Carter Subgroups and Carter's Theorem

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ABSTRACT

In 1961 Roger W. Carter proved a theorem about solvable groups similar to Sylow's theorem. He proved that if a group is solvable then it always contains a nilpotent, self-normalizing subgroup called a Carter subgroup, and that all such subgroups are conjugate to each other by an element of the group. The objective of this thesis is to present a proof of Carter's theorem.

Dedication

To my husband, Ishahu Abubakar.

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CONTENTS

1 Introduction

Let G be a finite group, p be a prime, and $n \in \mathbb{Z}^+ \cup \{0\}$ such that p^n divides $|G|$ but p^{n+1} does not divide $|G|$. In 1872 Ludwig Sylow proved that there is a subgroup P of G such that $|P| = p^n$ and that all such subgroups are conjugate to each other by an element of G. Such a subgroup P is called a Sylow p-subgroup, named after Ludwig Sylow. If G has only one Sylow p-subgroup for each prime p, then G is called a nilpotent group. Now if $H \leq G$ then it is well known that the set

$$N_G(H) = \{g \in G | gHg^{-1} = H\}$$

is a subgroup of G.

Roger W. Carter obtained his PhD in 1960 and his dissertation was entitled "Some Contributions to the Theory of Finite Soluble Groups". He worked as a professor at the University of Warwick in the United Kingdom. He defined Carter subgroups and wrote the standard reference *Simple Groups of Lie Type.* Roger W. Carter in mid 1900s wondered if all groups contained a subgroup H that was nilpotent with the property that H is self-normalizing(ie $= H = N_G(H)$). Well it turns out that not all groups have a nilpotent, self-normalizing subgroup. For example, the alternating group A_5 of order 60 has no such subgroup. A group G is **solvable** if there exists a subnormal series

$$G = G_0 \trianglerighteq G_1 \trianglerighteq G_2 \trianglerighteq ... \trianglerighteq G_n = 1$$

such that the factors

$$\frac{G_i}{G_{i+1}}$$

are abelian, for all $0 \leq i \leq n-1$.

In 1961 Roger W. Carter showed a theorem about these subgroups similar to Sylow's theorem. He proved that if a group is solvable then it always contains a nilpotent, self-normalizing subgroup, and that all such subgroups are conjugate to each other by an element of the group [1]. These subgroups have been named Carter subgroups and the theorem, Carter's theorem. The objective of this thesis is to present a proof of Carter's theorem.

2 Preliminaries

Definition A **group** is a non empty set G along with a binary operation $*$ such that the following axioms are satisfied:

1. **Closed** $a * b \in G$ for all a, $b \in G$.
2. **Associativity** $(a * b) * c = a * (b * c)$ for all a, b, $c \in G$.
3. **Identity** There exists $e \in G$ such that for all $a \in G$, $e * a = a * e = a$.
4. **Inverses** For all $a \in G$ there exists $b \in G$ such that $a * b = b * a = e$.

We will write ab instead of $a * b$, 1 instead of e, and a^{-1} instead of b.

Definition A group G is called **abelian** if $ab = ba$ for all a, $b \in G$.

Definition Let G be a group and H be a non empty subset of G. Then H is a subgroup of G if H is a group. We write $H \leq G$.

Theorem 2.1. *(Subgroup test): Let G be a group and H be a non-empty subset of G. Then $H \leq G$ if and only if $ab^{-1} \in H$ for all a, $b \in H$.*

Proof

Suppose $H \leq G$. Let a, $b \in H$. Since $H \leq G$ and $b \in H$, we know $b^{-1} \in H$, and so $ab^{-1} \in H$ by closure. Suppose $ab^{-1} \in H$ for all a, $b \in H$. Let $a \in H$. Then $aa^{-1} \in H$, so $1 \in H$. Now $1a^{-1} \in H$ and so $a^{-1} \in H$ for all $a \in H$. Let a,$b \in H$. Then $b^{-1} \in H$ from above, and so $a(b^{-1})^{-1} \in H$. Thus $ab \in H$ and so H is closed. Since G is associative and $H \subseteq G$, we know H is associative. Therefore H is a group and so $H \leq G$. $\square$

Definition Let G be a group, the **center** of G is

$$Z(G) = \{g \in G | gx = xg \text{ for all } x \in G\}$$

Theorem 2.2. *Let G be a group. Then $Z(G) \leq G$.*

Proof

Now $1x = x$ and $x1 = x$ and so $1x = x1$ for all $x \in G$. Therefore $1 \in Z(G)$ and so $Z(G) \neq \emptyset$. Let a, $b \in Z(G)$ and let $x \in G$ then

$$\begin{aligned} xab^{-1} &= axb^{-1} \text{ since } a, b \in Z(G) \\ &= ab^{-1}bxb^{-1} \\ &= ab^{-1}xbb^{-1} \\ &= ab^{-1}x. \end{aligned}$$

Thus $ab^{-1} \in Z(G)$ and so $Z(G) \leq G$ by the Subgroup test. $\square$

Definition Let G be a group and $a \in G$. Define the **cyclic subgroup generated by** a by

$$\langle a \rangle = \{a^k | k \in Z\}.$$

Theorem 2.3. *Let G be a group and $a \in G$ then $\langle a \rangle \leq G$.*

Proof

Since $1 = a^0 \in \langle a \rangle$ then $\langle a \rangle \neq \emptyset$. Let a^m, $a^n \in \langle a \rangle$. Then $a^m(a^n)^{-1} = a^m a^{-n} =$

$a^{m-n} \in \langle a \rangle$ since $m - n \in \mathbb{Z}$. Therefore $\langle a \rangle \leq G$ by the Subgroup test. $\square$

Definition Let G be a group, $H \leq G$ and $g \in G$. Then the **left coset of** H **in** G **containing** g is the set

$$gH = \{gh | h \in H\}.$$

A number of theorems will be listed for (informational purposes) whose proofs are not given here.

Theorem 2.4. *Let G be a group, $H \leq G$, and a, $b \in G$. Then*

1. $|aH| = |H|$.
2. $aH = bH$ *if and only if* $b^{-1}a \in H$.

Theorem 2.5. *(Lagrange): Let G be a group and $H \leq G$. Then $|H|$ divides $|G|$ and*

$$\frac{|G|}{|H|} = \text{ number of left cosets of } H \text{ in } G$$

.

Definition Let G_1 and G_2 be groups and $\phi : G_1 \longrightarrow G_2$. Then ϕ is a **homomorphism** if $\phi(ab) = \phi(a)\phi(b)$ for all a, $b \in G_1$. If, in addition, ϕ is one-to-one and onto, we call ϕ an **isomorphism** and write $G_1 \cong G_2$.

Theorem 2.6. *Let $\phi : G_1 \longrightarrow G_2$ be a homomorphism and $a \in G_1$. Then*

1. $\phi(1) = 1$.

2. $\phi(a^{-1}) = (\phi(a))^{-1}$.

3. $\phi(a^n) = \phi(a)^n$ *for any* $n \in \mathbb{Z}$.

4. *If* $|a|$ *is finite, then* $|\phi(a)|$ *divides* $|a|$.

5. *If* $H \leq G_1$, *then* $\phi(H) \leq G_2$.

6. *If* $K \leq G_2$, *then* $\phi^{-1}(K) \leq G_1$.

Definition Let G_1 and G_2 be groups and $\phi : G_1 \longrightarrow G_2$ be a homomorphism. Define the **kernel** of ϕ by

$$\text{Kern } \phi = \{g \in G_1 | \phi(g) = 1\}.$$

Theorem 2.7. *Let* $\phi : G_1 \longrightarrow G_2$ *be a homomorphism. Then Kern* $\phi \trianglelefteq G_1$.

Definition Let G be a group and $H \leq G$. Then H is a **normal subgroup** of G if $ghg^{-1} \in H$ for all $g \in G$ and for all $h \in H$. We write $H \trianglelefteq G$.

Theorem 2.8. *Let* G *be a group and* $H \trianglelefteq G$. *Define the set* G/H *by*

$$G/H = \{gH | g \in G\}.$$

Then G/H *is a group under the operation* $aHbH = abH$ *for all* aH, $bH \in G/H$. *The group* G/H *is called the quotient group, the factor group, or* G *mod* H.

Theorem 2.9. *(First Isomorphism Theorem): Let G_1 and G_2 be groups and $\phi : G_1 \longrightarrow G_2$ be a homomorphism with Kern $\phi = K$. Then*

$$G_1/K \cong \phi(G_1).$$

Proof

Define a map $\chi : G_1/K \longrightarrow \phi(G_1)$ by $\chi(gK) = \phi(g)$ for all $g \in G$. Let $g_1, g_2 \in G_1$. Suppose $g_1K = g_2K$ then $g_2^{-1}g_1 \in K =$ Kern ϕ and so $\phi(g_2^{-1}g_1) = 1$ or $\phi(g_2^{-1})\phi(g_1) = 1$ since ϕ is a homomorphism. Therefore $\phi(g_2)^{-1}\phi(g_1) = 1$ and so $\phi(g_1) = \phi(g_2)$. Therefore $\chi(g_1K) = \chi(g_2K)$. This implies χ is well defined. Now let $g_1K, g_2K \in G_1/K$. Since ϕ is a homomorphism

$$\chi((g_1K)(g_2K)) = \chi((g_1g_2)K) = \phi(g_1g_2) = \phi(g_1)\phi(g_2) = \chi(g_1K)\chi(g_2K)$$

