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I, Xin Wang , hereby submit this original work as part of the requirements for the degree of Doctor of Philosophy in Business Administration.

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Bounded Multiattribute Utility in Behavior Decision Research: Theory, Estimation and Experimental Tests

A dissertation submitted to the

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Doctor of Philosophy

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ABSTRACT

Mounting evidence suggests that subjective value is encoded neurologically using a neural common scale. We address the question of how encoded part-values are integrated to form overall value. Our results are based on three broad biological considerations that impact this process – positivity, detectability, and boundedness; mechanisms that are strongly supported in the literatures of neuroscience, psychology, and anthropology. We show that these three conditions lead to a specific functional form to represent the value integration process; a form we refer to as neuroaddition. We show deductively that the neuroadditive theory validates many published results from experimental programs in behavioral decision theory, including loss aversion, the endowment effect, and non-compensatory information processing. The dissertation capitalizes on these deductive results to develop statistical estimation routines – maximum likelihood and MCMC hBayes – for a multinomial logit model that uses the new operator in place of standard addition. The dissertation offers experimental tests of predicted consequences and draws strategic implications from the new theory. Positive outcomes in these tests support the integration of known behavioral effects into the new product design process.

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DEDICATIONS

For Xiaoke, Dinghong and Haili

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CHAPTER 1 INTRODUCTION

1.1 Overview

Theories of multiattribute utility inform our understanding of consumer preferences and choices. Extant theories in the traditions of conjoint analysis and choice-based conjoint analysis presume that the composition of holistic value judgments from parts is additive, hence unbounded. We develop a new theory of bounded utility, based on a new binary operator to define “additive”. We show deductively that the theory validates many published results from experimental programs in behavioral decision theory, including loss aversion, the endowment effect, and non-compensatory information processing. The dissertation proposes to capitalize on the deductive results to develop statistical estimation routines – maximum likelihood and MCMC hBayes – for a multinomial logit model that uses the new operator in place of standard addition. The proposed operator explains loss aversion as an emergent property of brain biology and shows how a fundamental asymmetry in neural processing creates the endowment effect. The dissertation offers experimental tests of predicted consequences and draws strategic implications from the new theory. Positive outcomes in these tests will support the integration of known behavioral effects into the new product design process.

1.2 Past Research

The question of how individuals form holistic preferences from parts – preferences for various levels of attributes comprising a whole – is fundamental to the study of consumer behavior. In marketing this question is most often addressed by theories of multiattribute utility as operationalized by conjoint analysis and discrete-choice modeling. Since its introduction in marketing, conjoint analysis (Green and Rao 1971) has become the most frequently used commercial tool to assess how consumers trade off attribute levels among competing products. Conjoint analysis – especially choice-based methods – has helped many firms make decisions in new product design, pricing, segmentation, positioning, and advertising (Cattin and Wittink 1982, Wittink and Cattin 1989). Bryan Orme, CEO of Sawtooth Software, estimates that upwards of 18,000 conjoint studies are conducted annually (Orme 2009, p.143). Many of these applications use estimated partworth utilities in strategic follow-up analyses, called "choice simulators", to identify the combination of attribute levels that will appeal to the most individuals in a target market. An internal survey at Sawtooth found that 65-80% of choice-based conjoint studies use the firm's product optimization simulators. Thus with Sawtooth alone, roughly 6,500 new product optimizations are conducted each year, and about 4,500 of these are done by U.S.-based firms (B. Orme, personal communication with thesis committee chair, May 2013).

This dissertation has its origins in preference measurement, an interdisciplinary field with foundations in mathematical psychology (Luce and Tukey 1964) and economics (e.g., Arrow 1959, McFadden 1974). Incorporating psychological effects into preference measurement has become of increasing interest to quantitative marketing scholars. Few conjoint studies have attempted to integrate well-known behavioral phenomena, such as loss-aversion, into the design and analysis of discrete-choice data. Some notable exceptions include Bradlow et al (2004), who modeled behavioral context effects caused by omitted product attributes in partial profile designs and Kivetz et al (2004), who captured compromise effects using alternative preference measurement tasks. Adamowicz et al (2008) provide a detailed overview of behavioral effects in choice modeling. From an academic perspective, I suggest that incorporating cognitive and behavioral effects should improve prediction accuracy in preference measurement and in choice models and as a consequence should increase the success rate of new products.

An area ripe for improvement in most conjoint-style preference measurements is to relax the standard assumption that overall utility is linear and additive in partworth utilities. Linear-additive theory presumes that the composition of holistic value is unbounded. However, evidence from neuro-cognitive research suggests that value may be bounded from above and below. Because standard addition is unbounded, the dissertation uses a new binary operator, called neuroaddition, to impose these properties. Deductive analyses using the new operator show that it mimics psychological mechanisms that support established findings from behavioral decision theory. My dissertation capitalizes on these deductive results to develop a statistical estimation routine for the conditional multinomial logit model that uses the new operator in place of standard addition. In the process, I collect experimental data to test consequences of the theory, and to elaborate its strategic implications. For example, I explore the impact of the new "algebra" on choice optimization routines. Using the new algebra, a multinomial logit model will naturally exhibit loss aversion and mimic both compensatory and non-compensatory decision rules (Gilbride and Allenby 2004). The shift between rules will be moderated by the total utility of the choice options under consideration, a property sought by Allenby and Rossi (1991) and Hardie et al (1993).

1.3 Abstracts of Essays

1.3.1 Essay 1 (Chapter 2)

Mounting evidence suggests that subjective value is encoded neurologically using a neural common scale. We address the question of how encoded part-values are integrated to form holistic values. Results are based on three broad biological considerations that impact this process – positivity, detectability, and boundedness; mechanisms that are strongly supported in the literatures of neuroscience, psychology, and anthropology. We show that these three conditions lead to a specific functional form to represent the value integration process; a form we refer to as neuroaddition. We offer three main results; first, diminishing sensitivity is a natural by-product in a neuroadditive system; second, neuroadditivity is a sufficient condition for loss aversion, hence implies referent dependent behaviors associated with this mechanism; and third, neuroadditivity is sufficient to generate non-compensatory behaviors with the degree of non-compensation moderated by total value.

1.3.2 Essay 2 (Chapter 3)

Because the neuroadditive model generates a variety of experimentally testable hypotheses, it would be useful to have a routine that can estimate the model's parameters from data. This essay develops such routines for discrete-choice data. Both pooled and individual models are developed. The maximum likelihood approach is applied for pooled estimation, supplemented by a grid search routine. The grid search permits the multinomial logit model (MNL) to compensate for heterogeneity in the variance of parameter estimates. To include consumer heterogeneity in the model and achieve estimates at the individual level, I propose a modified MCMC algorithm. Results from this algorithm are compared to results from the standard model on both simulated and real data.

1.3.3 Essay 3 (Chapter 4)

The endowment effect is typically interpreted as a direct manifestation of loss aversion. In this research, we suggest that loss aversion itself is a consequence of more fundamental conditions of human cognitive processing by linking it to neural primitives. We use these primitives to derive a new representation of how holistic subjective value judgments emerge as the integration of neural signatures of partworth values. A specific feature of the resulting integration model – a processing asymmetry – suggests a direct causal link to the endowment effect. We test this link in a large-scale empirical study.

CHAPTER 2 THEORETICAL FOUNDATION OF NEUROADDITIVE MODEL[1]

2.1 Introduction

"The assignment of subjective values during decision making constitutes one of the fundamental computations supporting human behavior." (Yoon et al 2012, p 478). Mounting evidence now suggests a clear neurological basis for value computations. In their meta-analysis of thirteen recent fMRI studies, Levy and Glimcher (2012 p 1027) conclude that the evidence suggests "…the existence of a small group of specific brain sites that appear to encode the subjective values of different types of rewards on a neural common scale, almost exactly as predicted by theory." (See also Glimcher and Fehr 2014, ch 20.) Today's standard theory – as referenced in this quote and illustrated in Table 1 – is an amalgam of two fundamental components, subjective value and decision weights, that are intended to predict choices (Kahneman and Tversky 1979; see Stott 2006). For example, a dm might choose between receiving option **x** with probability *p* or nothing (with probability 1-*p*). This choice, denoted $V(\mathbf{x}, p) = w(p)v(\mathbf{x})$ in prospect theory, involves option **x**; with holistic value $v(\mathbf{x})$ and option "nothing" with holistic value $v(0) \equiv 0$. More generally modern *choice theory* concerns the (subjective view of) realizations of a random process where each outcome is holistically valued. Recent theory development has emphasized the probability weighting function (Prelec 1998, Wu and Gonzalez 1996). As Glimcher stresses; "... models of the valuation circuit are much less well developed than models of the choice circuit." (Glimcher p 380)

[1] This chapter is largely based on co-authored; Curry and Wang (2014)

Table 1: What we mean by "the standard model" of value integration [a]

Description	*Algebraic Form*		*Definitions*
Option	$\mathbf{z} = (z_1, z_2, \ldots$	:	Levels of a multi-attribute choice option **z**.
Value Phase: holistic value from parts			
Variant 1: unweighted	$V(\mathbf{z}) = \sum_{i=1}^{m} v_i(z_i)$	:	$v_i(z_i)$ value of level z_i on attribute i
Variant 2: non-parameterized weights	$V(\mathbf{z}) = \sum_{i=1}^{m} w_i \cdot v_i(z_i)$	:	$w_i; i = 1, m$ is a constant, usually constrained to be positive and normalized to add to 1.
Variant 3: parameterized weights			
3.1: Prospect theory	$w_1 = \pi(p_1)$ $w_i = \pi\left(\sum_{j=1}^{i} p_j\right) - \pi\left(\sum_{j=1}^{i-1} p_j\right)$	:	where $0 \le \pi(p) \le 1$ is monotonically increasing in p, the subjective probability of an outcome.
3.2: Other variations (See Stott 2006, table 3.)	*Example*: Prelec-I : $\pi(p) = e^{-(-\ln p)^{\alpha}}$	:	$0 < \alpha < 1$ (See Prelec 1998, p 503.)
Choice Phase: Linking holistic values to choice probabilities			
Three prominent examples where overall value serves as an argument in the choice phase.			
Luce Choice	$P[V(\mathbf{x}) \mid \{V(\mathbf{x}), V(\mathbf{y})\}] = \dfrac{V(\mathbf{x})}{V(\mathbf{x}) + V(\mathbf{y})}$		
Logit Choice	$P[V(\mathbf{x}) \mid \{V(\mathbf{x}), V(\mathbf{y})\}] = \dfrac{e^{V(\mathbf{x})}}{e^{V(\mathbf{x})} + e^{V(\mathbf{y})}}$		
Probit Choice	$P[V(\mathbf{x}) \mid \{V(\mathbf{x}), V(\mathbf{y})\}] = \Phi[V(\mathbf{x}) - V(\mathbf{y})]$		$\Phi[x, \mu, \sigma]$: cumulative Normal density, mean μ, standard deviation σ at point x.

[a] Fox and Poldrack (2014) provide an excellent historical overview of the evolution of choice modeling leading to today's current iterations. They distinguish three major periods involving: (1) the classic models of Von Neumann and Morgenstern (1947) and Savage (1954), (2) the neo-classical models inspired by Allais' paradox (Allais 1953) and Ellsberg (1961). Finally (3) is today's era initiated by prospect theory (Kahneman and Tversky 1979) and cumulative prospect theory (Tversky and Kahneman 1992). McFadden's econometric implementation of random utility theory as the conditional multinomial logit model preceded Kahneman and Tversky's work (McFadden 1974) and seeded extensive further developments in the arena of estimation of RUMs.

This section focuses on the valuation circuit; the integration process that gives rise to a choice option's holistic subjective value. This process – whatever form it may take neurologically – must occur prior to the application of decision weights to generate choice probabilities among options. Options as wholes comprise aspects instantiated at levels. The overall value of a single option must consider the option's many different attributes (e.g., color, size, etc) and assess the value of these; then combine these part-values into one coherent overall value representation (Levy and Glimcher 2012, p 1027). Methodologies such as conjoint analysis and more generally random utility models (RUMs) such as the multinomial logit and probit models do this, but as suggested in Table 1 their value integration rule is invariably an additive in parameters rule. In this research we question whether this is correct; or more specifically, whether such a rule follows from neuro-biological considerations.

Historically, decision scientists have rarely invoked explicit considerations of human biology when developing today's standard model. Key assumptions have primarily concerned either mental operations as behavior (e.g., transitive preferences, Luce 1992) or mental states (e.g., perceived risk, Slovic et al 2005). These assumptions address the abstract world of the mind, not the analogue world of the brain where processing is largely beyond conscious control. Indeed, there is little to suggest that the loss averse nature of human decision-making results from premeditated considerations. Rather, the asymmetry emerges unconsciously much like cardiac function or pupil dilation. The same can be said for a large class of associated referent dependent behaviors, such as extremeness aversion (Simonson and Tversky 1992), the compromise effect (Thaler 1980), the attraction effect (Huber et al 1982; Wedell and Pettibone 1996), and others. These effects counter standard logic yet emerge instinctively in human decision-making (Tversky and Simonson 1993; Tsetso et al 2010; Tversky et al 1990). Other mechanisms such as construal (Trope and Lieberman 2010) and the broader elements of preference construction (Payne et al 1992, Warren et al 2011) clearly influence choice criteria, information salience, and overall evaluation principally via automatic rather than conscious processes (Dijksterhuis et al 2006). Finally, neither prospect theory, cumulative prospect theory, nor neoclassical theories (RUMs) adequately address the important question of value integration rules that yield non-compensatory behaviors.

In this section we seek to understand relevant automatic processes of value integration in the brain by developing a new model. We develop the model analytically using biological/neurological considerations that serve as constraints on a proposed numerical representation of the value integration process. Two of these constraints are regular features of connectionist models; i.e., the leaky competing accumulator model (LCA) of Usher et al (2001) and decision field theory (DFT) (Roe et al 2001), used to study perceptual choice (see Gold and Heekeren 2014 for a review). We argue that these constraints also play a role in the emerging class of valuative choice models with neurological foundations (Soltani, De Martino, Camerer 2012; see Glimcher 2014). The third constraint is unique to our theory but has strong underpinnings from psychophysics.

Our primary (technical) result shows that the integration process has a unique functional form regardless of contextual reference. We propose this form as a new binary operator that we label *neuroaddition* (n^+) as a way to algebraically represent the integration of neural part-values. We offer three main substantive results: first, in an algebra where n^+ defines "addition", value representations will naturally exhibit diminishing sensitivity; second, loss aversion is an emergent property of n^+, and finally that n^+ supports both compensatory and non-compensatory behaviors with the degree of non-compensatory activity moderated by an option's overall value.

Following our deductive results, we outline several hypotheses generated by the theory that extend published work.

2.2 General Development

Classical theories of utility have three aspects: a model of choice, a set of assumptions (=axioms) that characterize preference relations in the choice domain, and a representation theorem into the real numbers (Luce 1992). Our approach differs from this pattern. First, our assumptions, *positivity*, *detectability*, and *boundedness*, address automatic activities of the brain rather than thoughtful activities, such as stated preferences, hence we do not invoke axioms such as transitivity. Second, the term *utility* sometimes means numerical representation but it is also used as a label for an internal state of felt sensations; valuation, worth, joy, or satisfaction, as in the notions of *experienced utility* and *decision utility*. (See Berridge and O'Doherty 2014, Fox and Poldrack 2014, and Warren et al 2011.) To avoid confusion, we use the term *subjective value* to describe the hypothetical construct of central interest to our development and reserve the word *utility* exclusively for numerical representations. As Table 1 suggests, our interest lies in the integration of value, or more precisely, the integration of neural signatures of value, not in the use of the result (holistic value) in a choice model. Finally, we avoid the term information integration, using instead the phrase *neural integration* to emphasize the subconscious, analog nature of the signals being integrated. Because our assumptions are somewhat novel, we clarify their meaning in an intuitive way before delving into formal results.[2]

[2] Readers familiar with the neuroscience literature on perceptual choice will recognize our positivity and boundedness axioms. To our knowledge only one evaluative (vs. perceptual) choice model evokes these constraints, Soltani et al (2012). These authors offer a novel explanation for context effects. In the process, they critique connectionist models as involving neuronal operations that are unrealistically demanding (e.g., comparing all options on all attributes then integrating these differences). They also note that LCA must presume loss aversion and CDA cannot explain experimental results when decoys dominate original choice options or are symmetrically dominated by them (i.e., similarity effects) (Soltani et al 2012 p 5). The theory developed here offers simpler explanations for these effects.

2.2.1 Positivity

Positivity is best explained by analogy. Consider the physical interaction among particles (atoms/molecules) in an object. These kinetic interactions manifest macroscopically as – or, more accurately, are labeled by humans as – the *temperature* of the object. Although we represent outcomes numerically using temperature scales, thermodynamic physical reality is in no way a function of (humanly contrived) numerical representation. Today we realize there is a theoretical limit at which thermodynamic activity ceases. Clearly, there is no physical state "beyond" that limit; i.e., a state corresponding to the idea of "molecules negatively bumping into one another". Using the negative registers of the Celsius and Fahrenheit scales may be convenient numerically, but doing so is misleading scientifically. This realization led Lord Kelvin to use kinetic principles in his study of thermometry, which bounds numerical representations of temperature from below by zero (Chang and Yi 2005). This limit imposes real constraints on the overall structure of an algebra designed to represent behaviors of the system and on meaningful transformations of scale values. And because this algebra informs scientific conclusions, an accurate representation of this lower bound is important.

In the same way, the brain as a physical system cannot register a negative state. The registered signatures of stored information must exist physically, electrically, or chemically. [3] This claim is far different from stating that the mind cannot process negative numbers; clearly it can. However, to do so, the mind must work within the brain's capabilities. The mental encoding of what we interpret as a negative quantity is a derivative signal that combines information at least one level of abstraction beyond the purely analog capabilities of the brain.

[3] The brain (selected neural clusters) can register a decrease in average activity (neural firing rate), but this is not equivalent to registering a negative state.

On this point, it is important to distinguish the use of mental rules for numerical processing – an activity that comprises both conscious and memorized activity – from the brain's analog activities. Evidence suggests that conscious numerical processing– which we are not modeling – is performed in the region of the inferior parietal lobule (IPL) known as the angular gyrus (AG) (Gullick et al 2012). In contrast, the areas associated with internal signatures of subjective value are the ventral striatum (vSTR) and the ventromedial prefrontal cortex (vmPFC) (Ernst and Paulus 2005; Plassmann et al 2007, 2010; Kringleback 2005; Kable and Glimcher 2009; Yoon et al 2012, Levy and Glimcher 2012). (Table 2 is included to help sort out the names of brain regions.) The physical constraints imposed by non-negative signatures emerge in decision-making behaviors and require practiced dedication and conscious reflection on the part of humans to overcome.[4] Modern fMRI studies show that mathematically proficient adults invoke a variety of conscious strategies to deal with negatives (Gullick 2012, p 553), indicating the role of conscious reflection to overcome analog constraints. (See Tom et al 2007; Knutson et al 2008; and Fischer 2003, among others.) Assuredly as solutions to equations, negatives, like imaginary numbers, have great power to aid human understanding via symbol manipulation. But this is a far cry from claiming that these constructs are present in our neural circuitry. In summary, we assume that the internal signatures of magnitudes – including those of subjective value – are non-negative. These signatures exist physically and are subject to the laws of physics. This assumption leads to axiom one.

A1: **Positivity**: Numerical representations of subjective value must map into the positive real numbers.

[4] For example, consider the processing of negative numbers. Negative numbers were not even invented until the 12th century (Gullick 2012, p 2) and as recently as the 17th century, iconic mathematicians did not believe in negative quantities. Descartes labeled negative square roots as "false roots" and his contemporary Pascal declared "I know people who cannot understand that when you subtract four from zero what is left is zero" (Hefendel-Hebeker 1991).

Table 2: Brain Regions Mentioned in This Paper

Central Front Region [a]		*As used in this paper*	
Name	**Abbreviation**	**Name**	**Abbreviation**
Medial prefrontal cortex	mPFC	ventromedial prefrontal cortex	vmPFC
Orbitofrontal cortex	mOFC		
Ventromedial frontal cortex	vmPFC		
Anterior border of the anterior cingulate cortex	ACC		
Striatum			
Name	**Abbreviation**	**Name**	**Abbreviation**
Caudate	CN	ventral striatum	vSTR
Putamen	Pu		
Ventral Striatum	vSTR		
Other Areas			
Inferior parietal lobe	IPL	Inferior parietal lobe	IPL
Gyrus (anterior Gyrus)	AG	Gyrus (anterior Gyrus)	AG

[a] The central frontal area of the brain goes by a number of different names in the existing literature. The many-one mapping used in this table is suggested by Glimcher (2014, p 386).

