UNIVERSITY OF CINCINNATI

Date: 3-Nov-2009

I, Eugene Kramer **,**

hereby submit this original work as part of the requirements for the degree of:

Doctor of Philosophy

in Mathematical Sciences

It is entitled:

Nonhomogeneous Boundary Value Problems for the Korteweg-de Vries Equation on a Bounded Domain

Student Signature: Eugene Kramer

This work and its defense approved by:

Committee Chair: Bingyu Zhang, PhD
Bingyu Zhang, PhD

H Dumas, PhD
H Dumas, PhD

Anthony Leung, PhD
Anthony Leung, PhD

Philip Korman, PhD
Philip Korman, PhD

Nonhomogeneous Boundary Value Problems for the Korteweg-de Vries Equation on a Bounded Domain

A dissertation submitted to the

Division of Research and Advanced Studies
of the University of Cincinnati

in partial fulfillment of the
requirements for the degree of

Doctor of Philosophy (Ph.D.)

in the Department of Mathematical Sciences
of the McMicken College of Arts & Sciences

2009

by

Eugene Kramer

M.S., University of Cincinnati,1998
B.S., University of South Florida, 1996

Committee Chair: Prof. Bing-Yu Zhang

Abstract

The Korteweg- de Vries equation models unidirectional propagation of small finite amplitude long waves in a non-dispersive medium. The well-posedness, that is the existence, uniqueness of the solution, and continuous dependence on data, has been studied on unbounded, periodic, and bounded domains.

This research focuses on an initial and boundary value problem (IBVP) for the Korteweg-de Vries (KdV) equation posed on a bounded interval with general nonhomogeneous boundary conditions. Using Kato smoothing properties of an associated linear problem and the contraction mapping principle, the IBVP is shown to be locally well-posed given several conditions on the parameters for the boundary conditions, in the L^2-based Sobolev space $H^s(0,1)$ for any $s \geq 0$.

Acknowledgements

I am grateful for the support and encouragement of the faculty, staff, and fellow students of the Department of Mathematical Sciences at the University of Cincinnati. I especially wish to thank my advisor, Dr. B. Zhang, for sharing his love of mathematics as well as his great patience, guidance, and constant encouragement. I would also like to express my gratitude to the other members of my doctoral committee, Dr. Anthony Leung, Dr. H. Scott Dumas, and Dr. Philip Korman. In addition, I would like to thank my fellow faculty members, the staff, and students of the University of Cincinnati—Blue Ash, Raymond Walters College. I am indebted to you for your kindness.

To my friends and family, a thank you is not enough. I could not have completed this without your prayers and endless support. I am eternally grateful to my parents unwavering belief in my abilities. To my wife and children, I offer my heartfelt thanks, you have made this journey all the richer.

Contents

CHAPTER 1

Introduction

The research presented here is concerned with the Korteweg-de Vries equation (KdV-equation henceforth)

$$u_t + u_x + uu_x + u_{xxx} = 0 \tag{1.1}$$

posed as an initial- and boundary-value problem. In the conception pursued here, one asks for a solution of (1.1) for $(x,t) \in \Omega \times R^+$, where $\Omega = (a,b)$ is a finite interval in R, subject to an initial condition

$$u(x,0) = \phi(x) \tag{1.2}$$

and the following general boundary conditions at the ends of the interval (without loss of generality, we choose $(a,b) = (0,1)$):

$$\begin{cases} B_1 u := \alpha_1 u_{xx}(0,t) + \alpha_2 u_x(0,t) + \alpha_3 u(0,t) = h_1(t), \\ B_2 u := \beta_1 u_{xx}(1,t) + \beta_2 u_x(1,t) + \beta_3 u(1,t) = h_2(t), \\ B_3 u := \xi_1 u_x(1,t) + \xi_2 u(1,t) = h_3(t) \end{cases} \tag{1.3}$$

where α_i, β_i, ξ_j for $i = 1,2,3$ $j = 1,2$ are real constants.

It is shown that this problem has a unique solution that depends continuously on initial data in the L^2-based Sobolev space $H^s(0,1)$ for any $s \geq 0$ when given conditions on the parameters for the boundary conditions are imposed.

The well-posedness will be proven using the approach developed by Bona, Sun and Zhang in [8]. It relies heavily on the smoothing properties of the associated linear problem

$$\begin{cases} u_t + u_{xxx} = f, \qquad u(x,0) = \phi(x), \\ B_1 u = h_1(t), \;\; B_2 u = h_2(t), \;\; B_3 u = h_3(t) \end{cases} \tag{1.4}$$

There are three types of smoothing associated with solving this problem (1.4); the smoothing effects of the solution u with respect to the forcing f, the initial value ϕ and the boundary data $h_j = 0,\ j = 1, 2, 3$, respectively. It will be demonstrated that

(i) For $\phi \in L^2(0,1)$ with $f \equiv 0,\ h_j \equiv 0, j = 1, 2,,$ the solution u of (1.4) belongs to the space $C(R^+; L^2(0,1)) \cap L^2(R^+; H^1(0,1))$;

(ii) For $f \in L^1(R^+; L^2(0,1))$ with $\phi \equiv 0,\ h_j \equiv 0, j = 1, 2, 3,$ the solution u of (1.4) belongs to the space $C(R^+; L^2(0,1)) \cap L^2(R^+; H^1(0,1))$;

(iii) For $h_1, h_2 \in H^{\frac{1}{3}}_{loc}(R^+),\ h_3 \in L^2_{loc}(R^+)$ with $f \equiv 0, \phi \equiv 0,$ the solution u of (1.4) belongs to the space $C(R^+; L^2(0,1)) \cap L^2(R^+; H^1(0,1))$.

With the aid of those smoothing properties of the associated linear system, the well-posedness of the nonlinear IBVP (1.1)-(1.3) will be established by the contraction mapping principle.

The organization of this paper is as follows: in chapter one, a brief history of the development and renewed interest in the study of the Korteweg-de Vries equation is presented. The second section details the derivation of the KdV equation. Chapter two is devoted to relevant notation, definitions, and the motivation for this research. The second section of the chapter is an overview from a purely mathematical perspective of the well-posedness of initial and boundary value problems of the KdV equation on infinite, periodic and bounded domains.

Chapters three and four contain the main results of the dissertation. The estimates for the associated linear problem using homogeneous and nonhomogeneous generalized boundary conditions are developed in chapter three. The well-posedness of the problem is considered in chapter four. Conclusions and future research are presented in chapter five. The appendix contains relevant theorems and a brief review of the semigroup theory used in chapters three and four.

1.1. History of Korteweg-de Vries Equation

During the nineteenth century, the study of water waves was of great interest due to applications for naval architecture and engineering as well as for the knowledge of tides and floods in regards to commercial and industrial growth. John Scott Russell (1808–1882), a Scottish engineer, investigated the feasibility of steam-powered canal transport and, as part of his research, studied the connection between resistance to motion and wave-generation. He presented these findings to the British Association for the Advancement of Science (BAAS). Due to this work the association appointed Russell and Scottish physicist, Sir John Robison, to a 'Committee on Waves' in 1837. The report generated by the committee was published in 1844 and contained observations of waves at sea, in rivers and canals, and in Russell's own wave tank constructed for experiments. Included in the report was the observation of a 'Great Wave of Translation', now termed a solitary wave. While observing a canal boat at the Edinburgh-Glasgow canal, Russell noticed a wave moving in front of a canal boat.

I was observing the motion of a boat which was rapidly drawn along a narrow channel by a pair of horses, when the boat suddenly stopped - not so the mass of water in the channel which it had put in motion; it accumulated round the prow of a vessel in a state of violent agitation, then suddenly leaving it behind, rolled forward with great velocity, assuming the form of a large solitary elevation, a rounded, smooth and well-defined heap of water, which continued its course along the channel apparently without change of form or diminution of speed. I followed it on horseback, and overtook it still rolling on at a rate of some eight or nine miles an hour, preserving its original figure some thirty feet long and a foot to a foot and a half in height. Its height gradually diminished, and after a chase of one or two miles I lost it in the windings of the channel. Such, in the month of August 1834, was my first chance interview with that singular and beautiful phenomenon which I have called the Wave of Translation.

In his observations of the 'Wave of Translation', both in the field and his experiments conducted in his homemade wave tank, he noted that these waves were stable and could travel over very large distances. This conflicted with the contemporary wave theory, as normal waves would tend to either flatten out or rise and topple over. Russell also noted that the speed of the solitary wave depended on the size of the wave, and its width depended on the depth of water. Another curious observation was that these solitary waves never merged. When two moved in the same direction, and the large one overtook the slower, smaller wave ahead of it, a nonlinear interaction occurred, after which both waves returned to their original shape. If one of these solitary waves was too big for a particular depth of water, the wave split into two, one component with larger amplitude than the other. Russell's observations challenged the accepted theories of Newton and Bernoulli regarding hydrodynamics. Mathematician Sir George Biddell Airy objected to the emphasis placed by Russell on his 'Great Wave', arguing that it was just one consequence of linear shallow water wave theory. Airy also raised doubts that the solitary wave could propagate without change in form. George Gabriel Stokes submitted that the only permanent wave would be sinusoidal, and the solitary wave would eventually dissipate. Despite these objections, experimental results for the existence of the solitary wave remained strong, and it was Joseph Boussinesq who first developed the mathematical theory to support Russell's observations.

There were also several publications during the later part of the nineteenth century that offered mathematical theory allowing for the observed solitary waves. In an 1871 publication, Boussinesq considered long shallow waves in a canal with regular cross section and found a partial differential equation that allowed for the existence of solitary waves. In his approach, he used a fixed coordinate system and the assumption that the potential and its derivatives with respect to x, y, and t vanish for $x \to \pm\infty$. Independently, in 1876, Lord Rayleigh also proposed an equation that allowed for solitary waves. He proposed a constant basic velocity equal and opposite to the wave to eliminate the dependence on time. In addition, he assumed the existence of a solitary wave vanishing at infinity. The issue was not truly settled until 1895 when Korteweg and de Vries derived and published a model equation for the motion of waves on the surface of a layer of fluid above a flat bottom:

$$\frac{\partial \eta}{\partial t} = \frac{3}{2}\sqrt{\frac{g}{h}}\frac{\partial}{\partial x}\left(\frac{1}{2}\eta^2 + \frac{2}{3}\alpha\eta + \frac{1}{3}\sigma\frac{\partial^2 \eta}{\partial x^2}\right),$$

where η is the surface elevation above the equilibrium level h, α is small arbitrary constant related to the uniform motion of the liquid, g is the gravitational acceleration, and $\sigma = h^3/3 - Th/\rho g$ with surface capillary tension T and density ρ. This equation is known as the *Korteweg-de Vries equation* (KdV) equation.

In the paper, Korteweg and de Vries used an independent approach to develop their theory of the behavior of long waves in shallow water. Their approach used a coordinate system moving with the wave and did not assume that the potential and its derivatives vanish, giving a new free surface condition.

Re-scaling and translating the dependent and independent variables to eliminate the physical constants using the transformations

$$t' \equiv \frac{1}{2}\sqrt{\frac{g}{h\sigma}}t, \quad x' \equiv -\frac{x}{\sqrt{\sigma}}, \quad u \equiv -\frac{1}{2}\eta - \frac{1}{3}\alpha,$$

The KdV equation becomes

$$u_t + 6uu_x + u_{xxx} = 0.$$

This model is known to have the solitary wave solution and describes the phenomenon observed by Russell. After the KdV equation was presented in 1865, the solitary wave was not considered significant. It was a small aspect of the mathematical structure of nonlinear wave theory. The dispute had been settled, and the equation and research in the area faded. However, within the next 50 years, interest in the equation would become much greater.

It was not until 1965 that the full significance of the solitary wave and its generalization would be revealed. Palais [57] describes the events leading to the renewed interest in his 1997 paper. In the early 1950's, Enrico Fermi, John Pasta, and Stanislaw Ulam (FPU) [32] were exploring the heat-transfer in crystal lattices with nonlinear interactions and made an interesting discovery. They wanted to verify numerically the conjecture that if a mechanical system has many degrees of freedom and is close to a stable equilibrium, a generic nonlinear interaction would cause the energy to become equidistributed among the normal modes of the corresponding linearized system. The computer simulations surprised them because the system showed very little tendency towards equidistribution of energy. Ten years later Zabuski and Kruskal [71], at the Plasma Physics Laboratory in Princeton University, decided to re-investigate the problem. They demonstrated that certain solutions of the FPU Lattice Equations could be described in terms of solutions of the KdV equation. Zabusky and Kruskal coined the term *soliton* to describe these particle-like solitary waves. In fact, it was quickly recognized that the soliton was a vital new feature of nonlinear dynamics. The KdV equation established a mathematical basis for the study of the phenomena.

Understanding nonlinear wave equations that had soliton solutions became a primary focus for research in both pure and applied mathematics. Many equations exhibit solitonic behavior, examples include the Schrödinger equation, the sine-Gordon, Born-Infeld, and the Boussinesq equation. The KdV equation has been found to have applications in the studies of plasma physics, anharmonic lattices, elastic rods, shock waves, nonlinear optics, superconductivity, blood flow in the body, and protein folding. Several far reaching results have been discovered while investigating the KdV equation. In 1967, Gardner, Greene, Kruskal and Miura [37] devised a method for exact solution of the initial value problem of the KdV equation through a sequence of linear problems. This method, called *Inverse Scattering Transform*, is known as the nonlinear Fourier transform. In 1968, Peter Lax [55] made a fundamental step forward by providing a mathematical framework to apply the inverse-scattering theory to initial-value problems for partial differential equations. Research regarding the behavior of the KdV equation solutions has yielded many real world applications and methods applicable to the study of ordinary and partial differential equations.

1.2. Derivation of the Korteweg-de Vries Equation

From conservation laws we will derive the KdV equation. First recall that the conservation of mass is given by

$$\partial_t \rho + \nabla \cdot (\rho \vec{v}) = 0 \tag{1.5}$$

where ρ is the density, and $\vec{v}$ is the velocity of the fluid. The conservation of momentum is given by

$$\rho(\partial_t + \vec{v} \cdot \nabla)\vec{v} = -\nabla P + \vec{f} \tag{1.6}$$

where P is the internal pressure and $\vec{f}$ is the external force density.

We wish to model long waves in a shallow canal with rectangular cross section, where the length of the channel is far greater than the width. The comparison between the length and width allow the flow to be considered one dimensional. Let the x-direction be in the direction of flow (the length of the canal) and y-axis be oriented vertically. Friction for the fluid and along the boundaries of the canal will also be neglected, and the flow is assumed to have no viscosity (called inviscid flow). In order to derive the model we will assume that the fluid is incompressible and homogeneous, therefore the density is constant. We then have

$$\rho_t = 0 \qquad \text{and} \qquad \nabla \rho = 0 \tag{1.7}$$

The vorticity of a fluid is the circulation per unit area at a point in the fluid flow, the curl of the fluid velocity. We can assume that the flow being considered is rotation free. For irrotational flow we have

$$\nabla \times \vec{v} = 0, \tag{1.8}$$

In addition we will look at the velocity field as the gradient of a scalar function called the velocity potential, ϕ, that is $\vec{v} = \nabla \phi \equiv u\vec{x} + v\vec{y}$. Combining the potential function and the conditions given by incompressible and irrotational flow we can rewrite the conservation of mass equation (1.5)

$$\nabla \cdot \vec{v} = \nabla^2 \phi = \Delta \phi = 0 \tag{1.9}$$

For the momentum equation first recall that

$$\vec{v} \times (\nabla \times \vec{v}) = \nabla\Big(\frac{1}{2}\vec{v} \cdot \vec{v}\Big) - \vec{v} \cdot \nabla \vec{v}$$

Using this with the fact that the fluid is rotation free (1.8) and the velocity potential we have that

$$\vec{v} \cdot \nabla \vec{v} = \nabla\Big(\frac{1}{2}\nabla\phi \cdot \nabla\phi\Big)$$

We will also assume that the external force in the momentum equation is gravity acting in the negative y-direction, $\vec{f} = \nabla(-gy)$. Assuming that $\phi = \phi(x, y, t)$ is smooth enough, the momentum equation (1.6) becomes

$$\nabla\Big(\phi_t + \frac{1}{2}\nabla\phi \cdot \nabla\phi + \frac{P}{\rho} + gy\Big) = 0$$

therefore

$$\phi_t + \frac{1}{2}\nabla\phi \cdot \nabla\phi + \frac{P}{\rho} + gy + \chi(t) = 0 \tag{1.10}$$

Suppose that the depth of water in the channel at rest is given by $y = h$. The bottom of the channel is rigid so the water cannot move it, therefore we have the boundary condition $\phi_y\Big|_{(x,0,t)} = 0$. Also, assume that the amplitude of the traveling wave is given by $\eta = \eta(x, t)$. We then have at the free surface that $y = h + \eta(x, t)$. Using the velocity potential ϕ, we have at the surface

$$v\Big|_{surface} = \frac{dy}{dt}\Big|_{surface} = \eta_t + \eta_x \frac{dx}{dt}\Big|_{surface}$$

which gives the kinematic condition

$$\phi_y = \eta_t + \phi_x \eta_x \qquad \text{at } y = h + \eta(x,t) \tag{1.11}$$

Next consider the surface tension of the water. Let T represent the surface tension, looking at the force balance over a small element of the free surface gives

$$p_f \delta s = p_{atm} \delta s + 2T sin(\delta\theta/2)$$

where p_f is the pressure of the fluid at the surface, δs is the arc length associated with the angle $\delta\theta$, and p_{atm} is the atmospheric pressure at the surface. Since $\delta s \sim R\delta\theta$ where R is the radius of curvature, we have

$$p_f - p_{atm} = \frac{T}{R}$$

We have the following estimate involving the curvature

$$\frac{1}{R} = -\frac{\partial^2 y_s}{\partial x^2}$$

where $y_s = h + \eta(x,t)$ is the surface of the water. Therefore we find that

$$p_f - p_{atm} = -T\frac{\partial^2 \eta}{\partial x^2}$$

Using this in equation (1.10) for $P = p_f - p_{atm}$ at the surface we find

$$\phi_t + \frac{1}{2}\nabla\phi \cdot \nabla\phi + g\eta + \chi(t) = \frac{T}{\rho}\frac{\partial^2 \eta}{\partial x^2} \tag{1.12}$$

which can be rewritten

$$\phi_t + \frac{1}{2}(\phi_x^2 + \phi_y^2) + g\eta + \chi(t) = \phi_t + \frac{1}{2}(u^2 + v^2) + g\eta + \chi(t) = \frac{T}{\rho}\frac{\partial^2 \eta}{\partial x^2} \qquad \text{at } y = h + \eta(x,t) \tag{1.13}$$

This is where an important difference occurs in the approach of Korteweg and de Vries. Since Boussinesq assumed that f and its derivatives vanish for $x \to \pm\infty$, the arbitrary function $\chi(t)$ can be eliminated from the equation. Korteweg and de Vries do not make this assumption, and the function $\chi(t)$ is eliminated by taking the derivative of the equation with respect to x. Differentiating (1.13) along the channel flow, that is with respect to x, and evaluating at the surface, we find the following free surface condition

$$\phi_{xt} + \phi_x\phi_{xx} + \phi_y\phi_{xy} + g\eta_x - \frac{T}{\rho}\frac{\partial^3\eta}{\partial x^3} = u_t + uu_x + vv_x + g\eta_x - \frac{T}{\rho}\frac{\partial^3\eta}{\partial x^3} = 0 \tag{1.14}$$

Next we consider ϕ as a power series in y,

$$\phi = \sum_{n=0}^{\infty} y^n \phi_n(x,t)$$

then substitute this into (1.9),

$$\begin{aligned}
&\Delta\phi = \phi_{xx} + \phi_{yy} = 0\\
&\left(\textstyle\sum_{n=0}^{\infty} y^n\phi_n\right)_{xx} + \left(\textstyle\sum_{n=0}^{\infty} y^n\phi_n\right)_{yy} = 0\\
&\textstyle\sum_{n=0}^{\infty} y^n\phi_{n,xx} + \sum_{n=2}^{\infty} y^{n-2}(n)(n-2)\phi_n = 0\\
&\textstyle\sum_{n=0}^{\infty} y^n\{\phi_{n,xx} + (n+2)(n+1)\phi_{n+2}\} = 0\\
&\phi_{n,xx} + (n+2)(n+1)\phi_{n+2} = 0
\end{aligned} \tag{1.15}$$

we also have $\phi_y = \sum_{n=1}^{\infty} ny^{n-1}\phi_n(x,t)$. Recall that $\phi_y\Big|_{(x,0,t)} = 0$, so we have that $\phi_1 = 0$. Using this with (1.15), we find that the terms with odd values of n vanish. Letting $f \equiv \phi_0(x,t)$ we now have

$$\phi = f - \frac{y^2}{2!}f_{xx} + \frac{y^4}{4!}f_{xxxx} - \frac{y^6}{6!}f^{(6)} + \cdots = \sum_{n=0}^{\infty}\frac{(-1)^n y^{2n}}{(2n)!}f^{(2n)} \tag{1.16}$$

where $f^{(2n)}$ is the $2n$-derivative of f with respect to x.

Therefore we have

$$
\begin{aligned}
u &= \phi_x = f_x - \tfrac{y^2}{2!} f_{xxx} + \tfrac{y^4}{4!} f^{(5)} - \cdots \\
v &= \phi_y = -y f_{xx} + \tfrac{y^3}{3!} f_{xxxx} + \cdots
\end{aligned}
\tag{1.17}
$$

Using (1.17), in the free surface condition (1.14) and the kinematic condition (1.11) we can develop approximations for the waves. For the approximations we will also use that $y = h + \eta(x,t)$ and $f(x,t) = \phi_0(x,t) = q_0 + \beta(x,t)$ where q_0 is an undetermined constant velocity. Also note that $\sqrt{gh}$ is the approximate linear, non-dispersive wave speed for small amplitude disturbances. The first order approximation for small η and for a wave progressing in the positive x-direction is the expression

$$\eta = \eta(x - (q_0 + \sqrt{gh})t)$$

If we let $q_0 = -\sqrt{gh}$, then we have

$$\frac{\partial \eta}{\partial t} = 0, \qquad \frac{\partial \beta}{\partial t} = 0$$

and

$$\frac{\partial \beta}{\partial x} = -\frac{q_0}{h}\frac{\partial \eta}{\partial x} = -\frac{g}{q_0}\frac{\partial \eta}{\partial x}$$

therefore

$$\beta = -\frac{g}{q_0}(\eta + a)$$

where a is an undetermined constant. The next approximation is obtained by

$$f(x,t) = q_0 - \frac{g}{q_0}(\eta + \alpha + \gamma(x,t))$$

with γ small in comparison with η and a.

Substitution into the free surface condition (1.14) and the kinematic condition (1.11) gives two equations for η and γ. We can then rewrite into a single equation eliminating γ to find

$$\frac{\partial \eta}{\partial t} = \frac{3}{2}\frac{g}{q_0}\frac{\partial}{\partial x}\Big(\frac{1}{2}\eta^2 + \frac{2}{3}\alpha\eta + \frac{1}{3}\sigma\frac{\partial^2 \eta}{\partial x^2}\Big) \tag{1.18}$$

with $\sigma = \frac{1}{3}h^3 - \frac{Th}{\rho g}$. Consider the transformation from a fixed coordinate system (x, y) to a moving frame where

$$\xi = x - \Big(\sqrt{gh} - \alpha\sqrt{\frac{g}{h}}\Big)t, \qquad \tau = t \tag{1.19}$$

Applying this to (1.18) we have the Korteweg-de Vries equation as described in section one

$$\frac{\partial \eta}{\partial t} = \frac{3}{2}\sqrt{\frac{g}{h}}\frac{\partial}{\partial x}\left(\frac{1}{2}\eta^2 + \frac{2}{3}\alpha\eta + \frac{1}{3}\sigma\frac{\partial^2 \eta}{\partial x^2}\right) \tag{1.20}$$

If we view this over the entire real line, this can be rewritten using the following transformations

$$t' \equiv \frac{1}{2}\sqrt{\frac{g}{h\sigma}}t, \qquad x' \equiv -\frac{x}{\sqrt{\sigma}}, \qquad u \equiv -\frac{1}{2}\eta - \frac{1}{3}\alpha \tag{1.21}$$

We then have(leaving off the primes),

$$u_t - 6uu_x + u_{xxx} = 0 \tag{1.22}$$

The factor of six is left in for reasons of complete integrability, but can easily be scaled out. This transform is not available on the half line or the finite domain, so the drift term (derived from taking $\frac{\partial}{\partial x}\Big(\frac{2}{3}\alpha\eta\Big)$) would not be eliminated.

