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I, Brandon S Sweeting, hereby submit this original work as part of the requirements for the degree of Doctor of Philosophy in Mathematical Sciences.

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Student's name: **Brandon S Sweeting**

This work and its defense approved by:

Committee chair: Leonid Slavin, Ph.D.

Committee member: Michael Goldberg, Ph.D.

Committee member: Nageswari Shanmugalingam, Ph.D.

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Novel Bellman Estimates for A_p Weights

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Author: Brandon Sweeting
Degrees: B.A. Mathematics, 2013, University of South Florida

Committee: Dr. Leonid Slavin
Dr. Michael Goldberg
Dr. Nageswari Shanmugalingam

Abstract

This thesis presents two sets of new sharp estimates in harmonic analysis united by a common theme: they are both related to A_p weights, they are both treated with the use of Bellman functions, and they both develop new technical tools that go beyond established theory. The first part concerns estimates for a family of Carleson sequences related to dyadic A_2 weights; the corresponding Bellman functions do not arise as solutions of a PDE and have a new, non-infinitesimal kind of optimizers. The second part deals with lower L^p-estimates for logarithms of A_∞ weights as an indirect way of estimating the constant of exponential integrability of BMO^p for $0 < p < 1$. The geometry of the underlying PDE is significantly complicated by a lack of regularity in the boundary condition, and the known mechanism relating A_∞ estimates to BMO^p has to be modified to work for this range of p.

Throughout this work we maintain the perspective on Bellman functions as stand-alone objects of study, and emphasize the connection among sharp constants, optimizing functions or sequences, and, in the second part, the structure of developable surfaces.

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Chapter 1

Introduction

This thesis covers a variety of sharp estimates for A_p weights in the dyadic and continuous settings, obtained using Bellman functions. It consists of two parts, each of which presents a joint result with Dr. Leonid Slavin. In addition to obtaining new sharp results, in each part we develop new technical tools to overcome limitations of existing Bellman machinery.

The first part deals with sharp estimates for a family of Carleson sequences related to dyadic A_2 weights in dimension 1. These are accomplished by constructing the corresponding Bellman functions or Bellman majorants. An in-depth earlier study left a prominent special case open. This was because the standard Bellman approach to estimating dyadic sums seeks Bellman functions as solutions of certain PDEs. An equivalent way of putting it is that the optimizing sequences in the resulting sharp inequalities are obtained using infinitesimal extremal splits. However, the solutions of all applicable PDEs were found in the earlier work and none of them covered the open special case. In this part of the thesis, we develop a new non-infinitesimal splitting procedure, which yields new optimizing sequences and new Bellman majorants. These coincide with the actual Bellman functions for a key selection of points in the Bellman domain, which yields the desired sharp estimates.

The second part of the thesis deals with sharp estimates for continuous (i.e., non-dyadic) A_∞ weights on the line. These estimates are used to obtain new best bounds for the John–Nirenberg constant of BMO^p, $0 < p < 1$. Bellman functions on this space cannot be evaluated directly, so one instead estimates p-oscillations of logarithms of A_∞ weights from below and examines the behavior of the corresponding Bellman function defined on the A_∞ domain. Our work here is an extension of earlier studies that dealt with $p \geqslant 1$. However, while the generalized Monge-Ampère PDE our Bellman functions solve is the same as before, the geometry of the solution is much more complicated for our range of p, due to a lack of regularity in the boundary condition. This produces new geometrical configurations that go beyond established theory. Furthermore, an earlier theorem that allowed one to obtain estimates on BMO^p as certain inverses of estimates on A_∞ fails to yield non-trivial results; we prove a new, sharper version that does yield new best estimates.

Chapter 2

Dyadic A_2 weights, Carleson Sequences, and Non-infinitesimal Bellman functions

2.1 Setting, preliminaries, and main questions

We begin with some definitions. Let $\mathcal{D}$ be the collection of all open dyadic intervals on $\mathbb{R}$, i.e., intervals of the form $(2^{-k}j,\, 2^{-k}(j+1)),\ j,k \in \mathbb{Z}$. For an interval $I \in \mathcal{D}$, let $\mathcal{D}(I)$ be the collection of all dyadic subintervals of I, and $\mathcal{D}_n(I)$ be the collection of the dyadic subintervals of I of the n-th generation, $\mathcal{D}_n(I) = \{J : J \in \mathcal{D}(I), |J| = 2^{-n}|I|\}$.

A weight on $\mathbb{R}$ is a locally integrable function that is positive almost everywhere. Our weights will be assumed to belong to the dyadic Muckenhoupt class A_2^d. Let $\langle w\rangle_J$ be the average of a weight w over an interval J,

$$\langle w\rangle_J := \frac{1}{|J|}\int_J w(t)\,dt.$$

A weight w is said to belong to A_2^d, written $w \in A_2^d$, if

$$[w]_{A_2^d} := \sup_{J\in\mathcal{D}} \langle w\rangle_J \langle w^{-1}\rangle_J < \infty, \tag{2.1.1}$$

where w^{-1} denotes the reciprocal of w. The quantity $[w]_{A_2^d}$ is called the A_2^d-characteristic of w. Observe that $[w]_{A_2^d} \geqslant 1$ by Jensen's inequality, and $w \in A_2^d$ if and only if $w^{-1} \in A_2^d$, in which case $[w]_{A_2^d} = [w^{-1}]_{A_2^d}$. For $Q \geqslant 1$, let $A_2^{d,Q}$ be the set of all A_2^d weights w with characteristic at most Q:

$$A_2^{d,Q} = \{w \in A_2^d \colon [w]_{A_2^d} \leqslant Q\}. \tag{2.1.2}$$

If $I \in \mathcal{D}$ and the supremum in (2.1.1) is taken over all $J \in \mathcal{D}(I)$ instead of all $J \in \mathcal{D}$, we will write $A_2^d(I)$ and $A_2^{d,Q}(I)$, as appropriate.

A non-negative sequence $\{c_J\}_{J\in\mathcal{D}}$ is called a Carleson sequence if

$$\|\{c_J\}\|_{\mathcal{C}} := \sup_{I\in\mathcal{D}} \frac{1}{|I|} \sum_{J\in\mathcal{D}(I)} c_J < \infty. \tag{2.1.3}$$

The quantity $\|\{c_J\}\|_{\mathcal{C}}$ is called the Carleson norm of $\{c_J\}$.

Take a weight $w \in A_2^d$. Let Φ be a non-negative increasing function on $[1,\infty)$. Consider the following sequence $\{c_J^\Phi(w)\}_{J\in D}$:

$$c_J^\Phi(w) = |J|\, \Phi\big(\langle w\rangle_J \langle w^{-1}\rangle_J\big) \left[\frac{(\Delta_J w)^2}{\langle w\rangle_J^2} + \frac{(\Delta_J w^{-1})^2}{\langle w^{-1}\rangle_J^2}\right], \tag{2.1.4}$$

where $\Delta_J w = \langle w\rangle_{J_-} - \langle w\rangle_{J_+}$, and $J^\pm$ are the two halves of J. We are looking to estimate sharply $\|\{c_J^\Phi(w)\}\|_{\mathcal{C}}$ for $w \in A_2^d$. Specifically, the goal is to find the sharp upper bounds in terms of the characteristic $[w]_{A_2^d}$, or, equivalently, find the best (smallest) function

$K_\Phi(\cdot)$ in the inequality:

$$\sup_{I\in D}\frac{1}{|I|}\sum_{J\in D(I)} c_J^\Phi(w) \leqslant K_\Phi\big([w]_{A_2^d}\big). \tag{2.1.5}$$

The need for such sharp estimates arises in applications when one uses the Carleson embedding theorem in its various forms. One such form is given by the so-called Carleson Lemma, whose proof can be found in [MP13]:

Lemma 2.1.1 (Carleson Lemma)**.** *A sequence $\{c_J\}_{J\in\mathcal{D}}$ is a Carleson sequence with norm B if and only if for all non-negative, measurable functions F on the line,*

$$\sum_{J\in\mathcal{D}} c_J \inf_{x\in J} F(x) \leqslant B\int_{\mathbb{R}} F(x)\,dx.$$

Sequences similar to (2.1.4) have been used to obtain weighted bounds for dyadic paraproducts or square functions ([Bez08; MP13; Wit00]); in such situations Φ is usually a power function, i.e. $\Phi(t) = t^\alpha$ for some $\alpha \geqslant 0$. In [Sla16] Lemma 2.1.1 was used in conjunction with sharp estimate (2.1.5) for a class of functions Φ to obtain new bounds in a larger family of inequalities for dyadic weights. In addition, these sequences provide equivalent definitions of A_2^d: as shown in [Sla16], for any increasing Φ, $w \in A_2^d$ if and only if $\{c_J^\Phi(w)\}$ is Carleson, and inequality (2.1.5) quantifies one side of this relationship.

Beyond immediate applications, given modern tools and specifically Bellman functions, the family $\{c_J^\Phi(w)\}$ presents a fascinating stand-alone object of study, and the corresponding sharp bounds (2.1.5) yield a rich picture worthy of a detailed investigation. (As one manifestation of this, it is shown in [Sla16] that for large classes of functions Φ, inequalities (2.1.5) are extremized by the same sequences of weights, which allows for preservation of sharp estimates under polynomial algebra. The new results presented here provide additional support for this notion.) This stand-alone study was initiated and partly carried out in [Sla16]. To compute the function K_Φ one defines the corresponding upper Bellman function:

$$\boldsymbol{B}_{Q,\Phi}(x_1,x_2)=\sup\Big\{\frac{1}{|I|}\sum_{J\in\mathcal{D}(I)}c_J^{\Phi}(w):\ w\in E_{Q,x,I}\Big\}\tag{2.1.6}$$

where $x=(x_1,x_2)$ and

$$E_{Q,x,I}=\big\{w\colon\ w\in A_2^{d,Q}(I),\ \langle w\rangle_I=x_1,\ \langle w^{-1}\rangle_I=x_2\big\}.\tag{2.1.7}$$

The function $\boldsymbol{B}_{Q,\Phi}$ is naturally defined on the planar domain

$$\Lambda_Q=\{(x_1,x_2)\colon\ 1\leqslant x_1x_2\leqslant Q\}.\tag{2.1.8}$$

That is to say, Λ_Q is precisely the set of $x\in\mathbb{R}^2$ such that each $E_{Q,x,I}$ is non-empty and which contains the point $(\langle w\rangle_I,\langle w^{-1}\rangle_I)$ for all $w\in A_2^{d,Q}(I)$. The inequality on the left in (2.1.8) is given by Jensen's inequality, while the one on the right is due to the assumption $w\in A_2^{d,Q}(I)$. The elements of $E_{Q,x,I}$ are referred to as admissible or test functions. Furthermore, $\boldsymbol{B}_{Q,\Phi}$ satisfies the natural boundary condition:

$$\boldsymbol{B}_{Q,\Phi}(x_1,x_1^{-1})=0,\tag{2.1.9}$$

since if w is a weight such that $\langle w\rangle_I\langle w^{-1}\rangle_I=1$, then w is constant almost everywhere on I, meaning $c_J^{\Phi}(w)$ given by (2.1.4) is zero for each $I\in\mathcal{D}(I)$. We also note that the Bellman function (2.1.6) does not actually depend on I. This can be shown by a rescaling argument, which we will present later in section 2.3.

If $\boldsymbol{B}_{Q,\Phi}$ is known, then we immediately obtain the value $K_\Phi(Q)$ in (2.1.5).

Theorem 2.1.2. *For $Q\geqslant 1$,*

$$K_\Phi(Q)=\sup_{x\in\Lambda_Q}\boldsymbol{B}_{Q,\Phi}(x).\tag{2.1.10}$$

Proof. We adapt an argument from [Sla16]. Take $w \in A_2^d$ such that $[w]_{A_2^d} \leqslant Q$. For any $I \in \mathcal{D}$, by the definition of $\boldsymbol{B}_{Q,\Phi}$,

$$\frac{1}{|I|} \sum_{J \in \mathcal{D}(I)} c_J^\Phi(w) \leqslant \boldsymbol{B}_{Q,\Phi}\big(\langle w\rangle_I, \langle w^{-1}\rangle_I\big) \leqslant \sup_{x \in \Lambda_Q} \boldsymbol{B}_{Q,\Phi}(x).$$

Therefore, $\|\{c_J^\Phi(w)\}\|_{\mathcal{C}} \leqslant \sup_{x \in \Lambda_Q} \boldsymbol{B}_{Q,\Phi}(x)$ and, thus,

$$K_\Phi(Q) = \sup_{[w]_{A_2^d} = Q} \|\{c_J^\Phi(w)\}\|_{\mathcal{C}} \leqslant \sup_{x \in \Lambda_Q} \boldsymbol{B}_{Q,\Phi}(x).$$

To prove the converse inequality, note that by the definition of $\boldsymbol{B}_{Q,\Phi}$, for each $x \in \Lambda_Q$ and any interval $I \in \mathcal{D}$, there exists a sequence of $A_2^d(I)$-weights $\{w_n\}_n$ such that for each n, $[w_n]_{A_2^d(I)} \leqslant Q$, $\langle w_n\rangle_I = x_1$, $\langle w_n^{-1}\rangle_I = x_2$, and

$$\lim_{n \to \infty} \frac{1}{|I|} \sum_{J \in \mathcal{D}(I)} c_J^\Phi(w_n) = \boldsymbol{B}_{Q,\Phi}(x).$$

Let us extend each w_n to all of $\mathbb{R}$; call the extension $\tilde{w}_n$. Let $\tilde{w}_n$ be the appropriately translated copy of w_n on each dyadic interval of length $|I|$, except for one such interval, say $\tilde{I}$, on which change w_n to any A_2^d-weight $\tilde{w}$ such that $\langle \tilde{w}\rangle_{\tilde{I}} = x_1$, $\langle \tilde{w}^{-1}\rangle_{\tilde{I}} = x_2$, and $[\tilde{w}]_{A_2^d(\tilde{I})} = Q$. (Such a weight can be constructed to take only two constant values on $\tilde{I}$; section 2.6 has more complicated examples.) Clearly, $\tilde{w}_n \in A_2^d(\mathbb{R})$ and $[\tilde{w}_n^k]_{A_2^d} = Q$, thus,

$$K_\Phi(Q) \geqslant \frac{1}{|I|} \sum_{J \in \mathcal{D}(I)} c_J^\Phi(\tilde{w}_n) = \frac{1}{|I|} \sum_{J \in \mathcal{D}(I)} c_J^\Phi(w_n) \xrightarrow[n \to \infty]{} \boldsymbol{B}_{Q,\Phi}(x).$$

It remains to take the supremum on the right. $\square$

In section 2.3, we show that our Bellman function has certain homogeneity:

$$\boldsymbol{B}_{Q,\Phi}(x_1, x_2) = \boldsymbol{B}_{Q,\Phi}(\sqrt{x_1 x_2}, \sqrt{x_1 x_2}),$$

thus,

$$\sup_{x \in \Lambda_Q} \boldsymbol{B}_{Q,\Phi}(x) = \sup_{1 \leqslant s \leqslant \sqrt{Q}} \boldsymbol{B}_{Q,\Phi}(s,s).$$

While we are looking for a sharp *upper* bound K_Φ in (2.1.5), one can similarly ask for a sharp lower bound, which would naturally lead to the definition of the lower Bellman function, with infimum replacing supremum in (2.1.6). Such sharp lower estimates were obtained in [Sla16] for any Φ for which the function Ψ given by

$$\Psi(s) = \frac{\Phi(s^2)}{s^2}, \ s \geqslant 1,$$

is either increasing or decreasing, covering most cases of interest. Thus, we focus on upper estimates in this thesis.

In [Sla16], the upper Bellman function $\boldsymbol{B}_{Q,\Phi}$ was computed for all increasing Φ for which the function Ψ is convex on $[1,\infty)$. When $\Phi(t) = t^\alpha$, let us write K_α and $\boldsymbol{B}_{Q,\alpha}$ for K_Φ and $\boldsymbol{B}_{Q,\Phi}$, respectively. The following theorem was proven in [Sla16].

Theorem 2.1.3 ([Sla16])**.** *If Φ is increasing and Ψ is convex, then*

$$K_\Phi(Q) = 16\Phi(Q)\Big(1 - \frac{1}{\sqrt{Q}}\Big) + 8\int_1^Q \frac{\Phi(y)}{y}\Big(1 - \frac{1}{\sqrt{y}}\Big)\,dy.$$

In particular,

$$K_\alpha(Q) = \begin{cases} \dfrac{8(2\alpha+1)}{\alpha}\,Q^\alpha - \dfrac{32\alpha}{2\alpha-1}\,Q^{\alpha-1/2} + \dfrac{8}{\alpha(2\alpha-1)}, & \alpha \in \big(0,\tfrac{1}{2}\big) \cup \big(\tfrac{1}{2},1\big] \cup \big[\tfrac{3}{2},\infty\big); \\ 32Q^{1/2} - 8\log Q - 32, & \alpha = 1/2; \\ 8\log Q, & \alpha = 0. \end{cases} \tag{2.1.11}$$

This left unanswered the question of what happens for Φ with concave Ψ and, in particular, for $\Phi(t) = t^\alpha, \alpha \in (1, \frac{3}{2})$. One can obtain suboptimal estimates for this case, by replacing such a Ψ with a tangential majorant; for $\alpha \in (1, \frac{3}{2})$ this amounts to interpolation between the sharp results for $\alpha = 1$ and $\alpha = \frac{3}{2}$. However, the correct value for K_α was not known until now.

Let us explain the reason. Directly from its definition, the Bellman function $\boldsymbol{B}_{Q,\Phi}$ satisfies a so-called main inequality: for all $x^-, x^+ \in \Lambda_Q$ such that $x := \frac{1}{2}(x^- + x^+) \in \Lambda_Q$,

$$\boldsymbol{B}_{Q,\Phi}(x) \geqslant \frac{1}{2}\boldsymbol{B}_{Q,\Phi}(x^-) + \frac{1}{2}\boldsymbol{B}_{Q,\Phi}(x^+) + \Phi(x_1 x_2)\left[\frac{(x_1^- - x_1^+)^2}{x_1^2} + \frac{(x_2^- - x_2^+)^2}{x_2^2}\right]. \quad (2.1.12)$$

Furthermore, for each $x \in \Lambda_Q$ there exists a split $x = \frac{1}{2}(x^- + x^+)$ such that this inequality becomes equality, or approximate equality, when x^- and x^+ are infinitesimally close. The standard approach is to focus on such infinitesimal splits. This is accomplished by expanding both sides of the inequality to the second order of smallness, turning the resulting differential inequality into a PDE, and then solving this PDE subject to the boundary condition (2.1.9). While the PDE here is non-linear, it is easy to solve due to the homogeneity of the problem. In fact, we have two distinct families of solutions and one expects $\boldsymbol{B}_{Q,\Phi}$ to be found as an element of one of these families. That is exactly what happens for convex Ψ. However, one can show (see section 2.4) that for concave Ψ neither of these solutions of PDE can be the Bellman function (for such Ψ a particular step in the process, the Bellman induction, reverses in direction). The standard Bellman arsenal for dyadic estimates, which relies on infinitesimal extremal splits, is thus unavailable, and a new, non-infinitesimal splitting procedure is necessary. This leads us to a series of questions.

Question 1. What is $K_\alpha(Q)$ for $\alpha \in (1, \frac{3}{2})$?

Question 2. What is the Bellman function $\boldsymbol{B}_{Q,\Phi}$ for $\Phi(t) = t^\alpha$ and $\alpha \in (1, \frac{3}{2})$?

Question 3. Describe the non-infinitesimal splitting procedure and optimizing sequence(s).

Question 4. What about other increasing Φ with concave Ψ, different from power functions?

2.2 Main results

Answer 1. We have obtained a complete answer to Question 1.

Theorem 2.2.1. *For* $\alpha \in (1, \frac{3}{2})$,

$$K_\alpha(Q) = 16\sqrt{Q}\big(\sqrt{Q}-1\big)Q^{\alpha-1} + 8\big(\sqrt{Q}-1\big)^2 \sum_{k=1}^{\infty} 2^{-k}\big((1-2^{-k})\sqrt{Q}+2^{-k}\big)^{2\alpha-2}. \quad (2.2.1)$$

It is easy to check that this expression is strictly larger than what formula (2.1.11) would give for this range of α. On the other hand, for other $\alpha \geqslant 0$, (2.1.11) gives a larger value than (2.2.1). This is illustrated in Figure 2.1 and is, of course, expected behavior for sharp estimates.

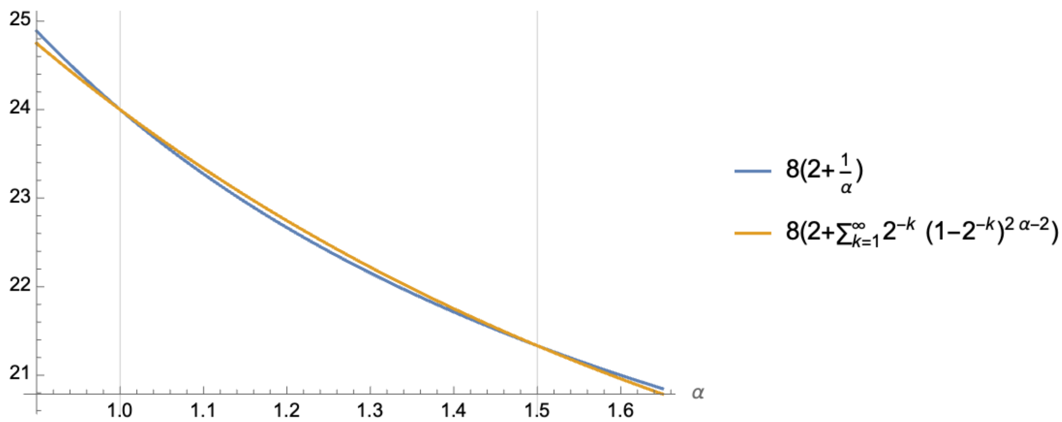

Figure 2.1: Coefficient comparison for dominant terms of $K_\alpha(Q)$.

Formula (2.2.1) was obtained with the use of a *Bellman majorant*. This is a common term for functions that satisfy the main inequality such as (2.1.12), and thus can be used in induction arguments to estimate the original sum, but also may be larger than the

Bellman function itself at some or all points of the domain. Let $s = \sqrt{x_1 x_2}$ and define

$$L = \sqrt{Q}; \quad s_k = s + (1 - 2^{-k})(L - s); \quad r_k = 1 + (1 - 2^{-k})(L - 1).$$

Here is our majorant:

$$B_{Q,\alpha}(x_1, x_2) = 16L(s-1)L^{2\alpha-2} + 8\sum_{k=1}^{\infty} 2^{-k}\big[\big(L-1\big)^2 (r_k)^{2\alpha-2} - \big(L-s\big)^2 (s_k)^{2\alpha-2}\big]. \quad (2.2.2)$$

This function arises as a result of a purported splitting procedure that is, in a way, the opposite of infinitesimal splits. To describe it, first observe that due to the homogeneity it is enough to construct the Bellman function, or a Bellman majorant, only at the points (s, s), $1 \leqslant s \leqslant L$. We postulate that each $s \geqslant \frac{1}{2}(L+1)$ splits so that the right endpoint is at L. Along this split, we require that the the main inequality (2.1.12) hold with equality. Turning this around, each point $s \in [1, L]$ is now the left endpoint of such a split, meaning that the value at that point of the Bellman candidate being constructed is prescribed by the values of the candidate at $\frac{1}{2}(s+L)$ and L. Continuing this procedure and coupling it with the zero boundary condition at (2.1.9) and a natural condition on the derivative at the point L, we obtain the complete candidate (2.2.2). The details of this construction are given in section 2.4. We note that this function satisfies the main inequality (2.1.12), meaning it indeed majorates the actual Bellman function $\boldsymbol{B}_{Q,\alpha}$.

However, it turns out that the procedure described above is overdetermined and can be realized as an actual sequence of weights only for the points r_k. Indeed, it is clear that the candidate (2.2.2) does not even use any values of Φ on the interval $(1, \frac{1}{2}(L+1))$. For $s = r_k$ such a weight sequence does exist and we have a partial answer to Question 2.

Answer 2. For $Q > 1$, let $L = \sqrt{Q}$, $r_k = 2^{-k} + (1 - 2^{-k})L$, $k \geqslant 1$, and $r_\infty = L$.

Theorem 2.2.2. *If* $\alpha \in (1, \frac{3}{2})$*, then*

$$\boldsymbol{B}_{Q,\alpha}(s, s) \leqslant B_{Q,\alpha}(s, s), \quad 1 \leqslant s \leqslant L$$

and

$$\boldsymbol{B}_{Q,\alpha}(r_k, r_k) = B_{Q,\alpha}(r_k, r_k), \quad 0 \leqslant k \leqslant \infty.$$

Answer 3. As mentioned above, for each $s = r_k$ we have an actual optimizing sequence of weights $\{w_n^{(k)}\}_n$ on the interval $I := (0, 1)$. This means that $\langle w_n^{(k)} \rangle_I = \langle (w_n^{(k)})^{-1} \rangle_I = r_k$, $w_n^{(k)} \in A_2^{d,Q}(I)$, and

$$\lim_{n\to\infty} \frac{1}{|I|} \sum_{J \in \mathcal{D}(I)} c_J^{(\alpha)}\big(w_n^{(k)}\big) = \boldsymbol{B}_{Q,\alpha}(r_k, r_k).$$

The weights $w_n^{(k)}$ are defined recursively, in a manner similar to that used for infinitesimal splits in [Sla16], but with important differences accounting for non-infinitesimal nature of the splits.

Answer 4. Our majorant actually satisfies the main inequality (2.1.12) for a larger class of Φ with concave Ψ than just $\Phi(t) = t^\alpha$, $\alpha \in (1, \frac{3}{2})$, though not for every such Φ. Let

$$B_{Q,\Phi}(x_1, x_2) = 16L(s-1)\Psi(L) + 8\sum_{k=1}^{\infty} 2^{-k}\big[\big(L-1\big)^2\Psi(r_k) - \big(L-s\big)^2\Psi\big(s_k\big)\big)\big].$$

Theorem 2.2.3. *If $\Psi \in C^4([1, L])$, such that Ψ and Ψ'' are increasing and concave, then*

$$\boldsymbol{B}_{Q,\Phi}(s, s) \leqslant B_{Q,\Phi}(s, s), \quad 1 \leqslant s \leqslant L$$

and

$$\boldsymbol{B}_{Q,\Phi}(r_k, r_k) = B_{Q,\Phi}(r_k, r_k), \quad 0 \leqslant k \leqslant \infty.$$

This theorem applies to some Φ's that are not power functions. As an example, consider $\Phi(t) = t\log(t)$, then $\Psi(t) = 2\log(t)$. Clearly, Ψ satisfies the conditions of this theorem, and we thus have the following corollary.

Corollary 2.2.4. *For $\Phi(t) = t\log(t)$,*

$$K_\Phi(Q) = 32L\log(L)\big(L-1\big) + 16\big(L-1\big)^2 \sum_{k=1}^{\infty} 2^{-k}\log(r_k).$$

2.3 Necessary conditions on the Bellman candidate

To find a Bellman function such as (2.1.6), one typically determines key properties that it possesses directly from its definition. These properties become conditions imposed on any candidate function being constructed. Once a candidate is found, it is then shown to be equal to the true Bellman function, or at least to bound it from above or below, as appropriate. In this section, we first derive the independence of $\boldsymbol{B}_{Q,\Phi}$ with respect to the interval I. This property is crucial for induction arguments involving $\boldsymbol{B}_{Q,\Phi}$. We then proceed as in [Sla16] and derive three conditions and an induction-on-scales result which together make up the essential properties of $\boldsymbol{B}_{Q,\Phi}$ on Λ_Q. Following this, we use these properties to translate the main inequality from a condition on Λ_Q to one on the interval $[1, L]$, $L = \sqrt{Q}$.

Lemma 2.3.1 (Interval Independence)**.** *Let $\tilde{I}$ be an interval with $A_2^d(\tilde{I})$, $\tilde{\boldsymbol{B}}_{Q,\Phi}$, $E_{Q,x,\tilde{I}}$ and $\tilde{\Lambda}_Q$ the respective analogues of $A_2^d(I)$, $\boldsymbol{B}_{Q,\Phi}$, $E_{Q,x,I}$ and Λ_Q. Then*

$$\forall x \in \Lambda_Q = \tilde{\Lambda}_Q,\ \boldsymbol{B}_{Q,\Phi}(x) = \tilde{\boldsymbol{B}}_{Q,\Phi}(x).$$

Proof. This follows directly from the fact that a linear change of variables $\phi : I \to \tilde{I}$ will preserve both dyadic lattices and integral averages. Indeed, let ϕ be such a function and define:

$$\tilde{w} = w \circ \phi; \qquad \tilde{J} = \phi(J),\ J \subseteq I.$$

If we assume, without loss of generality, that ϕ is increasing, i.e. $\phi'(t) \equiv c > 0$, then

$$\forall J = (a,b) \subseteq I, \quad |\tilde{J}| = \phi(b) - \phi(a) = c(b-a) = c|J|.$$

Therefore, if $J \in \mathcal{D}(I)$ then there exists an $n \geqslant 0$ such that J partitions I with $2^n - 1$ other intervals of equal measure. From the above, we see that the same will hold for $\tilde{J}$ within $\tilde{I}$, and with the same relative position; hence, $\tilde{J} \in \mathcal{D}(\tilde{I})$ and $\tilde{J}^{\pm} = \phi(J^{\pm})$. Consequently, given any weight $w \in A_2^d(I)$ and interval $J \subseteq I$,

$$\langle w \rangle_J = \frac{1}{|J|} \int_J w(t)\ dt = \frac{1}{c|J|} \int_{\tilde{J}} w(\phi(t))\ dt = \langle \tilde{w} \rangle_{\tilde{J}};$$

thus $J \overset{\phi}{\longleftrightarrow} \tilde{J}$ is a bijection from the dyadic intervals of $\mathcal{D}(I)$ to $\mathcal{D}(\tilde{I})$ on which w and $\tilde{w}$ (along with their reciprocals) have the same average. Since it clearly follows that

$$\frac{1}{|I|} c_J^{\Phi}(w) = \frac{1}{|\tilde{I}|} c_{\tilde{J}}^{\Phi}(\tilde{w}),$$

we have $w \overset{\phi}{\longleftrightarrow} \tilde{w}$ is a bijection from $E_{Q,x,I}$ to $E_{Q,x,\tilde{I}}$ which preserves the value of the sum in definition (2.1.6); hence, $\Lambda_Q = \tilde{\Lambda}_Q$ and

$$\boldsymbol{B}_{Q,\Phi}(x) = \sup_{w \in E_{Q,x,I}} \frac{1}{|I|} \sum_{J \in \mathcal{D}(I)} c_J^{\Phi}(w) = \sup_{\tilde{w} \in E_{Q,x,\tilde{I}}} \frac{1}{|\tilde{I}|} \sum_{\tilde{J} \in \mathcal{D}(\tilde{I})} c_J^{\Phi}(\tilde{w}) = \tilde{\boldsymbol{B}}_{Q,\Phi}(x). \qquad \square$$

With this in hand, we will now prove the following key properties:

Lemma 2.3.2. *The function $\boldsymbol{B}_{Q,\Phi}$ satisfies the following*

1. ***Main Inequality.*** *For all points $x^-, x^+ \in \Lambda_Q$ such that $x = \frac{1}{2}(x^- + x^+) \in \Lambda_Q$,*

$$\boldsymbol{B}_{Q,\Phi}(x) \geqslant \frac{1}{2}\boldsymbol{B}_{Q,\Phi}(x^-) + \frac{1}{2}\boldsymbol{B}_{Q,\Phi}(x^+) + \Phi(x_1 x_2)\left[\frac{(x_1^- - x_1^+)^2}{x_1^2} + \frac{(x_2^- - x_2^+)^2}{x_2^2}\right].$$

2. ***Boundary Condition.*** *For all $x_1 \in (0, \infty)$,*

$$\boldsymbol{B}_{Q,\Phi}(x_1, x_1^{-1}) = 0.$$

3. ***Homogeneity.*** *For all* $x \in \Lambda_Q$,

$$\boldsymbol{B}_{Q,\Phi}(x_1, x_2) = \boldsymbol{B}_{Q,\Phi}(\sqrt{x_1 x_2}, \sqrt{x_1 x_2}).$$

Proof. Fix $x^-, x^+ \in \Lambda_Q$ such that $x = \frac{1}{2}(x^- + x^+) \in \Lambda_Q$ and let $w^\pm \in A_2^{d,Q}(I^\pm)$ be admissible weights for $x^\pm$ for which the supremum of the sum in the Bellman function is almost attained, i.e. for some small, positive η:

$$\frac{1}{2|I^\pm|} \sum_{J \in \mathcal{D}(I^\pm)} c_J^\Phi(w^\pm) \geqslant \boldsymbol{B}_{Q,\Phi}(x^\pm) - \eta.$$

Such a selection is possible since, as previously mentioned, given any interval I and $x \in \Lambda_Q$ the corresponding set $E_{Q,x,I}$ is non-empty; furthermore, it should be noted that there are no constraints on the relationship between these weights (i.e. we choose them independently). We construct a new weight, w, on I by concatenating $w^\pm$, i.e. we define w on $I^\pm$ as $w^\pm$. It follows that w is an admissible weight for $x = (x_1, x_2)$ since

$$\langle w \rangle_I = \frac{1}{2}\langle w^- \rangle_{I^-} + \frac{1}{2}\langle w^+ \rangle_{I^+} = \frac{1}{2}(x_1^- + x_1^+) = x_1, \tag{2.3.1}$$
$$\langle w^{-1} \rangle_I = \frac{1}{2}\langle (w^-)^{-1} \rangle_{I^-} + \frac{1}{2}\langle (w^+)^{-1} \rangle_{I^+} = \frac{1}{2}(x_2^- + x_2^+) = x_2 \tag{2.3.2}$$

and $w \in A_2^{d,Q}(I)$, which follows from the fact that $w^\pm \in A_2^{d,Q}(I^\pm)$ and $x \in \Lambda_Q$. We can split the quantity in the definition of $\boldsymbol{B}_{Q,\Phi}$ as follows:

$$\begin{aligned}\frac{1}{|I|} \sum_{J \in \mathcal{D}(I)} c_J^\Phi(w) &= c_I^\Phi(w) + \frac{1}{2|I^-|} \sum_{J \in \mathcal{D}(I^-)} c_J^\Phi(w) + \frac{1}{2|I^+|} \sum_{J \in \mathcal{D}(I^+)} c_J^\Phi(w) \\ &= c_I^\Phi(w) + \frac{1}{2|I^-|} \sum_{J \in \mathcal{D}(I^-)} c_J^\Phi(w^-) + \frac{1}{2|I^+|} \sum_{J \in \mathcal{D}(I^+)} c_J^\Phi(w^+).\end{aligned}$$

Consequently, for our constructed weight w, we'll have

$$\begin{aligned}\frac{1}{|I|}\sum_{J\in\mathcal{D}(I)} c_J^{\Phi}(w) &\geqslant \frac{1}{2}\boldsymbol{B}_{Q,\Phi}(x^-) + \frac{1}{2}\boldsymbol{B}_{Q,\Phi}(x^-) - \eta + \Phi(\langle w\rangle_I\langle w^{-1}\rangle_I)R_I(w)\\ &= \frac{1}{2}\boldsymbol{B}_{Q,\Phi}(x^-) + \frac{1}{2}\boldsymbol{B}_{Q,\Phi}(x^-) - \eta + \Phi(x_1x_2)\left[\frac{(x_1^- - x_1^+)^2}{x_1^2} + \frac{(x_2^- - x_2^+)^2}{x_2^2}\right].\end{aligned}$$

Taking the supremum over all $w \in E_{Q,x,I}$ gives

$$\boldsymbol{B}_{Q,\Phi}(x) \geqslant \frac{1}{2}\boldsymbol{B}_{Q,\Phi}(x^-) + \frac{1}{2}\boldsymbol{B}_{Q,\Phi}(x^+) - \eta + \Phi(x_1x_2)\left[\frac{(x_1^- - x_1^+)^2}{x_1^2} + \frac{(x_2^- - x_2^+)^2}{x_2^2}\right].$$

Since η can be made arbitrarily small, the main inequality follows.

Now, fix $x \in \Lambda_Q$ with $x_2 = x_1^{-1}$. If is w an admissible weight for this point, i.e. $w \in E_{Q,x,I}$, then $\langle w^{-1}\rangle_I = \langle w\rangle_I^{-1}$ and so w satisfies Jensen's inequality with equality. This can only occur if w is constant almost everywhere, and there is only one such constant function admissible for x, namely $w \equiv x_1$. Therefore, $\forall J \in \mathcal{D}(I)$ we have $\Delta_J(w) := \langle w\rangle_{J^-} - \langle w\rangle_{J^+} = 0$ and thus $\boldsymbol{B}_{Q,\Phi}(x) = 0$. All that remains to show now is homogeneity.