2.2.2 Detectability

Detectability means that the brain as a human organ can only process signals it can detect. More importantly, the brain cannot exhibit any differential reaction to neural signatures it cannot distinguish from one another. The idea rests on traditions from psychophysics (Fechner 1966, Weber (see Ross and Murray 1996), Stevens 1936, 1970). For example, in the human ear the cochlea converts sound pressure patterns from the outer ear into electrochemical impulses that are passed to the brain via the auditory nerve (Standring, 2004; Figures 922-28). The mechanical systems involved in this conversion, the malleus (hammer), incus (anvil) and stapes (stirrup), among others, are subject to the laws of physics, hence can only detect pressures over a limited range of sounds (Ehret 2010, Luce and Galenter 1963, Stevens 1936).

There is no reason to believe that the brain is an exception to this rule; that it somehow can detect signals over an unlimited range. Decades of work in psychophysics have established algebraic representations ("laws"); such as Steven's Law (Stevens 1961), Fechner's Law (Fechner 1966), and the psychophysical law (Norwich 1993) to model the relationship between a physical stimulus and a corresponding psychological sensation. Norwich (1987, 1993) synthesizes this work to derive the Weber fraction law from information theory. (See also Falmagne 1974.) For a physical stimulus, such as an aural tone, this law states that the ratio of a detectable difference in two stimuli to their total physical magnitude is the Weber fraction for that category of stimuli (e.g., tones). Different categories – visual, tactile, etc. – are characterized by parameters that are experimentally calibrated.

In the context of the neuro signature(s) of subjective value, this idea establishes a necessary condition for behaviors, such as option $\mathbf{x}$ is preferred to option $\mathbf{y}$. Let $u(\mathbf{x})$ and $u(\mathbf{y})$ be the unknown (but to be determined) numerical representations of the subjective value of options $\mathbf{x}$ and $\mathbf{y}$, respectively. Detectability requires that any numerical representation of separate signatures, $sig[u(\mathbf{x})], sig[u(\mathbf{y})]$, relative to a combined signature, $sig[u(\mathbf{x},\mathbf{y})]$ satisfy expression (1), where ϖ is an appropriate threshold for the neural signatures involved. There are two important aspects of (1); first it does not commit to any empirical parameters and second as a binary relation it establishes an implicit reference frame for valuation; i.e., the two options involved and, by extension, the aspects of each option that are stored neurologically, hence have associated neurological value signatures.[5]

A2: **Detectability**: Any numerical representation of separate neural signatures – relative to a combined signature – must satisfy the Weber fraction law:

$$u(\mathbf{x},\mathbf{y}) \equiv \frac{u(\mathbf{x})-u(\mathbf{y})}{u(\mathbf{x})+u(\mathbf{y})} > \varpi \tag{1}$$

2.2.3 Boundedness

Lastly, we assert that there is an upper bound to the internal representation of magnitude. This bound must hold for any numerical representation of subjective value. We arrive at this principle using both behavioral and neurological evidence from an extensive series of anthropological and associated fMRI studies. The behavioral studies are conducted with neurologically normal adults not exposed to formal arithmetic (see Pica et al 2004, Dehaene 2006, Iszard et al 2011, among others) and with pre-school children (see Booth and Siegler 2008, Dehaene 2003, among others). These studies overwhelmingly support what is termed the *numerical magnitude effect*. The effect has two major components. First, although the brain naturally recognizes ordinality; i.e., individuals can identify which side of a computer screen shows more randomly displayed "dots", judgments of magnitude are increasingly compressed as magnitude increases. Thus a one unit change among the small counting numbers, say from 2 to 3, is processed as a larger magnitude than a one unit change at a higher absolute level, e.g., 22 vs. 23. Second, the neuronal signatures of discrete quantities, "two-ness, three-ness" etc., exhibit

[5] Glimcher (2014, p 386) notes that literally dozens of studies now make it clear that neurons responsible for encoding subjective value encode the difference between expected and obtained reward; called the *reward prediction error*. Therefore (1) may also be interpreted as an algebraic summary of the differential levels of dopamine when one option is noticeably better than the other option; divisively normalized by total expected reward.

spread in the topography of brain scans, such that adjacent small magnitudes have distinct fMRI footprints with little overlap, while adjacent larger magnitudes have indistinct footprints with large overlap (Merton and Nieder 2009). In these studies "large" refers to magnitudes typically no greater than thirty.

Thus, as magnitudes grow they have increasingly indistinguishable internal representations. This means that however subjective value is stored and represented in the brain's neural circuitry, at some point large values become undifferentiated, hence form an – albeit vague – upper limit. This is not surprising. The phenomenon manifests itself in known ways in economics and psychology. The most basic example is decreasing marginal utility. A second example is construal theory, which asserts diminishing sensitivity as we move further from a natural boundary, such as "none" (Trope & Lieberman 2010). A third, related area is the notion of subadditivity in subjective value (see Fox and See 2003), which Read (2001) and Scholten and Read (2006) use to explain time-discounted utility. This leads to axiom three.

A3: **Boundedness**: Numerical representations of subjective value are bounded from above; therefore, if $u(\mathbf{z}) \in \mathbb{R}$ with $u(\mathbf{z}) \leq u_{\max}$ is mapped to a new scale by the positive transformation *f*, then $f\left[u(\mathbf{z})\right] \leq u_{\max}$.

2.2.4 Summary of Assumptions

We have presented three very general assumptions about value signatures in the human brain. In the next few sections, we use these assumptions to draw a series of conclusions about an appropriate numerical representation of signal integration. Subsequently we explore the implications of this composition rule for human decision behavior. We emphasize that our axioms concern the brain not the mind. *Positivity* is about neuro signatures, not human invented numerical systems or conscious reasoning with numbers. *Detectability* simply requires that two internal representations be electrically, chemically, and topographically distinct enough to register as different in the brain. And *boundedness* questions the face validity of any representation that would permit signatures leading to infinite sensations of value, joy, or satisfaction, or any other manifestation of subjective value. We believe an infinitely high subjective value to be impossible or at the very least, other worldly.[6]

2.3 Notation

Our notation qualitatively differentiates the following general classes of "objects". *Options*, denoted by lower case boldface letters such as **x**, **y**, and **z**, are things such as insurance policies and automobiles that exist in the real world and are available for human decision-makers (dm's) to choose, buy, or consume. We distinguish an option, **z**, from its neural signature, denoted *sig*(**z**). By *sig*(**z**) we mean the neural circuitry that permits a dm to recognize the object, for example as a member of the class "car". (See Deppe et al 2005 and Ernst and Paulus 2005 for extensive reviews relevant to this idea.) We use *u*(**z**) to mean an analyst's numerical representation of an individual's subjective value, denoted *v*(**z**) for an object **z,** while *sig*[*v*(**z**)] means the neural signature of the object's encoded subjective value. We presume the latter is close to what Louie and Glimcher (2012) mean by *action-independent value* and what Yoon et al (2012) refer to when they use phrases such as a brain encoded "common currency" (p 478) and "subjective value signals" (p 479). Any proposed numerical representation of the brain's version

[6] Our logic for boundedness invokes macro considerations as opposed to the micro constraints on firing rates of single neurons; the source of bounds in connectionist models. But these viewpoints converge. As Soltani et al note (2012 p 2): "The guiding presumption in our range-normalization (RN) model is that subjective values of option attributes are encoded in the firing rate of neural populations, rather than other aspects of neural firing. If so, mental representations of subjective value will be bound by the same biophysical limits that govern neural representations. Namely, neural responses are bound from below by zero and from above by a few hundred spikes per second, and therefore, neurons can represent a set of stimuli using a limited range of firing rates."

of subjective value is a mapping from the physical domain where $sig[v(\mathbf{z})]$ resides to the real numbers where $u(\mathbf{z})$ resides. We seek a high fidelity mapping to paramorphically mimic important properties of these values held in the brain and moderated by the mind.

Real-world options comprise multiple aspects which are abstracted and idealized by analysts and referenced using labels such as *attributes* and *levels* (among many others). We assume m attributes with $m \geq 2$ and let $M = \{1, \ldots$. Attribute $Q(i)$, $i \in M$ has $q(i)$ levels. A *profile* is an m-component vector with the i^{th} component taking on one of the $q(i)$ levels for that component. For a given level, $z_i \in \mathbf{z}$, we make the same distinctions as made for the whole; $sig(z_i)$ is a valence neutral neural encoding of the "level"; e.g., **z**'s color is blue, its price is \$6.95; $u_i(z_i)$ is a numerical representation of the subjective value of this level (or "part" of the object); i.e., $sig[v(z_i)]$ the neural encoding of subjective (part) value.

We use the option pair (vs. three or more options) as the atomic unit to establish referential context for a value-driven choice. "Context-dependence means people compare choices with a set rather than assigning separate numerical utilities" (Camerer et al 2004, p 569). The idea has robust support in the literature and will not be reviewed here (see Warren et al 2011). The neuroscience evidence suggests that pairwise comparisons (vs. the ranking of 3+ options) establish reference in automatic processes. As Deppe et al (2005, p 180) state: "Ranking orders can be established only by multiple time and neuronal resource-consuming brand-to-brand comparisons. From a neuroeconomical and evolutionary point of view, reasoning-based rating scales are ineffective in the sense of *periculum in mora* (danger in delay)." (The authors refer to the need for automatic processing to be as fast and effortless as possible. See also Levy and Glimcher 2012 and Glimcher 2014.) To indicate context, we subscript reference frames established by an option pair; for example detectability in the reference frame defined by options **x** and **y** is denoted $u_{\alpha,\beta}(\mathbf{x},\mathbf{y})$. These subscripts implicitly refer to key aspects one may associate with reference dependency; e.g., status-quo points, attributes triggered when considering a choice between **x** and **y**, salient levels, and so on. Although leaving these aspects implicit would seem to indicate less than thorough modeling, this apparent weakness ultimately becomes a strength of our model. We are able to show that our main result – the algebraic form of n^+ – holds independent of referential context. In other words, integrated neural signatures of values will exhibit all the important effects of reference dependency despite this global property of signal integration. When explaining our propositions, theorems and corollaries, we emphasize substance, not mathematical formalism. However, each result is proved rigorously for the general *m*-dimensional case in the appendix.

2.4 Definitions

2.4.1 Reference Frames and the Integration of Subjective Values

Our formal development proceeds in two parts, technical foundations and substantive implications. The present section – technical foundations – uses functional equations to show that if positivity, detectability, and boundedness obtain, then so too does a particular algebraic form n^+. Subsequently, we explore the implications of n^+ as a binary operator to replace standard addition in utility representations of subjective value. We establish three main results; n^+ utility representations will naturally exhibit diminishing sensitivity, loss aversion, and moderated compensatory behavior.

2.4.2 Referentially-Based Value

Proposition 1 explores relationships among utility representations as the reference frame shifts. Specifically, detectability is a statement about the numerical representations of value for two options. But what happens when a third option enters the picture? There are now three pairwise reference frames; how do the utilities involved in each frame relate to one another? Proposition 1 uses positivity, detectability, and the notion of meaningfulness to answer this question; i.e., that the truth or falsity of any scientific statement about subjective value – based on a given numerical representation (utility scale) – should remain invariant if that scale undergoes a permissible transformation (Krantz et al 1971).

Consider options **x** and **y** which have detectably distinct value signatures represented on the utility scale *u*. By positivity, we know that $u(\bullet$ for all arguments. Consider a positive transformation of *u*; i.e., the function $f\left[u(\bullet \quad \mathbb{R}\right.$. Since the difference $u_{\alpha\beta}(\mathbf{x},\mathbf{y})$ is detectable in *u*, then for *f* to be permissible $u_{\alpha\beta}(\mathbf{x},\mathbf{y})$ must be detectable in the transformed utility scale or:

$$u_{\alpha\beta}(\mathbf{x},\mathbf{y}) = \frac{f\left[u_\alpha(\mathbf{x})\right] - f\left[u_\beta(\mathbf{y})\right]}{f\left[u_\alpha(\mathbf{x})\right] + f\left[u_\beta(\mathbf{y})\right]} \tag{2}$$

The appendix shows that relation (2) implies relation (3), which we state as Proposition 1.

Proposition 1: Detectability and positivity together imply a law of relative value

$$u_{\beta\gamma}(\mathbf{y},\mathbf{z}) = \frac{u_{\beta\alpha}(\mathbf{y},\mathbf{x}) - u_{\gamma\alpha}(\mathbf{z},\mathbf{x})}{1 - u_{\beta\gamma}(\mathbf{y},\mathbf{x})u_{\gamma\alpha}(\mathbf{z},\mathbf{x})} \tag{3}$$

Thus, if options **y** and **z** are each detectably different from a common referent option, **x**, then a detectable difference in their subjective values must assume the functional form on the RHS of (3). The assertion is simultaneously referent free and referent dependent. It postulates that the values of options **y** and **z** must be calibrated relative to some baseline, which is referent **x** in (3). However, it also asserts that even though any specific numerical representation will depend on the baseline referent **x**, the functional form of the law does not. As shown in the substantive implications section, this property leads to indifference curves that travel with the choice set as suggested by Drolet et al (2000) and used by Orhun (2009) and others (Masatlioglu and Uler 2013).[7]

Corollary P1.1: A coherent positive algebra in which relative values obey (3) will have a summative law for subjective value of the form:

$$u_{\beta\gamma}(\mathbf{y},\mathbf{z}) = \frac{u_{\beta\alpha}(\mathbf{y},\mathbf{x}) + u_{\gamma\alpha}(\mathbf{z},\mathbf{x})}{1 + u_{\beta\gamma}(\mathbf{y},\mathbf{x})u_{\gamma\alpha}(\mathbf{z},\mathbf{x})} \tag{4}$$

Corollary P1.1 asserts two key points. First, if **y** and **z** are both detectably different in subjective value from a referent **x**, then the sum of these subjective values must take the functional form on the RHS of (4). More subtly, in a positive system, relative differences behave as the distance or gap between objects, not as a signed quantity. Therefore, expressions (3) and (4) are notationally consistent despite the inclination to think of (3) as a "difference" and (4) as a "sum". Understanding the relationship between (3) and (4) requires the abstract idea of "betweenness" (see Suppes et al 1989, Ch. 13, Tarski 1959). If we envision options **x** and **z** on a continuum with **y** "between **x** and **z** in subjective value", then the endpoints of the continuum must be the "sum" of the unsigned distances of the intermediate segments **x** to **y** and **y** to **z**.

[7] The idea implicit in (3) is not inconsistent with cumulative prospect theory (CPT). As Wu and Markle (2008, p. 1322) aptly summarize, the key argument of CPT is that change in wealth (not absolute wealth) is the relevant carrier of utility. Although this change is typically modelled using deviations from a specific reference point; i.e., [*u*(new)-*u*(ref)], the construct of a reference option (or status-quo option) is not the kernel of the idea. There are an infinite number of "start points" and "end points" that can yield the same wealth (i.e., utility) difference; it is not these points that matter but the fact that a utility differential arises; i.e., reward prediction error. This idea is implicitly captured by making the option pair the atomic unit so that any option can serve as the reference for any other. The real question then becomes, can we make any reasonable statements about the properties of such a system?

Although the simplest picture places **x**, **y** and **z** on a straight line, this notion of simple order – as a binary relation that is transitive, connected, and anti-symmetric – is misleading. The generalization embodied in a positive system eliminates the sense of direction inherent on the real line. There is no preferred direction and no meaningful way to fix direction (Suppes et al, 1989, ch. 13). This last point bears emphasizing because researchers often visualize referential context effects, such as the attraction effect, using 2d geometrical figures. Such representations can be highly misleading when modelling neuronal activity. Neuronal activity is "directionless" by nature. Using a 2d Euclidean representation of, say, the attraction effect is a useful way to visually communicate ideas but the analogue neuronal representation has no such geometric basis.

Corollary P1.2: If a positive system obeys the law of summative value (4) and has supremum *c*, then subjective value obeys the divisively normalized law of summation (5).

$$r_{\beta\gamma}(\mathbf{y},\mathbf{z}) = \frac{u_{\beta\alpha}(\mathbf{y},\mathbf{x}) + u_{\gamma\alpha}(\mathbf{z},\mathbf{x})}{1 + \frac{u_{\beta\gamma}(\mathbf{y},\mathbf{x}) \cdot u_{\gamma\alpha}(\mathbf{z},\mathbf{x})}{c^2}} \tag{5}$$

2.4.3 Neuroadditivity

Because the functional form of the summation of (relative) values does not depend on the baseline option, the form (5), defines a referent-free abstract binary operator. We formally define this operator in (6) and refer to it as *neuroaddition*. The relative utilities $u_{\beta\alpha}(\mathbf{y},\mathbf{x})$ and $u_{\gamma\alpha}(\mathbf{z},\mathbf{x})$, which sum as shown in (5) are represented in referent-free format as the (dummy) addends *s* and *t* in (6). Thus, using positivity, detectability, and boundedness, we assert that (6) describes the summative operation that should be used to integrate (numerical representations of) subjective value rather than standard addition as exemplified in Table 1.

Neuroaddition: Let $\Omega_c \equiv \{\mathbb{R}$ with elements $s, t \in \Omega_c$ have supremum $0 < c < \infty$ and define the binary operation: $n^+(s,t)$: $\Omega_c \times \Omega_c \to \Omega_c$ as:

$$n^+(s,t) \equiv s \oplus t = \frac{s+t}{1+\frac{s \cdot t}{c^2}} \tag{6}$$

The appendix shows that ⊕ is commutative, associative, distributive with respect to induced multiplication, and is closed. The operator also has a natural identity, and is summand-wise compressive with respect to standard addition; a property similar to sub-additivity (Fox and See 2003).

Importantly, (6) does not assert a unique zero even though for all $s, t \geq 0$ we have $0 \leq s \oplus t \leq c$. Therefore, our model differs from those of Bodner and Prelec (1994), Kivetz et al (2004a,b), Hardie (1993), Orhun (2009), where the referent is associated with either a specific option or a generic option constructed as a combination of options in a choice set. Paradoxically, the algebra (6) is referent dependent, but sums in the system always have the same (referent independent) functional form. Mapping this idea to the brain and subjective value, the model asserts that a dm must always baseline subjective values of options somehow. However, conditional on the (subconsciously) selected baseline, subjective values must sum as per (6).

2.5 Substantive Implications

2.5.1 Diminishing Sensitivity

Diminishing sensitivity is "a cornerstone of economic theory" (Baucells and Sarin 2010, p 288) and "a robust empirical generalization about human perception and decision making…. " Kivetz et al (2004a, p 240). It is important to bear in mind that our modeling approach does not depend on a specific functional form for utility, a form rendered in standard arithmetic. Our assertion is more general. We assert that any value integration model using n^+ as its "addition operator" will naturally exhibit diminishing sensitivity. This is so because n^+ itself exhibits this property; hence the property emerges throughout the abstract algebra of the value space. To verify this property, it suffices to show that n^+ increases in each argument at a decreasing rate. This result is Theorem 1.

Theorem 1: (*monotonicity and diminishing sensitivity*): Neuroaddition is monotone increasing in each argument $\frac{\partial n^+(s,t)}{\partial s} \geq 0$ at a decreasing rate $\frac{\partial^2 n^+(s,t)}{\partial s^2} < 0$.

Theorem 1is not the idea that "the next piece of cake won't generate as much utility as the one you are eating now"; i.e., diminishing sensitivity that refers to satiation at the consumption level for a single good. Rather it is the idea that however value is signalized by the brain, the relationship between increases in any underlying attribute (or generalized option schema) and the subjective value of that increase is sub-proportional (Fox and See 2003).

2.5.2 Loss Aversion

Kahneman and Tversky (1991) found that individuals should be *loss averse*; changes equal in face value from an initial state, but in opposite directions will engender different cognitive values, i.e., a smaller absolute utility for the gain. Accumulated empirical evidence strongly supports the loss aversion hypothesis, both behaviorally (Tversky and Fox 1995; Tversky and Wakker 1995; Wu and Gonzalez 1996) as well as neurally (Dickhaut et al 2003, Tom et al 2007). (See also Novemsky and Kahneman 2005, Bateman et al 2005, Brenner et al 2007, and Tversky and Kahneman 1991, among many others). We show next that a numerical representation that integrates subjective values according to (6) will exhibit loss aversion. Thus loss aversion will hold for any particular – referent bound – instance of the operator (5).