In general, the family of KdV equations is given by

$$u_t + u_{xxx} + (P(u))_x = 0 \tag{1.23}$$

$P(u)$ is a polynomial in terms of u. When $P(u) = cu^{k+1}$, then the equation is referred to as generalized KdV of order k. The original KdV equation is the generalized KdV of order one.

CHAPTER 2

Problems and Motivation

2.1. Notation and Definitions

Definition 2.1. *If X is a Banach space, the continuous mappings $w : [a,b] \to X$, equipped with the maximum norm*

$$\max_{a\le t\le b} ||w(t)||_X ,$$

is again a Banach space denoted by $C(a,b;X)$.

Definition 2.2. *Let X be a Banach space, $1 \le p \le \infty$ and $-\infty \le a < b \le \infty$. Then $L^p(a,b;X)$ is a class of L^p functions from (a,b) into X which is a Banach space with the norm*

$$||f||_{L^p(a,b;X)} = \left(\int_a^b |f(t)|_X^p dt\right)^{\frac{1}{p}} .$$

Let $H^s(R)$, where $s \ge 0$, be the Sobolev-class of L^2-functions whose first s (s is not necessarily an integer) derivatives belongs to L^2. Then for $f \in H^s(R)$

$$\|f\|_{H^s(R)} = \left(\int_{-\infty}^{\infty} (1+\xi^2)^s |\hat{f}(\xi)|^2 d\xi\right)^{\frac{1}{2}} , \tag{2.1}$$

is the norm on $H^s(R)$. Given the Parseval formula,

$$\int_{-\infty}^{\infty} |f(t)|^2 dt = \int_{-\infty}^{\infty} |\hat{f}(\xi)|^2 d\xi$$

We have that for $s = 0$, $H^0(R) = L^2(R)$.

In addition the norm on $H^s(R)$ is equivalent to the usual norm

$$\left(\sum_{j=0}^{s}\|f^{(j)}(x)\|_{L^2}^2\right)^{\frac{1}{2}},$$

when s is an positive integer.

Definition 2.3. *Let $B(X,Y)$ denote the set of all bounded linear operators from X and Y. The associated norm is denoted by $||\cdot||_{X,Y}$. The domain of an operator A is written as $\mathcal{D}(A)$.*

The following are useful spaces for this research

Definition 2.4. *For any $T > 0$ and $s \geq 0$ define the space,*

$$X_{s,T} = H^s(0,1) \times H^{(s+1)/3}(0,T) \times H^{(s+1)/3}(0,T) \times H^{s/3}(0,T)$$

with norm

$$\|(\phi,\vec{h})\|_{X_{s,T}} := \left(\|\phi\|_{H^s(0,1)}^2 + \|h_1\|_{H^{(s+1)/3}(0,T)}^2 + \|h_2\|_{H^{(s+1)/3}(0,T)}^2 + \|h_3\|_{H^{s/3}(0,T)}^2\right)^{\frac{1}{2}}$$

Definition 2.5. *For $s \geq 0$, and $T > 0$, let*

$$\mathcal{H}_{s,T} = H^{(s+1)/3}(0,T) \times H^{(s+1)/3}(0,T) \times H^{s/3}(0,T)$$

If $T = \infty$, denote $\mathcal{H}_{s,T}$ by $\mathcal{H}_s$. The norm on the space $\mathcal{H}_{s,T}$ is defined as

$$\|\vec{h}\|_{\mathcal{H}_{s,T}} \equiv \left(\|h_1\|_{H^{(s+1)/3}(0,T)}^2 + \|h_2\|_{H^{(s+1)/3}(0,T)}^2 + \|h_3\|_{H^{s/3}(0,T)}^2\right)^{1/2}$$

Definition 2.6. *Let $Y_{s,T}$ be the space of functions $v(x,t)$ such that $v \in C([0,T]; H^s(0,1)) \cap L_2([0,T]; H^{s+1}(0,1))$ with $v_x \in C([0,1]; L_2(0,T))$ The norm for $v \in Y_{s,T}$ is defined as*

$$\|v\|_{Y_{s,T}} := \left(\|v\|_{C([0,T];H^s(0,1))}^2 + \|v\|_{L^2([0,T];H^{s+1}(0,1))}^2\right)^{\frac{1}{2}}$$

In addition , let

$$\mathcal{Y}_{s,T} = Y_{s,T} \cap H^{s/3}(0,T); H^1(0,1))$$

with its norm defined as

$$\|v\|_{\mathcal{Y}_{s,T}} := \left(\|v\|^2_{Y_{s,T}} + \|v\|^2_{H^{s/3}(0,T;H^1(0,1))} \right)^{\frac{1}{2}}$$

Note that if u is a C^∞-smooth solution of the IBVP(1.1)-(1.3) then the initial data $u(x,0) = \phi(x)$ and its boundary values $h_j(t), j = 1,2,3$ must satisfy the following compatibility conditions:

$$B_1\phi_k = h_1^{(k)}(0),\ B_2\phi_k = h_2^{(k)}(0),\ B_3\phi_k = h_3^{(k)}(0)$$

for $k = 0, 1, \ldots$, where $h_j^{(k)}(t)$ is the $k-$th order derivative of h_j and

$$\begin{cases} \phi_0(x) = \phi(x), \\ \phi_k(x) = -(\phi'''_{k-1}(x) - \phi'_{k-1}(x) + \Sigma_{j=0}^{k-1}(\phi_j(x)\phi_{k-j-1}(x))'). \end{cases} \tag{2.2}$$

Definition 2.7 (**Compatibility Conditions**). *Let $s \geq 0$ be given. For any $\phi \in H^s(0,1)$ and*

$$\vec{h} = (h_1, h_2, h_3) \in H_{loc}^{\frac{s+1}{3}}(R^+) \times H_{loc}^{\frac{s+1}{3}}(R^+) \times H_{loc}^{\frac{s}{3}}(R^+),$$

we say that $(\phi, \vec{h})$ is $s-$compatible if

$$B_1(\phi_k) = h_1^{(k)}(0), \quad B_2(\phi_k) = h_2^{(k)}(0), \quad B_3(\phi_k) = h_3^{(k)}(0) \quad \textit{in the space } H^{s-3k}(0,1) \tag{2.3}$$

for $k = 0, 1, 2, \cdots, [\frac{s}{3}] - 1$.

Note here when we say that $af''(0) + bf'(0) + cf(0) = 0$ in the space $H^\mu(0,1)$ with $f \in H^\mu(0,1)$ it means for real $\mu \geq 0$, a, b, and c,

$$\begin{cases} cf(0) = 0 & if\ \frac{1}{2} < s \leq \frac{3}{2}; \\ bf'(0) + cf(0) = 0 & if\ \frac{3}{2} < s \leq \frac{5}{2}; \\ af''(0) + bf'(0) + cf(0) = 0 & if\ s > \frac{5}{2}. \end{cases}$$

2.2. Well-Posedness of the Boundary Value Problem

The modern definition of a well-posed problem is based on a classification proposed by J. Hadamard. He proposed that for mathematical models of physical phenomena to be well-posed, the problem should have the following properties: A solution exists, the solution is unique, and the solution depends continuously on the data in some topology. The solutions can exist in the classical (or strong) sense or in the weak (or mild) sense. Existence of a solution may also only be local, over a time interval to some time T, or there a global solution as $T \to \infty$. The remainder of this section will review the research conducted on the well-posedness of the initial value problem (IVP) for the KdV equation posed on the real line or periodic domain, and the IBVP problem posed on the half line and bounded domain.

2.2.1. Infinite or Periodic Domain. The first natural problem to consider for the KdV equation is the Cauchy problem, in which the initial position $u(x, 0)$ is specified.

$$u_t + u_x + uu_x + u_{xxx} = 0, \qquad u(x, 0) = \phi(x), \qquad x \in \Omega,\ t \in R^+ \tag{2.4}$$

The solution of (2.4) depends on $(x, t) \in \Omega \times R^+$. The space Ω for the initial value problem is either the real line R, where there is the assumption of some decay at infinity, or on a periodic domain S, where the initial data ϕ is periodic.

The study of the Cauchy problem where $\Omega = R$, was initiated by Gardner et al. [37] and Lax [55] with the development of inverse scattering theory. Later this problem was investigated by Sjöberg [66] and Temam [68] using new methods for the analysis of nonlinear partial differential equations. The well-posedness where ϕ lies in an $L^2(R)$-based Sobolev space $H^s(R)$ has received much attention. Various smoothing properties have been discovered in researching the pure initial value problem (2.4) that enable one to prove that it is (analytically) well-posed in the space $H^s(R)$. A brief review follows.

Bona and Smith [5] showed the Cauchy problem

$$\begin{cases} u_t + uu_x + u_{xxx} = 0, & x \in R,\ t \in R \\ u(x,0) = \phi(x) \end{cases}$$

was well-posed in the space $H^s(R)$ for $s \geq 2$ using a regularization approach. In 1979 this was improved by T. Kato [45] to $s > \frac{3}{2}$ using a semingroup approach developed by Kato for abstract evolution equations in Banach spaces in the late 1960's.

Remark 2.8. *The use of semigroups will play an important role in the development of the research presented in this dissertation. More information regarding semigroup theory is located in the appendix.*

Using a contraction mapping approach, Kening, Ponce and Vega ([47],[48], and [49]) improved the results to $s > \frac{3}{4}$. It was thought that this would be a sharp result for this approach, however Bourgain [15], used the contraction mapping principle and what is now termed the Bourgain space

$$X_{s,b} = \{f \in L^2(R; H^s(R)); \|f\|_{X_{s,b}} := \Lambda(f) < \infty\},$$

where

$$\Lambda(f) = \left(\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (1 + |\xi|)^{2s} (1 + |\tau - \xi^3|)^{2b} |\hat{f}(\xi,\tau)|^2 d\xi d\tau \right)^{\frac{1}{2}}$$

to improve the value to $s \geq 0$. This led to further research and smoothing properties shown by Kenig et al. ([50], [52]) that show that the Cauchy problem for $\Omega = R$ is well-posed for $s > -\frac{3}{4}$.

On the periodic domain S, (e.g., the unit circle in the plane), the well-posedness has also been well researched using similar methods. In 1979 Kato [45] proved that the problem was well-posed for $s > \frac{3}{2}$. Bourgain [15] improved this to $s \geq 0$ in 1993. The problem (2.4) on the periodic domain has been shown to globally well-posed in $H^s(S)$ for $s \geq -1$, in the real case by Kappeler and Topalov [44]. For more details on the IVP for both R and S domains please see [4], [5], [14], [15], [26], [40], [41], [45], [46], [48], [49], [50], [51], [52], [55], [56], [61], [62], [63], [65], [67], [68], [72], [73], and [74].

2.2.2. Half Line. For the one point boundary value problem of the KdV equation, or the KdV equation posed on the half line, the corresponding solution is the wave propagating to right. The problem is as follows,

$$\begin{cases} u_t + u_x + uu_x + u_{xxx} = 0, \quad u(x,0) = \phi(x), \\ u(0,t) = h(t) \end{cases} \tag{2.5}$$

where $x \in R^+$ and $t < \infty$. This model with $\phi(x) \equiv 0$ corresponds to experimental data with the waves generated by a wave-maker at the left-hand end of the channel and monitored as they propagate down the channel, with the experiment ceasing as soon as the waves reach the other end to leave out reflected components ([3], [40], [41], and [70]). Therefore the KdV equation no longer models the behavior of the wave once the incoming wave encounters the boundary reflections. Another assumption for the half line domain problem is that the zero boundary condition holds as $x \to +\infty$. The study of its well-posedness was initiated by Ton in [69], where existence and uniqueness were established assuming that the initial data ϕ is smooth and the boundary data, $h \equiv 0$. The first well-posedness result for the IBVP (2.5) was presented by Bona and Winther ([12],[13]); they showed that the IBVP (2.5) is (globally) well-posed in the space $H^{3k+1}(R^+)$ with $(\phi, h) \in H^{3k+1}(R^+) \times H^{k+1}_{loc}(R^+)$ for $k = 1, 2, \cdots$ There have been many works on the well-posedness of (2.5) since then. The reader is referred to [2], [6], [7], [9], [10], [25], [28],[29], [30], [31], [33], [34], [35], [36], [43] and the references therein for an overall literature review. In particular Bona, Sun and Zhang in [6] extended the theory of Kenig, Ponce and Vega ([48],[51]) on the initial value problem (IVP) for the KdV equation posed on the whole line R to the IBVP (2.5), showing that it is well-posed in the space $H^s(R^+) \times H^{(s+1)/3}_{loc}(R^+)$ for $s > 3/4$.

In [25], Colliander and Kenig demonstrated how the powerful theory developed by Kenig, Ponce and Vega, Bourgain and others for the pure initial value problems for nonlinear dispersive wave equations can be adapted to deal with initial boundary value problems for the same equations. They showed in [25] that for a given $s-$compatible pair $(\phi, h) \in H^s(R^+) \times H_{loc}^{(s+1)/3}(R^+)$ with $0 \leq s \leq 1$, $s \neq \frac{1}{2}$, the IBVP (2.5) admits a solution $u \in C([0,T]; H^s(R^+))$ which depends continuously on (ϕ, h). This result was strengthened later by Holmer [43] to include the case $-3/4 < s < 0$. In a recent paper [10], Bona, Sun and Zhang showed that the IBVP (2.5) possesses a strong global smoothing property that comes about because of the dissipative mechanism introduced through imposition of the boundary condition at $x = 0$. With the aid of this boundary smoothing property and the use of restricted Bourgain spaces, they resolved the uniqueness issue left open in [25] and showed that the IBVP (2.5) is unconditionally well-posed in the space $H^s(R^+) \times H^{s+1}3_{loc}(R+)$ for any $s > -\frac{3}{4}$. More recently, they showed that the (2.5) well-posed in a weighted Sobolev space $H^s_\nu(R^+)$ for any $s > -1$ where

$$H^s_\nu(R^+) = \{f \in H^s((R^+); e^\nu f \in H^s(R^+)\}.$$

2.2.3. Bounded Domain. Perhaps the most realistic application of the KdV equation is the two point boundary value problem. As mentioned earlier the KdV equation is used to model unidirectional propagation of small finite amplitude long waves in canals, and in most physical applications, fluid dynamical experiments, and numerical computations, the region is finite. Therefore it is realistic to consider the KdV equation in a bounded domain, namely an interval in R. Traditionally the problem is viewed in one of two ways

$$\begin{cases} u_t + u_x + uu_x + u_{xxx} = 0, \quad 0 < x < 1, \ t > 0 \\ u(x,0) = \phi(x), \\ u(0,t) = h_1(t), \quad u(1,t) = h_2(t), \quad u_x(1,t) = h_3(t) \end{cases} \tag{2.6}$$

$$\begin{cases} u_t + u_x + uu_x - u_{xxx} = 0, \quad 0 < x < 1, \ t > 0 \\ u(x,0) = \phi(x), \\ u(0,t) = h_1(t), \quad u(1,t) = h_2(t), \quad u_x(0,t) = h_3(t) \end{cases} \tag{2.7}$$

the only difference being the direction of the wave. For our discussions we will refer to problem (2.6).

Bubnov ([20], [21]) studied the general two point boundary value problem

$$\begin{cases} u_t + uu_x + u_{xxx} = f, \quad x \in (0,1), t \geq 0, \\ u(x,0) = 0, \end{cases} \tag{2.8}$$

with boundary conditions

$$\begin{cases} \alpha_1 u_{xx}(0,t) + \alpha_2 u_x(0,t) + \alpha_3 u(0,t) = 0, \\ \beta_1 u_{xx}(1,t) + \beta_2 u_x(1,t) + \beta_3 u(1,t) = 0, \\ \xi_1 u_x(1,t) + \xi_2 u(1,t) = 0 \end{cases} \tag{2.9}$$

where α_i, β_i, ξ_j for $i = 1,2,3$ $j = 1,2$ are real constants. In order to ensure that the energy would decay and that there would be three distinct boundary conditions, Bubnov developed the following conditions on the parameters α_i, β_i, ξ_j for $i = 1,2,3$ $j = 1,2$.

$$\begin{cases} (i.) \text{ If } \alpha_1\beta_1\xi_1 \neq 0, \text{ then } F_1 > 0, F_2 > 0 \\ (ii.) \text{ If } \beta_1 \neq 0, \xi_1 \neq 0, \alpha_1 = 0, \text{ then } F_2 > 0, \alpha_2 = 0, \alpha_3 \neq 0 \\ (iii.) \text{ If } \beta_1 = 0, \xi_1 \neq 0, \alpha_1 \neq 0, \text{ then } F_1 > 0, F_3 \neq 0 \\ (iv.) \text{ If } \alpha_1 = \beta_1 = 0, \xi_1 \neq 0, \text{ then } F_3 \neq 0, \alpha_2 = 0, \alpha_3 \neq 0 \\ (v.) \text{ If } \beta_1 = 0, \alpha_1 \neq 0, \xi_1 = 0, \text{ then } F_1 > 0, F_3 \neq 0 \\ (vi.) \text{ If } \alpha_1 = \beta_1 = \xi_1 = 0, \text{ then } F_3 \neq 0, \alpha_2 = 0, \alpha_3 \neq 0 \end{cases} \tag{2.10}$$

where $F_1 = \frac{\alpha_3}{\alpha_1} - \frac{\alpha_2^2}{2\alpha_1^2}$, $F_2 = \frac{\beta_2\xi_2}{\beta_1\xi_1} - \frac{\beta_3}{\beta_1} - \frac{{\xi_2}^2}{2{\xi_1}^2}$ and $F_3 = \beta_2\xi_2 - \beta_3\xi_1$.

Given on of these conditions held, Bubnov proved the problem was well-posed.

Theorem 2.9 (Bubnov)**.** *Assume that one of the conditions of (3.2) is satisfied. Let $T > 0$ be given and*

$$\phi \equiv 0, \;\; h_j \equiv 0 \; \textit{for} \; j = 1,2,3.$$

For any $f \in H^1([0,T];L^2(0,1))$, there exists a $T^* > 0$ depending on $||f||_{H^1([0,T];L^2(0,1))}$ such that (2.8)-(2.9) admits a unique solution solution

$$u \in L^2([0,T^*];H^3(0,1)),\ u_t \in L_\infty([0,T^*];L^2(0,1)) \cap L^2([0,T^*];H^1(0,1)).$$

In proving the theorem Bubnov used the linear form of the partial differential equation and considered the following equation along with the conditions (3.2):

$$u_t + u_{xxx} + \nu u u_x = F, \quad 0 \le \nu \le 1$$

Defining the set $A = \nu \in [0,1]$ as the values for which the equation had a solution, he proved that the set A was closed using the linear approximation and that the set was open using Shauder's principle.

Zhang [72] considered boundary control of KdV equation posed on a finite interval $(0,1)$ with the Dirichlet boundary conditions

$$\begin{cases} u_t + uu_x + u_{xxx} = 0, \quad x \in (0,1), t \ge 0, \\ u(x,0) = \phi(x), \\ u(0,t) = 0,\ u(1,t) = 0,\ u_x(1,t) = 0, \end{cases} \tag{2.11}$$

He showed that the IBVP (2.11) is globally well-posed in the space $H^{3k+1}(0,1)$ for $k = 0, 1, \ldots$.

Colin and Ghidaglia [23] considered the following initial-boundary problem

$$\begin{cases} u_t + uu_x + u_{xxx} = 0, \quad x \in (0,1), t \ge 0, \\ u(x,0) = \phi(x), \\ u(0,t) = h_1(t),\ u_x(1,t) = h_2(t),\ u_{xx}(1,t) = h_3(t) \end{cases} \tag{2.12}$$

and showed that the IBVP (2.12) is locally well-posed in the space $H^1(0,1)$ with the initial data $\phi \in H^1(0,1)$ and boundary data from the product space $C^1[0,T] \times C^1[0,T) \times C^1[0,T)$.

Bona, Sun and Zhang [8], considered the following initial-boundary value problem with nonhomogeneous boundary conditions

$$\begin{cases} u_t + u_x + uu_x + u_{xxx} = 0, \quad x \in (0,1), t \geq 0, \\ u(x,0) = \phi(x), \\ u(0,t) = h_1(t), \ u(1,t) = h_2(t), \ u_x(1,t) = h_3(t). \end{cases} \tag{2.13}$$

They proved using smoothing properties of the associated linear problem, various linear estimates, the contraction mapping principle, and finding a priori estimates for the smooth solution of (2.13) as well as nonlinear interpolation theory. that the IBVP (2.13) is locally well-posed in the space $H^s(0,1)$ for any $s \geq 0$ with $s-$compatible $\phi \in H^s(0,1)$, $h_1, \ h_2 \in H_{loc}^{\frac{s+1}{3}}(R^+)$ and $h_3 \in H_{loc}^{\frac{s}{3}}(R^+)$.

This well-posedness result in $H^s(0,1)$ was extended later to the case of $s > -\frac{3}{4}$ by Homler [43], and then by Bona, Sun and Zhang [11], to reach $s > -1$.

In this dissertation, we continue Bubnov's work to study the general two-point boundary value problem with *nonhomogeneous* boundary conditions for its well-posedness in the space $H^s(0,1)$.

2.3. Statement of Results

The focus of this dissertation is the well-posedness of

$$u_t + u_x + uu_x + u_{xxx} = 0 \tag{2.14}$$

subject to an initial condition

$$u(x,0) = \phi(x) \tag{2.15}$$

and the following general boundary conditions at the ends of the interval (without loss of generality, we choose $(a,b) = (0,1)$):

$$\begin{cases} B_1u := \alpha_1 u_{xx}(0,t) + \alpha_2 u_x(0,t) + \alpha_3 u(0,t) = h_1(t), \\ B_2u := \beta_1 u_{xx}(1,t) + \beta_2 u_x(1,t) + \beta_3 u(1,t) = h_2(t), \\ B_3u := \xi_1 u_x(1,t) + \xi_2 u(1,t) = h_3(t) \end{cases} \tag{2.16}$$

where α_i, β_i, ξ_j for $i = 1, 2, 3$ $j = 1, 2$ are real constants.

The IBVP (2.14)-(2.16) will be considered with the following assumptions imposed on the coefficients of the boundary conditions:

$$\begin{cases} (a) \text{ If } \alpha_1\beta_1\xi_1 \neq 0, \text{ then } F_1 \geq 0, F_2 \geq 0 \\ (b) \text{ If } \beta_1 \neq 0, \xi_1 \neq 0, \alpha_1 = 0 \text{ then } F_2 \geq 0, \alpha_2 = 0, \alpha_3 \neq 0 \\ (c) \text{ If } \beta_1 = 0, \xi_1 \neq 0, \alpha_1 \neq 0, \text{ then } F_1 \geq 0, \quad F_3 \neq 0 \\ (d) \text{ If } \beta_1 = 0, \alpha_1 \neq 0, \xi_1 = 0, \text{ then } F_1 \geq 0, \quad F_3 \neq 0 \end{cases} \tag{2.17}$$

where,

$$F_1 = \frac{\alpha_3}{\alpha_1} - \frac{\alpha_2^2}{2\alpha_1^2}, F_2 = \frac{\beta_2\xi_2}{\beta_1\xi_1} - \frac{\beta_3}{\beta_1} - \frac{{\xi_2}^2}{2{\xi_1}^2} \text{ and } F_3 = \beta_2\xi_2 - \beta_3\xi_1.$$

Theorem 2.10 (**Well-Posedness**). *Assume one of the conditions of (2.17) is satisfied. Let $T > 0$, $s \geq 0$ and $\eta > 0$ be given. For any $s-$compatible*

$$(\phi, \vec{h}) \in H^s(0,1) \times H^{\frac{s+1}{3}}(0,T) \times H^{\frac{s+1}{3}}(0,T) \times H^{\frac{s}{3}}(0,T))$$

with

$$\|(\phi, \vec{h})\|_{H^s(0,1)\times H^{\frac{s+1}{3}}(0,T)\times H^{\frac{s+1}{3}}(0,T)\times H^{\frac{s}{3}}(0,T)} \leq \eta,$$

there exists a $T^* \in (0, T]$ depending only on η such that the IBVP (2.14)-(2.16) admits a unique solution

$$u \in C([0,T^*]; H^s(0,1)) \cap L^2(0,T^*; H^{s+1}(0,1)).$$

Moreover, the solution map is real analytic in the corresponding spaces.