Implies χ is a homomorphism. Let $g_1K, g_2K \in G_1/K$, suppose $\chi(g_1K) = \chi(g_2K)$. Then $\phi(g_1) = \phi(g_2)$ or $(\phi(g_2))^{-1}\phi(g_1) = 1$ or $\phi(g_2^{-1})\phi(g_1) = 1$ since ϕ is a homomorphism. Hence $\phi(g_2^{-1})\phi(g_1) = \phi(g_2^{-1}g_1) = 1$ since ϕ is a homomorphism. Therefore $g_2^{-1}g_1 \in$ Kern $\phi = K$; hence $g_1K = g_2K$. So χ is one-to-one. Let $y \in \phi(G_1)$. Then there exists $x \in G_1$ such that $y = \phi(x)$. But then $xK \in G/K$ and $\chi(xK) = \phi(x) = y$. Hence χ is onto. Therefore $G_1/K \cong \phi(G_1)$. $\square$

Theorem 2.10. *(Second Isomorphism Theorem): Let G be a group, $H \leq G$, and $N \trianglelefteq G$. Then*

$$\frac{HN}{N} \cong \frac{H}{H \cap N}$$

Proof

Define a map $\phi : H \longrightarrow HN/N$ by $\phi(h) = hN$ for $h \in H$. Let $h_1,\ h_2 \in H$. Then $\phi(h_1h_2) = (h_1h_2)N = h_1Nh_2N = \phi(h_1)\phi(h_2)$. Hence ϕ is a homomorphism. Let $h_1 \in H$. Then

$$\begin{aligned} h_1 &\in \text{ Kern } \phi \\ \text{if and only if } \phi(h_1) = h_1N &= 1N \\ \text{if and only if } 1^{-1}h_1 &\in N \\ \text{if and only if } h_1 &\in H \cap N. \end{aligned}$$

Hence $H \cap N = $ Kern ϕ. Let $hnN \in HN/N$ where $h \in H$ and $n \in N$. Then $\phi(h) = hN = hnN$ since $(hn)^{-1}h = n^{-1} \in N$ and so χ is onto. Now by the First Isomorphism Theorem

$$\frac{H}{\text{Kern } \phi} \cong \phi(H)$$

which implies

$$\frac{HN}{N} \cong \frac{H}{H \cap N}.$$

□

Theorem 2.11. *(Third Isomorphism Theorem): Let G be a group, $N \leq H \leq G$, $N \trianglelefteq G$, and $H \trianglelefteq G$. Then*

$$\frac{G/N}{H/N} \cong G/H.$$

Proof

Define $\phi : G/N \longrightarrow G/H$ by $\phi(gN) = gH$ for all $gN \in G/N$. Let $g_1N,\ g_2N \in G/N$ for $g_1, g_2 \in G$. Suppose $g_1N = g_2N$. Then $g_2^{-1}g_1 \in N$. Also $g_2^{-1}g_1 \in H$ since $N \leq H$

and so $g_1H = g_2H$. Therefore $\phi(g_1N) = \phi(g_2N)$ and ϕ is well-defined. Now let g_1N, $g_2N \in G/N$ for some $g_1, g_2 \in G$. Then

$$\phi(g_1Ng_2N) = \phi(g_1g_2N) = g_1g_2H = g_1Hg_2H = \phi(g_1N)\phi(g_2N),$$

and so ϕ is a homomorphism. Let $gH \in G/H$. Then $gN \in G/N$ and so $\phi(gN) = gH$. Therefore ϕ is onto. Let $g_1N \in G/N$. Then

$$\begin{aligned} g_1N &\in Kern\phi \\ \text{if and only if } \phi(g_1N) &= 1H \\ \text{if and only if } g_1H &= 1H \\ \text{if and only if } 1^{-1}g_1 &\in H \\ \text{if and only if } g_1 &\in H \\ \text{if and only if } g_1N &\in H/N. \end{aligned}$$

Thus Kern $\phi = H/N$. Now by the First Isomorphism Theorem

$$\frac{G/N}{\text{Kern } \phi} \cong \phi(G/N);$$

hence

$$\frac{G/N}{H/N} \cong G/H.$$

□

Definition Let G be a group and $S \subseteq G$ be a nonempty subset of G. Then the

subgroup generated by S is

$$\langle S\rangle = \bigcap_{S\subseteq H\leq G} H.$$

Theorem 2.12. *Let G be a group and $S \subseteq G$ be a nonempty subset. Then*

$$\langle S\rangle = \{s_1^{n_1}s_2^{n_2}\cdots s_k^{n_k} | s_i \in S \textit{ and } n_i \in \mathbb{Z} \textit{ for all } 1 \leq i \leq k\}.$$

Proof

Let $T = \{s_1^{n_1}s_2^{n_2}\cdots s_k^{n_k} | s_i \in S \text{ and } n_i \in Z \text{ for all } 1 \leq i \leq k\}$. We claim that $T \leq G$ Since S is nonempty there exists $s_1 \in S$. Then $1 = s_1^0 \in T$ and so $T \neq \emptyset$. Now let $s_1^{n_1}s_2^{n_2}\cdots s_k^{n_k}$, $r_1^{m_1}r_2^{m_2}\cdots r_l^{m_l} \in T$ where s_i, $r_j \in S$ and n_i, $m_j \in \mathbb{Z}$ for $1 \leq i \leq k$ and $1 \leq j \leq l$. Then

$$\begin{aligned}(s_1^{n_1}s_2^{n_2}\cdots s_k^{n_k})(r_1^{m_1}r_2^{m_2}\cdots r_l^{m_l})^{-1} &= (s_1^{n_1}s_2^{n_2}\cdots s_k^{n_k})(r_l^{-m_l}r_{l-1}^{-m_{l-1}}\cdots r_2^{-m_2}r_1^{-m_1}) \\ &= s_1^{n_1}s_2^{n_2}\cdots s_k^{n_k}r_l^{-m_l}r_{l-1}^{-m_{l-1}}\cdots r_1^{-m_1} \in T.\end{aligned}$$

Thus $T \leq G$ by the subgroup test. Let $s \in S$. Then $s = s^1 \in T$ and so $S \subseteq T \leq G$. Therefore $\langle S\rangle = \bigcap_{S\subseteq H\leq G} H \leq T$. Let $s_1^{n_1}s_2^{n_2}\cdots s_k^{n_k} \in T$ where $k \in \mathbb{Z}^+, s_i \in S$, and $n_i \in \mathbb{Z}$ for all $1 \leq i \leq k$. Suppose that $S \subseteq H \leq G$. Since $s_i \in S \subseteq H$ for all i we know $s_i^{n_i} \in H$ for all i since $H \leq G$. Therefore $s_1^{n_1}s_2^{n_2}\cdots s_k^{n_k} \in H$ since $H \leq G$. Thus $T \leq H$ and so $T \leq \langle S\rangle$ and we have $\langle S\rangle = T$. $\square$

Theorem 2.13. *Let G be a group, $N \trianglelefteq G$, $H \leq G$ and let $\phi : G \longrightarrow G/N$ be defined by $\phi(g) = gN$ for all $g \in G$. Then*

1. *ϕ is a homomorphism;*

2. *Kern $\phi = N$;*

3. *$\phi(H) = HN/N$;*

4. *$\phi^{-1}(HN/N) = HN$;*

5. *if $L \leq G/N$ then $L = K/N$ where $N \leq K \leq G$.*

Proof

For (1), let $g_1, g_2 \in G$. Then $\phi(g_1 g_2) = g_1 g_2 N = g_1 N g_2 N$, so ϕ is a homomorphism. For (2), let $g \in G$. Then

$$\begin{aligned} g &\in \text{Kern } \phi \\ \text{if and only if } \phi(g) &= 1N \\ \text{if and only if } gN &= 1N \\ \text{if and only if } 1^{-1}g &\in N \\ \text{if and only if } g &\in N. \end{aligned}$$

So Kern $\phi = N$. For (3), let $hnN \in HN/N$ for $h \in H$, $n \in N$. Then $hnN = hN$ since $(hn)^{-1}h = n^{-1} \in N$. Therefore $hnN = \phi(h) \in \phi(H)$ and so $HN/N \subseteq \phi(H)$. Let $x \in \phi(H)$. There exists $h \in H$ such that $x = \phi(h)$. Then $x = \phi(h) = hN \in HN/N$. Thus

$$\phi(H) = \frac{HN}{N}.$$

For (4), let $g \in \phi^{-1}(HN/N)$. Then there exists $hnN \in HN/N$ such that $\phi(g) = hnN = hN$. Hence $gN = hN$ and so $h^{-1}g \in N$. But then there exists $n_1 \in N$ such that $h^{-1}g = n_1$ and so $g = hn_1 \in HN$. Hence

$$\phi^{-1}\left(\frac{HN}{N}\right) \subseteq HN.$$

Now let $hn \in HN$. Then $\phi(hn) = hnN \in HN/N$ and so $hn \in \phi^{-1}(HN/N)$. Thus $HN \subseteq \phi^{-1}(HN/N)$, so $\phi^{-1}(HN/N) = HN$. Finally, consider $\phi^{-1}(L) = K$. Since $L \leq G/N$ we know $\phi^{-1}(L) \leq G$. Let $n \in N$, then $\phi(n) = nN = 1N \in L$ since $L \leq G/N$. Hence $n \in \phi^{-1}(L)$ and so $N \leq \phi^{-1}(L)$. We claim that

$$L = \frac{\phi^{-1}(L)}{N}.$$

Let $gN \in L$. Then $\phi(g) = gN \in L$. Hence $g \in \phi^{-1}(L)$ and so $gN \in \phi^{-1}(L)/N$. Therefore $L \leq \phi^{-1}(L)/N$. Let $gN \in \phi^{-1}(L)/N$. Then $g \in \phi^{-1}(L)$ and so $gN = \phi(g) \in L$. Thus $\phi^{-1}(L)/N \leq L$ and so $L = \phi^{-1}(L)/N$. □

Definition Let G be a finite group, p be a prime, and $n \in \mathbb{Z}^+ \cup \{0\}$ such that p^n divides $|G|$ but p^{n+1} does not divide $|G|$. Then

1. A subgroup $P \leq G$ is called a **Sylow p-subgroup** if $|P| = p^n$.

2. $\mathrm{Syl}_p(G) = \{P \leq G | P \text{ is a Sylow } p\text{-subgroup of } G\}$.