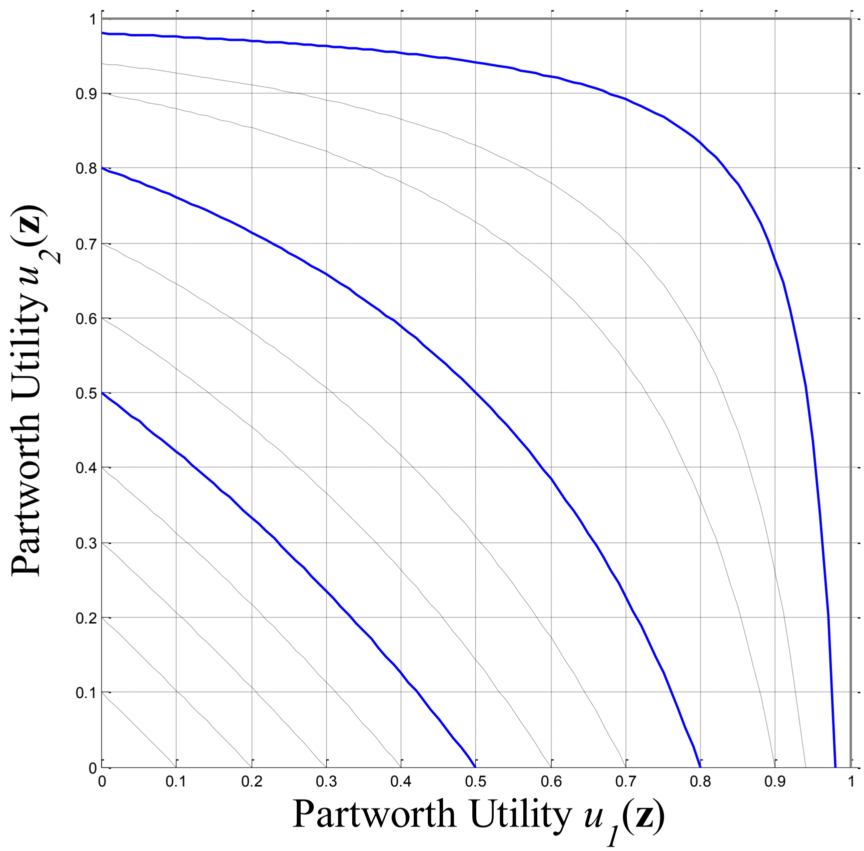
The loss aversion property is most easily seen graphically in two dimensions. Figure 1 shows the iso-value contours for overall subjective value comprising two addends using n^+. (Without loss of generality, we set the upper bound c to 1 in all figures.) A few of these are rendered in thicker lines to emphasize how the shape shifts as value increases. With two attributes, each curve is the locus of points satisfying $u_1(z) \oplus u_2(z) = \kappa$, which from (6) yields the relationship between subjective value on attributes one and two as (7).

$$u_2(z) = \frac{c^2\left[u_1(z) - \kappa\right]}{u_1(z)\kappa - c^2} \tag{7}$$

Movement along any ray outbound from the origin – a common referent on the RHS of (5) – represents constant face value of options because the marginal rate of substitution among part values is constant in this direction. However, on any such ray the distance required to increase subjective value by a fixed amount moving "outbound" is always more than that required moving "inbound" to decrease subjective value by the same amount. This is so because the iso-value contours are concave; all points on an iso-value contour lie "outside" any line segment connecting two points on that contour. Furthermore, this asymmetry increases as total utility increases. Theorem 2 formalizes this property using the following definitions. For any option $\mathbf{x} \in \Omega_c$ on a ray from the origin in Figure 1, we define a *loss* relative to an option $\mathbf{y}$ as an option that lies on an iso-value contour with $u(\mathbf{x}) < \kappa_y$. In other words, $\mathbf{x}$ simply has less subjective value than $\mathbf{y}$. Similarly a *gain* is any $\mathbf{x} \in \Omega_c$; such that $u(\mathbf{x}) > \kappa_y$; with equivalent subjective value occurring when $u(\mathbf{x}) = \kappa_y$.

Figure 1: ISO-Utility Contours for the Neuroadditive Model

Utility integration takes the standard additive form when overall utility is low but takes a convex form when overall utility is high.



Theorem 2: *Loss Aversion*: Let $\mathbf{z}_L$, $\mathbf{z}_R$, $\mathbf{z}_G$ be three points with $u(\mathbf{z}_L) < u(\mathbf{z}_R) < u(\mathbf{z}_G)$ that lie on any ray from the origin of the two-dimensional utility space such that the (Euclidean) distances satisfy $d(\mathbf{z}_L, \mathbf{z}_R) = d(\mathbf{z}_G, \mathbf{z}_R)$; i.e., point $\mathbf{z}_R$ lies mid-distance between the other two points on the ray. Then $G \equiv d(\mathbf{z}_G, \mathbf{z}_R) = d(\mathbf{z}_R, \mathbf{z}_L) \equiv |L|$, but $|u(\mathbf{z}_L) - \kappa_R| > u(\mathbf{z}_G) - \kappa_R$ where $\kappa_R = u(\mathbf{z}_R)$.

2.5.3 Moderated Compensatory Behavior

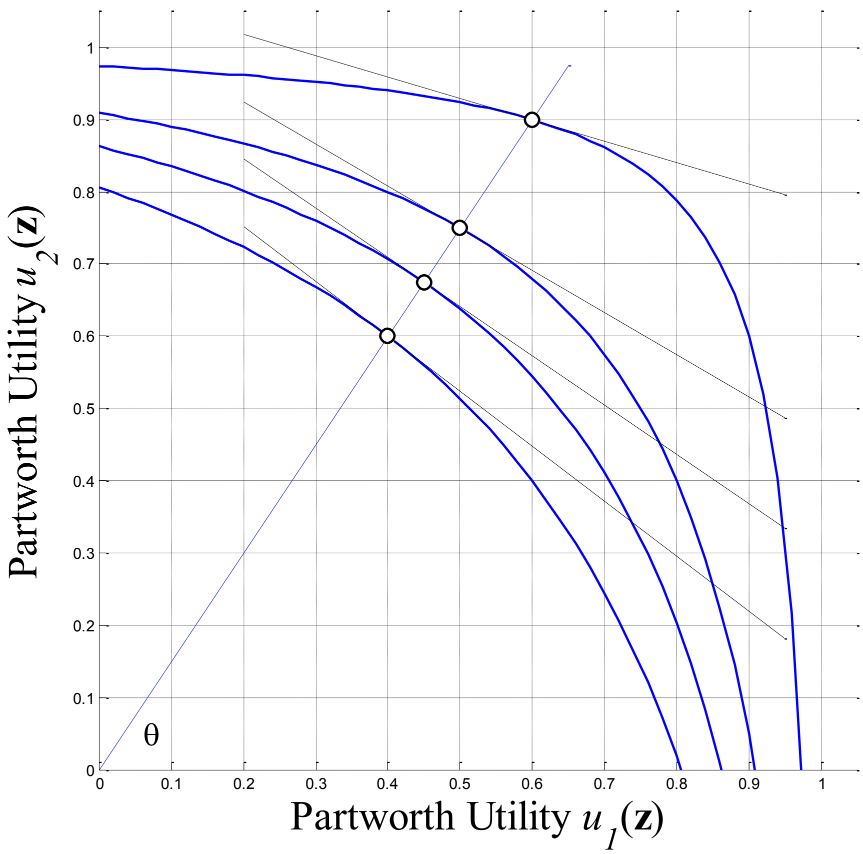
The literature on decision-making is replete with debates about whether and when decisions will be compensatory. (See Hastie and Dawes 2001, Montgomery 1976, Tversky 1972.) The n^+ representation offers new insights on this issue and makes precise predictions about what conditions should yield which types of behaviors. The conventional understanding of compensatory information processing is the capacity to experience more value on one attribute and less on another while still experiencing the same overall value. In the standard additive model, this tradeoff is zero-sum. In contrast, the n^+ representation supports the idea of trade-offs that are soft-edged and variable. The marginal rate of substitution required to maintain constant overall subjective value changes as a function of both the relative contributions of each attribute and the magnitude of total value as illustrated in figure 2.[8] When overall value is low, iso-value contours are nearly linear, corresponding to the assumption of the standard model. However, as overall value increases, iso-value contours become increasingly convex to the origin. We call this *moderated compensatory behavior*; a phenomenon modeled exogenously by Allenby and Rossi (1991) and Gilbride and Allenby (2004). A decision-maker considering just two attributes will increasingly favor one or the other depending on the overall value of options under consideration. In other words, if you like something (certain levels of attribute A vs. those of attribute B), then the more you experience it, the more you like it. The moderator effect accords naturally with models of heuristic behaviors. When faced with options with many attributes, a decision-maker's "mental search" for higher value will naturally lead him or her to focus on

[8] Note the distinction between iso-value contours and the classic economic theory of indifference curves for bundles of goods. Goods – outdoor grills, pearl necklaces – are traded via explicit transactions in the marketplace. Thus, prices are set exogenously to any given individual's neural processes and serve as a natural common denominator across goods, which encourages the use of standard arithmetical cognitive processing (Merton and Nieder 2009, p 345). Baucells and Sarin (2010, section 4) provide a prototypical example when extending the HS model to multiple goods.

fewer attributes, thus decisions tend to become less compensatory. (See Gigerenzer and Gaissmaier 2011.)

Figure 2: Neuroadditive Utility Generates Moderated Compensatory Behavior

As overall utility increases, a decision-maker will increasingly resist "giving up" value on a subset of attributes.



Theorem 3 (*moderated compensatory behavior*): As subjective value increases, that is as $\kappa \to c$, a decision-maker's marginal rate of substitution between attributes – his or her willingness to make trade-offs – diminishes.

Theorem 3 can be restated as follows: the n^+ representation of subjective value is both non-Fechnerian and non-homothetic. These properties link neuroadditivity to several practical applications in the marketing and psychology literatures. Fechnerian iso-value contours are "parallel". They can be parallel straight lines – as per RUMs – and for deterministic utility in additive conjoint measurement. Or they can be parallel curves. For example, the Luce (1959) function has radial iso-utility contours consistent with its dependence on the ratio (vs. the difference) of prospect values (Stott 2006 p 124). Fechnerian iso-value contours preclude phenomena such as asymmetric price switching, which occurs when a higher quality brand steals share from a lower quality brand as the price gap between the brands shrinks. In other words, when considering options with higher overall utility, an individual may trade off attribute combinations differently than when considering options with lower levels of overall utility (Kamakura and Du 2012). Such behavior supports a non-Fechnerian utility representation as per n^+.

Homotheticity is parallelism at the margin. It obtains for iso-value contours with constant first derivatives. To combat homotheticity in the standard MNL, Allenby and Rossi (1991 p 187) developed a special MNL model that made : "… marginal utility a function of the overall level of utility." Their non-homothetic model significantly outperformed the standard model; it fit better in-sample and yielded a log-likelihood over 200 points better. The n^+ representation does precisely this; the marginal rate of substitution changes as a function of total utility. In Figure 2, attribute 1 is increasingly dominating substitution at the margin. If θ were less than 45^o then attribute 2 would increasingly dominate the marginal rate of substitution as overall utility increases. Orhun (2009) also invoked non-homotheticity in her analytic deductions.

2.6 Conclusion

We employ three very general ideas about the brain to deduce relational laws for a numerical representation of multiattribute utility. Our assumptions – positivity, detectability, and boundedness – converge with those in the neuroscience literature despite the fact that our arguments are only partially based on direct neurological considerations. These ideas are widely accepted among neuroscientists, e.g., "… neural activity is physically constrained to minimal and maximal levels and has very limited precision" (Louie and De Martino 2014, p 455); "… neural responses are bound from below and above, therefore can represent information using a limited range of firing rates" (p 473).

The most important of our relational laws is the neuroadditive law, which yields a binary operator to represent value integration to wholes from parts. The representation naturally embodies diminishing sensitivity, loss aversion, and moderated compensatory behavior. These properties are empirically supported by decades of laboratory experiments and large-scale field studies. Beyond existing support, the neuroadditive representation generates novel hypotheses in areas connected to loss aversion and context dependent decision-making. We outline three of these hypotheses selected from a longer list of possibilities.

The neuroadditive representation adds just one parameter to standard additivity, hence is a logical candidate for a RUM that unifies previous efforts to imbue the MNL with reference dependency (Hardie et al 1993, Orhun 2009, Kivetz et al 2004a), loss aversion (Kivetz et al 2004a,b; Orhun 2009), non-homotheticity (Allenby and Rossi 1991), and non-compensatory behavior (Swait 2001; Gilbride and Allenby 2004). The representation takes a divisive normalized form, making it a candidate for the role envisioned by Louie and De Martino (2014 p 390): "A growing body of evidence, summarized in Chapter 24, now suggests that the representation of decision variables in the choice areas is neither a ratio of choice-variables nor a difference-of-choice-variables as originally proposed for perceptually-based models. Instead it appears that these networks employ a divisively normalized representation that can accommodate both ratio-like and difference-like behavior. While the details of that representation are beyond the scope of this chapter, what is important is that these two classes of models (perceptual choice models and value-based choice models) are now beginning to fit together in our understanding of human decision-making."

CHAPTER 3 ECONOMETRIC MODELING AND STATISTICAL ESTIMATION

Because the neuroadditive model generates a variety of experimentally testable hypotheses, it would be useful to have a routine that can estimate the model's parameters from data. This section proposes to develop such routines for discrete-choice data. Both pooled and individual models will be developed. The section is organized as follows. We first generate simulation data (section 3.1) for which the true parameter values are pre-specified. In section 3.2, the maximum likelihood approach is applied for pooled estimation – treating individual differences in partworth utilities as random variation. Reasons for using a grid search procedure for certain parameters of the models compared here are explained in section 3.3. The chapter concludes with section 3.4, which compares results from the standard and neuroadditive MNL choice models fitted to data from a discrete-choice experiment concerning services in the airline industry.

3.1 Data generation: simulators

In this section, we describe the simulation methods used to create choice data for certain model comparisons. The goal is to simulate data from a discrete choice experiment involving (hypothetical) individuals who choose options from choice sets and do so according to different modelling assumptions; e.g., standard vs. neuroadditive processes. The simulator for the standard model assumes that partworth utilities are drawn from a normal distribution. We refer generically to the resulting dataset as the "normal distribution set." In cases where positivity is imposed on partworth utilities, we draw these partworth utility vectors ("betas") from a log-normal distribution and refer generically to the resulting data as the "log-normal distribution set". Table 3 summarizes the scenarios used in the simulation studies.

Table 3: Simulators

Model	Attribute	Attribute Levels	No Choice Option	Prior Distribution
Pooled Standard (SP)	A, B	A=3, B=4	Not included	Lognormal
Pooled Neuroadditive (NP)	A, B	A=3, B=4	Not included	Lognormal
Individual Standard 1 (SI1)	A, B	A=3, B=4	Included	Normal
Individual Neuroadditive 1(NI1)	A, B	A=3, B=4	Included	Normal
Individual Standard 2 (SI2)	A, B	A=3, B=4	Included	Lognormal
Individual Neuroadditive 2 (NI2)	A, B	A=3, B=4	Included	Lognormal

When a single set of partworth utilities is presumed to govern the behavior of an entire population of respondents; i.e. a pooled model, the true partworths of each respondent are nevertheless generated separately from a multivariate log-normal distribution, s.t.,

$$\log\left(\boldsymbol{\beta}_{7\times1}^{LN}\right) \sim MVN\left(\boldsymbol{\mu}_{7\times1}^{LN}, \boldsymbol{\Sigma}_{7\times7}^{LN}\right)$$

In this case the model's single parameter $\boldsymbol{\beta}^{LN}$ includes seven entries, which are partworth utilities corresponding to three levels of attribute A and four levels of attribute B. We do not include a "No Choice" option in this simulation routine. We set the true value of the parameter to: $\boldsymbol{\mu}^{N} = \left[8\ 11\ 16\ 1\ 2\ 3\ 4\right]$. When the model is neuroadditive, the maximum utility parameter is set to $\mu_{\log(c)} = 20$.

For models that explicitly include respondent heterogeneity; i.e., where the partworth utility vector is unique for each person and drives that person's choices (vs. as random variation in a pooled model), we generate both normal and log-normal versions. For the normal model, the true partworths of each respondent are generated as a single random draw from a multivariate normal distribution, s.t.,

$$\boldsymbol{\beta}_{8\times1}^{N} \sim MVN\left(\boldsymbol{\mu}_{8\times1}^{N}, \boldsymbol{\Sigma}_{8\times8}^{N}\right).$$

The first seven entries of $\boldsymbol{\beta}^N$ are parameters corresponding to three levels of attribute A and four levels of attribute B. The last entry corresponds to the "No Choice" option, which is included to mimic the typical setup in a discrete-choice experiment. The mean vector (true partworth values) is set to: $\boldsymbol{\mu}^N = \left[2\ 3\ 5\ 1\ 3\ 4\ 10\ 5\right]$, and the variance-covariance matrix $\boldsymbol{\Sigma}^N$ is generated using the approach suggested by Train (2001), which generates a diagonal iid vcv matrix where the expected variance is calculated from [illegible] ; i.e., $E($ [illegible] , therefore, $V($ [illegible] . Once the true partworths have been generated, the neuroadditive coefficient for respondent r, c_r, is generated from a uniform distribution, i.e. $c_r \sim \text{Uniform}\left(LC_r, UC\right)$, where $LC_r = \max\left(\boldsymbol{\beta}_r\right)$ and $UC = \max_{r=1,\ldots}\left\{\max\left(\boldsymbol{\beta}_r\right)\right\} + 0.5$. Each respondent has his/her own lower bound but respondents share the same upper bound. This procedure guarantees that the neuroadditive condition $c_r > \max\left(\boldsymbol{\beta}_r\right)$ is satisfied.

The lognormal model is created by simply taking the log of the normal model so that the data generation process follows a multivariate log-normal distribution, s.t,

$$\log\left(\boldsymbol{\beta}_{8\times1}^{LN}\right) \sim MVN\left(\boldsymbol{\mu}_{8\times1}^{LN}, \boldsymbol{\Sigma}_{8\times8}^{LN}\right),$$

Because lognormal densities are notoriously finicky, we set the mean vector to $\boldsymbol{\mu}^{LN} = \left[0.4\ 0.6\ 1.0\ 0.2\ 0.6\ 0.8\ 2.0\ 1.0\right]$. The covariance matrix $\boldsymbol{\Sigma}^{LN}$ is generated using Train's approach described above. Once the true partworths have been generated, the neuroadditive coefficient of the r^{th} respondent's c_r is a generated as a random draw from a univariate log-normal distribution, i.e., $\log(c) \sim N\left(\mu_{c,r}, \sigma_{c,r}^2\right)$, where $\mu_{c,r} = \max\left(\boldsymbol{\beta}_r\right) + 0.5$ and $\sigma_{c,r} = 0.5$ for all r. A valid c_r must satisfy the condition that $c_r > \max\left(\boldsymbol{\beta}_r\right)$.

Once the true partworths and neuroadditive coefficients have been generated, the choice probabilities for each option in any given choice set can be calculated using both standard additivity and neuroadditivity in the MNL rule for value integration. The option selected in a given choice set is determined by a single draw from a multinomial distribution with these choice probabilities as its parameter vector.

3.2 Estimation of the Pooled Neuroadditive Model (maxLK)

To achieve maximum likelihood estimates for the pooled neuroadditive model, we jointly estimate $\boldsymbol{\beta}$ and c using Aczél's (1966) functional equation (8) which yields a monotone transformation that renders the neuroadditive model linear and closed form.

$$u(\mathbf{x}) = f\left[\sum_i f^{-1}\left(u_i\left(x_i\right)\right)\right] \; or \; f^{-1}\left[u(\mathbf{x})\right] = \left[\sum_i f^{-1}\left(u_i\left(x_i\right)\right)\right] \tag{8}$$

The symbol x_i indexes levels of attribute i in profile $\mathbf{x}$ and $u_i\left(x_i\right)$ are the partworth values for profile $\mathbf{x}$. Note that when f is the identity function, (1) is the standard additive model. I provide another solution below to yield the neuroadditive model.

The log likelihood of the conditional MNL is shown in equation (9).

$$L^* = \log\left(L^{na}\right) = \sum_{q=1}^{Q}\sum_{k=1}^{K}\sum_{j=1}^{J} y_{jk}^{na} \log\left(P_{jk}^{na}\right) \tag{9}$$

L^* denotes the overall log-likelihood function for the standard MNL; q indexes respondents; k indexes choice sets; j indexes options within each choice set; and y_{jk} is the binary indicator variable indicating whether option j is chosen in set k. $\mathbf{x}_j$ is the dummy vector representation of option j; i.e., with a 1 in each entry where the option possesses the attribute level signaled by that entry.