CHAPTER 3

Linear Estimates

In this chapter the following initial boundary value problem of the linear KdV equation posed on the finite interval $(0,1)$ is considered

$$\begin{cases} u_t + u_{xxx} = f, \quad u(x,0) = \phi(x), \quad x \in (0,1),\ t \geq 0, \\ B_1 u = h_1(t), \qquad B_2 u = h_2(t), \qquad B_3 u = h_3(t) \end{cases} \tag{3.1}$$

for various estimates and smoothing properties of its solutions.

In continuing Bubnov's work to study the general two-point boundary value problem with nonhomogeneous boundary conditions for its well-posedness in the space $H^s(0,1)$, consider the assumptions Bubnov [20] imposed on the parameters for the boundary conditions (mentioned in section 2.3 of chapter two), namely,

$$\begin{cases} (i.)\ \text{If } \alpha_1\beta_1\xi_1 \neq 0,\ \text{then } F_1 > 0, F_2 > 0 \\ (ii.)\ \text{If } \beta_1 \neq 0, \xi_1 \neq 0, \alpha_1 = 0,\ \text{then } F_2 > 0, \alpha_2 = 0, \alpha_3 \neq 0 \\ (iii.)\ \text{If } \beta_1 = 0, \xi_1 \neq 0, \alpha_1 \neq 0,\ \text{then } F_1 > 0, F_3 \neq 0 \\ (iv.)\ \text{If } \alpha_1 = \beta_1 = 0, \xi_1 \neq 0,\ \text{then } F_3 \neq 0, \alpha_2 = 0, \alpha_3 \neq 0 \\ (v.)\ \text{If } \beta_1 = 0, \alpha_1 \neq 0, \xi_1 = 0,\ \text{then } F_1 > 0, F_3 \neq 0 \\ (vi.)\ \text{If } \alpha_1 = \beta_1 = \xi_1 = 0,\ \text{then } F_3 \neq 0, \alpha_2 = 0, \alpha_3 \neq 0 \end{cases} \tag{3.2}$$

where,

$$F_1 = \frac{\alpha_3}{\alpha_1} - \frac{\alpha_2^2}{2\alpha_1^2}, F_2 = \frac{\beta_2\xi_2}{\beta_1\xi_1} - \frac{\beta_3}{\beta_1} - \frac{{\xi_2}^2}{2{\xi_1}^2} \text{ and } F_3 = \beta_2\xi_2 - \beta_3\xi_1.$$

For this research the IBVP (3.1) will be considered under the following assumptions imposed on the coefficients of the boundary conditions

$$\begin{cases} (a) \text{ If } \alpha_1\beta_1\xi_1 \neq 0, \text{ then } F_1 \geq 0, F_2 \geq 0 \\ (b) \text{ If } \beta_1 \neq 0, \xi_1 \neq 0, \alpha_1 = 0 \text{ then } F_2 \geq 0, \alpha_2 = 0, \alpha_3 \neq 0 \\ (c) \text{ If } \beta_1 = 0, \xi_1 \neq 0, \alpha_1 \neq 0, \text{ then } F_1 \geq 0, \quad F_3 \neq 0 \\ (d) \text{ If } \beta_1 = 0, \alpha_1 \neq 0, \xi_1 = 0, \text{ then } F_1 \geq 0, \quad F_3 \neq 0 \end{cases} \tag{3.3}$$

where,

$$F_1 = \frac{\alpha_3}{\alpha_1} - \frac{\alpha_2^2}{2\alpha_1^2}, F_2 = \frac{\beta_2\xi_2}{\beta_1\xi_1} - \frac{\beta_3}{\beta_1} - \frac{{\xi_2}^2}{2{\xi_1}^2} \text{ and } F_3 = \beta_2\xi_2 - \beta_3\xi_1.$$

The difference in the inequalities for (3.2) to (3.3) is due the difference in the equation under consideration. Bubnov considered $u_t + u_{xxx} + uu_x = f$ with generalized homogeneous boundary conditions, however in the research presented here the equation under consideration is $u_t + u_{xxx} = f$ with nonhomogeneous boundary conditions. In addition the cases (iv) and (vi) presented in (3.2) are shown below to be equivalent to the Dirichlet boundary conditions for the linear partial differential equation studied for this research.

First consider case (iv), where we have $\alpha_1 = \beta_1 = 0, \xi_1 \neq 0$. This reduces the boundary conditions of (3.1) to

$$\alpha_2 g_x(0) = h_1(t) - \alpha_3 g(0), \qquad g_x(1) = \frac{1}{\beta_2}h_2(t) - \frac{\beta_3}{\beta_2}g(1), \qquad g_x(1) = \frac{1}{\xi_1}h_3(t) - \frac{\xi_2}{\xi_1}g(1)$$

In this case we also have $F_3 \neq 0$ and $\alpha_2 \neq 0$ and $\alpha_3 = 0$, so the boundary conditions can be rewritten further resulting in

$$g_x(0) = \frac{1}{\alpha_2}h_1(t), \qquad g(1) = \frac{1}{\beta_2\xi_2 - \beta_3\xi_1}(\xi_1 h_2(t) + \beta_2 h_3(t)), \qquad g_x(1) = \frac{1}{\xi_1}h_3(t) - \frac{\xi_2}{\xi_1}g(1)$$

Now consider case (vi), we have $\alpha_1 = \beta_1 = \xi_1 = 0$ which gives the following boundary conditions

$$\alpha_2 g_x(0) = h_1(t) - \alpha_3 g(0), \qquad \beta_2 g_x(1) = h_2(t) - \beta_3 g(1), \qquad g(1) = h_3(t)$$

For (vi) we also have that $F_3 \neq 0$ and $\alpha_2 \neq 0$ and $\alpha_3 = 0$, so the boundary conditions become

$$g_x(0) = \frac{1}{\alpha_2} h_1(t), \qquad g_x(1) = \frac{1}{\beta_2} h_2(t) - \frac{\beta_3}{\beta_2} g(1), \qquad g(1) = h_3(t)$$

Therefore given cases (iv) and (vi) the boundary conditions can be rewritten in the following form:

$$g(0,t) = f_1(t), \quad g(1,t) = f_2(t), \quad g_x(1,t) = f_3(t)$$

whose corresponding well-posedness in $H^s(0,1)$ has been thoroughly studied as reviewed in chapter two.

The IBVP (3.1) will broken up for analysis. First the linear problem with homogeneous boundary conditions is considered, followed by the nonhomogeneous boundary value problem.

3.1. Homogeneous Boundary Conditions

Consideration in this section is given to the following problem:

$$\begin{cases} u_t + u_{xxx} = f, \quad u(x,0) = \phi(x), \quad x \in (0,1),\ t \geq 0, \\ B_1 u = 0, \qquad B_2 u = 0, \qquad B_3 u = 0. \end{cases} \tag{3.4}$$

Let A be the linear operator defined in the space $L^2(0,1)$ by

$$Ag = -g_{xxx}$$

with domain

$$\mathcal{D}(A) = \{g \in H^3(0,1) | \quad B_1 g = 0, B_2 g = 0 \text{ and } B_3 g = 0\}$$

Then we also have

$$A^* = -A$$

with $\mathcal{D}(A) = \mathcal{D}(A^*)$.

Lemma 3.1. *Assume that one of the conditions from (3.3) is satisfied, both operator A and A^* are dissipative, i.e., for any $g \in \mathcal{D}(A)$,*

$$\Big\langle Au, u \Big\rangle_{L^2(0,1)} \leq 0, \qquad \Big\langle A^*v, v \Big\rangle_{L^2(0,1)} \leq 0.$$

Proof. Consider $g \in \mathcal{D}(A)$ in $\Big\langle Ag, g \Big\rangle_{L^2(0,1)}$ using integration by parts we have,

$$\begin{aligned} \Big\langle Ag, g \Big\rangle_{L^2(0,1)} &= -\int_0^1 g_{xxx} g dx \\ &= -g_{xx} g\Big|_0^1 + \frac{1}{2}(g_x)^2\Big|_0^1 \\ &= -g_{xx}(1)g(1) + g_{xx}(0)g(0) + \frac{1}{2}g_x^2(1) - \frac{1}{2}g_x^2(0) \end{aligned} \tag{3.5}$$

Suppose case (a) from (3.3) is satisfied. That is, for the given boundary conditions $\alpha_1\beta_1\xi_1 \neq 0$. We can then rewrite the boundary conditions as

$$g_{xx}(0) = \frac{-\alpha_2}{\alpha_1} g_x(0) - \frac{\alpha_3}{\alpha_1} g(0), \qquad g_{xx}(1) = \Big(\frac{\beta_2\xi_2}{\beta_1\xi_1} - \frac{\beta_3}{\beta_1}\Big) g(1), \qquad g_x(1) = \frac{-\xi_2}{\xi_1} g(1) \tag{3.6}$$

Using the rewritten boundary conditions and Young's inequality where

$$-\frac{\alpha_2}{\alpha_1} g_x(0)g(0) \leq \frac{1}{2}g_x^2(0) + \frac{{\alpha_2}^2}{2{\alpha_1}^2} g^2(0) \tag{3.7}$$

we find that

$$\begin{aligned} \Big\langle Ag, g \Big\rangle_{L^2(0,1)} &= -g_{xx}(1)g(1) + g_{xx}(0)g(0) + \frac{1}{2}g_x^2(1) - \frac{1}{2}g_x^2(0) \\ &= -\Big(\frac{\beta_2\xi_2}{\beta_1\xi_1} - \frac{\beta_3}{\beta_1}\Big) g^2(1) - \frac{\alpha_2}{\alpha_1} g_x(0)g(0) - \frac{\alpha_3}{\alpha_1} g^2(0) + \frac{{\xi_2}^2}{2{\xi_1}^2} g^2(1) - \frac{1}{2}{g_x}^2(0) \\ &= -\Big(\frac{\beta_2\xi_2}{\beta_1\xi_1} - \frac{\beta_3}{\beta_1} - \frac{{\xi_2}^2}{2{\xi_1}^2}\Big) g^2(1) - \frac{\alpha_2}{\alpha_1} g_x(0)g(0) - \frac{\alpha_3}{\alpha_1} g^2(0) \\ &\leq -F_2 g^2(1) - F_1 g^2(0) \leq 0 \end{aligned}$$

as $F_1 \geq 0$ and $F_2 \geq 0$. Therefore the operator A is dissipative.

Next we need to consider the adjoint operator $A^*g = g_{xxx}$. Using integration by parts it can be determined that,

$$\Big\langle Ag, v\Big\rangle_{L^2(0,1)} = -g_{xx}v\Big|_0^1 + g_x v_x\Big|_0^1 - g v_{xx}\Big|_0^1 + \Big\langle g, A^*v\Big\rangle_{L^2(0,1)}$$

Given that (a) holds, we can then rewrite the boundary conditions and substitute to determine,

$$\begin{aligned} &- g_{xx}v\Big|_0^1 + g_x v_x\Big|_0^1 - g v_{xx}\Big|_0^1 \\ &= -\Big(\frac{\beta_2\xi_2}{\beta_1\xi_1} - \frac{\beta_3}{\beta_1}\Big)g(1)v(1) - \frac{\alpha_2}{\alpha_1}g_x(0)v(0) - \frac{\alpha_3}{\alpha_1}g(0)v(0) - \frac{\xi_2}{\xi_1}g(1)v_x(1) \\ &\qquad - g_x(0)v_x(0) - g(1)v_{xx}(1) + g(0)v_{xx}(0) \end{aligned} \tag{3.8}$$

Reorganizing the terms and setting equal to zero gives the following boundary conditions for the adjoint problem.

$$v_{xx}(1) = -\Big(\frac{\beta_2\xi_2}{\beta_1\xi_1} - \frac{\beta_3}{\beta_1}\Big)v(1) - \frac{\xi_2}{\xi_1}v_x(1), \qquad v_{xx}(0) = \frac{\alpha_3}{\alpha_1}v(0), \qquad v_x(0) = -\frac{\alpha_3}{\alpha_1}v(0) \tag{3.9}$$

Then for $v \in \mathcal{D}(A^*)$ we find that

$$\begin{aligned} \Big\langle A^*v, v\Big\rangle_{L^2(0,1)} &= \int_0^1 v_{xxx}v \, dx \\ &= v_{xx}v\Big|_0^1 - \frac{1}{2}v_x^2\Big|_0^1 \\ &= -\Big(\frac{\beta_2\xi_2}{\beta_1\xi_1} - \frac{\beta_3}{\beta_1}\Big)v^2(1) - \frac{\xi_2}{\xi_1}v_x(1)v(1) - \Big(\frac{\alpha_3}{\alpha_1} - \frac{{\alpha_2}^2}{2{\alpha_1}^2}\Big)v^2(0) - \frac{1}{2}v_x^2(1) \\ &\leq -\Big(\frac{\beta_2\xi_2}{\beta_1\xi_1} - \frac{\beta_3}{\beta_1} - \frac{{\xi_2}^2}{2{\xi_1}^2}\Big)v^2(1) - \Big(\frac{\alpha_3}{\alpha_1} - \frac{{\alpha_2}^2}{2{\alpha_1}^2}\Big)v^2(0) \\ &= -F_2v^2(1) - F_1v^2(0) \leq 0 \end{aligned}$$

where we have used the inequality,

$$-\frac{\xi_2}{\xi_1}v_x(1)v(1) \leq \frac{1}{2}v_x^2(1) + \frac{{\xi_2}^2}{2{\xi_1}^2}v^2(1) \tag{3.10}$$

and the assumptions for (a) that $F_1 \geq 0$ and $F_2 \geq 0$.

Therefore both the operator A and the adjoint A^* are dissipative for case (a).

Next consider case (b) of (3.3). If $\beta_1 \neq 0, \xi_1 \neq 0, \alpha_1 = 0$, the boundary conditions for (3.4) can be rewritten:

$$g_x(0) = -\frac{\alpha_3}{\alpha_2}g(0), \qquad g_{xx}(1) = \Big(\frac{\beta_2\xi_2}{\beta_1\xi_1} - \frac{\beta_3}{\beta_1}\Big)g(1), \qquad g_x(1) = \frac{-\xi_2}{\xi_1}g(1)$$

Then we have

$$\begin{aligned}\Big\langle Ag, g\Big\rangle_{L^2(0,1)} &= -g_{xx}g\Big|_0^1 + \frac{1}{2}(g_x)^2\Big|_0^1 \\ &= -g_{xx}(1)g(1) + g_{xx}(0)g(0) + \frac{1}{2}g_x^2(1) - \frac{1}{2}g_x^2(0) \\ &= -F_2g^2(1) + g_{xx}(0)g(0) + \Big(\frac{\alpha_3}{\alpha_2} + \frac{1}{2}\Big)g_x^2(0)\end{aligned}$$

Case (b) also requires $\alpha_2 = 0$ and $\alpha_3 \neq 0$. These conditions combined with the boundary condition $\alpha_2 g_x(0) = -\alpha_3 g(0)$ forces $g(0) = 0$. In addition we have that, $F_2 \geq 0$. Hence the operator A is dissipative.

For the adjoint boundary conditions we have, $-g_{xx}v\Big|_0^1 + g_xv_x\Big|_0^1 - gv_{xx}\Big|_0^1 = 0$, which leads to the following adjoint boundary conditions for case (b),

$$v(0) = 0, \qquad v_x(0) = 0, \qquad v_{xx}(1) + \frac{\xi_2}{\xi_1}v_x(1) + \Big(\frac{\beta_2\xi_2}{\beta_1\xi_1} - \frac{\beta_3}{\beta_1}\Big)v(1) = 0$$

Then

$$\begin{aligned}\left\langle A^*v, v\right\rangle_{L^2(0,1)} &= v_{xx}v\Big|_0^1 - \frac{1}{2}(v_x)^2\Big|_0^1 \\ &= -\frac{\xi_2}{\xi_1}v_x(1)v(1) - \Big(\frac{\beta_2\xi_2}{\beta_1\xi_1} - \frac{\beta_3}{\beta_1}\Big)v^2(1) - \frac{1}{2}v_x^2(1) \\ &\le -F_2v^2(1) \le 0\end{aligned}$$

where we have used inequality (3.10) and $F_2 \ge 0$. The adjoint operator is therefore dissipative.

For cases (c) and (d) of (3.3), the boundary conditions for (3.4) can be rewritten to the same result. First consider case (c), where we have $\beta_1 = 0, \xi_1 \neq 0, \alpha_1 \neq 0$. This reduces the boundary conditions to

$$g_{xx}(0) = \frac{-\alpha_2}{\alpha_1}g_x(0) - \frac{\alpha_3}{\alpha_1}g(0), \qquad g_x(1) = -\frac{\beta_3}{\beta_2}g(1), \qquad g_x(1) = -\frac{\xi_2}{\xi_1}g(1)$$

In order to ensure there are three distinct boundary conditions, it is required in case (c) that $F_3 = \beta_2\xi_2 - \beta_3\xi_1 \neq 0$. Looking at the boundary conditions $g_x(1) = -\frac{\beta_3}{\beta_2}g(1)$ and $g_x(1) = -\frac{\xi_2}{\xi_1}g(1)$ we see that $(\beta_2\xi_2 - \beta_3\xi_1)g(1) = 0$. As a result $g_x(1) = g(1) = 0$ since $F_3 \neq 0$. The argument is similar for case (d).

The assumptions $\beta_1 = 0, \alpha_1 \neq 0, \xi_1 = 0$ for (d), give the following

$$g_{xx}(0) = -\frac{\alpha_2}{\alpha_1}g_x(0) - \frac{\alpha_3}{\alpha_1}g(0), \qquad g_x(1) = -\frac{\beta_3}{\beta_2}g(1), \qquad \xi_2 g(1) = 0$$

Again to ensure there are three distinct boundary conditions we require $F_3 \neq 0$. This means that $\xi_2 \neq 0$. Therefore the resulting boundary conditions are the same as case (c).

$$g_{xx}(0) = \frac{-\alpha_2}{\alpha_1}g_x(0) - \frac{\alpha_3}{\alpha_1}g(0), \qquad g_x(1) = 0, \qquad g(1) = 0$$

Using these boundary conditions we can prove that A and A^* are dissipative.

$$\Big\langle Ag, g \Big\rangle_{L^2(0,1)} = -g_{xx}g\Big|_0^1 + \frac{1}{2}(g_x)^2\Big|_0^1$$
$$= -\frac{\alpha_2}{\alpha_1}g_x(0)g(0) - \frac{\alpha_3}{\alpha_1}g^2(0) - \frac{1}{2}g_x^2(0)$$

given the inequality (3.7) we have that A is dissipative,

$$\Big\langle Ag, g \Big\rangle_{L^2(0,1)} \leq -F_1 g^2(0) \leq 0$$

since $F_1 \geq 0$.

For the adjoint boundary conditions again consider, $-g_{xx}v\Big|_0^1 + g_x v_x\Big|_0^1 - g v_{xx}\Big|_0^1$. The adjoint boundary conditions assuming the conditions for case (c) or (d) are

$$v(1) = 0, \qquad v_x(0) = -\frac{\alpha_2}{\alpha_1}v(0), \qquad v_{xx}(0) = \Big(\frac{\alpha_3}{\alpha_1}\Big)v(0)$$

Therefore for the adjoint we have

$$\Big\langle A^* v, v \Big\rangle_{L^2(0,1)} = v_{xx}v\Big|_0^1 - \frac{1}{2}v_x^2\Big|_0^1$$

$$= -\Big(\frac{\alpha_3}{\alpha_1} - \frac{{\alpha_2}^2}{2{\alpha_1}^2}\Big)v^2(0) - \frac{1}{2}v_x^2(1)$$

since $F_1 \geq 0$ then $\Big\langle A^* v, v \Big\rangle_{L^2(0,1)} \leq 0$.
The proof is complete.

□

The following corollary then follows from the standard semi-group theory [58].

Corollary 3.2. *The operator A is the infinitesimal generator of a C_0 semigroup $W_0(t)$ in the space $L^2(0,1)$. Furthermore, for given $\phi \in L^2(0,1)$ and $f \in L^1_{loc}(R^+; L^2(0,1))$, there* *is a unique solution $u \in C(R^+; L^2(0,1))$ to (3.4) which can be written in the form of .*

$$u(x,t) = W_0(t)\phi(x) + \int_0^t W(t-\tau)f(\tau)d\tau. \tag{3.11}$$

Next we demonstrate the solution u of (3.4) also possesses a global Kato smoothing property.

Proposition 3.3. *Let $T > 0$ be given and assume one of the conditions of (3.3) is satisfied. There exists a constant $C > 0$ such that for any $\phi \in L^2(0,1)$ and $f \in L^1(0,T; L^2(0,1))$, the corresponding solution $u \in C([0,T]; L^2(0,1))$ of the IBVP (3.4) also belongs to the space $L^2(0,T; H^1(0,1))$ and satisfies*

$$\sup_{0\le t\le T} \|u(\cdot,t)\|_{L^2(0,1)} + \|u(\cdot,t)\|_{L^2(0,T;H^1(0,1))} \le C\left(\|\phi\|_{L^2(0,1)} + \|f\|_{L^1(0,T;L^2(0,1))}\right)$$

Proof. Without loss of generality we may assume that $\phi \in \mathcal{D}(A)$ and $f \in L^2(0,T; \mathcal{D}(A))$. From corollary (3.2)the solution u then belongs to the space $C([0,T]; \mathcal{D}(A)) \cap C^1(0,T; L^2(0,1))$. Using the fact that we have u in the form (3.11) consider problem (3.4).