For a fixed $x \in \Lambda_Q$ and $\tau > 0$, let $w \in E_{Q,x,I}$ and define the new weight $w_\tau(t) = \tau w(t)$. It's readily seen that $\langle w_\tau\rangle_J = \tau\langle w\rangle_J$ and $\langle w_\tau^{-1}\rangle_J = \tau^{-1}\langle w^{-1}\rangle_J$, $\forall J \in \mathcal{D}(I)$. Consequently, $w_\tau \in A_2^{d,Q}(I)$, w_τ is admissible for $x_\tau = (\tau x_1, \tau^{-1}x_2)$ and $c_J^{\Phi}(w) = c_J^{\Phi}(w_\tau)$, $\forall J \in \mathcal{D}(I)$. It follows that $w \to w_\tau$ is a bijection from $E_{Q,x,I}$ to $E_{Q,x_\tau,I}$ which preserves the value of the sum in the definition of $\boldsymbol{B}_{Q,\Phi}$; hence, $\boldsymbol{B}_{Q,\Phi}(x) = \boldsymbol{B}_{Q,\Phi}(x_\tau)$. Taking $\tau = \sqrt{x_2/x_1}$ gives $\boldsymbol{B}_{Q,\Phi}(x) = \boldsymbol{B}_{Q,\Phi}(\sqrt{x_1x_2}, \sqrt{x_1x_2})$. □

Lastly, we will need the following lemma.

Lemma 2.3.3 (Bellman Induction)**.** *Let B be a function satisfying the main inequality and boundary condition on Λ_Q, i.e. for all $x_1 \in (0,\infty)$:*

$$B(x_1, x_1^{-1}) = 0,$$

and for all points $x^{\pm} \in \Lambda_Q$ *such that* $\frac{1}{2}(x^- + x^+) \in \Lambda_Q$*,*

$$B(x) \geqslant \frac{1}{2}B(x^-) + \frac{1}{2}B(x^+) + \Phi(x_1x_2)\left[\frac{(x_1^- - x_1^+)^2}{x_1^2} + \frac{(x_2^- - x_2^+)^2}{x_2^2}\right].$$

Then B *majorates the Bellman function* $\boldsymbol{B}_{Q,\Phi}$ *on* Λ_Q*, i.e.*

$$\boldsymbol{B}_{Q,\Phi}(x) \leqslant B(x),\ \forall x \in \Lambda_Q.$$

Proof. We begin by proving a simple estimate on B, namely $B(x) \geqslant 0$, $\forall x \in \Lambda_Q$. Fix $x = (x_1, x_2) \in \Lambda_Q$ and define the points $x^{\pm}$:

$$x_1^{\pm} = x_1\Big(1 \pm \sqrt{1 - \frac{1}{x_1x_2}}\Big), \quad x_2^{\pm} = x_2\Big(1 \mp \sqrt{1 - \frac{1}{x_1x_2}}\Big).$$

Clearly, $x_1^{\pm}x_2^{\pm} = 1$ and $x = \frac{1}{2}(x^+ + x^-)$; thus, by our assumptions on B, we have

$$\begin{aligned} B(x) &\geqslant \frac{1}{2}B(x^-) + \frac{1}{2}B(x^+) + \Phi(x_1x_2)\left[\frac{(x_1^- - x_1^+)^2}{x_1^2} + \frac{(x_2^- - x_2^+)^2}{x_2^2}\right] \\ &= \Phi(x_1x_2)\left[\frac{(x_1^- - x_1^+)^2}{x_1^2} + \frac{(x_2^- - x_2^+)^2}{x_2^2}\right] \geqslant 0. \end{aligned}$$

Now, choose any admissible weight $w \in E_{Q,x,I}$. For $J \in \mathcal{D}(I)$ we define the associated point $b_J(w) := (\langle w\rangle_J, \langle w^{-1}\rangle_J)$. Clearly, $b_I(w) = x$. Since $w \in A_2^{d,Q}(I)$, it follows that $b_J(w) \in \Lambda_Q$ for any $J \in \mathcal{D}(I)$. Furthermore, by (2.3.1) and (2.3.2), we have

$$b_J(w) = \frac{1}{2}b_{J^-}(w) + \frac{1}{2}b_{J^+}(w),\ \forall J \in D(I). \tag{2.3.3}$$

Observe, $b_{I^-}(w)$ and $b_{I^+}(w)$ are two points in Λ_Q whose midpoint $b_I(w) \in \Lambda_Q$. Since B satisfies the main inequality we have

$$B(b_I(w)) \geqslant \frac{1}{2}B(b_{I^-}(w)) + \frac{1}{2}B(b_{I^+}(w)) + \frac{1}{|I|}c_I^{\Phi}(w).$$

Equivalently,

$$|I|B(b_I(w)) \geqslant |I^-|B(b_{I^-}(w)) + |I^+|B(b_{I^+}(w)) + c_I^\Phi(w). \tag{2.3.4}$$

We can iterate this argument by splitting subsequent $b_J(w)$ as in (2.3.3) then applying (2.3.4). Doing so until the n-th generation, i.e. $J \in D_n := \{J \in \mathcal{D}(I) : \; |J| = 2^{-n}|I|\}$, gives:

$$|I|B(b_I(w)) \geqslant \sum_{J \in \mathcal{D}_n(I)} |J|B(b_J(w)) + \sum_{k=0}^{n-1} \sum_{J \in D_k(I)} c_J^\Phi(w),$$

from which it follows that

$$B(x) \geqslant \frac{1}{|I|} \sum_{k=0}^{n-1} \sum_{J \in D_k(I)} c_I^\Phi(w)$$

since $B(x) \geqslant 0$. Taking the limit as $n \to \infty$ gives

$$B(x) \geqslant \frac{1}{|I|} \sum_{J \in \mathcal{D}(I)} c_I^\Phi(w).$$

As w was chosen arbitrarily, this holds over all weights in $E_{Q,x,I}$. Taking the supremum of the right hand side expression over all such weights gives the desired result. □

Let B be a function on Λ_Q and define $A(s) = B(s,s)$ for $s \in [1, L]$. Suppose B satisfies the homogeneity property on Λ_Q and fix $x^\pm \in \Lambda_Q$ such that $x = \frac{1}{2}(x^- + x^+) \in \Lambda_Q$. If we let $s = s(x) = \sqrt{x_1 x_2}$ and $s^\pm = s(x^\pm)$, a quick calculation gives:

$$\frac{(x_1^- - x_1^+)^2}{x_1^2} + \frac{(x_2^- - x_2^+)^2}{x_2^2} = 8 - 4\,\frac{(s^-)^2 + (s^+)^2}{s^2} + \frac{((s^-)^2 - (s^+)^2)^2}{s^4}.$$

The main inequality for $x^\pm$ becomes

$$\begin{aligned} B(x) &\geqslant \frac{1}{2}B(x^-) + \frac{1}{2}B(x^+) + \Phi(x_1x_2)\left[8 - 4\,\frac{(s^-)^2+(s^+)^2}{s^2} + \frac{((s^-)^2-(s^+)^2)^2}{s^4}\right] \\ &= \frac{1}{2}B(x^-) + \frac{1}{2}B(x^+) + \Psi(s)\left[8\,s^2 - 4\,((s^-)^2+(s^+)^2) + \frac{((s^-)^2-(s^+)^2)^2}{s^2}\right]; \end{aligned}$$

or, equivalently,

$$A(s) \geqslant \frac{1}{2}A(s^-) + \frac{1}{2}A(s^+) + \Psi(s)\left[8\,s^2 - 4\,((s^-)^2+(s^+)^2) + \frac{((s^-)^2-(s^+)^2)^2}{s^2}\right]. \quad (2.3.5)$$

It follows that for B to satisfy the main inequality on Λ_Q it is both necessary and sufficient for the function A to satisfy (2.3.5) for some selection of $s, s^\pm \in [1, L]$; this latter observation follows from the fact that the constraints on x, namely $x = \frac{1}{2}(x^- + x^-)$, will result in constraints on s. Let ω_L denote this collection of triples $(s^-, s^+, s) \in [1, L]^3$. We will specify conditions for ω_L so that it contains all (s^-, s^+, s) arising from $x, x^\pm \in \Lambda_Q$ with $x = \frac{1}{2}(x^- + x^+)$. If we freely choose $s^\pm \in [1, L]$, it is clear that s can be made as large as desired in the interval $[1, L]$. (Consider $x^\pm$ on the tangent to the upper boundary of Λ_Q such that $x = (L, L)$.) We are therefore interested in the lower estimate:

$$\min_{x^\pm \in \Lambda_Q} \left\{ \sqrt{\left(\frac{x_1^- + x_1^+}{2}\right)\left(\frac{x_2^- + x_2^+}{2}\right)} : x_1^- x_2^- = (s^-)^2,\ x_1^+ x_2^+ = (s^+)^2 \right\}.$$

We can see that this minimum is attained when all three points $x, x^\pm$ lie on the same line through the origin. Indeed

$$\begin{aligned} \left(\frac{x_1^- + x_1^+}{2}\right)\left(\frac{x_2^- + x_2^+}{2}\right) &= \frac{1}{4}((s^+)^2 + (s^-)^2) + \frac{1}{2}(x_1^- x_2^+ + x_2^- x_1^+) \\ &= \frac{1}{4}((s^+)^2 + (s^-)^2) + \frac{1}{2}\langle x^-,\, \theta(x^+)\rangle, \end{aligned}$$

where θ denotes the reflection through the line $x_2 = x_1$ and $\langle \cdot, \cdot \rangle$ the inner product on $\mathbb{R}^2$; this product is minimized precisely when the vectors x^- and $\theta(x^+)$ are perpendicular, i.e. when $x^\pm$ are parallel. It follows then that $s = \frac{1}{2}(s^- + s^+)$. We therefore take

$$\omega_L = \left\{ (s^-, s^+, s) : \quad 1 \leqslant s^- \leqslant L; \quad 1 \leqslant s^+ \leqslant L; \quad \frac{1}{2}(s^- + s^+) \leqslant s \leqslant L \right\}.$$

If we define

$$P(s^-, s^+, s) = A(s) - \frac{1}{2}\left[A(s^-) + A(s^+)\right] - \Psi(s)\left[8s^2 - 4((s^-)^2 + (s^+)^2) + \frac{((s^-)^2 - (s^+)^2)^2}{s^2}\right],$$

then by the above discussion, to show B satisfies the main inequality on Λ_Q it suffices to show that $P \geqslant 0$ on ω_L. We make a further modification to this problem by defining

$$U(s^-, s^+, s) = A(s) - \frac{1}{2}\left[A(s^+) + A(s^-)\right] - 8\Psi(s)\left[s^2 - s^- s^+\right].$$

It is clear that $P \geqslant U$ on ω_L and that $P = U$ when $s = \frac{1}{2}(s^- + s^+)$. Though a seemingly harder task, we will prove that $U \geqslant 0$ on ω_L as this function is computationally easier to work with.

2.4 Non-infinitesimal candidate

As discussed in the introduction, the standard approach to finding the Bellman function, or a majorant, is to assume the Bellman function satisfies the main inequality with equality, or approximate equality, whenever $x^\pm \in \Lambda_Q$ are infinitesimally close. This gives a PDE which we then solve to produce a candidate function B. This function is then shown to satisfy the main inequality for all other $x^\pm \in \Lambda_Q$, from which it follows that it majorates the Bellman function by Bellman induction. Optimizers, which are sequences that support the value of the candidate, are then constructed for each $x \in \Lambda_Q$ to give the converse inequality $B(x) \leqslant \boldsymbol{B}_{Q,\Phi}(x)$.

However, this approach fails when Ψ is concave. The function U, derived in the previous section, fails to be positive for any triple $(s^-, s^+, \frac{s^-+s^+}{2}) \in \omega_L$—a necessary condition for the main inequality to hold. Specifically, in [Sla16] it was shown that for any candidate arising from this PDE, we have for all $s^\pm \in [1, L]$:

$$A\Big(\frac{s^+ + s^-}{2}\Big) - \frac{1}{2}\left(A(s^+) + A(s^-)\right) - 2\,\Psi\Big(\frac{s^+ + s^-}{2}\Big)(s^+ - s^-)^2 \geqslant 0 \iff$$
$$8\int_{-\Delta}^{\Delta}(\Delta - |t|)\left(\frac{1}{2}\Psi(s+t) + \frac{1}{2}\Psi(s-t) - \Psi(s)\right)\, dt \geqslant 0$$

where $\Delta = \frac{1}{2}(s^+ - s^-)$; for concave Ψ the latter inequality is clearly reversed. Therefore, since Bellman induction implies the existence of a direction in which the Bellman function splits optimally, we must have that these splits are non-infimitesimal.

In this section we construct a special Bellman candidate via a non-infinitesimal splitting procedure which we will show to be equal to the Bellman function at a key selection of points in Λ_Q. By homogeneity, it suffices to construct our candidate as a function of one variable on the interval $[1, L]$ which we will then extend to Λ_Q as follows: if A is our candidate on $[1, L]$, we define the extension $B_{Q,\Phi}(x_1, x_2) := A(\sqrt{x_1 x_2})$.

Our candidate A will be defined as the pointwise limit of a sequence of functions $\{A_n\}$. For every $s \in [1, L]$, each A_n will be constructed so as to satisfy equality in the main equality for a prescribed number of splits to the boundary point L. We recall the notation:

$$s_k = s + (1 - 2^{-k})(L - s); \quad r_k = 1 + (1 - 2^{-k})(L - 1).$$

Fix $s \in \left[1, \frac{L-1}{2}\right)$ and $n \geqslant 1$. We define A_n at the points $\{s_k\}_{k=0}^n$ simultaneously as the solution of the system of equations:

$$A_n(s_k) = \frac{1}{2}A_n(s_{k-1}) + \frac{1}{2}A_n(L) + 2\Psi(s_k)\left[L - s_{k-1}\right]^2, \quad 1 \leqslant k \leqslant n \tag{2.4.1}$$

$$A_n(L) = A_n(s_n) + 8\Psi(s_n)\left[L^2 - s_n^2\right]. \tag{2.4.2}$$

This definition is well-defined in the sense that each $s \in [1, L)$ will belong to a unique system, so there are no redefinitions. (The case of L will be addressed shortly.) Furthermore, as each system consists of $n+1$ equations in $n+2$ unknowns, A_n is fully determined at each $s \in [1, L)$ up to knowledge of $A_n(L)$. When $s = 1$, i.e. $\{s_k\}_{k=0}^n = \{r_k\}_{k=0}^n$, we set $A_n(1) = 0$ to satisfy the boundary condition and solve the resulting fully-determined system, thus fixing $A_n(L)$. This splitting procedure is illustrated below in Figure 2.2.

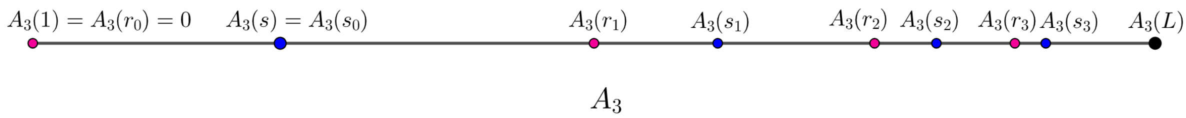

Figure 2.2: The splitting procedure for the intermediate function A_3

We now proceed to solve (2.4.1) and (2.4.2) for A_n and $\{r_k\}_{k=0}^n$. Define:

$$P(s_k) = 2\Psi(s_k)\,[L - s_{k-1}]^2\,; \qquad T(s_k) = 8\Psi(s_k)\,\left[L^2 - s_k^2\right].$$

Then for $1 \leqslant k \leqslant n$, using (2.4.1) repeatedly, we have:

$$\begin{aligned}
A_n(r_k) &= \frac{1}{2}A_n(r_{k-1}) + \frac{1}{2}A_n(L) + P(r_k) \\
&= \frac{1}{2}\left[\frac{1}{2}A_n(r_{k-2}) + \frac{1}{2}A_n(L) + P(r_{k-1})\right] + \frac{1}{2}A_n(L) + P(r_k) \\
&= \frac{1}{4}A_n(r_{k-2}) + \frac{3}{4}A_n(L) + \frac{1}{2}P(r_{k-1}) + P(r_k) \\
&\vdots \\
&= 2^{-k}A_n(r_0) + (1 - 2^{-k})A_n(L) + \sum_{j=1}^{k} 2^{j-k}P(r_j) \\
&= (1 - 2^{-k})A_n(L) + \sum_{j=1}^{k} 2^{j-k}P(r_j).
\end{aligned}$$

Therefore, by (2.4.2)

$$A_n(L) = (1-2^{-n})A_n(L) + \sum_{j=1}^{n} 2^{j-n}P(r_j) + T(r_n),$$

so that,

$$\begin{aligned} A_n(L) &= 2^n T(r_n) + \sum_{j=1}^{n} 2^j P(r_j), \\ A_n(r_k) &= (1-2^{-k})\left[2^n T(r_n) + \sum_{j=1}^{n} 2^j P(r_j)\right] + \sum_{j=1}^{k} 2^{j-k}P(r_j), \quad 1 \leqslant k \leqslant n. \end{aligned}$$

As mentioned, this fixes $A_n(L)$ for all other systems. By a similar argment, we have:

$$A_n(s_k) = 2^{-k}A_n(s_0) + (1-2^{-k})A_n(L) + \sum_{j=1}^{k} 2^{j-k}P(s_j), \quad 1 \leqslant k \leqslant n.$$

We may then solve for $A_n(s) = A_n(s_0)$ explicitly using (2.4.2):

$$\begin{aligned} A_n(s) &= 2^n A_n(s_n) - (2^n - 1)A_n(L) - \sum_{j=1}^{n} 2^j P(s_j) \\ &= 2^n \left[A_n(L) - T(s_n)\right] - (2^n - 1)A_n(L) - \sum_{j=1}^{n} 2^j P(s_j) \\ &= A_n(L) - 2^n T(s_n) - \sum_{j=1}^{n} 2^j P(s_j) \\ &= 2^n \left[T(r_n) - T(s_n)\right] + \sum_{j=1}^{n} 2^j P(r_j) - \sum_{j=1}^{n} 2^j P(s_j). \end{aligned}$$

Noting that $L - s_k = 2^{-k}(L-s)$, we have the general formula for A_n on $[1, L]$:

$$A_n(s) = 16L(s-1)\Psi(L) + 8\sum_{j=1}^{n} 2^{-j}\left[(L-1)^2\Psi(r_j) - (L-s)^2\Psi(s_j)\right]$$

and thus for A as well,

$$A(s)=\lim_{n\to\infty}A_n(s)=16L(s-1)\Psi(L)+8\sum_{j=1}^{\infty}2^{-j}\left[(L-1)^2\Psi(r_j)-(L-s)^2\Psi(s_j)\right].$$

Therefore, our Bellman candidate on Λ_Q is:

$$B_{Q,\Phi}(x_1,x_2):=A(\sqrt{x_1x_2}). \tag{2.4.3}$$

2.5 Main inequality verification

In this section, we prove that the Bellman candidate $B_{Q,\Phi}$ given by (2.4.3) satisfies the main inequality. In section 2.3, it was established that for a function B with homogeneity on Λ_Q, to prove the main inequality it was sufficient to show that the function

$$U(s^-,s^+,s)=A(s)-\frac{1}{2}\left[A(s^+)+A(s^-)\right]-8\Psi(s)\left[s^2-s^-s^+\right]$$

is non-negative on the domain

$$\omega_L=\left\{(s^-,s^+,s):\quad 1\leqslant s^-\leqslant L;\quad 1\leqslant s^+\leqslant L;\quad \frac{1}{2}(s^-+s^+)\leqslant s\leqslant L\right\},$$

where $A(\sqrt{x_1x_2}):=B(x_1,x_2)$. This task is made easier by the following lemma.

Lemma 2.5.1. *Let $\Phi\in C^{(4)}([1,L])$ such that Ψ and Ψ'' are concave and increasing. Then $U\geqslant 0$ on ω_L if and only if*

$$U(s_1,s_2,\frac{1}{2}(s_1+s_2))\geqslant 0,\ \forall\, s_1,s_2\in[1,L] \tag{2.5.1}$$

$$U(s_1,s_1,s_2)\geqslant 0,\ \forall\, 1\leqslant s_1\leqslant s_2\leqslant L \tag{2.5.2}$$

Proof. Both inequalities are clearly necessary. To prove sufficiency, assume (2.5.1) and (2.5.2) hold. We will first prove a simple pointwise estimate for A'. For notational convenience, let $G(s) := \sum_{j=1}^{\infty} 2^{-j}\Psi(s_j)$ so that

$$A(s) = 16L(s-1)\Psi(L) + 8(L-1)^2 G(1) - 8(L-s)^2 G(s).$$

For $j \geqslant 1$, we have $s_{j-1} = s_j - (L - s_j) = s_j - 2^{-j}(L-s)$. Thus, by concavity of Ψ,

$$\Psi(s_{j-1}) \leqslant \Psi(s_j) - (s_j - s_{j-1})\Psi'(s_j) = \Psi(s_j) - 2^{-j}(L-s)\Psi'(s_j).$$

Since $G'(s) = \sum_{j=1}^{\infty} 2^{-2j}\Psi'(s_j)$, the above estimate gives

$$\begin{aligned} G(s) - (L-s)G'(s) &= \sum_{j=1}^{\infty} 2^{-j}\left[\Psi(s_j) - 2^{-j}(L-s)\Psi'(s_j)\right] \\ &\geqslant \sum_{j=1}^{\infty} 2^{-j}\Psi(s_{j-1}) = \frac{1}{2}\left[\Psi(s) + G(s)\right]. \end{aligned}$$

Therefore, since Ψ is positive and increasing, we have

$$\begin{aligned} A'(s) &= 16L\Psi(L) + 16(L-s)G(s) - 8(L-s)^2 G'(s) \\ &\geqslant 16L\Psi(L) + 12(L-s)G(s) + 4(L-s)\Psi(s) \\ &\geqslant 16L\Psi(L). \end{aligned} \tag{2.5.3}$$

Let $(s^-, s^+, s) \in \omega_L$; we may assume, without loss of generality, that $s^- \leqslant s^+$. Since $\frac{1}{2}(s^- + s^+) \leqslant s$, we must have either $s^- \leqslant s \leqslant s^+$ or $s^- \leqslant s^+ \leqslant s$. Let us consider these cases separately.

Case 1: $s^- \leqslant s \leqslant s^+$. Since $\frac{1}{2}(s^- + s^+) \leqslant s$, we have $s^- \leqslant 2s - s^+ \leqslant s \leqslant s^+$. Differentiating U with respect to s^-, we have

$$\frac{\partial U}{\partial s^-} = -\frac{1}{2}A'(s^-) + 8\,\Psi(s)\,s^+ \leqslant 8L\left[\Psi(s) - \Psi(L)\right] \leqslant 0. \tag{2.5.4}$$

Therefore, $U(s^-, s^+, s) \geqslant U(2s - s^+, s^+, s)$, which is non-negative by (2.5.1).

Case 2: $s^- \leqslant s^+ \leqslant s$. This also follows from (2.5.4). We'll have $U(s^-, s^+, s) \geqslant U(s^+, s^+, s)$, which is non-negative by (2.5.2). $\square$

It remains to verify (2.5.1) and (2.5.2).

Lemma 2.5.2. *Under the assumptions of Lemma 2.5.1, inequalities* (2.5.1) *and* (2.5.2) *hold.*

Proof. To prove (2.5.1), we must show that for all $s^\pm \in [1, L]$

$$A\left(\frac{s^+ + s^-}{2}\right) - \frac{1}{2}\left(A(s^-) + A(s^+)\right) - 2\Psi\left(\frac{s^+ + s^-}{2}\right)(s^+ - s^-)^2 \geqslant 0.$$

We first introduce some notation. Let $s = \frac{1}{2}(s^+ + s^-)$, $\Delta = \frac{1}{2}(s^+ - s^-)$ and for $k \geqslant 1$

$$s_k = s + (1 - 2^{-k})(L - s); \qquad s_k^\pm = s^\pm + (1 - 2^{-k})(L - s^\pm).$$

After cancellation of first order terms, inequality (2.5.1) becomes

$$4\sum_{j=1}^{\infty} 2^{-j}\left[\Psi(s_j^-)(L - s^-)^2 + \Psi(s_j^+)(L - s^+)^2 - 2\Psi(s_j)(L - s)^2\right] - 8\Psi(s)\Delta^2 \geqslant 0.$$

Let $H(t) = \Psi(t)(L - t)^2$. Since $L - s_k = 2^{-k}(L - s)$, we can rewrite the above as

$$\sum_{j=1}^{\infty} 2^j\left[H(s_j^-) + H(s_j^+) - 2H(s_j)\right] - 2\Psi(s)\Delta^2 \geqslant 0.$$

Finally, we define the function

$$S(s, \Delta) = \sum_{j=1}^{\infty} 2^j\left[H(s_j^-) + H(s_j^+) - 2H(s_j)\right] - 2\Psi(s)\Delta^2.$$

To prove (2.5.1), it suffices to prove S is non-negative on the domain

$$D := \{(s, \Delta) :\ 1 \leqslant s \leqslant L,\ 0 \leqslant \Delta \leqslant \min(s,\, L - s)\}.$$

To this end, we prove a stronger result: that S is non-negative on the enlarged domain

$$\bar{D} := \{(s, \Delta) : \ 1 \leqslant s \leqslant L, \ 0 \leqslant \Delta \leqslant L - s\}.$$

Clearly, S is identically zero on the line segment $\Delta = 0$ in $\bar{D}$, as for such points we'll have $s_j = s_j^{\pm}$, $\forall j \geqslant 1$. We can also readily deduce the behavior of S along the line segment $\Delta = L - s$, since $s_1^- = s$ and for $j \geqslant 1$, $s_j = s_{j+1}^-$ and $s_j^+ = L$. It follows that,

$$H(s_j) = H(s_{j+1}^-), \quad H(s_j^+) = 0, \quad \forall j \geqslant 1.$$

Therefore,

$$\begin{aligned} S(s, L-s) &= \sum_{j=1}^{\infty} 2^j \left[H(s_j^-) + H(s_j^+) - 2H(s_j) \right] - 2\Psi(s)(L-s)^2 \\ &= 2H(s_1^-) + \sum_{j=1}^{\infty} 2^{j+1} \left[H(s_{j+1}^-) + H(s_j) \right] - 2\Psi(s)(L-s)^2 \\ &= 2H(s) - 2\Psi(s)(L-s)^2 = 0. \end{aligned}$$

We use the behavior on these two line segments to prove positivity for all points in the domain via an argument involving the partial derivatives of S in Δ. We first compute these derivatives and note some of their properties on $\bar{D}$, making use of our assumptions on Ψ:

$$\begin{aligned} S_\Delta(s, \Delta) &= \sum_{j=1}^{\infty} \left[H'(s_j^+) - H'(s_j^-) \right] - 4\Psi(s)\Delta, \\ S_{\Delta\Delta}(s, \Delta) &= \sum_{j=1}^{\infty} 2^{-j} \left[H''(s_j^+) + H''(s_j^-) \right] - 4\Psi(s), \\ S_{\Delta\Delta\Delta}(s, \Delta) &= \sum_{j=1}^{\infty} 2^{-2j} \left[H'''(s_j^+) - H'''(s_j^-) \right]. \end{aligned}$$

Clearly, $S_{\Delta\Delta\Delta}(s,0) = S_\Delta(s,0) = 0$. Furthermore, $S_{\Delta\Delta\Delta} \leqslant 0$ since

$$H^{(4)}(t) = \Psi^{(4)}(t)(L-t)^2 - 8\Psi'''(t)(L-t) + 12\Psi''(t) \leqslant 0.$$

Fix $s \in (1, L)$ and suppose that $\min\{S(s,\Delta) :\ 0 < \Delta < L - s\} = S(s,\Delta') < 0$. Clearly then $S_{\Delta\Delta}(s,\Delta') > 0$. Since $S_{\Delta\Delta\Delta} \leqslant 0$, we must have that $S_{\Delta\Delta\Delta}$ is monotonically decreasing, and thus positive, on $[0,\Delta']$. Given that $S(s,0) = S_\Delta(s,0) = 0$, there exists a neighborhood of 0 on which S is strictly positive (except at 0). Let $[0,\alpha)$ be the largest such neighborhood. By the Extreme Value Theorem, S attains its maximum on $[0,\alpha]$, say at Δ''. Consequently, $S_{\Delta\Delta}(s,\Delta'') < 0$. However, by the Mean Value Theorem (MVT), this implies the existence of some $\Delta''' \in [0,\Delta']$ such that $S_{\Delta\Delta\Delta}(s,\Delta''') > 0$, a contradiction. Therefore, no such point exists, i.e. inequality (2.5.1) holds.

To prove (2.5.2), we must show that for $1 \leqslant s^- \leqslant s^+ \leqslant L$:

$$A(s^+) - A(s^-) \geqslant 8\Psi(s^+)\left[(s^+)^2 - (s^-)^2\right].$$

This follows directly from (2.5.3) and the MVT, since for some $c \in (s^-, s^+)$ we'll have

$$\frac{A(s^+) - A(s^-)}{s^+ - s^-} = A'(c) \geqslant 16L\Psi(L) \geqslant 8\Psi(s^+)\left[s^+ + s^-\right].$$

□

As a consequence of Lemma 2.5.2, we have that $U \geqslant 0$ on ω_L and thus $B_{Q,\Phi}$ satisfies the main inequality. Therefore, $B_{Q,\Phi}$ majorates the Bellman function:

$$\boldsymbol{B}_{Q,\Phi}(x) \leqslant B_{Q,\Phi}(x),\ \forall x \in \Lambda_Q.$$

The opposite inequality is taken up in the next section.

2.6 Optimizers

In this section, we construct special sequences of functions which will prove the converse inequality for $B_{Q,\Phi}$. As mentioned in the introduction, we are only able to do so for those points (r_k, r_k) and (L, L), as $B_{Q,\Phi}$ is not the true Bellman function but a majorant. Naturally, the construction of these sequences for $B_{Q,\Phi}$ will reflect the splitting procedure of section 2.4. Recall that a sequence of functions $\{w_n^x\}$ on I is called an *optimizing sequence for* $B_{Q,\Phi}(x)$, if $\{w_n^x\}$ satisfies the following three conditions:

$$\forall n, \quad w_n^x \in A_2^d(I); \tag{2.6.1}$$

$$\forall n, \quad \langle w_n^x \rangle_I = x_1; \ \langle (w_n^x)^{-1} \rangle_I = x_2; \tag{2.6.2}$$

$$\frac{1}{|I|} \sum_{J \in \mathcal{D}(I)} c_J^\Phi(w_n^x) \longrightarrow B_{Q,\Phi}(s), \ \text{ as } n \to \infty. \tag{2.6.3}$$

Once optimizers are constructed for these points, we will have finished the proof of Theorems 2.2.2 and 2.2.3. By the independence of the Bellman function on the interval (Lemma 2.3.1), we may take $I = (0, 1)$.

Let $k \geqslant 1$. We construct for each (r_k, r_k) an optimizing sequence $\{w_n^{(k)}\}_{n=k}^\infty$ recursively. The n-th optimizer in the sequence at (r_k, r_k) will be construted in terms of the $(n-k)$-th optimizer at the points $\{(r_k, r_k)\}_{k=0}^{n-1}$ and (L, L). Specifically, we define n-th element of the optimizing sequence at (r_k, r_k) as the concatenation of the n-th element of the optimizing sequence at (r_{k-1}, r_{k-1}) and (L, L), i.e.

$$w_n^{(k)}(t) = \begin{cases} w_n^{(k-1)}(2t), & t \in \left(0, \frac{1}{2}\right) \\ w_n^L(2t-1), & t \in \left(\frac{1}{2}, 1\right). \end{cases} \tag{2.6.4}$$

Naturally, for (r_0, r_0) we take the constant optimizer $w_n^{(0)} := 1$ and we close the recursion at the n-th level by homogeneity: let $\Delta_n = \sqrt{L^2 - r_n^2}$, we define

$$w_n^{(L)}(t) = \begin{cases} \frac{L-\Delta_n}{r_n}\, w_n^{(n)}(2t), & t \in (0, \frac{1}{2}) \\ \frac{L+\Delta_n}{r_n}\, w_n^{(n)}(2t-1), & t \in (\frac{1}{2}, 1) \end{cases}. \tag{2.6.5}$$

This construction is illustrated in Figure 2.3. Before we proceed to verify conditions

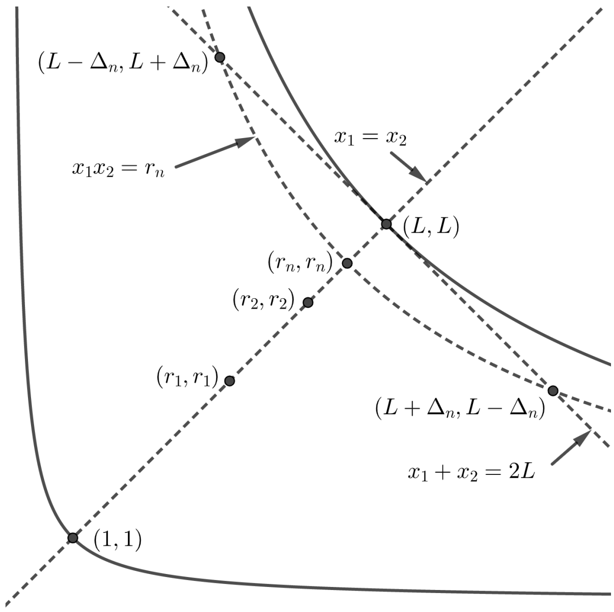

Figure 2.3: The optimizer for A : normal splits to the boundary $r_k = \frac{1}{2}(r_{k-1} + L)$ inside the domain; tangential split on the boundary at (L, L).

