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I, Benjamin Barr, hereby submit this original work as part of the requirements for the degree of Master of Science in Mathematical Sciences.

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On three theorems for extensions of functions

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Abstract

Much of the current literature about the Heisenberg group $\mathbb{H}$ is difficult for those who are in the beginning of their mathematical careers, yet $\mathbb{H}$ is endowed with an interesting structure which allows for the generalization of many aspects of analysis in Euclidean space. Such topics include continuity and stronger forms of the same, integral calculus, restrictions and extensions of functions, and Taylor's theorem. The goal of this thesis is to make more accessible a combination of these tenets and others, through examining a Whitney extension theorem in $\mathbb{H}$.

We start by building the fundamentals in a more familiar setting, namely in Euclidean 3-space. We then discuss $\mathbb{H}$ and its properties, including the notion of horizontality of curves in $\mathbb{H}$. The concept of horizontality provides a natural segue to a version of Whitney's extension theorem for horizontal curves in $\mathbb{H}$; we discuss the necessity and sufficiency of three criteria a curve in $\mathbb{H}$ must satisfy in order to have a smooth horizontal extension. We conclude by examining two other types of extension theorems, namely Lipschitz maps on metric spaces and continuous maps on normal topological spaces.

Ackowledgments

To enumerate the people I would like to thank for their guidance and patience with me during my M.S. program and the writing of this thesis, would be nearly impossible. Nonetheless, I wish to thank in particular my advisor, Professor Gareth Speight, who has gracefully guided me through my first "real" mathematical endeavor, and my committee, Professor Nageswari Shanmugalingam, who has taught me more about mathematics than any other mathematician with whom I have interacted. (Prof. Nages will be pleased to know that there are no pictures in this thesis.)

Dedication

It is my pleasure and honor to dedicate this thesis to Mrs. Donna Hoeing, who is not only a wonderful person but also is the reason I am where I am today.

Contents

Introduction

A useful tool in analysis is the notion of restriction of a function to a subset of its domain. One sees such utility, for example, in finding the inverse of a function which is bijective only on a proper subset of said function's domain. Moreover, some important mathematical results, such as the pasting lemma in topology, require the ability to restrict a function. Perhaps more interesting is a "converse" to restricting a function: *extending* a function from its source to a superset of its source.

In attempting to extend a function, immediately one encounters important questions. When does a function f defined on a subset S of a set A have an extension to the entire space A? What properties can such an extension have? Is there a "natural" way to define such an extension? Of course, the answers to these questions depend on the type of function in question and its domain and codomain. In this thesis, we seek to answer those questions for three different types of functions: continuous, Lipschitz continuous, and continuously m-times differentiable.

An interesting example of such a question is: *When does a mapping from a subset K of the real numbers into the Heisenberg group $\mathbb{H}$ have a continuously m-times differentiable, horizontal extension?*

An important area of research seeks to extend classical analysis to more general spaces. The Heisenberg group is an example of such a space, with a structure similar to $\mathbb{R}^3$ in the first two components of its elements but very different in the third. Given the structure of the Heisenberg group, it is no surprise that an answer to the above question is not so straightforward. There exists a considerable amount of literature on this specific topic, but much of it requires prerequisite knowledge in Lebesgue integration and differential geometry, for example, which beginning graduate students may not be properly equipped to handle. One goal of this thesis is to make more accessible the building blocks of this topic.

We start in Section 1 by easing into the topic in a Euclidean setting. We exploit the familiarity of Euclidean space to define important preliminaries which will be the objects of study in later sections. We define jets and some related ideas, and we state the classical Whitney extension theorem. We discuss the notion of absolute continuity which is stronger than uniform continuity yet weaker than continuous differentiability, and we conclude the section by discussing some ideas that will help our intuition when we arrive at the main theorem in Section 3.

In Section 2, we define and discuss some properties of the Heisenberg group. Perhaps the most significant topic in this section is the idea of horizontality of curves. We define a horizontal curve in the Heisenberg group and discuss some properties of horizontal curves before moving into the main theorem in Section 3.

In Section 3, we prove the necessity of three important conditions for a horizontal curve in the Heisenberg group to have a continuously m-times differentiable extension. We outline the proof of the sufficiency of the criteria, but we note that this is much more difficult than that of the necessity.

In Section 4, we compare with the Whitney extension theorem two extension theorems for Lipschitz continuous functions in metric spaces and for continuous functions in normal topological spaces.

0. Notation and basic terminology

(1) If $m \in \mathbb{N}$, we denote by $\mathbb{N}_m$ the set
$$\{k \in \mathbb{N} \cup \{0\} : k \leq m\}.$$
We shall denote by $\mathbb{N}_m^*$ the set $\mathbb{N}_m \setminus \{0\}$.

(2) Let $k \in \mathbb{N}$, let $U \subset \mathbb{R}$ be a nonempty open set, and let $f : U \to \mathbb{R}$ be a function which is continuous on U. If the kth derivative of f exists on the interior of U, it shall be denoted $D^k f$. Moreover, for each $x \in U$ which belongs to the boundary of U, we define $D^k f(x)$ by the appropriate left- or right-handed limit.

(3) For each $m \in \mathbb{N}$ and any nonempty open subset $U \subset \mathbb{R}$, denote by $C^m(U)$ the class of continuous functions $f : U \to \mathbb{R}$ such that for any $k \in \mathbb{N}_m^*$, $D^k f$ exists and is continuous on U. We shall call an element of $C^m(U)$ a *function of class* $C^m(U)$, or, if U is implicitly understood, a C^m *function.* In the case that $m = 1$, we shall call said element a *continuously differentiable function.*

(4) For any nonempty open subset $U \subset \mathbb{R}$, denote by $C^\infty(U)$ the class of infinitely differentiable, continuous functions on U, i.e. the set of functions which has derivatives of order m for any $m \in \mathbb{N}$. An element of $C^\infty(U)$ shall be known as a *function of class* $C^\infty(U)$, or more simply, a C^∞ function.

(5) Let $E \subset \mathbb{R}$, let $K \subset E$, and let $f : E \to \mathbb{R}$. The *restriction of* f *to* K is a function, which shall be denoted $f|_K$ and shall be defined, for any $x \in K$, by $f|_K(x) := f(x)$. Alternatively, we say $f|_K$ *extends to* f.

1. Classical Whitney extension theorem and other preliminaries

In this section, we seek to answer the following question. Given a function (or a collection of functions) on a subset K of a nonempty open set $U \subset \mathbb{R}$, when may we extend the function to a C^m function, and how must the functions in such a collection be related to each other?

1.1. Jets.
Throughout this section let $m \in \mathbb{N}$, let $U \subset \mathbb{R}$ be a nonempty open set, and let $K \subset U$.

Definition 1.1. A *jet of order* m *on* K is a collection $(F^k)_{k=0}^m$ of functions such that for any $i \in \mathbb{N}_m$, we have that F^i is a continuous function from K into $\mathbb{R}$. Occasionally, we shall denote a jet $(F^k)_{k=0}^m$ simply by a boldface $\boldsymbol{F}$. In addition we shall denote the space of all jets of order m on K by $\mathcal{J}^m(K)$.

Example 1.2. Define respectively four functions $F^0, F^1, F^2, F^3 : [0,1] \to \mathbb{R}$ by
$$F^0(x) = x^3, \quad F^1(x) = 3x^2, \quad F^2(x) = 6x, \quad F^3(x) = 6.$$
Then for any $i \in \mathbb{N}_3$ we have that F^i is a continuous function on $[0,1]$, so that $(F^k)_{k=0}^3$ is a jet of order 3 on $[0,1]$. △

Example 1.3. Define respectively two functions $G^0, G^1 : [0,1] \to \mathbb{R}$ by
$$G^0(x) = x^2 \quad \text{and} \quad G^1(x) = 0.$$
Then, by reasoning similar to that in the previous example, $(G^k)_{k=0}^1$ is a jet of order 1 on $[0,1]$. △

Now, define a mapping

$$J^m : C^m(U) \to \mathcal{J}^m(K) \quad \text{by} \quad F \mapsto \left(D^k F|_K\right)_{k=0}^m .$$

In other words, the mapping J^m takes as argument a C^m function on U and produces the jet on K which constitutes the derivatives of F, restricted to K, up to order m. By definition of J^m, for every function $F : U \to \mathbb{R}$ which is of Class $C^m(U)$ there exists a jet $(F^k)_{k=0}^m$ of order m on K. This jet is defined exactly by $J^m(F)$.

Now, a key question is: *which jets arise in this way?* To answer this question, we need some sort of "compatibility" between the different maps in the jet. We will see this can be understood using the Taylor polynomial and remainder, which we define shortly.

First, we note that the classical Whitney Extension Theorem (Theorem 1.10) acts as a sort of converse to Taylor's Theorem. We recall as a preliminary this theorem, which is covered in most advanced calculus textbooks.

Theorem 1.4 (Taylor's Theorem)**.** *Let $m \in \mathbb{N}$ and let $U \subset \mathbb{R}$ be open. Let also $f : U \to \mathbb{R}$ be m-times differentiable on U. Then there exists a function $h_m : U \to \mathbb{R}$ such that, for any $x \in U$, we have*

$$f(x) = \sum_{k=0}^{m} \frac{D^k f(a)}{k!}(x-a)^k + h_m(x)(x-a)^m \quad \text{and} \quad \lim_{x\to a} h_m(x) = 0.$$

Definition 1.5. Let $a \in K$. The *Taylor polynomial of degree m of $\boldsymbol{F} \in \mathcal{J}^m(K)$ at a* is a function $T_a^m \boldsymbol{F} : \mathbb{R} \to \mathbb{R}$ defined by

$$(T_a^m \boldsymbol{F})(x) := \sum_{k=0}^{m} \frac{F^k(a)}{k!}(x-a)^k.$$

We note that if $\boldsymbol{F}, \boldsymbol{G} \in J^m(K)$ are two jets of the same order on K then we can define addition of such jets by

$$\forall x \in K \quad (\boldsymbol{F} + \boldsymbol{G})(x) = \left(F^0(x) + G^0(x), F^1(x) + G^1(x), \dots, F^m(x) + G^m(x)\right).$$

Notice that the addition operation admits a group structure on $\mathcal{J}^m(K)$ so that for any $\boldsymbol{F}, \boldsymbol{G} \in J^m(K)$ the jet $\boldsymbol{F} - \boldsymbol{G}$ is well defined.

Definition 1.6. Let $a \in K$. The *remainder polynomial of degree m of $\boldsymbol{F} \in \mathcal{J}^m(K)$ at a* is a jet $\boldsymbol{R_a^m F}$ defined by

$$\boldsymbol{R_a^m F} := \boldsymbol{F} - \boldsymbol{J^m(T_a^m F)} \tag{1.1}$$

In particular, for each $k \in \mathbb{N}_m$ and each $x \in \mathbb{R}$, we have

$$(\boldsymbol{R_a^m F})^k(x) = F^k(x) - \sum_{\ell=0}^{m-k} \frac{F^{k+\ell}(a)}{\ell!}(x-a)^\ell. \tag{1.2}$$

Said another way, for each $k \in \mathbb{N}_m$, the function $(\boldsymbol{R_a^m F})^k$ is defined by the difference of the kth term of the jet $\boldsymbol{F}$ and the Taylor polynomial of the "subjet" $(F^j)_{j=k}^m$. Intuitively, we want this remainder to be small; how the remainder is controlled leads us to the discussion of Whitney fields.

1.2. **Whitney fields and the classical Whitney extension theorem.** Throughout this section let $m \in \mathbb{N}$, let $U \subset \mathbb{R}$ be nonempty and open, and let $K \subset U$ be compact.

Definition 1.7. A jet $(F^k)_{k=0}^m$ on K is a *Whitney field of class* C^m *on* K if for every $k \in \mathbb{N}_m$,

$$\lim_{\substack{|a-b|\to 0,\\ a,b\in K}} \frac{(\boldsymbol{R_a^m F})^k(b)}{|a-b|^{m-k}} = 0.$$

We shall denote by $\mathcal{W}^m(K)$ the set of Whitney fields of class C^m on K.