Multiply both sides of the differential equation in (3.4) by $ue^{\lambda x}$, with $0 \le \lambda \le \frac{1}{2}$,and integrate over $(0,1)$.

$$\int_0^1 (u_t u e^{\lambda x} + u_{xxx} u e^{\lambda x})dx = \int_0^1 f u e^{\lambda x} dx$$

Consider the terms on the left side of the equation,

$$\int_0^1 u_t u e^{\lambda x}\, dx = \frac{1}{2}\frac{d}{dt}\int_0^1 u^2 e^{\lambda x}\, dx$$

and using integration by parts,

$$\int_0^1 u_{xxx}ue^{\lambda x}\ dx = u_{xx}ue^{\lambda x}\Big|_0^1 - \frac{1}{2}{u_x}^2e^{\lambda x}\Big|_0^1 + \frac{\lambda^2}{2}u^2e^{\lambda x}\Big|_0^1 - \lambda u_x ue^{\lambda x}\Big|_0^1 + \frac{3\lambda}{2}\int_0^1 u_x^2e^{\lambda x}dx - \frac{\lambda^3}{2}\int_0^1 u^2e^{\lambda x}\ dx$$

Rewriting the original equation gives,

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_0^1 u^2e^{\lambda x}\ dx + u_{xx}ue^{\lambda x}\Big|_0^1 - \frac{1}{2}{u_x}^2e^{\lambda x}\Big|_0^1 + \frac{\lambda^2}{2}u^2e^{\lambda x}\Big|_0^1 \\ - \lambda u_x ue^{\lambda x}\Big|_0^1 + \frac{3\lambda}{2}\int_0^1 u_x^2e^{\lambda x}dx - \frac{\lambda^3}{2}\int_0^1 u^2e^{\lambda x}\ dx = \int_0^1 fue^{\lambda x}dx\end{aligned} \tag{3.12}$$

Suppose that (a) of (3.3) is satisfied. Since $\alpha_1\beta_1\xi_1 \neq 0$, (3.12) can be rewritten using the boundary conditions as

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_0^1 u^2e^{\lambda x}\ dx + \frac{3\lambda}{2}\int_0^1 u_x^2e^{\lambda x}dx - \frac{\lambda^3}{2}\int_0^1 u^2e^{\lambda x}\ dx + \left[F_2 + \frac{\lambda\xi_2}{\xi_1} + \frac{\lambda^2}{2}\right]e^{\lambda}u^2(1) \\ + \left[\frac{\alpha_2}{\alpha_1} + \lambda\right]u_x(0)u(0) + \left[\frac{\alpha_3}{\alpha_1} - \frac{\lambda^2}{2}\right]u^2(0) + \frac{1}{2}u_x^2(0) = \int_0^1 fue^{\lambda x}dx\end{aligned} \tag{3.13}$$

From the inequality above consider $\int_0^1 u^2e^{\lambda x}dx$. For the following recall $x \in (0, 1)$

$$\begin{aligned}
u^2(x)e^{\lambda x} - u^2(0) &= \int_0^x (u^2 e^{\lambda x})_x dx \\
&= \int_0^x (2uu_x e^{\lambda x} + \lambda u^2 e^{\lambda x}) dx \\
&= 2\int_0^x uu_x e^{\lambda x} dx + \lambda \int_0^x u^2 e^{\lambda x} dx \\
&\leq 2\int_0^x |u||u_x| e^{\lambda x} dx + \lambda \int_0^x u^2 e^{\lambda x} dx \\
&\leq 2\int_0^1 |u||u_x| e^{\lambda x} dx + \lambda \int_0^1 u^2 e^{\lambda x} dx
\end{aligned}$$

Integrating both sides over $(0, 1)$,

$$\begin{aligned}
\int_0^1 u^2(x)e^{\lambda x} dx - u^2(0) &\leq \int_0^1 \Big(2\int_0^1 |u||u_x| e^{\lambda x} dx + \lambda \int_0^1 u^2 e^{\lambda x} dx\Big) dx \\
&\leq 2\int_0^1 |u||u_x| e^{\lambda x} dx + \lambda \int_0^1 u^2 e^{\lambda x} dx
\end{aligned}$$

Rearranging the terms and using Young's inequality $|u||u_x|e^{\lambda x} \leq \varepsilon u^2 e^{\lambda x} + \frac{1}{\varepsilon} {u_x}^2 e^{\lambda x}$, where $\varepsilon > 0$

$$\begin{aligned}
(1-\lambda)\int_0^1 u^2 e^{\lambda x} dx &\leq 2\int_0^1 |u||u_x| e^{\lambda x} dx + u^2(0) \\
&\leq \varepsilon \int_0^1 u^2 e^{\lambda x} dx + \frac{1}{\varepsilon}\int_0^1 u_x^2 e^{\lambda x} + u^2(0)
\end{aligned}$$

Therefore,

$$(1-\lambda-\varepsilon)\int_0^1 u^2 e^{\lambda x}dx \leq \frac{1}{\varepsilon}\int_0^1 u_x^2 e^{\lambda x} + u^2(0)$$

One can choose λ and ε such that $1-\lambda-\varepsilon > 0$, which gives

$$\int_0^1 u^2 e^{\lambda x}dx \leq \frac{1}{\varepsilon(1-\lambda-\varepsilon)}\int_0^1 u_x^2 e^{\lambda x} + \frac{1}{1-\lambda-\varepsilon}u^2(0)$$

Then given $0 \leq \lambda \leq \frac{1}{2}$,

$$-\Big(\frac{\lambda^3}{2}\Big)\Big(\frac{1}{\varepsilon(1-\lambda-\varepsilon)}\Big)\int_0^1 u_x^2 e^{\lambda x}dx - \Big(\frac{\lambda^3}{2}\Big)\Big(\frac{1}{1-\lambda-\varepsilon}\Big)u^2(0) \leq -\Big(\frac{\lambda^3}{2}\Big)\int_0^1 u^2 e^{\lambda x}dx \quad (3.14)$$

With this result and the following inequality,

$$-\frac{1}{2}u_x^2(0) - \frac{1}{2}\Big[\frac{\alpha_2}{\alpha_1}+\lambda\Big]^2 u^2(0) \leq \Big[\frac{\alpha_2}{\alpha_1}+\lambda\Big]u_x(0)u(0)$$

We can now rewrite (3.13) as follows,

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_0^1 u^2 e^{\lambda x}dx + \Big[\frac{3\lambda}{2} - \frac{\lambda^3}{2\varepsilon(1-\lambda-\varepsilon)}\Big]\int_0^1 u_x^2 e^{\lambda x}dx + e^{\lambda}\Big[F_2 + \frac{\lambda\xi_2}{\xi_1} + \frac{\lambda^2}{2}\Big]u^2(1) \\ + \Big[\frac{\alpha_3}{\alpha_1} - \frac{\lambda^2}{2} - \frac{\lambda^3}{2(1-\lambda-\varepsilon)} - \frac{1}{2}\Big[\frac{\alpha_2}{\alpha_1}+\lambda\Big]^2\Big]u^2(0) \leq \int_0^1 fue^{\lambda x}dx\end{aligned} \quad (3.15)$$

For (a) we have that $F_1 \geq 0$and $F_2 \geq 0$. We can choose $0 \leq \lambda \leq \frac{1}{2}$ and $\varepsilon \geq 0$ such that

$$\tfrac{3\lambda}{2} - \tfrac{\lambda^3}{2\varepsilon(1-\lambda-\varepsilon)} \geq 0, \qquad F_2 + \tfrac{\lambda\xi_2}{\xi_1} + \tfrac{\lambda^2}{2} \geq 0, \qquad \tfrac{\alpha_3}{\alpha_1} - \tfrac{\lambda^2}{2} - + \tfrac{\lambda^3}{2(1-\lambda-\varepsilon)} - \tfrac{1}{2}\Big[\tfrac{\alpha_2}{\alpha_1}+\lambda\Big]^2 \geq 0$$

Let $C_0 = \frac{3\lambda}{2} - \frac{\lambda^3}{2\varepsilon(1-\lambda-\varepsilon)}$, the inequality becomes,

$$\frac{1}{2}\frac{d}{dt}\int_0^1 u^2 e^{\lambda x}dx + C_0\int_0^1 u_x^2 e^{\lambda x}dx \leq \int_0^1 fue^{\lambda x}dx \quad (3.16)$$

Multiplying by two and using $C_0 \geq 0$ we have

$$\frac{d}{dt}\int_0^1 u^2 e^{\lambda x}dx \leq 2\int_0^1 f u e^{\lambda x}dx$$

$$\leq 2\int_0^1 \Big(\frac{1}{2}f^2 e^{\lambda x} + \frac{1}{2}u^2 e^{\lambda x}\Big)dx$$

$$\leq \int_0^1 f^2 e^{\lambda x}dx + \int_0^1 u^2 e^{\lambda x}dx$$

By Gronwall's Inequality we have

$$\int_0^1 u^2 e^{\lambda x}dx \leq e^t\Big[\int_0^1 \Big(u(x,0)\Big)^2 e^{\lambda x}dx + \int_0^t\int_0^1 f^2 e^{\lambda x}dxd\tau\Big]$$

$$\leq e^t\int_0^1 \phi^2 e^{\lambda x}\ dx + e^t\int_0^t\int_0^1 f^2 e^{\lambda x}dxd\tau$$

With $\phi \in L^2(0,1), f \in H^1([0,T];L_2(0,1))$ and $e^{\lambda x}$ bounded when $0 \leq \lambda \leq \frac{1}{2}$, $x \in (0,1)$ we find that

$$\int_0^1 u^2 dx \leq \int_0^1 u^2 e^{\lambda x}dx \leq e^t\int_0^1 \phi^2 e^{\lambda x}\ dx + \int_0^t\int_0^1 f^2 e^{\lambda x}dxd\tau$$

with,

$$\sup_{0\leq t\leq T}\int_0^1 u^2 dx \leq e^T\int_0^1 \phi^2 e^{\lambda x}\ dx + e^T\int_0^T\int_0^1 f^2 e^{\lambda x}dxd\tau$$

$$\leq M\int_0^1 \phi^2\ dx + M\int_0^T\int_0^1 f^2 dxd\tau$$

Therefore,

$$\sup_{0\le t\le T}\|u\|_{L^2(0,1)} \le M\int_0^1 \phi^2\,dx + M\int_0^T\int_0^1 f^2 dx d\tau \tag{3.17}$$

From inequality (3.16) will now consider

$$C_0\int_0^1 u_x^2 e^{\lambda x}dx \le \int_0^1 fue^{\lambda x}dx$$

Combing the properties that $C_0 > 0$, $e^{\lambda x}$ is bounded when $0 \le \lambda \le \frac{1}{2}$ and $x \in [0,1]$, we find that

$$\begin{aligned}
\int_0^1 u_x^2 dx \le \int_0^1 u_x^2 e^{\lambda x}dx &\le \frac{1}{C_0}\int_0^1 fue^{\lambda x}dx \\
&\le \frac{1}{C_0}\Big(\int_0^1 f^2 dx \int_0^1 \left(ue^{\lambda x}\right)^2 dx\Big) \\
&\le \frac{1}{C_0}\Big(\frac{1}{2}\int_0^1 f^2 e^{\lambda x}dx + \frac{1}{2}\int_0^1 u^2 e^{\lambda x}dx\Big) \\
&\le \frac{M}{2C_0}\Big(\int_0^1 f^2 dx + \int_0^1 u^2 dx\Big) \\
&\le \frac{M}{2C_0}\Big(\int_0^1 f^2 dx + M\int_0^1 \phi^2\,dx + M\int_0^T\int_0^1 f^2 dx d\tau\Big) \\
&\le \frac{M^2}{2C_0}\int_0^1 \phi^2\,dx + \frac{M}{2C_0}\int_0^1 f^2 dx + \frac{M^2}{2C_0}\int_0^T\int_0^1 f^2 dx d\tau
\end{aligned}$$

Therefore,

$$\|u_x\|^2_{L^2(0,T,L^2(0,1))} \le \frac{M^2}{2C_0}\int_0^1 \phi^2\,dx + \frac{M}{2C_0}\int_0^1 f^2 dx + \frac{M^2}{2C_0}\int_0^T\int_0^1 f^2 dx d\tau \tag{3.18}$$

Combining (3.17) and (3.18) we have,

$$
\begin{aligned}
&\sup_{0\le t\le T} \|u\|_{L^2(0,1)} + \|u_x\|^2_{L^2(0,T,L^2(0,1))} \\
&\le M\int_0^1 \phi^2\, dx + M\int_0^T\int_0^1 f^2 dx d\tau + \frac{M^2}{2C_0}\int_0^1 \phi^2\, dx + \frac{M}{2C_0}\int_0^1 f^2 dx + \frac{M^2}{2C_0}\int_0^T\int_0^1 f^2 dx d\tau \\
&\le \Big(M + \frac{M^2}{2C_0}\Big)\int_0^1 \phi^2\, dx + \frac{M}{2C_0}\int_0^1 f^2 dx + \Big(M + \frac{M^2}{2C_0}\Big)\int_0^T\int_0^1 f^2 dx d\tau \\
&\le C_1\|\phi\|_{L^2(0,1)} + C_2\|f\|_{L^2(0,T;L^2(0,1))}
\end{aligned} \tag{3.19}
$$

This proves the Proposition for case (a).

Starting with (3.12) assume that case (b) of (3.3) is satisfied. Using the boundary conditions with the assumptions that $\beta_1 \ne 0$, $\xi_1 \ne 0$, $\alpha_1 = 0$, $\alpha_2 = 0$, and $\alpha_3 \ne 0$ we have that $u(0) = 0$ (as shown in the proof of Lemma (3.1)). with these boundary conditions and inequality (3.14), (3.12) can be rewritten as:

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_0^1 u^2 e^{\lambda x}\, dx + \Big[\frac{3\lambda}{2} - \frac{\lambda^3}{2\varepsilon(1-\lambda-\varepsilon)}\Big]\int_0^1 u_x^2 e^{\lambda x} dx \\
&\qquad + \Big[F_2 + \frac{\lambda\xi_2}{\xi_1} + \frac{\lambda^2}{2}\Big] e^{\lambda} u^2(1) + \frac{1}{2}u_x^2(0) \le \int_0^1 f u e^{\lambda x} dx
\end{aligned}
$$

where $\varepsilon \ge 0$. As with case (a) we can choose $0 \le \lambda \le \frac{1}{2}$ and $\varepsilon \ge 0$ such that $C_0 = \frac{3\lambda}{2} - \frac{\lambda^3}{2\varepsilon(1-\lambda-\varepsilon)} \ge 0$, thereby reducing the above inequality to (3.16).

The remainder of the proof is the same as for case (a).

Lastly we consider cases (c) and (d) together. Recall that the boundary conditions determined for cases (c) and (d) reduced to the same result. These boundary conditions along with the inequalities (3.7) and (3.14) allow (3.12) to become:

$$\frac{1}{2}\frac{d}{dt}\int_0^1 u^2 e^{\lambda x}dx + \left[\frac{3\lambda}{2} - \frac{\lambda^3}{2\varepsilon(1-\lambda-\varepsilon)}\right]\int_0^1 u_x^2 e^{\lambda x}dx$$

$$+\left[\frac{\alpha_3}{\alpha_1} - \frac{\lambda^2}{2} - +\frac{\lambda^3}{2(1-\lambda-\varepsilon)} - \frac{1}{2}\left[\frac{\alpha_2}{\alpha_1}+\lambda\right]^2\right]u^2(0) \leq \int_0^1 fue^{\lambda x}dx$$

Furthermore with $F_1 \geq 0$ for both (c) and (d) it is clear that $\frac{\alpha_3}{\alpha_1} \geq 0$. Then it is possible to choose $0 \leq \lambda \leq \frac{1}{2}$ and $\varepsilon \geq 0$ such that $C_0 = \frac{3\lambda}{2} - \frac{\lambda^3}{2\varepsilon(1-\lambda-\varepsilon)} \geq 0$ and $\left[\frac{\alpha_3}{\alpha_1} - \frac{\lambda^2}{2} - \frac{\lambda^3}{2(1-\lambda-\varepsilon)} - \frac{1}{2}\left[\frac{\alpha_2}{\alpha_1}+\lambda\right]^2\right] \geq 0$. Thereby reducing the inequality to (3.16).

The remainder of the proof is the same as for case (a), so the proposition holds in all cases. □

3.2. Nonhomogeneous Boundary Value Problem

In this subsection we turn to consider the following nonhomogeneous boundary value problem of the linear KdV equation.

$$\begin{cases} u_t + u_{xxx} = 0, \quad u(x,0) = 0, \\ B_1 u = h_1(t), \quad B_2 u = h_2(t), \quad B_3 u = h_3(t) \end{cases} \tag{3.20}$$

Using the Laplace transform an explicit solution formula for (3.20) will be determined in terms of the boundary values. Applying the Laplace transform with respect to t, (3.20) is converted to

$$\begin{cases} s\hat{u}(x,s) + \hat{u}_{xxx}(x,s) = 0, \\ B_1\hat{u} = \hat{h}_1(s), \quad B_2\hat{u} = \hat{h}_2(s), \quad B_3\hat{u} = \hat{h}_3(s) \end{cases} \tag{3.21}$$

where

$$\hat{u}(x,s) = \int_0^\infty e^{st}u(x,t)dt, \quad \hat{h}_j(s) = \int_0^\infty e^{st}h_j(t)dt \text{ for } j = 1,2,3.$$

The solution of (3.21) can be written in the form

$$\hat{u}(x,s) = c_1(s)e^{\lambda_1(s)x} + c_2(s)e^{\lambda_2(s)x} + c_3(s)e^{\lambda_3(s)x} = \sum_{j=1}^{3} c_j(s)e^{\lambda_j(s)x}$$

where $\lambda_j = \lambda_j(s)$, j=1,2,3, are the solutions for the characteristic equation

$$\lambda^3 + s = 0$$

For the $c_j = c_j(s)$, j=1,2,3, are constants depending only on s. Notice that

$$\hat{u}_x(x,s) = \sum_{j=1}^{3} c_j\lambda_j e^{\lambda_j x}, \text{ and } \hat{u}_{xx}(x,s) = \sum_{j=1}^{3} c_j{\lambda_j}^2 e^{\lambda_j x}$$

The c_j, j=1,2,3, then solve the system

$$\begin{cases} \alpha_1 \sum_{j=1}^{3} c_j{\lambda_j}^2 + \alpha_2 \sum_{j=1}^{3} c_j\lambda_j + \alpha_3 \sum_{j=1}^{3} c_j = \hat{h}_1(s), \\ \beta_1 \sum_{j=1}^{3} c_j{\lambda_j}^2 e^{\lambda_j} + \beta_2 \sum_{j=1}^{3} c_j\lambda_j e^{\lambda_j} + \beta_3 \sum_{j=1}^{3} c_j e^{\lambda_j} = \hat{h}_2(s) \\ \xi_1 \sum_{j=1}^{3} c_j\lambda_j e^{\lambda_j} + \xi_2 \sum_{j=1}^{3} c_j = \hat{h}_3(s) \end{cases}$$

Rewriting the system we have

$$\begin{cases} c_1\Big(\alpha_1{\lambda_1}^2 + \alpha_2\lambda_1 + \alpha_3\Big) + c_2\Big(\alpha_1{\lambda_2}^2 + \alpha_2\lambda_2 + \alpha_3\Big) + c_3\Big(\alpha_1{\lambda_3}^2 + \alpha_2\lambda_3 + \alpha_3\Big) = \hat{h}_1(s), \\ c_1e^{\lambda_1}\Big(\beta_1{\lambda_1}^2 + \beta_2\lambda_1 + \beta_3\Big) + c_2e^{\lambda_2}\Big(\beta_1{\lambda_2}^2 + \beta_2\lambda_2 + \beta_3\Big) \\ \qquad\qquad\qquad + c_3e^{\lambda_3}\Big(\beta_1{\lambda_3}^2 + \beta_2\lambda_3 + \beta_3\Big) = \hat{h}_2(s), \\ c_1e^{\lambda_1}\Big(\xi_1\lambda_1 + \xi_2\Big) + c_2e^{\lambda_2}\Big(\xi_1\lambda_2 + \xi_2\Big) + c_3e^{\lambda_3}\Big(\xi_1\lambda_3 + \xi_2\Big) = \hat{h}_3(s), \end{cases}$$

Let $\Delta(s)$ be the determinant of the coefficient matrix and $\Delta_j(s)$ be the determinants of the matrices that are obtained by replacing the jth-column of $\Delta(s)$ by the column vector $\langle \hat{h}_1(s), \hat{h}_2(s), \hat{h}_3(s)\rangle^T$, $j = 1,2,3$ Cramer's rule implies that

$$c_j = \frac{\Delta_j(s)}{\Delta(s)}, \quad j = 1,2,3$$

for $\Delta(s) \neq 0$. For $\Delta(s) = 0$ we can look at homogeneous boundary conditions. Recall the linear operator A,

$$Ag = -g_{xxx}, \qquad \mathcal{D}(A) = \{g \in H^3(0,1) | B_1 g = 0, B_2 g = 0 \text{ and } B_3 g = 0\}$$

Consider

$$Av = \lambda v, \qquad v \in \mathcal{D}(A)$$

then we have

$$\begin{cases} -v''' = \lambda v \\ B_1 v = 0, \quad B_2 v = 0, \quad B_3 v = 0 \end{cases}$$

The solutions μ_1,μ_2 and μ_3 for the characteristic equation $-\mu^3 = \lambda$ lead to the following

$$v = c_1(\lambda)e^{\mu_1(\lambda)x} + c_2(\lambda)e^{\mu_2(\lambda)x} + c_3(\lambda)e^{\mu_3(\lambda)x}$$

Since $B_1 v = 0$, $B_2 v = 0$, and $B_3 v = 0$, we have the following system of linear equations

$$\begin{cases} c_1\Big(\alpha_1{\mu_1}^2 + \alpha_2\mu_1 + \alpha_3\Big) + c_2\Big(\alpha_1{\mu_2}^2 + \alpha_2\mu_2 + \alpha_3\Big) + c_3\Big(\alpha_1{\mu_3}^2 + \alpha_2\mu_3 + \alpha_3\Big) = 0, \\ c_1e^{\mu_1}\Big(\beta_1{\mu_1}^2 + \beta_2\mu_1 + \beta_3\Big) + c_2e^{\mu_2}\Big(\beta_1{\mu_2}^2 + \beta_2\mu_2 + \beta_3\Big) \\ \qquad\qquad\qquad\qquad + c_3e^{\mu_3}\Big(\beta_1{\mu_3}^2 + \beta_2\mu_3 + \beta_3\Big) = 0, \\ c_1e^{\mu_1}\Big(\xi_1\mu_1 + \xi_2\Big) + c_2e^{\mu_2}\Big(\xi_1\mu_2 + \xi_2\Big) + c_3e^{\mu_3}\Big(\xi_1\mu_3 + \xi_2\Big) = 0, \end{cases}$$

Let $\Delta(\lambda) = \Delta(s)$ be the determinant of the coefficient matrix. Hence, there will be a nontrivial solution v, if and only if $\Delta(\lambda) = 0$.

The following proposition has now been proven.

Proposition 3.4. *λ is an eigenvalue of A if and only if $\Delta(\lambda) = 0$.*

Next we look more closely at the possible eigenvalues of A.