Theorem 2.14. *(Sylow's) Let G be a finite group, with $|G| = p^n m$, where p is prime, $n \geq 1$ and p does not divide m. Then*

1. For each i, $1 \leq i \leq n$. There is a subgroup of G of order p^i. Every subgroup *of order p^i is a normal subgroup of some subgroup of order p^{i+1} for all $1 \leq i \leq n-1$;*

2. *Any two Sylow p-subgroups of G are conjugate in G;*

3. *The number n_p of Sylow p-subgroups of G divides $|G|$ and is congruent to 1 mod p.*

Theorem 2.15. *Let G be a group, $H \leq G$, $K \leq G$ and $L \leq G$ such that $K \leq H$. Then,*

$$H \cap KL = K(H \cap L)$$

Proof

Let $x \in K(H \cap L)$. Then there exist $k \in K \leq H$ and also $n \in H \cap L$ such that $x = kn$. Since $n \in H \cap L, n \in H$ and $n \in L$. Therefore $x = kn \in H$ by closure. Also $x = kn \in KL$. Hence $x \in H \cap KL$ and so $K(H \cap L) \subseteq H \cap KL$. Now let $y \in H \cap KL$. Then $y \in H$ and $y \in KL$. Therefore there exist $k \in K$ and $l \in L$ such that $y = kl$. Since $y \in H$ we have $kl \in H$. But since $k \in K \leq H$ and $H \leq G$ we know $k^{-1} \in H$. Thus $l = k^{-1}kl \in H$, and so $l \in H \cap L$. Thus $y = kl \in K(H \cap L)$. Therefore $H \cap KL \subseteq K(H \cap L)$ and so $H \cap KL = K(H \cap L)$. $\square$

3 Solvable Groups

Definition A **subnormal series** of a group G is a sequence of subgroups, each a normal subgroup of the next one. In a standard notation

$$G = G_0 \trianglerighteq G_1 \trianglerighteq G_2 \trianglerighteq \cdots \trianglerighteq G_n = 1.$$

Definition A group G is **solvable** if there exists a subnormal series

$$G = G_0 \trianglerighteq G_1 \trianglerighteq G_2 \trianglerighteq \cdots \trianglerighteq G_n = 1$$

such that the factors

$$\frac{G_i}{G_{i+1}}$$

are abelian for all $0 \leq i \leq n-1$.

Lemma 3.1. *If G is an abelian group then G is solvable.*

Proof

Consider the subnormal series $G = G_0 \trianglerighteq G_1 = 1$. Then $G_0/G_1 = G/1 \cong G$ is abelian.

□

Examples.

$\mathbb{Z}_n$ and $\mathbb{Z}_m \times \mathbb{Z}_n$ are solvable for all m, $n \in \mathbb{Z}^+$ by Lemma 3.1.

Lemma 3.2. *If G is a p-group then G is solvable.*

Proof

We use induction on $|G|$. If $|G| = p^0 = 1$ then $G = \{1\}$. Hence G is abelian and so G is solvable by Lemma 3.1. Suppose the lemma holds for all p-groups of order less than $|G|$. Since G is a p-group we know $1 \neq Z(G) \trianglelefteq G$. Then $|G/Z(G)| < |G|$ and $G/Z(G)$ is a p-group. Hence $G/Z(G)$ is solvable and so there exists a subnormal series

$$G/Z(G) = G_0/Z(G) \trianglerighteq G_1/Z(G) \trianglerighteq G_2/Z(G) \trianglerighteq \cdots \trianglerighteq G_n/Z(G) = 1$$

such that

$$\frac{G_i/Z(G)}{G_{i+1}/Z(G)}$$

is abelian for all $0 \leq i \leq n-1$. Taking preimages we get

$$G = G_0 \trianglerighteq G_1 \trianglerighteq G_2 \trianglerighteq \cdots \trianglerighteq Z(G) \trianglerighteq 1,$$

a subnormal series. By the Third Isomorphism Theorem

$$\frac{G_i}{G_{i+1}} \cong \frac{G_i/Z(G)}{G_{i+1}/Z(G)}$$

and so G_i/G_{i+1} is abelian for all $0 \leq i \leq n-1$. Finally, $Z(G)/1 \cong Z(G)$ is abelian and so G is solvable. $\square$

Examples. D_4, Q_8, $\mathbb{Z}_{16} \times D_8$ are all solvable groups.

Theorem 3.3. *Let G be a solvable group and $H \leq G$. Then H is solvable.*

Proof

Since G is solvable, there exists a subnormal series

$$G = G_0 \trianglerighteq G_1 \trianglerighteq G_2 \trianglerighteq \cdots \trianglerighteq G_n = 1$$

such that G_i/G_{i+1} is abelian for all $0 \leq i \leq n-1$. Now we have the series

$$H = H \cap G \geq H \cap G_1 \geq H \cap G_2 \geq \cdots \geq H \cap G_n = 1.$$

If $g \in H \cap G_{i+1}$ and $x \in H \cap G_i$, then $xgx^{-1} \in H$ since $g, x \in H$ and $H \leq G$. Also since $g \in G_{i+1}$, $x \in G_i$ and $G_{i+1} \trianglelefteq G_i$, we get $xgx^{-1} \in G_{i+1}$. Thus $xgx^{-1} \in H \cap G_{i+1}$; so $H \cap G_{i+1} \trianglelefteq H \cap G_i$ for all $0 \leq i \leq n-1$. Therefore we have a subnormal series

$$H = H \cap G_0 \trianglerighteq H \cap G_1 \trianglerighteq H \cap G_2 \trianglerighteq \cdots \trianglerighteq H \cap G_n = 1.$$

Also

$$\frac{H \cap G_i}{H \cap G_{i+1}} = \frac{H \cap G_i}{H \cap G_i \cap G_{i+1}} \cong \frac{(H \cap G_i)G_{i+1}}{G_{i+1}}$$

by the Second Isomorphism Theorem. Now

$$\frac{(H \cap G_i)G_{i+1}}{G_{i+1}} \leq \frac{G_i}{G_{i+1}}$$

and G_i/G_{i+1} is abelian. Therefore $H \cap G_i / H \cap G_{i+1}$ is abelian and so H is solvable. □

Theorem 3.4. *If G is solvable and $N \trianglelefteq G$ then G/N is solvable.*

Proof

Since G is solvable, there exists a subnormal series $G = G_0 \trianglerighteq G_1 \trianglerighteq G_2 \trianglerighteq \cdots \trianglerighteq G_n = 1$ such that G_i/G_{i+1} is abelian for all $0 \leq i \leq n-1$. Taking the image of this series under the natural map $\phi : G \longrightarrow G/N$ we get

$$\frac{G}{N} = \frac{G_0}{N} \trianglerighteq \frac{G_1 N}{N} \trianglerighteq \cdots \trianglerighteq \frac{G_n N}{N} = N.$$

Now by the Second and Third Isomorphism Theorems,

$$\frac{G_i N/N}{G_{i+1}N/N} \cong \frac{G_i N}{G_{i+1}N} = \frac{G_i G_{i+1} N}{G_{i+1}N} \cong \frac{G_i}{G_i \cap G_{i+1}N} \cong \frac{G_i/G_{i+1}}{(G_i \cap G_{i+1}N)/G_{i+1}}.$$

Since G_i/G_{i+1} is abelian we get

$$\frac{G_i N/N}{G_{i+1}N/N}$$

is abelian for all $0 \leq i \leq n-1$. Therefore G/N is solvable. □

Theorem 3.5. *Let G be a solvable group and $N \trianglelefteq G$. If N is solvable and G/N is solvable then G is solvable.*

Proof

Since N is solvable there exists a subnormal series $N = N_0 \trianglerighteq N_1 \trianglerighteq N_2 \trianglerighteq \cdots \trianglerighteq N_n = 1$ such that N_i/N_{i+1} is abelian for all $0 \leq i \leq n-1$. Also since G/N is solvable then there exists a subnormal series

$$\frac{G}{N} = \frac{G_0}{N} \trianglerighteq \frac{G_1}{N} \trianglerighteq \frac{G_2}{N} \trianglerighteq \cdots \trianglerighteq \frac{G_m}{N} = N$$

such that

$$\frac{G_i/N}{G_{i+1}/N}$$

is abelian for all $0 \leq i \leq m-1$. Taking preimages we get

$$G = G_0 \trianglerighteq G_1 \trianglerighteq G_2 \trianglerighteq \cdots \trianglerighteq N = N_0 \trianglerighteq N_1 \trianglerighteq N_2 \trianglerighteq \cdots \trianglerighteq N_n = 1$$

. By the Third Isomorphism Theorem

$$\frac{G_i}{G_{i+1}} \cong \frac{G_i/N}{G_{i+1}/N}$$

and so G_i/G_{i+1} is abelian for all $0 \leq i \leq m-1$. Therefore G is solvable. □

Definition Let G be a group, $H \leq G$, $K \leq G$ and a, $b \in G$. Then

1. $[a,b] = aba^{-1}b^{-1}$ is called the **commutator** of a and b.

2. $[H,K] = \langle [h,k] | h \in H, k \in K \rangle$.

3. $G' = \langle [x,y] | x, y \in G \rangle$ is called the **commutator subgroup of** G.

Theorem 3.6. *Let G be a group, $N \trianglelefteq G$, $H \leq G$ and a, $b \in G$. Then*

1. $[a,b] = 1$ *if and only if* $ab = ba$.