Example 1.8. Consider the jet $(F^k)_{k=0}^3$ as in Example 1.2, and let $a, b \in [0,1]$. Assume without loss of generality that $a < b$. Then for any $x \in [0,1]$, we have by (1.2) that

$$\begin{aligned}(\boldsymbol{R_a^3 F})^0(x) &= F^0(x) - \sum_{\ell=0}^{3} \frac{F^\ell(a)}{\ell!}(x-a)^\ell \\ &= x^3 - a^3 - 3a^2(x-a) - \frac{1}{2}\big(6a(x-a)^2\big) - \frac{1}{6}\big(6(x-a)^3\big) \\ &= x^3 - a^3 - 3a^2(x-a) - 3a(x-a)^2 - (x-a)^3,\end{aligned}$$

so that

$$(\boldsymbol{R_a^3 F})^0(b) = b^3 - a^3 - 3a^2(b-a) - 3a(b-a)^2 - (b-a)^3 \tag{1.3}$$

Expanding the right hand side of (1.3), we see that $(\boldsymbol{R_a^3 F})^0(b)$ is identically 0 for all $a, b \in [0,1]$. Hence, for any $\varepsilon > 0$ there is $\delta_0 > 0$ (namely $\delta_0 = \varepsilon$) such that for each $a, b \in [0,1]$,

$$|a-b| < \delta_0 \implies \left| \frac{(\boldsymbol{R_a^3 F})^0(b)}{|a-b|^3} \right| = 0 < \varepsilon.$$

It follows that

$$\lim_{\substack{|a-b|\to 0,\\ a,b\in[0,1]}} \frac{(\boldsymbol{R_a^3 F})^0(b)}{|a-b|^3} = 0.$$

Again by (1.2), we see for each $a, b \in [0,1]$ that

$$\begin{aligned}(\boldsymbol{R_a^3 F})^1(b) &= F^1(b) - \sum_{\ell=0}^{2} \frac{F^{1+\ell}(a)}{\ell!}(b-a)^\ell \\ &= 3b^2 - 3a^2 - 6a(b-a) - \frac{6}{2}(b-a)^2, \qquad (1.4)\end{aligned}$$

that

$$\begin{aligned}(\boldsymbol{R_a^3 F})^2(b) &= F^2(b) - \sum_{\ell=0}^{1} \frac{F^{2+\ell}(a)}{\ell!}(b-a)^\ell \\ &= 6b - 6a - 6(b-a), \qquad (1.5)\end{aligned}$$

and that

$$(\boldsymbol{R_a^3 F})^3(b) = F^3(b) - F^3(a), \tag{1.6}$$

and the quantities (1.4), (1.5), (1.6) are identically 0 as $|a-b| \to 0$ with $a, b \in K$.

Therefore, for each $i \in \mathbb{N}_3$ and each $\varepsilon > 0$ there is $\delta_i > 0$ (namely, for each $i \in \mathbb{N}_3$ take $\delta_i = \varepsilon$) such that for any $a, b \in [0,1]$

$$|a-b| < \delta \implies \left| \frac{(\boldsymbol{R_a^3 F})^i(b)}{|a-b|^{3-i}} \right| = 0 < \varepsilon.$$

Hence the jet $(F^k)_{k=0}^3$ satisfies Definition 1.7 of Whitney field and so is a Whitney field of Class C^3 on $[0,1]$.

△

More generally, if $F : U \to \mathbb{R}$ is of class $C^m(U)$, and if $K \subset U$ is compact, then $J^m(F)$ is a Whitney field of class C^m on K; this is due to the fact that for each $k \in \mathbb{N}_m$, the function $F^k|_K$ must satisfy Taylor's theorem. In fact, in order to obtain the uniform limit in Definition 1.7, rather than the pointwise limit in Theorem 1.4, one must use a version of Taylor's Theorem with a more precise form of the remainder [see Fol90, p. 234].

Example 1.9. Consider the jet $(G^k)_{k=0}^1$ defined for any $x \in [0,1]$ by

$$G^0(x) = x \quad \text{and} \quad G^1(x) = 0.$$

Then for any $a, b \in [0,1]$, we have that

$$(\boldsymbol{R_a^1 G})^0(b) = G^0(b) - \sum_{\ell=0}^{1} \frac{G^\ell(a)}{\ell!}(b-a)^\ell = b - a. \tag{1.7}$$

We claim that $(G^k)_{k=0}^1$ is not a Whitney field of Class C^1 on $[0,1]$. To see this, let $\varepsilon_0 = \frac{1}{2}$, let $\delta > 0$, and choose any $a \in [0,1]$. Then, for every $b \in [0,1]$ with $0 < |b-a| < \delta$, we have by (1.7) that

$$\left| \frac{(\boldsymbol{R_a^1 G})^0(b)}{|a-b|} \right| = 1 \geq \varepsilon_0.$$

Hence there is $\varepsilon_0 > 0$ such that for any $\delta > 0$ there exist $a, b \in [0,1]$ such that

$$|a-b| < \delta \quad \text{and} \quad \left| \frac{(\boldsymbol{R_a^1 G})^0(b)}{|a-b|} \right| \geq \varepsilon_0.$$

Therefore, $(G^k)_{k=0}^1$ is not a Whitney field of Class C^1 on $[0,1]$. △

For any compact set $K \subset \mathbb{R}$, the set of Whitney fields $\mathcal{W}^m(K)$ is a vector space over the field of real numbers: closure and the other vector space axioms are verified easily, using the laws of limits one learns in an elementary calculus course. Additionally, it is well known from an elementary linear algebra course that $C^m(U)$ is also a vector space over the field of real numbers.

We now state the classical Whitney extension theorem. For the proof of this theorem, see [Whi34].

Theorem 1.10 (Classical Whitney extension theorem)**.** *Let K be a compact subset of a nonempty open subset $U \subset \mathbb{R}$, let $m \in \mathbb{N}$, and let $\boldsymbol{F} \in \mathcal{J}^m(K)$. Then there exists a linear transformation $W : \mathcal{W}^m(K) \to C^m(U)$ such that for any $k \in \mathbb{N}_m$ and any $x \in K$, we have that $D^k(WF)(x) = F^k(x)$, and we have that $WF \in C^\infty(U \backslash K)$.*

Later we will be concerned with a version of the Whitney extension theorem specific to the Heisenberg group.

1.3. **Absolute continuity.** The notion of absolute continuity is important to our discussion of the Heisenberg group. We define absolute continuity and explain its connection to continuous differentiability.

Definition 1.11. Let (X, d_X) be a metric space, and let $I \subset \mathbb{R}$ be an interval. A function $f : I \to X$ is *absolutely continuous on* I if for any $\varepsilon > 0$ there is a $\delta > 0$ such that, whenever $n \in \mathbb{N}$ and $((x_k, y_k))_{k=1}^n$ is a finite sequence of pairwise disjoint subintervals of I, with $x_j < y_j$ for any $j \in \mathbb{N}_n^*$, we have that

$$\text{if} \quad \sum_{k=1}^{n} |y_k - x_k| < \delta, \quad \text{then} \quad \sum_{k=1}^{n} d_X(f(y_k), f(x_k)) < \varepsilon.$$

We note that any absolutely continuous function is continuous, and that on any interval of the real line, an absolutely continuous function is uniformly continuous.

Recall that if $N \in \mathbb{N}$ and $I \subset \mathbb{R}$ is an interval, then a *curve* in $\mathbb{R}^N$ is a continuous function $\gamma : I \to \mathbb{R}^N$.

Lemma 1.12. *Let $N \in \mathbb{N}$, let $I \subset \mathbb{R}$ be an interval, and let $\gamma : I \to \mathbb{R}^N$ be a curve given by*

$$\gamma(t) = (\gamma_1(t), \gamma_2(t), \ldots, \gamma_N(t)).$$

Then γ is absolutely continuous if and only if each component $\gamma_1, \gamma_2, \ldots, \gamma_N : I \to \mathbb{R}$ is absolutely continuous.

Proof. Suppose first that for each $i \in \mathbb{N}_N^*$, γ_i is absolutely continuous. Then for any $i \in \mathbb{N}_N^*$ and any $\varepsilon > 0$, there exist $\delta_i > 0$ such that for any $n \in \mathbb{N}$ and any finite sequence $((x_k, y_k))_{k=1}^n$ of pairwise disjoint subintervals of I,

$$\text{if} \quad \sum_{k=1}^{n} |y_k - x_k| < \delta_i, \quad \text{then} \quad \sum_{k=1}^{n} |\gamma_i(y_k) - \gamma_i(x_k)| < \frac{\varepsilon}{N}.$$

So, let $n \in \mathbb{N}$ and let $((x_k, y_k))_{k=1}^n$ be a finite sequence of pairwise disjoint subintervals of I. Additionally, let

$$\delta := \min_{i \in \mathbb{N}_N^*} \{\delta_i\} > 0.$$

Then, if

$$\sum_{k=1}^{n} |y_k - x_k| < \delta,$$

then we have

$$\begin{aligned}
\sum_{k=1}^{n} d_{\mathbb{R}^N}(\gamma(y_k), \gamma(x_k)) &= \sum_{k=1}^{n} ||\gamma(y_k) - \gamma(x_k)|| \\
&= \sum_{k=1}^{n} \sqrt{\sum_{i=1}^{N} |\gamma_i(y_k) - \gamma_i(x_k)|^2} \\
&\leq \sum_{k=1}^{n} \sum_{i=1}^{N} \sqrt{|\gamma_i(y_k) - \gamma_i(x_k)|^2} \qquad \text{by Hölder} \\
&= \sum_{i=1}^{N} \sum_{k=1}^{n} |\gamma_i(y_k) - \gamma_i(x_k)| \\
&< N\delta = \varepsilon.
\end{aligned}$$

We conclude that γ is absolutely continuous.

Conversely, suppose that γ is absolutely continuous. Then for any $\varepsilon > 0$ there exists $\delta > 0$ such that for any $m \in \mathbb{N}$ and any finite sequence $((x_k, y_k))_{k=1}^m$ with $x_k < y_k$ for each $k \in \mathbb{N}_m^*$,

$$\text{if} \quad \sum_{k=1}^m |y_k - x_k| < \delta, \quad \text{then} \quad \sum_{k=1}^m ||\gamma(y_k) - \gamma(x_k)|| < \varepsilon.$$

It follows that if

$$\sum_{k=1}^m |y_k - x_k| < \delta,$$

then we have that

$$\begin{aligned} \sum_{k=1}^m \sqrt{\sum_{i=1}^N |\gamma_i(y_k) - \gamma_i(x_k)|^2} < \varepsilon &\implies \forall i \in \mathbb{N}_N^* \quad \sum_{k=1}^m \sqrt{|\gamma_i(y_k) - \gamma_i(x_k)|^2} < \varepsilon \\ &\implies \forall i \in \mathbb{N}_N^* \quad \sum_{k=1}^m |\gamma_i(y_k) - \gamma_i(x_k)| < \varepsilon. \end{aligned}$$

We conclude that the three components of γ are absolutely continuous. ■

Recall that a real-valued function f on any subset $S \subset \mathbb{R}$ is *Lipschitz continuous on S* if there exists $C \geq 0$ such that for any $x, y \in S$,

$$|f(x) - f(y)| \leq C\,|x - y|\,.$$

Lemma 1.13. *A continuously differentiable, real-valued function on a compact interval of the real line is absolutely continuous.*

Proof. Let $a, b \in \mathbb{R}$ such that $a < b$, and let $f : [a, b] \to \mathbb{R}$ be continuously differentiable. Since f is continuously differentiable, its derivative f' is continuous. Hence, by the mean value theorem, we have for any $x, y \in [a, b]$ with $x < y$ that there exists $c \in [a, b]$ such that

$$|f(y) - f(x)| = |f'(c)| \cdot |y - x|\,.$$

Because f' is defined on a compact interval and is continuous, it follows that f' is bounded, so that there exists $M > 0$ such that for every $c \in [a, b]$, we have $|f'(c)| \leq M$. It follows for any $x, y \in [a, b]$ that

$$x < y \implies |f(y) - f(x)| \leq M\,|y - x|\,, \tag{1.8}$$

whence f is Lipschitz continuous on $[a, b]$.

Now, let $n \in \mathbb{N}$, and for each $k \in \mathbb{N}_n^*$ let $x_k, y_k \in [a, b]$ such that $x_k < y_k$ and that $((x_k, y_k))_{k=1}^n$ is a finite sequence of pairwise disjoint subintervals of $[a, b]$. Let $\varepsilon > 0$, and choose $\delta := \dfrac{\varepsilon}{M} > 0$. Then if

$$\sum_{k=1}^n |y_k - x_k| < \delta,$$

then

$$\begin{aligned}\sum_{k=1}^{n}|f(y_k)-f(x_k)| &\le \sum_{k=1}^{n} M\,|y_k-x_k| \qquad\qquad \text{by (1.8)}\\ &= M\sum_{k=1}^{n}|y_k-x_k|\\ &< M\delta\\ &= \varepsilon.\end{aligned}$$

Therefore, f is absolutely continuous. ∎

We remark that absolute continuity is related to the fundamental theorem of calculus by way of the Lebesgue integral: in particular, if $I \subset \mathbb{R}$ is a compact interval and $f : I \to \mathbb{R}$ is absolutely continuous on I, then f' exists exists almost everywhere on I and for any $a, b \in I$,

$$f(b) - f(a) = \int_a^b f'\,dx.$$

Since in this paper we do not assume familiarity with the Lebesgue theory of integration, and since continuous differentiability implies absolute continuity, we shall restrict our attention almost exclusively to continuously differentiable functions.

1.4. **Signed area.** The following lemma gives some intuition about signed area in the plane for later discussion of horizontal curves in the Heisenberg group.

Recall that given $N \in \mathbb{N}$ and $a, b \in \mathbb{R}$ such that $a < b$, a curve $\gamma : [a, b] \to \mathbb{R}^N$ is *simple* if it it does not intersect itself except possibly at the endpoints a and b; in particular, the restriction $\gamma|_{[a,b)}$ is injective. Recall also that given a bounded subset $S \subset \mathbb{R}^2$, its *signed area* is defined to be the area of S with the counterclockwise orientation.