(2.6.1)-(2.6.3), we will give an alternate presentation to help illustrate the behavior of these functions. We will express each $w_n^{(k)}$ as a sequence of simple functions defined on a strictly increasing subset of $(0, 1)$. For a fixed $n \geqslant 1$, we introduce the notation

$$a_n = \frac{L - \Delta_n}{r_n}, \qquad b_n = \frac{L + \Delta_n}{r_n},$$

and define the sequence of functions $\left\{v_m^{(L)}\right\}_{m=0}^{\infty}$ on an increasing subset of $(0,1)$ as follows:

$$v_0^{(L)}(t) = \begin{cases} a_n, & t \in (0, 2^{-(n+1)}) \\ Undefined & t \in [2^{-(n+1)}, \frac{1}{2}) \\ b_n, & t \in [\frac{1}{2}, \frac{1}{2} + 2^{-(n+1)}) \\ Undefined & t \in [\frac{1}{2} + 2^{-(n+1)}, 1) \end{cases} \tag{2.6.6}$$

and for $m \geqslant 1$,

$$v_m^{(L)}(t) = \begin{cases} a_n, & t \in (0, 2^{-(n+1)}) \\ a_n\, v_{m-1}^{(L)}(2^i t - 1), & t \in [2^{-(i+1)}, 2^{-i}),\ 1 \leqslant i \leqslant n \\ b_n, & t \in [\frac{1}{2}, \frac{1}{2} + 2^{-(n+1)}) \\ b_n\, v_{m-1}^{(L)}(2^i t - 1), & t \in [\frac{1}{2} + 2^{-(i+1)}, \frac{1}{2} + 2^{-i}),\ 1 \leqslant i \leqslant n \end{cases} . \tag{2.6.7}$$

Observe that the measure of the set on which $v_m^{(L)}$ is undefined is $(1 - 2^{-n})$ times that of the set on which $v_{m-1}^{(L)}$ is undefined. This follows since there are $2n$ intervals on which $v_m^{(L)}$ is not explicitly given; on each of these intervals $v_m^{(L)}$ is just a rescaling of $v_{m-1}^{(L)}$ and the total measure of these intervals is $2(\frac{1}{2} - 2^{-(n+1)}) = (1 - 2^{-n})$; hence, the measure of the set on which $v_m^{(L)}$ is undefined is $(1 - 2^{-n})^m$. Furthermore, we have $v_m^{(L)}$ agrees with $v_{m-1}^{(L)}$ everywhere $v_{m-1}^{(L)}$ is defined, i.e. there are no redefinitions. Therefore, this sequence converges to a function defined almost everywhere on $(0,1)$. Specifically, $\lim\limits_{m\to\infty} v_m^{(L)} = w_n^{(L)}$. This can be easily seen when we express the original recursive definition of $w_n^{(n)}$ as follows:

$$w_n^{(n)}(t) = \begin{cases} 1, & t \in \left(0, 2^{-n}\right) \\ w_n^{(L)}(2^i t - 1), & t \in [2^{-i}, 2^{1-i}),\ 1 \leqslant i \leqslant n \end{cases} . \tag{2.6.8}$$

Hence, by (2.6.5), we have

$$w_n^{(L)}(t) = \begin{cases} a_n, & t \in (0, 2^{-(n+1)}) \\ a_n\, w_n^{(L)}(2^i t - 1), & t \in [2^{-(i+1)}, 2^{-i}),\ 1 \leqslant i \leqslant n \\ b_n, & t \in [\frac{1}{2}, \frac{1}{2} + 2^{-(n+1)}) \\ b_n\, w_n^{(L)}(2^i t - 1), & t \in [\frac{1}{2} + 2^{-(i+1)}, \frac{1}{2} + 2^{-i}),\ 1 \leqslant i \leqslant n \end{cases}. \tag{2.6.9}$$

Therefore, $v_m^{(L)}$ is defined precisely for those values such that the recursion given above terminates within m recursive steps; thus, for the intervals on which it is defined, $v_m^{(L)}$ agrees with $w_n^{(L)}$. Recursing as we had to obtain (2.6.8) gives $\forall\, 1 \leqslant k \leqslant n$:

$$w_n^{(k)}(t) = \begin{cases} 1, & t \in \big(0, 2^{-k}\big) \\ w_n^{(L)}(2^i t - 1), & t \in [2^{-i}, 2^{1-i}),\ 1 \leqslant i \leqslant k \end{cases}. \tag{2.6.10}$$

Therefore, for a fixed $n \geqslant 1$, we can also represent each $w_n^{(k)}$ as the limit of the simple functions $\{v_m^{(L)}\}_{m=0}^{\infty}$ defined on an increasing subset of $(0,1)$ as follows:

$$v_0^{(k)}(t) = \begin{cases} 1, & t \in (0, 2^{-k}) \\ Undefined & t \in [2^{-k}, 1) \end{cases}, \tag{2.6.11}$$

and for $m \geqslant 1$,

$$v_m^{(k)}(t) = \begin{cases} 1, & t \in \big(0, 2^{-k}\big) \\ v_m^{(L)}(2^i t - 1), & t \in [2^{-i}, 2^{1-i}),\ 1 \leqslant i \leqslant k \end{cases}. \tag{2.6.12}$$

To complete this presentation, we fix $L = 1.1$ and plot the first few functions from the sequence $\{v_m^{(L)}\}_{m=0}^{\infty}$, first for $n = 1$ (Figure 2.4) then for $n = 2$ (Figure 2.5). Recall

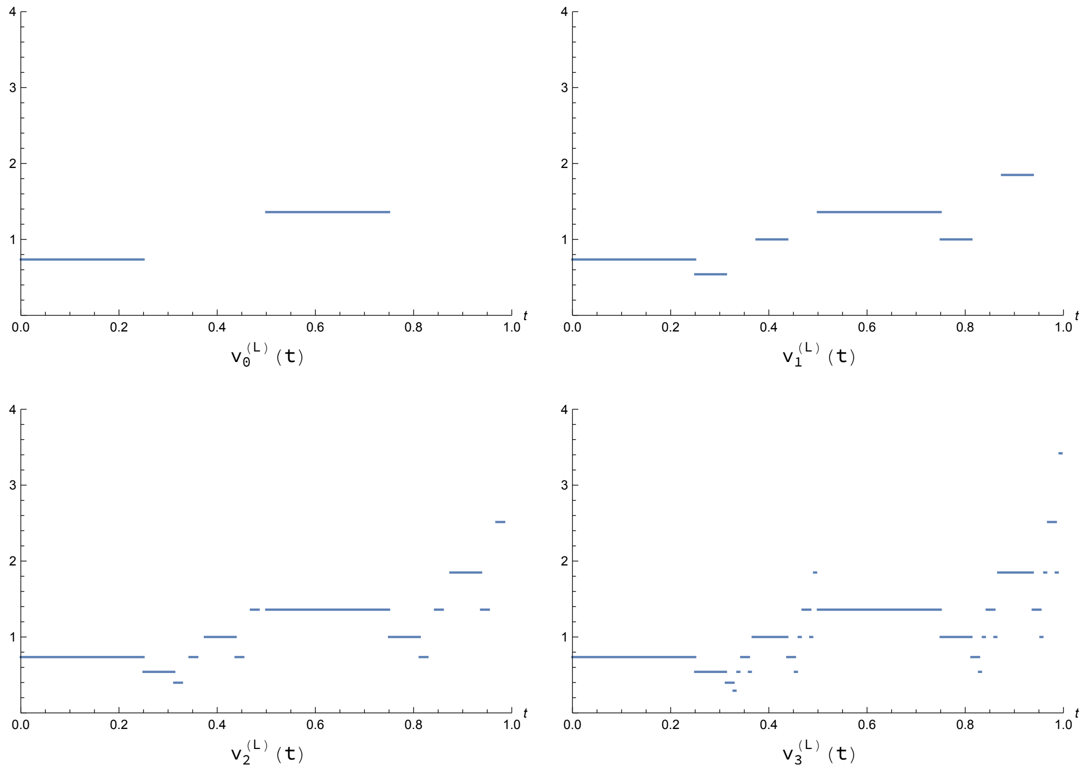

Figure 2.4: The first four functions of the sequence $\{v_m^{(L)}\}_{m=0}^{\infty}$ defining $w_1^{(L)}$

.

that the sequence $\{v_m^{(L)}\}_{m=0}^{\infty}$ for a fixed n only approximates one element of the sequence $\{w_n^{(L)}\}_{n=1}^{\infty}$, so each optimizer in this sequence becomes quite convoluted.

Lemma 2.6.1. *The sequence $\{w_n^{(k)}\}_{n=k}^{\infty}$ is an optimizing sequence for $B_{Q,\Phi}$ at (r_k, r_k) and $\{w_n^{(L)}\}_{n=1}^{\infty}$ for $B_{Q,\Phi}$ at (L, L). Therefore, if $\Phi \in C^{(4)}([1, L])$ such that Ψ, Ψ'' are concave and increasing:*

$$B_{Q,\Phi}(L, L) \leqslant \boldsymbol{B}_{Q,\Phi}(L, L); \qquad B_{Q,\Phi}(r_k, r_k) \leqslant \boldsymbol{B}_{Q,\Phi}(r_k, r_k), \quad k \geqslant 1.$$

Proof. Fix $n \geqslant 1$. We proceed to show that $w_n^{(L)}$ is admissible for (L, L) from which conditions (2.6.1) and (2.6.2) will follow. From the above discussion, we have that $w_n^{(L)}$ is the limit of the sequence of functions $v_m^{(L)}$ given by (2.6.6) and (2.6.7). If we let I_m

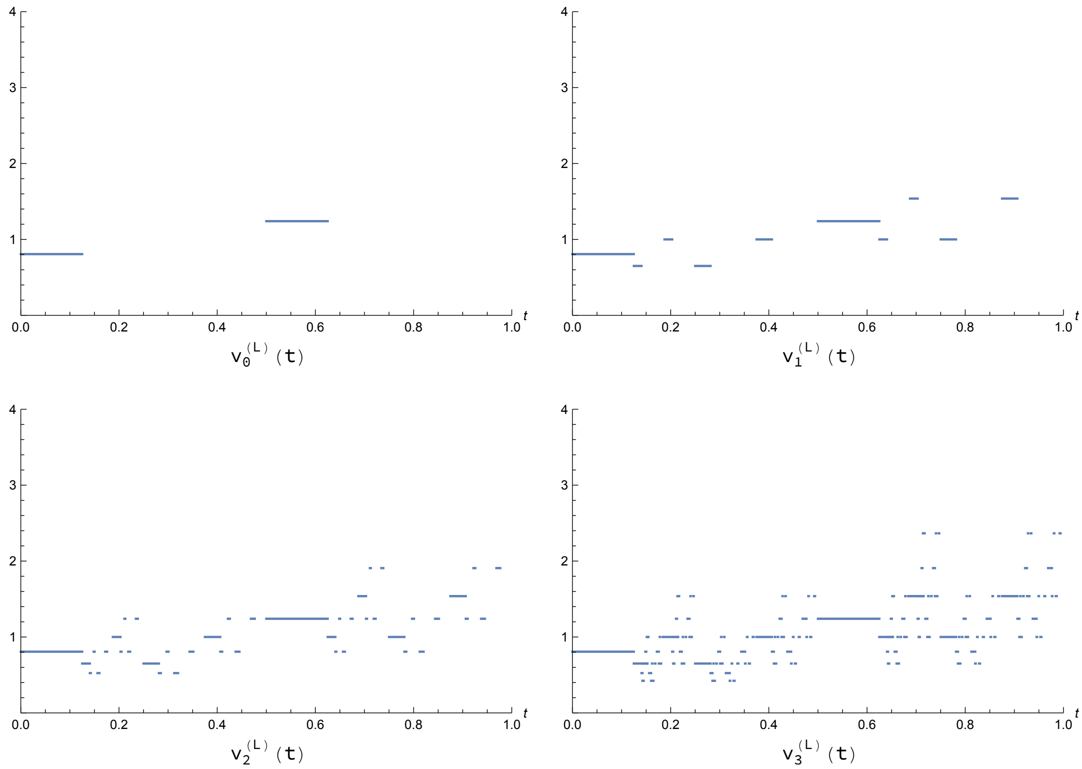

Figure 2.5: The first four functions of the sequence $\{v_m^{(L)}\}_{m=0}^{\infty}$ defining $w_2^{(L)}$

denote the set on which $v_m^{(L)}$ is defined, noting that $a_n + b_n = 2\frac{L}{r_n}$, we have for $m \geqslant 1$:

$$\begin{aligned}
\int_{I_m} v_m^{(L)}(t)\,dt &= 2^{-(n+1)}(a_n+b_n) + \left(\frac{1}{2} - 2^{-(n+1)}\right)(a_n+b_n)\int_{I_{m-1}} v_{m-1}^{(L)}(t)\,dt \\
&= 2^{-n}\frac{L}{r_n} + (1-2^{-n})\frac{L}{r_n}\int_{I_{m-1}} v_{m-1}^{(L)}(t)\,dt \\
&= 2^{-n}\frac{L}{r_n} + (1-2^{-n})\frac{L}{r_n}\left[2^{-n}\frac{L}{r_n} + (1-2^{-n})\frac{L}{r_n}\int_{I_{m-1}} v_{m-1}^{(L)}(t)\,dt\right] \\
&= 2^{-n}\frac{L}{r_n} + 2^{-n}(1-2^{-n})\left(\frac{L}{r_n}\right)^2 + (1-2^{-n})^2\left(\frac{L}{r_n}\right)^2\left[\int_{I_{m-2}} v_{m-2}^{(L)}(t)\,dt\right] \\
&\vdots \\
&= 2^{-n}\frac{L}{r_n}\sum_{i=0}^{m}\left[(1-2^{-n})\frac{L}{r_n}\right]^i \\
&= L - 2^{-(m+1)}\frac{L^2}{r_n}.
\end{aligned}$$

Therefore, by the monotone convergence theorem,

$$\int_0^1 w_n^{(L)}(t)\,dt = \lim_{m\to\infty} L - 2^{-(m+1)}\frac{L^2}{r_n} = L.$$

A similar argument can be applied to $(w_n^{(L)})^{-1}$. This function is evidently the limit of the sequence of step functions $\{(v_m^{(L)})^{-1}\}_{m=0}^{\infty}$. Since $a_n^{-1}+b_n^{-1} = (a_n+b_n)(a_nb_n)^{-1} = a_n+b_n$, we obtain the same recursion for $\int_{I_m}(v_m^{(L)}(t))^{-1}\,dt$ as that for $\int_{I_m} v_m^{(L)}(t)\,dt$; thus,

$$\int_0^1 (w_n^{(L)}(t))^{-1}\,dt = \lim_{m\to\infty} L - 2^{-(m+1)}\frac{L^2}{r_n} = L.$$

Furthermore, by (2.6.11) and (2.6.12), we have for $k \geqslant 1$

$$\begin{aligned}\int_0^1 w_n^{(k)}(t)\,dt &= 2^{-k} + (1-2^{-k})\int_0^1 w_n^{(L)}(t)\,dt \\ &= 2^{-k} + (1-2^{-k})L \\ &= r_k.\end{aligned}$$

Arguing as we had for $(w_n^{(L)})^{-1}$ gives $\int_0^1 (w_n^{(k)}(t))^{-1}\,dt = r_k$; thus, (2.6.1) is satisfied. To show (2.6.2), it suffices to remark that on every $J \in \mathcal{D}(I)$, $w_n^{(k)}$ is, by construction, a constant multiple of an appropriately scaled $w_n^{(L)}$ or $w_n^{(k)}$. Let c be this constant and w this function. It follows then that, on J, $(w_n^{(k)})^{-1}$ is just a rescaling of $c^{-1}w^{-1}$. Consequently,

$$\langle w_n^{(k)}\rangle_J\langle (w_n^{(k)})^{-1}\rangle_J = \langle w\rangle_I\langle w^{-1}\rangle_I \leqslant L^2 = Q.$$

It remains to verify condition (2.6.3). Fix $n \geqslant 1$ and for each $k \in \{0,1,2,...,n\}$, let

$$\Sigma^{(k)} = \sum_{J\in\mathcal{D}(I)} c_J^{\Phi}(w_n^{(k)}).$$

Then by our conditions on each $w_n^{(k)}$, we have the following system of equations for the associated sums:

$$\Sigma^{(k)} = \frac{1}{2}\Sigma^{(k-1)} + \frac{1}{2}\Sigma^{(L)} + c_I^{\Phi}(w_n^{(k)}); \quad \Sigma^{(L)} = \Sigma^{(n)} + 8\Psi(L)\left[L^2 - r_n^2\right]; \quad \Sigma^{(0)} = 0. \tag{2.6.13}$$

Since $|I| = 1$, we have

$$\begin{aligned}
c_I^{\Phi}(w_n^{(k)}) &= |I|\Phi(\langle w_n^{(k)}\rangle_I \langle (w_n^{(k)})^{-1}\rangle_I) R_I(w_n^{(k)}) \\
&= \Psi(r_k)\left[\left[\langle w_n^{(k)}\rangle_{I^-} - \langle w_n^{(k)}\rangle_{I^+}\right]^2 + \left[\langle (w_n^{(k)})^{-1}\rangle_{I^-} - \langle (w_n^{(k)})^{-1}\rangle_{I^+}\right]^2\right] \\
&= \Psi(r_k)\left[\left[\langle w_n^{(k-1)}\rangle_I - \langle w_n^{(L)}\rangle_I\right]^2 + \left[\langle (w_n^{(k-1)})^{-1}\rangle_I - \langle (w_n^{(L)})^{-1}\rangle_I\right]^2\right] \\
&= 2\Psi(r_k)\left[L - r_{k-1}\right]^2.
\end{aligned}$$

Therefore, the system of equations (2.6.13) is identical to that defining the intermediate function A_n at the points $\{r_k\}_{k=0}^n$ and L. That is to say, we are again solving the system

$$\Sigma^{(k)} = \frac{1}{2}\Sigma^{(k-1)} + \frac{1}{2}\Sigma^{(L)} + P(r_k); \quad \Sigma^{(L)} = \Sigma^{(n)} + T(r_n); \quad \Sigma^{(0)} = 0$$

where

$$P(r_k) = 2\Psi(r_k)\left[L - r_{k-1}\right]^2, \qquad T(r_k) = 8\Psi(r_k)\left[L^2 - r_k^2\right].$$

Consequently, $\Sigma^{(k)} = A_n(r_k)$ and so

$$\lim_{n\to\infty} \sum_{J\in D(I)} c_J^{\Phi}(w_n^{(k)}) = \lim_{n\to\infty} A_n(r_k) = B_{Q,\Phi}(r_k, r_k)$$

which proves (2.6.3). $\square$

Chapter 3

The John-Nirenberg Constant of BMO^p, $0 < p < 1$

3.1 Introduction, preliminaries, and main results

Here we are taking up a study begun in [Sla15] and continued in [VS16]. We are concerned with obtaining sharp estimates for BMO norms of logarithms of A_∞ weights, and in so doing estimating the so-called John–Nirenberg constant of the space BMO^p. For $p > 0$ and a fixed interval I this space is defined as follows:

$$\mathrm{BMO}^p(I) = \{\varphi \in L^1(I)\colon \quad \|\varphi\|_{\mathrm{BMO}^p(I)} := \sup_{\text{interval } J\subset I} \langle|\varphi - \langle\varphi\rangle_J|^p\rangle_J^{1/p} < \infty\}.$$

It is classical that BMO^p norms are equivalent for all $p > 0$; the paper [SV12] contains a series of sharp results concerning this equivalence.

Recall that a weight is an almost everywhere positive function. For a fixed interval I we say that a weight w on I belongs to $A_\infty(I)$, $w \in A_\infty(I)$, if both w and $\log w$ are integrable and the following condition holds:

$$[w]_{A_\infty(I)} := \sup_{\text{interval } J\subset I} \langle w\rangle_J e^{-\langle \log w\rangle_J} < \infty. \tag{3.1.1}$$

Observe that $[w]_{A_\infty(I)} \geqslant 1$ by Jensen's inequality. For $C \geqslant 1$, we will also use the notation $A_\infty^C(I)$ for the set of those $A_\infty(I)$ weights with the characteristics bounded by C:

$$A_\infty^C(I) := \{w \in A_\infty(I)\colon\ [w]_{A_\infty(I)} \leqslant C\}. \tag{3.1.2}$$

Let us also recall the definition of A_2, which we gave earlier in a dyadic setting. We say that w belongs to $A_2(I)$, $w \in A_2(I)$, if both w and w^{-1} are locally integrable and

$$[w]_{A_2(I)} := \sup_{\text{interval } J\subset I} \langle w\rangle_J \langle w^{-1}\rangle_J < \infty. \tag{3.1.3}$$

The quantities $[w]_{A_\infty(I)}$ and $[w]_{A_2(I)}$ are called the A_∞- and the A_2-characteristics of w, respectively. Clearly, $A_2 \subset A_\infty$. In fact, it is easy to show that w is in A_2 if and only if both w and w^{-1} are in A_∞.

The John–Nirenberg constant of BMO^p, denoted by $\varepsilon_0(p)$, is the supremum of all $\varepsilon > 0$ such that for any function $\varphi \in \mathrm{BMO}^p(I)$ with BMO^p-norm ε we have $e^\varphi \in A_2(I)$. Since a weight $w \in A_2$ if and only if $w, w^{-1} \in A_\infty$, we have the following equivalent definition of $\varepsilon_0(p)$:

$$\varepsilon_0(p) = \sup\{\varepsilon > 0\colon\ \forall\varphi \in \mathrm{BMO}^p, \|\varphi\|_{\mathrm{BMO}^p} = \varepsilon \implies e^\varphi \in A_\infty\}$$

This constant was computed for $1 \leqslant p \leqslant 2$ in [Sla15] and for $p > 2$ in [VS16]. Here is the combined main result of those studies:

Theorem 3.1.1 ([Sla15; VS16]). *If $p \geqslant 1$, then*

$$\varepsilon_0(p) = \left[\frac{p}{e}\Big(\Gamma(p) - \int_0^1 t^{p-1}e^t\,dt\Big) + 1\right]^{1/p}. \tag{3.1.4}$$

Let us briefly explain how $\varepsilon_0(p)$ may be computed or estimated. A straightforward approach is to compute the supremum of the A_∞ "oscillation" $\langle e^{\varphi - \langle\varphi\rangle_J}\rangle_J$ for all φ such that $\|\varphi\|_{\mathrm{BMO}^p} = \varepsilon$. The value of ε for which this supremum becomes infinite is precisely $\varepsilon_0(p)$. As discussed in [Sla15], in trying to put this in practice one might consider the following upper Bellman function:

$$\boldsymbol{\mathcal{B}}_{p,\varepsilon}(x_1, x_2) = \sup\{\langle e^{\varphi}\rangle_I :\ \langle\varphi\rangle_I = x_1, \langle|\varphi - \langle\varphi\rangle_I|^p\rangle_I = x_2,\ \|\varphi\|_{\mathrm{BMO}^p(I)} \leqslant \varepsilon\}. \tag{3.1.5}$$

This works well for $p = 2$, since

$$\langle|\varphi - \langle\varphi\rangle_I|^2\rangle_I = \langle\varphi^2\rangle_I - \langle\varphi\rangle_I^2,$$

and one can then describe the dynamics of the function $\boldsymbol{\mathcal{B}}_{2,\varepsilon}$ in terms of the variables x_1 and x_2. Indeed, this is precisely what was done in [SV11], where it was first proved that $\varepsilon_0(2) = 1$. However, this direct approach no longer works when $p \neq 2$, since one cannot describe the dynamics of the oscillation $\langle|\varphi - \langle\varphi\rangle_I|^p\rangle_I = x_2$ when the interval I is split into subintervals. In particular, one cannot obtain a PDE for this function.

For $p \neq 2$ an inverse approach was developed in [Sla15]: instead of fixing $\langle\varphi\rangle_I$ and $\|\varphi\|_{\mathrm{BMO}^p(I)}$ and estimating $\langle e^{\varphi}\rangle_I$ from above, one fixes $\langle\varphi\rangle_I$, $\langle e^{\varphi}\rangle_I$, and $[e^{\varphi}]_{A_\infty(I)}$ and estimates $\langle|\varphi|^p\rangle_I$ from below. Thus, one aims to compute the following lower Bellman function:

$$\boldsymbol{b}_{p,C}(x) = \inf\{\langle|\varphi|^p\rangle_I :\ \varphi \in E_{x,C,I}\}. \tag{3.1.6}$$

where $E_{x,C,I}$ is the set of admissible (or test) functions φ on I:

$$E_{x,C,I} = \{\varphi\colon\ e^{\varphi} \in A_{\infty}^{C}(I),\ \langle\varphi\rangle_I = x_1,\ \langle e^{\varphi}\rangle_I = x_2\}.$$

As in the case of dyadic A_2, this function can be shown by a simple rescaling argument not to depend on I. It is naturally defined on the planar domain

$$\Omega_C = \{x \in \mathbb{R}^2 : e^{x_1} \leqslant x_2 \leqslant C\, e^{x_1}\}, \tag{3.1.7}$$

where the left inequality is Jensen's and the right one follows because $e^{\varphi} \in A_{\infty}^{C}(I)$. Since the only test functions for which the left inequality in (3.1.7) holds with equality are constants almost everywhere on I, we also have the following boundary condition:

$$\boldsymbol{b}_{p,C}(x_1, e^{x_1}) = |x_1|^p. \tag{3.1.8}$$

We are interested in computing this function for all values of parameters and variables. This will allow us to obtain sharp lower estimates for p-averages of logarithms of A_∞ weights, and also sharply estimate their BMO^p norms. It will also yield new best estimates for $\varepsilon_0(p)$, $0 < p < 1$. Finally, as we will see later, the graph of such a function is a special degenerately-convex surface, and this computation will allow us to extend the existing geometric techniques for building such surfaces to the settings with lower boundary regularity. We thus arrive at our first question.

Question 1. Compute $\boldsymbol{b}_{p,C}$ for $0 < p < 1$.

If $\boldsymbol{b}_{p,C}$ is at hand, one can estimate $\varepsilon_0(p)$ using a theorem from [Sla15]:

Theorem 3.1.2 ([Sla15]).

$$\varepsilon_0^p(p) \geqslant \limsup_{C\to\infty} \boldsymbol{b}_{p,C}(0, C) \tag{3.1.9}$$

Note that this theorem does not guarantee equality, but if equality does hold the converse estimate is obtained by presenting an explicit function $\varphi \in \mathrm{BMO}^p$ such that $\|\varphi\|_{\mathrm{BMO}^p} = \varepsilon_0(p)$ and $e^\varphi \notin A_\infty$. This is precisely what was done in [Sla15] and [VS16]. Unfortunately, for $0 < p < 1$ Theorem 3.1.2 fails to yield a non-trivial result: even without knowing $\boldsymbol{b}_{p,C}$ it easy to show, using a simple test function, that for this range of p we have $\limsup_{C\to\infty} \boldsymbol{b}_{p,C}(0, C) = 0$ (this is done in section 3.7.1). We thus arrive at our second question.

Question 2. Replace Theorem 3.1.2 with a sharper result that allows one to estimate $\varepsilon_0(p)$ non-trivially using the formula for $\boldsymbol{b}_{p,C}$.

Remark 3.1.3. *By "nontrivial" we mean an estimate that couldn't be obtained from an earlier result. Specifically, it is shown in [SV12] that for* $0 < p < 1$ *one has* $\|\varphi\|_{BMO^p} \geqslant 2^{1-2/p}\|\varphi\|_{BMO^2}$. *Since* $\varepsilon_0(2) = 1$, *we immediately have* $\varepsilon_0(p) \geqslant 2^{1-2/p}$. *We are looking for a better estimate.*

Next, it is easy to show that $\varepsilon_0(p)$ coincides with the supremum of all constants c_0 in the weak-form John–Nirenberg inequality,

$$\frac{1}{|J|}|\{t \in J : |\varphi(t) - \langle\varphi\rangle_J| \geqslant \lambda\}| \leqslant C_1 e^{-c_0\lambda/\|\varphi\|_{\mathrm{BMO}^p}}. \tag{3.1.10}$$

However, it's not obvious that one can replace c_0 with $\varepsilon_0(p)$. In [Sla15], it was shown to be the case for $1 < p \leqslant 2$ (the cases $p = 1$ and $p = 2$ had been established earlier in [Kor92] and [VV13], respectively). However, the argument used in [Sla15] did not extend to the range $p > 2$, so this question was left open for that range. We ask the same question for $0 < p < 1$.

Question 3. Can the value obtained while answering Question 2 replace c_0 in (3.1.10)?

3.1.1 The answers.

Answer 1. We have been able to fully compute the function $\boldsymbol{b}_{p,C}$. At its core, this is a purely differential-geometric task, though a challenging one. Indeed, the main theorem from [SZ16] asserts that $\boldsymbol{b}_{p,C}$ is simply the largest locally convex function on Ω_C satisfying the boundary condition (3.1.8). ("Locally convex" means convex along any line segment contained in Ω_C.) Therefore, wherever $\boldsymbol{b}_{p,C}$ is second differentiable, it satisfies the homogeneous Monge-Ampère equation. Its graph is a surface ruled by lines of zero curvature. The projections of these rulings – referred to as Monge-Ampère characteristics – foliate the domain and do not intersect unless the function is affine in a neighborhood of the point of intersection.

The Monge-Ampère geometry of $\boldsymbol{b}_{p,C}$ is the simplest for $1 < p \leqslant 2$: for large enough C (and these are the only values of C that inequality (3.1.9) is concerned with) the foliation consists solely of one-sided tangents to the upper boundary of the Bellman domain. For $p > 2$, the geometry is more complicated: the tangential foliation splits into two parts, separated by a so-called "trolleybus" region (see [Iva+18] for the BMO analog of this region); however, as $C \to \infty$, the width of this region goes to 0, which effectively yields the same result.

For $0 < p < 1$, the geometry is more difficult still, due to the lack of differentiability of the boundary function (3.1.8) at $x_1 = 0$. In particular, the foliation contains several non-tangential elements, none of which disappear or become infinitesimally thin as $C \to \infty$. Furthermore, the nature of these regions changes as C increases, a process referred to as evolution. Our solution draws inspiration and basic understanding from the monograph [Iva+18] and the next (as yet unpublished) iteration in the modern-Bellman series, [Iva+a]. However, both of those sources treat Bellman functions with C^{2+} boundary conditions, while our boundary function is not even differentiable. As a result, we obtain foliations that are partly outside the established theory. In addition, we compute

$\boldsymbol{b}_{p,C}$ for all possible C and not just large ones; as we will see below, this is needed in Theorem 3.1.4, which is our replacement for Theorem 3.1.2.

Our focus is on the value $\boldsymbol{b}_{p,C}(0,C)$. It turns out that there exists a certain threshold $C_* = C_*(p)$ such that for $C \in [1, C_*]$ this value is unaffected by the parts of the foliation outside of a fixed central configuration; for such C the formula for $\boldsymbol{b}_{p,C}(0,C)$ is relatively simple. However, for $C > C_*$ this central configuration collides with another evolving region, which makes the formula for, and subsequent analysis of, $\boldsymbol{b}_{p,C}(0,C)$ much more complicated. The complete answer to Question 1 requires much additional notation to state. This result is contained in Theorem 3.4.5. With $\boldsymbol{b}_{p,C}$ in hand, we also record a number of immediate corollaries, concerned with sharp estimates for p-averages, p-oscillations, and BMO^p norms of logarithms of A_∞ weights. All of these are new in literature.

Answer 2. We have indeed managed to find a useful substitute for Theorem 3.1.2.

Theorem 3.1.4.

$$\varepsilon_0^p(p) \geqslant \sup_{1<C<\infty} \frac{\boldsymbol{b}_{p,C}(0,C)}{(\xi^+(C))^p}$$

Of course, this estimate is formally stronger than (3.1.9), but what matters is that it does produce new estimates on $\varepsilon_0(p)$. As a simple test, if we take the limit of the expression under supremum as $C \to 1^+$, we get $\varepsilon_0^p(p) \geqslant 2^{p-2}$, thus recovering what we could have obtained from [VS16]; see Remark 3.1.3 above. In section 3.7.3 we expand on this calculation and show that in fact $\varepsilon_0(p) > 2^{p-2}$ for every $p \in (0,1)$; we then illustrate this fact quantitatively.

Answer 3. We also have the positive answer to Question 3. In fact, the following key theorem has Th. 3.1.4 as a corollary.

Theorem 3.1.5. *Take* $p \in (0,1)$. *Let*

$$\varepsilon_*(p) = \max_{1<C<\infty} \frac{(\boldsymbol{b}_{p,C}(0,C))^{1/p}}{\xi^+(C)}.$$

Then there exists a constant $K = K(p)$ such that for all $\varphi \in BMO^p(I)$ and any subinterval J of I we have

$$\frac{1}{|J|}|\{t \in J : |\varphi(t) - \langle\varphi\rangle_J| \geqslant \lambda\}| \leqslant K\, e^{-\lambda\varepsilon_*(p)/\|\varphi\|_{BMO^p}} \tag{3.1.11}$$

The proofs of Theorems 3.1.4 and 3.1.5 are given in section 3.7.

The rest of this chapter is organized as follows. In section 3.2 we describe a conjectured geometry of our Bellman function, i.e., the splitting of the domain Ω_C into specific regions, and Monge-Ampèrefoliation in each region. In section 3.3, we construct an explicit function in each region according to the conjectured foliation. In section 3.4, we formulate compatibility conditions for these regions, prove that they are fulfilled, and thus obtain a so-called *Bellman candidate* in Ω_C. We then state our main Bellman-function result, which asserts that the candidate is in fact equal to the Bellman function, and record several key corollaries. In section 3.5, we demonstrate the local convexity of the candidate, which proves that it bounds the Bellman function from below. In section 3.6, we present explicit optimizers for each point of the domain, thus sowing the the candidate bounds the function from above. Finally, in section 3.7, we prove the main theorems relating the Bellman function and the John–Nirenberg constant and explore specific estimates.

3.2 The conjectured foliation for a Bellman candidate

As pointed out in the introduction, the Bellman function $\boldsymbol{b}_{p,C}$ defined by (3.1.6) is the largest locally convex function on the domain Ω_C. Accordingly, we aim to build a locally convex Bellman candidate $b_{p,C}$ on Ω_C and then rigorously show that the candidate and the function are one and the same.

While there is currently no definitive algorithm for constructing such functions on non-convex planar domains for arbitrary boundary conditions, the current state of the

Bellman studies does provide important insights as to what our function might look like. In this section, we first introduce important notation and collect several facts about the geometry of the domain Ω_C. We then synthesize a conjectured recipe for $b_{p,C}$ from several recent studies and implement this recipe piece-by-piece, splitting Ω_C into subdomains and constructing $b_{p,C}$ in each piece. Finally, we prove that the pieces can be glued together in a continuous — in fact, C^1, with a couple of important exceptions — fashion.

3.2.1 The domain Ω_C and the numbers $\xi^\pm$

Fix $C \geqslant 1$ and let $\xi^\pm = \xi^\pm(C)$ be the two solutions of the equation

$$e^{-\xi} = C(1-\xi): \quad -\infty < \xi^- \leqslant 0 \leqslant \xi^+ < 1. \tag{3.2.1}$$

The numbers $\xi^\pm$ are variants of the so-called product logarithm functions. They are well-studied, and we collect several of their key properties without proof.

$$\xi^\pm(1) = 0, \qquad \lim_{C\to\infty} \xi^+(C) = 1, \qquad \lim_{C\to\infty} \xi^-(C) = -\infty \tag{3.2.2}$$

$$(\xi^\pm)'(C) = \frac{1-\xi^\pm}{\xi^\pm}\,\frac{1}{C} \tag{3.2.3}$$

Thus, ξ^+ is strictly increasing in C, while ξ^- is strictly decreasing.

$$\lim_{C\to 1} \frac{\xi^+(C)}{|\xi^-(C)|} = 1, \qquad \xi^- + \xi^+ < 0 \tag{3.2.4}$$

We will often encounter the difference $\xi^- - \xi^+$, so we will give it a separate name:

$$w_0 = \xi^- - \xi^+. \tag{3.2.5}$$

We can readily express $\xi^{\pm}$ through w_0:

$$\xi^- = 1 + \frac{w_0}{1 - e^{w_0}}, \qquad \xi^+ = 1 + \frac{w_0 e^{w_0}}{1 - e^{w_0}}. \tag{3.2.6}$$

For $R > 0$, let

$$\Gamma_R = \{x \in \mathbb{R}^2 : \; x_2 = Re^{x_1}\};$$

then Ω_C is the region in the plane bounded below by Γ_1 and above by Γ_C. We can interpret $\xi^{\pm}$ as the x_1-coordinates of the two points of tangency when two tangents to Γ_C are drawn from the point $(0, 1)$. In relation to that, let us define two new functions on Ω_C, u^+ and u^-, by the implicit formula

$$x_2 = e^{u^{\pm}} \Big(\frac{x_1 - u^{\pm}}{1 - \xi^{\pm}} + 1 \Big). \tag{3.2.7}$$

To illustrate their geometric meaning, take a point $x \in \Omega_C$ and draw two one-sided tangents to Γ_C, so that each tangent starts at Γ_1, passes through x, and terminates at the point of tangency. One of these tangents will have its point of tangency to the right of x; the other has the point of tangency to the left of x. In the first case, the horizontal coordinate of the initial point is $u^+(x)$ and that of the point of tangency is $u^+(x) + \xi^+$; in the second case, these are $u^-(x)$ and $u^-(x) + \xi^-$, respectively; see Figure 3.1, which is reproduced from [Sla15].

3.2.2 Conjectured foliation for the Bellman function

Perhaps the most important inspiration for our conjectured foliation comes from [SV12], where the lower Bellman function was found for the functional $\langle |\varphi|^p \rangle_I$, $0 < p < 1$, for $\varphi \in \mathrm{BMO}^2(I)$. Thus, this is exactly the same functional as ours, but the domain is a symmetric parabolic strip. Figure 3.2, reproduced from [SV12], shows the key elements of interest to us: There are five regions in total. Two are so-called cups – the regions

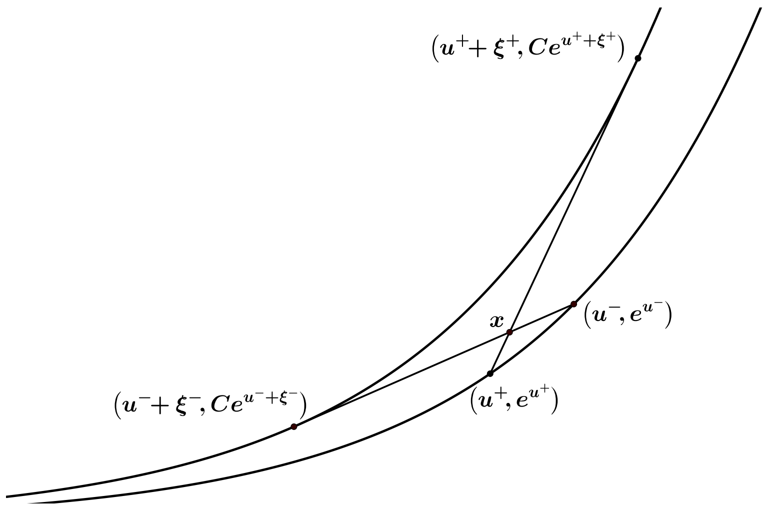

Figure 3.1: The geometric meaning of $\xi^{\pm}$ and $u^{\pm}(x)$.