2. $G' \trianglelefteq G$.

3. G/G' *is abelian.*

4. *If* $G' \leq H$ *then* $H \trianglelefteq G$.

Proof

For (1): Now $[a,b] = 1$ if and only if $aba^{-1}b^{-1} = 1$ if and only if $ab = ba$. For (2) :

We know that $G' \leq G$. Now let $g \in G$ and $\prod_{i=1}^{n}[a_i, b_i] \in G'$. Since conjugation is a homomorphism,

$$\begin{aligned} g(\prod_{i=1}^{n}[a_i, b_i])g^{-1} &= \prod_{i=1}^{n} g[a_i, b_i]g^{-1} \\ &= \prod_{i=1}^{n}[ga_ig^{-1}, gb_ig^{-1}] \in G'. \end{aligned}$$

Hence $G' \trianglelefteq G$. For (3): Let aG', $bG' \in G/G'$. Then $(ba)^{-1}ab = a^{-1}b^{-1}ab = [a^{-1}, b^{-1}] \in G'$. Therefore $abG' = baG'$ and so $aG'bG' = bG'aG'$. Hence G/G' is abelian. For (4): Let $h \in H$ and $g \in G$. Then $[h^{-1}, g] \in G' \leq H$ and so $[h^{-1}, g] \in H$. Now since $h \in H$ and $H \leq G$ we get $h(h^{-1}ghg^{-1}) \in H$. Therefore $H \trianglelefteq G$. □

Lemma 3.7. *Let G be a group and $N \trianglelefteq G$ such that G/N is abelian. Then $G' \leq N$.*

Let a, $b \in G$. Then $a^{-1}N$, $b^{-1}N \in G/N$. Since G/N is abelian, $a^{-1}Nb^{-1}N = b^{-1}Na^{-1}N$ and so $a^{-1}b^{-1}N = b^{-1}a^{-1}N$. Hence $(b^{-1}a^{-1})^{-1}a^{-1}b^{-1} \in N$ and so $aba^{-1}b^{-1} \in N$ or $[a, b] \in N$. Now since $N \leq G$ we get $G' \leq N$. □

Definition Let G be a group. Define the **derived series** of G by
$G^{(0)} = G$, $G^{(1)} = (G^{(0)})' = G'$, $G^{(2)} = (G^{(1)})' = G''$, and inductively by $G^{(n)} = (G^{(n-1)})'$.

Lemma 3.8. *Let G be a group. Then*

1. *$G^{(i+1)} \leq G^{(i)}$ for all i.*

2. $G^{(i)} \trianglelefteq G$ *for all* i.

3. G *is solvable if and only if there exists* $n \in \mathbb{Z}^+ \cup \{0\}$ *such that* $G^{(n)} = 1$.

Proof

By definition of derived series, $G^{(i+1)} = (G^{(i)})' \leq G^{(i)}$ for all $i \in \mathbb{Z}^+$. Statement (2) is true for $i = 1$ since $G^{(1)} = (G^{(0)})' = (G)' = G' \trianglelefteq G$. Suppose the statement is true for i i.e $G^{(i)} \trianglelefteq G$. Let $g \in G$. then

$$\begin{aligned} gG^{(i+1)}g^{-1} &= g(G^{(i)})'g^{-1} \\ &= g[G^{(i)}, G^{(i)}]g^{-1} \\ &= [gG^{(i)}g^{-1}, gG^{(i)}g^{-1}] \\ &= [G^{(i)}, G^{(i)}] \\ &= G^{(i+1)}. \end{aligned}$$

And (2) is proven. Therefore $G^{(i+1)} \trianglelefteq G$. Suppose $G^{(n)} = 1$. Then we have

$$G = G^{(0)} \trianglerighteq G^{(1)} \trianglerighteq \cdots \trianglerighteq G^{(n)} = 1.$$

Also

$$\frac{G^{(i)}}{G^{(i+1)}} = \frac{G^{(i)}}{(G^{(i)})'}$$

is abelian for $0 \leq i \leq n-1$. Thus G is solvable. Next suppose G is solvable. Then there exists a subnormal series $G = G_0 \trianglerighteq G_1 \trianglerighteq G_2 \trianglerighteq \cdots \trianglerighteq G_n = 1$ such that G_i/G_{i+1} is abelian for all $0 \leq i \leq n-1$. We claim that $G^{(i)} \leq G_i$ for all $0 \leq i \leq n-1$. If $i = 0$ then $G^{(0)} = G \leq G = G_0$ and so $G^{(0)} \leq G_0$. Suppose $G^{(i)} \leq G_i$. Then $G^{(i+1)} = (G^{(i)})' \leq G_i' \leq G_{i+1}$ since G_i/G_{i+1} is abelian. Therefore $G^{(n)} \leq G_n = 1$ and so $G^{(n)} = 1$. □

Definition Let G be a group. Then $\phi : G \longrightarrow G$ is a **automorphism** if ϕ is one-to-one, onto, and a homomorphism.

Definition Let G be a group and $H \leq G$. Then H is a **characteristic subgroup** if $\phi(H) \leq H$ for all automorphisms ϕ of G. We write H char G.

Theorem 3.9. *Let G be a group. Then*

1. *$Z(G)$ char G.*
2. *G' char G.*
3. *If $P \in Syl_p(G)$ such that $P \trianglelefteq G$, then P char G.*

Proof

Let ϕ be a automorphism of G, $x \in Z(G)$, and $g \in G$. Since ϕ is onto, there exists $y \in G$ such that $\phi(y) = g$. Then

$$\phi(x)g = \phi(x)\phi(y) = \phi(xy) = \phi(yx) = \phi(y)\phi(x) = g\phi(x)$$

since $x \in Z(G)$ and ϕ is a homomorphism. Therefore $\phi(x) \in Z(G)$ and so $\phi(Z(G)) \leq Z(G)$. Hence $Z(G)$ char G. Next let ϕ be a automorphism of G and $\prod_{i=1}^{n}[a_i, b_i] \in G'$. Then

$$\phi(\prod_{i=1}^{n}[a_i, b_i]) = \prod_{i=1}^{n}\phi([a_i, b_i]) = \prod_{i=1}^{n}[\phi(a_i), \phi(b_i)] \in G'.$$

Thus $\phi(G') \leq G'$ and so G' char G. Finally, since $P \trianglelefteq G$ we know $N_G(P) = \{g \in G | gPg^{-1} = P\} = G$. Thus by Sylow's Theorem,

$$n_p = \frac{|G|}{|N_G(P)|} = 1.$$

Since ϕ is one-to-one and onto, $|\phi(P)| = |P|$. Hence $\phi(P) \in \mathrm{Syl}_p(G)$. Therefore $\phi(P) = P$ which implies P char G. $\square$

Definition Let G be a group and $N \trianglelefteq G$. Then N is a **minimal normal subgroup** if whenever there exist $M \leq N$ such that $M \trianglelefteq G$ then $M = 1$ or $M = N$.

Example. Note $A_3 \trianglelefteq S_3$ and $|A_3| = 3$. Hence A_3 has no nontrivial subgroups and so A_3 is a minimal normal subgroup of S_3.

Example. Let $H = \{1, (13), (24), (13)(24)\}$. Then $|D_4|/|H| = 8/4 = 2$ and so $H \trianglelefteq D_4$. But H is not a minimal normal subgroup since $1 \neq Z(D_4) \leq H$ and $Z(D_4) \trianglelefteq D_4$.

Theorem 3.10. *Let G be a group and $H \leq K \leq G$. If H char K and K char G. Then H char G.*

Proof

Let ϕ be a automorphism of G. Then since K char G we have $\phi(K) \leq K$. Also since ϕ is one-to-one, $|\phi(K)| = |K|$ and so $\phi(K) = K$. Hence $\phi|_K$ is a automorphism of K. Since H char K we get $\phi|_K(H) \leq H$ or $\phi(H) \leq H$. Hence H char G. $\square$

Theorem 3.11. *Let G be a group, H char K, and $K \trianglelefteq G$. Then $H \trianglelefteq G$.*

Proof

For $g \in G$ define $\phi : K \longrightarrow K$ by $\phi(k) = gkg^{-1}$ for all $k \in K$. Then ϕ is a homomorphism and ϕ is one-to-one. If $k \in K$, and $K \trianglelefteq G$ we have $g^{-1}kg \in K$. Also $\phi(g^{-1}kg) = g(g^{-1}kg)g^{-1} = k$ and so ϕ is onto. Thus ϕ is a automorphism of K. Since H char K we get $\phi(H) \leq H$. But $|gHg^{-}| \leq |H|$. Now since $|gHg^{-1}| = |H|$ we get $gHg^{-1} = H$ and so $H \trianglelefteq G$. $\square$

Definition A group G is called **characteristically simple** if 1 and G are its only characteristic subgroups.

Theorem 3.12. *Let G be a characteristically simple group. Then*

$$G \cong G_1 \times G_2 \times \cdots \times G_n$$

such that $G_i s$ are simple isomorphic groups.

Proof

Let $G_1 \trianglelefteq G$ such that $G_1 \neq 1$ and $|G_1|$ is minimal. Also let $H = \prod_{i=1}^{s} G_i$ such that

1. $G_i \trianglelefteq G$ for all $1 \leq i \leq s$;

2. $G_i \cong G_1$ for all $1 \leq i \leq s$;

3. $G_i \bigcap \prod_{j \neq i} G_j = 1$ for all $1 \leq i \leq s$;

4. s is maximal.

Since $G_i \trianglelefteq G$ for all $1 \leq i \leq s$, we get $H = \prod_{i=1}^{s} G_i \trianglelefteq G$. We claim that H char G. If not, there exists an automorphism ϕ of G and $1 \leq i \leq s$ such that $\phi(G_i) \not\leq H$.