Lemma 1.14. *Let $a, b \in \mathbb{R}$ such that $a < b$, and let $\sigma : [a, b] \to \mathbb{R}^2$ given by*

$$\sigma(t) = (\sigma_1(t), \sigma_2(t))$$

be a simple, closed, piecewise smooth curve with $\sigma(a) = (0, 0)$. Denote by A_σ the signed area of the region enclosed by σ. Then

$$A_\sigma = \frac{1}{2}\int_a^b \left(\sigma_1(t)\sigma_2'(t) - \sigma_1'(t)\sigma_2(t)\right)\,dt. \tag{1.9}$$

Proof. Define C to be the contour

$$C := \left\{(\sigma_1(t), \sigma_2(t)) \in \mathbb{R}^2 \colon t \in [a, b]\right\},$$

and let D be the region of $\mathbb{R}^2$ enclosed by C. Notice that C is a piecewise smooth, simple, closed curve. Thus we may apply a form Green's theorem to find the area of D, i.e., to find A_σ. The form of Green's theorem we shall use is

$$A_\sigma = \frac{1}{2}\oint_C \left(\sigma_1 \frac{d\sigma_2}{dt}\,dt - \sigma_2\frac{d\sigma_1}{dt}\,dt\right). \tag{1.10}$$

Applying this to the piecewise smooth, simple, closed curve C , the integral in (1.10) and the area of D are equal to A_σ in (1.9). ∎

1.5. **Inequalities for polynomials.** We state some results about polynomials that will be useful to our discussion. The first lemma is the Markov inequality, whose proof can be found in [Sha04].

Lemma 1.15 (Markov inequality)**.** *Let $n \in \mathbb{N}$, and let $P : \mathbb{R} \to \mathbb{R}$ be a polynomial of degree n. Let also $a, b \in \mathbb{R}$ such that $a < b$. Then*

$$\max\{|P'(x)| : x \in [a,b]\} \leq \frac{2n^2}{b-a} \max\{|P(x)| : x \in [a,b]\}.$$

The following lemma and corollary and their proofs are due to [PSZ19].

Lemma 1.16. *Let $n \in \mathbb{N}$. Let $P : \mathbb{R} \to \mathbb{R}$ be a polynomial of degree n, and let $a, b \in \mathbb{R}$ such that $a < b$. Let*

$$M := \max\{|P(x)| : x \in [a,b]\}.$$

Then there exists a closed subinterval $I \subset [a,b]$ of length at least $(b-a)/4n^2$ such that for every $x \in I$, we have $|P(x)| \geq M/2$.

Proof. Let $x_0 \in [a,b]$ such that $P(x_0) = M$. Let $I \subset [a,b]$ be an interval of length $(b-a)/4n^2$ such that $I = [x_0, b_0]$ for some $b_0 \in \mathbb{R}$ with $x_0 < b_0$ or that $I = [a_0, x_0]$ for some $a_0 \in \mathbb{R}$ with $a_0 < x_0$. Without loss of generality, assume I is of the first form in the previous sentence.

Now, suppose there exists $y \in I$ such that $|P(y)| < M/2$, and let

$$M' := \max\{|P'(x)| : x \in [a,b]\}.$$

Then by Lemma 1.15, we have

$$\begin{aligned}
\frac{M}{2} &< |P(y) - P(x_0)| \\
&\leq \int_{x_0}^{y} |P'(x)|\, dx && \text{by the Fundamental Theorem of Calculus} \\
&\leq \frac{b-a}{4n^2} \max\{|P'(x)| : x \in [a,b]\} \\
&\leq \frac{1}{2} \max\{|P(x)| : x \in [a,b]\} \\
&= \frac{M}{2}.
\end{aligned}$$

This is impossible, so there is no such y. ■

Application of Lemma 1.16 yields the following.

Corollary 1.17. *Let n, a, b, P, and M be as in Lemma 1.16. Then*

$$\frac{M(b-a)}{8n^2} \leq \int_a^b |P(x)|\, dx \leq M(b-a). \tag{1.11}$$

Proof. By Lemma 1.16, we have that

$$\int_a^b |P(x)|\, dx \geq \int_a^b \frac{M}{2}\, dx = \frac{M(b-a)}{2} \geq \frac{M(b-a)}{8n^2},$$

since $m \in \mathbb{N}$. Moreover, we have that

$$M(b-a) = \int_a^b M\, dx \geq \int_a^b |P(x)|\, dx.$$

■

2. The Heisenberg group

In this thesis we are interested in the relationship between the Whitney extension theorem and the Heisenberg group. Thus, we pause our discussion purely on the Whitney extension theorem, and move to discuss some preliminaries regarding the Heisenberg group.

Technically speaking, there are many Heisenberg groups, in fact one for each positive integer. We shall be concerned only with the first Heisenberg group.

2.1. **Definition of the Heisenberg group.**

Definition 2.1. The *(first) Heisenberg group* $\mathbb{H}^1$ (or simply $\mathbb{H}$) is the set $\mathbb{R}^3$ equipped with a binary operation $\star$ defined for each $(x_1, y_1, t_1), (x_2, y_2, t_2) \in \mathbb{R}^3$ by

$$(x_1, y_1, t_1) \star (x_2, y_2, t_2) := (x_1 + x_2, y_1 + y_2, t_1 + t_2 + 2(y_1 x_2 - x_1 y_2)) .$$

That $\mathbb{H}$ indeed is a group follows from the usual closure, associativity, identity, and inverse properties of vector addition. In particular, the identity of $\mathbb{H}$ is the vector $(0, 0, 0)$ and for any $(x, y, t) \in \mathbb{H}$ its inverse element is given by $(-x, -y, -t)$.

Notice that $(\mathbb{H}, \star)$ is not group-isomorphic to $(\mathbb{R}^3, +)$, where $+$ represents normal vector addition. Although $(\mathbb{R}^3, +)$ is abelian, we have that

$$(1, 2, 3) \star (4, 5, 6) = (5, 7, 15), \quad \text{while} \quad (4, 5, 6) \star (1, 2, 3) = (5, 7, 3);$$

hence, $(\mathbb{H}, \star)$ is non-abelian.

2.2. **Vector fields and left invariance.**

Recall that given a subset S of $\mathbb{R}^3$, a *vector field on* S is a function $V : S \to \mathbb{R}^3$. In the first Heisenberg group, we shall denote by ∂_x the coordinate vector $(1, 0, 0)$, by ∂_y the coordinate vector $(0, 1, 0)$, and by ∂_t the coordinate vector $(0, 0, 1)$. From these we form the vector fields X and Y defined by

$$\begin{aligned} X(x, y, t) &= \partial_x + 2y\partial_t = (1, 0, 2y), \\ Y(x, y, t) &= \partial_y - 2x\partial_t = (0, 1, -2x). \end{aligned} \tag{2.1}$$

For fixed $p := (p_1, p_2, p_3) \in \mathbb{R}^3$ we define a map

$$L_p : \mathbb{R}^3 \to \mathbb{R}^3 \quad \text{by} \quad L_p(x, y, t) := (p_1, p_2, p_3) \star (x, y, t),$$

so that for any $(x, y, t) \in \mathbb{R}^3$, we have

$$L_p(x, y, t) = (p_1 + x, p_2 + y, p_3 + t + 2(p_2 x - p_1 y)) \tag{2.2}$$

Since we have for any $(p_1, p_2, p_3) \in \mathbb{R}^3$ that $(p_1, p_2, p_3) \star (x, y, t)$ comprises coordinates which are polynomials in x, y, and t, we have that L_p is differentiable so that its derivative DL_p exists. Moreover, for any differentiable vector field $V : \mathbb{R}^3 \to \mathbb{R}^3$ given by

$$V(x, y, t) = (V_1(x, y, t), V_2(x, y, t), V_3(x, y, t))$$

the derivative DV is represented by a matrix

$$(DV)(x,y,t) = \begin{pmatrix} \frac{\partial V_1}{\partial x}(x,y,t) & \frac{\partial V_1}{\partial y}(x,y,t) & \frac{\partial V_1}{\partial t}(x,y,t) \\ \frac{\partial V_2}{\partial x}(x,y,t) & \frac{\partial V_2}{\partial y}(x,y,t) & \frac{\partial V_2}{\partial t}(x,y,t) \\ \frac{\partial V_3}{\partial x}(x,y,t) & \frac{\partial V_3}{\partial y}(x,y,t) & \frac{\partial V_3}{\partial t}(x,y,t) \end{pmatrix},$$

so that, for fixed $p = (p_1, p_2, p_3) \in \mathbb{R}^3$,

$$DL_p = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2p_2 & -2p_1 & 1 \end{pmatrix} \tag{2.3}$$

We will see that the function DL_p is significant to our discussion.

Definition 2.2. Let $p = (p_1, p_2, p_3) \in \mathbb{R}^3$. A vector field $V : \mathbb{R}^3 \to \mathbb{R}^3$ on $\mathbb{H}$ is *left invariant* if for any $(x, y, t) \in \mathbb{R}^3$ we have

$$(DL_p)(V) = V((p_1, p_2, p_3) \star (x, y, t)).$$

With this definition we see that the vector fields X and Y are left invariant. For, fixing $p = (p_1, p_2, p_3) \in \mathbb{R}^3$ and viewing the row vectors in (2.1) as column vectors, we have by (2.1) and (2.3) for any $(x, y, t) \in \mathbb{R}^3$ that

$$\begin{aligned} (DL_p)(X) &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2p_2 & -2p_1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 2y \end{pmatrix} \\ &= \begin{pmatrix} 1 \\ 0 \\ 2p_2 + 2y \end{pmatrix}, \end{aligned} \tag{2.4}$$

while also for any $(x, y, t) \in \mathbb{R}^3$,

$$X((p_1, p_2, p_3) \star (x, y, t)) = (1, 0, 2(p_2 + y)). \tag{2.5}$$

Comparing the vectors in (2.4) and in (2.5) yields that the definition of left invariance is satisfied. By similar reasoning, Y is also left invariant.

2.3. **Horizontality of curves.** We now return to our discussion of the Heisenberg group and a special class of functions in the Heisenberg group.

Definition 2.3. A vector $(x, y, t) \in \mathbb{R}^3$ is *horizontal at* $p = (p_1, p_2, p_3) \in \mathbb{R}^3$ if there exist $\alpha, \beta \in \mathbb{R}$ such that

$$(x, y, t) = \alpha X(p_1, p_2, p_3) + \beta Y(p_1, p_2, p_3),$$

i.e., if (x, y, t) is a $\mathbb{R}$-linear combination of the vectors $(1, 0, 2p_2)$ and $(0, 1, -2p_1)$.

Definition 2.4. Let $a, b \in \mathbb{R}$ such that $a < b$. A continuously differentiable curve $\gamma : [a, b] \to \mathbb{R}^3$ defined by

$$\gamma(t) = (\gamma_1(t), \gamma_2(t), \gamma_3(t))$$

is *horizontal (in the Heisenberg group)* if for any $t \in [a, b]$, the derivative vector $\gamma'(t)$ is horizontal at $\gamma(t)$.

Lemma 2.5. *Let γ be a continuously differentiable curve as in the previous definition. Then γ is horizontal in the Heisenberg group if and only if for any $t \in [a,b]$,*

$$\gamma_3(t) = \gamma_3(a) + 2\int_a^t (\gamma_1'(\tau)\gamma_2(\tau) - \gamma_2'(\tau)\gamma_1(\tau))\, d\tau. \tag{2.6}$$

Proof. Suppose $\gamma : [a,b] \to \mathbb{R}^3$ is a C^1 horizontal curve in the Heisenberg group. Then γ is continuously differentiable, and, for any $t \in [a,b]$, we have that $\gamma'(t)$ is horizontal at $\gamma(t)$. In particular, there exist two functions $\alpha, \beta : [a,b] \to \mathbb{R}$ such that, for any $t \in [a,b]$,

$$\begin{aligned}\gamma'(t) &= \alpha(t)(1,0,2\gamma_2(t)) + \beta(t)(0,1,-2\gamma_1(t))\\ &= (\alpha(t), 0, 2\alpha(t)\gamma_2(t)) + (0, \beta(t), -2\beta(t)\gamma_1(t))\\ &= (\alpha(t), \beta(t), 2\alpha(t)\gamma_2(t) - 2\beta(t)\gamma_1(t)).\end{aligned}$$

Thus, for any $t \in [a,b]$,

$$\gamma_1'(t) = \alpha(t), \quad \gamma_2'(t) = \beta(t), \quad \text{and} \quad \gamma_3'(t) = 2[\alpha(t)\gamma_2(t) - \beta(t)\gamma_1(t)]. \tag{2.7}$$

Since γ is continuously differentiable on $[a,b]$, we have by the fundamental theorem of calculus that, for any $t \in [a,b]$,

$$\begin{aligned}\gamma_3(t) - \gamma_3(a) &= 2\int_a^t [\alpha(\tau)\gamma_2(\tau) - \beta(\tau)\gamma_1(\tau)]\, d\tau\\ &= 2\int_a^t [\gamma_1'(\tau)\gamma_2(\tau) - \gamma_2'(\tau)\gamma_1(\tau)]\, d\tau \qquad \text{by (2.7).}\end{aligned}$$

It follows that $\gamma_3(t)$ satisfies (2.6) for any $t \in [a,b]$.