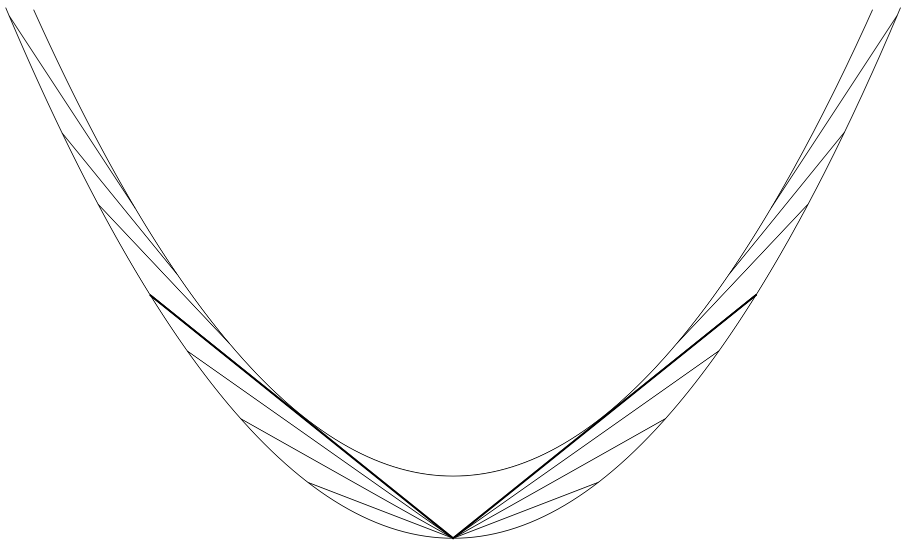

Figure 3.2: The foliation for the boundary function $f(t) = |t|^p$ on the BMO^2 domain.

under a single chord connecting two points on the lower boundary. Though cups can be foliated in any number of ways, in these cups all extremal trajectories emanate from a single corner, the point $(0, 0)$. Such cups result from insufficient differentiability of the boundary functions and are called singular. Two other regions are foliated by tangents to the upper boundary. Lastly, there is a central region in which the function is affine.

The biggest difference between that setting and ours is, of course, the lack of symmetry in our domain. Still, it is reasonable to conjecture that we do have the central aggregate consisting of two cups and the “triangle" with the vertex at $(0,1)$ (the equivalent of $(0,0)$ in the BMO formulation). It turns out upon inspection that we cannot combine this conjectured configuration with the two regions of tangent as was done in [SV12]; the resulting function will not be locally convex in the whole domain.

However, the difficulties lie away from the point $(0,1)$, meaning away from the point $t=0$ for the boundary function $f(t)=|t|^p$. And such smooth situations are completely described in a pair of recent monographs, [Iva+18] and [Iva+a]. The first one lays out a complete theory for BMO; the second one, still in preprint form as of this writing, deals with general non-convex planar domains, of which Ω_C is an example. Both references assume C^{2+} boundary conditions, which we do have away from $t=0$. Without going into details, the main, beautiful idea is that the sign changes of the so-called torsion function determine the configuration of various foliation blocks for small values of the domain constant – in our case, for C close to 1. After that, the configuration starts to evolve, often in complicated ways. We will see how the evolution proceeds in our case.

For a domain with the boundary curve $(g_1(t), g_2(t))$ (that boundary on which test functions are constants) and the boundary function $f(t)$, the torsion function is given by

$$T(t) := \begin{vmatrix} g_1' & g_1'' & g_1''' \\ g_2' & g_2'' & g_2''' \\ f' & f'' & f''' \end{vmatrix}.$$

We have $g_1(t)=t$, $g_2(t)=e^t$, and $f(t)=|t|^p$, so $T(t)=e^t p(p-1)|t|^{p-4}t(t-p+2)$. This function changes sign from negative to positive at $t=p-2$, which for the lower Bellman function indicates the presence of a triangular region between two counter-directed one-sided tangents; the function is affine in the region. The lower vertex of this region starts at the point $(p-2, e^{p-2})$ and then moves away from it as C increases, though it is not

a priori clear in what direction. Let us refer to the horizontal coordinate of this point as w. Upon extensive experimentation, we have determined that w moves to the left from the starting point $p-2$. Here is our conjectured foliation for sufficiently small values of C. It has a fixed configuration of four regions: the left singular cup P_2, the right singular cup P_4, the triangular region P_3 at the center, and a tangentially foliated region to the right of P_2, called P_1. This fixed configuration is complemented by the moving triangular region P_6 and two regions of opposite tangential foliations on its left and on its right, called P_7 and P_5 respectively. This foliation is presented in Figure 3.3.

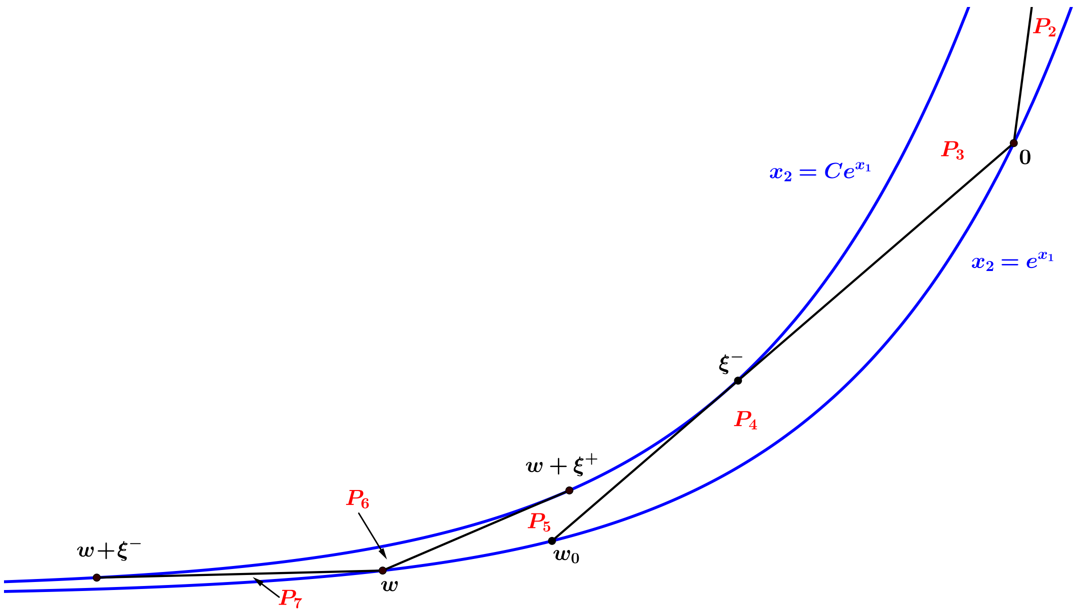

Figure 3.3: The global foliation, pre-collision; P_1 is not shown.

We now explore what happens with the moving region P_6 as C increases. Note that the left-most point of the cup P_4, (w_0, e^{w_0}), is moving to the left as C grows. Three possibilities now present themselves: P_6 could be moving to the left and disappear for all large enough C; it could be moving to the left and be present for all C, but never collide with the P_4; or it could be moving to the left and eventually collide with P_4. The first and second possibilities can be ruled out with calculation, and we conjecture that the last one is the one that actually happens. We will refer to the value of C when P_4 and P_6 collide as *the moment of collision; to the smaller values of* C *as being in the pre-collision range; and to the larger value of* C *as being in the post-collision range. In later sections we will formally prove that the collision happens. At the moment of collision,* P_5 is reduced to a single half-tangent; see Figure 3.4.

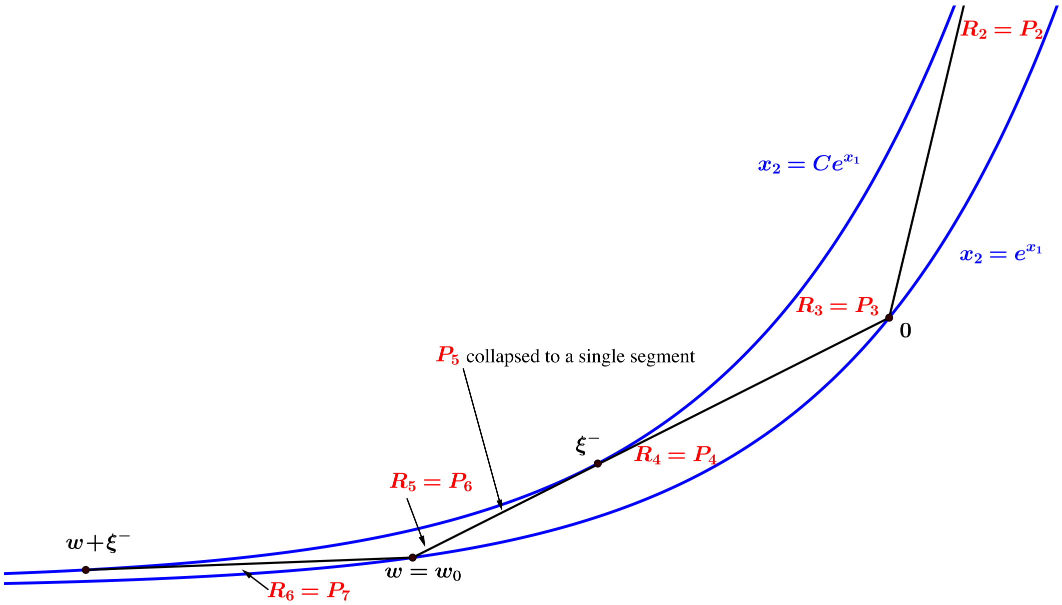

Figure 3.4: The global foliation at collision; $P_1 = R_1$ is not shown.

After collision there are only six regions instead of the original seven; these are now called $R_1, ..., R_6$. We have $R_1 = P_1$, $R_2 = P_2$, and $R_3 = P_3$. The cup P_4, now reduced by collision, is called R_4, and the left-most region of one-sided left-leaning tangents is called R_6. The "quadrilateral" region R_5 is referred to as "trolleybus" in Bellman literature. The candidate b is affine in R_5, as it is in the adjacent R_3. However, it is not affine in $R_5 \cup R_3$. One usually does not see two regions of affinity next to each other; the reason we see it here is that the vertex of the central triangle R_3 is fixed at $(0, 1)$ because of the singularity. See Figure 3.5 for the post-collision foliation.

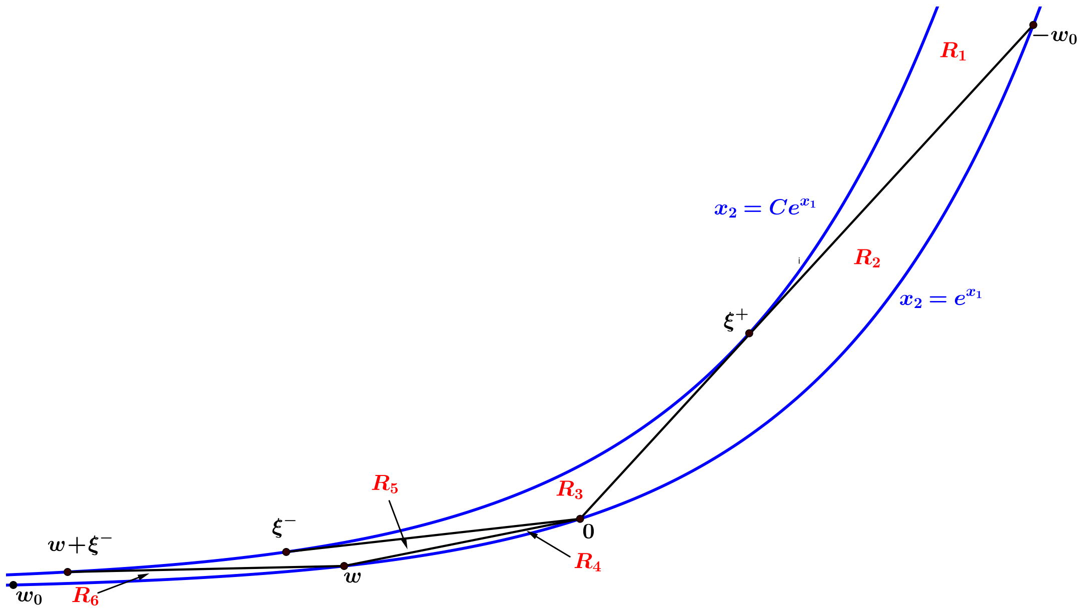

Figure 3.5: The global foliation, post-collision.

3.3 Construction and properties of local Bellman candidates

We are now starting to build the global foliation conjectured in the previous section. To that end, we construct the unique Bellman candidate in each region. Of our regions, cups are entirely self-contained (or "complete," in the language of [SV12]), being independent of the values of the candidate outside; regions foliated by tangents are "half-complete," reading information from either left or right, depending on their lean; and regions of affinity are "incomplete," requiring the knowledge of the candidate on both sides (or on two of three sides in the case of R_5).

3.3.1 Candidates in cups

Bellman candidates in cups depend only on the location of the cup and the conjectured foliation within it. As pointed out earlier, all of our cup foliations will be of the so-called singular variety, meaning all foliating chords emanate from one corner of the cup. Furthermore, that corner will be the point $(0, 1)$ in all cases.

Specifically, one of our cups is the region called P_2 pre-collision and R_2 post-collision. In this region all extremal chords start at $(0, 1)$ and terminate at a point (v^+, e^{v^+}) on the lower boundary, for some $v^+ \in (0, -w_0]$. Its top chord is tangent to the upper boundary at the point (ξ^+, Ce^{ξ^+}); see Figure 3.3.1.

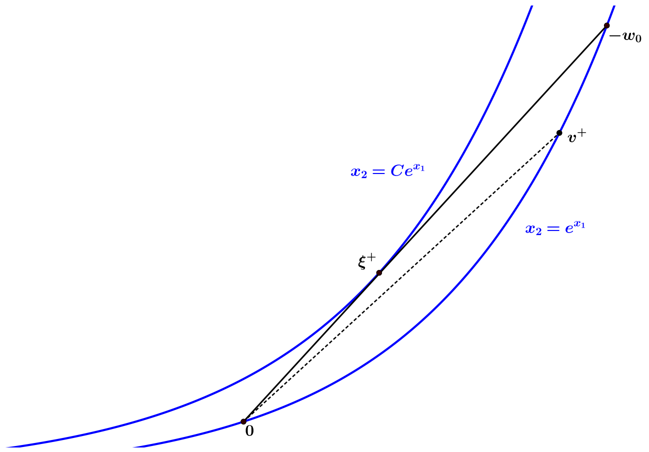

Figure 3.6: The cup $P_2 = R_2$ and a typical element of the foliation.

Another cup region is P_4, existing only pre-collision. It is algebraically symmetric to P_2; see Figure 3.3.1. In this region all extremal chords start at $(0, 1)$ and terminate at a point (v^-, e^{v^-}) on the lower boundary, for some $v^- \in [w_0, 0)$. The top chord is tangent to the upper boundary at the point (ξ^-, Ce^{ξ^-}).

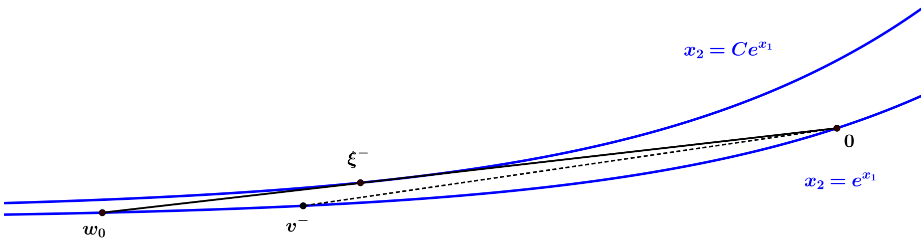

Figure 3.7: The cup P_4 and a typical element of the foliation.

Lastly, we have the cup R_4, which is P_4 partially reduced by the collision. Unlike in the previous two cases, its top chord does not reach the upper boundary, and it terminates at the points (w, e^w), where $w > w_0$; see Figure 3.3.1.

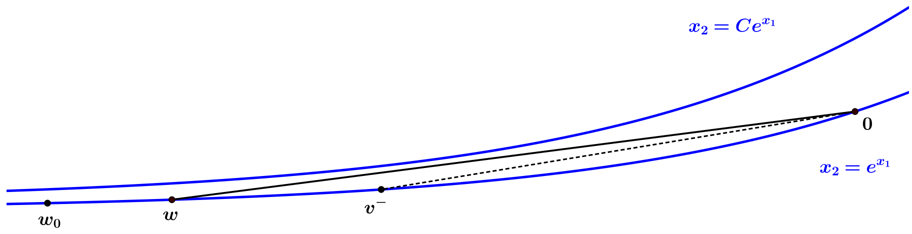

Figure 3.8: The cup R_4 and a typical element of the foliation.

For each x in each of our cups, there is a unique chord containing the points $(0, 1)$, x, and (v, e^v) (here v stands for either v^- or v^+). This v is given by the following equation:

$$\frac{x_2 - 1}{x_1} = \frac{e^v - 1}{v} \tag{3.3.1}$$

We will also need v_{x_2}:

$$v_{x_2} = \frac{v^2}{ve^v - e^v + 1} \frac{1}{x_1} \tag{3.3.2}$$

It clear both from geometry and from this formula that $v_{x_2} \geqslant 0$ when $v = v^+$ and $v_{x_2} \leqslant 0$ when $v = v^-$. We require that the candidate b be affine along each such chord:

$$b(x) = m(v)(x_1 - v) + f(v).$$

This implies that b satisfies the homogeneous Monge-Ampère equation in the interior of the cup. Since we also have $b(0, 1) = 0$, we conclude that $m(x) = \frac{f(v)}{v} = \mathrm{sgn}(v)|v|^{p-1}$. Thus,

$$b(x) = \mathrm{sgn}(v)|v|^{p-1} x_1. \tag{3.3.3}$$

We will need the first partial derivative of b with respect to x_2:

$$b_{x_2} = \frac{(p-1)|v|^p}{ve^v - e^v + 1} \tag{3.3.4}$$

3.3.2 Tangential candidates

General considerations

We are constructing a locally convex function b in a subdomain of Ω_C between two one-sided tangents of the same orientation (both left-leaning or both right-leaning). If the orientation is chosen, each point x in such a region corresponds to a unique one-sided tangent connecting the points $(u, e^u) \in \Gamma_1$ and $(u+\xi, Ce^{u+\xi}) \in \Gamma_C$. Here u is $u^-(x)$ for left-leaning tangents and $u^+(x)$ for right-leaning ones; see Figure 3.1. Recall equation (3.2.7):

$$x_2 = e^{u^\pm}\Big(\frac{x_1 - u^\pm}{1-\xi^\pm} + 1\Big).$$

In calculations below that apply to both u^- and u^+ (and, thus, to ξ^- and ξ^+), we simply use u and ξ.

We require that the function b have constant first partial derivatives for each fixed u, which will automatically imply that it satisfies the Monge-Ampère equation in the interior of the region in question. Thus, b is affine along each tangent and we have

$$b(x) = m(u)(x_1 - u) + f(u), \tag{3.3.5}$$

where $f(u) = |u|^p$.

From (3.2.7) we obtain

$$u_{x_2} = \frac{e^{-u}(1-\xi)}{x_1 - u - \xi}.$$

Then

$$\begin{aligned}
b_{x_2} &= \big(m'(u)(x_1-u) - m(u) + f'(u)\big)u_{x_2} \\
&= \frac{m'(u)(x_1-u-\xi) + m'(u)\xi - m(u) + f'(u)}{x_1-u-\xi}\, e^{-u}(1-\xi) \\
&= m'(u)e^{-u}(1-\xi) + \frac{m'(u)\xi - m(u) + f'(u)}{x_1-u-\xi}\, e^{-u}(1-\xi)
\end{aligned}$$

Since for a fixed u, b_{x_2} is constant with respect to x_1, we have

$$b_{x_2} = m'(u)e^{-u}(1-\xi) \tag{3.3.6}$$

and

$$m'(u)\xi - m(u) + f'(u) = 0. \tag{3.3.7}$$

Though we will mostly be dealing with b_{x_2}, in the same fashion we can obtain a formula for b_{x_1}:

$$b_{x_1} = m(u) - m'(u), \tag{3.3.8}$$

Solving the linear differential equation (3.3.7) gives

$$m(u) = e^{u/\xi}\left(A - \frac{1}{\xi}\int_{u_0}^{u} e^{-s/\xi} f'(s)\, ds\right). \tag{3.3.9}$$

For a left-leaning tangential foliation, u_0 is the horizontal coordinate of the right-most point on the lower boundary Γ_1 to which it extends (including the possibility $u_0 = -\infty$), while for a right-leaning tangential foliation, it is the left-most point. We have three (pre-collision) or two (post-collision) different tangential foliations. Let us consider them separately.

Candidate in P_7 (pre-collision) and R_6 (post-collision)

Recall that the regions P_7 and R_6 have identical definitions in terms of the number w. While we will see that w itself is computed differently pre- and post-collision, the formula for b stays exactly the same.

Specifically, in this case we have $u = u^-$, $\xi = \xi^-$, $u_0 = -\infty$, and $A = 0$. Hence,

$$b(x) = M(u^-)(x_1 - u^-) + f(u^-), \quad -\infty < u^- \leqslant w, \tag{3.3.10}$$

where, from (3.3.9),

$$M(u) = -\frac{1}{\xi^-} e^{u/\xi^-} \int_{-\infty}^{u} e^{-s/\xi^-} f'(s)\, ds. \tag{3.3.11}$$

Candidate in P_5

Here we have $u = u^+$ and $\xi = \xi^+$. To determine u_0 and A, recall that in the cup P_4 we have

$$b(x) = -|v|^{p-1} x_1,$$

where (v, e^v) is the left end-point of the chord from $(0, 1)$ passing through x. We must match this expression with out desired formula $b(x) = m(x)(x_1 - u) + f(u)$ along the boundary between P_4 and P_5. Thus, we set

$$u_0 = v = w_0.$$

Now, from (3.3.9),

$$A = -e^{-w_0/\xi^+} |w_0|^{p-1}.$$

and so,

$$b(x) = \mu(u^+)(x_1 - u^+) + f(u^+), \quad w \leqslant u^+ \leqslant w_0, \tag{3.3.12}$$

where

$$\mu(u) = e^{u/\xi^+} \left(-e^{-w_0/\xi^+} |w_0|^{p-1} + \frac{1}{\xi^+} \int_u^{w_0} e^{-s/\xi^+} f'(s)\, ds \right). \tag{3.3.13}$$

Candidate in $P_1 = R_1$

Recall that P_1 and R_1 denote the same region, pre- and post-collision, respectively. Here we have $u = u^-$ and $\xi = \xi^-$. By an entirely symmetric reasoning to the previous case (matching in this case with the cup $P_2 = R_2$, we obtain

$$u_0 = -w_0, \qquad A = e^{w_0/\xi^-} |w_0|^{p-1}$$

and

$$b(x) = \mathcal{M}(u^-)(x_1 - u^-) + f(u^-), \quad -w_0 \leqslant u^- < \infty, \tag{3.3.14}$$

where

$$\mathcal{M}(u) = e^{u/\xi^-} \left(e^{w_0/\xi^-} |w_0|^{p-1} - \frac{1}{\xi^-} \int_{-w_0}^{u} e^{-s/\xi} f'(s)\, ds \right) \tag{3.3.15}$$

3.3.3 Candidates in regions of affinity

General considerations

Affine regions occur when two candidates with different foliations must be glued together. The boundaries of the regions being glued meet at a single point on Γ_1. For regions P_3 and R_3 this point is fixed; for regions P_6 and R_5, this point evolves with the parameter C to satisfy a compatibility condition imposed by our conjectured foliation. Since the function we are constructing is affine along each boundary subject to gluing, it must be affine in he whole region between these two boundaries. That is to say,

$$b(x) = \gamma_1\, x_1 + \gamma_2\, x_2 + \gamma_3.$$

The coefficients γ_i are determined by the values from the two boundaries. We will now consider each case separately.

Candidate in P_6

$$b(x) = \beta_1(x_1 - w) + \beta_2(x_2 - e^w) + f(w)$$

Since w is chosen so that b is continuously differentiable in the interior of $P_7 \cup P_6 \cup P_5$, we have, using (3.3.6) and (3.3.8):

$$\beta_1 = b_{x_1}(w)\Big|_{P_7} = M(w) - M'(w), \qquad \beta_2 = b_{x_2}(w)\Big|_{P_7} = M'(w)e^{-w}(1 - \xi^-).$$

Once we have established the defining equations for w we will be able to rewrite these coefficients more explicitly.

Candidate in P_3

$$b(x) = \alpha_1 x_1 + \alpha_2(x_2 - 1),$$

where

$$\alpha_1 + \alpha_2 \frac{1}{1 - \xi^-} = -|w_0|^{p-1}, \qquad \alpha_1 + \alpha \frac{1}{1 - \xi^+} = |w_0|^{p-1}$$

So, solving for α_1 and α_2,

$$\alpha_1 = |w_0|^{p-2}(\xi^+ + \xi^- - 2), \quad \alpha_2 = 2|w_0|^{p-2}(1 - \xi^-)(1 - \xi^+) \tag{3.3.16}$$

Candidate in R_5

$$b(x) = \beta_1(x_1 - w) + \beta_2(x_2 - e^w) + f(w).$$

Since, again, w is chosen so that b_{x_1} and b_{x_2} are continuous across the boundary between R_6 and R_5, we again have the same formulas for β_1 and β_2 as in section 3.3.3 above:

$$\beta_1 = b_{x_1}(w)\Big|_{R_6} = M(w) - M'(w), \qquad \beta_2 = b_{x_2}(w)\Big|_{R_6} = M'(w)e^{-w}(1-\xi^-). \quad (3.3.17)$$

Candidate in R_3

$$b(x) = \alpha_1 x_1 + \alpha_2(x_2 - 1),$$

where

$$\alpha_1+\alpha_2\,\frac{1}{1-\xi^-} = \beta_1+\beta_2\,\frac{1}{1-\xi^-} = M(w)+M'(w)(e^{-w}-1) = M'(w)(e^{-w}-1+\xi^-)-p|w|^{p-1},$$

where we have used β_1 and β_2 are from (3.3.17) and the equation $M = \xi^- M' + f'$ (cf. (3.3.7)), and

$$\alpha_1 + \alpha_2\,\frac{1}{1-\xi^+} = |w_0|^{p-1}.$$

Solving for α_1 and α_2 we have

$$\alpha_1 = |w_0|^{p-1} - \frac{|w_0|^{p-1} - M'(w)(e^{-w} - 1 + \xi^-) + p|w|^{p-1}}{|w_0|}\,(1-\xi^-) \quad (3.3.18)$$

and

$$\alpha_2 = \frac{|w_0|^{p-1} - M'(w)(e^{-w} - 1 + \xi^-) + p|w|^{p-1}}{|w_0|}\,(1-\xi^-)(1-\xi^+). \quad (3.3.19)$$

3.4 The global candidate and the main Bellman theorem

In this section, we prove that the individual regions and foliations conjectured in section 3.2 and constructed in section 3.3 can be glued together using certain compatibility conditions. We then prove that these conditions are fulfilled, which gives the formula for the Bellman candidate in the whole domain Ω_C. We then state the main theorem, asserting that the candidate and function coincide, as well as several corollaries.

3.4.1 Compatibility equations for C_* and w

Equation for w in the pre-collision case

When $C < C_*$, w is defined to be the unique number so that the Bellman candidate b so far defined individually in the regions P_7, P_6, and P_5 is continuously differentiable in the interior in the interior of their union. Since b is affine in P_6 by construction, this happens if and only if

$$b_{x_2}\Big|_{u^-=w} = b_{x_2}\Big|_{u^+=w}$$

Using (3.3.6), we have

$$M'(w)(1-\xi^-) = \mu'(w)(1-\xi^+), \tag{3.4.1}$$

where the functions M and μ are given, respectively, by (3.3.11) and (3.3.13).

Using the differential equation (3.3.7) and integrating by parts, we have

$$\begin{aligned} M'(w) &= \frac{1}{\xi^-}\left(M(w) - f'(w)\right) = \frac{1}{\xi^-}\Big(-\frac{1}{\xi^-}\, e^{w/\xi^-} \int_{-\infty}^{w} e^{-s/\xi^-} f'(s)\, ds - f(w)\Big) \\ &= -\frac{1}{\xi^-}\, e^{w/\xi^-} \int_{-\infty}^{w} e^{-s/\xi^-} f''(s)\, ds \end{aligned}$$

and

$$\begin{aligned} \mu'(w) &= \frac{1}{\xi^+}\left(\mu(w) - f'(w)\right) \\ &= \frac{1}{\xi^+}\Big(e^{w/\xi^+}\Big(-e^{-w_0/\xi^+}|w_0|^{p-1} + \frac{1}{\xi^+}\int_{w}^{w_0} e^{-s/\xi^+} f'(s)\, ds\Big) - f'(w)\Big) \\ &= \frac{p-1}{\xi^+}\, e^{(w-w_0)/\xi^+}|w_0|^{p-1} + \frac{1}{\xi^+}\, e^{w/\xi^+}\int_{w}^{w_0} e^{-s/\xi^+} f''(s)\, ds \end{aligned}$$

Substituting these two expressions into (3.4.1), writing f'' explicitly, changing the variable in both integrals to $y = \frac{s}{w}$ and cancelling where appropriate, we have

$$-\frac{1-\xi^-}{\xi^-}\, p\, e^{\frac{w}{\xi^-}} \int_1^\infty e^{-\frac{w}{\xi^-}\, y}\, y^{p-2}\, dy = \frac{1-\xi^+}{\xi^+}\, e^{\frac{w}{\xi^+}} \left(e^{-\frac{w_0}{\xi^+}} + p \int_{\frac{w_0}{w}}^1 e^{-\frac{w}{\xi^+}\, y}\, y^{p-2}\, dy \right) \tag{3.4.2}$$

Equation for w in the post-collision case

Similarly to the previous case, when $C > C_*$, w is defined to be the unique number so that the Bellman candidate b is continuously differentiable in the interior $R_6 \cup R_5 \cup R_4$. Since b is affine in R_5, this happens if and only if

$$b_{x_2}\Big|_{u^-=w} = b_{x_2}\Big|_{v=w}$$

Recall that in the cup R_4, we have

$$b_{x_2} = \frac{(p-1)|v|^p}{ve^v - e^v + 1}.$$

Using (3.3.6) we thus obtain the following equation for w:

$$M'(w)(1-\xi^-) = \frac{(p-1)|w|^p}{e^{-w} - 1 + w}, \tag{3.4.3}$$

where the function M is given by (3.3.11). Rewriting $M'(w)$ as in the previous section and rearranging (3.4.3) slightly, we have

$$pe^{w/\xi^-} \int_1^\infty e^{-\frac{w}{\xi^-}\, y}\, y^{p-2}\, dy = -\frac{\xi^-}{1-\xi^-}\, \frac{|w|}{e^{-w} - 1 + w} \tag{3.4.4}$$

Equation for C_*

Finally, $C_* = C_*(p)$ by design is the value such that

$$w(C_*) = w_0(C_*),$$

where $w(C)$ is given by either (3.4.1) or (3.4.4). These two equations become the same at the moment of collision, thus, using (3.4.4), we have the following equation for C_*:

$$pe^{w_0/\xi^-}\int_1^\infty e^{-\frac{w_0}{\xi^-}y}\,y^{p-2}\,dy = -\frac{\xi^-}{1-\xi^-}\,\frac{|w_0|}{e^{-w_0}-1+w_0}. \tag{3.4.5}$$

From the definition of $\xi^\pm$,

$$e^{-w_0}-1+w_0 = e^{-\xi^-+\xi_0}-1+\xi^- - \xi^+ = \frac{1-\xi^-}{1-\xi^+}-1+\xi^- - \xi^+ = \frac{\xi^+(\xi^+-\xi^-)}{1-\xi^+}, \tag{3.4.6}$$

and, thus, (3.4.5) becomes

$$pe^{w_0/\xi^-}\int_1^\infty e^{-\frac{w_0}{\xi^-}y}\,y^{p-2}\,dy = -\frac{(1-\xi^+)\xi^-}{(1-\xi^-)\xi^+} \tag{3.4.7}$$

3.4.2 Key lemmas about C_* and w

Here we prove that the compatibility equations stated stated above can be resolved uniquely and also obtain key bounds on the numbers C_* and $w(C)$.

Lemma 3.4.1. *Fix $p \in (0,1)$.*

1. *There exists a unique $w_1 = w_1(p) < 0$ such that*

$$\frac{w_1^2}{e^{-w_1}-1+w_1} = p, \tag{3.4.8}$$

2. *There exists a unique $w_2 = w_2(p) < 0$ such that*

$$\frac{w_2}{1-e^{-w_2}} = p. \tag{3.4.9}$$

3. *The following inequalities are true:*

$$w_1(p) < \min\{p-2, w_2(p)\} \tag{3.4.10}$$

and

$$w_2(p) < p - 1. \tag{3.4.11}$$

Proof. To prove statement (1), let us rewrite (3.4.8) by flipping the fractions and expanding the exponent in a power series:

$$\sum_{k=0}^{\infty} \frac{(-w_1)^k}{(k+2)!} = \frac{1}{p} \tag{3.4.12}$$

The left-hand side of this equation is strictly increasing in $|w_1|$; it is also equal to $\frac{1}{2}$ and thus less than $\frac{1}{p}$ when $w_1 = 0$; and it is clearly larger than $\frac{1}{p}$ for all sufficiently large $|w_1|$. This proves the existence of a unique root $w_1 \in (-\infty, 0)$, as required.

To prove statement (2), let us similarly rewrite (3.4.9):

$$\sum_{k=0}^{\infty} \frac{(-w_2)^k}{(k+1)!} = \frac{1}{p} \tag{3.4.13}$$

The left-hand side is strictly increasing in $|w_2|$; is equal to 1 and thus less than $\frac{1}{p}$ for $w_1 = 0$; and is larger than $\frac{1}{p}$ for all sufficiently large $|w_2|$. This proves the existence of a unique root $w_2 \in (-\infty, 0)$.

To prove that $w_1 < w_2$, compare the left-hand sides of (3.4.12) and (3.4.13) when evaluated at the same negative number w. We clearly have:

$$\sum_{k=0}^{\infty} \frac{(-w)^k}{(k+2)!} < \sum_{k=0}^{\infty} \frac{(-w)^k}{(k+1)!}$$

When $w = w_1$, the left-hand side equals $\frac{1}{p}$, thus the right-hand side is greater than $\frac{1}{p}$. Since the right-hand side is strictly increasing in $|w|$ and equals $\frac{1}{p}$ when $w = w_2$, we conclude that $|w_1| > |w_2|$ or, equivalently, that $w_1 < w_2$.

Proving that $w_1 < p-2$ is equivalent to showing that the left-hand side of (3.4.12) is less than $\frac{1}{p}$ when we replace w_1 with $p-2$. Equivalently, we want to show that

$$\frac{e^{2-p}-1+(p-2)}{(p-2)^2} < \frac{1}{p} \qquad \Longleftrightarrow \qquad pe^{2-p}+p-4<0$$

The last inequality is true at $p=1$, and its left-hand side is increasing in p. This means it is true of all $p \in (0,1)$, and (3.4.10) is thus proved.

Similarly, to show (3.4.11) is equivalent to showing that

$$\frac{1-e^{1-p}}{1-p} < \frac{1}{p} \qquad \Longleftrightarrow \qquad pe^{1-p} < 1.$$

The last inequality becomes equality when $p=1$, and its left-hand side is again increasing in p. The proof of the lemma is complete. □

Lemma 3.4.2. *For any $\theta > 0$ and $q > \max\{0, 1-\theta\}$ we have*

$$\frac{1}{\theta+q} \leqslant \int_1^\infty e^{-\theta(y-1)}\, y^{-q}\, dy \leqslant \frac{1}{\theta+q-1}. \tag{3.4.14}$$

Proof. We make use of the elementary inequality

$$e^{-(y-1)} \leqslant \frac{1}{y},$$

valid for all $y \geqslant 1$. With it, we immediately obtain

$$\int_1^\infty e^{-(\theta+q)(y-1)}\, dy \leqslant \int_1^\infty e^{-\theta(y-1)}\, y^{-q}\, dy \leqslant \int_1^\infty y^{-\theta-q}\, dy,$$

and, after integration, the bounds in (3.4.14). □

Lemma 3.4.3.

(i) For every $p \in (0,1)$, there exists a unique $C_* = C_*(p) > 1$ such that equation (3.4.7) holds with $w_0 = w_0(C_*)$ and $\xi^{\pm} = \xi^{\pm}(C_*)$.

(ii) Let $w_*(p) = w_0(C_*(p))$. Then

$$w_1(p) < w_*(p) < \min\{p-2, w_2(p)\}, \tag{3.4.15}$$

where $w_1(p)$ and $w_2(p)$ are the numbers given by Lemma 3.4.1. Consequently,

$$\frac{w^2}{e^{-w}-1+w} \geqslant p, \text{ for all } w \geqslant w_*(p). \tag{3.4.16}$$

Proof. We first prove statement (i). Let us refer to the left- and right-hand sides of (3.4.7) as $L(C)$ and $R(C)$, respectively. Let $\tau(C) = 1 - \frac{\xi^+(C)}{\xi^-(C)}$; thus,

$$L(C) = p\int_1^\infty y^{p-2}e^{-\tau(y-1)}\,dy \tag{3.4.17}$$

From (3.2.4), we have $\tau(C) \to 2$ as $C \to 1$; from (3.2.2), we have $\tau(C) \to 1$ as $C \to \infty$. Thus,

$$\lim_{C\to 1} L(C) = p\int_1^\infty y^{p-2}e^{-2(y-1)}\,dy \leqslant \int_1^\infty e^{-2(y-1)}\,ds = \frac{1}{2}.$$

From (3.2.4) we also have

$$\lim_{C\to 1} R(C) = 1.$$

At the other endpoint we have

$$\lim_{C\to\infty} L(C) = p\int_1^\infty y^{p-2}e^{-(y-1)}\,dy > 0,$$

and, from (3.2.2),

$$\lim_{C\to\infty} R(C) = 0.$$

In shorthand, $L(1) < R(1)$ and $L(\infty) > R(\infty)$. This proves the existence of a root C_* in the interval $(1, \infty)$.