Proposition 3.5. *Suppose that one of the conditions from (3.3) holds. If $Re(\lambda) > 0$, then λ is not an eigenvalue of A.*

Proof. Recall that the operator A has been shown to be dissipative if one of the conditions of (3.3) holds, $\left\langle Av, v\right\rangle_{L^2(0,1)} \leq 0$. Suppose there is an eigenvalue for A such that $Re(\lambda) > 0$. Then

$$\begin{aligned} Re\Big(\left\langle Av, v\right\rangle_{L^2(0,1)}\Big) &= Re\Big(\left\langle \lambda v, v\right\rangle_{L^2(0,1)}\Big) \\ &= Re\Big(\lambda\left\langle v, v\right\rangle_{L^2(0,1)}\Big) \geq 0 \end{aligned}$$

This contradicts the fact that A is dissipative. □

Consider

$$Av = \lambda v, \qquad v \in \mathcal{D}(A)$$

with $\lambda = i\rho$, $\rho \in R$. Then we have

$$\begin{cases} -v''' = i\rho v \\ B_1 v = 0, \quad B_2 v = 0, \quad B_3 v = 0 \end{cases}$$

The solutions for the characteristic equation $-\mu^3 = i\rho$, give the following

$$v = c_1(\rho)e^{\mu_1(\rho)x} + c_2(\rho)e^{\mu_2(\rho)x} + c_3(\rho)e^{\mu_3(\rho)x}$$

Since $B_1 v = 0$, $B_2 v = 0$, and $B_3 v = 0$, we have the following system of linear equations

$$\begin{cases} c_1\Big(\alpha_1{\mu_1}^2 + \alpha_2\mu_1 + \alpha_3\Big) + c_2\Big(\alpha_1{\mu_2}^2 + \alpha_2\mu_2 + \alpha_3\Big) + c_3\Big(\alpha_1{\mu_3}^2 + \alpha_2\mu_3 + \alpha_3\Big) = 0, \\ c_1 e^{\mu_1}\Big(\beta_1{\mu_1}^2 + \beta_2\mu_1 + \beta_3\Big) + c_2 e^{\mu_2}\Big(\beta_1{\mu_2}^2 + \beta_2\mu_2 + \beta_3\Big) \\ \qquad\qquad + c_3 e^{\mu_3}\Big(\beta_1{\mu_3}^2 + \beta_2\mu_3 + \beta_3\Big) = 0, \\ c_1 e^{\mu_1}\Big(\xi_1\mu_1 + \xi_2\Big) + c_2 e^{\mu_2}\Big(\xi_1\mu_2 + \xi_2\Big) + c_3 e^{\mu_3}\Big(\xi_1\mu_3 + \xi_2\Big) = 0, \end{cases}$$

Let $\Delta(\rho)$ be the determinant of the coefficient matrix. Given the solutions for the characteristic equation are

$$\mu_1 = i\rho^{\frac{1}{3}}, \quad \mu_2 = -\frac{1}{2}\rho^{\frac{1}{3}}(i + \sqrt{3}), \quad \mu_3 = \frac{1}{2}\rho^{\frac{1}{3}}(-i + \sqrt{3})$$

we can rearranging the terms for $\Delta(\rho)$ into the real and imaginary parts by powers of ρ. For $Re(\Delta(\rho))$ we have

$$
\begin{aligned}
&\rho^{\frac{5}{3}}\alpha_1\beta_1\xi_1(-\sqrt{3}e^{-\mu_1}-\frac{\sqrt{3}}{2}e^{-\mu_2}+\frac{\sqrt{3}}{2}e^{-\mu_3})\\
&+\rho^{\frac{4}{3}}[\alpha_2\beta_1\xi_1(\frac{3}{2}e^{-\mu_2}+\frac{3}{2}e^{-\mu_3})-\alpha_1\beta_1\xi_2(\frac{3}{2}e^{-\mu_2}+\frac{3}{2}e^{-\mu_3})]\\
&+\rho[\alpha_3\beta_1\xi_1(\sqrt{3}e^{-\mu_1}-\sqrt{3}e^{-\mu_2}+\sqrt{3}e^{-\mu_3})+\alpha_1\beta_3\xi_1(\sqrt{3}e^{-\mu_1}-\sqrt{3}e^{-\mu_2}+\sqrt{3}e^{-\mu_3})\\
&\qquad-\alpha_2\beta_1\xi_2(\sqrt{3}e^{-\mu_1}-\frac{\sqrt{3}}{2}e^{-\mu_2}+\frac{\sqrt{3}}{2}e^{-\mu_3})+\alpha_1\beta_2\xi_2(\sqrt{3}e^{-\mu_1}-\sqrt{3}e^{-\mu_2}+\sqrt{3}e^{-\mu_3})]\\
&+\rho^{\frac{2}{3}}[\alpha_2\beta_3\xi_1(\frac{3}{2}e^{-\mu_2}+\frac{3}{2}e^{-\mu_3})-\alpha_3\beta_1\xi_2(\frac{3}{2}e^{-\mu_2}+\frac{3}{2}e^{-\mu_3})+\alpha_2\beta_2\xi_2(\frac{3}{2}e^{-\mu_2}+\frac{3}{2}e^{-\mu_3})]\\
&+\rho^{\frac{1}{3}}[\alpha_3\beta_3\xi_1(-\sqrt{3}e^{-\mu_1}-\frac{\sqrt{3}}{2}e^{-\mu_2}+\frac{\sqrt{3}}{2}e^{-\mu_3})-\alpha_3\beta_2\xi_2(\sqrt{3}e^{-\mu_1}+\frac{\sqrt{3}}{2}e^{-\mu_2}-\frac{\sqrt{3}}{2}e^{-\mu_3})]
\end{aligned}
$$

For $Im(\Delta(\rho))$

$$
\begin{aligned}
&\rho^{\frac{5}{3}}\alpha_1\beta_1\xi_1(-\frac{3}{2}e^{-\mu_2}-\frac{3}{2}e^{-\mu_3})\\
&+\rho^{\frac{4}{3}}[\alpha_2\beta_1\xi_1(\sqrt{3}e^{-\mu_1}+\frac{\sqrt{3}}{2}e^{-\mu_2}-\frac{\sqrt{3}}{2}e^{-\mu_3})-\alpha_1\beta_1\xi_2(\sqrt{3}e^{-\mu_1}+\frac{\sqrt{3}}{2}e^{-\mu_2}-\frac{\sqrt{3}}{2}e^{-\mu_3})]\\
&+\rho^{\frac{2}{3}}[\alpha_2\beta_3\xi_1(-\sqrt{3}e^{-\mu_1}-\frac{\sqrt{3}}{2}e^{-\mu_2}+\frac{\sqrt{3}}{2}e^{-\mu_3})+\alpha_3\beta_1\xi_2(\sqrt{3}e^{-\mu_1}+\frac{\sqrt{3}}{2}e^{-\mu_2}-\frac{\sqrt{3}}{2}e^{-\mu_3})\\
&\qquad-\alpha_2\beta_2\xi_2(\sqrt{3}e^{-\mu_1}+\frac{\sqrt{3}}{2}e^{-\mu_2}-\frac{\sqrt{3}}{2}e^{-\mu_3})]\\
&+\rho^{\frac{1}{3}}[\alpha_3\beta_3\xi_1(-\frac{3}{2}e^{-\mu_2}-\frac{3}{2}e^{-\mu_3})-\alpha_3\beta_2\xi_2(-\frac{3}{2}e^{-\mu_2}-\frac{3}{2}e^{-\mu_3})]
\end{aligned}
$$

Proposition 3.6. *Suppose that one of the conditions from (3.3) holds. There are finitely many $\lambda = i\rho$, $\rho \in R$ which are eigenvalues. That is, there exists an N such that if $|\rho| > N$, then $\Delta(\rho) \neq 0$.*

Proof. As $\rho \to \infty$ we find for $Re(\Delta(\rho))$

Given case (a), with $\alpha_1\beta_1\xi_1 \neq 0$

$$\Delta(\rho) \sim -\rho^{\frac{5}{3}} e^{\frac{\sqrt{3}}{2}\rho^{\frac{1}{3}}}$$

For the remaining cases: case (b), with $\alpha_1, \alpha_2 = 0$, $\alpha_3\beta_1\xi_1 \neq 0$, case (c), with $\beta_1 = 0$, $\alpha_1\xi_1 \neq 0$ and $\beta_2\xi_2 - \beta_3\xi_1 \neq 0$ and case (d), with $\beta_1, \xi_1 = 0$, $\alpha_1 \neq 0$, and $\beta_2\xi_2 - \beta_3\xi_1 \neq 0$ we have

$$\Delta(\rho) \sim -\rho e^{\frac{\sqrt{3}}{2}\rho^{\frac{1}{3}}}$$

As $\rho \to \infty$ we find for $Im(\Delta(\rho))$

Given case (a), with $\alpha_1\beta_1\xi_1 \neq 0$

$$\Delta(\rho) \sim -\rho^{\frac{5}{3}} e^{\frac{\sqrt{3}}{2}\rho^{\frac{1}{3}}}$$

For case (b), with $\alpha_1, \alpha_2 = 0$, $\alpha_3\beta_1\xi_1 \neq 0$

$$\Delta(\rho) \sim \rho^{\frac{2}{3}} e^{\frac{\sqrt{3}}{2}\rho^{\frac{1}{3}}}$$

For cases (c), with $\beta_1 = 0$, $\alpha_1\xi_1 \neq 0$ and $\beta_2\xi_2 - \beta_3\xi_1 \neq 0$ and $(d)$with $\beta_1, \xi_1 = 0$, $\alpha_1 \neq 0$, and $\beta_2\xi_2 - \beta_3\xi_1 \neq 0$ we have

$$\Delta(\rho) \sim -\rho^{\frac{2}{3}} e^{\frac{\sqrt{3}}{2}\rho^{\frac{1}{3}}}$$

Given these results we can find for $\rho >> 0$, that $\Delta(\rho) \neq 0$. □

Now we will go back to considering the solution $u(x,t)$ for (3.20). It can be written in the form

$$u(x,t) = u_1(x,t) + u_2(x,t) + u_3(x,t) = \sum_{n=1}^{3} u_n(x,t) \tag{3.22}$$

where $u_n(x,t)$ solves (3.20) with $h_j \equiv 0$ when $j \neq n$, $j,n = 1,2,3$. Using the inverse Laplace transform yields

$$u(x,t) = \frac{1}{2\pi i}\int_{r-i\infty}^{r+i\infty} e^{st}\hat{u}(x,s)ds = \sum_{j=1}^{3}\frac{1}{2\pi i}\int_{r-i\infty}^{r+i\infty} e^{st}\frac{\Delta_j(s)}{\Delta(s)}e^{\lambda_j(s)x}ds \tag{3.23}$$

for any $r > 0$. Combining this with (3.22) we can write the values of u_n as follows for $n = 1,2,3$

$$u_n(x,t) = \sum_{j=1}^{3}\frac{1}{2\pi i}\int_{r-i\infty}^{r+i\infty} e^{st}\frac{\Delta_{j,n}(s)}{\Delta(s)}e^{\lambda_j(s)x}\hat{h}_n(s)ds \equiv [W_n(t)h_n](x) \tag{3.24}$$

where $\Delta_{j,n}(s)$ is obtained from $\Delta_j(s)$ by letting $\hat{h}_n(t) = 1$ and $\hat{h}_k(t) \equiv 0$ for $k \neq n$, $k,n = 1,2,3$. In the last two formulas, the right-hand sides are continuous with respect to r for $r > 0$. For $r = 0$ we have shown that there are finitely many eigenvalues (3.6). Without loss of generality we can assume there are no eigenvalues since these can be easily estimated. Then, as the left-hand sides do not depend on r, we can take $r = 0$ in these formulas.

$$\begin{aligned} u_n(x,t) &= \sum_{j=1}^{3}\frac{1}{2\pi i}\int_{0}^{+i\infty} e^{st}\frac{\Delta_{j,n}(s)}{\Delta(s)}e^{\lambda_j(s)x}\hat{h}_n(s)ds \\ &\quad + \sum_{j=1}^{3}\frac{1}{2\pi i}\int_{-i\infty}^{0} e^{st}\frac{\Delta_{j,n}(s)}{\Delta(s)}e^{\lambda_j(s)x}\hat{h}_n(s)ds \\ &\equiv I_n(x,t) + II_n(x,t), \end{aligned}$$

for $n = 1,2,3$. Letting $s = (-i\rho)^3 = i\rho^3$ with $0 \leq \rho < +\infty$ in the characteristic equation $\lambda^3 + s = 0$ the three roots are given in terms of ρ by

$$\lambda_1^+(\rho) = i\rho, \quad \lambda_2^+(\rho) = -i\rho\Big(\frac{1+i\sqrt{3}}{2}\Big), \quad \lambda_3^+(\rho) = -i\rho\Big(\frac{1-i\sqrt{3}}{2}\Big) \tag{3.25}$$

therefore $I_n(x,t)$ and $II_n(x,t)$ may be written in the form

$$I_n = \sum_{j=1}^{3} \frac{1}{2\pi} \int_0^{+\infty} e^{i\rho^3 t} e^{\lambda_j^+(\rho)x} \frac{\Delta_{j,n}^+(\rho)}{\Delta^+(\rho)} 3\rho^2 \hat{h}_n^+(\rho) d\rho$$

and

$$II_n = \sum_{j=1}^{3} \frac{1}{2\pi} \int_0^{+\infty} e^{-i\rho^3 t} e^{\lambda_j^-(\rho)x} \frac{\Delta_{j,n}^-(\rho)}{\Delta^-(\rho)} 3\rho^2 \hat{h}_n^-(\rho) d\rho$$

for $n = 1, 2, 3$, where $\hat{h}_n^+(\rho) = \hat{h}_n(i\rho^3)$, $\Delta^+(\rho)$ and $\Delta_{j,n}^+(\rho)$ are obtained from $\Delta(s)$ and $\Delta_{j,n}(s)$, respectively, by replacing s with $i\rho^3$ and $\lambda_j(s)$ with $\lambda_j^+(\rho)$, for $j = 1, 2, 3$.

Notice that we can rewrite, $\Delta^-(\rho) = \overline{\Delta^+}(\rho)$ and $\Delta_{j,n}^-(\rho) = \overline{\Delta_{j,n}^+}(\rho)$ for $j = 1, 2, 3$, and $\hat{h}_n^-(\rho) = \overline{\hat{h}_n^+}(\rho)$.

Let $\vec{h}(t) = \langle h_1(t), h_2(t), h_3(t) \rangle$ and write the solution u of (3.20) using (3.24) as

$$u(x,t) = [W_b(t)\vec{h}](x) = \sum_{n=1}^{3} [W_n(t)h_n](x) \tag{3.26}$$

The following lemma from Bona, Sun and Zhang [8] is also used in proving the propositions presented in this section:

Lemma 3.7. *For any $f \in L^2(0,\infty)$, let Kf be the function defined by*

$$Kf(x) = \int_0^{+\infty} e^{\gamma(\mu)x} f(\mu) d\mu$$

where $\gamma(\mu)$ is a continuous complex-valued function defined on $(0,\infty)$ satisfying the following two conditions:

(i) *there exist $\delta > 0$ and $b > 0$ such that*

$$\sup_{0<\mu<\delta} \frac{|Re\,\gamma(\mu)|}{\mu} \geq b;$$

(ii) *there exists a complex number* $\alpha + i\beta$ *such that*

$$\lim_{\mu\to+\infty} \frac{\gamma(\mu)}{\mu} = \alpha + i\beta.$$

Then there exists a constant C *such that for all* $f \in L^2(0,\infty)$,

$$\|Kf\|_{L^2(0,1)} \le C(\|e^{Re\gamma(\cdot)} f(\cdot)\|_{L^2(R^+)} + \|f(\cdot)\|_{L^2(R^+)}) \tag{3.27}$$

The following three propositions will be used to prove the theorem at the end of this section by providing estimates of u_1, u_2, and u_3.

Proposition 3.8. *Assume that one of the conditions from (3.3) is satisfied. There exists a constant* C *such that*

$$\|u_1\|_{L^2(R^+;H^1(0,1))} + \sup_{0\le t\le\infty} \|u_1(\cdot,t)\|_{L^2(0,1)} \le C\|h_1\|_{H^{1/3}(R^+)} \tag{3.28}$$

and $\partial_x u_1 \in C_b([0,1]; L^2(R^+))$ *with*

$$\sup_{x\in(0,1)} \|\partial_x u_1(x,\cdot)\|_{L^2(R^+)} \le C\|h_1\|_{H^{1/3}(R^+)} \tag{3.29}$$

for all $h_1 \in H^{1/3}(R^+)$.

Proof. First note that $\lambda_1(s) + \lambda_2(s) + \lambda_3(s) = 0$. Next the determinant of the matrices for $\Delta_{j,1}(s)$ with $j = 1, 2, 3$ will be determined. To determine $\Delta_{1,1}(s)$ consider the matrix

$$\begin{bmatrix} 1 & \left(\alpha_1{\lambda_2}^2 + \alpha_2\lambda_2 + \alpha_3\right) & \left(\alpha_1{\lambda_3}^2 + \alpha_2\lambda_3 + \alpha_3\right) \\ 0 & e^{\lambda_2}\left(\beta_1{\lambda_2}^2 + \beta_2\lambda_2 + \beta_3\right) & e^{\lambda_3}\left(\beta_1{\lambda_3}^2 + \beta_2\lambda_3 + \beta_3\right) \\ 0 & e^{\lambda_2}\left(\xi_1\lambda_2 + \xi_2\right) & e^{\lambda_3}\left(\xi_1\lambda_3 + \xi_2\right) \end{bmatrix}$$

we have

$$\Delta_{1,1}(s) = e^{-\lambda_1}(\lambda_2 - \lambda_3)[\beta_1\xi_1\lambda_2\lambda_3 - \beta_1\xi_2\lambda_1 + \beta_2\xi_2 - \beta_3\xi_1]$$

For $\Delta_{2,1}(s)$ consider the matrix

$$\begin{bmatrix} \Big(\alpha_1{\lambda_1}^2+\alpha_2\lambda_1+\alpha_3\Big) & 1 & \Big(\alpha_1{\lambda_3}^2+\alpha_2\lambda_3+\alpha_3\Big) \\ e^{\lambda_1}\Big(\beta_1{\lambda_1}^2+\beta_2\lambda_1+\beta_3\Big) & 0 & e^{\lambda_3}\Big(\beta_1{\lambda_3}^2+\beta_2\lambda_3+\beta_3\Big) \\ e^{\lambda_1}\Big(\xi_1\lambda_1+\xi_2\Big) & 0 & e^{\lambda_3}\Big(\xi_1\lambda_3+\xi_2\Big) \end{bmatrix}$$

we have

$$\Delta_{2,1}(s) = e^{-\lambda_2}(\lambda_3-\lambda_1)[\beta_1\xi_1\lambda_1\lambda_3-\beta_1\xi_2\lambda_2+\beta_2\xi_2-\beta_3\xi_1]$$

For $\Delta_{3,1}(s)$ consider the matrix

$$\begin{bmatrix} \Big(\alpha_1{\lambda_1}^2+\alpha_2\lambda_1+\alpha_3\Big) & \Big(\alpha_1{\lambda_2}^2+\alpha_2\lambda_2+\alpha_3\Big) & 1 \\ e^{\lambda_1}\Big(\beta_1{\lambda_1}^2+\beta_2\lambda_1+\beta_3\Big) & e^{\lambda_2}\Big(\beta_1{\lambda_2}^2+\beta_2\lambda_2+\beta_3\Big) & 0 \\ e^{\lambda_1}\Big(\xi_1\lambda_1+\xi_2\Big) & e^{\lambda_2}\Big(\xi_1\lambda_2+\xi_2\Big) & 0 \end{bmatrix}$$

we have

$$\Delta_{3,1}(s) = e^{-\lambda_3}(\lambda_1-\lambda_2)[\beta_1\xi_1\lambda_1\lambda_2-\beta_1\xi_2\lambda_3+\beta_2\xi_2-\beta_3\xi_1]$$

Therefore for $\Delta(s)$ we have

$$\Delta(s) = \Big(\alpha_1{\lambda_1}^2+\alpha_2\lambda_1+\alpha_3\Big)\Delta_{1,1}-\Big(\alpha_1{\lambda_2}^2+\alpha_2\lambda_2+\alpha_3\Big)\Delta_{2,1}+\Big(\alpha_1{\lambda_3}^2+\alpha_2\lambda_3+\alpha_3\Big)\Delta_{3,1} \quad (3.30)$$

Notice that for $\frac{\Delta^+_{1,1}(\rho)}{\Delta^+(\rho)}$

$$\frac{\Delta^+_{1,1}(\rho)}{\Delta^+(\rho)} = \frac{e^{-\lambda_1}(\lambda_2-\lambda_3)[\beta_1\xi_1\lambda_2\lambda_3-\beta_1\xi_2\lambda_1+\beta_2\xi_2-\beta_3\xi_1]}{\Big(\alpha_1{\lambda_1}^2+\alpha_2\lambda_1+\alpha_3\Big)\Delta_{1,1}-\Big(\alpha_1{\lambda_2}^2+\alpha_2\lambda_2+\alpha_3\Big)\Delta_{2,1}+\Big(\alpha_1{\lambda_3}^2+\alpha_2\lambda_3+\alpha_3\Big)\Delta_{3,1}}$$

multiply by $\frac{e^{\lambda_1}}{e^{\lambda_1}}$ and substituting the values for $\Delta_{j,1},\ j=1,2,3$

$$\frac{\Delta_{1,1}^{+}(\rho)}{\Delta^{+}(\rho)} = \frac{(\lambda_2-\lambda_3)\gamma_1^{(1)}}{\Big(\alpha_1{\lambda_1}^2+\alpha_2\lambda_1+\alpha_3\Big)(\lambda_2-\lambda_3)\gamma_1^{(1)} - e^{\lambda_1-\lambda_2}(\lambda_3-\lambda_1)\gamma_2^{(1)} + e^{\lambda_1-\lambda_3}(\lambda_1-\lambda_2)\gamma_3^{(1)}}$$

where

$$\gamma_1^{(1)} = [\beta_1\xi_1\lambda_2\lambda_3 - \beta_1\xi_2\lambda_1 + \beta_2\xi_2 - \beta_3\xi_1]$$
$$\gamma_2^{(1)} = \Big(\alpha_1{\lambda_2}^2+\alpha_2\lambda_2+\alpha_3\Big)[\beta_1\xi_1\lambda_1\lambda_3 - \beta_1\xi_2\lambda_2 + \beta_2\xi_2 - \beta_3\xi_1]$$
$$\gamma_3^{(1)} = \Big(\alpha_1{\lambda_3}^2+\alpha_2\lambda_3+\alpha_3\Big)[\beta_1\xi_1\lambda_1\lambda_2 - \beta_1\xi_2\lambda_3 + \beta_2\xi_2 - \beta_3\xi_1]$$

Using (3.25) we have

$$e^{\lambda_1-\lambda_2} = e^{\frac{-3i\rho}{2}} e^{-\frac{\sqrt{3}}{2}\rho} \qquad e^{\lambda_1-\lambda_3} = e^{\frac{-3i\rho}{2}} e^{\frac{\sqrt{3}}{2}\rho}$$

As $\rho \to \infty$ we find that

$$\frac{\Delta_{1,1}^{+}(\rho)}{\Delta^{+}(\rho)} \sim e^{-\frac{\sqrt{3}}{2}\rho} \tag{3.31}$$

holds for all cases (a)-(d).

Next consider $\frac{\Delta_{2,1}^{+}(\rho)}{\Delta^{+}(\rho)}$

$$\frac{\Delta_{2,1}^{+}(\rho)}{\Delta^{+}(\rho)} = \frac{e^{-\lambda_2}(\lambda_3-\lambda_1)[\beta_1\xi_1\lambda_1\lambda_3 - \beta_1\xi_2\lambda_2 + \beta_2\xi_2 - \beta_3\xi_1]}{\Big(\alpha_1{\lambda_1}^2+\alpha_2\lambda_1+\alpha_3\Big)\Delta_{1,1} - \Big(\alpha_1{\lambda_2}^2+\alpha_2\lambda_2+\alpha_3\Big)\Delta_{2,1} + \Big(\alpha_1{\lambda_3}^2+\alpha_2\lambda_3+\alpha_3\Big)\Delta_{3,1}}$$

multiply by $\frac{e^{\lambda_2}}{e^{\lambda_2}}$ and substituting the values for $\Delta_{j,1},\ j=1,2,3$

$$\frac{\Delta_{2,1}^{+}(\rho)}{\Delta^{+}(\rho)} = \frac{(\lambda_3-\lambda_1)\gamma_2^{(2)}}{e^{\lambda_2-\lambda_1}(\lambda_2-\lambda_3)\gamma_1^{(2)} - \Big(\alpha_1{\lambda_2}^2+\alpha_2\lambda_2+\alpha_3\Big)(\lambda_3-\lambda_1)\gamma_2^{(2)} + e^{\lambda_2-\lambda_3}(\lambda_1-\lambda_1)\gamma_3^{(2)}}$$

where

$$\gamma_1^{(2)} = \Big(\alpha_1{\lambda_1}^2+\alpha_2\lambda_1+\alpha_3\Big)[\beta_1\xi_1\lambda_2\lambda_3 - \beta_1\xi_2\lambda_1 + \beta_2\xi_2 - \beta_3\xi_1]$$
$$\gamma_2^{(2)} = [\beta_1\xi_1\lambda_1\lambda_3 - \beta_1\xi_2\lambda_2 + \beta_2\xi_2 - \beta_3\xi_1]$$
$$\gamma_3^{(2)} = \Big(\alpha_1{\lambda_3}^2+\alpha_2\lambda_3+\alpha_3\Big)[\beta_1\xi_1\lambda_1\lambda_2 - \beta_1\xi_2\lambda_3 + \beta_2\xi_2 - \beta_3\xi_1]$$

Using (3.25) we have

$$e^{\lambda_2-\lambda_1} = e^{\frac{-3i\rho}{2}}e^{-\frac{\sqrt{3}}{2}\rho} \qquad\qquad e^{\lambda_2-\lambda_3} = e^{\sqrt{3}\rho}$$

As $\rho \to \infty$ we find that

$$\frac{\Delta^+_{2,1}(\rho)}{\Delta^+(\rho)} \sim e^{-\sqrt{3}\rho} \tag{3.32}$$

holds for all cases (a)-(d).