Then $\phi(G_i) \bigcap H < \phi(G_i)$. Since $G_i \trianglelefteq G$ we get $\phi(G_i) \trianglelefteq G$. But then $H \trianglelefteq G$ implies $\phi(G_i) \bigcap H \trianglelefteq G$. Since ϕ is an automorphism of G we get $G_i \cong \phi(G_i)$, so $|\phi(G_i) \bigcap H| < |\phi(G_i)| = |G_i| = |G_1|$. Therefore by the minimality of G_1 we get $\phi(G_i) \bigcap H = 1$. Now $\phi(G_i) \trianglelefteq G, \phi(G_i) \cong G_i \cong G_1$, and $\phi(G_i) \bigcap \prod_{i=1}^{s} G_i \leq \phi(G_i) \bigcap H = 1$. But then we get $H = \prod_{i=1}^{s} G_i < \phi(G_i) \prod_{i=1}^{s} G_i$ a contradiction to the maximality of s. Therefore H char G. Since G is characteristically simple, $H = 1$ or $H = G$. But $1 \neq G_1 \leq H$ and so $H \neq 1$. Thus $G = H = \prod_{i=1}^{s} G_i$ and G_is are isomorphic groups. Let $1 \leq i \leq s$ and $N \trianglelefteq G_i$. If $1 \leq j \leq s$ and $j \neq i$ then $[G_j, N] \leq [G_j, G_i] \leq G_j \bigcap G_i \leq G_i \bigcap \prod_{j \neq i} G_j = 1$ and so $[G_j, N] = 1$. Hence $G_j \leq N_G(N)$ for all $1 \leq j \leq s$ such that $j \neq i$. Also, since $N \trianglelefteq G_i$ we know $G_i \leq N_G(N)$. Hence $G = \prod_{i=1}^{s} G_i \leq N_G(N)$ and so $N = 1$ or $|N| = |G_1|$ by the minimality of G_1. Thus $N = 1$ or $N = G_i$ and so G_i is simple for all $1 \leq i \leq s$. But then $G = \prod_{i=1}^{s} G_i \cong G_1 \times G_2 \times \cdots \times G_s$ when we consider the map $\theta : G \longrightarrow G_1 \times G_2 \times \cdots \times G_s$ defined by

$$\theta(g_1 g_2 \cdots g_s) = (g_1, g_2, \cdots, g_s)$$

Let $g_1 g_2 \cdots g_s$, $h_1 h_2 \cdots h_s \in G$. Then

$$\begin{aligned} \theta((g_1 g_2 \cdots g_s)(h_1 h_2 \cdots h_s)) &= \theta(g_1 g_2 \cdots g_s h_1 h_2 \cdots h_s) \\ &= \theta(g_1 h_1 g_2 h_2 \cdots g_s h_s) \\ &= (g_1, g_2, \cdots, g_s)(h_1, h_2, \cdots, h_s) \\ &= \theta(g_1 g_2 \cdots g_s)\theta(h_1 h_2 \cdots h_s). \end{aligned}$$

Hence θ is homomorphism. Let $g_1 g_2 \cdots g_s$, $h_1 h_2 \cdots h_s \in G$ Now $\theta(g_1 g_2 \cdots g_s) = \theta(h_1 h_2 \cdots h_s)$. This implies that $(g_1, g_2, \cdots, g_s) = (h_1, h_2, \cdots, h_s)$ or $g_i = h_i$ for all $1 \leq i \leq s$. Hence θ is one-to-one. Let $(g_1, g_2, \cdots, g_s) \in G_1 \times G_2 \times \cdots \times G_s$. Since $g_i \in G_i$ for each i we know $(g_1 g_2 \cdots g_s) \in G$ and $\theta(g_1 g_2 \cdots g_s) = (g_1, g_2, \cdots, g_s)$. Therefore θ is onto and so $G \cong G_1 \times G_2 \times \cdots \times G_n$ where the G_is are simple isomorphic groups. □

Theorem 3.13. *Let G be a group and N be a minimal normal subgroup of G. Then*

$$N \cong N_1 \times N_2 \times \cdots \times N_n$$

such that the N_is are simple non-abelian isomorphic groups or $N_i \cong \mathbb{Z}_p$ for all $1 \leq i \leq n$, and for some prime p.

Proof

If M char N then, since $N \trianglelefteq G$, we get $M \trianglelefteq G$. Hence $M = 1$ or $M = N$ by the minimality of N. Therefore N is characteristically simple and so by previous theorem $N \cong N_1 \times N_2 \times \cdots \times N_n$, where the N_is are simple isomorphic groups.

Case 1: N_i is abelian for all $1 \leq i \leq n$. Since N_i is simple we get 1 and N_i as the only subgroups of N_i. By Cauchy's theorem there exist a prime p such that $|N_i| = p^m$. But then by Sylow's theorem $m = 1$ and so $|N_i| = p$; hence $N_i \cong \mathbb{Z}_p$ for all $1 \leq i \leq n$.

Case 2: N_i is non abelian for all $1 \leq i \leq n$. Then $N \cong N_1 \times N_2 \times \cdots \times N_n$ is the direct product of simple non-abelian isomorphic groups. □

Definition Let G be a group. Define the **lower central series** of G by $K_0(G) =$

$G, K_1(G) = [K_0(G), G] = [G, G] = G', K_2(G) = [K_1(G), G] = [[G, G], G]$, and inductively by $K_n(G) = [K_{n-1}(G), G]$.

Theorem 3.14. *Let G be a group. Then*

1. $K_i(G) \trianglelefteq G$ *for all* i.

2. $K_{i+1}(G) \leq K_i(G)$ *for all* i.

Proof

Proceed by using induction on i. If $i = 0$, then $K_0(G) = G \trianglelefteq G$. Assume $K_i(G) \trianglelefteq G$ and let $g \in G$. Then

$$\begin{aligned} gK_{i+1}(G)g^{-1} &= g[K_i(G), G]g^{-1} \\ &= [gK_i(G)g^{-1}, gGg^{-1}] \\ &= [K_i(G), G] \\ &= K_{i+1}(G). \end{aligned}$$

Thus, $K_{i+1}(G) \trianglelefteq G$ and we have (1) by induction. Now $K_{i+1}(G) = [K_i(G), G] \leq K_i(G)$, since $K_i(G) \trianglelefteq G$. Hence we get $K_{i+1}(G) \leq K_i(G)$ for all i. $\square$

4 Nilpotent Groups

Definition A group G is called **nilpotent** if there exists $n \in \mathbb{Z}^+ \cup \{0\}$ such that $K_n(G) = 1$.

Lemma 4.1. *If G is abelian, then $K_1(G) = [K_0(G), G] = [G, G] = 1$. Hence G is nilpotent.*

Example $\mathbb{Z}_{10}$, $\mathbb{Z}_8 \times \mathbb{Z}_{12}$, $\mathbb{R}$, $\mathbb{Q}$ *are nilpotent groups.*

Theorem 4.2. *Let G be a p-group. Then G is nilpotent.*

Proof

We use induction on $|G|$. If $|G| = p$ then G is cyclic. It fellows that G is abelian and by Lemma 4.1 G is nilpotent. Suppose all p-groups of order less than $|G|$ are nilpotent. We claim G is nilpotent. Since G is a p-group, we know $1 \neq Z(G) \trianglelefteq G$. So $G/Z(G)$ is a p-group and $|G/Z(G)| < |G|$. Then by assumption $G/Z(G)$ is nilpotent. So there exists $n \in \mathbb{Z}^+ \cup \{0\}$ such that

$$K_n\left(\frac{G}{Z(G)}\right) = 1.$$

We claim

$$\frac{K_i(G)Z(G)}{Z(G)} \leq K_i\left(\frac{G}{Z(G)}\right) \text{ for all } i$$

Use induction on i. If $i = 0$ then

$$\frac{K_0(G)Z(G)}{Z(G)} = \frac{GZ(G)}{Z(G)} = \frac{G}{Z(G)} \leq K_0\left(\frac{G}{Z(G)}\right) = \frac{G}{Z(G)}.$$

Suppose $K_i(G)Z(G)/Z(G) \leq K_i(G/Z(G))$. Then

$$\begin{aligned}
\frac{K_{i+1}(G)Z(G)}{Z(G)} &= \frac{[K_i(G), G]Z(G)}{Z(G)} \\
&\leq \left[\frac{K_i(G)Z(G)}{Z(G)}, \frac{G}{Z(G)}\right] \\
&\leq \left[K_i(\frac{G}{Z(G)}), \frac{G}{Z(G)}\right] \\
&= K_{i+1}\left(\frac{G}{Z(G)}\right).
\end{aligned}$$

Thus

$$\frac{K_i(G)Z(G)}{Z(G)} \leq K_i\left(\frac{G}{Z(G)}\right)$$

for all i. Hence

$$\frac{K_n(G)Z(G)}{Z(G)} \leq K_n\left(\frac{G}{Z(G)}\right) = 1Z(G)$$

. And so $K_n(G) \leq Z(G)$. Then $K_{n+1}(G) = [K_n(G), G] \leq [Z(G), G] = 1$. Therefore $K_{n+1}(G) = 1$ and so G is nilpotent. □

Theorem 4.3. *Let G be a nilpotent group and $H \leq G$. Then H is nilpotent.*

Proof

Since G is nilpotent there exists $n \in \mathbb{Z}^+ \cup \{0\}$ such that $K_n(G) = 1$. Claim: $K_i(H) \leq K_i(G)$ for all i. We use induction on i. If $i = 0$ then $K_0(H) = H \leq G = K_0(G)$. Suppose $K_i(H) \leq K_i(G)$. Then $K_{i+1}(H) = [K_i(H), H] \leq [K_i(G), G] = K_{i+1}(G)$, which implies $K_{i+1}(H) \leq K_{i+1}(G)$, and so $K_i(H) \leq K_i(G)$ for all i. Hence $K_n(H) \leq K_n(G) = 1$ and so H is nilpotent. □