Conversely, suppose that γ_3 satisfies (2.6). Then for any $t \in [a,b]$,

$$\gamma_3'(t) = 2(\gamma_1'(t)\gamma_2(t) - \gamma_1(t)\gamma_2'(t)).$$

It follows that

$$\begin{aligned}\gamma'(t) &= (\gamma_1'(t), \gamma_2'(t), 2(\gamma_1'(t)\gamma_2(t) - \gamma_1(t)\gamma_2'(t)))\\ &= \gamma_1'(t)(1,0,2\gamma_2(t)) + \gamma_2'(t)(0,1,-2\gamma_1(t)).\end{aligned}$$

Because $\gamma(t) \in \mathbb{R}$ for any $t \in [a,b]$, we conclude that $\gamma'(t)$ is an $\mathbb{R}$-linear combination of the vectors $(1,0,2\gamma_2(t))$ and $(0,1,-2\gamma_1(t))$; that is, γ is horizontal in the Heisenberg group.

■

Example 2.6. Define a curve $\gamma : [0,1] \to \mathbb{R}^3$ by

$$\gamma(t) = \left(t^2, t^3, -\frac{2}{5}t^5\right).$$

Then for each $i \in \mathbb{N}_3^*$, we have that γ_i is continuously differentiable on $[0,1]$. Thus, γ is continuously differentiable on $[0,1]$.

Note that $\gamma_3(0) = 0$, so that for any $t \in [0,1]$,

$$\begin{aligned}2\int_0^t (\gamma_1'(\tau)\gamma_2(\tau) - \gamma_1(\tau)\gamma_2'(\tau))\, d\tau &= 2\int_0^t (2\tau\tau^3 - 3\tau^2\tau^2)\\ &= -2\int_0^t \tau^4\, d\tau = \gamma_3(t).\end{aligned}$$

It follows that γ_3 satisfies equation (2.6). Consequently, γ is horizontal in the Heisenberg group. △

Example 2.7. Define another curve $\sigma : [0,1] \to \mathbb{R}^3$ by

$$\sigma(t) := \left(t^2, t^3, e^{t^2}\right).$$

Although σ is absolutely continuous, by reasoning similar to that in the previous example, it is not the case that σ satisfies (2.6), so σ is not horizontal. △

Example 2.8. We construct an example of a curve which is horizontal in the Heisenberg group and explore an interesting property of such a curve. Let $r > 0$, and define the curve $\gamma := (\gamma_1, \gamma_2, \gamma_3) : [0, 2\pi] \to \mathbb{R}^3$ by

$$\gamma(t) = \big(r + r\cos(t-\pi), r\sin(t-\pi), -2r^2 t\big).$$

Notice that $\gamma(0) = (0,0,0)$, and that the projection $\Pi_\gamma : [0,2\pi] \to \mathbb{R}^2$ defined by

$$\Pi_\gamma(t) = \Pi((\gamma_1(t), \gamma_2(t), \gamma_3(t)) := (\gamma_1(t), \gamma_2(t))$$

of γ onto the $\gamma_1\gamma_2$-plane is represented by a circle with center $(r, 0)$ and radius r.

Notice also that γ is horizontal in the Heisenberg group. For, it is continuously differentiable on $[0, 2\pi]$, and, with $\gamma_3(0) = 0$, we have for any $t \in [0, 2\pi]$ that

$$\begin{aligned}
2\int_0^t (\gamma_1'(\tau)\gamma_2(\tau) - \gamma_2'(\tau)\gamma_1(\tau))\, d\tau &= 2\int_0^t \left[-r^2\sin^2(\tau-\pi) - r^2\cos^2(\tau-\pi)\right] d\tau \\
&= -2r^2\int_0^t d\tau \\
&= -2r^2 t \\
&= \gamma_3(t).
\end{aligned}$$

Hence γ_3 satisfies the conclusion of Lemma 2.5 and so is horizontal in the Heisenberg group.

Now, we define the $\mathbb{H}$-*length* $L_{[a,b]}$ of a horizontal curve σ over an interval $[a, b]$ by

$$L_{[a,b]}(\sigma) = \int_a^b \sqrt{(\sigma_1'(t))^2 + (\sigma_2'(t))^2}\, dt.$$

Thus,

$$\begin{aligned}
L_{[0,2\pi]}(\gamma) &= \int_0^{2\pi} \sqrt{r^2\sin^2(t-\pi) + r^2\cos^2(t-\pi)}\, dt \\
&= \int_0^{2\pi} r\, dt \\
&= 2\pi r.
\end{aligned}$$

Since $|\gamma_3(2\pi)| = 4\pi r^2$, we see that the $\mathbb{H}$-length of the curve is proportional to the square root of the horizontal lift of $\Pi_\gamma(t)$ for $t \in [0, 2\pi]$. △

Example 2.8 hints at a metric used to measure distances in the Heisenberg group. This metric is known as the *Carnot-Carathéodory distance* d_{CC} and is defined, for any $p, q \in \mathbb{R}^3$ and any $a, b \in \mathbb{R}$ such that $a < b$, by

$$d_{CC}(p,q) := \inf_{[a,b]} \left\{L_{[a,b]}(\gamma) \colon \gamma \text{ is a horizontal curve connecting } p \text{ and } q\right\}.$$

So that the definition of d_{CC} makes sense, we now show that any two points in $\mathbb{H}$ can be connected by a horizontal curve.

Lemma 2.9. *The origin in $\mathbb{H}$ may be connected by a horizontal curve to any other point in $\mathbb{H}$.*

Proof. Let $(a,b,c) \in \mathbb{H}$. Define the curve $\eta = (\eta_1, \eta_2, \eta_3) : [0,1] \to \mathbb{H}$ by

$$\eta(t) := \begin{cases} \left(ct^2(t-1),\, 15t(t-1),\, ct^3\left(10-15t+6t^2\right)\right), & \text{if } a=b=0, \\[2ex] \left(-\dfrac{3c}{2b}t(1-t),\, tb + \dfrac{3c}{2b}t(1-t),\, ct^3\right), & \text{if } a=0 \text{ and } b\neq 0, \\[2ex] \left(ta,\, tb + \dfrac{3c}{2a}t(1-t),\, ct^3\right), & \text{if } a\neq 0 \text{ and } b \neq 0. \end{cases}$$

In each case of the definition of η, we have that $\eta(0) = (0,0,0)$, that $\eta(1) = (a,b,c)$, that η is continuously differentiable, and by (2.6), that η is horizontal in the Heisenberg group. We conclude that the statement of the lemma is true. ■

Corollary 2.10. *Any two points in $\mathbb{H}$ may be connected by a horizontal curve.*

Proof. First, given two points $(a,b,c),(p,q,r) \in \mathbb{H}$, use Lemma 2.9 to choose a horizontal curve $\eta : [0,1] \to \mathbb{H}$ with $\eta(0) = (0,0,0)$ and $\eta(1) = (a,b,c)$. Then, define $\hat{\eta} = (\hat{\eta}_1, \hat{\eta}_2, \hat{\eta}_3) : [0,1] \to \mathbb{H}$ by

$$\hat{\eta}(t) := (p,q,r) \star (\eta_1(t), \eta_2(t), \eta_3(t)).$$

In fact, by Definition 2.1 of $\star$ and (2.6), we have for any $t \in [0,1]$ that

$$\hat{\eta}(t) = \left(p + \eta_1(t),\, q + \eta_2(t),\, r + \eta_3(0) + 2\int_0^t \left(\eta_1'(\tau)\eta_2(\tau) - \eta_2'(\tau)\eta_1(\tau)\right)\, d\tau\right).$$

Because additionally, $\hat{\eta}_3(0) = r$, we have by definition of $\hat{\eta}$ that

$$\begin{aligned} \hat{\eta}_1(t) &= p + \eta_1(t), \\ \hat{\eta}_2(t) &= q + \eta_2(t), \\ \hat{\eta}_3(t) &= 2r + 2\int_0^t \left(q\eta_1'(\tau) + \eta_1'(\tau)\eta_2(\tau) - p\eta_2'(\tau) - \eta_1(\tau)\eta_2'(\tau)\right)\, d\tau. \end{aligned}$$

Hence,

$$\begin{aligned} \hat{\eta}_1'(t) &= \eta_1'(t), \\ \hat{\eta}_2'(t) &= \eta_2'(t), \\ \hat{\eta}_3'(t) &= 2\eta_1'(t)(q + \eta_2(t)) - 2\eta_2'(t)(p + \eta_1(t)) = 2\hat{\eta}_1'(t)\hat{\eta}_2(t) - 2\hat{\eta}_2'(t)\hat{\eta}_1(t). \end{aligned}$$

Therefore, for any $t \in [0,1]$,

$$\hat{\eta}'(t) = \hat{\eta}_1'(t)(1,0,2\hat{\eta}_2(t)) + \hat{\eta}_2'(t)(0,1,-2\hat{\eta}_1(t)).$$

Since $\hat{\eta}_1'(t), \hat{\eta}_2'(t) \in \mathbb{R}$ for any $t \in [0,1]$, we have by Definition 2.3 that $\hat{\eta}'(t)$ is horizontal at $\hat{\eta}(t)$ for all $t \in [0,1]$. Therefore, by Definition 2.4, we conclude that $\hat{\eta}$ is horizontal in the Heisenberg group.

Notice that $\hat{\eta}(0) = (p, q, r)$ and that $\hat{\eta}(1) = (p, q, r) \star (a, b, c)$. Hence, to travel from (p, q, r) to another point $(x, y, z) \in \mathbb{H}$ via $\hat{\eta}$, we solve the equation

$$(p, q, r) \star (x, y, z) = (a, b, c),$$

or equivalently,

$$(a, b, c) = (p, q, r)^{-1} \star (x, y, z).$$

We conclude that the statement of the corollary is true. ∎

We now prove two lemmas regarding a necessary condition for horizontality of a curve in the Heisenberg group.

Lemma 2.11 (Product rule for higher-order derivatives)**.** *Let $n \in \mathbb{N}$, let $D \subset \mathbb{R}$ be an open interval, and let $f, g : D \to \mathbb{R}$ be two functions of class $C^n(D)$. Then*

$$\frac{d^n}{dx^n}(f(x) \cdot g(x)) = \sum_{k=0}^{n} \binom{n}{k} \frac{d^k}{dx^k}(f(x)) \cdot \frac{d^{n-k}}{dx^{n-k}}(g(x)) \,. \tag{2.8}$$

Proof. We prove this lemma by mathematical induction on n. For notational convenience, we say $\dfrac{d^0}{dx^0}(f(x)) := f(x)$ for all $x \in D$. Since

$$\begin{aligned} \frac{d}{dx}(f(x) \cdot g(x)) &= f(x) \cdot \frac{d}{dx}(g(x)) + g(x)\frac{d}{dx}(f(x)) \\ &= \binom{1}{0} f(x) \frac{d}{dx}(g(x)) + \binom{1}{1} g(x) \frac{d}{dx}(f(x)) \,, \end{aligned}$$

we have that the case $n = 1$ is true.

