reduces further to

$$r_1(x + \tilde{x}) - r_1(x) + e\tilde{x}t_1(x, z_1, z_2) + \frac{1}{2}e\tilde{x}^2e_1.$$

As we did in chapter 3, we can now apply 3.2.1 to get

$$\frac{|r_1(x + \tilde{x}) - r_1(x) + e\tilde{x}t_1(x, z_1, z_2) + \frac{1}{2}e\tilde{x}^2e_1|^{\frac{1}{2}}}{d(s, 0)} \longrightarrow 0 \text{ as } s \longrightarrow 0,$$

which implies

$$\frac{r_1(x + \tilde{x}) - r_1(x)}{\tilde{x}} \longrightarrow -et_1(x, z_1, z_2) \text{ as } \tilde{x} \longrightarrow 0.$$

Hence, $r_1'(x) = -et_1(x, z_1, z_2)$. However, r_1 only depends on x , so $t_1(x, z_1, z_2) = t_1(x)$ is actually just a function of x as well.

We can follow similar steps while considering the Z_2 component of $-dF(p)(s) * (-F(p)) * F(p * s)$, and we find that $r_2'(x) = -et_2(x)$. So, both t_1 and t_2 are functions of x .

Proposition 4.3.2. *The functions $t_1(x), t_2(x), r_1(x), r_2(x)$ given in 4.1.1 and 4.2.1 are such that*

$$\begin{aligned} r_1(x) &= -e \int_0^x t_1(s) ds + r_1(0), \\ r_2(x) &= -e \int_0^x t_2(s) ds + r_2(0). \end{aligned}$$

Proposition 4.3.3. *t_1, t_2 are Lipschitz functions.*

Proof. Let $p = xX$ and $q = \tilde{x}X$. We apply that F is biLipschitz to get the following inequality.

$$|t_1(\tilde{x}) - t_1(x)| \leq d(F(p), F(q)) \leq L \cdot d(p, q) = L \cdot |\tilde{x} - x|$$

□

4.4 Rigidity of quasiconformal maps

We now consolidate the information gained about the component functions to give the form of any quasiconformal map on $\mathfrak{n}$.

Theorem 4.4.1. *Let $F : \mathfrak{n} \longrightarrow \mathfrak{n}$ be a quasiconformal map. Then, for $p \in \mathfrak{n}$, where*

$p = xX * (y_1Y_1 + y_2Y_2 + z_1Z_1 + z_2Z_2)$, we get

$$F(p) = fX * (g_1Y_1 + g_2Y_2 + h_1Z_1 + h_2Z_2),$$

where the component functions f, g_1, g_2, h_1, h_2 are given by

$$f = f(x) = ex + C,$$

$$g_1 = g_1(x, y_1, y_2) = ay_1 + cy_2 + t_1(x),$$

$$g_2 = g_2(x, y_1, y_2) = by_1 + dy_2 + t_2(x),$$

$$h_1 = h_1(x, z_1, z_2) = eaz_1 + ecz_2 - e \int_0^x t_1(s) ds + C_1,$$

$$h_2 = h_2(x, z_1, z_2) = ebz_1 + edz_2 - e \int_0^x t_2(s) ds + C_2,$$

where t_1, t_2 are Lipschitz functions, $a, b, c, d, e, C, C_1, C_2 \in \mathbb{R}$, $e \neq 0$, and $ad - bc \neq 0$.

Now that we have the form of any quasiconformal map on $\mathfrak{n}$, it's apparent that left cosets of V_2 are mapped to left cosets V_2 . (V_2 can easily be shown to be a subgroup in the same manner as W and $N(W)$.) Hence, quasiconformal maps on $\mathfrak{n}$ preserve the vertical direction.

Corollary 4.4.2. *If $F : \mathfrak{n} \longrightarrow \mathfrak{n}$ is a quasiconformal map, then F permutes left cosets of V_2 .*

Let $\mathfrak{g} = V_1 \oplus V_2 \oplus \dots \oplus V_r$ be a Carnot group and $H_i = V_i \oplus V_{i+1} \oplus \dots \oplus V_r$ where $1 \leq i \leq r$. Xie conjectured that if $\mathfrak{g}$ doesn't have a Heisenberg group factor, then every quasiconformal map on $\mathfrak{g}$ permutes left cosets of H_i , for $1 \leq i \leq r$ (see [5] Question 4.3). Corollary 4.4.2 verifies this conjecture at least in the case of the Carnot group $\mathfrak{n}$.

Taking the initial problem of finding the form of a quasiconformal map on $\mathfrak{n}$ one step further, we find that the reverse implication of Theorem 4.4.1 is also true.

Theorem 4.4.3. *Let $F : \mathfrak{n} \longrightarrow \mathfrak{n}$ be a map that has the form given in Theorem 4.4.1, then F is quasiconformal.*

Proof. See [6] Theorem 1.1. □

Hence, we get the following, stronger, theorem.

Theorem 4.4.4. *Let $F : \mathfrak{n} \longrightarrow \mathfrak{n}$ be a map. Then F is quasiconformal if and only if F has the form given in Theorem 4.4.1.*

In a general metric space, quasiconformal maps are not always as rigid as the case we've examined. In Euclidean space, for example, not all quasiconformal maps are biLipschitz. On the other hand, every quasiconformal map of the quaternion Heisenberg group is a similarity (see [4]). In our case, not only are all quasiconformal maps biLipschitz (see 2.3.5), they're even more rigid as exhibited by permuting certain left cosets (see 2.3.7, 2.3.11, 4.4.2) and having a specific form (see 4.4.4).

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