Theorem 4.4. *Let G be a nilpotent group and $N \trianglelefteq G$. Then G/N is nilpotent.*

Proof

Since G is nilpotent there exists $n \in \mathbb{Z}^+ \cup \{0\}$ such that $K_n(G) = 1$. As before

$$K_i\left(\frac{G}{N}\right) \leq \frac{K_i(G)N}{N} \text{ for all } i.$$

Thus

$$K_n\left(\frac{G}{N}\right) \leq \frac{K_n(G)N}{N} = \frac{1N}{N} = 1N.$$

Hence G/N is nilpotent. □

Lemma 4.5. *Let G be a nilpotent group and $H < G$. Then $H < N_G(H)$*

Proof

Clearly $H \leq N_G(H)$. Since G is nilpotent there exists $n \in \mathbb{Z}^+$ such that $K_n(G) = 1$. Since $H \neq G$ there exists a maximal i such that $K_i(G)$ is not contained in H. Then

$$[K_i(G), H] \leq [K_i(G), G] = K_{i+1}(G) \leq H$$

by the maximality of i. Let $k \in K_i(G)$ and $h \in H$. Then $[k, h] \in [K_i(G), H] \leq H$ and so $[k, h] \in H$. But $h \in H$ and so $[k, h]h = khk^{-1} \in H$. Thus, $K_i(G) \leq N_G(H)$. Therefore, since $K_i(G)$ is not contained in H, $H < N_G(H)$. □

Definition Let G be a group and $M \leq G$. Then M is a **maximal subgroup** of G if $M \neq G$ and, whenever there exists a subgroup H of G such that $M \leq H \leq G$, then $H = M$ or $H = G$.

Example $\langle(12)\rangle, \langle(13)\rangle, \langle(23)\rangle$, and $\langle(123)\rangle$ are all maximal subgroups of S_3.

Lemma 4.6. *Let G be a nilpotent group and M be a maximal subgroup of G. Then $M \trianglelefteq G$.*

Proof

Now since M is maximal we know $M < G$. Hence, by Lemma 4.5 $M < N_G(M) \leq G$. Thus, $G = N_G(M)$ by the maximality of M. Hence $M \trianglelefteq G$. □

Theorem 4.7. Frattini's argument *Let G be a group, $H \trianglelefteq G$, and $P \in Syl_p(H)$, then $G = N_G(P)H$.*

Proof

Clearly, $N_G(P)H \subseteq G$. Let $g \in G$. Then since $P \leq H$ we get $gPg^{-1} \leq gHg^{-1}$. But since $H \trianglelefteq G$, we have $gHg^{-1} = H$. Thus, $gPg^{-1} \leq H$. Now since $P \in \mathrm{Syl}_p(H)$ and $|gPg^{-1}| = |P|$ we get $gPg^{-1} \in \mathrm{Syl}_p(H)$. Then by Sylow's theorem $gPg^{-1} = hPh^{-1}$ for some $h \in H$. So $h^{-1}gPg^{-1}h = P$, or $hgP(hg)^{-1} = P$. But then $hg \in N_G(P)$ and so $g \in N_G(P)H$. Therefore $G = N_G(P)H$. □

Lemma 4.8. *Let G be a nilpotent group and $P \in Syl_p(G)$. Then $P \trianglelefteq G$.*

Proof

If P is not normal in G then $N_G(P) < G$. Let M be a maximal subgroup of G such that $N_G(P) \leq M$. Since G is nilpotent, by maximality of M, we know $M \trianglelefteq G$. Now $P \leq N_G(P) \leq M$ and $P \in \mathrm{Syl}_p(G)$ implies $P \in \mathrm{Syl}_p(M)$. Therefore by the Frattini Argument $G = N_G(P)M = M$. This is a contradiction to the maximality of M.

Therefore $P \trianglelefteq G$. $\square$

Theorem 4.9. *Let G be a nilpotent group. Then G is solvable.*

Proof

Since G is a nilpotent group, there exists $n \in \mathbb{Z}^+ \cup \{0\}$ such that $K_n(G) = 1$. We know from Theorem 3.15 that $K_i(G) \trianglelefteq G$ for all i and $K_{i+1}(G) \leq K_i(G)$ for all i. Then we have a subnormal series

$$G = K_0(G) \trianglerighteq K_2(G) \trianglerighteq \cdots \trianglerighteq K_n(G) = 1.$$

We claim that $K_i(G)/K_{i+1}(G)$ is abelian for all $1 \leq i \leq n-1$. Let x^{-1}, $y^{-1} \in K_i(G)$. Now $K_i(G)/K_{i+1}(G)$ is abelian if and only if

$$\begin{aligned}
x^{-1}K_{i+1}(G)y^{-1}K_{i+1}(G) &= y^{-1}K_{i+1}(G)x^{-1}K_{i+1}(G) \\
x^{-1}y^{-1}K_{i+1}(G) &= y^{-1}x^{-1}K_{i+1}(G) \\
xyx^{-1}y^{-1} = [x,y] &\in K_{i+1}(G) \\
[K_i(G), K_i(G)] &\leq K_{i+1}(G) \\
K_i(G)' = [K_i(G), K_i(G)] &\leq K_{i+1}(G)
\end{aligned}$$

So by Theorem 3.6 $K_{i+1}(G) \trianglelefteq K_i(G)$ and $K_i(G)/K_{i+1}(G)$ is abelian for all $0 \leq i \leq n-1$. $\square$

Lemma 4.10. *Let G be a nilpotent group such that $G \neq 1$. Then $Z(G) \neq 1$.*

Proof

Since G is nilpotent, there exists a minimal $n \in \mathbb{Z}^+$ such that $K_n(G) = 1$. Then

$$1 = K_n(G) = [K_{n-1}(G), G],$$

and so $K_{n-1}(G) \leq Z(G)$. But $1 \neq K_{n-1}(G)$ by the minimality of n and so $Z(G) \neq 1$.□

Lemma 4.11. *Let G be a nilpotent group and $1 \neq N \trianglelefteq G$. Then $N \cap Z(G) \neq 1$.*

Proof

Since G is nilpotent, there exists $n \in \mathbb{Z}^+$ such that $K_n(G) = 1$. Define $N_0 = N, N_1 = [N_0, G] = [N, G]$, and inductively by $N_k = [N_{k-1}, G]$ for all $k \in \mathbb{Z}^+ \cup \{0\}$. Then we have a normal series

$$N = N_0 \trianglelefteq N_1 \trianglelefteq N_2 \trianglelefteq \cdots$$

Claim $N_i \leq K_i(G)$ for all $i \in \mathbb{Z}^+ \cup \{0\}$. We use induction on i. If $i = 0$, then $N_0 = N \leq G = K_0(G)$. Now suppose $N_i \leq K_i(G)$. Then $N_{i+1} = [N_i, G] \leq [K_i(G), G] = K_{i+1}(G)$. Hence the claim holds by induction. Thus,

$$N_n \leq K_n(G) = 1 \text{ and so } N_n = 1.$$

Let $m \in \mathbb{Z}^+$ be minimal such that $N_m = 1$. Then $1 = N_m = [N_{m-1}, G]$ and so $N_{m-1} \leq Z(G)$. But $N_{m-1} \leq N$ and $N_{m-1} \neq 1$ by the minimality of m. Thus, $1 \neq N_{m-1} \leq N \cap Z(G)$. □

Lemma 4.12. *Let $G = HK$ be a group such that $H \trianglelefteq G, K \trianglelefteq G$ and H and K are nilpotent. Then G is nilpotent.*

Proof

Use induction on $|G|$. If $|G| = 1$ then $K_0(G) = G = 1$ and so G is nilpotent. Assume $|G| > 1$ and that the theorem holds for all groups of order less than $|G|$. We want to show the theorem holds for G. Since H is nilpotent, by Lemma 4.9 $Z(H) \neq 1$. Let $N = [Z(H), K]$. If $N = 1$ then $[Z(H), K] = 1$. Thus

$$1 \neq Z(H) \leq C_G(H) \cap C_G(K) = Z(G).$$

Now $Z(G) \trianglelefteq G$ and so

$$\frac{G}{Z(G)} = \frac{HZ(G)}{Z(G)}\frac{KZ(G)}{Z(G)}$$

is a group. Since $H \trianglelefteq G$ and $K \trianglelefteq G$ we know

$$\frac{HZ(G)}{Z(G)} \trianglelefteq \frac{G}{Z(G)} \text{ and } \frac{KZ(G)}{Z(G)} \trianglelefteq \frac{G}{Z(G)}.$$

Also since H is nilpotent, $\dfrac{HZ(G)}{Z(G)} \cong \dfrac{H}{H \cap Z(G)}$ is nilpotent and similarly $\dfrac{KZ(G)}{Z(G)}$ is nilpotent.