In the standard MNL, the conditional choice probabilities P_{jk} are computed from the closed form for the MNL; i.e., in the k^{th} choice set, we have

$$P_{jk} \equiv \Pr\left(j \text{ is selected} \mid j \in Set_k\right) = \Pr\left(y_{jk} = 1\right) = \frac{\exp\left(V_j\right)}{\sum_{l \in \text{set } k} \exp\left(V_k\right)}. \tag{10}$$

To use this logic for the neuroadditive MNL, we transform the neuroadditive binary operator to its linear additive form using the functional equation method of Aczél (1966). In this form, the overall value of option j is a functional composite of "partworths". In the standard additive case, it is the inner product $V_j \equiv \left\langle \mathbf{x}_j \cdot \boldsymbol{\beta} \right\rangle$. In the neuroadditive case, it is the non-linear function $V_j \equiv f\left(V_j^-\right)$, where $V_j^- \equiv \left\langle \mathbf{x}_j \cdot f^{-1}\left(\boldsymbol{\beta}, c\right) \right\rangle \equiv \left\langle \mathbf{x}_j \cdot \boldsymbol{\beta}^- \right\rangle$ with $\boldsymbol{\beta}^- \equiv f^{-1}\left(\boldsymbol{\beta}, c\right)$. The function $f\left(\cdot\right)$ that solves Aczél's equation (8) for neuroaddition is: $f\left(z\right) = c\left(\frac{e^{2\alpha z} - 1}{e^{2\alpha z} + 1}\right)$ with inverse $f^{-1}\left(z\right) = \frac{1}{2\alpha}\log\left(\frac{c + z}{c - z}\right)$. The joint likelihood of $\boldsymbol{\beta}$ and c is, therefore, the composite function;

$\log\left(P_{jk}\right) = g_5 \circ \quad \circ \quad \circ \quad \circ$, where $\boldsymbol{\beta}^- = g_1\left(\boldsymbol{\theta}\right) = f^{-1}\left(\boldsymbol{\beta}, c\right)$, $V_j^- = g_2\left(\boldsymbol{\beta}^-\right) = \mathbf{x}_j \cdot \boldsymbol{\beta}^-$,

$V_j = g_3\left(V_j^-\right) = f\left(V_j^-\right) = c\left(\frac{\mathrm{e}^{2\alpha V_j^-} - 1}{\mathrm{e}^{2\alpha V_j^-} + 1}\right)$, $P_{jk} = g_4\left(V_j\right) = \frac{\exp\left(V_j\right)}{\sum_{l \in \text{set } k} \exp\left(V_k\right)}$, and $z = g_5\left(P_{jk}\right) = \log\left(P_{jk}\right)$.

The gradient of this composite function with respect to both $\boldsymbol{\beta}$ and c, denoted as the single vector-partitioned parameter $\boldsymbol{\theta} \equiv \begin{pmatrix} \boldsymbol{\beta} \\ \cdots \\ c \end{pmatrix}$ is the partitioned matrix form shown in (11).

$$
\begin{aligned}
\frac{d\log\left(P_{jk}\right)}{d\boldsymbol{\theta}} = & \left[\frac{4c\alpha e^{2\alpha V_j^-}}{\left(e^{2\alpha V_j^-}+1\right)^2}\mathbf{x}_j \cdot \mathbf{f}_{\boldsymbol{\beta}^-}' \;\vdots\; \frac{4ce^{2\alpha V_j^-}}{\left(e^{2\alpha V_j^-}+1\right)^2}\sum_{n=1}^{N}\frac{x_{j,n}\beta_n}{\beta_n^2-c^2} \right] \\
& -\sum_{l=1}^{J} P_{lk}\left[\frac{4c\alpha e^{2\alpha V_l^-}}{\left(e^{2\alpha V_l^-}+1\right)^2}\mathbf{x}_l \cdot \mathbf{f}_{\boldsymbol{\beta}^-}' \;\vdots\; \frac{4ce^{2\alpha V_l^-}}{\left(e^{2\alpha V_l^-}+1\right)^2}\sum_{n=1}^{N}\frac{x_{l,n}\beta_n}{\beta_n^2-c^2} \right]
\end{aligned} \tag{11}
$$

where $\mathbf{f}_{\boldsymbol{\beta}^-}' = \frac{1}{2\alpha}\operatorname{diag}\left(\frac{2c}{c^2-\beta_1^2}, \frac{2c}{c^2-\beta_2^2}, \ldots \frac{2c}{c^2-\beta_N^2}\right)$, β_n is the n^{th} entry of $\boldsymbol{\beta}$.

A preliminary simulation study was conducted to test the ability of maximum likelihood estimation to recover the known (vector composite) parameter $\boldsymbol{\theta}$. Data were generated for 100 hypothetical respondents in a simulated discrete-choice experiment (20 choice sets of size 3) with two attributes at 3 and 4 levels, respectively. Each individual followed the neuroadditive multinomial logit model with unique parameters; i.e., $\boldsymbol{\beta}_q$ was drawn from a multivariate log-normal distribution (MVN); so that $\log\left(\boldsymbol{\beta}_q\right) \sim$; c was drawn from log-normal distribution truncated by the maximum $\boldsymbol{\beta}$; i.e., $\log(c) \sim \textit{Truncated } N\left(\max\left(\log\left(\boldsymbol{\beta}\right)\right)\right)$. The true (pooled) partworth vector $\boldsymbol{\beta}$ was set to [8 11 16 1 2 3 4] and the true (pooled) maximum utility parameter was set to $\mu_{\log(c)} = 20$.

The pooled maximum likelihood estimates for this problem are shown in Table 4 for both the (true) neuroadditive model and the standard additive model using maximum.

Table 4: Parameter estimates (Maximum Likelihood)

	$\boldsymbol{\beta}_1$	$\boldsymbol{\beta}_2$	$\boldsymbol{\beta}_3$	$\boldsymbol{\beta}_4$	$\boldsymbol{\beta}_5$	$\boldsymbol{\beta}_6$	$\boldsymbol{\beta}_7$	**c**	**log(L)**
True value	8.00	11.00	16.00	1.00	2.00	3.00	4.00	20.00	-744.00
neuroadditive estimate	8.12	11.26	15.75	1.02	1.99	3.04	3.85	19.92	-743.60
Standard additive estimate	0.97	3.84	7.68	0.57	1.15	1.59	1.96	-na-	-752.00

The partworth estimates for the neuroadditive (vs. additive) model are more accurate and the estimate of *c* is close to its true value. (Formal tests are forthcoming.) Estimates based on the standard additive model roughly preserve the pairwise distance between parameters, but are generally inaccurate even though the log-likelihood values are similar. Deeper arguments made elsewhere suggest that the maximum likelihood estimator has relatively low power to distinguish between these models. The additive model is known to be robust. The neuroadditive model produces overall utilities that tend to agree in rank order with those of a standard additive model. This is an interesting case where a theory driven model (neuroadditivity) and a relatively ad hoc model (additive) can yield similar in-sample fits even though one model is known to be wrong. (For other examples, see Cui and Curry 2005.) We believe these models can be empirically more accurately distinguished using MCMC estimation since this method can capitalize on theoretical arguments from the functional equation approach regarding the distribution of parameters. If MCMC estimates can be achieved, the neuroadditive MNL model will be compared to the standard additive MNL model using more extensive simulation tests underpinned by detailed statistical analyses. Finally, neuroadditive estimation will be applied to a real dataset to reveal strategic implications as outlined next.

3.3 Individual model (MCMC Estimation plus Grid search)

To achieve individual estimates for the neuroadditive model, we propose a three step process. As we shall see, this process is a reasonable starting point, but has one main deficiency; it treats the estimation of the partworth utilities and the neuroadditive upper limit (maximum utility) as independent problems. These parameters should be estimated jointly, but to date, efforts to achieve a jointly optimized solution have fallen short. These efforts will be continued in future research.

The proposed estimation process consists of three main steps as follows:

Step One: Estimate the partworth vector $\boldsymbol{\beta}_r$ for each respondent r

As noted earlier, partworths from the neuroadditive and standard additive models are typically very similar, sharing rank order and leading to choice probabilities of nearly the same magnitude. Hence, a reasonable first step in estimating neuroadditive partworths is to approximate them by partworths from the standard additive model. However, partworths generated using the normal distribution set are neither constrained to be positive nor do they have an upper bound. The positivity constraint is naturally satisfied using the log-normal distribution set. Thus, step one uses MCMC estimation code published by Kenneth Train (2001) adjusted to assume that the prior density is lognormal (vs. normal). To track improvements in model fit as n^+ is imposed on these partworths, we use this estimate of $\hat{\boldsymbol{\beta}}_r$ for each respondent to calculate a baseline value of the log-likelihood. We denote this value as $\hat{p}_r^S$. When summed over respondents, this is the baselined total likelihood of overall model fit. We wish to compare the fit of the neuroadditive model to this baseline. To approximate the neuroadditive parameters we next search for a reasonable estimate of the maximum utility parameter, by respondent, using a grid search procedure.

Step Two: Estimate the neuroadditive coefficients, c_r, for each respondent r

We set the lower and upper bound for grid search of c_r as $\left[LC, UC\right]$, where

$$LC = \min_r \left\{ \max\left(\hat{\boldsymbol{\beta}}_r \right) \right\},$$

and

$$UC = \max_r \left\{ \max\left(\hat{\boldsymbol{\beta}}_r \right) \right\} + 10\,.$$

A grid search is performed in the range of $\left[LC, UC\right]$ using a step-size of 0.1 for each respondent. The estimate of c_r is given as

$$\hat{c}_r = \underset{c \in [LC, UC]}{\arg\max}\, \hat{p}_r^N \left(\hat{\boldsymbol{\beta}}_r, c\right),\ \forall r$$

where $\hat{p}_r^N \left(\hat{\boldsymbol{\beta}}_r, c_r\right)$ is the likelihood calculating under neuroadditive condition with coefficient c_r given that finite c improves fit for a given respondent; i.e., $\hat{p}_r^N \left(\hat{\boldsymbol{\beta}}_r, c_r\right) > \hat{p}_r^S \left(\hat{\boldsymbol{\beta}}_r\right)$.

We emphasize that the foregoing two-step procedure treats the estimation of each respondent's partworth vector and his or her maximum utility parameter independently. Thus, we expect less than optimal performance of the overall procedure. Optimally, these parameters should be estimated jointly. An intermediate procedure is to use an alternating technique that permits separate updating of each parameter ($\hat{\boldsymbol{\beta}}_r$ and c_r). These approaches are beyond the scope of this dissertation.

Step Three: Estimate variance adjuster (optional)

The neuroadditive model is a non-homothetic utility model. Unlike the additive model in which the marginal rate of substitution between attributes is everywhere constant, in the neuroadditive model, the marginal rate of substitution among attributes changes as a function of overall utility level. Allenby and Rossi (1991) showed that in a non-homothetic model, one must control for the variance of the MNL probabilities in order for the standard MNL derivation to be correct. We adopt this idea in our neuroadditive addition setting. Choice probabilities in the standard (homoscedastic) MNL are given by:

$$P\left(\mathbf{x} \mid \left\{Set_{choice}\right\}\right) = \frac{\exp\left(V_{\mathbf{x}}^{n+}\right)}{\sum_{\mathbf{z} \in Set} \exp\left(V_{\mathbf{z}}^{n+}\right)} \tag{12}$$

For this rule to hold in a non-homothetic model, an additional parameter is required to relax the standard MNL's restriction on (unit) variance; i.e., the choice probability rule becomes:

$$P_\tau\left(\mathbf{x} \mid \left\{Set_{choice}\right\}\right) = \frac{\exp\left(\tau V_{\mathbf{x}}^{n+}\right)}{\sum_{\mathbf{z}\in Set}\exp\left(\tau V_{\mathbf{z}}^{n+}\right)} \tag{13}$$

where the inner product value rule is neuroadditive; i.e. $V_{\mathbf{x}}^{n+} = \left\langle \mathbf{X}\cdot\boldsymbol{\beta}\right\rangle_{\oplus}$. The likelihood of each respondent is then given by:

$$\hat{p}_r^N\left(\hat{\boldsymbol{\beta}}_r, \hat{c}_r, \hat{\tau}\right) = \prod_{Set}\hat{P}_\tau\left(\mathbf{x} \mid \left\{Set_{choice}\right\}\right) = \prod_{Set}\frac{\exp\left(\hat{\tau}\hat{V}_{r,\mathbf{x}}^{n+}\right)}{\sum_{\mathbf{z}\in Set}\exp\left(\hat{\tau}\hat{V}_{r,\mathbf{z}}^{n+}\right)}, \tag{14}$$

A grid search procedure is employed to find the $\hat{\tau}$ that maximizes the likelihood of the observed choice data for this respondent:

$$\hat{\tau}_r = \arg\max_{\tau}\left\{\hat{p}_r^N\left(\hat{\boldsymbol{\beta}}_r, \hat{c}_r, \tau\right)\right\},$$

and satisfying $\hat{p}_r^N\left(\hat{\boldsymbol{\beta}}_r, \hat{c}_r, \hat{\tau}\right) > \hat{p}_r^N\left(\hat{\boldsymbol{\beta}}_r, \hat{c}_r\right) = \hat{p}_r^N\left(\hat{\boldsymbol{\beta}}_r, \hat{c}_r, \hat{\tau} \equiv 1\right)$. The range of search is [0.1, 6] with step size 0.1. Results with and without this step are compared in the next section. In practice, steps 1-3 or 1-2 are repeated 100 times. The average values $\overline{\boldsymbol{\beta}}_r = \frac{1}{100}\sum_{i=1}^{100}\hat{\boldsymbol{\beta}}_{i,r}$, $\overline{c}_r = \frac{1}{100}\sum_{i=1}^{100}\hat{c}_{i,r}$, $\overline{\tau}_r = \frac{1}{100}\sum_{i=1}^{100}\hat{\tau}_{i,r}$ are the final estimates of partworths, neuroadditive coefficients, and variance adjusters. Table 5 summarizes all the results which will be discussed in details in the next sections.

Table 5: Estimation Results

Distribution	Model	Var. adj.	A1 [a]	A2	A3	B1	B2	B3	B4	NC	C	LogLK [b]	-2*ΔLogLK [c]	AIC [d]
Normal	True value		6.9455	8.0212	9.8797	5.9922	8.0160	8.9715	14.9354	5.1082	16.7837	-1736.9	N/A	5237.8
	SI1	No	6.3862	6.5871	7.1166	5.8996	6.6190	7.1794	9.5285	0.4562	N/A	-1866.6	N/A	5333.2
	NI1	No	6.3862	6.5871	7.1166	5.8996	6.6190	7.1794	9.5285	0.4562	50.8662	-1607.1	519 ($p<0.0001$)	5014.2
	NI1	Yes	6.3862	6.5871	7.1166	5.8996	6.6190	7.1794	9.5285	0.4562	50.8662	-1579.9	54.4 ($p>0.99$)	5159.9
Log-normal	True value		0.3982	0.6072	1.0023	0.2039	0.5986	0.7832	2.0258	1.0057	3.3285	-2708.5	N/A	7217.0
	SI2	No	0.0033	0.0781	0.3119	0.0027	0.1410	0.2240	1.2893	0.0282	N/A	-2736.9	N/A	7073.9
	NI2	No	0.0033	0.0781	0.3119	0.0027	0.1410	0.2240	1.2893	0.0282	8.2844	-2620.1	233.6 ($p<0.0001$)	7040.2
	NI2	Yes	0.0033	0.0781	0.3119	0.0027	0.1410	0.2240	1.2893	0.0282	8.2844	-2597.1	50 ($p>0.99$)	7194.2

a. Average over all 100 responders (A1-A3, B1-B4, NC, C)
b. Average across 100 iterations of overall log-likelihood of all 100 responders
c. Data is presented as -2*ΔLogLK(p-value). ΔLogLK= LogLK in the above line minus LogLK in the present line. P-values are obtained by comparing with a chi-square distribution with 100 degrees of freedom, i.e. 100 responders each with one parameter difference.
d. Average across 100 iterations of AIC using overall log-likelihood of all 100 responders (each with 8-10 parameters)

3.3.1 Normal without τ

In this section, the partworths and neuroadditive coefficients are generated using the normal distribution set. Steps one and two are applied to get the estimates, but variance is not adjusted to relax the assumption of homotheticity. Therefore, no estimate of τ is given. The results are shown in table 5. All β estimates are biased toward zero, while estimates of c are generally inflated with very large standard errors. In fact, about 58 of the 100 respondents have $\overline{c}$ equal to the upper bound, UC; i.e. their likelihood functions are monotonically increasing in the range of $\left[LC,UC\right]$. Under the standard-additive model, the average of overall log-likelihood across 100 iterations is -1866.6. And the average responder-wise log-likelihood range from -31.0157 to -8.8788. While under neuro-additive model, the mean overall log-likelihood increased to -1607.1. The range of average responder-wise log-likelihood moves to [-26.2218, -5.9868].

3.3.2 Normal with τ

In this comparison, we relax homotheticity; i.e., perform step three after reporting results for steps one and two. Since we do not update estimates of partworths and neuroadditive coefficients, these estimates are the same as in section 3.3.1. The mean estimate of τ is 1.2308±0.4578, with median at 1.0930 and range from 0.9910 to 3.6730. The average overall log-likelihood further increases to -1579.9, and the range moves again to [-26.2151, -2.5354].

3.3.3 LogN without τ

To satisfy positivity, the partworths and neuroadditive coefficients are generated using the log-normal distribution set. Steps one and two are applied to get the estimates, but variance is not adjusted. Therefore, no estimate of τ is given. All β estimates are biased toward zero, but c estimates are generally inflated with very large standard deviation. In fact, about half of the respondents always have $\bar{c}$ equal to the upper bound, UC; i.e., their likelihood functions are monotonically increasing in the range of $\left[LC, UC\right]$. And about 25% of the respondents always have $\bar{c}$ equal to the lower bound, LC; i.e., their likelihood functions are monotonically decreasing in the range of $\left[LC, UC\right]$. Only the remaining approximately 25% of respondents have a unimodal likelihood function in the range of $\left[LC, UC\right]$. Compared with standard-additive results, using the neuro-additive model increases the average overall log-likelihood from -2736.9 to -2620.1. While the range of average responder-wise log-likelihood over 100 iterations moves from [-31.9965, -18.3932] to [-30.7251, -16.0515].

3.3.4 LogN with τ

Step three is performed after steps one and two. Since we do not update estimates of partworths and neuroadditive coefficients, the results in this section are the same as in section 3.3.3. The mean estimate of τ is 1.1845±0.3003, with median at 1.2170 and range from 0.1 to 2.1290. However, permitting flexibility in the MNL variance increases the average overall log-likelihood further to -2597.1, and the responder-wise range moves again to [-29.9839, -15.6034].

3.4 Applying the Neuroadditive Model to Real Data (Empirical example)

The data come from a survey about airline services. The discrete-choice experiment manipulated levels of four attributes: 1) Base fare, 2) Baggage policy, 3) Ticket change fee, and 4) Airline reputation; each at three levels. The survey used a fractional design with nine choice sets. Each set contained three options from which a respondent selected his or her most preferred flight. A total of one hundred and seventy-one respondents participated in the online survey, one hundred and forty-two successfully submitted the survey. Out of these 142, n=105 respondents completed all choice sets. Since our estimation code requires complete data by person, only the 105 completed surveys are used. Results are summarized in table 6 below.

Table 6: Estimation Results

Model	Var. adj.	A1 [a]	A2	A3	P1	P2	P3	C1 [*]	C2	C3	D1	D2	D3	C	LogLK [b]	-2*ΔLogLK [c]	AIC [d]
SI1	No	7.18	2.86	0.029	0.50	3.39	0.03	1.81	0.20	0.03	0.03	1.41	1.92	N/A	-570.0	N/A	3660.0
NI1	No	7.18	2.86	0.029	0.50	3.39	0.03	1.81	0.20	0.03	0.03	1.41	1.92	43.53	-203.3	727.4 (p<0.0001)	3136.5
NI1	Yes	7.18	2.86	0.029	0.50	3.39	0.03	1.81	0.20	0.03	0.03	1.41	1.92	43.53	-159.0	88.6 (p=0.13)	3258.0

* C1-C3 are attribute levels, not neuroadditive coefficients, *c* is neuroadditive coefficient

a. Average over all 105 responders (A1-A3, P1-P3, C1-C3, D1-D3, C)
b. Average across 100 iterations of overall log-likelihood of all 105 responders
c. Data is presented as -2*ΔLogLK(p-value). ΔLogLK= LogLK in the above line minus LogLK in the present line. P-values are obtained by comparing with a chi-square distribution with 105 degrees of freedom, i.e. 105 responders each with one parameter difference.
d. Average across 100 iterations of AIC using overall log-likelihood of all 105 responders (each with 12-14 parameters)

3.4.1 LogN without τ

We fit the log-normal model and estimated $\boldsymbol{\theta}$ – the vector valued parameter of twelve partworths and the neuroadditive coefficient – for each respondent. We report results for both a standard model and a homothetic model to baseline the log-likelihood values for these data. Under standard-additive conditions, the average overall log-likelihood of all 105 responders is -570.0 with range of responder-wise log-likelihood at [-12.53, -1.64]. For the homothetic neuroadditive model, the overall log-likelihood increased to -203.27 and responder-wise these are in the range [-8.81, -0.01]. The results are compared to those for the full (non-homothetic) neuroadditive model in the next section.