Now, assume there exists $m \in \mathbb{N}$ such that $m > 1$ and

$$\frac{d^{m-1}}{dx^{m-1}}(f(x) \cdot g(x)) = \sum_{k=0}^{m-1} \binom{m-1}{k} \frac{d^k}{dx^k}(f(x)) \cdot \frac{d^{m-1-k}}{dx^{m-1-k}}(g(x)) \,.$$

Then,

$$\begin{aligned} \frac{d^m}{dx^m}(f(x) \cdot g(x)) &= \frac{d}{dx}\left[\frac{d^{m-1}}{dx^{m-1}}(f(x) \cdot g(x))\right] \\ &= \frac{d}{dx}\left[\sum_{k=0}^{m-1} \binom{m-1}{k} \frac{d^k}{dx^k}(f(x)) \cdot \frac{d^{m-1-k}}{dx^{m-1-k}}(g(x))\right] \\ &= \sum_{k=0}^{m-1} \binom{m-1}{k} \frac{d}{dx}\left[\frac{d^k}{dx^k}(f(x)) \cdot \frac{d^{m-1-k}}{dx^{m-1-k}}(g(x))\right], \end{aligned} \tag{2.9}$$

by linearity of $\dfrac{d}{dx}$. By the standard product rule, the sum in (2.9) is equal to

$$\begin{aligned} \sum_{k\,=0}^{m-1} \Bigg[\binom{m-1}{k} &\left(\frac{d^{k+1}}{dx^{k+1}}(f(x)) \cdot \frac{d^{m-(k+1)}}{dx^{m-(k+1)}}(g(x))\right) \\ &+ \binom{m-1}{k} \left(\frac{d^k}{dx^k}(f(x)) \cdot \frac{d^{m-k}}{dx^{n-k}}(g(x))\right)\Bigg], \end{aligned} \tag{2.10}$$

which by linearity of $\sum$ is the same as

$$\sum_{k=0}^{m-1}\binom{m-1}{k}\left(\frac{d^{k+1}}{dx^{k+1}}(f(x))\cdot\frac{d^{m-(k+1)}}{dx^{m-(k+1)}}(g(x))\right) + \sum_{k=0}^{m-1}\binom{m-1}{k}\left(\frac{d^{k}}{dx^{k}}(f(x))\cdot\frac{d^{m-k}}{dx^{m-k}}(g(x))\right). \tag{2.11}$$

Now, we reindex the first sum in (2.11) to get that (2.10) is equal to

$$\sum_{k=1}^{m}\binom{m-1}{k-1}\frac{d^{k}}{dx^{k}}(f(x))\cdot\frac{d^{m-k}}{dx^{m-k}}(g(x)) + \sum_{k=0}^{m-1}\binom{m-1}{k}\frac{d^{k}}{dx^{k}}(f(x))\cdot\frac{d^{m-k}}{dx^{m-k}}(g(x)) \tag{2.12}$$

Notice that, for each $k \in \mathbb{N}_m$, we have

$$\begin{aligned}\binom{m-1}{k-1}+\binom{m-1}{k} &= \frac{(m-1)!}{(k-1)!\,(m-k)!}+\frac{(m-1)!}{k!\,(m-1-k)!}\\ &= (m-1)!\left[\frac{k}{k!\,(m-k)!}+\frac{m-k}{k!\,(m-k)!}\right]\\ &= (m-1)!\,\frac{m}{k!\,(m-k)!}\\ &= \frac{m!}{k!\,(m-k)!}=\binom{m}{k}.\end{aligned}$$

(This is just Pascal's rule for binomial coefficients.) Hence, we have that the quantity in (2.12) is equal to

$$\sum_{k=1}^{m-1}\binom{m}{k}\frac{d^{k}}{dx^{k}}(f(x))\cdot\frac{d^{m-k}}{dx^{m-k}}(g(x))+f(x)\frac{d^{m}}{dx^{m}}(g(x))+g(x)\frac{d^{m}}{dx^{m}}(f(x)). \tag{2.13}$$

Rewriting (2.13) in one summation completes the induction. ■

Now, we introduce a notation as follows: let $m \in \mathbb{N}$, let $a, b \in \mathbb{R}$ such that $a < b$, and let $\gamma : [a, b] \to \mathbb{R}^3$ be a C^m curve which is horizontal in the Heisenberg group. Then for any $k \in \mathbb{N}_m^*$, we have, by application of (2.8) to each of the terms in the integrand of (2.6), that $D^k\gamma_3$ is a polynomial of the arguments γ_1, γ_2, and their derivatives up to order k; in particular, for any $t \in [a, b]$, we write

$$D^k\gamma_3(t) := \mathcal{P}^k\left(\gamma_1(t), \gamma_2(t), \gamma_1'(t), \gamma_2'(t), \ldots, D^k\gamma_1(t), D^k\gamma_2(t)\right).$$

Indeed, the polynomials $\mathcal{P}^k$ are exactly those determined by application of (2.8) to each of the terms in the integrand of (2.6). To be more explicit, for each $k \in \mathbb{N}_m^*$, the polynomial $\mathcal{P}^k$ is given by

$$
\begin{aligned}
&2\big[D^{k-1}(\gamma_1'\gamma_2) - D^{k-1}(\gamma_2'\gamma_1)\big] \\
&\quad = 2\left[\sum_{\ell=0}^{k-1}\binom{k-1}{\ell}D^\ell\gamma_1'\cdot D^{k-\ell-1}\gamma_2 - \sum_{\ell=0}^{k-1}\binom{k-1}{\ell}D^\ell\gamma_2'\cdot D^{k-\ell-1}\gamma_2\right] \\
&\quad = 2\left[\sum_{\ell=0}^{k-1}\binom{k-1}{\ell}D^{\ell+1}\gamma_1\cdot D^{k-\ell-1}\gamma_2 - \sum_{\ell=0}^{k-1}\binom{k-1}{\ell}D^{\ell+1}\gamma_2\cdot D^{k-\ell-1}\gamma_1\right]. \qquad (2.14)
\end{aligned}
$$

Henceforward it shall be understood that for each $k \in \mathbb{N}^m$, any use of a calligraphic $\mathcal{P}^k$ indicates this polynomial.

3. The main theorem

We move now to discuss the main topic of this paper: a Whitney extension theorem for curves in the Heisenberg group.

Definition 3.1. Let $m \in \mathbb{N}$ and let $K \subset \mathbb{R}$ be compact. Let also $\boldsymbol{F} := (F^k)_{k=0}^m$, $\boldsymbol{G} := (G^k)_{k=0}^m$, and $\boldsymbol{H} := (H^k)_{k=0}^m$ be three jets of order m on K. We say the triple $(\boldsymbol{F}, \boldsymbol{G}, \boldsymbol{H})$, *extends to the C^m horizontal curve* $(f, g, h) : \mathbb{R} \to \mathbb{H}$ if both of the following conditions are true.

(3.1.1) The triple $(f, g, h) : \mathbb{R} \to \mathbb{H}$ is a C^m horizontal curve
(3.1.2) For each $k \in \mathbb{N}_m$ and each $x \in K$, we have that

$$D^k f(x) = F^k(x), \quad D^k g(x) = G^k(x), \quad \text{and} \quad D^k h(x) = H^k(x).$$

The following is due to [PSZ19]. Let $m \in \mathbb{N}$; let $K \subset \mathbb{R}$; and let $\boldsymbol{F}$, $\boldsymbol{G}$, and $\boldsymbol{H}$ be as in Definition 3.1. We define, for any $a, b \in K$, the *area discrepancy* $A(a, b)$ by

$$
\begin{aligned}
&H^0(b) - H^0(a) - 2\int_a^b \left[(T_a^m\boldsymbol{F})'(x)(T_a^m\boldsymbol{G})(x) - (T_a^m\boldsymbol{G})'(x)(T_a^m\boldsymbol{F})(x)\right] dx \\
&\qquad + 2F^0(a)\big[\boldsymbol{G}^0(b) - (T_a^m\boldsymbol{G})(b)\big] - 2G^0(a)\big[F^0(b) - (T_a^m\boldsymbol{F})(b)\big], \qquad (3.1)
\end{aligned}
$$

and the *velocity* $V(a, b)$ by

$$(b-a)^{2m} + (b-a)^m \int_a^b \left(|(T_a^m F)'(x)| + |(T_a^m G)'(x)|\right) dx. \qquad (3.2)$$

Theorem 3.2. *Let $m \in \mathbb{N}$, let $K \subset \mathbb{R}$ be a compact set, and let $\boldsymbol{F} := (F^k)_{k=0}^m$, $\boldsymbol{G} := (G^k)_{k=0}^m$, and $\boldsymbol{H} := (H^k)_{k=0}^m$ be three jets of order m on K. Then the triple $(\boldsymbol{F}, \boldsymbol{G}, \boldsymbol{H})$ extends to a C^m horizontal curve $(f, g, h) : \mathbb{R} \to \mathbb{H}$ if and only if*

(3.2.1) $\boldsymbol{F}$, $\boldsymbol{G}$, and $\boldsymbol{H}$ are Whitney fields of class C^m on K.
(3.2.2) For any $k \in \mathbb{N}_m^*$ and any $t \in K$, we have that

$$H^k(t) = \mathcal{P}^k\big(F^0(t), G^0(t), F^1(t), G^1(t), \ldots, F^k(t), G^k(t)\big).$$

(3.2.3) We have $A(a, b)/V(a, b) \to 0$ uniformly as $(b - a) \searrow 0$ with $a, b \in K$.

3.1. **Necessity of the criteria.** Notice, in Theorem 3.2, that because $\boldsymbol{F}$, $\boldsymbol{G}$, and $\boldsymbol{H}$ are Whitney fields, by the classical Whitney extension Theorem 1.10 there exists a C^m extension of $(\boldsymbol{F}, \boldsymbol{G}, \boldsymbol{H})$ on K; shortly we will prove that such an extension is not necessarily horizontal in the Heisenberg group. The following remark and propositions and their proofs are due to [PSZ19].

Remark 3.3. Let $m \in \mathbb{N}$, let $K \subset \mathbb{R}$ be compact, and let $\boldsymbol{F} := (F^k)_{k=0}^m$, $\boldsymbol{G} := (G^k)_{k=0}^m$, and $\boldsymbol{H} := (H^k)_{k=0}^m$ be three jets of order m on K. For any $a, b \in K$, define three additional jets $\hat{\boldsymbol{F}}$, $\hat{\boldsymbol{G}}$, and $\hat{\boldsymbol{H}}$ on K such that

$$\left(\hat{\boldsymbol{F}}, \hat{\boldsymbol{G}}, \hat{\boldsymbol{H}}\right) = (\boldsymbol{F}(a), \boldsymbol{G}(a), \boldsymbol{H}(a))^{-1} \star (\boldsymbol{F}, \boldsymbol{G}, \boldsymbol{H}), \tag{3.3}$$

that $\hat{F}^k = F^k$ and $\hat{G}^k = G^k$ for every $k \in \mathbb{N}_m^*$, and that $\hat{H}^k$ is arbitrary for all $k \in \mathbb{N}_m^*$. Notice that by Definition (2.1) of the binary operation $\star$, we have for any $t \in K$ that

$$\hat{\boldsymbol{F}}(t) = \boldsymbol{F}(t) - \boldsymbol{F}(a), \quad \hat{\boldsymbol{G}}(t) = \boldsymbol{G}(t) - \boldsymbol{G}(a), \tag{3.4}$$

and that

$$\hat{\boldsymbol{H}}(t) = \boldsymbol{H}(t) - \boldsymbol{H}(a) + 2\boldsymbol{F}(a)\boldsymbol{G}(t) - 2\boldsymbol{G}(a)\boldsymbol{F}(t).$$

It follows that $\hat{\boldsymbol{F}}(a) = \hat{\boldsymbol{G}}(a) = \hat{\boldsymbol{H}}(a) = 0$, so by (3.4), we have that $\hat{V}(a,b) = V(a,b)$. Additionally,

$$\begin{aligned}
\hat{A}(a,b) &= \hat{\boldsymbol{H}}(b) - \hat{\boldsymbol{H}}(a) \\
&\qquad - 2\int_a^b \left((T_a^m \hat{\boldsymbol{F}})'(t)(T_a^m \hat{\boldsymbol{G}})(t) - (T_a^m \hat{\boldsymbol{F}})(t)(T_a^m \hat{\boldsymbol{G}})'(t) \right) dt \\
&= \boldsymbol{H}(b) - \boldsymbol{H}(a) - 2\int_a^b ((T_a^m \boldsymbol{F}'(t))(T_a^m \boldsymbol{G}(t)) - (T_a^m \boldsymbol{F}(t))(T_a^m \boldsymbol{G}'(t)))\, dt \\
&\qquad + 2\boldsymbol{F}(a)(\boldsymbol{G}(b) - (T_a^m \boldsymbol{G}(b)) - 2\boldsymbol{G}(a)(\boldsymbol{F}(b) - T_a^m \boldsymbol{F}(b)) \\
&= A(a,b).
\end{aligned}$$

Proposition 3.4. *Let $m \in \mathbb{N}$. Suppose that $(f,g,h) : \mathbb{R} \to \mathbb{H}$ is a C^m horizontal curve in the Heisenberg group and that $K \subset \mathbb{R}$ is compact. Let $\boldsymbol{F} := J^m(f)|_K$, $\boldsymbol{G} := J^m(g)|_K$, and $\boldsymbol{H} := J^m(h)|_K$; that is, $\boldsymbol{F}$, $\boldsymbol{G}$, and $\boldsymbol{H}$ are the jets of order m obtained by restricting to K the functions f, g, and h, and their derivatives up to order m. Then we have*

(3.4.1) $\boldsymbol{F}$, $\boldsymbol{G}$, and $\boldsymbol{H}$ are Whitney fields of class C^m on K; and
(3.4.2) For any $k \in \mathbb{N}_m^*$ and any $t \in K$, we have

$$H^k(t) = \mathcal{P}^k\left(F^0(t), G^0(t), F^1(t), G^1(t), \ldots, F^k(t), G^k(t)\right),$$

(3.4.3) We have $A(a,b)/V(a,b) \to 0$ uniformly as $(b-a) \searrow 0$ with $a, b \in K$.

Proof. Let f, g, h, $\boldsymbol{F}$, $\boldsymbol{G}$, $\boldsymbol{H}$, and K be as in the statement of the proposition. Without loss of generality, assume that there exist $A, B \in \mathbb{R}$ such that $K = [A,B]$, for if (3.4.1), (3.4.2), and (3.4.3) are true on the compact interval $[A,B]$, then they are true also on any compact subset of $[A,B]$. By Taylor's theorem with the Peano form of the remainder [see Fol90], it follows that F, G, and H are Whitney fields of class C^m on K, so that (3.4.1) is true. But also by Lemma **??**, we have for any $k \in \mathbb{N}_m^*$ and any $t \in K$ that

$$D^k h(t) = \mathcal{P}^k\left(f(t), g(t), f'(t), g'(t), \ldots, D^k f(t), D^k g(t)\right).$$

Hence (3.4.2) is also true.