The uniqueness of this root will follow if we show that $L(C)$ is strictly increasing in C and $R(C)$ is strictly decreasing in C. First, observe that τ is decreasing in C. Indeed, using (3.2.3) we have

$$\begin{aligned}
\tau' &= -\frac{(\xi^+)'\xi^- - (\xi^-)'\xi^+}{(\xi^-)^2} = -\frac{1}{C(\xi^-)^2}\Big(\frac{1-\xi^+}{\xi^+}\,\xi^- - \frac{1-\xi^-}{\xi^-}\,\xi^+\Big) \\
&= -\frac{(\xi^+ - \xi^-)(\xi^+ + \xi^- + \xi^+\xi^-)}{C(\xi^-)^3\xi^+} < 0,
\end{aligned}$$

since $\xi^+ + \xi^- + \xi^+\xi^- < 0$, which follows from the second inequality in (3.2.4) and the fact that $\xi^-\xi^+ < 0$. By (3.4.17) L is decreasing in τ, and we have thus reached the desired conclusion about $L(C)$.

To deal with $R(C)$, we differentiate directly:

$$\begin{aligned}
R'(C) &= \frac{(-(\xi^-)'\xi^+ + (1-\xi^-)(\xi^+)')(1-\xi^+)\xi^- - (-(\xi^+)'\xi^- + (1-\xi^+)(\xi^-)')(1-\xi^-)\xi^+}{(1-\xi^-)^2(\xi^+)^2} \\
&= \frac{(1-\xi^-)(1-\xi^+)((\xi^-)^2 - (\xi^+)^2)}{C(1-\xi^-)^2(\xi^+)^3\xi^-} < 0,
\end{aligned}$$

where we used (3.2.3) on two occasions. The last factor in the numerator is positive by (3.2.4); the denominator is obviously negative. This completes the proof of (i).

We turn to the proof of (ii). In light of the monotonicity in C of the two sides of (3.4.7) that we have just established, to prove the left inequality in (3.4.15) is the same as to demonstrate that the left-hand side of (3.4.7) is greater than the right-hand side, when both are evaluated at $w_0 = w_1(p)$ and the corresponding $\xi^\pm$. Similarly, to prove the right inequality in (3.4.15) it the same as to show that the left-hand side of (3.4.7) is smaller than the right-hand side, when both are evaluated at $w_0 = \min\{p-2, w_2(p)\}$.

Now, for $w_0 = w_1(p)$ and the corresponding C, we use Lemma 3.4.2 with $\theta = \tau$ and $q = 2 - p$ to estimate $L(C)$ from below:

$$p \int_1^\infty y^{p-2} e^{-\tau(y-1)}\, dy \geqslant \frac{p}{\tau + 2 - p}$$

On the other hand, using first (3.4.6) and then the definition of w_1,

$$\begin{aligned} R(C)\Big|_{w_0=w_1} &= -\frac{(1-\xi^+)\xi^-}{(1-\xi^-)\xi^+} = \frac{\xi^-}{1-\xi^-}\,\frac{w_1}{e^{-w_1} - 1 + w_1} \\ &= \frac{\xi^-}{w_1(1-\xi^-)}\,\frac{w_1^2}{e^{-w_1} - 1 + w_1} = p\,\frac{\xi^-}{w_1(1-\xi^-)} = \frac{p}{\tau(1-\xi^-)}. \end{aligned}$$

Hence, to show $L(C)\big|_{w_0=w_1} > R(C)\big|_{w_0=w_1}$ it suffices to show that

$$\frac{1}{\tau + 2 - p} > \frac{1}{\tau(1-\xi^-)},$$

or, equivalently, $-\tau\xi^- < 2 - p$, which is the same thing as $w_1 < p - 2$, which is true by Lemma 3.4.1.

For $w_0 = w_2(p)$, we estimate $L(C)$ from above using Lemma 3.4.2 with $\theta = \tau$ and $q = 2 - p$:

$$L(C) \leqslant \frac{p}{\tau + 1 - p}.$$

On the other hand, using (3.4.6) and then the definition of w_2,

$$\begin{aligned} R(C)\Big|_{w_0=w_2} &= -\frac{\xi^-}{1-\xi^-}\,\frac{-w_2}{e^{-w_2} - 1}\,\frac{e^{-w_2} - 1}{e^{-w_2} - 1 + w_2} \\ &= -p\,\frac{\xi^-}{1-\xi^-}\,\frac{e^{-w_2} - 1}{e^{-w_2} - 1 + w_2} \end{aligned}$$

Since $\frac{w_2}{1-e^{-w_2}} = p$ we have $e^{-w_2} - 1 = -\frac{p}{w_2}$, so

$$R(C)\Big|_{w_0=w_2} = -p\,\frac{\xi^-}{1-\xi^-}\,\frac{-\frac{w_2}{p}}{-\frac{w_2}{p} + w_2} = -\frac{p}{1-p}\,\frac{\xi^-}{1-\xi^-}.$$

Thus, to show that $L(C)\big|_{w_0=w_2} < R(C)\big|_{w_0=w_2}$ it suffices to show that

$$\frac{1}{\tau+1-p} < -\frac{1}{1-p}\frac{\xi^-}{1-\xi^-}.$$

Rewriting, we have

$$(1-p)(1-\xi^-) < -\xi^-(\tau+1-p) \quad\Longleftrightarrow\quad -\xi^-\tau > 1-p \quad\Longleftrightarrow\quad w_2 < p-1,$$

the last inequality being true by Lemma 3.4.1.

For $w_0 = p-2$ we have to work a bit harder. We take the integral in $L(C)$ by parts:

$$L(C) = \frac{p}{\tau}\left[1-(2-p)\int_1^\infty e^{-\tau(y-1)}y^{p-3}\,dy\right],$$

and now estimate the new integral (which has a negative multiple in front) from below using Lemma 3.4.2 with $\theta = \tau$ and $q = 3-p$, obtaining a new upper bound on $L(C)$:

$$L(C) \leqslant \frac{p}{\tau}\left[1-\frac{2-p}{\tau+3-p}\right] = \frac{p(\tau+1)}{\tau(\tau+3-p)}.$$

For the right-hand side we have from (3.2.6):

$$R(C) = \frac{e^{w_0}-1-w_0}{e^{-w_0}-1+w_0}. \tag{3.4.18}$$

We have $p = w_0+2$ and $\tau = \frac{w_0}{\xi^-} = \frac{w_0(e^{w_0}-1)}{e^{w_0}-1-w_0}$. Let us temporarily write w for w_0. After some elementary algebra, we conclude that to prove that $L(C)\big|_{w_0=p-2} < R(C)\big|_{w_0=p-2}$ for $0<p<1$ it suffices to show that, for all $w \in (-2,-1)$

$$\frac{\big((w+1)(e^w-1)-w\big)(w+2)}{e^w-1-w+w^2} < \frac{w(e^w-1)}{e^{-w}-1+w} \tag{3.4.19}$$

Define,

$$S(w) = w(e^w - 1)(e^w - 1 - w + w^2) - ((w+1)(e^w - 1) - w)(w+2)(e^{-w} - 1 + w).$$

To prove 3.4.19, it's sufficient to show S is positive on the interval $(-2, -1)$. Rearranging gives:

$$S(w) = \left[e^{-w}(2w^2 + 5w + 2) + w^3 + 3w^2 - 5w - 4\right] - e^w(3w^2 - we^w - 2 + w).$$

Additionally, we define the functions:

$$\begin{aligned} h(w) &= e^{-w}(2w^2 + 5w + 2) + w^3 + 3w^2 - 5w - 4, \\ g(w) &= e^w(3w^2 - we^w - 2 + w). \end{aligned}$$

We compute the following derivatives:

$$\begin{aligned} h'(w) &= e^{-w}(3 - 2w^2 - w) + (3w^2 + 6w - 5), \\ h''(w) &= e^{-w}(2w^2 - 3w - 4) + 6(w+1). \end{aligned}$$

Since $2w^2 - 3w - 4 > 0$ on $(-2, -1)$, we have:

$$h''(w) > 4w^2 - 6w - 8 + 6(w+1) = 4w^2 - 2 > 0.$$

Therefore, h is convex on $(-2, -1)$. Similarly, we compute:

$$\begin{aligned} g'(w) &= e^w(7w + 3w^2 - e^w(1 + 2w) - 1), \\ g''(w) &= e^w(6 + 13w + 3w^2 - 4e^w(1 + w)). \end{aligned}$$

Since $-(1+w) > 0$ on $(-2,-1)$, we have:

$$g''(w) < e^w(6 + 13w + 3w^2 - 4(1+w)) = e^w(9w + 3w^2 + 2) < 0.$$

Therefore, g is concave on $(-2,-1)$. It follows that h lies above its tangent at $w = -1$ and g lies below its tangent at this same point. Since:

$$h(-1) - g(-1) = 3 - e - e^{-2} > 0, \quad h'(-1) - g'(-1) = e^{-2}\left[2e^3 - 8e^2 + 5e - 1\right] < 0,$$

the tangent to h at $w = -1$ lies above the tangent to g at $w = -1$. Consequently, $h(w) > g(w)$ on $(-2,-1)$, from which the desired result for S immediately follows.

Finally, to see (3.4.16), we rewrite it as

$$\sum_{k=0}^{\infty} \frac{(-w)^k}{(k+2)!} \leqslant \frac{1}{p}.$$

The left-hand side is clearly increasing in $|w|$. Since for $w = w_1$ we have equality and if $w \geqslant w_*$ then $|w| < |w_1|$, (3.4.16) holds as claimed. □

Figure 3.9 shows the function $w_*(p)$ given by Lemma 3.4.3 and graphically illustrates inequality (3.4.15).

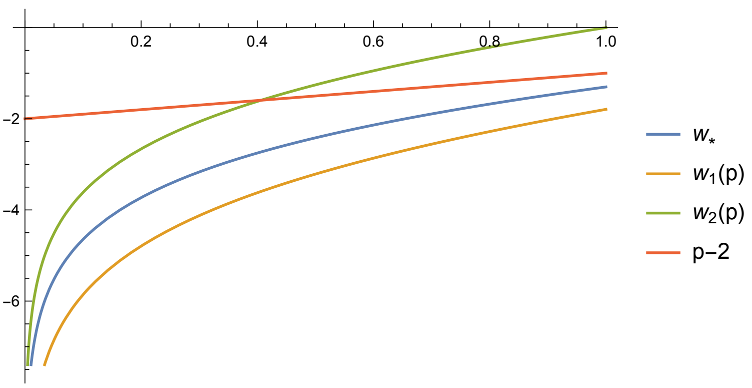

Figure 3.9: The relationship among w_*, w_1, w_2, and $p-2$.

Now, fix $p \in (0,1)$ and $C > 1$. The notation in the following statement is from Lemma 3.4.3.

Lemma 3.4.4. *Let C_* be given by Lemma 3.4.3.*

(i) If $1 < C < C_*$, then equation (3.4.2) *has a unique solution w in the interval $(-\infty, w_0)$ satisfying $w_*(p) < w < p-2$.*

(ii) If $C_* < C < \infty$, then (3.4.4) *has a unique solution w in the interval $(w_0, 0)$ satisfying $w_*(p) < w < w_1(p)$.*

Proof. We start with statement (i). Let us rewrite (3.4.2) in the form

$$l(w) = r(w),$$

where

$$l(w) = p\,\frac{w}{\xi^-}\int_1^\infty e^{-\frac{w}{\xi^-}(y-1)}\,y^{p-2}\,dy \tag{3.4.20}$$

and, after changing the variable in the integral,

$$r(w) = \frac{1-\xi^+}{1-\xi^-}\Big(-\frac{w}{\xi^+}\Big)e^{\frac{w}{\xi^+}}\left(e^{-\frac{w_0}{\xi^+}} + p\Big(-\frac{w}{\xi^+}\Big)^{1-p}\int_{\frac{-w_0}{\xi^+}}^{\frac{-w}{\xi^+}} e^z z^{p-2}\,dz\right)$$

First, note that with ξ^- fixed the function $l(w)$ is increasing in $\frac{w}{\xi^-}$. Since this fraction itself is decreasing in w, we conclude that l is decreasing in w. Likewise, the function

$$G(s) := se^{-s}\Big(e^{s_0} + ps^{1-p}\int_{s_0}^{s} e^z z^{p-2}\,dz\Big)$$

is decreasing in s for $2 < s_0 < s$. Note that we have $-\frac{w_0}{\xi^+} > 2$, meaning $r(w)$ is decreasing in $-\frac{w}{\xi^+}$. Since this fraction is decreasing in w, we conclude that r is increasing in w. Therefore, to prove the statement, we have to show that

$$l(w_*) > r(w_*) \qquad \text{and} \qquad l\big(\min\{p-2, w_0\}\big) < r\big(\min\{p-2, w_0\}\big).$$

The inequalities $l(w_*) > r(w_*)$ and $l(p-2) < r(p-2)$ are handled using similar arguments to those in the proof of Lemma 3.4.3. To show that $l(w_0) < r(w_0)$, recall the notation $L(C)$ and $R(C)$ from the proof in Lemma 3.4.3 for the left- and right-hand sides of equation (3.4.7), respectively and observe that since $C < C_*$, we have $L(C) < R(C)$. Then

$$l(w_0) = \Big(-\frac{w_0}{\xi^-}\Big)L(C) < \Big(-\frac{w_0}{\xi^-}\Big)R(C) = r(w_0),$$

as required.

To prove (ii), we rewrite (3.4.4) in the form

$$l(w) = r(w),$$

where $l(w)$ is the same as before and given by (3.4.20) and

$$r(w) = \frac{1}{1-\xi^-}\,\frac{w^2}{e^{-w} - 1 + w}.$$

First, as noted above, l is decreasing in w. Second, as we have seen already seen, the second fraction in r is decreasing in $|w|$, which means that r is increasing in w. Therefore, to prove the statement, we have to show that

$$l(w_*) > r(w_*) \qquad \text{and} \qquad l(w_2) < r(w_2).$$

Since $C > C_*$, we have $\xi^- < \xi^-(C_*)$ and so $\frac{w_*}{\xi^-} > \frac{w_*}{\xi^-(C_*)}$. Hence,

$$l(w_*) > p\,\frac{w_*}{\xi^-(C_*)} \int_1^\infty e^{-\frac{w_*}{\xi^-(C_*)}(y-1)}\, y^{p-2}\, dy = \frac{w_*}{\xi^-(C_*)}\, L(C_*).$$

On the other hand,

$$r(w_*) < \frac{1}{1-\xi^-(C_*)}\,\frac{w_*^2}{e^{-w_*} - 1 + w_*} = \frac{w_*}{\xi^-(C_*)}\, R(C_*).$$

Since $L(C_*) = R(C_*)$ and $\frac{w_*}{\xi^-(C_*} > 0$, we have established the inequality for the left endpoint.

When $w = w_2$, we estimate l from above using Lemma 3.4.2 with $\theta = \frac{w_2}{\xi^-}$ and $q = p-2$:

$$l(w_2) \leqslant \frac{p\,\frac{w_2}{\xi^-}}{\frac{w_2}{\xi^-} + 1 - p}.$$

Next, observe that

$$r(w_2) = \frac{1}{1-\xi^-}\,\frac{w_2^2}{e^{-w_2} - 1 + w_2} = \frac{1}{1-\xi^-}\,\frac{w_2^2}{-\frac{w_2}{p} + w_2} = \frac{1}{1-\xi^-}\,\frac{-w_2 p}{1-p}.$$

Thus, to show $l(w_2) < r(w_2)$ it suffices to show

$$\frac{p\,\frac{w_2}{\xi^-}}{\frac{w_2}{\xi^-} + 1 - p} < \frac{1}{1-\xi^-}\,\frac{-w_2 p}{1-p},$$

which is equivalent to the inequality $w_2 < p - 1$, which in turn is true by Lemma 3.4.1. This completes the proof of (ii) and the lemma. $\square$

Figures 3.10, 3.11, and 3.12 illustrate the results of Lemma 3.4.4 for $p = \frac{1}{4}$, $p = \frac{3}{4}$, and $p = \frac{9}{10}$, respectively. Shown are the starting position of w (the red line at $w = p-2$), its dynamics of w in both pre- and post-collision ranges (the blue line); the bounds w_* and w_2 (the green and dashed red lines, respectively); and the movement of w_0 (the yellow line). The moment of collision corresponds to the intersection of the w and w_0 graphs.

3.4.3 The expression for the global Bellman candidate

We will now assemble the pieces of the last two sections and present our full Bellman candidate $b_{p,C}$ on Ω_C with explicit expressions for each region of our proposed foliation.

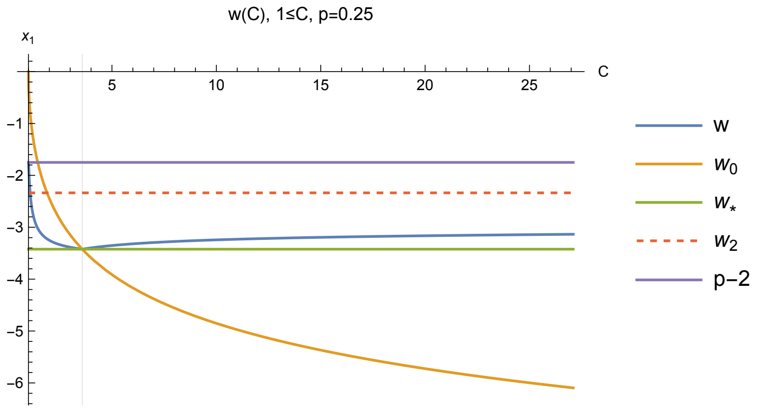

Figure 3.10: The dynamics of $w(C)$ for $p = \frac{1}{4}$.

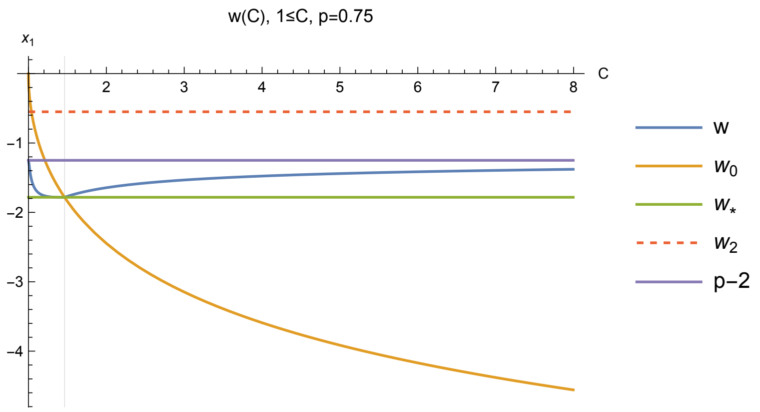

Figure 3.11: The dynamics of $w(C)$ for $p = \frac{3}{4}$.

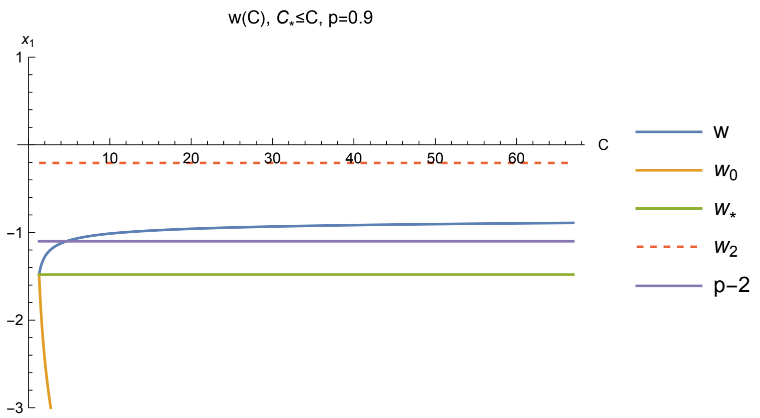

Figure 3.12: The dynamics of $w(C)$ for $p = \frac{9}{10}$, post-collision

In the pre-collision configuration, we have:

$$
\begin{aligned}
P_1 =&\{\xi^+ \leqslant x_1 \leqslant -w_0,\ Ce^{\xi^+} x_1 + 1 \leqslant x_2 \leqslant Ce^{x_1}\}\cup\\
&\{-w_0 \leqslant x_1 < \infty,\ e^{x_2} \leqslant x_2 \leqslant Ce^{x_1}\}\\
P_2 =&\{0 \leqslant x_1 \leqslant -w_0,\ e^{x_2} \leqslant x_2 \leqslant Ce^{x_1}\}\\
P_3 =&\{0 \leqslant x_1 \leqslant \xi^+,\ Ce^{\xi^+} x_1 + 1 \leqslant x_2 \leqslant Ce^{x_1}\}\cup\\
&\{\xi^- \leqslant x_1 \leqslant 0,\ Ce^{\xi^-} x_1 + 1 \leqslant x_2 \leqslant Ce^{x_1}\}\\
P_4 =&\{w_0 \leqslant x_1 \leqslant 0,\ e^{x_2} \leqslant x_2 \leqslant Ce^{\xi^-} x_1 + 1\}\\
P_5 =&\{w_0 \leqslant x_1 \leqslant \xi^-,\ Ce^{\xi^-} x_1 + 1 \leqslant x_2 \leqslant Ce^{x_1}\}\cup\\
&\{w + \xi^+ \leqslant x_1 \leqslant w_0,\ e^{x_2} \leqslant x_2 \leqslant Ce^{x_1}\}\cup\\
&\{w \leqslant x_1 \leqslant w + \xi^+,\ e^{x_2} \leqslant x_2 \leqslant Ce^{w+\xi^+}\left[x_1 - (w + \xi^+) + 1\right]\}\\
P_6 =&\{w \leqslant x_1 \leqslant w + \xi^+,\ Ce^{w+\xi^+}\left[x_1 - (w + \xi^+) + 1\right] \leqslant x_2 \leqslant Ce^{x_1}\}\cup\\
&\{w + \xi^- \leqslant x_1 \leqslant w,\ Ce^{w+\xi^-}\left[x_1 - (w + \xi^-) + 1\right] \leqslant x_2 \leqslant Ce^{x_1}\}\\
P_7 =&\{w + \xi^- \leqslant x_1 \leqslant w,\ e^{x_2} \leqslant x_2 \leqslant Ce^{w+\xi^-}\left[x_1 - (w + \xi^-) + 1\right]\}\cup\\
&\{-\infty < x_1 \leqslant w + \xi^-,\ e^{x_2} \leqslant x_2 \leqslant Ce^{x_1}\}
\end{aligned}
$$

The formula for the Bellman candidate is given by:

$$
b_{p,C}(x) = \begin{cases}
\mathcal{M}(u^-)(x_1 - u^-) + |u^-|^p, & x \in P_1\\
(v^+)^{p-1} x_1, & x \in P_2\\
\alpha_1\, x_1 + \alpha_2\,(x_2 - 1), & x \in P_3\\
-|v^-|^{p-1} x_1, & x \in P_4\\
\mu(u^+)(x_1 - u^+) + |u^+|^p, & x \in P_5\\
\beta_1\,(x_1 - w) + \beta_2\,(x_2 - e^w) + |w|^p, & x \in P_6\\
M(u^-)(x_1 - u^-) + |u^-|^p, & x \in P_7
\end{cases} \tag{3.4.21}
$$

where the slopes M, μ and $\mathcal{M}$ are given by formulas (3.3.11), (3.3.13) and (3.3.15), respectively, and the coefficients α_1, α_2 by the formula (3.3.16). The coefficients β_1, β_2 are given by:

$$\beta_1 = \frac{(1-\xi^-)M(w) - (1-\xi^+)\mu(w)}{|w_0|} \tag{3.4.22}$$

$$\beta_2 = [\mu(w) - M(w)]\,\frac{(1-\xi^-)(1-\xi^+)}{|w_0|}e^{-w} \tag{3.4.23}$$

In the post-collision configuration, we have:

$$\begin{aligned}
R_1 =& P_1\\
R_2 =& P_2\\
R_3 =& P_3\\
R_4 =& \{w \leqslant x_1 \leqslant 0,\ e^{x_2} \leqslant x_2 \leqslant w^{-1}[e^w - 1]x_1 + 1\}\\
R_5^* =& \{w + \xi^- \leqslant x_1 \leqslant w,\ Ce^{w+\xi^-}\left[x_1 - (w+\xi^-) + 1\right] \leqslant x_2 \leqslant Ce^{x_1}\}\cup\\
& \{w \leqslant x_1 \leqslant \xi^-,\ w^{-1}[e^w - 1]x_1 + 1 \leqslant x_2 \leqslant Ce^{x_1}\}\cup\\
& \{w \leqslant x_1 \leqslant 0,\ w^{-1}[e^w - 1]x_1 + 1 \leqslant x_2 \leqslant Ce^{\xi^-}x_1 + 1\}\\
R_5 =& \{w_0 \leqslant x_1 \leqslant \xi^-,\ Ce^{\xi^-}x_1 + 1 \leqslant x_2 \leqslant Ce^{x_1}\}\cup\\
& \{w + \xi^+ \leqslant x_1 \leqslant w_0,\ e^{x_2} \leqslant x_2 \leqslant Ce^{x_1}\}\cup\\
& \{w \leqslant x_1 \leqslant w + \xi^+,\ e^{x_2} \leqslant x_2 \leqslant Ce^{w+\xi^+}\left[x_1 - (w+\xi^+) + 1\right]\}\\
R_6 =& P_7
\end{aligned}$$

where R_5^* gives the region R_5 for those C such that $w < \xi^-$ (and R_5 afterwards). The formula for the Bellman candidate in this case is:

$$b_{p,C}(x) = \begin{cases} \mathcal{M}(u^-)(x_1 - u^-) + |u^-|^p, & x \in R_1 \\ (v^+)^{p-1} x_1, & x \in R_2 \\ \alpha_1\, x_1 + \alpha_2\, (x_2 - 1), & x \in R_3 \\ -|v^-|^{p-1} x_1, & x \in R_4 \\ \beta_1\, (x_1 - w) + \beta_2\, (x_2 - e^w) + |w|^p, & x \in R_5 \\ M(u^-)(x_1 - u^-) + |u^-|^p, & x \in R_6 \end{cases} \tag{3.4.24}$$

where M and $\mathcal{M}$ are the same as in the pre-collision case and the coefficients α_1 and α_2 are given by the formulas (3.3.18) and (3.3.19), respectively. The coefficients β_1 and β_2 are:

$$\beta_1 = \frac{(1 - e^{-w})(1 - \xi^-)\, M(w) - |w|^p}{(1 - e^{-w})(1 - \xi^-) - w} \tag{3.4.25}$$

$$\beta_2 = \frac{|w|^p - w\, M(w)}{(1 - e^{-w})(1 - \xi^-) - w}(1 - \xi^-) e^{-w} \tag{3.4.26}$$

3.4.4 The main Bellman theorem

Having constructed our candidate, we are in a position to state the main Bellman theorem. Recall the definition (3.1.6) of the Bellman function $\boldsymbol{b}_{p,C}$ and the definition of the candidate $b_{p,C}$ in the previous section.

Theorem 3.4.5. *Let $p \in (0, 1)$ and $C > 1$. Then*

$$\boldsymbol{b}_{p,C}(x) = b_{p,C}(x), \quad x \in \Omega_C.$$

The proof of the theorem consists of two parts: $\boldsymbol{b}_{p,C} \geqslant b_{p,C}$ and $\boldsymbol{b}_{p,C} \leqslant b_{p,C}$. As mentioned in the introduction, the first part has traditionally been carried out using the so-called Bellman induction. In our setting, one would first prove that the candidate b is locally convex on Ω_C and then use Vasyunin's lemma from [Vas03] to constructs a special quasi-dyadic martingale on which to run the induction on scales. However, the second part is no longer necessary. We have the following special case of a general theorem from [SZ16].

Theorem 3.4.6 ([SZ16])**.** *The Bellman function $\boldsymbol{b}_{p,C}$ is the largest locally convex function b on Ω_C satisfying the boundary condition $b(t, e^t) = |t|^p$.*

Since our candidate $b_{p,C}$ satisfies the boundary condition by construction, to show that $\boldsymbol{b}_{p,C} \geqslant b_{p,C}$ it is sufficient to show that $b_{p,C}$ is locally convex in Ω_C.

Lemma 3.4.7. *The function $b_{p,C}$ given by 3.4.21 and 3.4.24 is locally convex in Ω_C. Consequently,*

$$\boldsymbol{b}_{p,C} \geqslant b_{p,C}.$$

The proof of this lemma is given in section 3.5.

The converse inequality will be established by presenting explicit optimizers, i.e., appropriate test functions whose p-averages coincide with the values of $b_{p,C}$.

Lemma 3.4.8. *For each $x \in \Omega_C$, there exists a function $\varphi \in E_{x,C,(0,1)}$ such that $\langle |\varphi|^p \rangle_{(0,1)} = b_{p,C}(x)$. Consequently,*

$$\boldsymbol{b}_{p,C} \leqslant b_{p,C}.$$

The proof of this lemma is given in section 3.6.

We now collect several immediate corollaries of Theorem 3.4.5. The first corollary is the sharp estimate for the p-average of a logarithm of an A_∞ weight in terms of the appropriate averages and the A_∞-characteristics.

Corollary 3.4.9. *Let φ be a function on an interval I such that $e^\varphi \in A_\infty(I)$. Let $C = [e^\varphi]_{A_\infty(I)}$. Then, for each $p \in (0,1)$, we have the inequality*

$$\langle|\varphi|^p\rangle_I \geqslant b_{p,C}\big(\langle\varphi\rangle_I, \langle e^\varphi\rangle_I\big) \tag{3.4.27}$$

The inequality is sharp for each choice of $\langle\varphi\rangle_I$ and $\langle e^\varphi\rangle_I$.

The second corollary is the formula for $\boldsymbol{b}_{p,C}(0,C)$, which will be important in our main application to the John–Nirenberg constant.

Corollary 3.4.10. *Fix $p \in (0,1)$. Let $C_* = C_*(p)$ be given by (3.4.7). Then For $1 < C \leqslant C_*$,*

$$\boldsymbol{b}_{p,C}(0,C) = 2(\xi^+ - \xi^-)^{p-2}(1-\xi^-)(1-\xi^+)(C-1).$$

For $C > C_*$,

$$\boldsymbol{b}_{p,C}(0,C) = \frac{(\xi^+ - \xi^-)^{p-1} + |w|^{p-1}\Big(p - \frac{(1-p)w(e^{-w}+\xi^- -1)}{(1-\xi^-)(e^{-w}+w-1)}\Big)}{\xi^+ - \xi -}(1-\xi^-)(1-\xi^+)(C-1)$$

where $w = w(p,C)$ is given by (3.4.4).

Lastly, we have a sharp estimate on the BMO^p norm of logarithms of A_∞ weights.

Corollary 3.4.11. *Let $w \in A_\infty(I)$ and let $C = [w]_{A_\infty(I)}$. Then*

$$\|\log w\|_{BMO^p} \geqslant \big(\boldsymbol{b}_{p,C}(0,C)\big)^{1/p},$$

where $\boldsymbol{b}_{p,C}(0,C)$ is given in Corollary 3.4.10.

3.5 Local convexity of the Bellman candidate

In this section we prove Lemma 3.4.7. Our Bellman candidate b is by construction continuous on Ω_C. Furthermore, it is twice continuously differentiable and satisfies the homogeneous Monge-Ampère equation the interior of each subdomain. Therefore, as explained in [SV12] to check the local convexity of b in Ω_C, it is necessary and sufficient to check that $b_{x_2x_2}$ is non-negative in each individual subdomain and then verify that the jump in the derivative b_{x_2} (if any) across any shared boundary and in the direction of increasing x_2 is non-negative. We first collect several useful general calculations and then separate presentation into the the pre- and post-collision cases.

3.5.1 General calculations pertaining to $b_{x_2x_2}$

Tangential candidates

We examine sign of $b_{x_2x_2}$ for b given by (3.3.5) and (3.3.9), with u given by either u^- or u^+ from (3.2.7). Here b_{x_2} is given by (3.3.6). We differentiate further:

$$b_{x_2x_2} = (b_{x_2})_u u_{x_2} = \big(m''(u) - m'(u)\big)e^{-u}(1-\xi)u_{x_2}.$$

Since $u_{x_2} > 0$ when $u = u^-$ and $u_{x_2} < 0$ when $u = u^+$, we have

$$\operatorname{sgn} b_{x_2x_2} = \mp \operatorname{sgn}(m''(u^\pm) - m'(u^\pm)).$$

Using (3.3.7) we have

$$m' = \frac{m - f'}{\xi} \quad \Longrightarrow \quad m'' - m' = \frac{m' - f''}{\xi} - \frac{m - f'}{\xi} = \frac{m - f'}{\xi^2}(1-\xi) - \frac{f''}{\xi}$$

Thus,

$$\operatorname{sgn} b_{x_2x_2} = \mp \operatorname{sgn}\Big(m(u^\pm) - f'(u^\pm) - \frac{\xi^\pm}{1-\xi^\pm} f''(u^\pm)\Big) \tag{3.5.1}$$

Integrating in (3.3.9) by parts we get

$$m(u)e^{-u/\xi} = A + e^{-u/\xi}f'(u) - e^{-u_0/\xi}f'(u_0) - \int_{u_0}^{u} e^{-s/\xi}f''(s)\,ds.$$

Therefore,

$$S(u) := \Big(m - f' - \frac{\xi}{1-\xi}\, f''\Big) e^{-u/\xi} = A - e^{-u_0/\xi} f'(u_0) - \frac{\xi}{1-\xi}\, e^{-u/\xi} f''(u) - \int_{u_0}^{u} e^{-s/\xi} f''(s)\, ds \tag{3.5.2}$$

For future use, let us compute the derivative of this function:

$$S'(u) = \frac{\xi}{1-\xi}\, e^{-u/\xi}\, \big(f''(u) - f'''(u)\big) = \frac{\xi}{1-\xi}\, e^{-u/\xi} p(p-1)|u|^{p-3}\big(|u| - (p-2)\operatorname{sgn}(u)\big) \tag{3.5.3}$$

Candidates in cups

We compute $b_{x_2x_2}$ for b given by (3.3.3) and (3.3.1). Here, b_{x_2} is given by (3.3.4). We have the following formula for $b_{x_2x_2}$, obtained by direct differentiating and some rearranging:

$$b_{x_2x_2} = (p-1)\, \frac{v|v|^p}{(e^{-v} - 1 + v)^2}\, \frac{1}{x_1} \Big(p - \frac{v^2}{e^{-v} - 1 + v}\Big) \tag{3.5.4}$$

Since $p < 1$, and x_1 and v always have the same sign in all our cups, we have

$$\operatorname{sgn} b_{x_2x_2} = \operatorname{sgn}\Big(\frac{v^2}{e^{-v} - 1 + v} - p\Big) \tag{3.5.5}$$

Let us also observe that the expression in parentheses is increasing in v, as can be checked by direct differentiating (twice). Thus, if $v = v^+$, it suffices to check that the limit of this expression as $v \to 0^+$ is positive. Indeed,

$$\lim_{v \to 0^+} \Big(\frac{v^2}{e^{-v} - 1 + v} - p\Big) = 2 - p > 0.$$

The situation is more interesting when $v = v^-$. For this case, to show that $b_{x_2x_2} \geqslant 0$ in the cup, it is necessary and sufficient to show that the expression in parenthesis is non-negative for the left-most v^-.

3.5.2 Pre-collision case

We first show the local convexity in P_7, then in P_5, then in P_4. By our choice of w and the fact that the boundary between P_5 and P_4 is a part of the foliation for each region, the function b is continuously differentiable on the interior of $P_7 \cup P_6 \cup P_5 \cup P_4$. Since it is affine in P_6, it will automatically be locally convex in the union of these four regions. After this, we verify the local convexity in P_2, then in P_1. Again, since these are separated by a common extremal trajectory, b will be locally convex in their union. Lastly we verify that b_{x_2} is increasing in x_2 across the boundary between P_4 and P_3 and, separately, across the boundary between P_2 and P_3.