3.4.2 LogN with τ

Because we estimate partworths and the maximum utility parameter separately, these coefficients are the same as in the previous section. However, in this section we account for non-homotheticity by estimating τ for each respondent. The mean estimate of τ is 4.2346±2.0939, with median at 5.7410 and range from 0.8 to 6. For the non-homothetic neuroadditive model, the estimated overall log-likelihood further increased to -159.0, and the responder-wise range moved again to [-8.78, -0.00]. Thus, the heterogeneous neuroadditive model shows considerable improvement over the standard additive (comparing likelihoods as nested models) and the homothetic neuroadditive model (comparing likelihoods as nested models). However, the minimum AIC is again obtained without variance adjustment. In addition, the two overall log-likelihoods for the neuro-additive models are not significantly different.

For the individual models; i.e., those that support consumer heterogeneity in preference, results with simulated data and with real data show significant improvement when fitting the neuroadditive vs. standard-additive model. In the cases with simulated data, this statistical improvement is evidence that the estimation procedure is recovering appropriate parameters for each of the models used in the simulations. In the case with data from the airlines discrete-choice experiment, the dramatically improved fit of the neuroadditive models – both with and without variance adjustment parameters – is strong support for a neuroadditive representation of the partworth integration process. Modeling this process as bounded from above and below, hence as non-homothetic, pays significant dividends in modeling accuracy.

However, because our estimation procedures are not yet statistically optimal, we believe the foregoing results will improve as estimation techniques are refined. Furthermore, because the group of permissible transformations for the neuroadditive model has not yet been clearly worked out, raw estimates cannot be adequately judged by examining deviations from true values. Probability estimates from the standard model are invariant with respect to difference transformations of the partworth values. Hence, calculating transformations of raw estimates to match true values is a matter of linear model fitting. This is not so with the neuroadditive model. Future research will further investigate the problem of permissible transformations of neuroadditive scales and develop additional metrics for verifying that estimates match true values up to "scaling" indeterminacies. The constraints evoked by these transformations may also play a role in the estimation process itself, improving the in-sample fit of the neuroadditive representation.

In spite of these drawbacks, the large improvements in log-likelihoods in the simulation studies and with the airlines data strongly suggest the superiority of the neuro-additive model. The n^+ model uses just one additional degree of freedom per respondent (vs. the standard additive model) yet the fitted log-likelihoods improve by 5% to 50%.

Incorporating variance adjustment into the estimation process further increases the log-likelihoods of the n^+ models in both the simulated and real-data cases. These improvements do not compensate for the added degrees of freedom in either case, thus the AIC criterion selects the (simpler) neuroadditive model without variance adjustment. There are two (possibly interconnected) explanations for this result. First, variance adjustment is needed primarily in cases when there is considerable variation between respondents regarding the degree of non-homotheticity in their utility functions and within respondent when selecting from different choice sets. Neither the simulated nor real data necessarily infuse choices with sufficient variation to push the envelope between the linear-additive and non-homothetic regions of utility space. Secondly, because the current estimation routines are less than optimal, parameter estimates are not fully accurate and efficient. These inefficiencies lessen the model's fitted and predictive power; hence lower its ability to distinguish between various degrees of non-homotheticity.

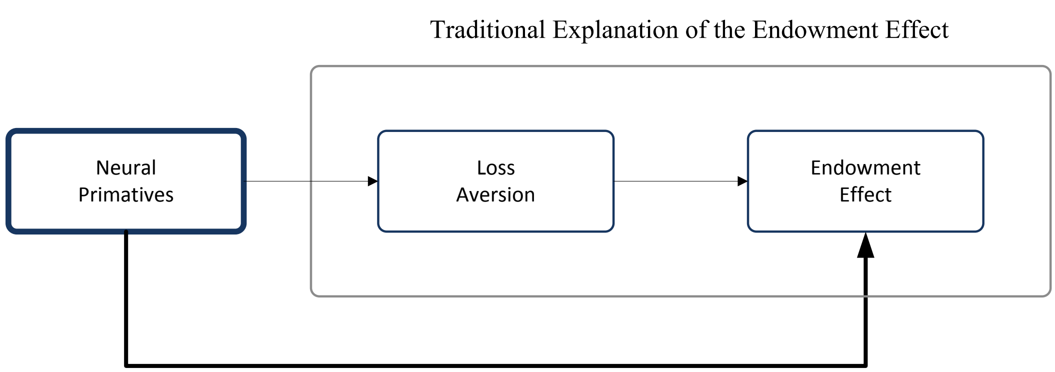
Results underscore the need for improved estimation techniques for the neuroadditive model. Results for a pooled neuroadditive model (Table 3) are excellent, so the potential for accurate estimation is present. However, the pooled model brings several orders of magnitude more statistical power to the problem than a model that supports heterogeneity. In a pooled model, only a single partworth parameter vector is estimated and this estimate uses – in the case of Table 1 – 100 times more data than is available at the individual level.

CHAPTER 4 BEHAVIORAL EXPERIMENT: RECONSIDERING THE ENDOWMENT EFFECT[9]

4.1 Introduction

Since the time that *loss aversion* was formalized as a component of prospect theory (Kahneman and Tversky 1979; Tversky and Kahneman 1991), it has been widely used as an explanatory construct in behavioral decision research. *Loss aversion* predicts that a loss decreases an individual's utility more than an equally sized gain increases it. Moreover, *loss aversion* produces an endowment effect; defined as the gap between the price buyers are willing to pay in order to acquire an object and the price sellers demand to part with the object (Carmon & Ariely, 2000; Kahneman et al., 1990; Thaler, 1980). As Novemsky and Kahneman note (2005, p. 120): "Most researchers accept loss aversion as both a description and an explanation of (the endowment effect) …" This view is suggested by the lightly boxed part of Figure 3.

Figure 3: Causal Links between Constructs



[9] This chapter is largely based on co-authored; Wang and Curry (2014)

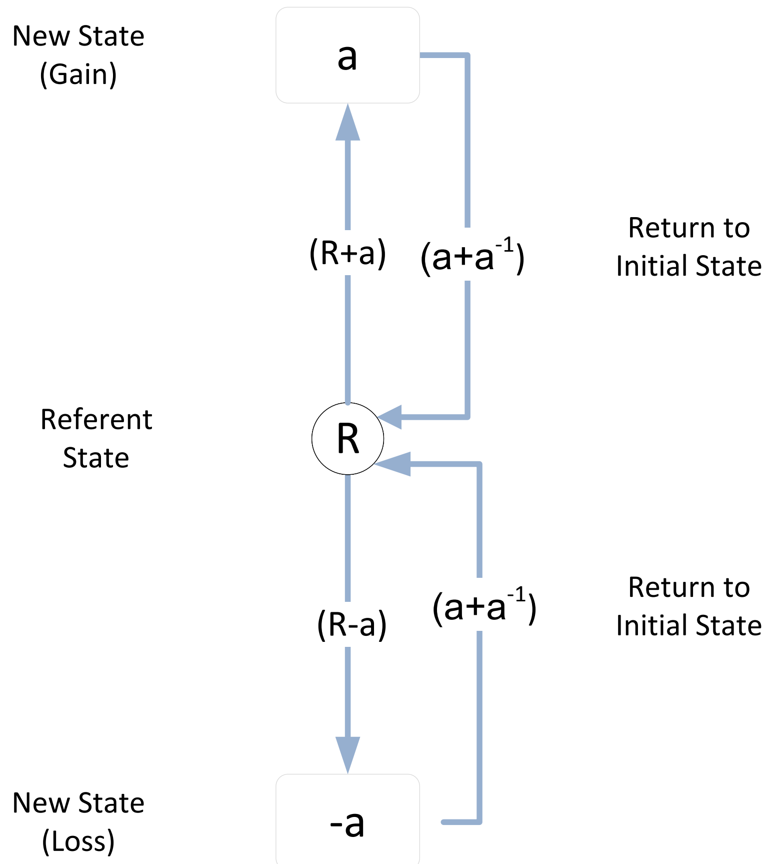
The findings supporting the endowment effect are robust (Kahneman, Knetsch, & Thaler, 1990; Knetsch & Sinden, 1984; Wu & Gonzalez 1996; Tversky & Wakker 1995), neural correlates of the effect have been found (Tom et al 2007, Camerer et al 1993), and replications and extensions suggest a wide range of constructs that moderate the effect (Nayakankuppam & Mishra, 2005; Brenner, Rottenstreich, Sood & Bilgin, 2007; Carmon & Ariely, 2000). By definition, both loss aversion and the endowment effect require an initial utility state, referred to as the *status-quo* or *reference point*. Somewhat surprisingly, this construct has been used without deep formalization. For example, in their original *Econometrica* article, Kahneman and Tversky (1979) simply say:

> Gains and losses, of course, are defined relative to some neutral reference point. The reference point usually corresponds to the current asset position, in which case gains and losses coincide with the actual amounts that are received or paid.

This definition has a quasi-accounting flavor that emphasizes financial interpretations of losses and gains. Subsequent research on the topic has broadened the idea of reference point beyond the notion of "current asset position."

In this chapter, we suggest that loss aversion is a consequence of a more fundamental condition of human information processing linked to neural primitives; hence it emerges from biological properties of the brain. We draw this conclusion from the fundamental theory presented in Chapter 2. The neuroadditive operator turns out to be directionally asymmetric, which in turn induces the asymmetry analysts observe in value judgments and label *loss aversion* (hence it induces the endowment effect as a special case). The asymmetry suggests a direct causal link to the endowment effect – shown as the bold arrow in Figure 3 – that "returning to a reference" (an inversion process as in $a+a^{-1}=0$, where "zero" is a status quo or initial state) is one-sided; losses can be inverted but gains cannot be.

Figure 4: Cognitive "Inversions" and the Endowment Effect



The contributions of this chapter are to carefully explain loss aversion as an emergent property of brain biology, to show how a fundamental asymmetry in neural processing creates the endowment effect, and to test these ideas experimentally. To these ends, we first review the subset of the literature on the endowment effect that is most relevant for our work. We then present our assumptions, explain them, and outline the argument that leads to our main conclusions. We then report findings from an experiment to test the idea of an asymmetry leading to the endowment effect. We conclude with a general discussion.

4.2 Relevant Literature

The *endowment effect* is the tendency to place a larger value on an item once it is in one's possession (Thaler 1980). The effect is signaled by individuals, who when randomly endowed with one of two items, demand more to sell their item than buyers are willing to pay, regardless of the item possessed. Previous research has replicated the endowment effect using a variety of manipulations. Researchers have also identified important moderators of the effect. For example, Zhang and Fishbach (2005) examine the possibility that evaluation of loss is not only a matter of subjective value, but also a matter of experience, thus depends on situational determinants such as mood state. Nayakankuppam and Mishra (2005) suggest that the mental processes a buyer invokes when exchanging money for an item are fundamentally different than those of a seller. They hypothesize an overrepresentation of the positive features of the item being traded and an underrepresentation of the negative features, by sellers, relative to buyers. Aggarwal and Zhang (2006) demonstrate that sellers have higher selling prices when norms of communal relationships (vs. exchange relationships) are salient. Saqib, Frohlich and Bruning (2010) show that the level of consumer involvement in a decision moderates the endowment effect. Shu and Peck (2011) propose that the concept of emotional attachment can help explain many of the endowment effect findings. Brenner et al (2007) explore the endowment effect by considering two distinct interpretations of the terms "loss" and "gain". One interpretation is that loss aversion is loss of (market) value per se, not the loss of the material possession and its detailed properties. The other view emphasizes real attachment to the object and its properties, not just its market value. The first interpretation builds on Bateman et al's (2005) distinction between the "currently endowed hypothesis" (CEH) and the "no loss in buying hypothesis" (NLIB), which proposes that money spent in buying goods is not "coded" as a loss, whereas loss of the material possession and its properties is.

Dommer and Swaminathan (2013) examine three moderators derived from theory that suggest a link between possession and the "self". They argue that the endowment effect results from identity associations on the possession-self link and, therefore, should vary by gender. Weaver and Frederick (2012) propose that the endowment effect is often better understood as reluctance to trade on unfavorable terms. Burson, Faro, and Rottenstreich (2013) show that multiple-unit holdings will yield attenuated endowment effects. Following the concern pointed out by Rick (2011) about whether loss aversion is the best explanation for the endowment effect, we show how the effect may result from a direct causal link in which value inversions are one-sided. In the next section, we explain the main assumptions of our model and develop the hypotheses tested in the present research.

4.3 Theoretical foundation

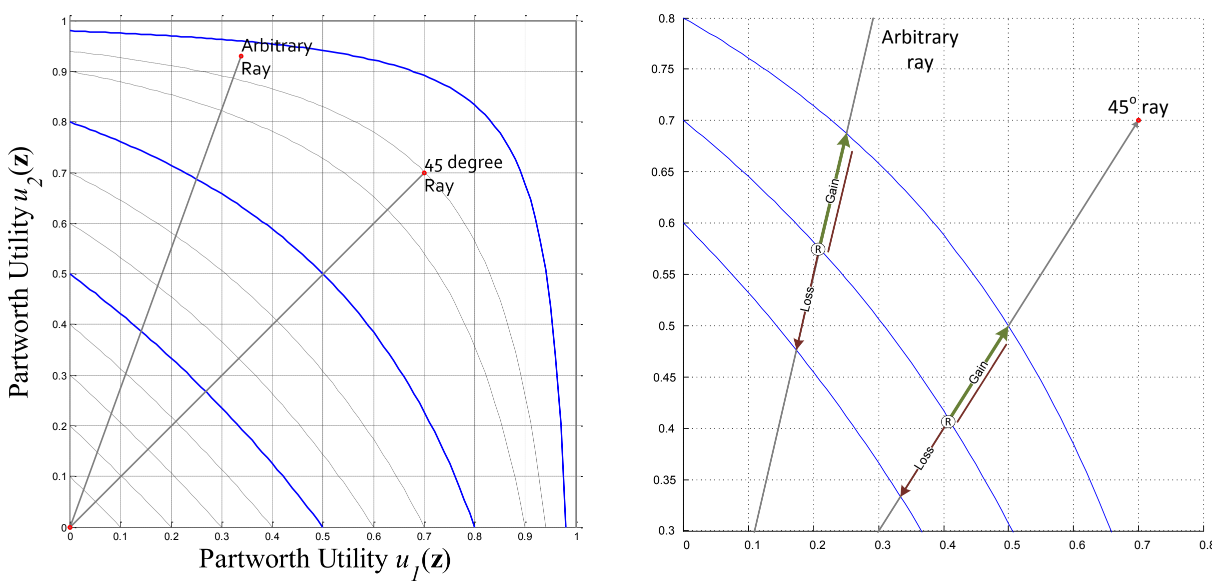
4.3.1 Loss Aversion as an Emergent Property

As shown in Chapter 2, neuroaddition follows from broad biological assumptions, resembles standard addition, and has many of its properties, such as commutativity, monotonicity, and associativity. However, neuroaddition generates iso-value contours wherein total value moderates marginal substitution rates between the "part-values" of choice options. Figure 5 (panel a) shows the operator's iso-value contours when choice options are characterized along two dimensions. Because the contours are concave – all points on a contour lie outside any line segment connecting two points on that contour – the distance required to increase value by a fixed amount moving "outbound" (the gain direction) is always more than that required moving "inbound" (the loss direction) to decrease value by the same amount.

Figure 5: Iso-Utility Contours for the r-Additive Model and Loss Aversion

(a) Rays from Origin

(b) Equi-Distant Rays (enlarged)



This property induces loss aversion as shown in (panel b). The two rays in panel (a) are enlarged in panel (b). They emanate from the origin, one at 45° where the effect is most prominent, the other at an arbitrary angle. In panel (b), the upper and lower iso-value contours represent equal gain (G) and loss (L) from any referent located on the mid iso-value contour. But note that the distance in subjective value one must travel outbound to equate $|G|=|L|$ always exceeds the distance traveling inward. This is shown by duplicating the line segment representing the loss in each case and physically setting it beside the gain segment. The distance to lose a given amount of value is always less than the distance to gain an equivalent amount of value. Put another way, a decision-maker must have disproportionately "more" of the underlying attributes if a value gain is to match a value loss.

This view suggests that loss aversion should not be interpreted as a conscious thought process; i.e., "I could gain such and such utility by choosing 'action 1', but lose this much utility by choosing 'action 2' and in weighing these two, although they equate in nominal value, I am averse to {the loss of utility associated with action 1}". Rather, asymmetry in the subconscious valuation of options emerges as a natural by-product of biology and is subsequently (mis-) interpreted by analysts as loss aversion. Put another way, autonomic signal integration in the brain – via the intuitive, rapid processing known as system 1 (Dhar and Gorlin 2013) – follows a specific non-linear "summative process" that, when translated to analysts' standard arithmetic, appears as loss aversion.

4.3.2 Asymmetry in Inversions

Besides offering a new explanation for and interpretation of loss aversion, the algebra of neuro-additivity (the positivity axiom) leads to asymmetry of inversion. This is not unusual in mathematical representations. For example, there are left and right inverses of non-square matrices in matrix algebra. In the present context, we find that a loss can be inverted, but a gain cannot be. A formal proof of this assertion is given in Appendix A (proposition 2). It depends on the concavity property stated above and the fact that the iso-value contours do not cross in the positive orthant, which is the monotonicity property. The behavioral counterpart of the proof leads to the core scientific hypothesis to be tested in this chapter.

H_S: It is possible to return a decision-maker to his or her status-quo after a loss, but not after a gain.

We translate $\mathbf{H}_S$ to specific statistical hypotheses in the context of our empirical study reported below.

Recent neural explanations of loss aversion lend support to our assumptions and to $\mathbf{H}_S$ as a conclusion. For example, Knutson et al (2008, p. 819) find that: 1) the aversion is to the loss of the object and its properties (a holistic, subconscious construct), not the loss of market value (a conscious weighing of exchange potential); 2) that gain computations exhibit a cognitive character distinct from loss computations; i.e., they activate different brain regions; and 3) that the endowment effect is strongest for more preferred products; i.e., the effect is moderated by an option's total utility, which is precisely so with neuro-additivity. (See also Knutson & Wimmer 2007 and Tom et al 2007.) The formal lack of a gain inverse in our proposed algebra was unanticipated. It complements published research on the endowment effect. Specific empirical tests to verify or refute the novel prediction are warranted. We present one such test next.

4.4 Study

4.4.1 Method

Participants. Participants (n = 179) completed the experiment online via Amazon's MTurk. Each participant received $0.15 in Amazon.com credit. Only participants who had an Amazon MTurk approval rating of 95% or higher and lived in the United States were permitted to participate.

Procedure. Participants were placed in a scenario in which they were a VIP member of an internet provider, GigaZoom. VIP members are eligible to participate in an annual rewards program offering free application programs (apps) for tablet computers. VIP members can select these apps from a wide variety of (paid) apps offered through an online menu. In this scenario, participants (Stage 0) were familiarized with the annual rewards program and knew that historically the program applied to all apps costing between \$0.99 and \$4.99. Participants were then randomly assigned to two conditions both of which contained two stages. In Condition-A, participants were first informed (mistakenly) by GigaZoom that the annual rewards program applied to apps costing between \$0.99 and \$9.99 (rather than the historical cut-off of \$4.99) (Stage 1). (Please refer to Figure B-1.) After a time lapse, these individuals then received a notification that the firm had mistakenly set the upper price limit at \$9.99 and that they were correcting this back to the historical precedent \$0.99 to \$4.99 (Stage 2). In Condition-B, participants were first notified that the program had been cancelled altogether this year due to budget cuts at the firm (Stage 1). Subsequently, participants were notified that the program would be reinstated due to an unexpected upturn in revenue (Stage 2). At each stage, participants were asked to rate the program on a scale from 1= Terrible to 9= Excellent.