Lastly consider $\frac{\Delta^+_{3,1}(\rho)}{\Delta^+(\rho)}$

$$\frac{\Delta^+_{3,1}(\rho)}{\Delta^+(\rho)} = \frac{e^{-\lambda_3}(\lambda_1-\lambda_2)[\beta_1\xi_1\lambda_1\lambda_2 - \beta_1\xi_2\lambda_3 + \beta_2\xi_2 - \beta_3\xi_1]}{\Big(\alpha_1{\lambda_1}^2+\alpha_2\lambda_1+\alpha_3\Big)\Delta_{1,1} - \Big(\alpha_1{\lambda_2}^2+\alpha_2\lambda_2+\alpha_3\Big)\Delta_{2,1} + \Big(\alpha_1{\lambda_3}^2+\alpha_2\lambda_3+\alpha_3\Big)\Delta_{3,1}}$$

multiply by $\frac{e^{\lambda_3}}{e^{\lambda_3}}$ and substituting the values for $\Delta_{j,1}$, $j = 1, 2, 3$

$$\frac{\Delta^+_{3,1}(\rho)}{\Delta^+(\rho)} = \frac{(\lambda_1-\lambda_2)\gamma_3^{(3)}}{e^{\lambda_3-\lambda_1}(\lambda_2-\lambda_3)\gamma_1^{(3)} - e^{\lambda_3-\lambda_2}(\lambda_3-\lambda_1)\gamma_2^{(3)} + \Big(\alpha_1{\lambda_3}^2+\alpha_2\lambda_3+\alpha_3\Big)(\lambda_1-\lambda_2)\gamma_3^{(3)}}$$

where

$$\begin{aligned}
\gamma_1^{(3)} &= \Big(\alpha_1{\lambda_1}^2+\alpha_2\lambda_1+\alpha_3\Big)[\beta_1\xi_1\lambda_2\lambda_3 - \beta_1\xi_2\lambda_1 + \beta_2\xi_2 - \beta_3\xi_1] \\
\gamma_2^{(3)} &= \Big(\alpha_1{\lambda_2}^2+\alpha_2\lambda_2+\alpha_3\Big)[\beta_1\xi_1\lambda_1\lambda_3 - \beta_1\xi_2\lambda_2 + \beta_2\xi_2 - \beta_3\xi_1] \\
\gamma_3^{(3)} &= [\beta_1\xi_1\lambda_1\lambda_2 - \beta_1\xi_2\lambda_3 + \beta_2\xi_2 - \beta_3\xi_1]
\end{aligned}$$

Using (3.25) we have

$$e^{\lambda_3-\lambda_1} = e^{\frac{-3i\rho}{2}}e^{-\frac{\sqrt{3}2}{\rho}} \qquad\qquad e^{\lambda_3-\lambda_2} = e^{-\sqrt{3}\rho}$$

As $\rho \to \infty$ we find that for cases (a), (c) and (d);

$$\frac{\Delta^+_{3,1}(\rho)}{\Delta^+(\rho)} \sim \rho^{-2} \tag{3.33}$$

For cases (b), since $\alpha_1 = \alpha_2 = 0$, we have

$$\frac{\Delta^+_{3,1}(\rho)}{\Delta^+(\rho)} \sim 1 \tag{3.34}$$

Now we are ready to estimate the solution $u_1(x,t) = I_1(x,t) + II_1(x,t)$. As

$$I_1 = \sum_{j=1}^{3} \frac{1}{2\pi} \int_0^{+\infty} e^{i\rho^3 t} e^{\lambda_j^+(\rho)x} \frac{\Delta^+_{j,1}(\rho)}{\Delta^+(\rho)} 3\rho^2 \hat{h}_1^+(\rho) d\rho$$

an application of Lemma (3.7) produces a constant C such that

$$\|I_1(\cdot,t)\|^2_{L^2(0,1)} \leq C \sum_{j=1}^{3} \int_0^{\infty} \left| \frac{\Delta^+_{j,1}(\rho)}{\Delta^+(\rho)} \right|^2 \left(e^{Re\lambda_j^+(\rho)} + 1 \right)^2 \left| \hat{h}_1^+(\rho) 3\rho^2 \right|^2 d\rho \tag{3.35}$$

Using the asymptotic behavior for each case, (3.31), (3.32), (3.33), or (3.34) consider $\left| \frac{\Delta^+_{j,1}(\rho)}{\Delta^+(\rho)} \right|^2 \left(e^{Re\lambda_j^+(\rho)} + 1 \right)^2$ from each integral in the sum.

For $\frac{\Delta^+_{1,1}(\rho)}{\Delta^+(\rho)}$

$$\left(e^{-\frac{\sqrt{3}}{2}\rho} \right)^2 \left(e^0 + 1 \right)^2$$

For $\frac{\Delta^+_{2,1}(\rho)}{\Delta^+(\rho)}$

$$\left(e^{-\sqrt{3}\rho} \right)^2 \left(e^{\frac{\sqrt{3}}{2}\rho} + 1 \right)^2$$

For $\frac{\Delta^+_{3,1}(\rho)}{\Delta^+(\rho)}$ we have two possibilities

$$\left(\rho^{-2} \right)^2 \left(e^{-\frac{\sqrt{3}}{2}\rho} + 1 \right)^2 \qquad \text{or} \qquad \left(1 \right)^2 \left(e^{-\frac{\sqrt{3}}{2}\rho} + 1 \right)^2$$

Hence in any of the cases there exists a constant C such that

$$\left| \frac{\Delta^+_{j,1}(\rho)}{\Delta^+(\rho)} \right|^2 \left(e^{Re\lambda_j^+(\rho)} + 1 \right)^2 < C \quad \text{for any } \rho \geq 1.$$

Therefore, for any $t \geq 0$,

$$\begin{aligned}\|I_1(\cdot,t)\|^2_{L^2(0,1)} &\leq C\int_0^\infty |\hat{h}_1^+(\rho)|^2(3\rho^2)^2 d\rho \\ &= C\int_0^\infty 3\rho^2|\hat{h}_1^+(\rho)|^2 3\rho^2 d\rho \\ &\leq C\int_0^\infty \mu^{2/3}\left|\int_0^\infty e^{-i\mu}h_1(\tau)d\tau\right|^2 d\mu \\ &\leq C\|h_1\|^2_{H^{1/3}(R^+)}.\end{aligned}$$

where we have rewritten the integral in terms of $\mu = \rho^3$. The same argument applied to $II_1(x,t)$ gives

$$\|II_1(\cdot,t)\|_{L^2(0,1)} \leq C\|h_1\|_{H^{1/3}(R^+)}.$$

Therefore (3.28) holds.

To prove (3.29), let $\theta(\mu)$ be the real solution of $\mu = \rho^3$ for $\rho \geq 1$. Then for $\partial_x I_1(x,t)$ we have

$$\begin{aligned}\partial_x I_1(x,t) &= \sum_{j=1}^3 \frac{1}{2\pi}\int_0^{+\infty} e^{i\rho^3 t}\lambda_j^+(\rho)e^{\lambda_j^+(\rho)x}\frac{\Delta_{j,1}^+(\rho)}{\Delta^+(\rho)}3\rho^2\hat{h}_1^+(\rho)d\rho \\ &= \sum_{j=1}^3 \frac{1}{2\pi}\int_0^{+\infty} e^{i\mu t}\lambda_j^+(\theta(\mu))e^{\lambda_j^+(\theta(\mu))x}\frac{\Delta_{j,1}^+(\theta(\mu))}{\Delta^+(\theta(\mu))}\hat{h}_1(i\mu)d\mu\end{aligned}$$

Using the Plancherel Theorem (with respect to t) yields that for any $x \in (0,1)$,

$$\|\partial_x I_1(x,\cdot)\|^2_{L^2(R^+)} \leq \sum_{j=1}^3 \frac{1}{2\pi}\int_0^{+\infty}\left|\lambda_j^+(\theta(\mu))e^{\lambda_j^+(\theta(\mu))x}\frac{\Delta_{j,1}^+(\theta(\mu))}{\Delta^+(\theta(\mu))}\right|^2|\hat{h}_1(i\mu)|^2 d\mu.$$

In addition,

$$\int_0^1 \|\partial_x I_1(x,\cdot)\|^2_{L^2(R^+)}dx \le \sup_{x\in(0,1)} \|\partial_x I_1(x,\cdot)\|^2_{L^2(R^+)}$$
$$\le C\sum_{j=1}^{3}\int_0^{+\infty} \left|\lambda_j^+(\theta(\mu))\right|^2 \sup_{x\in(0,1)}\left|e^{\lambda_j^+(\theta(\mu))x}\right|^2 \left|\frac{\Delta_{j,1}^+(\theta(\mu))}{\Delta^+(\theta(\mu))}\right|^2 |\hat{h}_1(i\mu)|^2 d\mu$$

Consider $\sup_{x\in(0,1)}\left|e^{\lambda_j^+(\theta(\mu))x}\right|^2$. Looking at each value of λ_j, j=1,2,3 we have the following estimates

$$\sup_{x\in(0,1)}\left|e^{\lambda_1(\rho)x}\right|^2 = \sup_{x\in(0,1)}\left|e^{i\rho x}\right|^2 \le C,$$

$$\sup_{x\in(0,1)}\left|e^{\lambda_2(\rho)x}\right|^2 = \sup_{x\in(0,1)}\left|e^{-i\rho\left(\frac{1+i\sqrt{3}}{2}\right)x}\right|^2 \le C\left(e^{\sqrt{3}\rho}+1\right),$$

$$\sup_{x\in(0,1)}\left|e^{\lambda_3(\rho)x}\right|^2 = \sup_{x\in(0,1)}\left|e^{-i\rho\left(\frac{1-i\sqrt{3}}{2}\right)x}\right|^2 \le C\left(e^{-\sqrt{3}\rho}+1\right)$$

Using these estimates with the results for the asymptotic behavior of $\left|\frac{\Delta_{j,1}^+(\theta(\mu))}{\Delta^+(\theta(\mu))}\right|^2$ namely (3.31), (3.32), (3.33), and (3.34), we have the following (recall that $\theta(\mu)$ is the real solution of $\mu=\rho^3$)

$$\sup_{x\in(0,1)}\left|e^{\lambda_1(\rho)x}\right|^2\left|\frac{\Delta_{1,1}^+(\theta(\mu))}{\Delta^+(\theta(\mu))}\right|^2 \le C\Big(e^{-\frac{\sqrt{3}}{2}\rho}\Big)^2,$$

$$\sup_{x\in(0,1)}\left|e^{\lambda_2(\rho)x}\right|^2\left|\frac{\Delta_{2,1}^+(\theta(\mu))}{\Delta^+(\theta(\mu))}\right|^2 \le C\left(e^{\sqrt{3}\rho}+1\right)\left(e^{-\sqrt{3}\rho}\right)^2,$$

$$\sup_{x\in(0,1)}\left|e^{\lambda_3(\rho)x}\right|^2\left|\frac{\Delta_{3,1}^+(\theta(\mu))}{\Delta^+(\theta(\mu))}\right|^2 \le C\left(e^{-\sqrt{3}\rho}+1\right)\left(\rho^{-2}\right) \text{ or } C\left(e^{-\sqrt{3}\rho}+1\right)$$

The two possibilities in the last inequality are due to the results for $\frac{\Delta_{3,1}^+(\rho)}{\Delta^+(\rho)}$ which depend on the assumptions for the boundary conditions. In any of the cases the expressions on the right hand side are convergent as $\rho\to\infty$, therefore we can find a constant C such that

$$\begin{aligned}\int_0^1 \|\partial_x I_1(x,\cdot)\|^2_{L^2(R^+)}dx &\le \sup_{x\in(0,1)} \|\partial_x I_1(x,\cdot)\|^2_{L^2(R^+)} \\ &\le C\sum_{j=1}^3 \int_0^{+\infty} \left|\lambda_j^+(\theta(\mu))\right|^2 \sup_{x\in(0,1)} \left|e^{\lambda_j^+(\theta(\mu))x}\right|^2 \left|\frac{\Delta^+_{j,1}(\theta(\mu))}{\Delta^+(\theta(\mu))}\right|^2 |\hat{h}_1(i\mu)|^2 d\mu \\ &\le C\int_0^{+\infty}(1+\mu)^{2/3}|\hat{h}_1(i\mu)|^2 d\mu \le C\|h_1\|^2_{H^{1/3}(R^+)}.\end{aligned}$$

where we have also used the definition of λ_j, j=1,2,3, rewritten in terms of μ.

To see $\partial_x I_1$ is continuous from $[0,1]$ to the space $L^2(R^+)$, choose any $x_0 \in [0,1]$ and $x \in (0,1)$ and observe that

$$\begin{aligned}&\partial_x I_1(x,t) - \partial_x I_1(x_0,t) \\ &\qquad = \sum_{j=1}^3 \frac{1}{2\pi}\int_0^{+\infty} e^{i\mu t}\lambda_j^+(\theta(\mu))\left(e^{\lambda_j^+(\theta(\mu))x} - e^{\lambda_j^+(\theta(\mu))x_0}\right)\frac{\Delta^+_{j,1}(\theta(\mu))}{\Delta^+(\theta(\mu))}\hat{h}_1(i\mu)d\mu.\end{aligned}$$

Using the Plancherel Theorem (with respect to t) as above yields

$$\begin{aligned}&\|\partial_x I_1(x,\cdot) - \partial_x I_1(x_0,\cdot)\|^2_{L^2(R^+)} \\ &\quad\le \sum_{j=1}^3 \frac{1}{2\pi}\int_0^{+\infty} \left|\lambda_j^+(\theta(\mu))\left(e^{\lambda_j^+(\theta(\mu))x} - e^{\lambda_j^+(\theta(\mu))x_0}\right)\frac{\Delta^+_{j,1}(\theta(\mu))}{\Delta^+(\theta(\mu))}\right|^2 |\hat{h}_1(i\mu)|^2 d\mu \\ &\quad\le C\int_0^{+\infty}(1+\mu)^{2/3}|\hat{h}_1(i\mu)|^2 d\mu.\end{aligned}$$

An application of Fatou's Lemma gives

$$\lim_{x\to x_0} \|\partial_x I_1(x,\cdot) - \partial_x I_1(x_0,\cdot)\|^2_{L^2(R^+)}$$
$$\leq \sum_{j=1}^{3} \frac{1}{2\pi}\int_0^{+\infty} \left| \lambda_j^+(\theta(\mu)) \lim_{x\to x_0}\left(e^{\lambda_j^+(\theta(\mu))x} - e^{\lambda_j^+(\theta(\mu))x_0}\right)\frac{\Delta^+_{j,1}(\theta(\mu))}{\Delta^+(\theta(\mu))}\right|^2 |\hat{h}_1(i\mu)|^2 d\mu$$
$$= 0$$

Continuity is established. Similarly we can show that

$$\int_0^1 \|\partial_x II_1(x,\cdot)\|^2_{L^2(R^+)}dx \leq \sup_{x\in(0,1)} \|\partial_x II_1(x,\cdot)\|^2_{L^2(R^+)} \leq C\|h_1\|^2_{H^{1/3}(R^+)}$$

and $II_1(x,\cdot) \in C_b([0,1]; L^2(R^+))$. Therefore since $u_1(x,t) = I_1(x,t) + II_1(x,t)$, the inequality (3.29) holds.

□

Next consider $u_2(x,t) = I_2(x,t) + II_2(x,t)$

Proposition 3.9. *Assume that one of the conditions from (3.3) is satisfied. There exists a constant C such that*

$$\|u_2\|_{L^2(R^+;H^1(0,1))} + \sup_{0\leq t\leq\infty} \|u_2(\cdot,t)\|_{L^2(0,1)} \leq C\|h_2\|_{H^{1/3}(R^+)} \tag{3.36}$$

and $\partial_x u_2 \in C_b([0,1]; L^2(R^+))$ with

$$\sup_{x\in(0,1)} \|\partial_x u_2(x,\cdot)\|_{L^2(R^+)} \leq C\|h_2\|_{H^{1/3}(R^+)} \tag{3.37}$$

for all $h_2 \in H^{1/3}(R^+)$.

Proof. The proof will be very similar to the proof for Proposition (3.8). First note that $\lambda_1(s) + \lambda_2(s) + \lambda_3(s) = 0$. Next the determinant of the matrices for $\Delta_{j,2}(s)$ with $j = 1,2,3$ will be determined. To determine $\Delta_{1,2}(s)$ consider the matrix

$$\begin{bmatrix} 0 & \left(\alpha_1{\lambda_2}^2+\alpha_2\lambda_2+\alpha_3\right) & \left(\alpha_1{\lambda_3}^2+\alpha_2\lambda_3+\alpha_3\right) \\ 1 & e^{\lambda_2}\left(\beta_1{\lambda_2}^2+\beta_2\lambda_2+\beta_3\right) & e^{\lambda_3}\left(\beta_1{\lambda_3}^2+\beta_2\lambda_3+\beta_3\right) \\ 0 & e^{\lambda_2}\left(\xi_1\lambda_2+\xi_2\right) & e^{\lambda_3}\left(\xi_1\lambda_3+\xi_2\right) \end{bmatrix}$$

we have

$$\Delta_{1,2}(s) = -e^{\lambda_3}\left(\alpha_1{\lambda_2}^2+\alpha_2\lambda_2+\alpha_3\right)\left(\xi_1\lambda_3+\xi_2\right)+e^{\lambda_2}\left(\alpha_1{\lambda_3}^2+\alpha_2\lambda_3+\alpha_3\right)\left(\xi_1\lambda_2+\xi_2\right)$$

For $\Delta_{2,2}(s)$ consider the matrix

$$\begin{bmatrix} \left(\alpha_1{\lambda_1}^2+\alpha_2\lambda_1+\alpha_3\right) & 0 & \left(\alpha_1{\lambda_3}^2+\alpha_2\lambda_3+\alpha_3\right) \\ e^{\lambda_1}\left(\beta_1{\lambda_1}^2+\beta_2\lambda_1+\beta_3\right) & 1 & e^{\lambda_3}\left(\beta_1{\lambda_3}^2+\beta_2\lambda_3+\beta_3\right) \\ e^{\lambda_1}\left(\xi_1\lambda_1+\xi_2\right) & 0 & e^{\lambda_3}\left(\xi_1\lambda_3+\xi_2\right) \end{bmatrix}$$

we have

$$\Delta_{2,2}(s) = e^{\lambda_3}\left(\alpha_1{\lambda_1}^2+\alpha_2\lambda_1+\alpha_3\right)\left(\xi_1\lambda_3+\xi_2\right)-e^{\lambda_1}\left(\alpha_1{\lambda_3}^2+\alpha_2\lambda_3+\alpha_3\right)\left(\xi_1\lambda_1+\xi_2\right)$$

For $\Delta_{3,2}(s)$ consider the matrix

$$\begin{bmatrix} \left(\alpha_1{\lambda_1}^2+\alpha_2\lambda_1+\alpha_3\right) & \left(\alpha_1{\lambda_2}^2+\alpha_2\lambda_2+\alpha_3\right) & 0 \\ e^{\lambda_1}\left(\beta_1{\lambda_1}^2+\beta_2\lambda_1+\beta_3\right) & e^{\lambda_2}\left(\beta_1{\lambda_2}^2+\beta_2\lambda_2+\beta_3\right) & 1 \\ e^{\lambda_1}\left(\xi_1\lambda_1+\xi_2\right) & e^{\lambda_2}\left(\xi_1\lambda_2+\xi_2\right) & 0 \end{bmatrix}$$

we have

$$\Delta_{3,2}(s) = -e^{\lambda_2}\left(\alpha_1{\lambda_1}^2+\alpha_2\lambda_1+\alpha_3\right)\left(\xi_1\lambda_2+\xi_2\right)+e^{\lambda_1}\left(\alpha_1{\lambda_2}^2+\alpha_2\lambda_2+\alpha_3\right)\left(\xi_1\lambda_1+\xi_2\right)$$

For $\Delta(s)$ we still have (3.30)

As in the proof for Proposition (3.8) using (3.25) we find the asymptotic behavior as $\rho \to \infty$. To determine the behavior we multiply $\frac{\Delta^+_{j,2}(\rho)}{\Delta^+(\rho)}$ by $\frac{e^{-\lambda_1}}{e^{-\lambda_1}}$ for $j = 1, 2, 3$. The results for $\frac{\Delta^+_{1,2}(\rho)}{\Delta^+(\rho)}$ are

$$\frac{\Delta^+_{1,2}(\rho)}{\Delta^+(\rho)} \sim \rho^{-2} \tag{3.38}$$

holds for cases (a) and (b).

$$\frac{\Delta^+_{1,2}(\rho)}{\Delta^+(\rho)} \sim 1 \tag{3.39}$$

holds for case (c) since $\beta_1 = 0$.

$$\frac{\Delta^+_{1,2}(\rho)}{\Delta^+(\rho)} \sim \rho^{-1} \tag{3.40}$$

holds for cases (d) since $\beta_1 = \xi_1 = 0$.

For $\frac{\Delta^+_{2,2}(\rho)}{\Delta^+(\rho)}$ we have

$$\frac{\Delta^+_{2,2}(\rho)}{\Delta^+(\rho)} \sim e^{-\frac{\sqrt{3}}{2}\rho} \tag{3.41}$$

holds for all cases (a)-(d).

The results for $\frac{\Delta^+_{3,2}(\rho)}{\Delta^+(\rho)}$ are

$$\frac{\Delta^+_{3,2}(\rho)}{\Delta^+(\rho)} \sim \rho^{-2} \tag{3.42}$$

holds for cases (a) and (b);

$$\frac{\Delta^+_{3,2}(\rho)}{\Delta^+(\rho)} \sim 1 \tag{3.43}$$

holds for case (c) since $\beta_1 = 0$;

$$\frac{\Delta^+_{3,2}(\rho)}{\Delta^+(\rho)} \sim \rho^{-1} \tag{3.44}$$

holds for cases (d) since $\beta_1 = \xi_1 = 0$.

Next, recall that $u_2(x,t) = I_2(x,t) + II_2(x,t)$, consider

$$I_2 = \sum_{j=1}^{3} \frac{1}{2\pi} \int_0^{+\infty} e^{i\rho^3 t} e^{\lambda_j^+(\rho)x} \frac{\Delta_{j,2}^+(\rho)}{\Delta^+(\rho)} 3\rho^2 \hat{h}_1^+(\rho) d\rho$$

An application of Lemma (3.7) produces a constant C such that

$$\|I_2(\cdot,t)\|_{L^2(0,1)}^2 \le C \sum_{j=1}^{3} \int_0^{\infty} \left| \frac{\Delta_{j,2}^+(\rho)}{\Delta^+(\rho)} \right|^2 \left(e^{Re\lambda_j^+(\rho)} + 1 \right)^2 \left| \hat{h}_2^+(\rho) 3\rho^2 \right|^2 d\rho$$

As in the proof of Proposition (3.8) the asymptotic behavior of (3.38), (3.39), (3.40), (3.41), (3.42), (3.43) or (3.44) allows the term $\left| \frac{\Delta_{j,2}^+(\rho)}{\Delta^+(\rho)} \right|^2 \left(e^{Re\lambda_j^+(\rho)} + 1 \right)^2$ to be bounded for each integral in the sum. The inequality becomes

$$\begin{aligned} \|I_2(\cdot,t)\|_{L^2(0,1)}^2 &\le C \int_0^{\infty} |\hat{h}_2^+(\rho)|^2 (3\rho^2)^2 d\rho \\ &\le C \int_0^{\infty} (1+\mu)^{2/3} \left| \int_0^{\infty} e^{-i\mu} h_2(\tau) d\tau \right|^2 d\mu \\ &\le C \|h_2\|_{H^{1/3}(R^+)}^2. \end{aligned}$$

where we have rewritten the integral in terms of $\mu = \rho^3$. The same argument applied to $II_2(x,t)$ gives

$$\|II_2(\cdot,t)\|_{L^2(0,1)} \le C\|h_2\|_{H^{1/3}(R^+)}.$$

Therefore (3.36) holds. To prove (3.37), we follow the same steps as in the proof of Proposition (3.8)

□

Lastly we consider $u_3(x,t) = I_3(x,t) + II_3(x,t)$

Proposition 3.10. *Assume that one of the conditions from (3.3) is satisfied. There exists a constant C such that*

$$\|u_3\|_{L^2(R^+;H^1(0,1))} + \sup_{0\le t\le\infty} \|u_3(\cdot,t)\|_{L^2(0,1)} \le C\|h_3\|_{H^{1/3}(R^+)} \tag{3.45}$$

and $\partial_x u_3 \in C_b([0,1];L^2(R^+))$ with

$$\sup_{x\in(0,1)} \|\partial_x u_3(x,\cdot)\|_{L^2(R^+)} \le C\|h_3\|_{H^{1/3}(R^+)} \tag{3.46}$$

for all $h_3 \in H^{1/3}(R^+)$.