Finally,

$$\left|\frac{G}{Z(G)}\right| = \frac{|G|}{|Z(G)|} < |G|$$

and so $G/Z(G)$ is nilpotent by induction. Therefore there exists $n \in \mathbb{Z}^+$ such that $K_n(G/Z(G)) = 1$. But then

$$\frac{K_n(G)Z(G)}{Z(G)} = K_n\left(\frac{G}{Z(G)}\right) = 1 \text{ and so } K_n(G) \leq Z(G).$$

Hence

$$K_{n+1}(G) = [K_n(G), G] \leq [Z(G), G] = 1 \text{ and so } G \text{ is nilpotent.}$$

If $N \neq 1$, as $K \trianglelefteq G$, we know $N \leq K$. Also since $H \trianglelefteq G, Z(H) \trianglelefteq G$. Thus, $N = [Z(H), K] \trianglelefteq K$. Now since K is nilpotent $N \cap Z(K) \neq 1$ by Lemma 4.10. Hence since $Z(H) \trianglelefteq G$ we get $1 \neq N \cap Z(K) \leq Z(H) \cap Z(K) \leq Z(G)$. Therefore $Z(G) \neq 1$ again and so G is nilpotent using the above argument. $\square$

Definition A group G is called an **elementary abelian p-group** if $G \cong \mathbb{Z}_p \times \mathbb{Z}_p \times \cdots \times \mathbb{Z}_p$ for some prime p.

Theorem 4.13. *Let G be a solvable group and N be a minimal normal subgroup of G. Then N is an elementary abelian p-group for some prime p.*

Proof

By Theorem **3.9**, $N \cong N_1 \times N_2 \times \cdots \times N_n$ where the N_is are non-abelian simple isomorphic groups or $N_i \cong \mathbb{Z}_p$ for all $1 \leq i \leq n$. If N_i is nonabelian for some $1 \leq i \leq n$ then $1 \neq N_i' \trianglelefteq N_i$ and so $N_i' = N_i^{(1)} = N_i$. Suppose $N_i^{(k)} = N_i$. Then $N_i^{(k+1)} = (N_i^{(k)})' = N_i' = N_i$. Thus, $N_i^{(k)} = N_i$ for all k by induction. But then N_i is not solvable. Now G is solvable and $N_i \leq G$ which implies that N_i is solvable, a contradiction. Hence there exists a prime p such that $N_i \cong \mathbb{Z}_p$ for all i and so

$$N \cong \mathbb{Z}_p \times \mathbb{Z}_p \times \cdots \times \mathbb{Z}_p$$

is a elementary abelian p-group. $\square$

5 The Hall and Schur-Zassenhaus Theorems

Definition Let G be a group and π be a set of primes. Then

1. $\pi' = \{p | p \text{ is prime and } p \notin \pi\}$.
2. $\pi(G) = \{p | p \text{ is prime and } p || G|\}$.
3. G is called a **π-group** if $\pi(G) \subseteq \pi$.
4. A subgroup $H \leq G$ is called a **Hall π-subgroup** if H is a π-group and $\pi(S) \subseteq \pi'$ where $S = \{gH | g \in G\}$.
5. $\text{Hall}_\pi(G) = \{H \leq G | H \text{ is a Hall } \pi - \text{subgroup of } G\}$.

Example 1 $|S_3| = 3 {=}\, 3 \cdot 2$ and $\pi(S_3) = \{2, 3\}$. Now $|A_3| = 3$; so A_3 is a 3-group and $\pi(S_3/A_3) \subseteq \{3\}'$. Hence $A_3 \in \text{ Hall}_{\{3\}}(S_3)$.

Example 2 $|A_5| = 5!/2 = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1/2 = 2^2 \cdot 3 \cdot 5$. Let $H = (A_5)_1$. Then $H \cong A_4$ and $|H| = 4!/2 = 2^2 \cdot 3$. Therefore H is a $\{2, 3\}$-group. Also $\pi(A_5/H) = 5 \in \{2, 3\}'$. Hence $H \in \text{ Hall}_{\{2,3\}}(A_5)$.

Example 3 If G is a group, p is a prime, and $\pi = \{p\}$, then $Syl_p(G) = \text{ Hall}_\pi(G)$. For some groups G and certain sets of primes π, $\text{Hall}_\pi(G) = \emptyset$.

Example $\text{Hall}_{\{2,5\}}(A_5) = \emptyset$.

Proof

Suppose $H \in \text{ Hall}_{\{2,5\}}(A_5)$. Then H is a $\{2, 5\}$-group and $\pi(A_5/H) \subseteq \{2, 5\}'$. Since

$|A_5| = 2^2 \cdot 3 \cdot 5$ we get $|H| = 2^2 \cdot 5$. Let A_5 act on $S = \{gH | g \in A_5\}$ by left multiplication via $\phi : A_5 \longrightarrow \text{Sym}(S)$, where ϕ is defined by $\phi(g)(xH) = gxH$ for all $g \in A_5$ and for all $xH \in S$. Now by Lagrange's Theorem $|S| = |A_5|/|H| = 3$ and so $\text{Sym}(S) \cong S_3$. Now $K = \text{Kern}\ \phi \trianglelefteq A_5$. Since A_5 is simple either $K = 1$ or $K = A_5$. If $K = A_5$ then

$$A_5 = K = \bigcap_{x \in A_5} xHx^{-1} \leq H$$

and we get $A_5 = H$, a contradiction. If $K = 1$ then, by the First Isomorphism Theorem,

$$A_5 \cong \frac{A_5}{1} = \frac{A_5}{K} \cong \phi(A_5) \leq \text{Sym}(S).$$

But then we get $60 = |A_5| = |\phi(A_5)|$ divides $|\text{Sym}(S)| = 6$, a contradiction. Thus $\text{Hall}_{\{2,5\}}(A_5) = \emptyset$. □

Theorem 5.1. *(Hall's): Let G be a solvable group and π be a set of primes. Then*

1. *$Hall_\pi(G) \neq \emptyset$*
2. *G acts transitively on $Hall_\pi(G)$ by conjugation.*

Definition Let G be a group and $H \leq G$. Then G splits over H if there exists $K < G$ such that $G = HK$ and $H \cap K = 1$. The subgroup K is called the complement of H in G.

Example: S_3 splits over A_3 since $S_3 = A_3\langle(12)\rangle$ and $A_3 \cap \langle(12)\rangle = 1$.

Theorem 5.2. *Let G be a solvable group, $H \in Hall_\pi(G)$, and suppose $N_G(H) \leq K \leq G$. Then $K = N_G(K)$.*

Proof

Clearly $K \leq N_G(K)$. Let $g \in N_G(K)$. Then $H \leq N_G(H) \leq K$; so $H \in$ Hall$_\pi(G)$, so $H \in$ Hall$_\pi(K)$. Now $H \leq K$ implies $gHg^{-1} \leq gKg^{-1} = K$. But $|gHg^{-1}| = |H|$ and so $gHg^{-1} \in$ Hall$_\pi(K)$. Now since G is solvable, K is also solvable. Thus by Hall's theorem there exists $k \in K$ such that $kgHg^{-1}k^{-1} = H$ or $kgH(kg)^{-1} = H$. But then $kg \in N_G(H)$ and so $g \in K$. Therefore $K = N_G(K)$. In this case we say K is self-normalizing. □

Theorem 5.3. *(Schur-Zassenhaus) Let G be a group and $H \in$ Hall$_\pi(G)$ such that $H \trianglelefteq G$. Then G splits over H. In addition if either H or G/H is solvable, then G acts transitively on the complements of H in G by conjugation.*

6 Carter's Theorem

Definition Let G be a group and $H \leq G$. Then H is a **Carter subgroup** of G if

1. H is nilpotent;

2. $N_G(H) = H$.

In this case we write H cart G. When condition (2) holds, we say H is self-normalizing.

Example Any nilpotent group G has a Carter subgroup, namely, G itself is a Carter subgroup since $N_G(G) = G$, and G is nilpotent.

Example $\langle(12)\rangle$ cart S_3 since $\langle(12)\rangle$ is abelian implies $\langle(12)\rangle$ is nilpotent. Also $\langle(12)\rangle \leq N_{S_3}(\langle(12)\rangle) \leq S_3$ and so $2 = |\langle(12)\rangle|$ which divide $|N_{S_3}(\langle(12)\rangle)|$ divides $|S_3| = 6$. Hence $|N_{S_3}(\langle(12)\rangle)| = 2$. But $N_{S_3}(\langle(12)\rangle) \neq S_3$ since $\langle(12)\rangle$ is not a normal subgroup of S_3. And so $\langle(12)\rangle = |N_{S_3}(\langle(12)\rangle)|$.

But not all groups have Carter subgroups.

Example A_5 has no Carter subgroups. $|A_5| = \frac{5!}{2} = 60 = 2^2 \cdot 3 \cdot 5$. A table showing 57 subgroups of A_5 is below.

Structure	*Subgroup*, H	*Number*	*Reason*
$\mathbb{Z}_2$	$\{1, (12)(34)\}$	15	$(13)(24) \in N_{A_5}(H) \setminus H$
$\mathbb{Z}_3$	$\{1, (123), (132)\}$	10	$(23)(45) \in N_{A_5}(H) \setminus H$
$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\{1, (12)(34), (14)(23), (13)(24)\}$	5	$(123) \in N_{A_5}(H) \setminus H$
$\mathbb{Z}_5$	$\{1, (12345), (13524), (14253), (15432)\}$	6	$(15)(24) \in N_{A_5}(H) \setminus H$
S_3	$\{1, (123), (132), (12)(45), (13)(45), (23)(45)\}$	10	Not nilpotent $n_2 = 3$
D_5	$\langle(12345), (15)(24)\rangle$	6	Not nilpotent $n_2 = 5$
A_4	$(A_5)_1$	5	Not nilpotent $n_3 = 4$

Theorem 6.1. *(Carter): Let G be a solvable group. Then*

1. *G has a Carter subgroup;*
2. *If $N \trianglelefteq G$ and H cart G then HN/N cart G/N;*
3. *If H_1 cart G and H_2 cart G then there exists $g \in G$ such that $H_2 = gH_1g^{-1}$.*