4.4.2 Expectations and Corresponding Statistical Hypotheses

Let $\mu_{sj}^{\mathbf{A}}, \mu_{sj}^{\mathbf{B}}$ be Ss *j*'s judged value of the program at stage *s* in treatments A and B, respectively. If superscripts are not present, then a statistic has been pooled over both treatments at a given stage. For example, $\mu_{0j}^{\mathbf{A}}$ is Ss *j*'s judged value of a rewards program with a higher end limit of \$9.99 (vs. \$4.99) after learning of the program change but before the Ss is told that a mistake has been made. The symbol μ_0 signifies a mean pooled over both populations of respondents at stage zero.

H1. (Main test) We expect an interaction between condition and stage.

a. Ss experiencing an initial gain *cannot* be returned to their referent value; i.e., their judged value at stage 2 will be lower than at stage 0. We state the null hypothesis at equality and examine the direction of difference if the null is rejected.

$$H_0 : \mu_2^{\mathbf{A}} = \mu_0^{\mathbf{A}} \qquad H_A : \mu_2^{\mathbf{A}} \neq \mu_0^{\mathbf{A}}$$

b. Ss experiencing an initial loss *can* be returned to their referent value; i.e., their judged value at stage 2 will not differ significantly from their judged value at stage 0.

$$H_0 : \mu_2^{\mathbf{B}} = \mu_0^{\mathbf{B}} \qquad H_A : \mu_2^{\mathbf{B}} \neq \mu_0^{\mathbf{B}}$$

H2. By random assignment to treatments, we expect the judged (overall) values of the rewards program at Stage 0 to be the same among those receiving treatments A and B; i.e.,

$$H_0 : \mu_0^{\mathbf{A}} = \mu_0^{\mathbf{B}} \qquad H_A : \mu_0^{\mathbf{A}} \neq \mu_0^{\mathbf{B}}$$

H3. We expect that – pooled across Ss – the judged value of the program at stage 2 will be lower than at stage 1 because both treatments cognitively disrupt respondents and suggest that GigaZoom is tainted by some managerial incompetence. We use a two-tailed test in which the null hypothesis is equality of (pooled) means.

$$H_0 : \mu_2 = \mu_0 \qquad H_A : \mu_2 \neq \mu_0$$

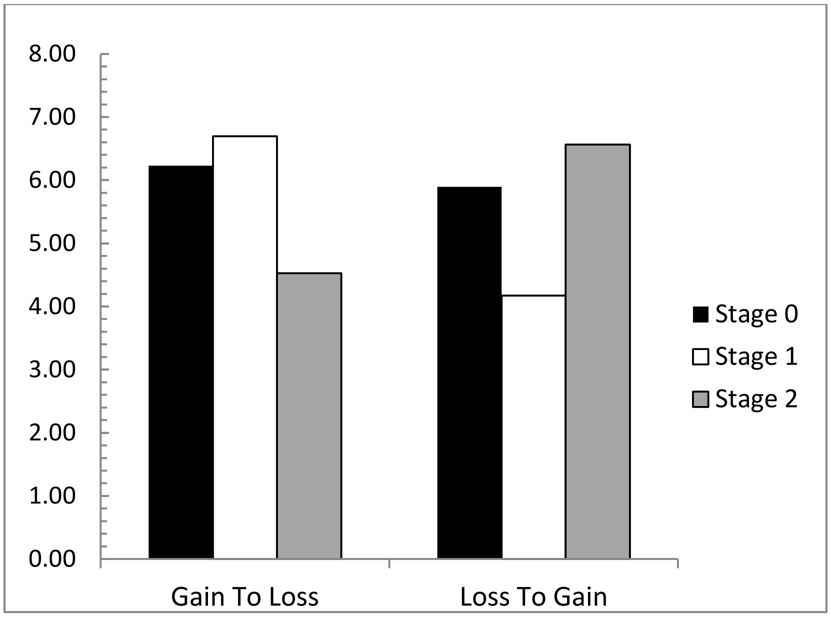
H4. From basic loss aversion, we expect the absolute magnitude of the value losses (Stage 0 to 1: B) to be larger than the absolute magnitude of the gains (Stage 0 to 1: A) because the absolute dollar losses (gains) are essentially equal in magnitude; i.e., |\$4.99 to \$0.00|= |\$4.99-\$9.99|.

$$H_0 : \left|\mu_0^{\mathbf{A}} - \mu_1^{\mathbf{A}}\right| = \left|\mu_0^{\mathbf{B}} - \mu_1^{\mathbf{B}}\right| \qquad H_A : \left|\mu_0^{\mathbf{A}} - \mu_1^{\mathbf{A}}\right| \neq \left|\mu_0^{\mathbf{B}} - \mu_1^{\mathbf{B}}\right|$$

4.4.3 Results

Participants had differing evaluations of the rewards program depending on their assigned condition (gain-to-loss vs. loss-to-gain) and stage (0 vs. 2). As expected, participants in both treatments evaluated the rewards program equivalently at stage 0, before knowing whether they would experience a gain or loss; H2: (M_0_gain=6.22 vs. M_0_loss=5.91, $t(177)=1.11$, $p<0.269$.) Participants evaluated the rewards program less positively at stage 2 than at stage 0 as per H3: ($M_2 = 5.47$ vs. $M_0 = 6.07$; $t(178) = 3.551$, $p<.001$). And participants exhibited loss aversion (H4); experiencing much higher subjective absolute loss than gain (M_Δ_loss=2.16 vs. M_Δ_gain=0.92, $t(177)=5.54$, $p<0.001$). (H1): Most importantly, the interaction between condition and stage (H1) is significant ($F(1, 177) = 62.92$, $p < .001$), such that participants in the gain-to-loss condition evaluated the rewards program less positively at stage 2 than at stage 0 ($M_2 = 4.51$ vs. $M_0 = 6.22$; $t(92) = 7.914$, $p <.001$), while participants in the loss-to-gain condition evaluated the rewards program more positively at stage 2 than at stage 0 ($M_2 = 6.51$ vs. $M_0 = 5.91$; $t(85) = -3.116$, $p = .003$).

Figure 6: Mean Subjective Values (by Stage)



4.5 General Discussion

Extant research on the endowment effect adopts traditional assumptions about mental operations and mental states. These assumptions are consistent with a standard additive information integration rule. We argue that analog activities of the brain provide an alternative foundation that leads to a "neural" integration rule that looks like addition – and shares most of its abstract properties – but is bounded above and below. We show that loss aversion is an emergent property of this rule. Empirical tests affirm this along with affirming several auxiliary hypotheses. More importantly, results support our main scientific hypothesis that people psychologically resist "returning to less after having more", but are much less resistant to "returning to more, after having less". The evidence suggests that the endowment effect is causally linked to a predicted asymmetry in cognitive states. This link becomes apparent in contexts in which a decision-maker is pushed to return to a status-quo subjective value. Relevant neuroscience research is gradually accumulating evidence in concert with our theoretical expectations and empirical findings. For example, Rick (2011) confirms the important role of distress in generating the endowment effect, and suggests that losses are experienced more intensely than comparable gains, which is precisely the expectation under neuro-additivity.

Moving from mathematical theory as a representation of human behavior to the behavior itself is rarely a cut-and-dry process. We agree that "Essentially, all models are wrong, but some are useful" (Box and Draper 1987, p. 424). In the present case, our primitives – positivity, detectability, and boundedness – seem reasonable as well as useful. But their implication of an irreversible cognitive state, which is a mathematical absolute, presents interesting interpretive problems behaviorally. If a cognitive state cannot obtain, then what takes its place? Thus, we wholeheartedly endorse additional interpretations and further empirical testing of our conclusions. Although within Ss designs are appropriate for testing individual-level theory, time must elapse for an individual to migrate from one stage to another of the gain (loss) process. Hence, there may be plausible alternative explanations for our results, unforeseen interventions as the process unfolds. However, if these "interventions" are emotional states (regret, fear, joy), then one could argue that rather than moderate the effect, they are in fact additional outputs of the process. For example, is mood state a la Zhang and Fishbach (2005) triggered by the asymmetry we identify?

4.6 Future Research

We encourage others to offer alternative explanations of our findings. Fortunately, the theory generates other testable hypotheses, many of which are unrelated to the endowment effect. Hence, we hope to triangulate results from endowment effect experiments with those from experiments in other areas to put the overall theory on sound footing. An unexpected, but highly confirmatory aspect of our work is that loss aversion emerges from our assumptions; hence, the theory offers a plausible new explanation for this well-documented phenomenon.

The asymmetry we isolate to create the endowment effect also offers intriguing insights about anecdotal phenomena. For example, it suggests that an individual will differentially value the same (baseline) level of wealth as a function of the path s/he took to reach baseline. The theory predicts disproportionately stressful consequences from the path *baseline-to-gain*, *then back to the status-quo*. Loss aversion only assesses one-way trips; it has little to say about different types of round trips to status-quo. And standard economic theory would suggest no difference in satisfaction.

CHAPTER 5 CONCLUSION

We employ three very general ideas about the brain to deduce relational laws for a numerical representation of multiattribute utility. Our assumptions – positivity, detectability, and boundedness – converge with those in the neuroscience literature despite the fact that our arguments are only partially based on direct neurological considerations. These ideas are widely accepted among neuroscientists, e.g., "… neural activity is physically constrained to minimal and maximal levels and has very limited precision" (Louie and De Martino 2014, p 455); "… neural responses are bound from below and above, therefore can represent information using a limited range of firing rates" (p 473).

The most important of our relational laws is the neuroadditive law, which yields a binary operator to represent value integration to wholes from parts. The representation naturally embodies diminishing sensitivity, loss aversion, and moderated compensatory behavior. These properties are empirically supported by decades of laboratory experiments and large-scale field studies. Beyond existing support, the neuroadditive representation generates novel hypotheses in areas connected to loss aversion and context dependent decision-making.

The neuroadditive representation adds just one parameter to standard additivity, hence is a logical candidate for a RUM that unifies previous efforts to imbue the MNL with reference dependency (Hardie et al 1993, Orhun 2009, Kivetz et al 2004a), loss aversion (Kivetz et al 2004a,b; Orhun 2009), non-homotheticity (Allenby and Rossi 1991), and non-compensatory behavior (Swait 2001; Gilbride and Allenby 2004). The representation takes a divisive normalized form, making it a candidate for the role envisioned by Louie and De Martino (2014 p 390): "A growing body of evidence, summarized in Chapter 24, now suggests that the representation of decision variables in the choice areas is neither a ratio of choice-variables nor a difference-of-choice-variables as originally proposed for perceptually-based models. Instead it appears that these networks employ a divisively normalized representation that can accommodate both ratio-like and difference-like behavior. While the details of that representation are beyond the scope of this chapter, what is important is that these two classes of models (perceptual choice models and value-based choice models) are now beginning to fit together in our understanding of human decision-making."

We propose an econometric form suitable for statistical estimation and apply developed routines to suggest strategic marketing implications. Because neuroadditivity works specifically with the deterministic part of utility, it attacks the problem using an approach that is atypical of the econometric literature, which emphasizes model generalizations via relaxing constraints on error structures. Thus, the neuro-additive hypothesis may help distinguish misspecification error from random error in RUM models.

We encourage others to offer alternative explanations of our findings. Fortunately, the theory generates other testable hypotheses, many of which are unrelated to the endowment effect tested in this dissertation, i.e., the dominance effect and the similarity effect among others. Hence, we hope to triangulate results from endowment effect experiments with those from experiments in other areas to put the overall theory on sound footing. An unexpected, but highly confirmatory aspect of our work is that loss aversion emerges from our assumptions; hence, the theory offers a plausible new explanation for this well-documented phenomenon. In the behavioral experiment from chapter 4, the asymmetry we isolate to create the endowment effect also offers intriguing insights about anecdotal phenomena. For example, it suggests that an individual will differentially value the same (baseline) level of wealth as a function of the path s/he took to reach baseline. The theory predicts disproportionately stressful consequences from the path *baseline-to-gain*, *then back to the status-quo*. Loss aversion only assesses one-way trips; it has little to say about different types of round trips to status-quo. And standard economic theory would suggest no difference in satisfaction.

The hope is that the dissertation will lead to a more productive blending of research in consumer behavior and in quantitative marketing and econometrics. Incorporating behavioral effects from consumer psychology may improve the accuracy of predictions in choice-based conjoint studies (Netzer et al 2008). From the practitioner's perspective, because choice simulators are sensitive to the predicted rank of new product profiles, integrating loss aversion and moderated compensatory behavior into the conjoint framework will help to advance innovation in new product design.

REFERENCES

Aggarwal, P., & Zhang, M. (2006). The Moderating Effect of Relationship Norm Salience on Consumers' Loss Aversion. *Journal of Consumer Research*, 33 (3), 413-419.

Agnew, R. E., J.F.W. Deakin. 2008. Medial orbitofrontal cortex codes relative preference value of financial rewards in humans. *European Journal of Neuroscience*, 27: 2213-18.

Allenby, G. M., P. E. Rossi. 1991. Quality Perceptions and Asymmetric Switching Between Brands. *Marketing Science,* 10 (3), 185-204.

Ashraf, N., C. F. Camerer, G. Loewenstein. 2005. Adam Smith, behavioral economist. *J. of Economic Perspectives*,19 (3), 131-45.

Bateman, I., D. Kahneman, A. Munro, C. Starmer, and R. Sugden. 2005. Testing Competing Models of Loss Aversion: An Adversarial Collaboration. *Journal of Public Economics,* 89 (8), 1561-80.

Baucells, M., R. K. Sarin. 2010. Predicting Utility under Satiation and Habit Formation. *Management Science*, 56(2):286-301.

Berridge, K., J. P. O'Doherty. 2014. From experienced Utility to Decision Utility. In P. Glimcher & E. Fehr (Eds.), *Handbook of neuroeconomics (2nd ed.).* New York, NY: Elsevier, 335-351.

Bodner, R., D. Prelec. 1994. The Centroid Model of Context-Dependent Choice. Working paper, Sloan School of Management, MIT.

Booth, J. L., R. S. Siegler. 2008. Numerical Magnitude Representations Influence Arithmetic Learning. *Child Development*, 79 (4), 1016-31.

Box, George E. P. and Norman R. Draper (1987), *Empirical Model-Building and Response Surfaces*, John Wiley and Sons, Inc.

Brenner, L., Y. Rottenstreich, S. Sood, B. Bilgin. 2007. On the Psychology of Loss Aversion: Possession, Valence, and Reversals of the Endowment Effect. *Journal of Consumer Research*, 34 (3), 369-76.

Burson, K., Faro, D., & Rottenstreich, Y. (2013). Multiple-Unit Holdings Yield Attenuated Endowment Effects. *Management Science*, 59 (3), 545-55.

Busemeyer, J. R., P. D. Bruza. 2012. Quantum models of cognition and decision. Cambridge University Press.

Camerer, C. F., Johnson, E., Rymon, T., & Sen. S. (1993). Cognition and Framjing in Sequential Bargaining for Gains and Losses. In K. Binmore, A. Kirman, & P. Tani (Eds.), *Frontiers in Game Theory* (pp. 27-48). Boston, MA: MIT Press.

Camerer, C.F., G. Loewenstein, D, Prelec. 2004. Neuroeconomics: Why economics needs brains. *Scandinavian Journal of Economics*. 106, 555–580.

Carmon, Z., & Ariely, D. (2000). Focusing on the Forgone: Why Value Can Appear So Different to Buyers and Sellers. *Journal of Consumer Research*, 27 (3), 360-70.

Curry, D., & Wang, X (2014). A Neuroeconomic Approach to Representation of Multiattribute Subjective Value. Working paper, University of Cincinnati.

Dhar, R., & Gorlin, M. (2013). A dual-system framework to understand preference construction processes in choice. *Journal of Consumer Psychology*, 23(4), 528-542

Dehaene, S. 2003. The Neural Basis of the Weber-Fechner Law: A Logarithmic Mental Number Line. *Trends in Cognitive Science*, 7 (4), 145-47.

Dahaene, S. 2006. Core Knowledge of Geometry in an Amazonian Indigene Group. *Science*, 311, 381-84.

Deppe, M., W. Schwindt, H. Kugel, H, Plassmann, P. Kenning. 2005. Nonlinear Responses Within the Medial Prefrontal Cortex Reveal When Specific Implicit Information Influences Economic Decision Making. *Journal of Neuroimaging,* 15 (2), 171-182.

Dickhaut, J., K. McCabe, J.C. Nagode, A. Rustichini, K. Smith, J.V. Pardo. 2003. *Proceedings of the National Academy of Science*, 100, 3536-3541

Dijksterhuis, A, M.W. Bos, L.F. Nordgren, R.B. van Baaren. 2006. On making the right choice: the deliberation-without-attention effect. *Science*, 311, 1005–7.

Drolet, A., I. Simonson, A. Tversky. 2000. Indifference Curves that Travel with the Choice Set. *Marketing Letters*, 11 (3), 199-209.

Dommer, S. L., Swaminathan, V. (2013). Explaining the Endowment Effect through Ownership: The Role of Identity, Gender, and Self-Threat. *Journal of Consumer Research*, 39 (5), 1034-50.

Ehret, G. 2010. Selective perception and recognition of vocal signals. Chapter 4.3, *Handbook of Behavioral Neuroscience*, 19, 1569-7339, pp. 125-134.

Elliot, R., Z. Agnew, and J.F.W. Deakin. 2008. Medial orbitofrontal cortex codes relative rather than absolute value of financial rewards in humans. European Journal of Neuroscience, 27 (9), 2213-18.

Elrod, T., R. Johnson, J. White. 2004. A new integrated model of noncompensatory and compensatory decision strategies. *Organizational Behavior and Human Decision Processes*, 95, 1-19.

Ernst, M., M.P. Paulus. 2005. Neurobiology of Decision Making: A Selective Review from a Neurocognitive and Clinical Perspective. *Biological Psychiatry,* 58 (8), 597-604.

Fechner, G.T. 1966. *Elements of Psychophysics*. New York, NY: Holt, Reinhart and Winston.

Fischer, M. H. 2003. Cognitive Representation of Negative Numbers. *Psychological Science*, 14 (3), 278-82.

Falmagne, J. C. 1974. Foundations of Fechnerian Psychophysics. In: *Contemporary Developments in Mathematical Psychology Vol.II, Measurement, Psychophysics and Neural Information Processing.* Eds. D.H. Krantz, R.C. Atkinson, R.D. Luce and P. Suppes. San Francisco: Freeman.

Fox, C. R., K. E. See. 2003. Belief and Preference in Decision under Uncertainty. Ch. 14, *Thinking: Psychological Perspectives on Reasoning, Judgment and Decision Making*, Ed. Hardman, D. and Macchi, L., John Wiley and Sons, New York, NY.

Fox, C.R., R. A. Poldrack, R.A. 2014. Prospect theory and the brain. In P. Glimcher & E. Fehr (Eds.), *Handbook of neuroeconomics (2nd ed.).* New York, NY: Elsevier.

Frisby, J.P., & Stone, J.V. (2010). *Seeing: The Computational Approach to Biological Vision*. Cambridge, MA: MIT Press.

Gigerenzer, G., W. Gaissmaier. 2011. Heuristic Decision Making. *Annual Review of Psychology,* 62 (1), 451-82.

Gilbride, T. J., G. M. Allenby. 2004. A Choice Model with Conjunctive, Disjunctive, and Compensatory Screening Rules. *Marketing Science*, 23 (3), 391-406.

Glimcher, P. W. (2014). Value-Based Decision Making. In Glimcher P. W., E. Fehr. 2014. Neuroeconomics (2^{nd} ed.). ch 20.

Gold, J. I., H. R. Heekeren. 2014. Neural Mechanisms for Perceptual Decision Making, In P. Glimcher & E. Fehr (Eds.), *Handbook of neuroeconomics (2nd ed.).* New York, NY: Elsevier.

Gold, J. I., H. R. Heekeren. 2014. Value-Based Decision Making, In P. Glimcher & E. Fehr (Eds.), *Handbook of neuroeconomics (2nd ed.).* New York, NY: Elsevier.

Gullick, M. M., G. Wolford, E. Temple. 2012. Understanding Less Than Nothing: Neural Distance Effects for Negative Numbers. *NeuroImage*, 62 (1), 542-54.

Hardie, B. G. S., E. J. Johnson, P. S. Fader. 1993. Modeling Loss Aversion and Reference Dependence Effects on Brand Choice. *Marketing Science*, 12 (4), 378-94.

Hastie, R., R. M. Dawes. 2001. *Rational Choice in an Uncertain World: The Psychology of Judgment and Decision Making*, Sage.