To prove (3.4.3), first fix $\varepsilon > 0$; then there exists $\delta > 0$ such that for any $a, b \in \mathbb{R}$ with $a < b$, if $[a, b] \subset K$ and $b - a < \delta$, then the following three items are true:

(i) For each $i \in \mathbb{N}_m$ and each $t \in [a, b]$, we have that

$$\left|D^i f(t) - D^i f(a)\right| \leq \varepsilon \quad \text{and} \quad \left|D^i g(t) - D^i g(a)\right| \leq \varepsilon. \tag{3.5}$$

To see this, observe that since f (respectively, g) is of class C^m on K, we have that $D^i f$ (respectively, $D^i g$) is continuous on K for each $i \in \mathbb{N}_m$; moreover, the Heine-Cantor Theorem assures that, because f (respectively, g) is continuous on the compact set K, it in fact is uniformly continuous. Therefore, (3.5) is true.

(ii) For each $t \in [a, b]$, we have that

$$|f(t) - (T_a^m f)(t)| \leq \varepsilon(b - a)^m \tag{3.6}$$

and that

$$|g(t) - (T_a^m g)(t)| \leq \varepsilon(b - a)^m. \tag{3.7}$$

These follow from the version of Taylor's Theorem explained on page 234 of [Fol90].

(iii) For each $t \in [a, b]$, we have that

$$|f'(t) - (T_a^m f)'(t)| \leq \varepsilon(b - a)^{m-1} \tag{3.8}$$

and that

$$|f'(t) - (T_a^m f)'(t)| \leq \varepsilon(b - a)^{m-1}. \tag{3.9}$$

This again follows from Taylor's Theorem in [Fol90], noting that since f is of class C^m on K (as is $T_a^m f$), we have that f' and $(T_a^m f)'$ are of class C^{m-1} on K.

Now, let $a \in K$, and fix $b \in K$ with $0 < b - a < \delta$.

Claim 3.5. *To finish proving (3.4.3), it is sufficient to consider the case $f(a) = g(a) = h(a) = 0$.*

Proof. If $a \in K$ and each of $f(a)$, $g(a)$, and $h(a)$ is arbitrary, define the C^m curve

$$(\hat{f}, \hat{g}, \hat{h}) := (f(a), g(a), h(a))^{-1} (f, g, h)$$

on $\mathbb{R}$, which is horizontal in the Heisenberg group, and notice, as in Remark 3.3, that $\hat{f}(a) = \hat{g}(a) = \hat{h}(a) = 0$. Set $\hat{\boldsymbol{F}} := J^m(\hat{f})|_K$, $\hat{\boldsymbol{G}} := J^m(\hat{g})|_K$, and $\hat{\boldsymbol{H}} := J^m(\hat{h})|_K$. Then, as in Remark 3.3, we have that

$$A(a, b)/V(a, b) = \hat{A}(a, b)/\hat{V}(a, b).$$

This proves the claim. ■

So, assume that $f(a) = g(a) = h(a) = 0$. Then by definition of the area discrepancy, we have

$$A(a, b) = h(b) - h(a) - 2 \int_a^b ((T_a^m f)'(t)(T_a^m g)(t) - (T_a^m f)(t)(T_a^m g)'(t)) \, dt. \tag{3.10}$$

But, since (f, g, h) is horizontal in the Heisenberg group, we have by (2.6) that

$$h(b) - h(a) = 2 \int_a^b (f'(t)g(t) - f(t)g'(t)) \, dt, \tag{3.11}$$

whence $|A(a,b)|$ is given, using the triangle inequality, by

$$\begin{aligned}
&\left| h(b) - h(a) - 2\int_a^b ((T_a^m f)'(t)(T_a^m g)(t) - (T_a^m f)(t)(T_a^m g)'(t))\ dt \right| \\
&\qquad \leq 2\int_a^b |f'(t)g(t) - (T_a^m f)'(t)(T_a^m g)(t)|\ dt \\
&\qquad\qquad + 2\int_a^b |f(t)g'(t) - (T_a^m f)(t)(T_a^m g)'(t)|\ dt.
\end{aligned} \tag{3.12}$$

But also, for any $t \in [a,b]$, we have that the quantity

$$f'(t)g(t) - (T_a^m f)'(t)(T_a^m g)(t)$$

may be rewritten in the following way:

$$\begin{aligned}
(f'(t) - (T_a^m f)'(t))(g(t) - (T_a^m g)(t)) &+ (f'(t) - (T_a^m f)'(t))((T_a^m g)(t)) \\
&+ (g - (T_a^m g)(t))((T_a^m f)'(t)).
\end{aligned} \tag{3.13}$$

It follows that

$$\begin{aligned}
\int_a^b |f'(t)g(t) - (T_a^m f)'(t)(T_a^m g)(t)|\ dt &\leq \int_a^b |f'(t) - (T_a^m f)'(t)|\,|g(t) - (T_a^m g)(t)|\ dt \\
&\quad + \int_a^b |f'(t) - (T_a^m f)'(t)|\,|(T_a^m g)(t)|\ dt \\
&\quad + \int_a^b |g(t) - (T_a^m g)(t)|\,|(T_a^m f)'(t)|\ dt \\
&\leq \varepsilon^2 (b-a)^{2m} + \varepsilon (b-a)^{m-1} \int_a^b |(T_a^m g)(t)|\ dt \\
&\quad + \varepsilon (b-a)^m \int_a^b |(T_a^m f)'(t)|\ dt,
\end{aligned} \tag{3.14}$$

where we attribute (3.14) to (3.6)–(3.9). By similar reasoning applied to the second term in the righthand side of (3.12), we have that

$$\begin{aligned}
&\left| h(b) - h(a) - 2\int_a^b ((T_a^m f)'(t)(T_a^m g)(t) - (T_a^m f)(t)(T_a^m g)'(t))\ dt \right| \\
&\quad \leq 4\varepsilon^2 (b-a)^{2m} \\
&\qquad + 2\varepsilon (b-a)^{m-1} \int_a^b (|(T_a^m f)(t)| + |(T_a^m g)(t)|)\ dt \\
&\qquad + 2\varepsilon (b-a)^m \int_a^b (|(T_a^m f)'(t)| + |(T_a^m g)'(t)|)\ dt.
\end{aligned} \tag{3.15}$$

Let

$$M := \max\left\{ |(T_a^m f)(x)| : x \in [a,b] \right\}.$$

Because $T_a^m f(a) = f(a) = 0$, it follows that $|(T_a^m)' f(x)| \leq M(b-a)$ for all $x \in [a,b]$. But also by Corollary 1.17 applied to $(T_a^m f)'$ gives that

$$\int_a^b |(T_a^m f)(x)|\ dx \leq M(b-a)^2 \leq 8m^2 (b-a) \int_a^b |(T_a^m f)'(x)|\ dx,$$

and similarly, that

$$\int_a^b |(T_a^m g)(x)| \leq M(b-a)^2 \leq 8m^2(b-a)\int_a^b |(T_a^m g)'(x)|\, dx.$$

Combining this with (3.15), we have

$$\begin{aligned}&\left| h(b) - h(a) - 2\int_a^b ((T_a^m f)'(t)(T_a^m g)(t) - (T_a^m f)(t)(T_a^m g)'(t))\, dt \right| \\ &\quad \leq 4\varepsilon^2(b-a)^{2m} \\ &\quad\quad + \left(2 + 16m^2\right)\varepsilon(b-a)^m \int_a^b (|(T_a^m f)'(t)| + |(T_a^m g)'(t)|)\, dt\end{aligned}$$

We have shown, for any $a, b \in K$ with $0 < b - a < \delta$, that

$$|A(a,b)| \leq \left(4\varepsilon^2 + \left(2 + 16m^2\right)\varepsilon\right) V(a,b).$$

Therefore, for any $a, b \in K$, we have

$$A(a,b)/V(a,b) \to 0 \quad \text{uniformly as } (b-a) \searrow 0,$$

This concludes the proof. ■

The next proposition describes the importance of the area discrepancy to Theorem 3.2; in fact, it shows that without it, there may not be a C^m horizontal extension of (F, G, H), where (F, G, H) is as in the theorem.

Proposition 3.6. *Let $m \in \mathbb{N}$. There exist a compact set $K \subset \mathbb{R}$ and three jets $\boldsymbol{F} := (F^k)_{k=0}^m$, $\boldsymbol{G} := (G^k)_{k=0}^m$, and $\boldsymbol{H} := (H^k)_{k=0}^m$ of order m on K such that:*
(3.6.1) $\boldsymbol{F}$, $\boldsymbol{G}$, and $\boldsymbol{H}$ are Whitney fields of class C^m on K; and
(3.6.2) For each $k \in \mathbb{N}_m^*$, we have

$$H^k(t) = \mathcal{P}^k\left(F^0(t), G^0(t), F^1(t), G^1(t), \ldots, F^k(t), G^k(t)\right).$$

However, there is no C^m horizontal curve $(f, g, h) : \mathbb{R} \to \mathbb{H}$ which extends $(\boldsymbol{F}, \boldsymbol{G}, \boldsymbol{H})$.

Proof. For each $n \in \mathbb{N}$ let $c_n := 1 - 2^{-n}$ and $d_n := 1 - \dfrac{3}{4}2^{-n}$. Then $c_n < d_n$ for each $n \in \mathbb{N}$. Let

$$K := \bigcup_{n=0}^{\infty} [c_n, d_n] \cup \{1\},$$

which is a compact subset of $\mathbb{R}$. For each $k \in \mathbb{N}_m$ and each $t \in K$, define $F^k(t) := 0$ and $G^k(t) := 0$. Define also

$$H^0(t) := \begin{cases} 3^{-mn}, & \text{if } t \in [c_n, d_n], \\ 0, & \text{if } t = 1, \end{cases}$$

and define $H^k(t) = 0$ for all $k \in \mathbb{N}_m^*$ and all $t \in K$. Since $\boldsymbol{F}$ and $\boldsymbol{G}$ are identically 0 on K, they satisfy the definition of Whitney field and (3.6.2).

For each $k \in \mathbb{N}_m^*$ and any $a, b \in K$, the remainder $(\boldsymbol{R_a^m H})^k$ is identically 0 on K. Moreover, we have that

$$\frac{(\boldsymbol{R_a^m H})^0(b)}{|b-a|^m} = \frac{H^0(b) - H^0(a)}{|b-a|^m}. \tag{3.16}$$

To finish showing that $\boldsymbol{H}$ is a Whitney field of class C^m on K, we must show that the quantity in the righthand side of (3.16) approaches 0 uniformly as $(b-a) \to 0$. Thus, fix $\varepsilon > 0$, and fix $n \in \mathbb{N}$ such that $4^m(2/3)^{mn} < \varepsilon$. Set

$$\delta := 2^{-(n+2)} > 0. \tag{3.17}$$

Suppose $a, b \in K$ such that $|b-a| < \delta$. If there exists $k \in \mathbb{N}$ such that $a, b \in [c_k, d_k]$ (i.e., a and b lie in the same interval), then $(R_a^m H)^0(b) = 0$. If a and b lie in different intervals, then there exist $k, \ell \in \mathbb{N}$ such that, say, $\ell > k$ with $a \in [c_k, d_k]$ and $b \in [c_\ell, d_\ell]$, then $k \geq n$. For, if $k < n$, then [see Zim18, p. 7]

$$|b-a| \geq c_\ell - d_k \geq c_{k+1} - d_k = 2^{-(k+2)} > 2^{-(n+2)},$$

which contradicts the assumption that $|b-a| < \delta$.

Therefore,

$$\frac{H^0(b) - H^0(a)}{|b-a|^m} \leq \frac{3^{-mk} - 3^{-m\ell}}{(c\ell - d_k)^m} \leq \frac{3^{-mk}}{(c_{k+1} - d_k)^m} = 4^m \left(\frac{2}{3}\right)^{mk} < \varepsilon,$$

and similarly if $a = 1$ or $b = 1$. We conclude that $\boldsymbol{H}$ is a Whitney field, so that (3.6.1) is true.

Now, suppose by the classical Whitney extension theorem that $(f, g, h) : \mathbb{R} \to \mathbb{H}$ is a C^m curve extending $(\boldsymbol{F}, \boldsymbol{G}, \boldsymbol{H})$ and that (f, g, h) is horizontal in the Heisenberg group. Then by Proposition 3.4, for any $a, b \in K$ we have that $A(a,b)/V(a,b) \to 0$ uniformly as $(b-a) \to 0$. But also, for any $a, b \in K$,

$$A(a,b) = H^0(b) - H^0(a) \quad \text{and} \quad V(a,b) = (b-a)^{2m}.$$

Therefore, we have that

$$|c_{n+1} - d_n| = 2^{-(n+2)} \to 0$$

but also, as $n \to \infty$, that

$$\frac{A(c_{n+1}, d_n)}{V(c_{n+1}, d_n)} = \frac{3^{-mn} - 3^{-m(n+1)}}{4^{-m(n+2)}} = \frac{(3^m - 1)\, 16^m}{3^m} \left(\frac{4}{3}\right)^{mn} \to \infty,$$

which contradicts Proposition 3.4. Therefore, there is no C^m horizontal curve extending $(\boldsymbol{F}, \boldsymbol{G}, \boldsymbol{H})$. ■

3.2. **Sufficiency of the criteria.** The proof of sufficiency of the criteria in (3.2.1), (3.2.2), and (3.2.3) in Theorem 3.2 is much more lengthy and difficult than the proof of the necessity. We shall state the sufficiency criteria as a theorem and then outline the proof.