Subdomain P_7

Here, b is given by (3.3.10), (3.3.11), and (3.2.7), with w given by (3.4.1) or (3.4.2). Using (3.5.1) and (3.5.2) with $A = 0$ and $u_0 = -\infty$, we have

$$b_{x_2x_2} \geqslant 0 \text{ in } P_7 \qquad \Longleftrightarrow \qquad S(u) \geqslant 0, \text{ for } u \leqslant w$$

where $S(u)$ is given by

$$S(u) = -\frac{\xi^-}{1-\xi^-}\, e^{-u/\xi^-} f''(u) - \int_{-\infty}^{u} e^{-s/\xi^-} f''(s)\, ds \tag{3.5.6}$$

and its derivative is given by (3.5.3):

$$S'(u) = \frac{\xi^-}{1-\xi^-}\, e^{-u/\xi^-} p(p-1)|u|^{p-3}\big(|u| - 2 + p\big)$$

Since ξ^- and $p-1$ are both negative, S has a single extremum – a maximum at $u = p-2$. Since $\lim_{u\to-\infty} S(u) = 0$, we have $S(u) \geqslant 0$ for all $u \leqslant p-2$. By Lemma 3.4.4, $w \leqslant p-2$, thus, we have $S(u) \geqslant 0$ for all $u \leqslant w$. Thus, we have shown the local convexity of b in the region P_7.

Subdomain P_5

Here, b is given by (3.3.12), (3.2.7), and (3.3.13). Using (3.5.1) and (3.5.2) with $A = -e^{-w_0/\xi^+}|w_0|^{p-1}$ and $u_0 = w_0$, we have

$$b_{x_2x_2} \geqslant 0 \text{ in } P_5 \qquad \Longleftrightarrow \qquad S(u) \leqslant 0, \text{ for } w \leqslant u \leqslant w_0$$

where $S(u)$ is given by

$$S(u) = -e^{-w_0/\xi^+}|w_0|^{p-1} - e^{-w_0/\xi^+} f'(w_0) - \frac{\xi^+}{1-\xi^+}\, e^{-u/\xi^+} f''(u) + \int_u^{w_0} e^{-s/\xi^+} f''(s)\, ds$$

and its derivative is given by (3.5.3):

$$S'(u) = \frac{\xi^+}{1-\xi^+}\, e^{-u/\xi^+} p(p-1)|u|^{p-3}\big(|u| - 2 + p\big)$$

Thus, S attains its minimum at $u = p - 2$. To show that $S(u) \leqslant 0$ for all $w \leqslant u \leqslant w_0$, it suffices to show that $S(w) \leqslant 0$ and $S(w_0) \leqslant 0$.

First, using differential equation (3.3.7) we see that equation (3.4.1), which defines w, is equivalent to

$$\frac{1-\xi^-}{\xi^-}\Big(M(w) - f'(w) - \frac{\xi^-}{1-\xi^-}\, f''(w)\Big) = \frac{1-\xi^+}{\xi^+}\Big(\mu(w) - f'(w) - \frac{\xi^+}{1-\xi^+}\, f''(w)\Big)$$

By (3.5.1), the expression in parentheses on the left has the same sign as $b_{x_2x_2}|_{u^-=w}$ in subdomain P_7, which was shown to be non-negative in the previous section. Since $\xi^- < 0$, the left-hand side is non-positive. On the other hand, again by (3.5.1), the expression in parentheses on the right has the opposite sign from that of $b_{x_2x_2}|_{u^+=w}$ in subdomain P_5, meaning the same sign as $S(w)$ above. Thus, $S(w) \leqslant 0$

For the other endpoint, we compute:

$$\begin{aligned}
S(w_0) &= e^{-w_0/\xi^+}\Big((p-1)|w_0|^{p-1} - p(p-1)\frac{\xi^+}{1-\xi^+}\,|w_0|^{p-2}\Big) \\
&= e^{-w_0/\xi^+}(p-1)|w_0|^{p-2}\frac{\xi^+}{1-\xi^+}\Big(\frac{(1-\xi^+)(\xi^+-\xi^-)}{\xi^+} - p\Big).
\end{aligned}$$

By (3.4.6), we have

$$\frac{(1-\xi^+)(\xi^+-\xi^-)}{\xi^+} - p = \frac{w_0^2}{e^{-w_0}-1+w_0} - p.$$

Since we are in the pre-collision case, $w_0 \geqslant w_*(p)$, which means that by Lemma 3.4.3 (inequality (3.4.16)), this expression is non-negative. Since $p < 1$, we have $S(w_0) \leqslant 0$ and and the consideration of this case is complete.

Subdomain P_4

By the remarks at the end of section 3.5.1, to show that b is locally concave in P_4 it suffices to show that

$$\frac{w_0^2}{e^{-w_0}-1+w_0} - p \geqslant 0,$$

which was shown immediately above, at the end of the consideration of the region P_5. This completes this case.

Subdomain P_2

This case was already addressed at the end of section 3.5.1.

Subdomain P_1

Here, b is given by (3.3.14), (3.2.7), and (3.3.15). Using (3.5.1) and (3.5.2) with $A = e^{-w_0/\xi^-}|w_0|^{p-1}$ and $u_0 = -w_0$, we have

$$b_{x_2x_2} \geqslant 0 \text{ in } P_1 \qquad \Longleftrightarrow \qquad S(u) \leqslant 0, \text{ for } u \geq -w_0,$$

where $S(u)$ is given by

$$S(u) = e^{-w_0/\xi^-}|w_0|^{p-1} - e^{w_0/\xi^-} f'(-w_0) - \frac{\xi^-}{1-\xi^-}\, e^{-u/\xi^-} f''(u) - \int_{-w_0}^{u} e^{-s/\xi^-} f''(s)\, ds$$

and its derivative is given by (3.5.3):

$$S'(u) = \frac{\xi^-}{1-\xi^-}\, e^{-u/\xi^-} p(p-1)|u|^{p-3}\big(u + 2 - p\big).$$

Since $\xi^- < 0$, $p < 1$, and $u+2-p > 0$, we conclude that S is increasing. Thus, it suffices to verify that $S(-w_0) \geqslant 0$. Note that $|w_0| \geqslant -\xi^-$. We then have

$$\begin{aligned}
S(-w_0) &= e^{-w_0/\xi^-}\Big(|w_0|^{p-1} - p|w_0|^{p-1} - \frac{\xi^-}{1-\xi^-}\, p(p-1)|w_0|^{p-2}\Big) \\
&= e^{-w_0/\xi^-}(1-p)|w_0|^{p-2}\frac{1}{1-\xi^-}\big(|w_0|(1-\xi^-) + p\xi^-\big) \\
&\geqslant e^{-w_0/\xi^-}(1-p)|w_0|^{p-2}\frac{1}{1-\xi^-}\big((-\xi^-)(1-p) + (\xi^-)^2\big) \geqslant 0.
\end{aligned}$$

Checking jumps in b_{x_2}

As mentioned earlier, we need only check that b_{x_2} is increasing in x_2 from P_4 into P_3 and, separately, from P_2 into P_3. Recall that in either cup we have b_{x_2} given by (3.3.4) with $v = v^-$. Since $p < 1$ and the denominator is positive for any v, we have $b_{x_2} \leqslant 0$

everywhere is each cup. On the other hand, in P_3, we have

$$b_{x_2}\Big|_{P_3} = \alpha_2$$

from (3.3.16), which is positive. Thus, the jumps have the correct signs.

3.5.3 Post-collision case

Since the collision affects neither the geometry of the regions $R_1 = P_1$ and $R_2 = P_2$, nor the values of the Bellman candidate b in them, they do not have to be rechecked in regard to the local convexity of b. Thus, going left to right, we have to check the local convexity in R_6 and R_4, and then the jumps in b_{x_2} across the boundary between R_4 and R_3, and between R_2 and R_3. (Note that while the region $R_3 = P_3$ remains the same geometrically, the expression for b is different in it, compared to the pre-collision case.)

Subdomain R_6

Here b is formally given by the same expression as in P_7, meaning by (3.3.10), (3.3.11), and (3.2.7). However, w is now given by (3.4.3). Exactly as in section 3.5.2, the sign of $b_{x_2x_2}$ is the same as that of the function S given by (3.5.6), S has a maximum at $u = p-2$, and $\lim_{u\to-\infty} S(u) = 0$. Thus, as before, it suffices to show that $S(w) \geqslant 0$. However, unlike in the previous case, it can be that $w > p-2$, meaning w can be to the right of the point of maximum. Thus, we must analyze equation (3.4.3). Using (3.3.7), we rewrite it as

$$\begin{aligned} -\frac{1-\xi^-}{\xi^-}\Big(M(w) - f'(w) - \frac{\xi^-}{1-\xi^-}\, f''(w)\Big) &= \frac{(1-p)|w|^p}{e^{-w}-1+w} - f''(w) \\ &= (1-p)|w|^{p-2}\Big(\frac{w^2}{e^{-w}-1+w} - p\Big) \end{aligned}$$

The left-most side has the same sign as $S(w)$, while the right-most side is non-negative: by Lemma 3.4.4, $w > w_1(p)$, and the conclusion follows by inequality (3.4.16) of Lemma 3.4.5. This case is thus complete.

Subdomain R_4

By (3.5.5),

$$\operatorname{sgn} b_{x_2x_2} = \operatorname{sgn}\Big(\frac{v^2}{e^{-v} - 1 + v} - p\Big)$$

Since the function $\frac{v^2}{e^{-v}-1+v}$ is increasing for all v, it suffices to show that the expression in parenthesis is non-negative for the left-most v, which for this cup is w. This was shown at the end of the previous section.

Checking jumps in b_{x_2}

As mentioned earlier we need to verify that b_{x_2} is increasing with respect to x_2 in two situations: from R_2 into R_3 and from R_5 into R_3. This is nearly as obvious as in the pre-collision case: in the cups R_2 and R_4 b_{x_2} is always non-positive, while in the affine region R_5, (the constant) b_{x_2} equals $b_{x_2}(w)|_{R_4}$ (see equation (3.4.3), thus it is again non-positive. Since

$$b_{x_2}\Big|_{R_3} = \alpha_2 \geqslant 0,$$

where α_2 is defined by (3.3.19); our verification is now complete.

3.6 Optimizers

In this section, we construct special functions, called optimizers, which will prove the converse inequality for our Bellman candidate. Specifically, given a fixed $x \in \Omega_C$, we say that a test function φ_x is an optimizer for $b(x)$ if the following three conditions are satisfied:

$$\langle\varphi_x\rangle_I = x_1;\ \langle e^{\varphi_x}\rangle_I = x_2 \tag{3.6.1}$$

$$e^{\varphi_x} \in A_\infty^C(I); \tag{3.6.2}$$

$$\langle|\varphi_x|^p\rangle_I = b(x); \tag{3.6.3}$$

Once optimizers are found for every point $x \in \Omega_C$, we'll have

$$\boldsymbol{b}_{p,C}(x) \leqslant \langle|\varphi_x|^p\rangle_I = b(x), \quad \forall x \in \Omega_C$$

By the interval independence of $\boldsymbol{b}_{p,C}$, we may construct these optimizers on any fixed interval; hence, unless otherwise noted, all function constructions will be made on $(0,1)$. Furthermore, for notational convenience, we will parameterize optimizers constructed for points on the upper boundary by their first coordinate: i.e. if (a, Ce^a) let $\psi_a := \varphi_{(a,Ce^a)}$.

Much like our candidate b, whose expression was determined locally for subregions and then glued together by choosing appropriate constants, our optimizers will be constructed from local variants, built for these same subregions then concatenated together in appropriate proportions.

Condition (3.6.2) will be verified via a geometric condition as follows. Let J be an interval and φ_x a test function. We call the point $x^J := (\langle\varphi_x\rangle_J, \langle e^{\varphi_x}\rangle_J)$ the Bellman point corresponding to φ_x and the interval J. Therefore, directly from the definition of Ω_C, we have $e^{\varphi_x} \in A_\infty^C(I)$ if and only if each Bellman point corresponding to φ_x and $J \subseteq I$ belongs to the domain Ω_C.

The main principle behind building optimizers for Bellman foliations is to take for each x the entire construction of φ_x to be along an extremal trajectory through x. This means that when the interval $I = (0,1)$ is split into two subintervals $I = I_+ \cup I_-$, the corresponding Bellman points of φ_x, $x^{I_\pm} = (\langle\varphi_x\rangle_{I_\pm}, \langle e^{\varphi_x}\rangle_{I_\pm})$, will also lie on the trajectory. Indeed, since $\boldsymbol{b}_{p,C}$ is linear along each trajectory, the local convexity inequality becomes equality. Conversely, if m and n are two points on the same trajectory for which we know the optimizers φ_m and φ_n, then the optimizer for every point x on the line segment $[m,n]$ can be obtained by concatenation of these two known optimizers:

$$\varphi_x(t) = \begin{cases} \varphi_m\left(\frac{t}{\gamma}\right), & t \in (0,\gamma), \\ \varphi_n\left(\frac{t-\gamma}{1-\gamma}\right), & t \in (\gamma,1) \end{cases} \tag{3.6.4}$$

where, $\gamma_- := \frac{x_1-n_1}{m_1-n_1}$.

Remark 3.6.1. *In general, the concatenation of two logarithms of A_∞ weights isn't necessarily a logarithm of an A_∞ weight itself; therefore, we must take care in how the functions φ_m, φ_n in the formula above are concatenated. In particular, we rearrange our constructions to ensure the smallest A_∞-characteristic of e^φ. So long as the proportions of φ_m and φ_n remain the same, rearrangements like swapping concatenation order won't affect conditions* (3.6.1) *and* (3.6.2).

We will need the following two lemmas.

Lemma 3.6.2 ([Sla15])**.** *Let φ be such that $e^\varphi \in A_\infty$. For $c,d \in \mathbb{R}$, such that $c<d$, let*

$$\varphi_{c,d} = c\chi_{\{\varphi\leqslant c\}} + \varphi\chi_{\{c<\varphi<d\}} + d\chi_{\{\varphi\geqslant d\}}. \tag{3.6.5}$$

Then for any interval J,

$$\langle e^{\varphi_{c,d}-\langle\varphi_{c,d}\rangle_J}\rangle_J \leqslant \langle e^{\varphi-\langle\varphi\rangle_J}\rangle_J$$

and, consequently,

$$[e^{\varphi_{c,d}}]_{A_\infty} \leqslant [e^{\varphi}]_{A_\infty}. \tag{3.6.6}$$

Lemma 3.6.3. *Let $q(t) = u + \xi\log\left(\frac{\mu}{t}\right)$ such that $u,\mu \in \mathbb{R}$, $0<\mu$, and ξ is one of $\xi^\pm$. Then $e^q \in A_\infty((0,1))$ with*

$$[e^{q(t)}]_{A_\infty} = C$$

and, for any interval $(0, d) \subseteq (0, 1)$, *we have*

$$x^{(0,d)} := (\langle q \rangle_{(0,d)}, \, \langle e^q \rangle_{(0,d)}) \in \Gamma_C$$

Proof. Let $0 < c < d \leqslant 1$. Integrating by parts, we compute

$$\langle q \rangle_{(c,d)} = u - \xi \langle \log{(t)} \rangle_{(\frac{c}{\mu}, \frac{d}{\mu})} = u - \frac{\xi}{d-c} \left[d \log{(d)} - c \log{(c)} \right] - \xi (1 + \log{(\mu)}),$$

thus

$$q - \langle q \rangle_{(c,d)} = \frac{\xi}{d-c} \left[d \log{(d)} - c \log{(c)} \right] - \xi \log{(t)} - \xi.$$

Therefore, we have

$$\langle e^{q - \langle q \rangle_J} \rangle_{(c,d)} = \frac{e^{-\xi}}{1-\xi} \left[\frac{d^{\frac{d}{d-c}}}{c^{\frac{c}{d-c}}} \, \frac{d^{1-\xi} - c^{1-\xi}}{d-c} \right] = \frac{e^{-\xi}}{1-\xi} \left[\theta^{-\frac{\xi\theta}{1-\theta}} \frac{1-\theta^{1-\xi}}{1-\theta} \right] \tag{3.6.7}$$

where we repeatedly used the identities $1 - \theta = \frac{d-c}{d}$ and $\theta^{-1} - 1 = \frac{d-c}{c}$ to obtain 3.6.7. We have that 3.6.7 is decreasing with a limit of $\frac{e^{-\xi}}{1-\xi}$ as $\theta \to 0$. Consequently,

$$[e^q]_{A_\infty} = \frac{e^{-\xi^-}}{1-\xi^-} = C$$

from which it follows that $e^q \in A_\infty((0,1))$. Taking the limit as $c \to 0$ of 3.6.7 shows that the Bellman point corresponding to q and any interval of the form $(0, d)$ will belong to the upper boundary, i.e. $x^{(0,d)} \in \Gamma_C$. □

3.6.1 Optimizers for Candidates in Cups

We recall that the cups consist of P_2 and P_4 (pre-collision) and R_2 and R_4 (post-collision). In these regions, the expression for the local candidate b was given by:

$$b(x) = \operatorname{sgn}(v) |v|^{p-1} x_1 \tag{3.6.8}$$

where v is either one of $v^{\pm}$. These regions are the portions of Ω_C lying entirely beneath a single extremal trajectory. In the case of P_2 and R_2, this trajectory is the full tangent to the upper boundary at (ξ^+, Ce^{ξ^+}); in the case of P_4, this trajectory is the full tangent to the upper boundary at (ξ^-, Ce^{ξ^-}); and, in the case of R_4, this trajectory is the line segment joining (w, e^w) and $(0, 1)$. Each of these regions are foliated by trajectories, also known as chords in this context, joining $(0, 1)$ to a point (u, e^u) on the lower boundary. Due to the identical nature of these foliations, the optimizers constructed for these regions will follow the same exposition. As such, we will address each case simultaneously, using D_C to refer to any one of these regions and $v = v^{\pm}$ as appropriate.

Given that the only admissible functions for points on the lower boundary are constants, the optimizers for the endpoints of each chord foliating D_C are already determined; specifically, we have $\varphi_{(0,1)} \equiv 0$ and $\varphi_{(u,e^u)} \equiv u$. Furthermore, through every point $x \in D_C - \{(0, 1)\}$ passes the unique chord which intersects the lower boundary at (v, e^v). Therefore, by the opening dicussion of this section, the optimizers for all $x \in D_C$ will be an appropriate concatentation of the optimizers for (0, 1) and $(v,\ e^v)$:

$$\varphi_x(t) = \begin{cases} 0, & t \in (0, \alpha) \\ v, & t \in (\alpha, 1) \end{cases} \tag{3.6.9}$$

where, $\alpha = (v - x_1)v^{-1} = 1 - x_1 v^{-1}$. We will now verify conditions (3.6.1) - (3.6.3).

Lemma 3.6.4. *Let x be a point in one of cups P_2, R_2, P_4 or R_4. Then the function φ_x given by* (3.6.9) *is an optimizer for $b(x)$ given by* (3.6.8).

Proof. For condition (3.6.1), we compute the following averages:

$$\langle \varphi_x \rangle_{(0,1)} = \alpha \langle \varphi_x \rangle_{(0,\alpha)} + (1 - \alpha) \langle \varphi_x \rangle_{(\alpha,1)} = (1 - \alpha)v = x_1,$$
$$\langle e^{\varphi_x} \rangle_{(0,1)} = \alpha \langle e^{\varphi_x} \rangle_{(0,\alpha)} + (1 - \alpha) \langle e^{\varphi_x} \rangle_{(\alpha,1)} = \alpha + (1 - \alpha)e^v = 1 + (e^v - 1)\frac{x_1}{v} = x_2.$$

For condition (3.6.2), given an interval $J = (c, d) \subseteq (0, 1)$, the corresponding Bellman point $x^J := (\langle \varphi_x \rangle_J, \langle e^{\varphi_x} \rangle_J)$ will be a convex combination of the points $(0, 1)$ and (v, e^v). This clearly follows if $d \leqslant \alpha$, in which case $x^J = (0, 1)$, or if $\alpha \leqslant c$, in which case $x^J = (v, e^v)$. For the remaining situation, when $c < \alpha < d$, we have

$$\langle \varphi_x \rangle_{(c,d)} = \frac{\alpha - c}{d - c} \langle \varphi_x \rangle_{(c,\alpha)} + \frac{d - \alpha}{d - c} \langle \varphi_x \rangle_{(\alpha,d)} = (1 - \lambda)\, v; \tag{3.6.10}$$

$$\langle \varphi_x \rangle_{(c,d)} = \frac{\alpha - c}{d - c} \langle e^{\varphi_x} \rangle_{(c,\alpha)} + \frac{d - \alpha}{d - c} \langle e^{\varphi_x} \rangle_{(\alpha,d)} = \lambda + (1 - \lambda) e^v \tag{3.6.11}$$

where $\lambda = \frac{\alpha - c}{d - c} < 1$. Clearly, $x^J = \lambda\,(0, 1) + (1 - \lambda)\,(v, e^v)$ and the claim follows. Since the interval J was arbitrary, this implies all Bellman points of φ_x lie on the chord joining $(0, 1)$ and (v, e^v)—which is fully contained in D. Therefore, $e^{\varphi_x} \in A_\infty^C(I)$. Lastly, for condition (3.6.3), we compute:

$$\langle |\varphi_x|^p \rangle_{(0,1)} = \alpha\, \langle |\varphi_x|^p \rangle_{0,\alpha} + (1 - \alpha)\, \langle |\varphi_x|^p \rangle_{(1,\alpha)} = \mathrm{sgn}(v)|v|^{p-1} x_1 = b(x)$$

□

3.6.2 Optimizers for Tangential Candidates

Optimizers for Candidate in P_1 and R_1

We recall the formula for the local candidate b in these regions:

$$b(x) = e^{\frac{u}{\xi^-}} \left[|u_0|^{p-1} e^{-\frac{u_0}{\xi^-}} - \frac{p}{\xi^-} \int_{u_0}^{u} s^{p-1} e^{-\frac{s}{\xi^-}}\, ds \right] (x_1 - u) + |u|^p \tag{3.6.12}$$

where $u_0 = \xi^+ - \xi^-$ and $u = u^-$ is the unique point satisfying:

$$x_2 = C e^{u + \xi^-} \left[x_1 - (u + \xi^-) + 1 \right] = \frac{e^u}{1 - \xi^-} \left[x_1 - (u + \xi^-) + 1 \right]$$

This candidate corresponds to the regions P_1 (pre-collision) and R_1 (post-collision). These regions are the portion of Ω_C sitting directly above the right-leaning tangent to the upper boundary at $(a_0, Ce^{a_0}) := (\xi^+, Ce^{\xi^+})$ and are foliated by right-leaning tangents to the upper boundary at points (a, Ce^a), $a_0 \leqslant a$. Since all considerations are identical for these regions in their respective configurations, we will only address the case for P_1. The case for R_1 will follow immediately. This candidate is incomplete, reading information from the local candidate in the adjacent cup, P_2, via the shared right-leaning tangent at (a_0, Ce^{a_0}). As discussed, the optimizers constructed for this candidate will thererfore depend on those of the candidate in this cup.

Clearly, the optimizers for this candidate are already determined for those x on the lower boundary, viz. $\varphi_x \equiv x_1$. Suppose we had already determined optimizers for points on the upper boundary. Then for a fixed $x \in P_1$ there passes a unique trajectory, i.e. a right-leaning tangent to the upper boundary, which will intersect the lower boundary at the point (u, e^u); the corresponding point of tangency is then $(a,\, Ce^a)$, where $a := u+\xi^-$. It follows from the opening discussion of this section, that an optimizer for x can then be constructed as an appropriate concatentation of $\varphi_{(u,\,e^u)}$ and ψ_a:

$$\varphi_x(t) = \begin{cases} \psi_a\big(\frac{t}{\alpha}\big), & t \in (0, \alpha), \\ u, & t \in (\alpha, 1) \end{cases} \tag{3.6.13}$$

where $\alpha = \frac{x_1 - u^-}{\xi^-}$. It therefore remains to construct optimizers ψ_a for points on the upper boundary. For such points, there is no extremal trajectory on which they lie whose endpoint optimizers are known. To overcome this limitation, an approximation procedure was developed in [SV12] which we shall present here, adapted to the domain Ω_C.

Let $(a,\, Ce^a)$ be a fixed point on the upper boundary Γ_C and Δ some small postive number such that $a_0 \leqslant a - \Delta$. Consider the point $(a - \Delta,\, Ce^{a-\Delta})$, also on the upper boundary. We can relate the optimizers ψ_a and $\psi_{a-\Delta}$ by considering the left and right tangents to Γ_C at $(a,\, Ce^a)$ and $(a-\Delta,\, Ce^{a-\Delta})$, respectively. Define the following function:

$$h(t) := 1 - t(e^t - 1)^{-1} \tag{3.6.14}$$

Then h gives the x_1 coordinate of the point of intersection of the aforementioned tangents. Let $\Delta_h := h(\Delta)$;. Observe that h is monotone increasing on $\mathbb{R}^+$ with $\lim_{t\to 0^+} h(t) = 0$, $\lim_{t\to\infty} h(t) = 1$ and $h'(t) \leqslant 1$; hence, $0 < \Delta_h < \Delta$. Since a_{Δ_h} lies on the extremal trajectory through $(a-\Delta,\, Ce^{a-\Delta})$, a corresponding optimizer is given by concatenation as follows:

$$\varphi_{a_{\Delta_h}}(t) = \begin{cases} \psi_{a-\Delta}\big(\frac{t}{\alpha}\big), & t \in (0,\alpha), \\ u - \Delta, & t \in (\alpha, 1) \end{cases} \tag{3.6.15}$$

where $\alpha = 1 + \frac{\Delta - \Delta_h}{\xi^-}$. We now have a trajectory, albeit not an extremal one, to which $(a,\, Ce^a)$ belongs and whose endpoint optimizers are known; thus, we can now build a test function for $(a,\, Ce^a)$ through concatenation:

$$\psi_a(t) \approx \begin{cases} \varphi_{a_{\Delta_h}}\big(\frac{t}{\beta}\big), & t \in (0,\beta), \\ u, & t \in (\beta, 1) \end{cases} \tag{3.6.16}$$

where $\beta = \frac{\xi^-}{\xi^- - \Delta_h}$. In all, combining (3.6.15) and (3.6.16), gives:

$$\psi_a(t) \approx \begin{cases} \psi_{a-\Delta}\big(\frac{t}{\gamma}\big), & t \in (0,\gamma), \\ u - \Delta, & t \in (\gamma,\beta) \\ u, & t \in (\beta, 1) \end{cases} \tag{3.6.17}$$

where $\gamma = \alpha\,\beta = 1 + \frac{\Delta}{\xi^- - \Delta_k}$. Repeating this procedure k more times, we obtain:

$$\psi_a(t) \approx \psi_a^{(k)}(t) := \begin{cases} \psi_{a-k\Delta}\left(\frac{t}{\gamma^k}\right), & t \in (0, \gamma^k), \\ u - k\Delta, & t \in (\gamma^k, \gamma^{k-1}\beta) \\ u - (k-1)\Delta, & t \in (\gamma^{k-1}\beta, \gamma^{k-2}\beta) \\ \quad\vdots & \quad\vdots \\ u - \Delta, & t \in (\gamma\beta, \beta) \\ u, & t \in (\beta, 1) \end{cases} \tag{3.6.18}$$

The function $\psi_a^{(k)}$ is completely determined up to knowledge of $\psi_{a-k\Delta}$. Since the optimizers are known for points on the shared boundary with P_2, choosing $\Delta = \frac{a-a_0}{k}$ completes the definition of (3.6.18), as we'll have $\psi_{a-k\Delta} = \psi_{a_0}$. To find $\psi_a = \lim_{k\to\infty} \psi_a^{(k)}$, we derive a basic differential equation which we then solve. Fix $t \in (0, 1)$; then, for large k, there exists $0 \leqslant j$ such that $t \in (\gamma^j\beta, \gamma^{j+1}\beta)$ and

$$\psi_a(t) - \psi_a(\gamma\, t) \approx \psi_a^{(k)}(t) - \psi_a^{(k)}(\gamma\, t) \approx (u - j\Delta) - (u - (j+1)\Delta) = \Delta$$

Furthermore, if ψ_a is assumed differentiable, for large k we have:

$$\psi_a(t) - \psi_a(\gamma\, t) \approx \psi_a'(t)t(1-\gamma) \approx \psi_a'(t)t\frac{\Delta}{\Delta_h - \xi^-}$$

Combining these approximate equalities yields the following differential equation:

$$\psi_a'(t) = -\xi^-\, t^{-1}$$

whose general solution is

$$\psi_a(t) = D - \xi^- \log{(t)}$$

Since $\Delta = \frac{a-a_1}{k}$, we have $k = \frac{a-a_1}{\Delta}$ and so

$$\lim_{k\to\infty} \gamma^k = \lim_{\Delta\to 0^+} \left(1 + \frac{\Delta}{\xi^- - h(\Delta)}\right)^{\frac{a-a_1}{\Delta}} = e^{(a-a_1)L}$$

where,

$$L := \lim_{\Delta\to 0^+} \frac{1}{\Delta} \log\left(\frac{\Delta}{\xi^- - h(\Delta)}\right)$$

Using l'Hôpitals rule, we have

$$\lim_{t\to 0^+} h'(t) = \lim_{t\to 0^+} \frac{te^t - (e^t - 1)}{(e^t - 1)^2} = \lim_{t\to 0^+} \frac{1}{2(e^t - 1)e^t + 2e^{2t}} = \frac{1}{2}$$

and again

$$L = \lim_{\Delta\to 0^+} \frac{\xi^- - h(\Delta) + \Delta\, h'(\Delta)}{(\xi^- - h(\Delta))^2 + \Delta(\xi^- - h(\Delta))} = \frac{\xi^-}{\xi^{-2}} = \frac{1}{\xi^-}$$

thus, $\lambda := \lim_{k\to\infty} \gamma^k = e^{\frac{a-a_0}{\xi^-}}$. In all, this gives,

$$\psi_a(t) = \begin{cases} \psi_{a_0}\left(\frac{t}{\lambda}\right), & t \in (0, \lambda), \\ D - \xi^- \log(t)\,, & t \in (\lambda, 1) \end{cases} \tag{3.6.19}$$

The constant D is determined by considering the imposed conditions on the averages of ψ_a and ψ_{a_0}. In particular, as optimizers for their respective points, we must have $\langle\psi_a\rangle_{(0,1)} = a$ and $\langle\psi_{a_0}\rangle_{(0,1)} = a_0$. We note the following integral for $0 \leqslant a < b$, obtained using integration by parts (and an appropriate limit if $a = 0$),

$$\int_a^b \log(t)\; dt = (b\log(b) - a\log(a)) - (b - a) \tag{3.6.20}$$

Therefore, since

$$\langle\psi_a\rangle_{(0,1)} = \langle\psi_{a_1}\rangle_{(0,\lambda)} + \langle D - \xi^- \log(t)\rangle_{(\lambda,1)}$$

It follows that

$$a = a_1\lambda + (D+\xi^-)(1-\lambda) + \lambda(a-a_1)$$

and so $D = a - \xi^- = u$. By our precending comments, we can now construct optimizers for all $x \in P_1$. Recalling formula (3.6.13), we can substitute (3.6.19) for ψ_a to obtain:

$$\varphi_x(t) = \begin{cases} \psi_{a_1}\left(\frac{t}{\mu\nu}\right), & t \in (0, \mu\nu), \\ u + \xi^- \log\left(\frac{\mu}{t}\right), & t \in (\mu\nu, \mu) \\ u, & t \in (\mu, 1) \end{cases} \tag{3.6.21}$$

where $\mu = \frac{x_1 - u}{\xi^-}$ and $\nu = e^{\frac{u-u_0}{\xi^-}}$. Lastly, we recall the formula for $\psi_{a_0} := \varphi_{(\xi^+, Ce^{\xi^+})}$. This optimizer for the candidate in the adjacent cup P_2 was derived in the last subsection. In this case, we have $v = u_0$ and $\alpha = \frac{\xi^-}{\xi^- - \xi^+}$:

$$\psi_{a_1}(t) = \begin{cases} 0, & t \in (0, \alpha) \\ u_0, & t \in (\alpha, 1) \end{cases}$$

which gives us the final formula

$$\varphi_x(t) = \begin{cases} 0, & t \in (0, \alpha\mu\nu) \\ u_0, & t \in (\alpha\mu\nu, \mu\nu) \\ u + \xi^- \log\left(\frac{\mu}{t}\right), & t \in (\mu\nu, \mu) \\ u, & t \in (\mu, 1) \end{cases} \tag{3.6.22}$$

Note that this formula will coincide with the one given for points on the upper and lower boundaries and is thus valid for all $x \in P_1$. It remains to show that each φ_x, so constructed, is an optimizer:

Lemma 3.6.5. *Let x be a point in the tangential region P_1 or R_1. Then the function φ_x given by* (3.6.22) *is an optimizer for $b(x)$ given by* (3.6.12).