Proof. The proof will be very similar to the proof for Proposition (3.8). First note that $\lambda_1(s)+\lambda_2(s)+\lambda_3(s)=0$. Next the determinant of the matrices for $\Delta_{j,3}(s)$ with $j=1,2,3$ will be determined. To determine $\Delta_{1,3}(s)$ consider the matrix

$$\begin{bmatrix} 0 & \Big(\alpha_1{\lambda_2}^2+\alpha_2\lambda_2+\alpha_3\Big) & \Big(\alpha_1{\lambda_3}^2+\alpha_2\lambda_3+\alpha_3\Big) \\ 0 & e^{\lambda_2}\Big(\beta_1{\lambda_2}^2+\beta_2\lambda_2+\beta_3\Big) & e^{\lambda_3}\Big(\beta_1{\lambda_3}^2+\beta_2\lambda_3+\beta_3\Big) \\ 1 & e^{\lambda_2}\Big(\xi_1\lambda_2+\xi_2\Big) & e^{\lambda_3}\Big(\xi_1\lambda_3+\xi_2\Big) \end{bmatrix}$$

we have

$$\Delta_{1,3}(s) = e^{\lambda_3}\Big(\alpha_1{\lambda_2}^2+\alpha_2\lambda_2+\alpha_3\Big)\Big(\beta_1{\lambda_3}^2+\beta_2\lambda_3+\beta_3\Big)-e^{\lambda_2}\Big(\alpha_1{\lambda_3}^2+\alpha_2\lambda_3+\alpha_3\Big)\Big(\beta_1{\lambda_2}^2+\beta_2\lambda_2+\beta_3\Big)$$

For $\Delta_{2,3}(s)$ consider the matrix

$$\begin{bmatrix} \Big(\alpha_1{\lambda_1}^2+\alpha_2\lambda_1+\alpha_3\Big) & 0 & \Big(\alpha_1{\lambda_3}^2+\alpha_2\lambda_3+\alpha_3\Big) \\ e^{\lambda_1}\Big(\beta_1{\lambda_1}^2+\beta_2\lambda_1+\beta_3\Big) & 0 & e^{\lambda_3}\Big(\beta_1{\lambda_3}^2+\beta_2\lambda_3+\beta_3\Big) \\ e^{\lambda_1}\Big(\xi_1\lambda_1+\xi_2\Big) & 1 & e^{\lambda_3}\Big(\xi_1\lambda_3+\xi_2\Big) \end{bmatrix}$$

we have

$$\Delta_{2,3}(s) = -e^{\lambda_3}\Big(\alpha_1{\lambda_1}^2+\alpha_2\lambda_1+\alpha_3\Big)\Big(\beta_1{\lambda_3}^2+\beta_2\lambda_3+\beta_3\Big)+e^{\lambda_1}\Big(\alpha_1{\lambda_3}^2+\alpha_2\lambda_3+\alpha_3\Big)\Big(\beta_1{\lambda_1}^2+\beta_2\lambda_1+\beta_3\Big)$$

For $\Delta_{3,3}(s)$ consider the matrix

$$\begin{bmatrix} \Big(\alpha_1{\lambda_1}^2 + \alpha_2\lambda_1 + \alpha_3\Big) & \Big(\alpha_1{\lambda_2}^2 + \alpha_2\lambda_2 + \alpha_3\Big) & 0 \\ e^{\lambda_1}\Big(\beta_1{\lambda_1}^2 + \beta_2\lambda_1 + \beta_3\Big) & e^{\lambda_2}\Big(\beta_1{\lambda_2}^2 + \beta_2\lambda_2 + \beta_3\Big) & 0 \\ e^{\lambda_1}\Big(\xi_1\lambda_1 + \xi_2\Big) & e^{\lambda_2}\Big(\xi_1\lambda_2 + \xi_2\Big) & 1 \end{bmatrix}$$

we have

$$\Delta_{3,3}(s) = e^{\lambda_2}\Big(\alpha_1{\lambda_1}^2+\alpha_2\lambda_1+\alpha_3\Big)\Big(\beta_1{\lambda_2}^2+\beta_2\lambda_2+\beta_3\Big)-e^{\lambda_1}\Big(\alpha_1{\lambda_2}^2+\alpha_2\lambda_2+\alpha_3\Big)\Big(\beta_1{\lambda_1}^2+\beta_2\lambda_1+\beta_3\Big)$$

For $\Delta(s)$ we still have (3.30)

As in the proof for Proposition (3.8) using (3.25) we find the asymptotic behavior as $\rho \to \infty$. To determine the behavior we multiply $\frac{\Delta^+_{j,3}(\rho)}{\Delta^+(\rho)}$ by $\frac{e^{-\lambda_1}}{e^{-\lambda_1}}$ for $j = 1, 2, 3$. The results for $\frac{\Delta^+_{1,3}(\rho)}{\Delta^+(\rho)}$ are

$$\frac{\Delta^+_{1,3}(\rho)}{\Delta^+(\rho)} \sim \rho^{-1} \tag{3.47}$$

holds for all cases (a)-(d).

For $\frac{\Delta^+_{2,3}(\rho)}{\Delta^+(\rho)}$ we have

$$\frac{\Delta^+_{2,3}(\rho)}{\Delta^+(\rho)} \sim \rho^{-1} e^{-\frac{\sqrt{3}}{2}\rho} \tag{3.48}$$

holds for all cases (a)-(d).

The results for $\frac{\Delta^+_{3,3}(\rho)}{\Delta^+(\rho)}$ are

$$\frac{\Delta^+_{3,3}(\rho)}{\Delta^+(\rho)} \sim \rho^{-1} \tag{3.49}$$

holds for all cases (a)-(d).

Next, recall that $u_3(x,t) = I_3(x,t) + II_3(x,t)$, consider

$$I_3 = \sum_{j=1}^{3} \frac{1}{2\pi} \int_0^{+\infty} e^{i\rho^3 t} e^{\lambda_j^+(\rho)x} \frac{\Delta_{j,3}^+(\rho)}{\Delta^+(\rho)} 3\rho^2 \hat{h}_1^+(\rho) d\rho$$

An application of Lemma (3.7) produces a constant C such that

$$\|I_3(\cdot,t)\|_{L^2(0,1)}^2 \leq C \sum_{j=1}^{3} \int_0^{\infty} \left| \frac{\Delta_{j,3}^+(\rho)}{\Delta^+(\rho)} \right|^2 \left(e^{Re\lambda_j^+(\rho)} + 1 \right)^2 \left| \hat{h}_3^+(\rho) 3\rho^2 \right|^2 d\rho$$

As in the proof for Proposition (3.8) the asymptotic behavior of (3.47), (3.48), or (3.49) allows the term $\left|\frac{\Delta_{j,3}^+(\rho)}{\Delta^+(\rho)}\right|^2 \left(e^{Re\lambda_j^+(\rho)} + 1\right)^2$ to be bounded for each integral in the sum. The inequality becomes

$$\begin{aligned} \|I_3(\cdot,t)\|_{L^2(0,1)}^2 &\leq C \int_0^{\infty} |\hat{h}_3^+(\rho)|^2 (3\rho^2)^2 d\rho \\ &\leq C \int_0^{\infty} (1+\mu)^{2/3} \left| \int_0^{\infty} e^{-i\mu} h_3(\tau) d\tau \right|^2 d\mu \\ &\leq C \|h_3\|_{H^{1/3}(R^+)}^2. \end{aligned}$$

where we have rewritten the integral in terms of $\mu = \rho^3$. The same argument applied to $II_3(x,t)$ gives

$$\|II_3(\cdot,t)\|_{L^2(0,1)} \leq C \|h_3\|_{H^{1/3}(R^+)}.$$

Therefore (3.45) holds. To prove (3.46), we follow the same steps as in the proof of Proposition (3.8)

□

For $s \geq 0$, and $T > 0$, let

$$\mathcal{H}_{s,T} = H^{(s+1)/3}(0,T) \times H^{(s+1)/3}(0,T) \times H^{s/3}(0,T)$$

If $T = \infty$, denote $\mathcal{H}_{s,T}$ by $\mathcal{H}_s$.

$$\|\vec{h}\|_{\mathcal{H}_{s,T}} \equiv \left(\|h_1\|^2_{H^{(s+1)/3}(0,T)} + \|h_2\|^2_{H^{(s+1)/3}(0,T)} + \|h_3\|^2_{H^{s/3}(0,T)}\right)^{1/2}$$

The following theorem regarding problem (3.20) has now been proven, combining the results from the beginning of the section with Propositions (3.8)-(3.10).

Theorem 3.11. *Assume that one of the conditions of (3.3) holds. For any* $\vec{h} \in \mathcal{H}_0$

$$\begin{cases} u_t + u_{xxx} = 0, \quad u(x,0) = 0, \\ B_1 u = h_1(t), \quad B_2 u = h_2(t), \quad B_3 u = h_3(t) \end{cases}$$

admits a unique solution

$$u(x,t) = [W_b(t)\vec{h}(t)](x)$$

which belongs to the space $C_b(R^+; L^2(0,1)) \cap L^2(R^+; H^1(0,1))$ *with* $u_x \in C_b([0,1]; L^2(R^+))$*, if one of the conditions from (3.3) holds. Moreover there exists a constant* C *such that*

$$\|u\|_{L^2(R^+;H^1(0,1))} + \sup_{0 \le t \le \infty} \|u(\cdot,t)\|_{L^2(0,1)} \le C\|\vec{h}\|_{\mathcal{H}_0}$$

and

$$\sup_{x \in (0,1)} \|u_x(x,\cdot)\|_{L^2(R^+)} \le C\|\vec{h}\|_{\mathcal{H}_0}$$

for all $\vec{h} \in \mathcal{H}_0$.

CHAPTER 4

Well-Posedness

In this chapter the well-posedness of the nonlinear IBVP

$$\begin{cases} u_t + u_x + uu_x + u_{xxx} = 0, \qquad u(x,0) = \phi(x), \\ B_1 u = h_1(t), \quad B_2 u = h_2(t), \quad B_3 u = h_3(t). \end{cases} \tag{4.1}$$

is considered. For any $T > 0$ and $s \geq 0$, let $X_{s,T}$ be the space defined by

$$X_{s,T} := H^s(0,1) \times H^{(s+1)/3}(0,T) \times H^{(s+1)/3}(0,T) \times H^{s/3}(0,T)$$

with norm

$$\|(\phi, \vec{h})\|_{X_{s,T}} := \left(\|\phi\|^2_{H^s(0,1)} + \|h_1\|^2_{H^{(s+1)/3}(0,T)} + \|h_2\|^2_{H^{(s+1)/3}(0,T)} + \|h_3\|^2_{H^{s/3}(0,T)} \right)^{\frac{1}{2}}$$

Let $Y_{s,T}$ be the space of functions $v(x,t)$ such that $v \in C([0,T]; H^s(0,1)) \cap L^2([0,T]; H^{s+1}(0,1))$ with its norm defined as

$$\|v\|_{Y_{s,T}} := \left(\|v\|^2_{C([0,T];H^s(0,1))} + \|v\|^2_{L^2([0,T];H^{s+1}(0,1))} \right)^{\frac{1}{2}}$$

In addition, let

$$\mathcal{Y}_{s,T} = Y_{s,T} \cap H^{s/3}(0,T; H^1(0,1))$$

with its norm defined as

$$\|v\|_{\mathcal{Y}_{s,T}} = \left(\|v\|^2_{Y_{s,T}} + \|v\|^2_{H^{\frac{s}{3}}(0,T;H^1(0,1))} \right)^{\frac{1}{2}}.$$

Note that $\mathcal{Y}_{0,T} = Y_{0,T}$.

The following two lemmas are helpful in establishing the well-posedness of (4.1).

Lemma 4.1. (i) *For $s \geq 0$ there exists a $C \geq 0$ such that for any $T > 0$ and* $u,\ v \in Y_{s,T}$,

$$\int_0^T \|uv_x\|_{H^s(0,1)}\, d\tau \leq C(T^{\frac{1}{2}} + T^{\frac{1}{3}})\|u\|_{Y_{s,T}}\|v\|_{Y_{s,T}} \tag{4.2}$$

(ii) *For $0 \leq s \leq 3$ there exists a $C \geq 0$ such that for any $T > 0$ and $u,\ v \in \mathcal{Y}_{s,T}$,*

$$\|uv_x\|_{W^{\frac{s}{3},1}(0,T;L^2(0,1))} \leq C(T^{\frac{1}{2}} + T^{\frac{1}{3}})\|u\|_{\mathcal{Y}_{s,T}}\|v\|_{\mathcal{Y}_{s,T}} \tag{4.3}$$

Proof. It is only necessary to prove (4.3) since (4.2) has already been established in [[8], Lemma 3.1]. Note first

$$\|uv_x\|_{W^{0,1}(0,T;L^2(0,1))} = \int_0^T \|uv_x\|_{L^2(0,1)} d\tau \leq C(T^{\frac{1}{2}} + T^{\frac{1}{3}})\|u\|_{\mathcal{Y}_{0,T}}\|v\|_{\mathcal{Y}_{0,T}}$$

according to (4.2) (with $s = 0$). Since

$$(uv_x)_t = u_t v_x + u v_{xt},$$

and $u_t,\ v_t \in \mathcal{Y}_{0,T}$ if $u,\ v \in \mathcal{Y}_{3,T}$, we have

$$\|(uv_x)_t\|_{W^{0,1}(0,T;L^2(0,1))} \leq C(T^{\frac{1}{2}} + T^{\frac{1}{3}})(\|u_t\|_{\mathcal{Y}_{0,T}}\|v\|_{\mathcal{Y}_{0,T}} + \|u\|_{\mathcal{Y}_{0,T}}\|v_t\|_{\mathcal{Y}_{0,T}})$$

Therefore

$$\|uv_x\|_{W^{1,1}(0,T;L^2(0,1))} \leq C_1(T^{\frac{1}{2}} + T^{\frac{1}{3}})\|u\|_{\mathcal{Y}_{3,T}}\|v\|_{\mathcal{Y}_{3,T}}$$

So estimate (4.3) is true for $s = 0$ and $s = 3$. For $0 < s < 3$ the result follows from the nonlinear interpolation theory developed in Bona and Scott [4]. □

Consider the following linear IBVP

$$\begin{cases} u_t + u_{xxx} = f, \qquad u(x,0) = \phi(x), \\ B_1 u = h_1(t), \quad B_2 u = h_2(t), \quad B_3 u = h_3(t). \end{cases} \tag{4.4}$$

Note that if u is a C^∞-smooth solution then the initial data $u(x,0) = \phi(x)$ and its boundary values $h_j(t), j = 1,2,3$ must satisfy the following compatibility conditions:

$$B_1\phi_k = h_1^{(k)}(0),\ B_2\phi_k = h_2^{(k)}(0),\ B_3\phi_k = h_3^{(k)}(0)$$

for $k = 0,1,\ldots$, where $h_j^{(k)}(t)$ is the $k-$th order derivative of h_j and

$$\begin{cases} \phi_0(x) = \phi(x), \\ \phi_k(x) = -(\phi'''_{k-1}(x) - \phi'_{k-1}(x) + \Sigma_{j=0}^{k-1}(\phi_j(x)\phi_{k-j-1}(x))'). \end{cases} \tag{4.5}$$

Therefore for any $s \geq 0$, $\phi \in H^s(0,1)$ and

$$\vec{h} = (h_1, h_2, h_3) \in H_{loc}^{\frac{s+1}{3}}(R^+) \times H_{loc}^{\frac{s+1}{3}}(R^+) \times H_{loc}^{\frac{s}{3}}(R^+),$$

$(\phi, \vec{h})$ is called $s-$compatible if

$$B_1(\phi_k) = h_1^{(k)}(0), \quad B_2(\phi_k) = h_2^{(k)}(0), \quad B_3(\phi_k) = h_3^{(k)}(0) \quad \text{in the space } H^{s-3k}(0,1) \tag{4.6}$$

for $k = 0, 1, 2, \cdots, [\frac{s}{3}] - 1$.

Lemma 4.2. *Let $T > 0$ and $0 \leq s \leq 3$ be given and assume that one of the conditions of (3.3) is satisfied. There exists a constant $C > 0$ such that for any $f \in W^{\frac{s}{3},1}(0,T; L^2(0,1))$ and $s-$compatible $(\phi, \vec{h}) \in X_{s,T}$, the IBVP (4.4) admits a unique solution $u \in \mathcal{Y}_{s,T}$ satisfying*

$$\|u\|_{\mathcal{Y}_{s,T}} \leq C \left(\|f\|_{W^{\frac{s}{3},1}(0,T;L^2(0,1))} + \|(\phi, \vec{h})\|_{X_{s,T}} \right). \tag{4.7}$$

Proof. When $s = 0$ the solution is established from the results proven in chapter three. In addition (4.7) follows from the linear estimates presented in chapter three. The remainder of the proof focuses on $s = 3$. The other cases follow from interpolation. Given that we have the solution u for $s = 0$, let $v = u_t$. Then v solves

$$\begin{cases} v_t + v_{xxx} = f_t, \qquad v(x,0) = f(x,0) - \phi'''(x), \\ B_1 v = h_1'(t), \quad B_2 v = h_2'(t), \quad B_3 v = h_3'(t). \end{cases} \tag{4.8}$$

Applying (4.7) for v gives

$$\|u_t\|_{\mathcal{Y}_{0,T}} = \|v\|_{\mathcal{Y}_{0,T}} \leq C\left(\|f\|_{W^{1,1}(0,T;L^2(0,1))} + \|(\phi,\vec{h})\|_{X_{3,T}}\right).$$

Note that

$$u_{xxx} = f - u_t,$$

and for $s = 3$, we have $f \in W^{1,1}(0,T;L^2(0,1))$ with $u, u_t \in Y_{0,t}$. It then follows for $s = 3$,

$$\|u\|_{\mathcal{Y}_{3,T}} \leq C\left(\|f\|_{W^{1,1}(0,T;L^2(0,1))} + \|(\phi,\vec{h})\|_{X_{3,T}}\right).$$

□

Next the well-posedness of the nonlinear IBVP (4.1) is established. First it is shown that the IBVP (4.1) is locally well-posed in the space $H^s(0,1)$ for $0 \leq s \leq 3$.

Theorem 4.3. *Assume one of the conditions of (3.3) is satisfied. Let $T > 0$, $0 \leq s \leq 3$ and $\eta > 0$ be given. There exists $T^* \in (0,T]$ such that for any $s-$compatible $(\phi,\vec{h}) \in X_{s,T}$ with*

$$\|(\phi,\vec{h})\|_{X_{s,T}} \leq \eta,$$

the IBVP (4.1) admits a unique solution $u \in \mathcal{Y}_{s,T^*}$. Moreover the solution depends Lipschitz continuously on $(\psi,\vec{h})$ in the corresponding spaces.

Proof. Let $(\phi,\vec{h}) \in X_{s,T}$ be as given and rewrite (4.1) as

$$\begin{cases} u_t + u_{xxx} = -u_x - uu_x, \\ u(x,0) = \phi(x), \\ B_1u = h_1(t), \quad B_2u = h_2(t), \quad B_3u = h_3(t) \end{cases}$$

Let $r > 0$ and $0 < \theta \leq \max\{1,T\}$ be constants to be determined. Set

$$S_{\theta,r} = \{v \in \mathcal{Y}_{s,\theta} \mid \|v\|_{\mathcal{Y}_{s,\theta}} \leq r, \}$$

which is a bounded closed convex subset of $\mathcal{Y}_{s,\theta}$. Define a map Γ on $S_{\theta,r}$ by

$$u = \Gamma(v)$$

where using the results from chapter three we have u being the unique solution of

$$\begin{cases} u_t + u_{xxx} = -v_x - vv_x, \\ u(x,0) = \phi(x), \\ B_1 u = h_1(t), \quad B_2 u = h_2(t), \quad B_3 u = h_3(t) \end{cases}$$

for $v \in S_{\theta,r}$. According to Lemma (4.2), for any $v \in S_{\theta,r}$, we have

$$\|\Gamma(v)\|_{\mathcal{Y}_{s,\theta}} \leq C_0 \|(\phi, \vec{h})\|_{X_{s,T}} + C\|(v+1)v_x\|_{W^{\frac{s}{3},1}(0,\theta;L^2(0,1))}$$

then using Lemma (4.1), and given $\|(\phi, \vec{h})\|_{X_{s,T}} \leq \eta$ with $0 < \theta \leq \max\{1, T\}$

$$\begin{aligned} \|\Gamma(v)\|_{\mathcal{Y}_{s,\theta}} &\leq C_0\eta + C\|(v+1)v_x\|_{W^{\frac{s}{3},1}(0,\theta;L^2(0,1))} \\ &\leq C_0\eta + C(\theta^{\frac{1}{2}} + \theta^{\frac{1}{3}})\|v+1\|_{\mathcal{Y}_{s,\theta}}\|v\|_{\mathcal{Y}_{s,\theta}} \\ &\leq C_0\eta + C_1\theta^{\frac{1}{3}}\|v\|_{\mathcal{Y}_{s,\theta}}(\|v\|_{\mathcal{Y}_{s,\theta}} + 1) \end{aligned}$$

For given $0 < \alpha < 1$, choose $r > 0$ and $0 < \theta \leq 1$ such that

$$r = 2C_0\eta \qquad \text{and} \qquad C_1\theta^{\frac{1}{3}}(r+1) \leq \frac{1}{2}\alpha \tag{4.9}$$

then for any $v \in S_{\theta,r}$

$$\begin{aligned} \|\Gamma(v)\|_{\mathcal{Y}_{s,\theta}} &\leq C_0\|(\phi, \vec{h})\|_{X_{3,T}} + C_1\theta^{\frac{1}{3}}\|v\|_{\mathcal{Y}_{s,\theta}}(\|v\|_{\mathcal{Y}_{s,\theta}} + 1) \\ &\leq \frac{r}{2} + C_1\theta^{\frac{1}{3}} r(r+1) \\ &\leq \frac{r}{2} + \frac{r}{2} = r \end{aligned}$$

Therefore $\Gamma : S_{\theta,r} \to S_{\theta,r}$. Recall that

$$u = \Gamma(v) = W_0(t)\phi(x) + [W_b(t)\vec{h}](x) + \int_0^\theta W_0(t-\tau)[(v+1)v_x]d\tau$$

. Then using Lemma (4.1), $0 < \theta \le \max\{1, T\}$ and for any $u, v \in S_{\theta,r}$;

$$\begin{aligned}
\|\Gamma(u) - \Gamma(v)\|_{\mathcal{Y}_{s,\theta}} &= \| \int_0^\theta W_0(t-\tau)[(u+1)u_x - (v+1)v_x]d\tau\|_{\mathcal{Y}_{s,\theta}} \\
&\le C\|(u+1)u_x - (v+1)v_x\|_{W^{\frac{s}{3},1}(0,\theta;L^2(0,1))} \\
&\le C\left(\|(u+1)(u-v)_x\|_{W^{\frac{s}{3},1}(0,\theta;L^2(0,1))} + \|(u-v)v_x\|_{W^{\frac{s}{3},1}(0,\theta;L^2(0,1))}\right) \\
&\le C_1\theta^{\frac{1}{3}}(\|u\|_{\mathcal{Y}_{s,\theta}} + \|v\|_{\mathcal{Y}_{s,\theta}} + 1)\|u-v\|_{\mathcal{Y}_{s,\theta}} \\
&\le 2C_1\theta^{\frac{1}{3}}(1+r)\|u-v\|_{\mathcal{Y}_{s,\theta}} \\
&\le \alpha\|(u-v)\|_{\mathcal{Y}_{s,\theta}}.
\end{aligned}$$

Therefore the map Γ is a contraction mapping on $S_{\theta,r}$. Its fixed point $u = \Gamma(u)$ is the desired solution. □

Next consider the following linearized IBVP associated to (4.1).

$$\begin{cases} u_t + u_x + (a(x,t)u)_x + u_{xxx} = f, \qquad u(x,0) = \phi(x), \\ B_1u = h_1(t), \quad B_2u = h_2(t), \quad B_3u = h_3(t) \end{cases} \tag{4.10}$$

where $a(x,t)$ is a given function.