Definition 3.7. A function $\alpha : [0, \infty) \to [0, \infty)$ is a *modulus of continuity* if α is increasing, $a(0) = 0$, and $a(t) \to 0$ as $t \searrow 0$.

For the proof of the following two lemmas, see Theorem 2.8 on page 150 of [Bie80].

Lemma 3.8. *Let $m \in \mathbb{N}$ and $\boldsymbol{F} := (F^k)_{k=0}^\infty$ be a C^m Whitney field on $K \subset \mathbb{R}$. Then there exists a modulus of continuity α such that, for any $k \in \mathbb{N}_m$ and any $a, x \in K$,*

$$\left|(R_a^m F)^k(x)\right| \leq \alpha(|x-a|)\, |x-a|^{m-k}$$

Lemma 3.9. *Let $m \in \mathbb{N}$, let $K \subset \mathbb{R}$, and let $U \subset \mathbb{R}$ be a bounded, open set such that $K \subset U$. Let also $f := WF$ be the Whitney extension constructed in Theorem 2.8 of [Bie80]. Then there exists $C \geq 0$ such that for all $a \in K$, all $x \in U$, and all $k \in \mathbb{N}_m$ that*

$$\left|D^k f(x) - D^k(T_a^m F)(x)\right| \leq C\alpha(|x-a|)\,|x-a|^{m-k}\,.$$

Theorem 3.10. *Let $m \in \mathbb{N}$, let $K \subset \mathbb{R}$ be a compact set, and let $\boldsymbol{F} := (F^k)_{k=0}^m$, $\boldsymbol{G} := (G^k)_{k=0}^m$, and $\boldsymbol{H} := (H^k)_{k=0}^m$ be three jets of order m on K. Assume that each of the following holds:*

(3.10.1) $\boldsymbol{F}$, $\boldsymbol{G}$, *and* $\boldsymbol{H}$ *are Whitney fields of class C^m on K.*

(3.10.2) For every $k \in \mathbb{N}_m^*$ at every $t \in K$, we have

$$H^k(t) = \mathcal{P}^k\left(F^0(t), G^0(t), F^1(t), G^1(t), \ldots, F^k(t), G^k(t)\right).$$

(3.10.3) We have $a, b \in K$, $A(a,b)/V(a,b) \to 0$ uniformly as $(b-a) \searrow 0$ with $a, b \in K$.

Then the triple $(\boldsymbol{F}, \boldsymbol{G}, \boldsymbol{H})$ extends to a C^m horizontal curve $(f,g,h) : \mathbb{R} \to \mathbb{H}$.

Sketch of proof. We proceed to outline the steps to prove this theorem.

Suppose that K, F, G, and H are as in the statement of the theorem and that they satisfy (3.10.1), (3.10.2), and (3.10.3). Let I be the compact interval

$$[\min(K), \max(K)].$$

Before continuing, we observe the following. Suppose $(f,g,h) : I \to \mathbb{H}$ is a C^m horizontal curve extending $(\boldsymbol{F}, \boldsymbol{G}, \boldsymbol{H})$. Then $f : I \to \mathbb{R}$ is C^m on I, so let $A := (-\infty, \min K)$, and define a function $f_{0,\min} : A \to \mathbb{R}$ by

$$f_{0,\min}(t) := D^m f(\min K).$$

Now, for every $k \in \mathbb{N}_m$, we have that $D^{m-k} f$ exists and is continuous on A; thus, for every $k \in \mathbb{N}_m^*$ we define $f_{k,\min} : A \to \mathbb{R}$ by

$$f_{k,\min}(t) = D^{m-k} f(\min K) + \int_{\min K}^{t} f_{k-1,\min}(\tau)\,d\tau.$$

Now, we define the set $B := (\max K, \infty)$ and the function $f_{0,\max} : B \to \mathbb{R}$ by

$$f_{0,\max}(t) = D^m f(\max K),$$

and for each $k \in \mathbb{N}_m^*$, we define the functions $f_{k,\max} : B \to \mathbb{R}$ similarly to how we defined $f_{k,\min}$ with all instances of $\min K$ replaced by $\max K$.

Now, we construct the function $f^* : \mathbb{R} \to \mathbb{R}$ defined by

$$f^*(t) = \begin{cases} f_{0,\min}(t), & \text{if } t \in A, \\ f(t), & \text{if } t \in I, \\ f_{0,\max}(t), & \text{if } t \in B. \end{cases}$$

Then f^* is a C^m extension of f to all of $\mathbb{R}$: for each $\ell \in \mathbb{N}_m^*$, we have that $D^\ell f^*$ exists and is continuous. In a similar manner, we may construct a C^m function g^* which extends the domain of g to all of $\mathbb{R}$. Finally, we may use Lemma 2.5 to construct a C^m function h^* which extends the domain of h to all of $\mathbb{R}$ and is such that $(f^*, g^*, h^*) : \mathbb{R} \to \mathbb{R}$ extends (f,g,h) and is C^m and horizontal. Hence, to finish proving the theorem, it suffices to find a C^m horizontal extension $(f,g,h) : I \to \mathbb{R}$.

Now, given the aforementioned construction of I, for each $i \in \mathbb{N}$ there exist $a_i, b_i \in K$ such that $a_i < b_i$, that (a_i, b_i) is disjoint from (a_j, b_j) for all $i, j \in \mathbb{N}$ with $i \neq j$, and that

$$I \setminus K = \bigcup_{i=1}^{\infty} (a_i, b_i).$$

Step 1. Using the classical Whitney Extension Theorem 1.10, choose two functions $f, g : I \to \mathbb{R}$ of class C^m on K, such that for every $k \in \mathbb{N}_m$ and every $x \in K$, we have that $D^k f(x) = F^k(x)$ and that $D^k g(x) = G^k(x)$; additionally, choose f and g such that they are of class $C^\infty(I \setminus K)$.

Step 2. Using Definition 1.5, we have for any $k \in \mathbb{N}_m^*$ that

$$D^k(T_a^m \boldsymbol{F})(x) = \sum_{\ell=0}^{m-k} \frac{F^{k+\ell}(a)}{\ell!}(x-a)^\ell,$$

and similarly for $D^k(T_a^m \boldsymbol{G})(x)$. Hence, by Lemma 3.8, we may assume that there exists a modulus of continuity α such that for every $k \in \mathbb{N}_m$, and any $a, x \in K$,

$$\left|F^k(x) - D^k(T_a^m \boldsymbol{F})(x)\right| \leq \alpha(|x-a|)\,|x-a|^{m-k}\,, \tag{3.18}$$

and similarly,

$$\left|G^k(x) - D^k(T_a^m \boldsymbol{G})(x)\right| \leq \alpha(|x-a|)\,|x-a|^{m-k}\,. \tag{3.19}$$

By Lemma 3.9, there exists $C \geq 0$ such that for every $a \in K$, every $x \in I$, and every $k \in \mathbb{N}_m$,

$$\left|D^k f(x) - D^k(T_a^m \boldsymbol{F})(x)\right| \leq C\alpha(|x-a|)\,|x-a|^{m-k}\,, \tag{3.20}$$

and similarly,

$$\left|D^k g(x) - D^k(T_a^m \boldsymbol{G})(x)\right| \leq C\alpha(|x-a|)\,|x-a|^{m-k}\,. \tag{3.21}$$

Combining (3.18)–(3.21), and scaling the value of α by some constant $\hat{C}_{F,G,K,m} \geq 0$ such that $\alpha(t) \to 0$ as $t \searrow 0$, we may assume for every $x \in K$ that

$$\left|D^k f(x) - D^k f(a)\right| \leq \alpha(|x-a|) \tag{3.22}$$

and that

$$\left|D^k g(x) - D^k g(a)\right| \leq \alpha(|x-a|)\,. \tag{3.23}$$

Additionally, by (3.10.3), choose α such that for any $a, b \in K$ with $a < b$,

$$A(a,b) \leq \alpha(b-a)V(a,b). \tag{3.24}$$

Step 3. We assert that there exists a modulus of continuity β satisfying $\beta \geq \alpha$ and that the following holds: for each interval $[a_i, b_i]$ there are two C^∞ functions $\varphi, \psi : [a_i, b_i] \to \mathbb{R}$ such that the following conditions are true:

(i) For each $k \in \mathbb{N}_m$, we have

$$D^k\varphi(a_i) = D^k\varphi(b_i) = D^k\psi(a_i) = D^k\psi(b_i) = 0,$$

and

(ii) For each $k \in \mathbb{N}_m$, we have that

$$\max\left(\left\{\left|D^k\varphi(x)\right| : x \in [a_i, b_i]\right\} \cup \left\{\left|D^k\psi(x)\right| : x \in [a_i, b_i]\right\}\right),$$

is at most $\beta(b_i - a_i)$, and

(iii) We have

$$H(b_i) - H(a_i) = 2\int_{a_i}^{b_i} [(f+\varphi)'(x)(g+\psi)(x) - (g+\psi)'(x)(f+\varphi)(x)]\ dx$$

Proving that it suffices to consider the case $F(a) = G(a) = H(a) = 0$ is analogous to the proof of Claim 3.5. We then construct the functions φ and ψ in two cases: one in which $(T_a^m f)'$ or $(T_a^m g)'$ is large enough on average to allow for the creation of a controlled perturbation of g or f, respectively, that encloses the prescribed area; and the other in which $(T_a^m f)'$ and $(T_a^m g)'$ are small on average so that A is also small. Then, both f and g may be perturbed slightly so that they satisfy (iii) of Step 3. The details of this construction are found in lemmas 6.5 and 6.6 of [PSZ19].

Step 4. For each $i \in \mathbb{N}$, we construct the C^m horizontal extension

$$(\mathcal{F}_i, \mathcal{G}_i, \mathcal{H}_i) : [a_i, b_i] \to \mathbb{H}$$

by defining, for each $x \in [a_i, b_i]$,

$$\begin{aligned}
\mathcal{F}_i &:= f + \varphi, \\
\mathcal{G}_i &:= g + \psi \\
\mathcal{H}_i(x) &:= H(a_i) + 2\int_{a_i}^{x} (\mathcal{F}_i'(t)\mathcal{G}_i(t) - \mathcal{F}_i(t)\mathcal{G}_i'(t))\, dt.
\end{aligned}$$

The horizontality of $(\mathcal{F}_i, \mathcal{G}_i, \mathcal{H}_i)$ follows from the definition of horizontal lift, and that $(\mathcal{F}_i, \mathcal{G}_i, \mathcal{H}_i)$ is C^m follows from the fact that f, g, φ, and ψ are at least C^m. We conclude also, for each $k \in \mathbb{N}_m$, that $D^k\mathcal{F}_i$ evaluated at the endpoint a_i (or b_i) takes the same value as does F^k evaluated at either endpoint, and similarly for $D^k\mathcal{G}_i$ and $D^k\mathcal{H}_i$. Also, we estimate, for each $k \in \mathbb{N}_m$ and each $x \in [a_i, b_i]$,

$$\left|D^k\mathcal{F}_i(x) - F^k(a_i)\right| \le 2\beta(b_i - a_i), \tag{3.25}$$

and similarly,

$$\left|D^k\mathcal{G}_i(x) - G^k(a_i)\right| \le 2\beta(b_i - a_i). \tag{3.26}$$

Step 5. Define the curve $\Gamma := (\mathcal{F}, \mathcal{G}, \mathcal{H}) : I \to \mathbb{H}$ by

$$\Gamma(x) := \begin{cases} (F(x), G(x), H(x)) & \text{if } x \in K, \\ (\mathcal{F}_i(x), \mathcal{G}_i(x), \mathcal{H}_i(x)) & \text{if } \exists i \in \mathbb{N} \text{ s.t. } x \in (a_i, b_i). \end{cases}$$

Then Γ is our desired C^m horizontal extension of $(\boldsymbol{F}, \boldsymbol{G}, \boldsymbol{H})$. Clearly it is C^m in each subinterval (a_i, b_i); that it is C^m elsewhere in I follows from the definitions of φ and of ψ and estimates (3.25) and (3.26). Moreover it is horizontal in the Heisenberg group, since (F, G, H) is horizontal by hypothesis and since $(\mathcal{F}_i, \mathcal{G}_i, \mathcal{H}_i)$ is horizontal on each subinterval (a_i, b_i).

■

Together, Proposition 3.4 and Theorem 3.10 prove the main result, Theorem 3.2.

4. Two more extension theorems

We move to discuss two extension theorems that hold in spaces which are more general than the Heisenberg group.