Proof. Recall that $\alpha = \frac{\xi^-}{\xi^- - \xi^+} = -\frac{\xi^-}{u_0}$, $\mu = \frac{x_1 - u}{\xi^-}$, $\nu = e^{\frac{u-u_0}{\xi^-}}$ and that u satisfies:

$$x_2 = \frac{e^u}{1-\xi^-}\left[x_1 - (u+\xi^-) + 1\right]$$

We first compute the following two integrals

$$\int_{\mu\nu}^{\mu} \log\left(\frac{\mu}{t}\right)\,dt = -\mu\int_{\nu}^{1}\log(t)\,dt = \mu\left[\nu\log(\nu) + (1-\nu)\right] \tag{3.6.23}$$

$$\int_{\mu\nu}^{\mu} e^{u+\xi^-\log\left(\frac{\mu}{t}\right)}\,dt = \mu\,e^u\int_{\nu}^{1} t^{-\xi^-}\,dt = \frac{e^u}{1-\xi^-}\left[\mu - \mu\nu\,e^{u_0-u}\right] \tag{3.6.24}$$

where in the second integral we used the fact that $\nu^{1-\xi^-} = \nu e^{u_0 - u}$. Therefore,

$$\begin{aligned}
\langle\varphi_x\rangle_{(0,1)} &= \mu\nu(1-\alpha)u_0 + \xi^-\mu\left[\nu\log(\nu) + (1-\nu)\right] + \mu(1-\nu)u + (1-\mu)u \\
&= \mu\nu(u_0+\xi^-) + \mu\nu(u-u_0) + \xi^-\mu(1-\nu) + (1-\mu\nu)u \\
&= \xi^-\mu + u \\
&= x_1
\end{aligned}$$

and,

$$\begin{aligned}
\langle e^{\varphi_x}\rangle_{(0,1)} &= \alpha\mu\nu + \mu\nu(1-\alpha)e^{u_0} + \frac{e^u}{1-\xi^-}\left[\mu - \mu\nu\,e^{u_0-u}\right] + (1-\mu)e^u \\
&= \mu\nu\left[\alpha + (1-\alpha)e^{u_0} - \frac{e^{u_0}}{1-\xi^-}\right] + \frac{e^u}{1-\xi^-}\left[(1-\xi^-)(1-\mu) + \mu\right] \\
&= \mu\nu\left[\frac{\xi^+(1-\xi^-) - \xi^-(1-\xi^+)}{(1-\xi^+)(\xi^+-\xi^-)} - \frac{1}{1-\xi^+}\right] + \frac{e^u}{1-\xi^-}\left[x_1 - (u+\xi^-) + 1\right] \\
&= x_2
\end{aligned}$$

where we used the fact that $e^{u_0} = e^{\xi^+ - \xi^-} = \frac{1-\xi^-}{1-\xi^+}$; this identity follows directly from the definitions of $\xi^{\pm}$. Therefore, condition (3.6.1) is satisfied. To show condition (3.6.2), we let $0 < c < d < 1$ and define

$$q(t) = u + \xi^- \log\left(\frac{\mu}{t}\right) \tag{3.6.25}$$

Note that q is the function in the statement of Lemma 3.6.3 with $\xi = \xi^-$. Therefore, q has all its Bellman points in Ω_C and, for intervals of the form $(0, c)$, $0 < c$, these points will belong to the upper boundary. Two additional results concerning q are needed; using (3.6.20), we compute

$$\langle q \rangle_{(0,\mu\nu)} = u - \xi^- \langle \log(t) \rangle_{(0,\nu)} = u - \xi^- [\log(\nu) - 1] = u_0 + \xi^- = a_1 \tag{3.6.26}$$

$$\langle e^q \rangle_{(0,\mu\nu)} = e^u \langle t^{-\xi^-} \rangle_{(0,\nu)} = \frac{e^u}{1-\xi^-}\nu^{-\xi^-} = \frac{e^u}{1-\xi^-}e^{u_0 - u} = Ce^{u_0 + \xi^-} = Ce^{a_1} \tag{3.6.27}$$

Since φ_x is the cutoff at height u of the following function:

$$\eta_x(t) = \begin{cases} 0, & t \in (0, \alpha\mu\nu) \\ u_0, & t \in (\alpha\mu\nu, \mu\nu) \\ q(t), & t \in (\mu\nu, 1) \end{cases}, \tag{3.6.28}$$

by Lemma 3.6.2, to show $e^{\varphi_x} \in A_\infty^C(I)$ it suffices to show $e^{\eta_x} \in A_\infty^C(I)$. We will do so by proving its Bellman point corresponding to an arbitrary interval $(c, d) \subseteq (0, 1)$:

$$x^{(c,d)} := (\langle \varphi_x \rangle_{(c,d)}, \langle e^{\varphi_x} \rangle_{(c,d)})$$

belongs to Ω_C. This is clearly the case if $d < \mu\nu$, since η_x restricted to $(0, \mu\nu)$ is simply the function ψ_{a_0} rescaled; the function ψ_{a_0}, an optimizer for the candidate in P_2, has already been shown to have its Bellman points in Ω_C. The case when $\alpha\mu\nu < c$ also readily follows, since η_x restricted to $(\alpha\mu\nu, 1)$ is the cutoff at height u_0 of the function q, given in (3.6.25), whose Bellman points we've also shown to be in Ω_C. Therefore, we only need to consider the case when $c < \alpha\mu\nu < \mu\nu < d$. In addition to $x^{(c,d)}$, we consider the Bellman points:

$$x^{(0,c)} = (\langle\varphi_x\rangle_{(0,c)}, \langle e^{\varphi_x}\rangle_{(0,c)}); \quad x^{(0,d)} = (\langle\varphi_x\rangle_{(0,d)}, \langle e^{\varphi_x}\rangle_{(0,d)})$$

Since η_x is constant on $(0, \alpha\mu\nu)$, the point $x^{(0,c)}$ lies on the lower boundary of Ω_C. Furthermore, by (3.6.26) and (3.6.27), we have:

$$(\langle q\rangle_{(0,\mu\nu)},\, \langle e^q\rangle_{(0,\mu\nu)}) = (\langle \eta_x\rangle_{(0,\mu\nu)},\, \langle e^{\eta_x}\rangle_{(0,\mu\nu)})$$

and therefore,

$$(\langle q\rangle_{(0,d)},\, \langle e^q\rangle_{(0,d)}) = (\langle \eta_x\rangle_{(0,d)},\, \langle e^{\eta_x}\rangle_{(0,d)})$$

for all $\mu\nu \leqslant d$. In particular, this means the point $x^{(0,d)}$ lies on the upper boundary of Ω_C. We consider the line through $x^{(0,c)}$ and $x^{(0,d)}$. Since q is increasing, $x^{(0,d)}$ lies to the right of the point (a_0, Ce^{a_0}) and $x^{(0,c)}$ to its left. Consequently, this line must exit the domain Ω_C and re-enter at $x^{(0,d)}$. Since $x^{(c,d)}$ lies on this line to the left of $x^{(0,d)}$, we must have that $x^{(c,d)} \in \Omega_C$. To show condition (3.6.3), we first compute the integral:

$$\int_{\mu\nu}^{\mu} \left|u + \xi^- \log\left(\frac{\mu}{t}\right)\right|^p dt = \mu \int_{\nu}^{1} \left|u - \xi^- \log(t)\right|^p dt = -\mu \int_{u_0}^{u} |s|^p\, e^{\frac{u-s}{\xi^-}}\, ds \tag{3.6.29}$$

where we used the substitution $s = u - \xi^- \log(t)$. Integrating by parts gives

$$-\mu \int_{u_0}^{u} |s|^p\, e^{\frac{u-s}{\xi^-}}\, ds = \mu|u|^p - \mu\nu|u_0|^p - p\,\mu \int_{u_0}^{u} |s|^{p-1} e^{\frac{u-s}{\xi^-}}\, ds \tag{3.6.30}$$

Therefore, since $u_0 = \xi^+ - \xi^- > 0$, using (3.6.29) and (3.6.30), we have

$$\begin{aligned}
\langle|\varphi_x|^p\rangle_{(0,1)} &= \mu\nu(1-\alpha)|u_0|^p + \int_\mu^{\mu\nu} \left|u + \xi^- \log\left(\frac{\mu}{t}\right)\right|^p dt + (1-\mu)|u|^p \\
&= \xi^+\mu\nu|u_0|^{p-1} - \mu\nu|u_0|^p - \mu\int_{u_0}^{u} |s|^{p-1} e^{\frac{u-s}{\xi^-}}\,ds + |u|^p \\
&= \mu\nu|u_0|^{p-1}(\xi^+ - u_0) - \mu\int_{u_0}^{u} |s|^{p-1} e^{\frac{u-s}{\xi^-}}\,ds + |u|^p \\
&= e^{\frac{u}{\xi^-}}\left[|u_0|^{p-1}e^{-\frac{u_0}{\xi^-}} - \int_{u_0}^{u} |s|^{p-1} e^{\frac{-s}{\xi^-}}\,ds\right]\xi^-\mu + |u|^p \\
&= e^{\frac{u}{\xi^-}}\left[|u_0|^{p-1}e^{-\frac{u_0}{\xi^-}} - \int_{u_0}^{u} |s|^{p-1} e^{\frac{-s}{\xi^-}}\,ds\right](x_1 - u) + |u|^p \\
&= b(x)
\end{aligned}$$

□

Optimizers for Candidate in P_7 and R_6

We recall the formula for the local candidate b in these regions:

$$b(x) = e^{\frac{u}{\xi^-}}\left[\frac{p}{\xi^-}\int_{-\infty}^{u} (-s)^{p-1} e^{-\frac{s}{\xi^-}}\,ds\right](x_1 - u) + |u|^p \tag{3.6.31}$$

where $u = u^-$ is the unique point satisfying:

$$x_2 = Ce^{u+\xi^-}\left[x_1 - (u + \xi^-) + 1\right] = \frac{e^u}{1-\xi^-}\left[x_1 - (u+\xi^-) + 1\right]$$

This candidate corresponds to the regions P_7 (pre-collision) and R_6 (post-collision). These regions are the portion of Ω_C sitting to the left of the right-leaning tangent to the upper boundary at $(w + \xi^-, Ce^{w+\xi^-})$ and are foliated by right-leaning tangents to the upper boundary at points (a, Ce^a), $a \leqslant w + \xi^-$. Since this sequence is identical among the two configurations, we will use P_7 to refer to both. This candidate is similar to the one for the tangential regions P_1 and R_1; however, this candidate is self-contained. Its expression is independent of those of candidates defined on adjacent regions. As in the last subsection, we first seek to construct optimizers for points on the upper boundary. Since optimizers on the lower boundary are known, we can obtain optimizers for arbitrary $x \in P_7$ by appropriately concatenating the boundary optimizers on the half-tangent through x.

To obtain optimizers for points on the upper boundary, we begin with the approximation procedure detailed in the last subsection. Specifically, fix $(a, Ce^a) \in P_7 \cap \Gamma_C$ and let $a_0 < a$. The approximation procedure gives a test function ψ_a, defined up to knowledge of the optimizer ψ_{a_0}:

$$\psi_a(t) = \begin{cases} \psi_{a_0}\left(\frac{t}{\lambda}\right), & t \in (0, \lambda), \\ u - \xi^- \log(t), & t \in (\lambda, 1) \end{cases} \tag{3.6.32}$$

where $\lambda = e^{\frac{u-u_0}{\xi^-}}$ and $u_0 := a_0 - \xi^-$. Since $\lim_{a_0 \to -\infty} \frac{u_0}{\xi^-} = \infty$, we have $\lim_{a_0 \to -\infty} \lambda = 0$; thus, taking the limit of this test function (in a_0) as $a_0 \to -\infty$ gives

$$\psi_a(t) = u - \xi^- \log(t)$$

For an arbitrary $x \in P_7$, we then concatenate the corresponding boundary optimizers (i.e. those for the points (a, Ce^a) and (u, Ce^u)) as mentioned:

$$\varphi_x(t) = \begin{cases} u + \xi^- \log\left(\frac{\mu}{t}\right), & t \in (0, \mu) \\ u, & t \in (\mu, 1) \end{cases} \tag{3.6.33}$$

where $\mu = \frac{x_1 - u}{\xi^-}$. To finish, we verify the following

Lemma 3.6.6. *Let x be a point in the tangential region P_7 or R_6. Then the function φ_x given by* (3.6.33) *is an optimizer for $b(x)$ given by* (3.6.31).

Proof. Using (3.6.23) and (3.6.24), noting that $\mu > 0$, we first compute

$$\int_0^\mu \log\left(\frac{\mu}{t}\right)\, dt = \lim_{x\to 0^+} \int_{\mu x}^\mu \log\left(\frac{\mu}{t}\right)\, dt = \mu \lim_{x\to 0^+} \left[x\log(x) + (1-x)\right] = \mu$$

$$\int_0^\mu e^{u+\xi^- \log\left(\frac{\mu}{t}\right)}\, dt = \lim_{x\to 0^+} \int_{\mu x}^\mu e^{u+\xi^- \log\left(\frac{\mu}{t}\right)}\, dt = \lim_{x\to 0^+} \frac{e^u}{1-\xi^-}\left[\mu - \mu x\, e^{u_0-u}\right] = \mu\frac{e^u}{1-\xi^-}$$

Therefore,

$$\langle \varphi_x\rangle_{(0,1)} = \mu u + \xi^-\mu + (1-\mu)u = \mu u + (x_1 - u) + (1-\mu)u = x_1,$$

$$\langle e^{\varphi_x}\rangle_{(0,1)} = \mu\frac{e^u}{1-\xi^-} + (1-\mu)e^u = \frac{e^u}{1-\xi^-}\left[\xi^-\mu + (1-\xi^-)\right] = x_2$$

and so condition (3.6.1) is satisfied. Condition (3.6.2) follows directly from Lemma 3.6.2 since φ_x is the cutoff at height u of the function q from the last section. Lastly, to show condition (3.6.3), we use (3.6.29) and (3.6.30) and compute

$$\begin{aligned}
\int_0^\mu \left|u + \xi^- \log\left(\frac{\mu}{t}\right)\right|^p dt &= \lim_{x\to 0^+} \int_{\mu x}^\mu \left|u + \xi^- \log\left(\frac{\mu}{t}\right)\right|^p dt \\
&= \mu \lim_{x\to 0^+} \left[|u|^p - x|\log(x)|^p + p\int_{\log x}^u |s|^{p-1} e^{\frac{u-s}{\xi^-}}\, ds\right] \\
&= \mu\left[|u|^p + p\int_{-\infty}^u |s|^{p-1} e^{\frac{u-s}{\xi^-}}\, ds\right]
\end{aligned}$$

where we used the fact that $\frac{d}{ds}|s|^p = -|s|^{p-1}$ for $s < 0$. Therefore, we have

$$\begin{aligned}
\langle |\varphi_x|^p\rangle_{(0,1)} &= \mu\left[|u|^p + p\int_{-\infty}^u |s|^{p-1} e^{\frac{u-s}{\xi^-}}\, ds\right] + (1-\mu)|u|^p \\
&= e^{\frac{u}{\xi^-}}\left[\frac{p}{\xi^-}\int_{-\infty}^u |s|^{p-1} e^{\frac{-s}{\xi^-}}\, ds\right](x_1 - u) + |u|^p \\
&= b(x)
\end{aligned}$$

□

Optimizers for Candidate in P_5

We recall the formula for the local candidate b in this region:

$$b(x) = e^{\frac{u}{\xi^+}} \left[-|w_0|^{p-1} e^{-\frac{w_0}{\xi^+}} + \frac{p}{\xi^+} \int_u^{w_0} (-s)^{p-1} e^{-\frac{s}{\xi^+}} \, ds \right] (x_1 - u) + |u|^p \tag{3.6.34}$$

where $w_0 = \xi^- - \xi^+$ and $u = u^+$ is the unique point satisfying:

$$x_2 = Ce^{u+\xi^+} \left[x_1 - (u + \xi^+) + 1 \right] = \frac{e^u}{1 - \xi^+} \left[x_1 - (u + \xi^+) + 1 \right]$$

The tangential region P_5 is the portion of Ω_C sitting between the left-leaning tangents to the upper boundary at the points $(w + \xi^+, Ce^{w+\xi^+})$ and (ξ^-, Ce^{ξ^-}) and is foliated by left-leaning tangents to the upper boundary at points (a, Ce^a), $w + \xi^- \leqslant a \leqslant \xi^-$. This candidate is incomplete, reading information from the candidate in the adjacent cup P_4 via their shared boundary (the left-leaning tangent to the upper boundary at (ξ^-, e^{ξ^-})).

The argument producing optimizers for this candidate is entirely symmetric to the one used for the candidate in regions P_1 and P_2. We again seek upper boundary optimizers which we then concatenate with the trivial lower boundary optimizers for general $x \in P_5$. However, instead of using the approximation procedure to relate the upper boundary optimizers ψ_a to those further to the left, i.e. $\psi_{a-\Delta}$ for some small $0 < \Delta$, we instead relate them to optimizers further to the right, i.e. $\psi_{a+\Delta}$. Since the optimizers are known for points on the shared boundary with P_4, we can complete the construction. The optimizer produced is the following:

$$\varphi_x(t) = \begin{cases} 0, & t \in (0, \alpha\mu\nu) \\ w_0, & t \in (\alpha\mu\nu, \mu\nu) \\ u + \xi^+ \log\left(\frac{\mu}{t}\right), & t \in (\mu\nu, \mu) \\ u, & t \in (\mu, 1) \end{cases} \tag{3.6.35}$$

where $\alpha = \frac{\xi^+}{\xi^+ - \xi^-}$, $\mu = \frac{x_1 - u}{\xi^+}$ and $\nu = e^{\frac{u-w_0}{\xi^+}}$. We record the following.

Lemma 3.6.7. *Let x be a point in the tangential region P_5. Then the function φ_x given by* (3.6.35) *is an optimizer for $b(x)$ given by* (3.6.34)*.*

3.6.3 Optimizers for Candidates in Affine Regions

Before addressing each situation explicitly, we will first discuss some similarities in the construction of optimizers for candidates in the affine regions P_3, P_6, R_3 and R_5. In the following discussion, we shall D_A to denote any one of the aforementioned regions. Therefore, D_A is the portion of Ω_C sitting between two half-tangents to the upper boundary (and above a third, non-tangent line segment in the case of R_5). The local candidate b is incomplete, reading information from two of its neighbors, $b^\pm$, each via a shared trajectory. These shared trajectories will each touch the upper boundary at some point $a^\pm$ and will intersect at a shared point u on the lower boundary; we'll use the notation $[a^\pm, u]$ to denote the trajectory shared with $b^\pm$ and we'll suppose, without loss of generality, that a^- lies to the left of a^+. As a consequence of b's dependence on $G^\pm$, the optimizers constructed for $x \in D_A$ will rely on those for $b^\pm$.

As mentioned in the opening discussion of this section, to construct optimizers for $x \in D_A$, we seek to concatenate (in an appropriate proportion) two known optimizers lying on either side of an extremal trajectory (entirely contained in D_A) through x. Since the local candidate b is affine, i.e.

$$b(x) = \alpha_1 \, x_1 + \alpha_2 \, x_2 + \alpha_3; \quad \alpha_i \in \mathbb{R}$$

every trajectory in D_A is extremal. This makes constructing optimizers straightforward. For any $x \in D_A$, we simply take any trajectory through x which intersects both boundaries from which b reads information, i.e. $[a^\pm, u]$. Provided such a trajectory exists, if the optimizers for $b^\pm$ are known along these boundaries, then we have our optimizers to concatenate. More specifically, fix $x \in D_A$ and let $[x^-, x^+]$ represent such a trajectory through x; thus $x^\pm$ lies on the boundary shared with $b^\pm$ and $[x^-, x^+] \subset D_A$. We can then define the corresponding test function φ_x as follows:

$$\varphi_x(t) = \begin{cases} \varphi_{x^-}\left(\frac{t}{\beta}\right), & t \in (0, \beta), \\ \varphi_{x^+}\left(\frac{t-\beta}{1-\beta}\right), & t \in (\beta, 1) \end{cases} \tag{3.6.36}$$

where,

$$\beta = \frac{x_1^+ - x_1}{x_1^+ - x_1^-} = \frac{x_2^+ - x_2}{x_2^+ - x_2^-} \tag{3.6.37}$$

Regarding the existence of such a trajectory through x, if x itself isn't on one of shared boundaries (and thus assumed to be known), we may take the right half-tangent to the upper boundary which passes through x.

Without explicit reference to either of the two adjacent candidates, $b^\pm$, we can show that the test function given by (3.6.36) and (3.6.37) satisfies conditions (3.6.1) and (3.6.3). We compute:

$$\langle \varphi_x \rangle_{(0,1)} = x_1^- \beta + x_1^+ (1-\beta) = x_1; \quad \langle e^{\varphi_x} \rangle_{(0,1)} = x_2^- \beta + x_2^+ (1-\beta) = x_2$$

which gives condition (3.6.1). For condition (3.6.2), since $\varphi_{x^\pm}$ are optimizers for $b^\pm(x^\pm)$, $x = \beta\, x^- + (1-\beta)\, x^+$ and b agrees with $b^\pm$ on their shared boundaries, we have

$$\begin{aligned} \langle |\varphi_x|^p \rangle_{(0,1)} &= \beta \,\langle \varphi_{x^-} \rangle_{(0,\beta)} + (1-\beta)\, \langle \varphi_{x^-} \rangle_{(\beta,1)} \\ &= \beta\, b^-(x^-) + (1-\beta)\, b^+(x^+) \\ &= \beta\, b(x^-) + (1-\beta)\, b(x^+) \\ &= \beta\, (\alpha_1\, x_1^- + \alpha_2\, x_2^- + \alpha_3) + (1-\beta)\, (\alpha_1\, x_1^+ + \alpha_2\, x_2^+ + \alpha_3) \\ &= \alpha_1\, x_1 + \alpha_2\, x_2 + \alpha_3 \\ &= b(x) \end{aligned}$$

Before we consider condition (3.6.2) for specific sequences, we will derive a new formula for the test function φ_x, given by (3.6.36) and (3.6.37), one that is independent of the trajectory $[x^-, x^+]$. Indeed, since $[a^\pm, u]$ are extremal trajectories for $b^\pm$, we can express the optimizers $\varphi_{x^\pm}$ as a concatenation of of the optimizers for $b^\pm(a^\pm)$ and $b^\pm(u)$ as follows:

$$\varphi_{x^-}(t) = \begin{cases} \varphi_{a^-}\left(\frac{t}{\gamma_-}\right), & t \in (0, \gamma_-), \\ u_1, & t \in (\gamma_-, 1) \end{cases} \qquad \varphi_{x^+}(t) = \begin{cases} u_1, & t \in (0, \gamma_+) \\ \varphi_{a^+}\left(\frac{t-\gamma_+}{1-\gamma_+}\right), & t \in (\gamma_+, 1) \end{cases} \tag{3.6.38}$$

where

$$\gamma_- := \frac{x_1 - u_1}{a_1^- - u_1}; \qquad \gamma_+ := \frac{x_1 - a_1^+}{u_1 - a_1^+} \tag{3.6.39}$$

Here, we rearranged the value of u_1 so as to follow the principle mentioned in remark 3.6.1. Substituting (3.6.38) and (3.6.39) into (3.6.36) gives:

$$\varphi_x(t) = \begin{cases} \varphi_{a^-}\left(\frac{t}{\beta\gamma_-}\right), & t \in (0, \beta\gamma_-), \\ u, & t \in (\beta\gamma_-, \beta + (1-\beta)\gamma_+) \\ \varphi_{a^+}\left(\frac{t-\beta-(1-\beta)\gamma_+}{(1-\beta)(1-\gamma_+)}\right), & t \in (\beta + (1-\beta)\gamma_+, 1) \end{cases} \tag{3.6.40}$$

Observe that D_A is contained in the triangle with vertices $a^\pm$ and u. This is clearly for the regions P_3, P_6 and R_3. In the remaining case of R_5, this holds since the line segment $[a^-, a^+]$ lies above the line segment through a^+ and (ξ^-, Ce^{ξ^-}) since $a^- < \xi^-$. As an immediate consequence of this containment, every $x \in D_A$ can be expressed as a unique convex combination of these corner points:

$$x = \delta_1\, a^- + \delta_2\, u + \delta_3\, a^+ \tag{3.6.41}$$

where $\delta_1, \delta_2, \delta_3$ are non-negative and $\delta_1 + \delta_2 + \delta_3 = 1$. Since,

$$(\beta\,\gamma_-)a^- + [\beta(1-\gamma_-) + (1-\beta)(\gamma_+)]\;u + (1-\beta)(1-\gamma_+)a^+ = x$$

It follows that these coefficients must all be independent of $x^{\pm}$. In particular,

$$\delta_1 \;= \beta\,\gamma_-; \quad \delta_2 = \beta(1-\gamma_-) + (1-\beta)(\gamma_+); \quad \delta_3 = (1-\beta)(1-\gamma_+)$$

Therefore, we have the final formula

$$\varphi_x(t) = \begin{cases} \varphi_{a^-}\left(\frac{t}{\delta_1}\right), & t \in (0, \delta_1) \\ u, & t \in (\delta_1,\, \delta_1 + \delta_2) \\ \varphi_{a^+}\left(\frac{t-\delta_1-\delta_2}{\delta_3}\right), & t \in (\delta_1 + \delta_2, 1) \end{cases} \tag{3.6.42}$$

We now verify condition (3.6.2) for each affine candidate

Optimizers for R_5

The region R_5 is the portion of Ω_C sitting between two right tangents to the upper boundary at the points (ξ^-, Ce^{ξ^-}) and $a^- = (w + \xi^-, Ce^{w+\xi^-})$ and above the chord connecting $u = (w, e^w)$ and $a^+ = (0, 1)$. This candidate reads information from its non-affine neighbors: the local candidate in the tangential region R_6 and the one in the cup R_4. In this case, the test function (3.6.42) becomes:

$$\varphi_x(t) = \begin{cases} w + \xi^- \log\left(\frac{\delta_1}{t}\right), & t \in (0,\, \delta_1), \\ w, & t \in (\delta_1,\, \delta_1 + \delta_2) \\ 0, & t \in (\delta_1 + \delta_2,\, 1) \end{cases} \tag{3.6.43}$$

To prove condition (3.6.2), we will show that the Bellman points of φ_x lie inside Ω_C. Fix $(c,d) \subseteq (0,1)$ and let $x^{(c,d)}$ be the corresponding Bellmain point of φ_x. By Lemmas 3.6.3 and 3.6.2, we can assume $\delta_1 + \delta_2 < d$. Furthermore, since φ_x restricted to $(\delta_1, 1)$ is a Bellman point for an optimizer for the candidate in the cup R_4, we can also assume $c < \delta_1$. Let β be given by (3.6.37) and $[x^-, x^+]$ its coresponding trajectory through x; note that $\delta_3 < \beta < \delta_2 + \delta_3$, otherwise one of $x^- = x^{(0,\beta)}$ and $x^+ = x^{(1,\beta)}$ will lie outside R_5. We want to locate the Bellman points $x^{(c,\beta)}$ and $x^{(d,\beta)}$. Since $x^{(\beta,d)}$ is a convex combination of $x^+ = x^{(\beta,1)}$ and $x^{(d,1)} = a^+$, we have that $x^{(\beta,d)}$ lies on the line segment $[u, a^+]$, since x^+ and a^+ do, and below x^+ since φ_x is increasing. For the point $x^{(c,\beta)}$, observe that Lemma 3.6.3 implies $x^{(0,c)}$ lies on the upper boundary, and to the left of a^- since $c < \delta_1$ and $w + \xi^- \log\left(\frac{\delta_1}{t}\right)$ is increasing. Since the upper boundary is the graph of a convex function, the line segment $[x^{(0,c)}, x^-]$ has slope no more than that of the line segment $[x^-, u]$, which itself has a slope no more than the line segment $[x^-, x^+]$. Therefore, points on $[x^{(0,c)}, x^-]$ to the right of x^- will lie below $[x^-, x^+]$; this includes $x^{(c,\beta)}$, since $x^- = x^{(0,\beta)}$ is a convex combination of $x^{(c,\beta)}$ and $x^{(0,c)}$. Since both $x^{(c,\beta)}$ and $x^{(\beta,d)}$ lie under $[x^-, x^+]$, so too does the line segment between them. It follows that $[x^{(c,\beta)}, x^{(\beta,d)}]$, and thus $x^{(c,d)}$, lies inside Ω_C. We have now proven the following.

Lemma 3.6.8. *Let x be a point in the affine region R_5. Then the function φ_x given by* (3.6.43) *is an optimizer for $b(x)$ given by* (3.6.42).

Optimizers for P_6

The region P_6 is the portion of Ω_C bounded on the left by the right-leaning tangent to the upper boundary at $a^- = (w + \xi^-, Ce^{w+\xi^-})$ and on the right be the left-leaning tangent to the upper boundary at $a^+ = (w + \xi^+, Ce^{w+\xi^+})$, which meet at the point $u = (w,\, e^w)$.

In this case, the test function (3.6.42) becomes:

$$\varphi_x(t) = \begin{cases} 0, & t \in (0,\, \alpha\nu\delta_3) \\ \xi^- - \xi^+, & t \in (\alpha\nu\delta_3,\, \nu\delta_3) \\ w + \xi^+ \log\left(\frac{\delta_3}{t}\right), & t \in (\nu\delta_3, \delta_3) \\ w, & t \in (\delta_3,\, \delta_2 + \delta_3) \\ w + \xi^- \log\left(\frac{\delta_1}{1-t}\right), & t \in (\delta_2 + \delta_3,\, 1), \end{cases} \tag{3.6.44}$$

where $\alpha = \frac{\xi^+}{\xi^+ - \xi^-}$ and $\nu = e^{1 + \frac{w - \xi^-}{\xi^+}}$. Here, we've rearranged the optimizers for $a^\pm$ in formula (3.6.42) to minimize the A_∞-characteristic of e^{φ_x} (see remark 3.6.1). To show condition (3.6.2), we first consider the function:

$$f(t) = \begin{cases} w + \xi^+ \log\left(\frac{\delta_3}{t}\right), & t \in (0, \delta_3) \\ w, & t \in (\delta_3,\, \delta_2 + \delta_3) \\ w + \xi^- \log\left(\frac{\delta_1}{1-t}\right), & t \in (\delta_2 + \delta_3,\, 1), \end{cases}$$

We will show that all Bellman points of f lie in Ω_C. Fix $(c, d) \subseteq (0, 1)$ and let $x^{(c,d)}$ be the corresponding Bellman point of f. If $\delta_3 < c$ or $d < \delta_2 + \delta_3$, then Lemmas 3.6.3 and 3.6.2 immediately imply $x^{(c,d)} \in \Omega_C$. For the remaining case, when $c < \delta_3 < \delta_2 + \delta_3 < d$, let $x^{(0,1)}$ be the Bellman point of f corresponding to $(0, 1)$. Since f and φ_x have the same average on $(0, \delta_3)$ and agree elsewhere, the Bellman point of f corresponding to $(0, 1)$ is precisely x and therefore lies in P_6. Let β be given by (3.6.37) and $[x^-,\, x^+]$ the corresponding trajectory through x; note that $\delta_3 < \beta < \delta_2 + \delta_3$, otherwise either $x^- = x^{(0,\beta)}$ or $x^+ = x^{(1,\beta)}$ will lie outside P_6. We would like to determine the location of $x^{(c,\beta)}$. By Lemma 3.6.3, we have that $x^{(0,c)}$ lies on the upper boundary, and to the right of a^+ since f is decreasing. Therefore, $x^{(0,c)}$ must lie above the line segment $[x^-, x^+]$. Since x^- is a convex combination of $x^{(0,c)}$ and $x^{(c,\beta)}$, we must have that $x^{(c,\beta)}$ lies beneath the line segment $[x^-, x^+]$. A similar as that used to show the Bellman point $x^{(c,\beta)}$ for the optimizer in R_5 can be applied to $x^{(\beta,d)}$. Therefore, $x^{(\beta,d)}$ also lies below $[x^-, x^+]$. It follows that the line segment $[x^-, x^+]$ lies above $[x^{(c,\beta)}, x^{(\beta,d)}]$; therefore, this latter segment, and thus $x^{(c,d)}$, must lie in P_6. So in all cases we have $x^{(c,d)} \in \Omega_C$.

We will now show the Bellman points of φ_x belong to Ω_C. Fix $(a, b) \subseteq (0, 1)$ and let $x^{(a,b)}$ be the corresponding Bellman point for φ_x. Since φ_x restricted to $(\alpha\nu\delta_3, 1)$ is the cutoff of f at height $\xi^- - \xi^+$, by Lemma 3.6.2 it suffices to consider the case when $a < \alpha\nu\delta$. Furthermore, if $d < \delta_2 + \delta_3$ then $x^{(a,b)}$ is a Bellman point for an optimizer for P_5. Therefore, we will assume that $a < \alpha\nu\delta_3 < \delta_2 + \delta_3 < d$. Arguing as we had for f, we let β be given by (3.6.37) and $[x^-,\, x^+]$ the corresponding trajectory through x. However, this time $x^{(0,a)} = (0, 1)$ and is therefore on the lower boundary and above the line segment $[x^-, x^+]$. Since $x^- = x^{(0,\beta)}$ is a convex combination of $x^{(0,a)}$ and $x^{(a,\beta)}$, we conclude that $x^{(a,\beta)}$ lies below the line segment $[x^-, x^+]$. From here, the argument identical as that for f. Thus we have proven the following.

Lemma 3.6.9. *Let x be a point in the affine region P_6. Then the function φ_x given by* (3.6.44) *is an optimizer for $b(x)$ given by* (3.6.42).

Optimizers for P_3

The region P_3 is the portion of Ω_C bounded on the left by the right-leaning tangent to the upper boundary at $a^- = (\xi^-, Ce^{\xi^-})$, and on the right be the left-leaning tangent to the upper boundary at $a^+ = (\xi^+, Ce^{\xi^+})$, which meet at the point $u = (0, 1)$. In this case, the test function (3.6.42) becomes:

$$\varphi_x(t) = \begin{cases} \xi^- - \xi^+, & t \in (0,\, \alpha\delta_1), \\ 0, & t \in (\alpha\delta_1,\, \delta_1 + \delta_2 + \alpha\delta_3) \\ \xi^+ - \xi^-, & t \in (\delta_1 + \delta_2 + \alpha\delta_3,\, 1) \end{cases} \tag{3.6.45}$$

where $\alpha = \frac{\xi^-}{\xi^- - \xi^+}$. Here, we've rearranged the optimizers for $a^\pm$ in formula (3.6.42) to ensure the smallest A_∞-characteristic of e^{φ_x} (see remark 3.6.1). To show condition (3.6.2), we fix $(c, d) \subseteq (0, 1)$. If $\alpha\delta_1 \leqslant c$ and/or $d \leqslant \delta_1 + \delta_2 + \alpha\delta_3$, the Bellman point $x^{(c,d)}$ will be a convex combination of u and one of a^- and a^+. Consequently, $x^{(c,d)}$ will lie on one of the two line segments $[a^\pm, u]$ and thus inside Ω_C. For the remaining case, when $c < \alpha\delta_1$ and $\delta_1 + \delta_2 + \alpha\delta_3 < d$, let β be given by (3.6.37) and $[x^-, x^+]$ the corresponding trajectory through x; note that $\alpha\delta_1 < \beta < \delta_1 + \delta_2 + \alpha\delta_3$, otherwise either $x^- = x^{(0,\beta)}$ or $x^+ = x^{(\beta,1)}$ will lie outside P_3. We have x^- is a convex combination of $x^{(0,c)}$ and $x^{(\beta,c)}$; therefore, since x^- and $x^{(0,c)}$ lie on the trajectory $[a^-, u]$, so too must $x^{(c,\beta)}$ and below x^-, since φ_x is increasing. A similar argument gives that $x^{(\beta,d)}$ lies on the trajectory $[a^+, u]$ and below x^+. It follows that the line segment $[x^-, x^+]$ lies above $[x^{(c,\beta)}, x^{(\beta,d)}]$; therefore, this latter segment, and thus $x^{(c,d)}$, must lie in P_3. We've now proven the following

Lemma 3.6.10. *Let x be a point in the affine region P_3. Then the function φ_x given by* (3.6.45) *is an optimizer for $b(x)$ given by* (3.6.42).

Optimizers for R_3

The region R_3, like P_3, is the portion of Ω_C bounded on the left by the right-leaning tangent to the upper boundary at $a^- = (\xi^-, Ce^{\xi^-})$ and on the right by the left-leaning tangent to the upper boundary at $a^+ = (\xi^+, Ce^{\xi^+})$, which meet at the point $u = (0, 1)$. In this case, the test function 3.6.40 becomes:

$$\varphi_x(t) = \begin{cases} w + \xi^- \log\left(\frac{\mu_1\delta_1}{t}\right), & t \in (0,\, \mu_1\delta_1), \\ w, & t \in (\mu_1\delta_1,\, (\mu_1+\mu_2)\delta_1) \\ 0, & t \in ((\mu_1+\mu_2)\delta_1,\, \delta_1+\delta_2+\alpha\delta_3) \\ \xi^+ - \xi^-, & t \in (\delta_1+\delta_2+\alpha\delta_3,\, 1) \end{cases} \tag{3.6.46}$$

where $\alpha = \frac{\xi^-}{\xi^- - \xi^+}$ and μ_1, μ_2, μ_3 are given by (3.6.41) with $x = (\xi^-,\, Ce^{\xi^-})$, $a^- = w + \xi^-$, $u = w$ and $a^+ = 0$. Fix $(c, d) \subseteq (0, 1)$ and let $x^{(c,d)}$ be the corresponding Bellman point of φ_x. Since the restriction of φ_x to the interval $(0, \delta_1 + \delta_2 + \alpha\delta_3)$ is a Bellman point for an optimizer for the candidate in R_5, we may assume $\delta_1 + \delta_2 + \alpha\delta_3 < d$. Furthermore, if $\mu_1\delta_1 < a$ then the restriction of φ_x to (a, b) will be a Bellman point of an optimizer like (3.6.45) from the previous section. So it suffices to show $x^{(a,b)} \in \Omega_C$ for the case when $a < \mu_1\delta < \delta_1 + \delta_2 + \alpha\delta_3 < d$. Let β be given by (3.6.37) and $[x^-, x^+]$ the corresponding trajectory through x; note that $\delta_1 < \beta < \delta_1 + \delta_2 + \alpha\delta_3$, otherwise one of $x^- = x^{(0,\beta)}$ and $x^+ = x^{(1,\beta)}$ will lie outside R_5. From here the proof follows, almost identically, the one given for optimizers for the local candidate in R_5. We have now finished proving:

Lemma 3.6.11. *Let x be a point in the affine region R_3. Then the function φ_x given by* (3.6.46) *is an optimizer for $b(x)$ given by* (3.6.42).

3.7 From the Bellman function to the John–Nirenberg constant

In this section we prove Theorems 3.1.4 and Theorem 3.1.5. We also obtain new best estimate on the John–Nirenberg constant $\varepsilon_0(p)$ using the latter result. However, let us first demonstrate that Theorem 3.1.2, for which Theorem 3.1.4 is a replacement, could not have provided any non-trivial estimates for $\varepsilon_0(p)$ in this range of p.

3.7.1 A simple example

This conclusion can be derived without an explicit formula for the Bellman function $\boldsymbol{b}_{p,C}$. Indeed, consider the following simple function on $(0,1)$:

$$\nu(t)=\begin{cases} w_0, & t\in(0,\gamma)\\ 0, & t\in(\gamma,1-\gamma)\\ -w_0, & t\in(1-\gamma,1)\end{cases},$$

where $\gamma=\frac{(1-\xi^-)(1-\xi^+)(C-1)}{w_0^2}$. It is an elementary matter to check that

$$\langle\nu\rangle_{(0,1)}=0,\qquad \langle e^{-\nu}\rangle_{(0,1)}=C,\qquad [e^{\nu}]_{A_\infty(0,1)}=C.$$

Thus, $\nu\in E_{(0,C),C,(0,1)}$ and by the very definition of $\boldsymbol{b}_{p,C}$,

$$\boldsymbol{b}_{p,C}(0,C)\leqslant\langle|\nu|^p\rangle_I=2|w_0|^p\gamma=2|w_0|^{p-2}(1-\xi^-)(1-\xi^+)(C-1).$$

Since $(1-\xi^+)(C-1)\to e$ and $|w_0|^{p-2}(1-\xi^-)\to 0$ as $C\to\infty$, we conclude that the limit in the right-hand side of (3.1.9) is 0.

3.7.2 Proofs of Theorems 3.1.4 and 3.1.5

We first prove Theorem 3.1.5 and then obtain Theorem 3.1.4 as an immediate corollary. We will need a definition and two further results from [Sla15]. For $\varphi\in\mathrm{BMO}$, let

$$\varepsilon_\varphi=\sup\{\varepsilon>0:\ e^{\varepsilon\varphi}\in A_\infty\}. \tag{3.7.1}$$

Lemma 3.7.1 ([Sla15])**.** *Let φ be a non-constant BMO function. For $\varepsilon\in[0,\varepsilon_\varphi)$, let $F(\varepsilon)=[e^{\varepsilon\varphi}]_{A_\infty}$. Then F is a strictly increasing, continuous function on $[0,\varepsilon_\varphi)$, and $\lim_{\varepsilon\to\varepsilon_\varphi}F(\varepsilon)=\infty$.*

Note that it is also clear that $\lim_{\varepsilon\to 0} F(\varepsilon) = 1$.