Hefendehl-Hebeker, L. 1991. Negative Numbers: Obstacles in Their Evolution from Intuitive to Intellectual Constructs. *For the Learning of Mathematics*, 11(1), 26-32.

Hosokawa, T., K., M. Kato, A. Inoue. 2007. Neurons in the macaque orbitofrontal cortex code relative preference of both rewarding and aversive outcomes. *Neuroscience Research*, 57, 434-45.

Huber, J., J. W. Payne, C. P. Puto. 1982. Adding Asymmetrically Dominated Alternatives: Violations of Regularity and the Similarity Hypothesis. *Journal of Consumer Research*, 9 (1), 90-98.

Izard, V., P. Pica, E. S. Spelke, S. Dehaene. 2011. Flexible Intuitions of Euclidean Geometry in an Amazonian Indigene Group. *Proceedings of the National Academy of Sciences*, 108 (24), 9782-87.

Kable, J.W., P.W. Glimcher. 2009. The neurobiology of decision: consensus and controversy. *Neuron*, 63, 733–745.

Kahneman, D., A. Tversky.1979. Prospect Theory: An Analysis of Decision Under Risk. *Econometrica*, 4 (2), 263-91.

Kahneman, D., A. Tversky.1991. Loss Aversion in Riskless Choice: A Reference-Dependent Model. *Quarterly Journal of Economics*, 106, 1039-61.

Kahneman, D. (2011). *Thinking, Fast and Slow*. New York, NY: Farrar, Straus and Giroux.

Kahneman, D., Knetsch, J. L., & Thaler R. (1990). Experimental Tests of the Endowment Effect and the Coase Theorem. *Journal of Political Economy*, 98 (6), 1325-48.

Kamakura, W.A., R.Y. Du. 2012. How Economic Contractions and Expansions Affect Expenditure Patterns. *Journal of Consumer Research*, 39 (2), 229-47.

Kivetz, R., O. Netzer, V. Srinivasan. 2004a. Alternative Models for Capturing the Compromise Effect. *Journal of Marketing Research*, 41(3), 237-257.

Kivetz, R., O. Netzer, V. Srinivasan. 2004b. Extending Compromise Effect Models to Complex Buying Situations and Other Context Effects. *Journal of Marketing Research*, 41(3), 262-268.

Knetsch, J. L., & Sinden, J.A. (1984). Willingness to pay and compensation demanded: Experimental evidence of an unexpected disparity in measures of value. *Quarterly Journal of Economics*, 99(3), 507−521.

Knutson, B., Wimmer, G. E. (2007). Splitting the difference: How does the brain code reward episodes? *Annals of the New York Academy of Sciences*, 1104, 54-69.

Knutson, B., G. E. Wimmer, S. Rick, N. G. Hollon, D. Prelec, G. Loewenstein. 2008. Neural Antecedents of the Endowment Effect. *Neuron*, 58 (5), 814-22.

Krantz, D.H., R. D. Luce, P. Suppes, A. Tversky. 1971. *Foundations of Measurement, Vol. I: Additive and polynomial representations*. New York: Academic Press.

Kringelbach, M.L. 2005. The human orbitofrontal cortex: linking reward to hedonic experience. *Nature Reviews Neuroscience*. 6(9), 691–702

Levy, D. J., P.W. Glimcher. 2012. The root of all value: a neural common currency for choice. *Current Opinions in Neurobiology*. 22, 1027-1038.

Louie, K., P. W. Glimcher. 2012. Efficient coding and the neural representation of value. *Annals of the New York Academy of Sciences*.1251, 13-32.

Louie, K., B, De Martino. 2014. The Neurobiology of Context-Dependent Valuation and Choice. In P. Glimcher & E. Fehr (Eds.), *Handbook of neuroeconomics* (2nd ed.). New York, NY: Elsevier.

Luce R. D. 1959. *Individual Choice Behavior: A Theoretical Analysis*, Wiley, New York, NY.

Luce, R. D. 1992. Where does subjective expected utility fail descriptively? *Journal of Risk and Uncertainty*, 5, 5-27.

Luce, R. D., E. D. Galanter. 1963. In R. D. Lute, R. R. Bush, & E. Galanter (Eds.), *Handbook of mathematical psychology*, Vol. 1. New York: Wiley, 193-243.

Masatlioglu, Y., N. Uler. 2013. Understanding the Reference Effect. *Games and Economic Behavior*. 82 (November), 403-23.

Merten, E. K., A. Nieder. 2009. Compressed Scaling of Abstract Numerosity Representations in Adult Humans and Monkeys. *Journal of Cognitive Neuroscience*, 21 (2), 333-46.

Montgomery, H. 1976. Decision Rules and the Search for a Dominance Structure: Towards a Process Model of Decision Making. *In Analysing and Aiding Decision Processes*, P. Humphres, Ola Svenson, and A. Vari (ed.) North Holland Publishers, Amsterdam (U.S.A. distributor: Elsevier Science Publishing Company, Inc. New York, NY 10017.)

Nayakankuppam, D., & Mishra, H. (2005). The endowment effect: rose-tinted and dark-tinted glasses. *Journal of Consumer Research*, 32(3), 390−395.

Norwich, K. H. 1987. On the Theory of Weber Fractions. *Perception & Psychophysics*. 42(3), 286-98.

Norwich, K. H. 1993. *Information, Sensation and Perception*. Academic Press, San Diego, CA.

Novemsky, N., D. Kahneman. 2005. The Boundaries of Loss Aversion. *Journal of Marketing Research*, 42 (2), 119-28.

Orhun, A. Y. 2009. Optimal Product Line Design When Consumers Exhibit Choice Set-Dependent Preference. *Marketing Science*, 28 (5), 868-86.

Padoa-Schioppa, C., J. A. Assad. 2008. The representation of economic value in the orbitofrontal cortex is invariant for changes of menu. *Nature Neuroscience*, 11: 95-102.

Payne, J. W. 1982. Contingent decision behavior. *Psychological Bulletin. 92*, 382–402.

Payne, J. W., J. R. Bettman, E.J. Johnson. 1992. Behavioral Decision Research: A Constructive Process Perspective. *Annual Review of Psychology*. 43, 87-131.

Petty, R. E., & Cacioppo, J. T. (1986). *Communication and Persuasion: Central and Peripheral Routes to Attitude Change*. New York: Springer-Verlag.

Pica, P., C. Lemer, V. Izard, S. Dehaene. 2004. Exact and Approximate Arithmetic in an Amazonian Indigene Group. *Science*, 306 (October), 499-503.

Plassmann H, J. P. O'Doherty, A. Rangel. 2007. Orbitofrontal cortex encodes willingness to pay in everyday economic transactions. *Journal of Neuroscience*. 27(37), 9984–9988

Plassmann H, J. P. O'Doherty, A. Rangel. 2010. Appetitive and aversive goal values are encoded in the medial orbitofrontal cortex at the time of decision making. *Journal of Neuroscience*. 30(32), 10799–10808.

Prelec, D. 1998. The Probability Weighting Function. *Econometrica*. 66 (3), 497-527.

Read, D. 2001. Is time-discounting hyperbolic or subadditive? *Journal of Risk Uncertainty.* 23(1), 5-32.

Rick, S. (2011). Losses, Gains, and Brains: Neuroeconomics Can Help Answer Open Questions about Loss Aversion. *Journal of Consumer Psychology*, 21(4), 453-63.

Roe, R.M., J. R.Busemeyer, J. T. Townsend. 2001. Multialternative Decision Field Theory: A Dynamic Connectionist Model of Decision-Making. *Psychological Review*. 108 (2), 370-92.

Ross, H.E., D. J. Murray. 1996. (Ed. and Transl.) *E.H.Weber on the Tactile Senses*. 2nd Ed. East Sussex, UK: Erlbaum (UK) Taylor & Francis.

Saqib, N., Frohlich, N., & Bruning, E. (2010). The influence of involvement on the endowment effect: The moveable value function. *Journal of Consumer Psychology*, 20 (3), 355-368.

Scholten, M., D. Read. 2006. Discounting by Intervals: A Generalized Model of Intertemporal Choice. *Management Science.* 52(9), 1424-1436.

Seymour, B., S. M. McClure. 2008. Anchors, scales and the relative coding of value in the brain. *Current Opinion in Neurobiology*, 18, 1-6.

Shu, S. B., & Peck, J. (2011). Psychological Ownership and Affective Reaction: Emotional Attachment Process Variables and the Endowment Effect. *Journal of Consumer Psychology*, 21 (4), 439-52.

Simonson, I. 1989. Choice Based on Reasons: The Case of Attraction and Compromise Effects. *Journal of Consumer Research*. 16 (2), 158-74.

Simonson, I., A. Tversky. 1992. Choice in Context: Tradeoff Contrast and Extremeness Aversion. *Journal of Marketing Research*, 29 (3), 281-95.

Sinn, F., S. J. Milberg, L. D. Epstein, R. C. Goodstein. 2007. Compromising the Compromise Effect: Brands Matter. *Marketing Letters*, 18 (3), 223-36.

Slovic, P., E. Peters, M. L. Finucane, D. G. MacGregor. 2005. Affect, Risk, and Decision Making. *Health Psychology*, 24 (4), 35–S40.

Soltani, A., B. De Martino, C. Camerer. 2012. A Range-Normalization Model of Context-Dependent Choice: A New Model and Evidence. PLoS *Computational Biology*. 8(7), 1-15.

Standring S. 2004 .*Gray's Anatomy: the Anatomical Basis of Clinical Practice*. 39th edition. Edinburgh: Elsevier/Churchill Livingstone.

Steenburgh, T. J. 2008. The Invariant Proportion of Substitution (IPS) Property of Discrete-Choice Models. *Marketing Science*, 27 (2), 300-307.

Stevens, S.S. 1936. A Scale for the Measurement of Psychological Magnitude: Loudness. *Psychological Review*, 43 (5), 405-16.

Stevens, S.S. 1961. To honor Fechner and repeal his law. *Science*. 13, 80-86.

Stevens, S.S. 1970. Neural Events and the Psychophysical Law. *Science*, 170, 1043-1050.

Stott, H. P. 2006. Cumulative prospect theory's functional menagerie. *Journal of Risk Uncertainty*. 32 (2), 101-130.

Suppes, P., D. M. Krantz, R. D. Luce, and A. Tversky. 1989. *Foundations of Measurement, Vol. 2: Geometrical, Threshold, and Probabilistic Representations.* New York, NY: Academic Press.

Swait, J. 2001. A non-compensatory choice model incorporating attribute cutoffs. *Transportation Research Part B: Methodological* , 35(10), 903–28.

Tarski, A. 1959. What is elementary geometry? In L. Henkin, P. Suppes and A. Tarski (Eds*.),* *The Axiomatic Method, With Special Reference to Geometry and Physics*. Amsterdam: hlorth-Holland Publishing Co.,16-29.

Thaler, R. H. 1980. Towards a Positive Theory of Consumer Choice. *Journal of Economic Behavior and Organization*, 1 (March), 39-60.

Tobler, P. N., C. D. Fiorillo, W. Schultz. 2005. Adaptive coding of reward value by dopamine neurons. *Science*, 307, (March), 1642-45.

Tom, S. M., C. R. Fox, C. Trepel, R. A. Poldrack. 2007. The Neural Basis of Loss Aversion in Decision-Making under Risk. *Science*, 315 (January), 515-18.

Tremblay, L., W. Schultz. 1999. Relative reward preference in primate orbitofrontal cortex. *Nature*, 398, 704–708.

Trope, Y., N. Liberman. 2010. Construal-Level Theory of Psychological Distance. *Psychological Review*. 117 (2), 440–63.

Tsetso, K., M. Usher, N. Chater. 2010. Preference Reversal in Multiattribute Choice. *Psychological Review*. 117 (4), 1275-1293.

Tversky, A. 1972. Elimination by Aspects: A Theory of Choice. *Psychology Review*. 79, 281-299.

Tversky, A., C. R. Fox. 1995. Weighing Risk and Uncertainty. *Psychological Review*. 102 (2), 269-83.

Tversky, A., D. Kahneman.1991. Loss Aversion in Riskless Choice: A Reference Dependent Model. *Quarterly Journal of Economics*. 106 (4), 1039-61.

Tversky A, I. Simonson. 1993. Context-dependent preferences. *Management Science*, 39 (10), 1179–89.

Tversky, A., P. P. Wakker. 1995. Risk Attitudes and Decision Weights. E*conometrica*. 63 (6), 1255-80.

Tversky, Amos, Paul Slovic and Daniel Kahneman (1990), "The Causes of Preference Reversals," *American Economic Review*, 80 (March), 204-17.

Tversky, A., I. Simonson. 1993. Context-dependent preferences. *Management Science*, 39 (10), 1179-1189.

Usher, M., J. L. McClelland. 2001. The Time Course of Perceptual Choice: The Leaky Competing Accumulator Model. *Psychological Review*. 108 (3), 550-592.

Usher, M., Elhalal, A., & McClelland, J. L. (2008). The Neurodynamics of Choice, Value-Based Decisions, and Preference Reversal. In N. Chater and M. Oaksford (Eds.), *The Probabilistic Mind: Prospects for Bayesian Cognitive Science* (pp 277-300). Oxford, England: Oxford University Press.

Warren, C., P. McGraw, L. Van Boven. 2011. Values and preferences: defining preference construction. *Cognitive Science*. 2 (2), 193-205.

Weaver, R., & Frederick, S. (2012). A Reference Price Theory of the Endowment Effect. *Journal of Marketing Research*, 49(5), 698-707.

Wedell, D. H., J. C. Pettibone. 1996. Using judgments to understand decoy effects in choice. *Organizational Behavior and Human Decision Processes*, 67 (3), 326-344.

Wang, X., & Curry, D. (2014). Reconsidering the Endowment Effect: A Neuro-Processing Perspective. Working paper, University of Cincinnati.

Wu, G., R. Gonzalez. 1996. Curvature of Probability Weighting Functions. *Management Science*. 42 (12), 1676-1690.

Wu, G., A. Markle. 2008. An empirical test of gain-loss separability in prospect theory. *Management Science*. 54 (7), 1322–1335.

Yoon, C., C. A. Cole, M. P. Lee. 2009. Consumer decision making and aging: current knowledge and future directions. *Journal of Consumer Psychology*. 19(1), 2-16.

Yoon, C., Bechara, A., Berns, G. S., Dagher, A. A., Dube, L., Huettel, S. A., Kable, J. W., Liberzon, I., Plassmann, H., Smidt, A., & Spence, C. (2012). Decision Neuroscience and Consumer Decision Making. *Marketing Letters*, 23 (2), 473-85.

Zhang, Y., & Fishbach, A. (2005). The role of anticipated emotions in the endowment effect. *Journal of Consumer Psychology*, 15(4), 316-324.

APPENDIX A: PROOFS OF PROPOSITIONS AND THEOREMS

Proposition 1: If the difference $u_{\alpha\beta}(\mathbf{x},\mathbf{y})$ is detectable in the αβ-frame, then for *u* to be meaningful, it must be detectable under any permissible (positive) transformation of the representative utilities. Let f be a positive function; i.e., $f(\mathbf{x}) \in \mathbb{R}$, yielding

$$u_{\alpha\beta}(\mathbf{x},\mathbf{y}) = \frac{f\left[u_\alpha(\mathbf{x})\right] - f\left[u_\beta(\mathbf{y})\right]}{f\left[u_\alpha(\mathbf{x})\right] + f\left[u_\beta(\mathbf{y})\right]}. \tag{a1}$$

Substituting (a1) into the LHS below and using the symmetry of detectability; i.e., $u_{\alpha\beta}(\mathbf{x},\mathbf{y}) = u_{\alpha\beta}(\mathbf{y},\mathbf{x})$ yields (a2).

$$\frac{1+u_{\alpha\beta}(\mathbf{y},\mathbf{x})}{1-u_{\alpha\beta}(\mathbf{y},\mathbf{x})} = \frac{\dfrac{\left(f\left[u_\alpha(\mathbf{x})\right]+f\left[u_\beta(\mathbf{y})\right]\right)+\left(f\left[u_\beta(\mathbf{y})\right]-f\left[u_\alpha(\mathbf{x})\right]\right)}{\left(f\left[\mu_\alpha(\mathbf{x})\right]+f\left[\mu_\beta(\mathbf{y})\right]\right)}}{\dfrac{\left(f\left[u_\alpha(\mathbf{x})\right]+f\left[u_\beta(\mathbf{y})\right]\right)-\left(f\left[u_\beta(\mathbf{y})\right]-f\left[u_\alpha(\mathbf{x})\right]\right)}{\left(f\left[u_\alpha(\mathbf{x})\right]+f\left[u_\beta(\mathbf{y})\right]\right)}} = \frac{f\left[u_\beta(\mathbf{y})\right]}{f\left[u_\alpha(\mathbf{x})\right]}. \tag{a2}$$

Multiplying numerator and denominator of the RHS of (a2) by $f\left[u_\gamma(\mathbf{z})\right]$, yields

$\dfrac{f\left[u_\beta(\mathbf{y})\right]\cdot f\left[u_\gamma(\mathbf{z})\right]}{f\left[u_\gamma(\mathbf{z})\right]\cdot f\left[u_\alpha(\mathbf{x})\right]} = \left[\dfrac{1+u_{\beta\gamma}(\mathbf{y},\mathbf{z})}{1-u_{\beta\gamma}(\mathbf{y},\mathbf{z})}\right]\cdot\left[\dfrac{1+u_{\gamma\alpha}(\mathbf{z},\mathbf{x})}{1-u_{\gamma\alpha}(\mathbf{z},\mathbf{x})}\right]$. Furthermore, because the expressions at the extremes of (a2) are equal, we have (a3):

$$\frac{1+u_{\beta\alpha}(\mathbf{y},\mathbf{x})}{1-u_{\beta\alpha}(\mathbf{y},\mathbf{x})} = \left[\frac{1+u_{\beta\gamma}(\mathbf{y},\mathbf{z})}{1-u_{\beta\gamma}(\mathbf{y},\mathbf{z})}\right]\cdot\left[\frac{1+u_{\gamma\alpha}(\mathbf{z},\mathbf{x})}{1-u_{\gamma\alpha}(\mathbf{z},\mathbf{x})}\right]. \tag{a3}$$

Cross-multiplying in (a3) yields the following equivalent expression,

$$\left[1+u_{\beta\alpha}(\mathbf{y},\mathbf{x})\right]\cdot\left[1-u_{\beta\gamma}(\mathbf{y},\mathbf{z})\right]\cdot\left[1-u_{\gamma\alpha}(\mathbf{z},\mathbf{x})\right] = \left[1-u_{\beta\alpha}(\mathbf{y},\mathbf{x})\right]\cdot\left[1+u_{\beta\gamma}(\mathbf{y},\mathbf{z})\right]\cdot\left[1+u_{\gamma\alpha}(\mathbf{z},\mathbf{x})\right]$$

which, after simplification yields (a4.1 and a4.2):

$$-u_{\beta\gamma}(\mathbf{y},\mathbf{z}) - u_{\gamma\alpha}(\mathbf{z},\mathbf{x}) + u_{\beta\alpha}(\mathbf{y},\mathbf{x}) + u_{\beta\alpha}(\mathbf{y},\mathbf{x})u_{\beta\gamma}(\mathbf{y},\mathbf{z})u_{\gamma\alpha}(\mathbf{z},\mathbf{x}) \tag{a4.1}$$

$$= u_{\beta\gamma}(\mathbf{y},\mathbf{z}) + u_{\gamma\alpha}(\mathbf{z},\mathbf{x}) - u_{\beta\alpha}(\mathbf{y},\mathbf{x}) - u_{\beta\alpha}(\mathbf{y},\mathbf{x})u_{\beta\gamma}(\mathbf{y},\mathbf{z})u_{\gamma\alpha}(\mathbf{z},\mathbf{x}) \tag{a4.2}$$

Rearranging (a4.2) and factoring out $u_{\beta\gamma}(\mathbf{y},\mathbf{z})$ yields;

$$u_{\beta\gamma}(\mathbf{y},\mathbf{z})-u_{\beta\alpha}(\mathbf{y},\mathbf{x})u_{\beta\gamma}(\mathbf{y},\mathbf{z})u_{\gamma\alpha}(\mathbf{z},\mathbf{x})=u_{\beta\alpha}(\mathbf{y},\mathbf{x})-u_{\gamma\alpha}(\mathbf{z},\mathbf{x})$$

$$\Leftrightarrow u_{\beta\gamma}(\mathbf{y},\mathbf{z})\left[1-u_{\beta\alpha}(\mathbf{y},\mathbf{x})u_{\gamma\alpha}(\mathbf{z},\mathbf{x})\right]=u_{\beta\alpha}(\mathbf{y},\mathbf{x})-u_{\gamma\alpha}(\mathbf{z},\mathbf{x})$$