4.1. **Lipschitz continuity and the McShane extension theorem for metric spaces.** Earlier in proving that continuous differentiability implies absolute continuity, we used Lipschitz continuity as an intermediate step. We now generalize our previous definition.

Definition 4.1. Let (X, d_X) and (Y, d_Y) be two metric spaces. We say a function $f : X \to Y$ is *Lipschitz continuous* if there exists $C \geq 0$ such that for any $x_1, x_2 \in X$, we have

$$d_Y(f(x_1), f(x_2)) \leq C \cdot d_X(x_1, x_2).$$

The number C is called a *Lipschitz constant for* f. If $\mathcal{C}$ is the collection of all Lipschitz constants for f, we say that $K := \min \mathcal{C}$ is the *optimal Lipschitz constant for* f or that f *is* C*-Lipschitz*.

Example 4.2. The function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = |x|$ is 1-Lipschitz: for every $x, y \in \mathbb{R}$, we have that

$$|f(x) - f(y)| = ||x| - |y|| \leq |x - y|$$

by the reverse triangle inequality.

Suppose $C > 0$ such that $C < 1$; then f is not C-Lipschitz. To see this, choose $x_0 := 2C \in \mathbb{R}$ and $y_0 := C \in \mathbb{R}$, so that $0 < |x_0 - y_0| = C < 1$ and

$$\begin{aligned} |f(x_0) - f(y_0)| &= ||2C| - |C|| \\ &= |2\,|C| - |C|| \\ &= |C| \\ &> C^2 && \text{because } 0 < C < 1 \\ &= C \cdot |x_0 - y_0|\,. \end{aligned}$$

Hence, for every C with $0 < C < 1$, there are $x_0, y_0 \in \mathbb{R}$ such that

$$|f(x_0) - f(y_0)| > C \cdot |x_0 - y_0|$$

We conclude that f is not C-Lipschitz for any $C > 0$ such that $C < 1$.

Therefore, f is 1-Lipschitz. △

Example 4.3. The function $g : [0, 1] \to \mathbb{R}$ defined by $g(x) = \sqrt{x}$ is *not* Lipschitz continuous on $[0, 1]$. To see this, first let $K > 0$. If also $K < 1$, then choose $x_0 := 0$ and $y_0 := 1$. Then

$$|g(x_0) - g(y_0)| = 1 > K = K \cdot |x - y|\,.$$

Otherwise, let $x_0 = 0$ and $y_0 := \dfrac{1}{4K^2} \in [0, 1]$. Then

$$|g(x_0) - g(y_0)| = \frac{1}{2K} > \frac{1}{4K} = \frac{K}{4K^2} = K \cdot |x_0 - y_0|\,.$$

△

Recall from advanced calculus that, if A is a set and if $f, g : A \to \mathbb{R}$ are two bounded functions, and if $c > 0$, then

$$\left|\sup_A f - \sup_A g\right| \leq \sup_A |f - g|\,, \tag{4.1}$$

and, if $c > 0$, then

$$\sup_A (cf) = c \sup_A f. \tag{4.2}$$

We now state and prove the McShane extension theorem, found in [McS34], which guarantees that a real-valued Lipschitz continuous function with Lipschitz constant $C > 0$ defined on a subset of a metric space can be extended to the entire space and that the extension has the same Lipschitz constant.

Theorem 4.4 (McShane extension theorem)**.** *Let (X, d) be a metric space, let $E \subset X$, and let $f : E \to \mathbb{R}$ be a C-Lipschitz function for some $C > 0$. Then there exists a C-Lipschitz function $\varphi : X \to \mathbb{R}$ such that for every $x \in E$, we have $\varphi|_E(x) = f(x)$.*

Proof. Define a function $\varphi : X \to \mathbb{R}$ by

$$\varphi(x) = \sup \{f(y) - C \cdot d(y, x) \colon y \in E\}. \tag{4.3}$$

Suppose first that $x \in E$. Then by Definition 4.1, we have for any $y \in E$ that

$$f(y) - C \cdot d(y, x) \leq f(x).$$

Moreover, if $y = x$, then $f(y) = f(x)$ so that, by (4.3), $\varphi(x) = f(x)$ for all $x \in E$.

Now, let $x_1, x_2 \in X$, and define two more functions $\psi_1, \psi_2 : E \to \mathbb{R}$ by

$$\psi_1(y) := f(y) - C \cdot d(y, x_1) \quad \text{and} \quad \psi_2(y) := f(y) - C \cdot d(y, x_2).$$

Without loss of generality, assume that $\varphi(x_2) \geq \varphi(x_1)$. It follows that $d(y, x_1) \geq d(y, x_2)$ for all $y \in E$. Moreover,

$$\begin{aligned} |\varphi(x_2) - \varphi(x_1)| &= \left| \sup_E \psi_2 - \sup_E \psi_1 \right| \\ &\leq \sup_E |\psi_2 - \psi_1| && \text{by (4.1)} \\ &= \sup \{C \cdot d(y, x_1) - C \cdot d(y, x_2) \colon y \in E\} \\ &= C \cdot \sup \{d(y, x_1) - d(y, x_2) \colon y \in E\} && \text{by (4.2).} \end{aligned} \tag{4.4}$$

Since $d(y, x_1) \geq d(y, x_2)$ and since

$$d(y, x_1) \leq d(x_1, x_2) + d(x_2, y)$$

by the triangle inequality, we have that

$$\begin{aligned} C \cdot \sup \{d(y, x_1) - d(y, x_2) \colon y \in E\} &\leq C \cdot \sup \{d(x_1, x_2) \colon y \in E\} \\ &= C \cdot d(x_1, x_2). \end{aligned}$$

We conclude that φ is a Lipschitz extension of f to all of X. ■

4.2. **The Tietze extension theorem for normal topological spaces.** Finally, we discuss another extension theorem for spaces more general than metric spaces.

Recall that if $(X, \mathcal{T}_X)$ is a topological space, we say it is *normal* if every two disjoint closed subsets of X have disjoint open neighborhoods. Interestingly, any metric space is a normal topological space. Recall that, if (X, d) is a metric space, $S \subset X$, and $x \in X$, then we define

$$d(x, S) := \inf \{d(x, s) \colon s \in S\}. \tag{4.5}$$

Proposition 4.5. *Every metric space is a normal topological space.*

Proof. Let (X, d) be a metric space. We start by proving that if C is a nonempty, closed subset of X, and if $x \in X$, then $d(x, C) = 0$ if and only if $x \in C$. For, if $x \in C$; then $d(x, C) = 0$ by (4.5). Conversely, let $C \subset X$ such that $C \neq \varnothing$ and that C is closed. Let also $x \in C$ with $d(x, C) = 0$. It follows for every $r > 0$ that $B(x, r) \cap C \neq \varnothing$, where $B(x, r)$ denotes the ball with center x and radius r. Therefore, $x \in \overline{C}$, where $\overline{C}$ denotes the closure of C in X. Since $\overline{C} \subset C$, we conclude that $x \in C$, and that $d(x, C) > 0$ for all $x \in X \setminus C$.

Now, let S and T be two nonempty, disjoint, closed subsets of X. For each $x \in S$, define

$$r_x := \frac{d(x, T)}{3},$$

and let $U_x := B(x, r_x)$. Additionally, for each $y \in T$, define

$$r_y := \frac{d(y, S)}{3}.$$

and let $V_y := B(y, r_y)$. Without loss of generality, assume $r_x \geq r_y$. Let also

$$U := \bigcup_{x \in S} U_x \quad \text{and} \quad V := \bigcup_{y \in T} V_y.$$

Then U and V, being unions of open balls, are open in X. Moreover, $S \subset U$ and $T \subset V$.

Now, suppose $U \cap V \neq \varnothing$; then there exists $z_0 \in U \cap V$. Then by definition of U, we have that there exists $x_0 \in S$ such that $z_0 \in U_{x_0}$ so that $d(x_0, z_0) < r_{x_0}$. Similarly, there exists $y_0 \in T$ such that $z_0 \in V_{y_0}$ so that $d(y_0, z_0) < r_{y_0}$. We then have that

$$\begin{aligned} 3r_{x_0} &= d(x_0, T) \\ &\leq d(x_0, y_0) && \text{by (4.5)} \\ &\leq d(x_0, z_0) + d(y_0, z_0) && \text{by the triangle inequality} \\ &< r_{x_0} + r_{y_0} \\ &\leq 2r_{x_0}, \end{aligned}$$

which is absurd. Hence, we have that $U \cap V = \varnothing$. Therefore, any two disjoint, closed subsets of X have disjoint open neighborhoods. We conclude that $(X, \mathcal{T}_d)$ is a normal topological space, where $\mathcal{T}_d$ represents the topology induced by the metric d. ■

We now state an important result known as Urysohn's Lemma, whose proof can be found many places, including in [Liu].

Lemma 4.6. *Let $(X, \mathcal{T}_X)$ be a normal topological space. Let $A, B \subset X$ be closed in X such that $A \cap B = \varnothing$. Then there exists a continuous function $f : X \to [0, 1]$ such that f is identically 0 on A and identically 1 on B.*

We now outline the proof of the Tietze extension theorem for normal topological spaces.

Theorem 4.7 (Tietze extension theorem)**.** *Let $(X, \mathcal{T}_X)$ be a normal topological space and let $A \subset X$ be closed. Let also $a < b$ and $f : A \to [a, b]$ be a continuous function. Then there exists a function $F : X \to [a, b]$ such that F is continuous and $F|_A = f$.*

Sketch of proof. Without loss of generality we may assume $[a,b]=[0,1]$, for otherwise, we may replace f with the function $\hat{f}$ defined by

$$\hat{f}(x)=\frac{f(x)-a}{b-a}.$$

Step 1. Let $k>0$ and let $h:A\to[0,k]$ be a continuous function. Define the sets

$$B:=h^{-1}\left(\left[0,\frac{k}{3}\right]\right) \quad \text{and} \quad C:=h^{-1}\left(\left[\frac{2k}{3},k\right]\right).$$

Then B and C are closed in A and so also in X. Therefore by Lemma 4.6, there exists a continuous function $g:X\to\left[0,\frac{k}{3}\right]$ such that $g(B)=\{0\}$ and $g(C)=\left\{\frac{k}{3}\right\}$.

Step 2. We start with the function $f:A\to[0,1]$; in particular, $k=1$. By the previous step, we have that there exists a continuous function $g_1:A\to\left[0,\frac{1}{3}\right]$, so that the function $f-g_1$ has domain A and codomain $\left[0,\frac{2}{3}\right]$. Apply Step 1 to the function $f-g_1$ with $k=\frac{2}{3}$ to obtain the continuous function g_2. Then $f-g_1-g_2$ has domain A and codomain $\left[0,\left(\frac{2}{3}\right)^2\right]$ with, for every $x\in A$, $g_2(x)\le\frac{1}{3}\cdot\frac{2}{3}=\frac{2}{9}$. Continue inductively to obtain a sequence $(g_\ell)_{\ell\in\mathbb{N}}$ of continuous functions such that, for every $x\in A$ and every $n\in\mathbb{N}$,

$$f(x)-\sum_{k=1}^{n}g_k(x)\le\left(\frac{2}{3}\right)^n$$

and

$$g_n(x)\le\frac{1}{3}\cdot\left(\frac{2}{3}\right)^{n-1}.$$

Step 3. For each $n\in\mathbb{N}$ define

$$s_n:=\sum_{k=1}^{n}g_k.$$

Then $(s_n)_{n\in\mathbb{N}}$ is a Cauchy sequence of continuous functions from X into $\mathbb{R}$. To see this, let $\varepsilon>0$ so that by the Archimedean property of the real numbers, there exists $N_\varepsilon\in\mathbb{N}$ such that $\varepsilon>\frac{4}{3^{N_\varepsilon}}$. Then, for each $n,m\in\mathbb{N}$

such that $m > n+1 > N$, we have for every $x \in X$ that

$$\begin{aligned}
||s_m(x) - s_n(x)|| &\leq \left|\left|\sum_{k=n+1}^{m} g_k(x)\right|\right| \\
&\leq \sum_{k=n+1}^{m} ||g_k(x)|| \\
&\leq \sum_{k=n+1}^{m} \left|\frac{1}{3}\cdot\left(\frac{2}{3}\right)^{k-1}\right| \\
&= \left|\left(\frac{2}{3}\right)^{n} - \left(\frac{2}{3}\right)^{m}\right| \\
&\leq 2\left(\frac{2}{3}\right)^{n} && \text{by the triangle inequality} \\
&\leq 2\left(\frac{2}{3}\right)^{N_\varepsilon} && \text{since } n+1 > N \\
&\leq \frac{4}{3^{N_\varepsilon}} \\
&< \varepsilon.
\end{aligned}$$

Now, since the space of continuous functions from X into $\mathbb{R}$ is complete, we have that there exists a continuous function $F : X \to \mathbb{R}$ such that $s_n \to F$ uniformly. Then F is the desired extension of F. For by step 2, we have for any $n \in \mathbb{N}$ that $||f - s_n||_A \leq \left(\frac{2}{3}\right)^n$; hence, $s_n \to f$ uniformly on A, so that $f = F$ on A.

■

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