Theorem 3.7.2 ([Sla15]). *If $C \geqslant 1$ and $e^{\varphi} \in A_{\infty}^{C}(I)$, then for any $\lambda \in \mathbb{R}$ and any subinterval J of I,*

$$\frac{1}{|J|}\left|\{t \in J:\ \varphi(t) - \langle\varphi\rangle_J \geqslant \lambda\}\right| \leqslant \frac{e^{-\xi^-/\xi^+}}{1-\xi^-/\xi^+}\, e^{-\frac{\lambda}{\xi^+}}. \tag{3.7.2}$$

Proof of Th. 3.1.5. Take $\varphi \in \mathrm{BMO}(I)$. For $\varepsilon \in [0, \varepsilon_\varphi)$, let $F(\varepsilon) = [e^{\varepsilon\varphi}]_{A_\infty(I)}$. Then

$$\|\varepsilon\varphi\|^p_{\mathrm{BMO}^p} \geqslant \boldsymbol{b}_{p,F(\varepsilon)}(0, F(\varepsilon)) = \xi^+(F(\varepsilon))^p\, \frac{\boldsymbol{b}_{p,F(\varepsilon)}(0, F(\varepsilon))}{\xi^+(F(\varepsilon))^p}. \tag{3.7.3}$$

By Lemma 3.7.1, F maps the interval $[0, \varepsilon_\varphi)$ onto $[1, \infty)$. Note that there exists $\tilde\varepsilon \in [1, \varepsilon_\varphi)$ such that

$$\varepsilon_*^p(p) = \frac{\boldsymbol{b}_{p,F(\tilde\varepsilon)}(0, F(\tilde\varepsilon))}{\xi^+(F(\tilde\varepsilon))^p}.$$

This is so because the fraction in the right-hand side is positive for all $\varepsilon \in [1, \varepsilon_\varphi)$ and its limit is 0 as $\varepsilon \to \varepsilon_\varphi$, as explained in section 3.7.1 above. Note that while $\tilde\varepsilon$ depends on φ, $F(\tilde\varepsilon)$ depends only on p. From (3.7.3),

$$\tilde\varepsilon\,\|\varphi\|_{\mathrm{BMO}^p} \geqslant \boldsymbol{b}_{p,F(\varepsilon)}(0, F(\varepsilon)) = \xi^+\big(F(\tilde\varepsilon)\big)\,\varepsilon_*(p)$$

and, thus, for any $\lambda \geqslant 0$,

$$-\frac{\lambda}{\xi^+(F(\tilde\varepsilon))} \leqslant -\frac{\lambda\varepsilon_*(p)}{\tilde\varepsilon\|\varphi\|_{\mathrm{BMO}^p}}.$$

Therefore, after applying (3.7.2) to $\tilde\varepsilon\varphi$ we get

$$\frac{1}{|J|}\left|\{t \in J:\ \tilde\varepsilon\big(\varphi(t) - \langle\varphi\rangle_J\big) \geqslant \lambda\}\right| \leqslant k\big(F(\tilde\varepsilon)\big)\, e^{-\frac{\lambda\varepsilon_*(p)}{\tilde\varepsilon\|\varphi\|_{\mathrm{BMO}^p}}},$$

where

$$k(C) := \frac{e^{-\xi^-/\xi^+}}{1-\xi^-/\xi^+}.$$

We can get the same inequality with $|\varphi(t)-\langle\varphi\rangle_J|$ in place of $\varphi(t)-\langle\varphi\rangle_J$ by doubling the constant in front of the exponent. Then, replacing $\frac{\lambda}{\tilde\varepsilon}$ with λ, we obtain (3.1.11) with $K(p)=2k\big(F(\tilde\varepsilon)\big)$. $\square$

Proof of Th. 3.1.4. For any $\varphi\in\mathrm{BMO}$ and any $\varepsilon<\varepsilon_*(p)$ inequality (3.1.11) can be integrated to bound the average $\big\langle e^{\varepsilon|\varphi-\langle\varphi\rangle_J|/\|\varphi\|_{\mathrm{BMO}^p}}\big\rangle_J$ uniformly with respect to J, and with the bound depending only on ε and p. Specifically, using the layer cake representation,

$$\begin{aligned}\Big\langle e^{\frac{\varepsilon|\varphi-\langle\varphi\rangle_J|}{\|\varphi\|_{\mathrm{BMO}^p}}}-1\Big\rangle_J &\leqslant \frac{\varepsilon}{\|\varphi\|_{\mathrm{BMO}^p}}\int_0^\infty e^{\frac{\lambda\varepsilon}{\|\varphi\|_{\mathrm{BMO}^p}}}\,\frac{1}{|J|}\,\big|\{t\in J:\ |\varphi(t)-\langle\varphi\rangle_J|\geqslant\lambda\}\big|\,d\lambda\\ &\leqslant K(p)\,\frac{\varepsilon}{\|\varphi\|_{\mathrm{BMO}^p}}\int_0^\infty e^{\frac{\lambda(\varepsilon-\varepsilon_*(p))}{\|\varphi\|_{\mathrm{BMO}^p}}}\,d\lambda=\frac{K(p)}{1-\frac{\varepsilon}{\varepsilon_*(p)}}.\end{aligned}$$

Thus, $e^{\varepsilon|\varphi-\langle\varphi\rangle_J|/\|\varphi\|_{\mathrm{BMO}^p}}\in A_2\in A_\infty$ and $\varepsilon_0(p)\geqslant\varepsilon_*(p)$. $\square$

3.7.3 Comparison to the estimate $\varepsilon_0(p)\geqslant 2^{1-2/p}$

Recall, from Remark 3.1.3, that we already have a useful estimate on the John–Nirenberg constant of BMO^p that follows from known results on BMO-norm equivalence: $\varepsilon_0(p)\geqslant 2^{1-2/p}$. Let us show that our new estimate is better for every p. We will do this by expanding the ratio $\frac{\boldsymbol{b}_{p,C}(0,C)}{(\xi^+)^p}$ up to first order in ξ^+ for C close to 1, meaning for ξ^+ close to 0.

Lemma 3.7.3. *For all* $p\in(0,1)$ *we have*

$$\frac{\boldsymbol{b}_{p,C}(0,C)}{(\xi^+)^p}=2^{p-2}\Big(1+\frac{p}{3}\,\xi^+\Big)+O\big((\xi^+)^2\big),\quad \text{as } \xi^+\to 0.$$

Consequently,

$$\varepsilon_0(p)>2^{1-2/p}. \tag{3.7.4}$$

Proof. By Corollary 3.4.10, for all C sufficiently close to 1 we have

$$\boldsymbol{b}_{p,C}(0,C)=2(\xi^+-\xi^-)^{p-2}(1-\xi^-)(1-\xi^+)(C-1).$$

Using this and the fact that $(1-\xi^+)(C-1) = e^{-\xi^+} - 1 + \xi^+$, we get

$$\frac{\boldsymbol{b}_{p,C}(0,C)}{(\xi^+)^p} = \frac{2\Big(1-\frac{\xi^-}{\xi^+}\Big)^p (1-\xi^-)(e^{-\xi^+}-1+\xi^+)}{(\xi^+-\xi^-)^2}.$$

We now expand every factor up to two terms in ξ^+. The key is to understand the relationship between ξ^+ and ξ^- when C is close to 1. Expanding the identity

$$\frac{e^{-\xi^+}}{1-\xi^+} = \frac{e^{-\xi^-}}{1-\xi^-}$$

we have

$$\frac{(\xi^+)^2}{2} + \frac{(\xi^+)^3}{3} \approx \frac{(\xi^-)^2}{2} + \frac{(\xi^-)^3}{3}$$

Canceling the common factor $\xi^+ - \xi^-$, we get $\xi^+ + \xi^- \approx -\frac{2}{3}(\xi^+)^2$, so

$$\xi^- \approx -\xi^+ - \frac{2}{3}(\xi^+)^2 \quad \Longrightarrow \quad \xi^+ - \xi^- \approx 2\xi^+ + \frac{2}{3}(\xi^+)^2 \quad \Longrightarrow \quad 1 - \frac{\xi^-}{\xi^+} \approx 2 + \frac{2}{3}\xi^+.$$

In addition, $1-\xi^- \approx 1+\xi^+$ and

$$e^{-\xi^+} - 1 + \xi^+ \approx \frac{(\xi^+)^2}{2} - \frac{(\xi^+)^3}{6}$$

Putting everything together,

$$\begin{aligned}
\frac{\boldsymbol{b}_{p,C}(0,C)}{(\xi^+)^p} &\approx \frac{2\big(2+\frac{2}{3}\xi^+\big)^p(1+\xi^+)\big(\frac{(\xi^+)^2}{2} - \frac{(\xi^+)^3}{6}\big)}{\big(2\xi^+ + \frac{2}{3}(\xi^+)^2\big)^2} \\
&= \frac{2^{p-2}\big(1+\frac{1}{3}\xi^+\big)^p(1+\xi^+)\big((\xi^+)^2 - \frac{1}{3}(\xi^+)^3\big)}{\big(\xi^+ + \frac{1}{3}(\xi^+)^2\big)^2} \\
&\approx \frac{2^{p-2}\big(1+\frac{p}{3}\xi^+\big)(1+\xi^+)\big(1-\frac{1}{3}\xi^+\big)}{1+\frac{2}{3}\xi^+} \\
&\approx 2^{p-2}\Big(1+\frac{p}{3}\xi^+\Big)(1+\xi^+)\Big(1-\frac{1}{3}\xi^+\Big)\Big(1-\frac{2}{3}\xi^+\Big) \approx 2^{p-2}\Big(1+\frac{p}{3}\xi^+\Big).
\end{aligned}$$

Note that the error in this compound approximation is of order $(\xi^+)^2$, as promised.

Therefore,

$$\lim_{C\to 1}\frac{\boldsymbol{b}_{p,C}(0,C)}{(\xi^+)^p}=2^{p-2}$$

and for every $p\in(0,1)$, there exists a number $\tilde{C}=\tilde{C}(p)>1$ such that the the function $\frac{\boldsymbol{b}_{p,C}(0,C)}{(\xi^+)^p}$ is increasing for $1<C\leqslant\tilde{C}$. Hence,

$$\varepsilon_0^p(p)\geqslant\varepsilon_*^p(p)=\sup_{1<C<\infty}\frac{\boldsymbol{b}_{p,C}(0,C)}{(\xi^+(C))^p}\geqslant\frac{\boldsymbol{b}_{p,\tilde{C}}(0,\tilde{C})}{(\xi^+(\tilde{C}))^p}>2^{p-2},$$

which gives (3.7.4). □

The lemma just proved is qualitative in nature. Let us illustrate this result. While at present we do not know the exact value of $\varepsilon_*(p)$ or whether that value is, in fact, equal to $\varepsilon_0(p)$, we can use Theorem 3.1.4 to produce any number of estimates for $\varepsilon_0(p)$, simply by substituting any specific C into the expression

$$\frac{\big(\boldsymbol{b}_{p,C}(0,C)\big)^{1/p}}{\xi^+(C)}.$$

We can do that because, unlike in the earlier studies for $p\geqslant 1$, for our range of p we know the Bellman function $\boldsymbol{b}_{p,C}$ for all values of C, and not just sufficiently large ones. To understand the nature of the estimate provided by Theorem 3.1.4, and also to see the subtle quantitative nature of the w-related correction that Corollary 3.4.10 introduces after the threshold $C=C_*$, compared to the behavior given by the relatively simple formula before the threshold, let us consider several numerical illustrations, given in Figures 3.13, 3.14, and 3.15. In all pictures in this group, the red curve is the ratio $\boldsymbol{b}_{p,C}(0,C)/(\xi^+)^p$; the green curve is what this ratio would have been without the correction that happens after $C=C_*$; the yellow curve is the function $\boldsymbol{b}_{p,C}(0,C)$ itself; and the blue curve is this function without the correction after $C=C_*$. The vertical black line is at $C=C_*$.

It view of the many implicit functions contained in the formula for $\boldsymbol{b}_{p,C}$ in Corollary 3.4.10 for $C>C_*$ it might be better to choose a value $C<C_*$ for an esti-

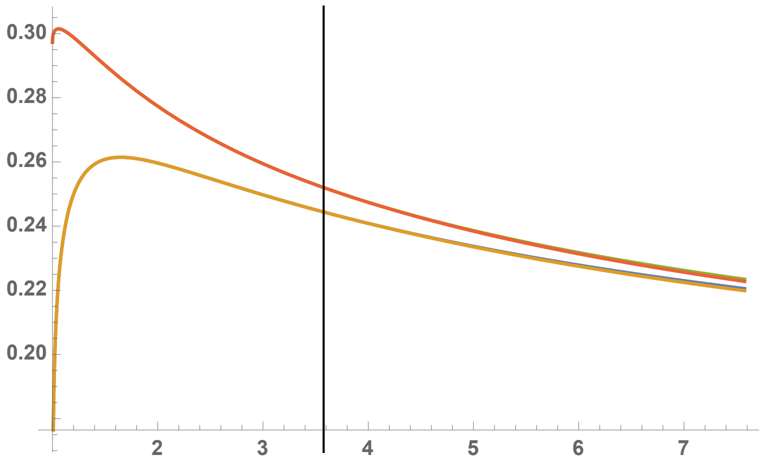

Figure 3.13: The behavior of $\frac{\boldsymbol{b}_{p,C}(0,C)}{(\xi^+)^p}$ for $p = \frac{1}{4}$

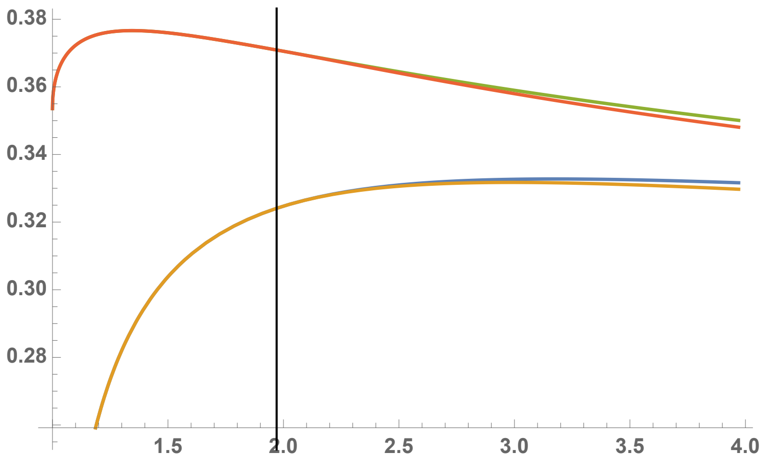

Figure 3.14: The behavior of $\frac{\boldsymbol{b}_{p,C}(0,C)}{(\xi^+)^p}$ for $p = \frac{1}{2}$

mate. Lemma 3.4.3 provides us with such a safe threshold: by (3.4.15) any C such that $w_0(C) > p-2$ will be in the pre-collision range. Figures 3.13, 3.14, and 3.15 suggest that the optimal w_0 decreases with p. In an admittedly simple-minded choice, we set $w_0 = -p$, which does, of course, satisfy $w_0 > p - 2$.

Having chosen w_0, we can easily compute $\xi^{\pm}$ using the formulas

$$\xi^+ = 1 + \frac{w_0 e^{w_0}}{1 - e^{w_0}}, \quad \xi^- = 1 + \frac{w_0}{1 - e^{w_0}},$$

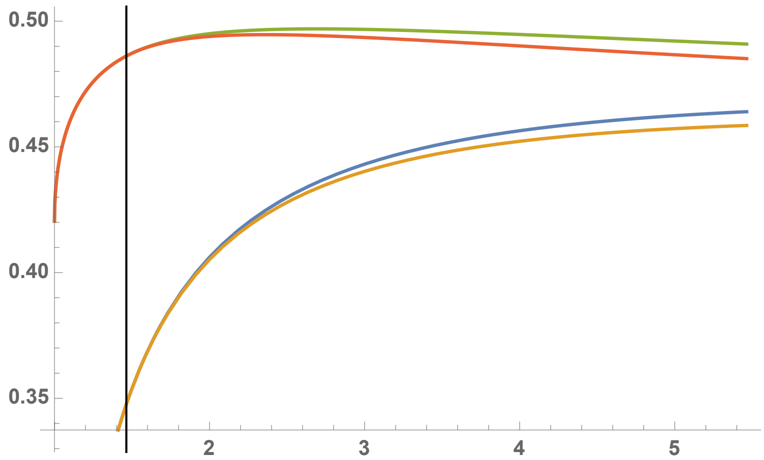

Figure 3.15: The behavior of $\frac{\boldsymbol{b}_{p,C}(0,C)}{(\xi^+)^p}$ for $p = \frac{3}{4}$

which then immediately gives C. Using these formulas with $w_0 = -p$, we can graph the quotient $\boldsymbol{b}_{p,C}(0,C)/(\xi^+)^p$ against p in lieu of a complicated and opaque analytic expression, and compare it to the function 2^{p-2}, which, as we know, corresponds to the limit of this quotient as $C \to 1^+$, meaning $w_0 \to 0^-$. The comparison is shown in Figure 3.16. We see that it is indeed the case that

$$\frac{\boldsymbol{b}_{p,C}(0,C)}{\big(\xi^+(C)\big)^p}\Big|_{w_0=-p} \geqslant 2^{p-2},$$

(blue curve for the left-hand side, yellow curve for the right-hand side) and, thus, our theorem readily yields better estimates than could be obtained from earlier results. Of course, much subtler and more deliberate choices of w_0 (and, thus, C) can be made, providing better – and perhaps optimal – values of this ratio for all p.

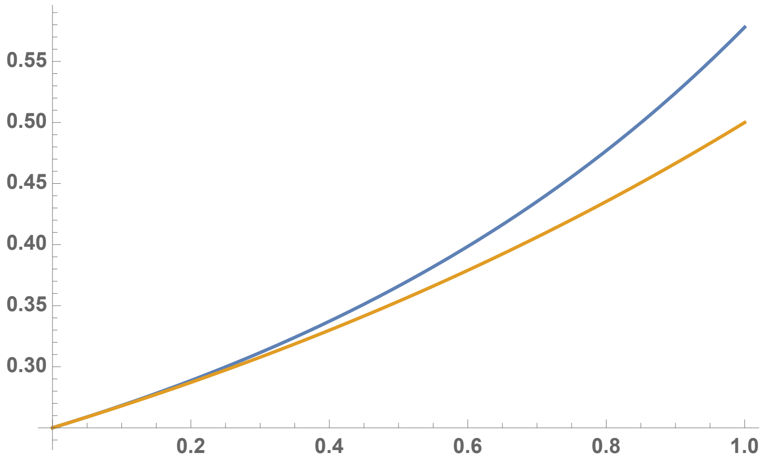

Figure 3.16: $w_0 = -p$ vs. $w_0 \to 0^-$

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Chapter 4

Appendix

4.1 Mathematica Code

The following pages contain all Mathematica code used for calculations and to generate the figures present in this thesis.

```
(*Used for Figure 2.1*)
Q := 1000
```

$$\text{LeadingK}_{\alpha_1}[\alpha\_] := 8 \times \left(2 + \frac{1}{\alpha}\right)$$

$$\text{LeadingK}_{\alpha_2}[\alpha\_] := 8 \times \left(2 + \sum_{k=1}^{\infty} 2^{-k} \left(1 - 2^{-k}\right)^{2\,\alpha-2}\right)$$

$$K_{\alpha_1}[\alpha\_,\ s\_] := 8 \times \left(2 + \frac{1}{\alpha}\right) Q^{\alpha} - \frac{32}{2\,\alpha - 1}\, Q^{\alpha-\frac{1}{2}} + \frac{8}{\alpha\,(2\,\alpha - 1)};$$

$$K_{\alpha_2}[\alpha\_,\ s\_] := 16 \left(Q^{\alpha} - Q^{\alpha-\frac{1}{2}}\right) + 8 \left(\sqrt{Q} - 1\right)^2 \sum_{j=1}^{\infty} 2^{-j} \left(s\, 2^{-j} + \sqrt{Q}\, \left(1 - 2^{-j}\right)\right)^{2\,\alpha-2};$$

$$\text{Plot}\Big[\left\{\text{LeadingK}_{\alpha_1}[a],\ \text{LeadingK}_{\alpha_2}[a]\right\},\ \{a,\ .9,\ 1.65\},$$
$$\text{AxesLabel} \to \{\text{"}\alpha\text{"},\ \text{""}\},\ \text{GridLines} \to \{\{1,\ 1.5\},\ \{\}\},$$
$$\text{PlotLegends} \to \left\{\text{"}8\,(2+\frac{1}{\alpha})\text{"},\ \text{"}8\,(2+\sum_{k=1}^{\infty} 2^{-k}\ (1-2^{-k})^{2\,\alpha-2})\text{"}\right\}\Big]$$

```
(*Used for Figures 2.4 & 2.5*)
L := 1.1
res := 28
r[n_] := 1 + (1 - Power[2, -n]) (L - 1)
b[n_] := L / r[n] + Sqrt[Power[L / r[n], 2] - 1]
a[n_] := L / r[n] - Sqrt[Power[L / r[n], 2] - 1]

(*Binary Expansion Termination Checking*)
digits[d_] := StringRiffle[RealDigits[d, 2, res, -1]⟦1⟧, ""]
counts[s_] := Count[StringCases[s, "10" | "11"], #] & /@ {"10", "11"}
zeros[n_, k_] := RegularExpression@
  ToString@StringForm["^(0{``}|(^(0{0,``}(10|11))+0{``}))", k, n - 1, n]
dStar[n_, k_, d_] := First[StringCases[digits[d], Shortest[zeros[n, k]]], ""]

(*Iterative Optimizers*)
w[n_, k_, d_] := If[dStar[n, k, d] == "", Null, powers[n, dStar[n, k, d]]]
powers[n_, s_] := Power[a[n], counts[s]⟦1⟧] Power[b[n], counts[s]]⟦2⟧

(*Recursive Optimizers*)
rL[n_, i_, d_] := If[i == res, Null,
  Piecewise[
   {
    {a[n] × r[n, n, i, 2 d], d ≤ .5},
    {b[n] × r[n, n, i, 2 d - 1], d > .5 }
   }
  ]
 ]

r[n_, k_, i_, d_] := If[k == 0, 1,
  If[i == res, Null,
   Piecewise[
    {{r[n, k - 1, i, 2 d], d ≤ .5},
     {rL[n, i + 1, 2 d - 1], d > .5}}
   ]]]

res = 1;
p1 = Plot[rL[1, 0, d], {d, 0, 1},
  PlotRange → {{0, 1}, {0, 4}},
  PlotPoints → 100, AxesLabel → {t, ""},
  PlotLegends → Placed[Style["v_0^(L) (t)", 14], Below]]
res = 2;
p2 = Plot[rL[1, 0, d], {d, 0, 1},
```

```
    PlotRange → {{0, 1}, {0, 4}},
    PlotPoints → 100, AxesLabel → {t, ""},
    PlotLegends → Placed[Style["v_1^(L)(t)", 14], Below]]
res = 3;
p3 = Plot[rL[1, 0, d], {d, 0, 1},
    PlotRange → {{0, 1}, {0, 4}},
    PlotPoints → 300, AxesLabel → {t, ""},
    PlotLegends → Placed[Style["v_2^(L)(t)", 14], Below]]
res = 4;
p4 = Plot[rL[1, 0, d], {d, 0, 1},
    PlotRange → {{0, 1}, {0, 4}},
    PlotPoints → 400, AxesLabel → {t, ""},
    PlotLegends → Placed[Style["v_3^(L)(t)", 14], Below]]

res = 1;
p5 = Plot[rL[2, 0, d], {d, 0, 1},
    PlotRange → {{0, 1}, {0, 4}},
    PlotPoints → 200, AxesLabel → {t, ""},
    PlotLegends → Placed[Style["v_0^(L)(t)", 14], Below]]
res = 2;
p6 = Plot[rL[2, 0, d], {d, 0, 1},
    PlotRange → {{0, 1}, {0, 4}},
    PlotPoints → 200, AxesLabel → {t, ""},
    PlotLegends → Placed[Style["v_1^(L)(t)", 14], Below]]
res = 3;
p7 = Plot[rL[2, 0, d], {d, 0, 1},
    PlotRange → {{0, 1}, {0, 4}},
    PlotPoints → 800, AxesLabel → {t, ""},
    PlotLegends → Placed[Style["v_2^(L)(t)", 14], Below]]
res = 4;
p8 = Plot[rL[2, 0, d], {d, 0, 1},
    PlotRange → {{0, 1}, {0, 4}},
    PlotPoints → 4200, AxesLabel → {t, ""},
    PlotLegends → Placed[Style["v_3^(L)(t)", 14], Below]]
```

```
(*Used for Figures 3.9-3.12*)
(*Computing C*)
```

$$C_*[p\_] := \frac{e^{-xm_*[w_*[p]]}}{1 - xm_*[w_*[p]]}$$

$$xm_*[w\_] := 1 + \frac{w}{1 - e^{w}};\ xp_*[w\_] := 1 + \frac{w\, e^{w}}{1 - e^{w}}$$

$$R_*[w\_?NumericQ, p\_] := -\frac{xm_*[w]}{1 - xm_*[w]}\ \frac{Abs[w]}{e^{-w} - 1 + w}$$

$$L_*[w\_?NumericQ, p\_] := p\, e^{\frac{w}{xm_*[w]}}\ ExpIntegralE\left[2 - p, \frac{w}{xm_*[w]}\right]$$

$$w_*[p\_] := w\ /.\ FindRoot[L_*[w, p] == R_*[w, p],\ \{w, p - 2\}];$$

```
(*Various definitions*)
```

$$w_1[p\_] := w\ /.\ FindRoot\left[\frac{w^2}{e^{-w} - 1 + w} == p, \{w, p - 2\}\right]$$

$$w_2[p\_] := w\ /.\ FindRoot\left[\frac{w}{1 - e^{-w}} == p, \{w, p - 2\}\right]$$

$$xm[C\_] := 1 + ProductLog\left[-1, -\frac{1}{C\, e}\right]$$

$$xp[C\_] := 1 + ProductLog\left[-\frac{1}{C\, e}\right]$$

$$w_0[C\_] := xm[C] - xp[C]$$

```
(*Computing w=wPre pre-collision; 1≤C≤C*)
```

$$L_{Pre}[w\_?NumericQ,\ p\_, C\_] := -\frac{1 - xm[C]}{xm[C]}\ p\, e^{\frac{w}{xm[C]}}\ ExpIntegralE\left[2 - p, \frac{w}{xm[C]}\right]$$

$$R_{Pre}[w\_?NumericQ, p\_, C\_] :=$$
$$\frac{1 - xp[C]}{xp[C]}\ e^{\frac{w}{xp[C]}}\ \left(e^{\frac{xp[C]-xm[C]}{xp[C]}} + p\ NIntegrate\left[e^{-\frac{w}{xp[C]}\, y}\ y^{p-2}, \left\{y, \frac{xm[C] - xp[C]}{w}, 1\right\}\right]\right)$$

$$w_{Pre}[p\_, C\_] := w\ /.\ FindRoot[L_{Pre}[w, p, C] == R_{Pre}[w, p, C], \{w, p - 2\}]$$

```
(*Pre-collision w0*)

PreCollisionPlot[p_] :=
 Plot[{wPre[p, C], w0[C], w*[p], p - 2}, {C, 1, C*[p] + 1}, GridLines → {{C*[p]}, None},
  AxesLabel → {"C", "x1"}, PlotLegends → {"w", "w0", "w*", "p-2"},
  PlotLabel → "w(C), 1≤C≤C*, p=" <> ToString[p]]


(*Computing w=wPost post-collision; C*≤C*)
```

```
R_Post[w_?NumericQ, p_, C_] := - xm[C]/(1 - xm[C]) Abs[w]/(ⅇ^-w - 1 + w)

L_Post[w_?NumericQ, p_, C_] := p ⅇ^(w/xm[C]) ExpIntegralE[2 - p, w/xm[C]]

w_Post[p_, C_] := w /. FindRoot[L_Post[w, p, C] == R_Post[w, p, C], {w, p - 2}]
(*Post-collision w_0*)

PostCollisionPlot[p_] :=
 Plot[{w_Post[p, C], w_0[C], w_*[p], w_2[p], p - 2}, {C, C_*[p], (50 + C_*[p]) C_*[p]},
  AxesLabel → {"C", "x_1"}, PlotLegends → {"w", "w_0", "w_*", "w_2", "p-2"},
  PlotLabel → "w(C), C_*≤C, p=" <> ToString[p],
  PlotStyle → {Automatic, Automatic, Automatic, Dashed, Automatic},
  PlotRange → {Automatic, {-3, 1}}]

(*Computing w=w_Full; 1≤C*)
w_Full[p_, K_] := Piecewise[{{w_Pre[p, K], 1 ≤ K ≤ C_*[p]}, {w_Post[p, K], C_*[p] ≤ K}}]
FullPlot[p_] := Plot[{w_Full[p, K], w_0[K], w_*[p], w_2[p], p - 2},
  {K, 1, (4 + C_*[p]) C_*[p]}, GridLines → {{C_*[p]}, None},
  AxesLabel → {"C", "x_1"}, PlotLegends → {"w", "w_0", "w_*", "w_2", "p-2"},
  PlotLabel → "w(C), 1≤C, p=" <> ToString[p],
  PlotStyle → {Automatic, Automatic, Automatic, Dashed, Automatic}]

(*Lemma 4.3 Plots*)
Plot[{w_*[p], w_1[p], w_2[p], p - 2}, {p, 0, 1},
 PlotLegends → {"w_*", "w_1(p)", "w_2(p)", "p-2"}]

(*Pre-collision plots for p=1/4, 1/2, 3/4*)
PreCollisionPlot[.25]
PreCollisionPlot[.50]
PreCollisionPlot[.75]

(*Post-collision plots for p=1/4, 1/2, 3/4*)
PostCollisionPlot[.25]
PostCollisionPlot[.50]
PostCollisionPlot[.75]
PostCollisionPlot[.9]

(*Full plots for p=1/4, 1/2, 3/4*)
p7 = FullPlot[.25]
p8 = FullPlot[.50]
p9 = FullPlot[.75]
```

(*Used for Figures 3.13–3.16*)

w[p_] := p - 2

$$K[p\_] := \frac{e^{-xp[p]}}{1 - xp[p]};$$

$$xm[p\_] := 1 + \frac{w[p]}{1 - e^{w[p]}};$$

$$xp[p\_] := 1 + \frac{w[p]\, e^{w[p]}}{1 - e^{w[p]}};$$

NMaxValue[{xp[p], 0 ≤ p ≤ 1}, p]

NMaxValue[{xm[p], 0 ≤ p ≤ 1}, p]

$$b[p\_] := 2\,(xp[p] - xm[p])^{p-2}\,(1 - xm[p]) \times (1 - xp[p])\,(K[p] - 1)\,xp[p]^{-p};$$

$$\text{NSolve}\left[b[p] == 2^{p-2}\ \&\&\ 0 \le p \le 1,\ p\right]$$

$$\text{Plot}\left[\left\{b[p],\ 2^{p-2}\right\},\ \{p, 0, 1\}\right]$$

(*For the purpose of computing C_0*)

$$\text{xm0[w\_]} := 1 + \frac{\text{w}}{1 - e^{\text{w}}};\ \text{xp0[w\_]} := 1 + \frac{\text{w}\, e^{\text{w}}}{1 - e^{\text{w}}};$$

$$\text{RHS1[w\_?NumericQ, p\_]} := -\frac{\text{xm0[w]}}{1 - \text{xm0[w]}} \frac{\text{Abs[w]}}{e^{-\text{w}} - 1 + \text{w}}$$

$$\text{LHS1[w\_?NumericQ, p\_]} := \text{p}\, e^{\frac{\text{w}}{\text{xm0[w]}}}\ \text{NIntegrate}\left[e^{-\frac{\text{w}}{\text{xm0[w]}}\text{y}}\ \text{y}^{\text{p}-2}, \{\text{y}, 1, \infty\}\right]$$

w0[p_] := w /. FindRoot[LHS1[w, p] == RHS1[w, p], {w, p − 2}]

$$\text{C0[p\_]} := \frac{e^{-\text{xm0[w0[p]]}}}{1 - \text{xm0[w0[p]]}};$$

(*For the purpose of computing w*)

$\text{xm[C\_]} := \text{x} /. \text{NSolve}[\text{C} * (1 - \text{x}) == e^{-\text{x}}\ \&\&\ \text{x} <= 0, \text{x}][[1]]$

$\text{xp[C\_]} := \text{x} /. \text{NSolve}[\text{C} * (1 - \text{x}) == e^{-\text{x}}\ \&\&\ 0 \le \text{x}, \text{x}][[1]]$

$$\text{RHS2[w\_?NumericQ, p\_, C\_]} := -\frac{\text{xm[C]}}{1 - \text{xm[C]}} \frac{\text{Abs[w]}}{e^{-\text{w}} - 1 + \text{w}}$$

$$\text{LHS2[w\_?NumericQ, p\_, C\_]} := \text{p}\, e^{\frac{\text{w}}{\text{xm[C]}}}\ \text{NIntegrate}\left[e^{-\frac{\text{w}}{\text{xm[C]}}\text{y}}\ \text{y}^{\text{p}-2}, \{\text{y}, 1, \infty\}\right]$$

w[p_, C_] := w /. FindRoot[LHS2[w, p, C] == RHS2[w, p, C], {w, p − 2}]

$$\text{Expr[p\_, C\_]} := \text{p} - \frac{(1 - \text{p})\ \text{w[p, C]}\ (e^{-\text{w[p,C]}} + \text{xm[C]} - 1)}{(1 - \text{xm[C]})\ (e^{-\text{w[p,C]}} + \text{w[p, C]} - 1)}$$

$$\text{b1[p\_, C\_]} := 2\ (\text{xp[C]} - \text{xm[C]})^{\text{p}-2}\ (1 - \text{xm[C]}) \times (1 - \text{xp[C]})\ (\text{C} - 1);$$

$$\text{b2[p\_, C\_]} := \frac{(\text{xp[C]} - \text{xm[C]})^{\text{p}-1} + \text{Abs[w[p, C]]}^{\text{p}-1}\ \text{Expr[p, C]}}{\text{xp[C]} - \text{xm[C]}}\ (1 - \text{xm[C]}) \times (1 - \text{xp[C]})\ (\text{C} - 1);$$

b[p_, C_] := Piecewise[{{b1[p, C], C ≤ C0[p]}}, b2[p, C]];
Plot[{b[.5, C], b1[1/2, C], b[.5, C]/xp[C]^(1/2)}, {C, 1, C0[.5] + 6}]

```
r = 1 / 4; Plot[{b[r, C], b1[r, C], b[r, C] / xp[C] ^ (r)}, {C, 1, C0[r] + 6}]

r = 3 / 4;
Plot[{b[r, C], b1[r, C], b[r, C] / xp[C] ^ (r)}, {C, 1, C0[r] + 2},
 Epilog → (*add vertical lines*)InfiniteLine[{C0[r], 0}, {0, 1}]]

r = 1 / 4;
Plot[{b[r, C], b1[r, C], b[r, C] / xp[C] ^ (r)}, {C, 1, C0[r] + 2},
 Epilog → (*add vertical lines*)InfiniteLine[{C0[r], 0}, {0, 1}]]
```

```
r = 1 / 2;
Plot[{b[r, C], b1[r, C], b[r, C] / xp[C] ^ (r)}, {C, 1, C0[r] + 2},
 Epilog → (*add vertical lines*)InfiniteLine[{C0[r], 0}, {0, 1}]]

r = 3 / 4;
Plot[{b[r, C], b1[r, C], b[r, C] / xp[C] ^ (r), b1[r, C] / xp[C] ^ (r)}, {C, 1, C0[r] + 4},
 Epilog → (*add vertical lines*)InfiniteLine[{C0[r], 0}, {0, 1}]]

r = 1 / 2;
Plot[{b1[r, C], b[r, C], b1[r, C] / xp[C] ^ (r), b[r, C] / xp[C] ^ (r)}, {C, 1, C0[r] + 2},
 Epilog → (*add vertical lines*)InfiniteLine[{C0[r], 0}, {0, 1}]]

r = 3 / 4;
Plot[{b1[r, C], b[r, C], b1[r, C] / xp[C] ^ (r), b[r, C] / xp[C] ^ (r)}, {C, 1, C0[r] + 4},
 Epilog → (*add vertical lines*)InfiniteLine[{C0[r], 0}, {0, 1}]]

r = 1 / 4;
Plot[{b1[r, C], b[r, C], b1[r, C] / xp[C] ^ (r), b[r, C] / xp[C] ^ (r)}, {C, 1, C0[r] + 6},
 Epilog → (*add vertical lines*)InfiniteLine[{C0[r], 0}, {0, 1}]]

r = 1 / 10;
Plot[{b1[r, C], b[r, C], b1[r, C] / xp[C] ^ (r), b[r, C] / xp[C] ^ (r)}, {C, 1, C0[r] + 6},
 PlotRange → Full, Epilog → (*add vertical lines*)InfiniteLine[{C0[r], 0}, {0, 1}]]

Plot[{w0[p], p - 2, w2[p], w3[p]}, {p, 0, 1}]
```