Therefore, $u_{\beta\gamma}(\mathbf{y},\mathbf{z})=\dfrac{u_{\beta\alpha}(\mathbf{y},\mathbf{x})-u_{\gamma\alpha}(\mathbf{z},\mathbf{x})}{1-u_{\beta\gamma}(\mathbf{y},\mathbf{x})\cdot u_{\gamma\alpha}(\mathbf{z},\mathbf{x})}$ ■

Corollary P1.1: Equation (a4) can also be derived from the relationship between **y** and **x**.

$$u_{\beta\alpha}(\mathbf{y},\mathbf{x})+u_{\beta\alpha}(\mathbf{y},\mathbf{x})u_{\beta\gamma}(\mathbf{y},\mathbf{z})u_{\gamma\alpha}(\mathbf{z},\mathbf{x})=u_{\beta\gamma}(\mathbf{y},\mathbf{z})+u_{\gamma\alpha}(\mathbf{z},\mathbf{x})$$
$$\Leftrightarrow u_{\beta\alpha}(\mathbf{y},\mathbf{x})\left[1+u_{\beta\gamma}(\mathbf{y},\mathbf{z})u_{\gamma\alpha}(\mathbf{z},\mathbf{x})\right]=u_{\beta\gamma}(\mathbf{y},\mathbf{z})+u_{\gamma\alpha}(\mathbf{z},\mathbf{x})$$

Therefore, $u_{\beta\alpha}(\mathbf{y},\mathbf{x})=\dfrac{u_{\beta\gamma}(\mathbf{y},\mathbf{z})+u_{\gamma\alpha}(\mathbf{z},\mathbf{x})}{1+u_{\beta\gamma}(\mathbf{y},\mathbf{z})u_{\gamma\alpha}(\mathbf{z},\mathbf{x})}$. When options **x, y**, and **z** share the same reference field(α,β,γ), their roles as possible reference options are interchangeable. Therefore, we have the summative law for subjective value of form $u_{\beta\gamma}(\mathbf{y},\mathbf{z})=\dfrac{u_{\beta\alpha}(\mathbf{y},\mathbf{x})+u_{\gamma\alpha}(\mathbf{z},\mathbf{x})}{1+u_{\beta\gamma}(\mathbf{y},\mathbf{x})u_{\gamma\alpha}(\mathbf{z},\mathbf{x})}$ ■

Corollary P1.2: Assume the system has supremum c. Expressing each value from P1.1 as a fraction of the upper bound and then multiplying both sides by the bound yields the normalized law of summative value, $\dfrac{u_{\beta\gamma}(\mathbf{y},\mathbf{z})}{c}=\dfrac{\dfrac{u_{\beta\alpha}(\mathbf{y},\mathbf{x})}{c}+\dfrac{u_{\gamma\alpha}(\mathbf{z},\mathbf{x})}{c}}{1+\dfrac{u_{\beta\gamma}(\mathbf{y},\mathbf{x})u_{\gamma\alpha}(\mathbf{z},\mathbf{x})}{c^2}}$. Multiplying both sides by c yields the desired result $r_{\beta\gamma}(\mathbf{y},\mathbf{z})=\dfrac{u_{\beta\alpha}(\mathbf{y},\mathbf{x})+u_{\gamma\alpha}(\mathbf{z},\mathbf{x})}{1+\dfrac{u_{\beta\gamma}(\mathbf{y},\mathbf{x})u_{\gamma\alpha}(\mathbf{z},\mathbf{x})}{c^2}}$ ■

Proofs of the properties of neuroaddition:

1. *Commutativity* follows from the symmetry of the function $n^+(s,t)$.

2. *Associativity*, we have:

$$(s\oplus t)\oplus w = \frac{s+t}{1+\frac{s\cdot t}{c^2}} \oplus w = \frac{c^2(s+t)}{c^2+st} + w \Bigg/ 1 + \frac{\left(\frac{c^2(s+t)}{c^2+st}\right)w}{c^2+st} = \left[\frac{c^2(s+t)+w(c^2+st)}{c^2+st}\right] \Bigg/ \left[1+\frac{w(s+t)}{c^2+st}\right]$$

$$= \left[\frac{c^2(s+t)+w(c^2+st)}{c^2+st}\right] \Bigg/ \left[\frac{c^2+st+w(s+t)}{c^2+st}\right] = \frac{c^2(s+t)+w(c^2+st)}{c^2+st+w(s+t)} = \frac{c^2s+c^2t+c^2w+stw}{c^2+st+sw+tw} = \frac{c^2(s+t)+s(c^2+tw)}{c^2+tw+s(t+w)}$$

$$= \left[\frac{c^2(t+w)+s(c^2+tw)}{c^2+tw}\right] \Bigg/ \left[\frac{c^2+tw+s(t+w)}{c^2+tw}\right] = \left[\frac{c^2(t+w)+s(c^2+tw)}{c^2+tw}\right] \Bigg/ \left[1+\frac{s(t+w)}{c^2+tw}\right] = \frac{c^2(t+w)}{c^2+tw} + u \Bigg/ 1 + \frac{\left(\frac{c^2(t+w)}{c^2+tw}\right)s}{c^2+tw}$$

$$= \frac{c^2(t+w)}{c^2+tw} \oplus s = s \oplus \frac{c^2(t+w)}{c^2+tw} = s\oplus(t\oplus w).$$

3. The *additive identity*, zero, follows by direct substitution into the expression defining ⊕.

4. *Closure* follows because $0 \le s \le c$ and $0 \le t \le c$ together imply that $0 \le \left(\frac{s\cdot t}{c\cdot c}\right) \le 1$, thus the

denominator of the function n^+ lies in the interval [1, 2]. Therefore, $n^+(s,t) \equiv (s\oplus t) \le (s+t) \le c$.

5. *Distributivity*: By definition we have $s\otimes(t\oplus w) = \underbrace{(t\oplus w)\oplus(t\oplus w)\oplus\cdots\quad w)}_{s-times}$. Associativity

yields $\Big(\underbrace{t\oplus t\oplus\cdots}_{s-times}\quad\cdots\quad\Big)$, which by definition is $s\otimes(w\oplus t)$ and by commutativity

equals $(t\oplus w)\otimes s$ ■

Proof Theorem 1: We have $\frac{\partial n^+(s,t)}{\partial s} = \frac{1+\left(\frac{1}{c^2}\right)(s\cdot t) - \left[(s+t)\left(\frac{1}{c^2}\right)t\right]}{\left[1+\left(\frac{1}{c^2}\right)(s\cdot t)\right]^2} = \frac{c^2(c^2-t^2)}{(c^2+st)^2} \ge 0$. The

denominator is strictly positive and since $c = Sup\ \Omega_c$ the numerator is also positive. Furthermore

$\frac{\partial^2 n^+(s,t)}{\partial s^2} = \frac{2tc^2(t^2-c^2)}{(c^2+st)^3} < 0$ because all terms in the denominator are positive but the numerator

is negative; i.e., the term $2tc^2$ is positive but t^2-c^2 is negative ■

Proof Theorem 2: Loss Aversion

We prove the assertion for two attributes. The generalization to higher dimensions follows by simple induction. We use the following lemma in the proof.

Lemma 1: κ-Indifference: With two attributes, the indifference curve $u_1(z) \oplus u_2(z) = \kappa$ is the locus of points with coordinates, $u_1(z)$ and $u_2(z) = \dfrac{c^2\left[u_1(z) - \kappa\right]}{u_1(z)\kappa - c^2}$.

Proof: By definition, $U_L \equiv \kappa_L - \kappa_R$ and $U_G \equiv \kappa_G - \kappa_R$, where κ_p is the iso-value contour through the point $p \in \{\mathbf{z}_L, \mathbf{z}_R, \mathbf{z}_G\}$. We capitalize on the fact that iso-utility contours under ⊕ increase in concavity on a given ray as we move further from the origin along the ray. Let θ be the angle of the ray with respect to the axis for attribute 1. With no loss of generality set *c*=1, then for any κ we have from Lemma 1, $u_2(z_2) = \dfrac{\kappa - u_1(z_1)}{1 - \kappa u_1(z_1)}$ with $u_2(z_2)/u_1(z_1) = \tan\theta \equiv \tau$, and

$\dfrac{d u_2(z_2)}{d u_1(z_1)} = \dfrac{\kappa^2 - 1}{\left[\kappa u_1(z_1) - 1\right]^2}$. Substituting for κ and for $u_2(z_2)$, yields a function in one variable;

i.e., substituting for κ yields; $\dfrac{\kappa^2 - 1}{\left[\kappa u_1(z_1) - 1\right]^2} = \dfrac{\left(\frac{u_1(z_1) + u_2(z_2)}{1 + u_1(z_1)u_2(z_2)}\right)^2 - 1}{\left[\left(\frac{u_1(z_1) + u_2(z_2)}{1 + u_1(z_1)u_2(z_2)}\right)u_1(z_1) - 1\right]^2}$, and substituting

$u_2(z_2) = u_1(z_1) \cdot \tau$, yields $f\left[u_1(z_1)\right] = \dfrac{\left(\frac{u_1(z_1) + u_1(z_1)\cdot\tau}{1 + u_1(z_1)^2\cdot\tau}\right)^2 - 1}{\left[\left(\frac{u_1(z_1)^2 + u_1(z_1)^2\cdot\tau}{1 + u_1(z_1)^2\cdot\tau}\right) - 1\right]^2}$. Note that $f\left[u_1(z_1)\right] < 0$ and that both the numerator and denominator tend to 0 as $u_1(z_1) \to 1$. The limit will exist only if the speed of convergence differs between numerator and denominator. After some algebra, we find that

$f\left[u_1(z_1)\right] = \dfrac{1 - u_1(z_1)^2 \cdot \tau^2}{u_1(z_1)^2 - 1}$. Differentiating the numerator and denominator separately to check the speed of convergence, we find that $\dfrac{d\left(1-u_1\left(z_1\right)^2\cdot\tau^2\right)}{du_1\left(z_1\right)}=-2u_1\left(z_1\right)\cdot\tau^2$ and

$\dfrac{d\left(u_1\left(z_1\right)^2-1\right)}{du_1\left(z_1\right)}=2u_1\left(z_1\right)$. Because $\tau>1$ for $\theta > 45^o$, it follows that $\left|-2u_1\left(z_1\right)\cdot\tau^2\right|>\left|2u_1\left(z_1\right)\right|$; i.e.,

the numerator is disappearing faster than the denominator. Thus, the $\lim\limits_{u_1(z_1)\to 1} f\left[u_1\left(z_1\right)\right]=0$ as desired.

Proof Theorem 3: The marginal rate of substitution between two attributes at a fixed point is the slope of the indifference curve at that point. But $\lim\limits_{u_1(z_1)\to 1} f\left[u_1\left(z_1\right)\right]=0$ indicates that, except at the point of intersection between the 45^o ray and an indifference curve (where the slope is always -1) this slope is decreasing as total value increases. ■

Non-Fechnarian Property

That neuroadditive utility is non-Fechnerian follows directly from the fact that the iso-utility contours are monotone increasing in convexity.

Non-Homotheticity Property

Let $F\left[u_i\left(z_i\right)\right]=\kappa,\ i=1,m$ be an iso-utility contour (line, surface, or hyper-surface) for an *m*-partworth utility function. The *utility expansion path* (or UEP) is the locus of points that have identical gradient as total utility (κ) increases. We show first that homogeneity if and only if the UEP is linear. We then show that the UEP for neuroadditivity is a non-linear function of total utility.

Lemma 2: Homogeneity holds if and only if the UEP is linear.

A ray from the origin is the infinite extension of a unit vector $\dfrac{\mathbf{x}}{|\mathbf{x}|}\equiv\bar{\mathbf{x}}=\left(\bar{x}_1,\bar{x}_2,\ldots\right.$, where $|\cdot|$ is the norm of $\mathbf{x}$. The directional derivative of *F* at any collection of partworths; i.e., the point

$(u_1, u_2, \ldots$ in the direction of **x** is the gradient $V_F \equiv \sum_{i=1}^{m} x_i \frac{\partial F}{\partial u_i}$. But if F is homogeneous, then

an increase in overall utility from κ to κ+k adds k utiles in the direction $\frac{\partial F}{\partial u_i} = k \frac{\partial F}{\partial u_i}$ $\forall i$. The

directional derivative of the expansion kF is $V_{k \cdot F} \equiv \sum_{i=1}^{m} \left(k \cdot x_i\right) \frac{\partial F}{\partial u_i} = k \cdot V_F$, which is in the same

direction as $\overline{\mathbf{x}}$ (and $\mathbf{x}$), hence is along the same ray, but is $\frac{k}{|\mathbf{x}|}$ longer. Conversely, if the UEP is

linear, then all directional derivatives are the same, which defines homogeneity. ■

Lemma 3: Additive F implies F is homothetic with partial gradient constant in the plane formed by any two attributes.

In m-dimensions, additive iso-utility contours take the form $\kappa = \sum_{i=1}^{m} u_i\left(z_i\right)$ and the relevant part of

the gradient in the ij^{th} plane is $u_j\left(z_j\right) = \kappa - \sum_{j \neq i}^{m} u_i\left(z_i\right) \Leftrightarrow \frac{\partial u_i\left(z_i\right)}{\partial u_j\left(z_j\right)} = -1$, which is constant as

required by homotheticity.

m-Attributes Case

Non-homotheticity follows if F from Lemmas 2 and 3 is non-homothetic. In m dimensions, let

the ratio of any two coordinates of a ray from the origin $\mathbf{x} = (x_1, x_2, \ldots$ be the constant $\frac{x_j}{x_i} = \tau_{ij}$

. The ratio τ_{ij} is the tangent of the angle θ between the ray and the axis for attribute i in the plane

formed by u_i and u_j. In this plane, the iso-utility contours are described by the equation

$\kappa_{ij} = u_i\left(z_i\right) \oplus u_j\left(z_j\right)$, where $\kappa_{ij} = \kappa - \sum_{k \neq i\, or\, j}^{m} u_k\left(z_j\right)$ is the part of total utility contributed by

$u_i\left(z_i\right)$ and $u_j\left(z_j\right)$. Setting c=1 (with no loss of generality), and dropping the functional dependence on levels to simplify notation, we have $u_j = \dfrac{\kappa_{ij} - u_i}{1 - \kappa_{ij} u_i}$. For F to be homothetic, the

(partial) gradient $\dfrac{\partial u_j}{\partial u_i} = \dfrac{\kappa_{ij} - u_i}{\left(\kappa_{ij} u_i - 1\right)^2}$ must be constant; that is $u_j = u_i \cdot \tau_{ij} \Leftrightarrow \dfrac{\partial u_j}{\partial u_i} = \tau_{ij}$. Setting equals

equal and substituting for κ_{ij}, we have $\tau_{ij} = \dfrac{\kappa_{ij} - u_i}{\left(\kappa_{ij} u_i - 1\right)^2} = \dfrac{\left(\kappa - \sum_{k \neq i/j} u_k\right) - u_i}{\left[\left(\kappa - \sum_{k \neq i/j} u_k\right) u_i - 1\right]^2}$, but this expression

shows that τ_{ij} is a function of total utility κ. Therefore, F is non-homothetic ■

Definition 1: Let an option **q** have overall value $u(\mathbf{q}) = \kappa_q > 0$. A G-inverse (or *gain* inverse) of

z∈G is an element $\mathbf{z}^{-1} \in \Omega_{c+}$ such that: $\mathbf{z} \oplus \mathbf{z}^{-1} \in \Omega_{c+}$ and $u\left(\mathbf{z} \oplus \mathbf{z}^{-1}\right) = \kappa_q$. An L-inverse (or *loss*

inverse) of **z**∈L is an element $\mathbf{z}_{-1} \in \Omega_{c+}$ such that: $u\left(\mathbf{z} \oplus \mathbf{z}_{-1}\right) = \kappa_q$.

Proposition 2 is the basis for the empirical tests reported in this research. It asserts that a loss can be inverted, but a gain cannot be.

Proposition 2 (Gain/Loss Inverses) The set $\left[\mathbf{z}^{-1}\right] = \varnothing$, but the set $\left[\mathbf{z}_{-1}\right] = \Omega_{c+}$; i.e., for every **z**∈L,

there exists a $\mathbf{z}_{-1} \in \Omega_{c+}$ such that $\mathbf{z} \oplus \mathbf{z}_{-1} \in \Omega_{c+}$ and $u\left(\mathbf{z} \oplus \mathbf{z}_{-1}\right) = \kappa_q$.

1. We prove the non-existence of G-inverses by contradiction. Suppose $\mathbf{x}^{-1} \in$ G and $u\left(\mathbf{x} \oplus \mathbf{x}^{-1}\right) = \kappa_q$. By monotonicity (property 6), we have $u\left(\mathbf{x} \oplus \mathbf{x}^{-1}\right) > u(\mathbf{x})$. But $u(\mathbf{x}) > u(\mathbf{q}) = \kappa_q$ for all $\mathbf{q} \in [\mathbf{q}]$ by definition of $\mathbf{x} \in$ G, which leads to the contradiction $u\left(\mathbf{x} \oplus \mathbf{x}^{-1}\right) > \kappa_q$.

We prove the existence of L-inverses by constructing $\mathbf{x}_{-1}$. Identify a $\mathbf{q}^* \in [\mathbf{q}]$ such that

$u_i\left(q_i^*\right) > u_i\left(x_i\right)\ \forall i$. This is always possible because the set $[\mathbf{q}]$ lies in a quadrant (orthant) of Ω_{c+}

to the "northeast" of **x**, hence it may be interpreted as the frontier of points dominating **x**. At least one of these points must dominate **x** in partworth utility on every attribute. Next, solve the following set of component-wise equations for the elements $u_i(z_i)$: $s.t. \dfrac{u_i(x_i)+u_i(z_i)}{1+\frac{u_i(x_i)u_i(z_i)}{c^{+2}}} = u_i(q_i^*)$.

The solutions are given by $u_i(z_i) = \dfrac{u_i(q_i^*)-u_i(x_i)}{1+\frac{u_i(q_i^*)u_i(x_i)}{c^{+2}}}$ for all i. Define $u_i(x_{-i}) \equiv u_{\ i}(z_i)$; i.e., $x_{-i} \equiv z_i$. It follows that $u_i(x_i) \oplus u_i(x_{-i}) = u_i(q_i^*) \Leftrightarrow u(\mathbf{x} \oplus \mathbf{x}_{-1}) = \kappa_q$. Note that $u_i(z_i) \in \Omega_{c+}$ because $q_i^* \succ$, which implies that $u_i(q_i^*) - u_i(x_i) > 0, \forall i$; i.e., positivity is a necessary condition for $u_i(x_{-i}) \in \Omega_{c+}$ ■

APPENDIX B: STIMULI FOR EXPERIMENT

Initial Condition: Status-quo

As a VIP member of GigaZoom, you are entitled to a limited-time rewards program that offers free downloads of up to 3 apps each year for your tablet computers. The rewards program occurs yearly on the firm's founding anniversary. Apps offered in the program typically retail for anywhere from $0.99 to $4.99.

How would you evaluate the rewards program based on the above information on the scale of 1 to 9?

Gain to Loss Scenario (Treatment A)

Stage 1 - Gain

You receive an email announcing that this year the annual rewards program includes free downloads of apps costing $9.99 or less (higher price than usual).

Please indicate how you feel about GigaZoom's rewards program after receiving this email on the scale of 1 to 9. (Keep in mind your ratings from the previous question.)

Stage 2 - Loss

After you have downloaded a few apps for free, you receive another email apologizing for a system error that misrepresented the promotion offer: Only apps in the $0.99 - $4.99 price range are included for the free downloads. You can either pay full price to keep apps that are not within that price range or "return" them online.

Please indicate how you feel about GigaZoom's rewards program after receiving this email on the scale of 1 to 9. (Keep in mind your ratings from the previous question.)

Loss to Gain Scenario (Treatment B)

Stage 1 - Loss

You receive an email announcing that this year the annual rewards program was suspended due to budget cuts.

Please indicate how you feel about GigaZoom's rewards program after receiving this email on the scale of 1 to 9. (Keep in mind your ratings from the previous question.)

Stage 2 - Gain

You receive another email announcing the re-launch of the rewards program. You are encouraged to download up to 5 apps free of charge. The offer applies to any app costing $4.99 or less.

Please indicate how you feel about GigaZoom's rewards program after receiving this email on the scale of 1 to 9. (Keep in mind your ratings from the previous question.)

Figure B-1 (next page) provides a visual that explains the stimuli.

Figure B-1: Treatments and Stages