Proof

We will use induction on $|G|$. If $|G| = 1$ then $\{1\}$ cart G and (1), (2) and (3) hold. Also if G is nilpotent, then G cart G and (1), (2) and (3) hold. Without loss of generality, assume that $|G| > 1, G$ is not nilpotent, and the result holds for all groups of order less than $|G|$. For (1): Let N be a minimal normal subgroup of G. Since G is solvable, N is an elementary p-group for some prime p. Since G is solvable, by Theorem 3.4 we know G/N is solvable. Also

$$|G/N| = \frac{|G|}{|N|} < |G|$$

and so by induction there exists K/N cart G/N. Now let $S/N \in \mathrm{Syl}_p(K/N)$. Since K/N cart G/N, we know K/N is nilpotent. Thus by Lemma 4.7, $S/N \trianglelefteq K/N$. But then $S \trianglelefteq K$. Now

$$\frac{|K|}{|S|} = \frac{|K|/|N|}{|S|/|N|} = \frac{|K/N|}{|S/N|}$$

and so p does not divide $|K|/|S|$ since $S/N \in \mathrm{Syl}_p(K/N)$. Also,

$$|S| = \frac{|S|}{|N|}|N| = |S/N||N|$$

is a power of p since $S/N \in \mathrm{Syl}_p(K/N)$ and N is an elementary p-group. Hence $S \in \mathrm{Syl}_p(K)$ and so K splits over S by the Schur-Zassenhaus Theorem. But then there exists $R \leq K$ such that $K = RS$ and $R \cap S = 1$. Now by the Second Isomorphism Theorem

$$R \cong \frac{R}{1} = \frac{R}{R \cap S} \cong \frac{RS}{S} = \frac{K}{S}.$$

From the above, p does not divide $|K/S|$ and so p does not divide $|R|$. Also

$$\frac{|K|}{|R|} = \frac{|RS|}{|R|} = \frac{|S|}{|R \cap S|} = |S|$$

is a power of p. Thus $R \in \mathrm{Hall}_{p'}(K)$. Let $H = N_K(R)$ and $g \in N_G(H)$. Now $N_K(R) \leq HN \leq K$, $R \in \mathrm{Hall}_{p'}(K)$, and K is solvable. Thus by Theorem 5.2 $HN = N_K(HN)$. But then

$$\frac{HN}{N} = \frac{N_K(HN)N}{N} = N_{K/N}\left(\frac{HN}{N}\right).$$

Now $HN/N \leq K/N$ and K/N is nilpotent. Hence we get $K/N = HN/N$ and so $K = HN$. Since $N \trianglelefteq G$ and $g \in N_G(H)$ we have $g \in N_G(HN) = N_G(K)$. Hence $gN \in N_{G/N}(K/N)$. But $K/N = N_{G/N}(K/N)$ since K/N cart G/N. Therefore $gN \in K/N$ and so $g \in K$. But then $g \in N_K(H)$. Also $N_K(R) \leq H \leq K$, $R \in \mathrm{Hall}_{p'}(K)$, and K is solvable. Thus by Theorem 5.2, $H = N_K(H)$. Therefore $g \in H$ and so

$H = N_G(H)$. Now

$$H = H \cap K = H \cap RS = R(H \cap S).$$

Since $S \trianglelefteq K$ and $H \leq K$ we know $S \cap H \trianglelefteq H$. Also since $R \leq H \leq N_G(R)$ we know $R \trianglelefteq H$. Since S is a p-group we know $S \cap H$ is a p-group. Therefore $S \cap H$ is nilpotent. Also by the Second and Third Isomorphism Theorems,

$$R \cong \frac{R}{1} = \frac{R}{R \cap S} \cong \frac{RS}{S} = \frac{K}{S} \cong \frac{K/N}{S/N}.$$

But since

$$\frac{K}{N} \text{ cart } \frac{G}{N}$$

K/N is nilpotent. Thus R is nilpotent by Theorem 4.4. Therefore $H = R(H \cap S)$ is nilpotent by Lemma 4.11 and so H cart G.

For (2) : Let H cart G and $N \trianglelefteq G$. Then

$$\frac{HN}{N} \leq \frac{G}{N}.$$

Also since H is nilpotent,

$$\frac{HN}{N} \cong \frac{H}{H \cap N}$$

is nilpotent. Clearly

$$\frac{HN}{N} \leq N_{G/N}\left(\frac{HN}{N}\right)$$

Let $gN \in N_{G/N}(HN/N)$. Then $g^{-1}N \in N_{G/N}(HN/N)$ and also

$$\frac{HN}{N} = g^{-1}N(\frac{HN}{N})gN = \frac{g^{-1}(HN)g}{N} = \frac{g^{-1}HgN}{N}.$$

By taking preimages we get $g^{-1}HgN = HN$. If $G = HN$ then $G/N = HN/N$. Hence

$$\frac{HN}{N} = \frac{G}{N} = N_{G/N}\left(\frac{G}{N}\right) = N_{G/N}\left(\frac{HN}{N}\right).$$

Therefore we may assume $HN < G$. Now $g^{-1}Hg \cong H$ and so $g^{-1}Hg$ is nilpotent. Also,

$$N_G(g^{-1}Hg) = g^{-1}N_G(H)g = g^{-1}Hg \text{ since } H \text{ cart } G.$$

Thus, $g^{-1}Hg$ cart HN and H cart HN. Therefore by induction there exists $n \in N$ such that $ng^{-1}Hgn^{-1} = H$. But then $ng^{-1} \in N_G(H) = H$ since H cart G. So $gn^{-1} \in H$ since $H \leq G$. Then $gN = gn^{-1}N \in HN/N$ and so

$$\frac{HN}{N} = N_{G/N}\left(\frac{HN}{N}\right) \text{ and so } \frac{HN}{N} \text{ cart } \frac{G}{N}.$$

For (3): Let H_1 cart G and H_2 cart G. Let N be a minimal normal subgroup of G. Since G is solvable, by Theorem 3.6, N is an elementary p-group. By (2),

$$\frac{H_1N}{N} \text{ cart } \frac{G}{N} \text{ and } \frac{H_2N}{N} \text{ cart } \frac{G}{N}.$$

Since $|G/N| < |G|$, by induction there exists $gN \in G/N$ such that

$$\frac{H_2N}{N} = gN\left(\frac{H_1N}{N}\right)g^{-1}N \quad = \quad \frac{gH_1g^{-1}N}{N}.$$

Therefore $gH_1g^{-1}N = H_2N$. If $H_2N < G$ then gH_1g^{-1} cart H_2N and H_2 cart H_2N. Hence by induction there exists $g_1 \in H_2N$ such that $g_1gH_1g^{-1}g_1^{-1} = H_2$. We may assume $G = gH_1g^{-1}N = H_2N$. Since gH_1g^{-1} and H_2 are nilpotent, there exist $gR_1g^{-1} \in \mathrm{Hall}_{p'}(gH_1g^{-1})$ and $R_2 \in \mathrm{Hall}_{p'}(H_2)$. Now

$$\frac{|G|}{|R_2|} = \frac{|G|}{|H_2|} \cdot \frac{|H_2|}{|R_2|} = \frac{|H_2N|}{|H_2|} \cdot \frac{|H_2|}{|R_2|} = \frac{|N|}{|N \cap H_2|} \cdot \frac{|H_2|}{|R_2|}$$

is a power of p. Thus $R_2 \in \mathrm{Hall}_{p'}(G)$ and similarly $gR_1g^{-1} \in \mathrm{Hall}_{p'}(G)$. Since G is solvable, by Hall's Theorem, there exists $g_2 \in G$ such that $g_2gR_1g^{-1}g_2^{-1} = R_2$. Now gR_1g^{-1} and H_2 are nilpotent implies $gR_1g^{-1} \trianglelefteq gH_1g^{-1}$ and $R_2 \trianglelefteq H_2$. Thus $g_2gR_1g^{-1}g_2^{-1} \trianglelefteq g_2gH_1g^{-1}g_2^{-1}$ and so

$$g_2gH_1g^{-1}g_2^{-1} \leq N_G(g_2gR_1g^{-1}g_2^{-1}) = N_G(R_2) \geq H_2.$$

Let $K = N_G(R_2)$. Now $R_2 \trianglelefteq K$ and so K/R_2 is a group. Since $g_2gH_1g^{-1}g_2^{-1}$ cart K, by part (2)

$$\frac{g_2gH_1g^{-1}g_2^{-1}R_2}{R_2} \text{ cart } \frac{K}{R_2} \text{ and } \frac{H_2}{R_2} \text{ cart } \frac{K}{R_2}.$$

If $R_2 = 1$ then H_2 is a p-group. Since N is a p-group, we get $G = H_2N$ is a p-group. Thus, G is nilpotent and so $G = H_1 = H_2$. We may assume $R_2 \neq 1$ and $|K/R_2| < |G|$.

So by induction there exists $kR_2 \in K/R_2$ such that

$$\frac{H_2}{R_2} = kR_2 \left(\frac{g_2 g H_1 g^{-1} g_2^{-1} R_2}{R_2} \right) k^{-1} R_2 = \frac{k g_2 g H_1 g^{-1} g_2^{-1} k^{-1} R_2}{R_2}.$$

Thus

$$k g_2 g H_1 g^{-1} g_2^{-1} k^{-1} R_2 = H_2.$$

Now $kR_2 \in K/R_2$ implies $k \in K = N_G(R_2)$. But

$$R_2 = g_2 g R_1 g^{-1} g_2^{-1} \leq g_2 g H_1 g^{-1} g_2^{-1}$$

and so

$$R_2 = k R_1 k^{-1} \leq k g_2 g H_1 g^{-1} g_2^{-1} k^{-1}.$$

Therefore

$$k g_2 g H_1 g^{-1} g_2^{-1} k^{-1} R_2 = H_2 = k g_2 g H_1 g^{-1} g_2^{-1} k^{-1} = H_2$$

and so we have (3). □

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