Proposition 4.4. *Let $T > 0$ and $0 \le s \le 3$ be given and assume that $a \in \mathcal{Y}_{s,T}$ and that one of the conditions of (3.3) is satisfied. Then for any $s-$compatible $(\phi, \vec{h}) \in X_{s,T}$ and $f \in W^{\frac{s}{3},1}(0,T;L^2(0,1))$, the IBVP (4.10) admits a unique solution $u \in \mathcal{Y}_{s,T}$. Moreover, there exists a constant $C > 0$ depending only on T and $\|a\|_{\mathcal{Y}_{s,T}}$ such that*

$$\|u\|_{\mathcal{Y}_{0,T}} \le C\left(\|(\phi,\vec{h})\|_{X_{s,T}} + \|f\|_{W^{\frac{s}{3},1}(0,T;H^s(0,1))}\right).$$

PROOF. For given $(\phi, \vec{h})$ and f, rewrite (4.10) as

$$\begin{cases} u_t + u_{xxx} = f - u_x - (au)_x, \\ u(x,0) = \phi(x), \\ B_1 u = h_1(t), \quad B_2 u = h_2(t), \quad B_3 u = h_3(t) \end{cases}$$

As in the proof of Theorem (4.3), consider the map

$$u = \Gamma(v)$$

for any $v \in \mathcal{Y}_{s,\theta}$ where $0 < \theta \leq \max\{1, T\}$ and u is the unique solution of

$$\begin{cases} u_t + u_{xxx} = f - v_x - (av)_x, \\ u(x,0) = \phi(x), \\ B_1 u = h_1(t), \quad B_2 u = h_2(t), \quad B_3 u = h_3(t) \end{cases}$$

Its norm in the space $\mathcal{Y}_{s,\theta}$ is estimated by

$$\begin{aligned} \|\Gamma(v)\|_{\mathcal{Y}_{s,\theta}} &\leq C(\|f - v_x - (av)_x\|_{W^{\frac{s}{3},1}(0,\theta;L^2(0,1))} + \|(\phi,\vec{h})\|_{X_{s,T}}) \\ &\leq C(\|f\|_{W^{\frac{s}{3},1}(0,\theta;L^2(0,1))} + \|(1+a)v_x\|_{W^{\frac{s}{3},1}(0,\theta;L^2(0,1))} + \|a_x v\|_{W^{\frac{s}{3},1}(0,\theta;L^2(0,1))} + \|(\phi,\vec{h})\|_{X_{s,T}}) \\ &\leq C_0(\|(\phi,\vec{h})\|_{X_{s,T}} + \|f\|_{W^{\frac{s}{3},1}(0,T;L^2(0,1))}) + C(\theta^{\frac{1}{2}} + \theta^{\frac{1}{3}})(\|1+a\|_{\mathcal{Y}_{s,T}} + \|a\|_{\mathcal{Y}_{s,T}})\|v\|_{\mathcal{Y}_{s,\theta}} \\ &\leq C_0(\|(\phi,\vec{h})\|_{X_{s,T}} + \|f\|_{W^{\frac{s}{3},1}(0,T;L^2(0,1))}) + C_1\theta^{\frac{1}{3}}(2\|a\|_{\mathcal{Y}_{s,T}} + 1)\|v\|_{\mathcal{Y}_{s,\theta}} \\ &\leq C_0(\|(\phi,\vec{h})\|_{X_{s,T}} + \|f\|_{W^{\frac{s}{3},1}(0,T;L^2(0,1))}) + 2C_1\theta^{\frac{1}{3}}(\|a\|_{\mathcal{Y}_{s,T}} + 1)\|v\|_{\mathcal{Y}_{s,\theta}} \end{aligned}$$

where we have used Lemma (4.1). Thus, if r and θ are chosen such that for $0 < \alpha < 1$

$$r = 2C_0(\|(\phi,\vec{h})\|_{X_{s,T}} + \|f\|_{W^{\frac{s}{3},1}(0,T;L^2(0,1))}), \quad C_1\theta^{\frac{1}{3}}(\|a\|_{\mathcal{Y}_{s,T}} + 1) = \frac{1}{2}\alpha \tag{4.11}$$

then for any $v,\ u \in S_{r,\theta}$ in the space $\mathcal{Y}_{s,\theta}$,

$$\|\Gamma(v)\|_{\mathcal{Y}_{s,\theta}} \leq \frac{r}{2} + \frac{r}{2} = r$$

and as in the previous proof

$$\begin{aligned}\|\Gamma(u)-\Gamma(v)\|_{\mathcal{Y}_{s,\theta}} &= \|\int_0^\theta W_0(t-\tau)[(f+u_x+(au)_x)-(f+v_x+(av)_x)]d\tau\|_{\mathcal{Y}_{s,\theta}} \\ &\leq C\left(\|(a+1)(u-v)_x\|_{W^{\frac{s}{3},1}(0,\theta;L^2(0,1))}+\|a_x(u-v)\|_{W^{\frac{s}{3},1}(0,\theta;L^2(0,1))}\right) \\ &\leq C_1\theta^{\frac{1}{3}}\left(2\|a\|_{\mathcal{Y}_{s,\theta}}+1\right)\|u-v\|_{\mathcal{Y}_{s,\theta}} \\ &\leq \alpha\|(u-v)\|_{\mathcal{Y}_{s,\theta}}.\end{aligned}$$

with $\alpha = C_1\theta^{\frac{1}{3}}(2\|a\|_{\mathcal{Y}_{s,T}}+1) < 1$. That is to say, Γ is a contraction mapping from $S_{r,\theta}$ to $S_{r,\theta}$ if r and θ are chosen according to (4.11). Its fixed point $u \in S_{r,\theta}$ solves the IBVP (4.10) in the time interval $[0,\theta]$. Note that θ depends only on $\|a\|_{\mathcal{Y}_{s,T}}$ (In fact, $\theta = \max\{1, \left(\frac{1}{2C_1(\|a\|_{\mathcal{Y}_{s,T}}+1)}\right)^3\}$). By the standard extension argument, the solution u can be extended to the time interval $[0,T]$ such that $u \in \mathcal{Y}_{s,T}$. The proof is complete. □

Theorem (4.3) can be extended to the case where $s > 3$ for the IBVP (4.1).

Theorem 4.5. *Assume one of the conditions of (3.3) is satisfied. Let $T > 0$, $s > 3$ and $\eta > 0$ be given. There exists $T^* \in (0,T]$ such that for any $s-$compatible $(\phi,\vec{h}) \in X_{s,T}$ with*

$$\|(\phi,\vec{h})\|_{X_{3,T}} \leq \eta,$$

the IBVP (4.1) admits a unique solution $u \in \mathcal{Y}_{s,T^*}$. Moreover the solution depends Lipschitz continuously on $(\psi,\vec{h})$ in the corresponding spaces.

Remark 4.6. *: In the above theorem, the length of the time interval $(0,T^*)$ depends only on $\|(\phi,\vec{h})\|_{X_{3,T}}$ instead of $\|(\phi,\vec{h})\|_{X_{s,T}}$.*

Proof. Consider the case $3 < s \leq 6$. The others can be proved similarly. First of all, according to Theorem (4.3), the IBVP (4.1) admits a unique solution $u \in \mathcal{Y}_{3,T^*}$. We just need to prove this solution u also belong to the space $\mathcal{Y}_{s,T^*}$. To see that, let $v = u_t$. Then v

solves the following linearized IBVP

$$\begin{cases} v_t + v_x + (a(x,t)v)_x + v_{xxx} = 0, \\ v(x,0) = \phi_1(x), \\ B_1 v = h_1^{(1)}(t), \quad B_2 v = h_2^{(1)}(t), \quad B_3 v = h_3^{(1)}(t) \end{cases}$$

where $a(x,t) = u(x,t) \in \mathcal{Y}_{3,T^*}$ and

$$\phi_1 \in H^{s-3}(0,1), \; h_1^{(1)}, \; h_2^{(1)} \in H^{\frac{s-2}{3}}(0,T^*), \quad h_3^{(1)} \in H^{\frac{s-3}{3}}(0,T^*).$$

It thus follows from Proposition (4.4) that

$$v = u_t \in \mathcal{Y}_{s-3,T^*}$$

and therefore

$$u \in \mathcal{Y}_{s,T^*}$$

since

$$u_{xxx} = -u_t - u_x - uu_x.$$

□

CHAPTER 5

Conclusions

The focus of this dissertation has been the the nonhomogeneous IBVP of the KdV equation posed on the finite interval $(0,1)$

$$\begin{cases} u_t + u_x + uu_x + u_{xxx} = 0, \qquad u(x,0) = \phi(x), \\ B_1 u = h_1(t), \quad B_2 u = h_2(t), \quad B_3 u = h_3(t) \end{cases} \tag{5.1}$$

for its well-posedness in the space $H^s(0,1)$ with initial data $\phi \in H^s(0,1)$ and the boundary value data

$$\vec{h} \in H_{loc}^{\frac{s+1}{3}}(R^+) \times H_{loc}^{\frac{s+1}{3}}(R^+) \times H_{loc}^{\frac{s}{3}}(R^+).$$

After establishing the global Kato smoothing property for the associated linear problem, the IBVP (5.1) is shown to be locally well-posed in the space $H^s(0,1)$ for any $s \geq 0$ via the *contraction mapping principle*. In particular, the life span $(0,T^*)$ of the solution depends only on the norm of auxiliary data $(\phi, \vec{h})$ in the space $X_{s,T}$ when $0 \leq s \leq 3$, but in the space $X_{3,T}$ when $s > 3$. The results from this research have improved the earlier works of Bubnov [20], [21] and Colin and Ghidaglia [23] and have extended the local well-posednes results of Bona, Sun and Zhang in [8] for the KdV equation posed on a finite domain with Dirichlet boundary conditions to the IBVP (5.1) where a class of general boundary conditions are imposed. There are still many problems left open for further study.

Problem 1: *Is the IBVP (5.1) globally well-posed in the space $H^s(0,1)$ for any $s \geq 0$?*

As indicated in the work of Bona, Sun and Zhang [8], it suffices establish the following *a priori* global estimate for solutions of the IBVP (5.1) with homogenous boundary data ($h_1 = h_2 = h_3 \equiv 0$) in the space $L^2(0,1)$:

for any $T > 0$, there exists a constant C_T such that

$$\sup_{0 \leq t \leq T} \|u(\cdot,t)\|_{L^2(0,1)} \leq C_T. \tag{5.2}$$

To try and establish this we will consider an energy estimate and determine where $\frac{dE(t)}{dt} \leq 0$. Consider the differential equation (5.1) and multiply both sides of the by u and integrate over $(0,1)$ with respect to x.

$$\int_0^1 u_t u dx = -\int_0^1 u^2 u_x dx - \int_0^1 u u_x dx - \int_0^1 u u_{xxx} dx$$

Using integration by parts we find that

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int_0^1 u^2 dx &= -\frac{1}{3}\int_0^1 (u^3)_x dx - \frac{1}{2}\int_0^1 (u^2)_x dx - \int_0^1 u u_{xxx} dx \\ &= -\frac{1}{3}u^3\Big|_0^1 - \frac{1}{2}u^2\Big|_0^1 - u_{xx}u\Big|_0^1 + \frac{1}{2}u_x^2\Big|_0^1 \end{aligned} \tag{5.3}$$

Consider the conditions placed on the parameters to establish local well-posedness, namely,

$$\begin{cases} (a) \text{ If } \alpha_1\beta_1\xi_1 \neq 0, \text{ then } F_1 \geq 0, F_2 \geq 0 \\ (b) \text{ If } \beta_1 \neq 0, \xi_1 \neq 0, \alpha_1 = 0 \text{ then } F_2 \geq 0, \alpha_2 = 0, \alpha_3 \neq 0 \\ (c) \text{ If } \beta_1 = 0, \xi_1 \neq 0, \alpha_1 \neq 0, \text{ then } F_1 \geq 0, \quad F_3 \neq 0 \\ (d) \text{ If } \beta_1 = 0, \alpha_1 \neq 0, \xi_1 = 0, \text{ then } F_1 \geq 0, \quad F_3 \neq 0 \end{cases} \tag{5.4}$$

where,

$$F_1 = \frac{\alpha_3}{\alpha_1} - \frac{\alpha_2^2}{2\alpha_1^2}, F_2 = \frac{\beta_2\xi_2}{\beta_1\xi_1} - \frac{\beta_3}{\beta_1} - \frac{{\xi_2}^2}{2{\xi_1}^2} \text{ and } F_3 = \beta_2\xi_2 - \beta_3\xi_1.$$

Consider case (a), where $\alpha_1\beta_1\xi_1 \neq 0$. As in the proof of lemma (3.1) in chapter three, the boundary conditions become

$$u_{xx}(0) = \frac{-\alpha_2}{\alpha_1}u_x(0) - \frac{\alpha_3}{\alpha_1}u(0), \qquad u_{xx}(1) = (\frac{\beta_2\xi_2}{\beta_1\xi_1} - \frac{\beta_3}{\beta_1})u(1), \qquad u_x(1) = \frac{-\xi_2}{\xi_1}u(1)$$

using these in (5.3), along with the inequality (3.7) we find that

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int_0^1 u^2 dx &= -\frac{1}{3}u^3(1) + \frac{1}{3}u^3(0) - \frac{1}{2}u^2(1) + \frac{1}{2}u^2(0) - \Big(\frac{\beta_2\xi_2}{\beta_1\xi_1} - \frac{\beta_3}{\beta_1}\Big)u^2(1) \\
&\quad - \frac{\alpha_2}{\alpha_1}u_x(0)u(0) - \frac{\alpha_3}{\alpha_1}u^2(0) + \frac{{\xi_2}^2}{2{\xi_1}^2}u^2(1) - \frac{1}{2}{u_x}^2(0) \\
&= -\frac{1}{3}u^3(1) + \frac{1}{3}u^3(0) - \Big(\frac{\beta_2\xi_2}{\beta_1\xi_1} - \frac{\beta_3}{\beta_1} - \frac{{\xi_2}^2}{2{\xi_1}^2} + \frac{1}{2}\Big)u^2(1) - \frac{\alpha_2}{\alpha_1}u_x(0)u(0) \\
&\quad - \frac{\alpha_3}{\alpha_1}u^2(0) - \frac{1}{2}{u_x}^2(0) + \frac{1}{2}u^2(0) \\
&\leq -\frac{1}{3}u^3(1) + \frac{1}{3}u^3(0) - \Big(\frac{\beta_2\xi_2}{\beta_1\xi_1} - \frac{\beta_3}{\beta_1} - \frac{{\xi_2}^2}{2{\xi_1}^2} + \frac{1}{2}\Big)u^2(1) - \Big(\frac{\alpha_3}{\alpha_1} - \frac{{\alpha_2}^2}{2{\alpha_1}^2} - \frac{1}{2}\Big)u^2(0)
\end{aligned}$$

Recall that for (a), $F_1 \geq 0, F_2 \geq 0$. However to control the first two terms, $-\frac{1}{3}u^3(1) + \frac{1}{3}u^3(0)$, we need to further assume that $u(0) = 0$ and $u(1) = 0$. Then the boundary conditions reduce to $u(0) = 0$, $u(1) = 0$, and $u_x(1) = 0$.

Consider the second case with $\beta_1 \neq 0, \xi_1 \neq 0, \alpha_1 = 0$. As before use the rewritten boundary conditions which gives:

$$u_x(0) = -\frac{\alpha_3}{\alpha_2}u(0), \qquad u_{xx}(1) = \Big(\frac{\beta_2\xi_2}{\beta_1\xi_1} - \frac{\beta_3}{\beta_1}\Big)u(1), \qquad u_x(1) = \frac{-\xi_2}{\xi_1}u(1)$$

using these in (5.3), we have

$$\frac{1}{2}\frac{d}{dt}\int_0^1 u^2 dx = -\frac{1}{3}u^3(1) + \frac{1}{3}u^3(0) - \Big(\frac{\beta_2\xi_2}{\beta_1\xi_1} - \frac{\beta_3}{\beta_1} - \frac{{\xi_2}^2}{2{\xi_1}^2} + \frac{1}{2}\Big)u^2(1)$$

$$u_{xx}(0)u(0) + \Big(\frac{\alpha_3}{\alpha_2} + \frac{1}{2}\Big)u^2(0)$$

Using $F_2 \geq 0, \alpha_2 = 0, \alpha_3 \neq 0$ from (b), we know that $u(0) = 0$. This still leaves the term $-\frac{1}{3}u^3(1)$. In order to control this term we will need to assume that $u(1) = 0$. This implies then that $u_x(1) = 0$, which leads us back to the reduced boundary conditions $u(0) = 0$, $u(1) = 0$, and $u_x(1) = 0$.

Lastly consider cases (c) and (d). Using the boundary conditions we have

$$u_{xx}(0) = \frac{-\alpha_2}{\alpha_1}u_x(0) - \frac{\alpha_3}{\alpha_1}u(0), \qquad u_x(1) = 0, \qquad u(1) = 0$$

using these in (5.3), along with the inequality (3.7) we find that

$$\frac{1}{2}\frac{d}{dt}\int_0^1 u^2 dx \leq \frac{1}{3}u^3(0) - \Big(\frac{\alpha_3}{\alpha_1} - \frac{{\alpha_2}^2}{2{\alpha_1}^2} - \frac{1}{2}\Big)u^2(0)$$

$$= -\Big(\frac{\alpha_3}{\alpha_1} - \frac{{\alpha_2}^2}{2{\alpha_1}^2} - \frac{1}{2} - \frac{1}{3}u(0)\Big)u^2(0)$$

As with previous cases we will have to assume that $u(0) = 0$.

Therefore, using this approach, $\frac{dE(t)}{dt} \leq 0$ is available only in the case of the Dirichlet boundary conditions

$$u(0,t) = 0, \quad u(1,t) = 0, \quad u_x(1,t) = 0.$$

Problem 2: *Is the IBVP (5.1) well-posed in the space $H^s(0,1)$ for some $s < 0$?*

The IBVP (5.1) with the Dirichlet boundary conditions

$$u(0,t) = h_1(t), \quad u(1,t) = h_2(t), \quad u_x(1,t) = h_3(t)$$

has been proved to be locally well-posed in the space $H^s(0,1)$ for any $-1 < s < 0$ [11]. It will be interesting to see if this will hold for the IBVP with general boundary conditions presented in this research.

One may also view the boundary value functions $h_j,\ j = 1,2,3$ as control input and study the IBVP (5.1) from control theory point of view as in [76].

Problem 3: *Is the system (5.1) exactly controllable?*

Control and stabilization of the KdV equation has been extensively studied in the past two decades (an overall view of the subject is presented in [60]). The control question posed here is given an initial state ϕ and a terminal state ψ in a certain space, is there an appropriate control input f so that the equation (5.1) admits a solution u which equals ϕ at time $t = 0$ and equals ϕ at time $t = T$? To be exactly controllable a control input f can always be found to guide the system from any given initial state to any given terminal state.

Appendix

Introduction to Semigroup Theory

Semigroup theory can be applied to solve many time-dependent partial differential equations as ordinary differential equations on a function space. This section contains a brief introduction to the elements of semigroup theory from Pazy [58] used in chapter three.
Let X be a Banach space with norm $\|\cdot\|$,

Definition 5.1. *A linear operator A with domain $\mathcal{D}(A) \subseteq X$ and range in the Banach space Y, is called a closed operator if it has the property that whenever $x_n \subseteq \mathcal{D}(A)$ satisfying $x_n \to x \in X$ and $Ax_n \to y \in Y$, then $x \in \mathcal{D}(A)$ and $Ax = y$.*

Definition 5.2. *An operator $(\mathcal{D}(A), A)$ is closable if for every sequence $x_n \in A$ such that $x_n \to 0$ we have either*

$$\text{(i) } Ax_n \to 0, \qquad \text{or} \qquad \text{(ii) } \lim_{n\to 0} Ax_n \text{ does not exist.}$$

We can conclude that all bounded operators are closed as well as differential operators.

Definition 5.3. *A linear operator A with domain $\mathcal{D}(A) \subseteq X$ is a densely defined operator for the Banach space X if $\overline{\mathcal{D}(A)} = X$.*

Definition 5.4. *A linear operator A is dissipative if for every $x \in \mathcal{D}(A)$ there is*

$$x^* \in F(x) = \left\{x^* : x^* \in X^* \text{ and } \langle x^*, x\rangle = \|x\|^2 = \|x^*\|^2\right\},$$

such that

$$Re\langle Ax, x^*\rangle \leq 0,$$

where $F(x) \subset X^*$.

Note that if the space is Hilbert, for the operator to be dissipative we need $\mathrm{Re}\langle Ax, x\rangle \leq 0$.

Definition 5.5. *A one-parameter family* $T(t), 0 \leq t \leq \infty$, *of bounded linear operators from* X *into* X *is a semigroup on* X *if*

(i) $T(0) = I$, *where* I *is the identity operator on* X;

(ii) $T(s+t) = T(s)T(t)$ *for every* $s, t \geq 0$ *(the semigroup property).*

A semigroup $T(t), 0 \leq t \leq \infty$ of bounded linear operators is a uniformly continuous semigroup if

$$\lim_{t \to 0} ||T(t) - I|| = 0. \tag{5.5}$$

From the definition, we can conclude that if $T(t)$ is a uniformly continuous semigroup of bounded linear operators, then

$$\lim_{x \to t} ||T(x) - T(t)|| = 0. \tag{5.6}$$

The linear operator A defined by

$$\begin{aligned} \mathcal{D}(A) &= \left\{ x \in X : \lim_{t \to 0} \frac{T(t)x - x}{t} \text{ exists} \right\}, \\ Ax &= \lim_{t \to 0} \frac{T(t)x - x}{t} = \frac{d^+ T(t)x}{dt} \Big|_{t=0}, \end{aligned} \tag{5.7}$$

is called the infinitesimal generator of the semigroup $T(t)$, and $\mathcal{D}(A)$ is the domain of the operator A.

Definition 5.6. *A semigroup* $T(t)$, *of bounded linear operators on* X *is strongly continuous semigroup of bounded operators if*

$$\lim_{t \to 0} T(t)x = x, \tag{5.8}$$

for every $x \in X$.

A strongly continuous semigroup of bounded linear operators on X will be called a semigroup of class C_0 or a C_0 semigroup.

Theorem 5.7. *Let $T(t)$ be a C_0 semigroup. There exist constants $\omega \geq 0$ and $M \geq 1$ such that*

$$\|T(t)\| \leq Me^{\omega t} \qquad for\ 0 \leq t < \infty \tag{5.9}$$

If $\omega = 0$, $T(t)$ is uniformly bounded and if moreover $M = 1$ it is called a C_0 semigroup of contractions. In other words if $||T(t)x|| \leq ||x||$ that is, $||T(t)|| \leq 1$ for each $t \geq 0$, then T is called a C_0 contraction semigroup.

Now we recall that if A is a linear, not necessarily bounded, operator in X, the resolvent set $\rho(A)$ of A is the set of all complex numbers λ for which $\lambda I - A$ is invertible,that is, $(\lambda I - A)^{-1}$ is a bounded operator in X. The family $R(\lambda : A) = (\lambda I - A)^{-1},\ \lambda \in \rho(A)$, of bounded linear operators is called the resolvent of A.

The following theorem is known as Hille–Yosida's theorem.

Theorem 5.8. *A linear (unbounded) operator A is the infinitesimal generator of a C_0 semigroup of contractions $T(t), t \geq 0$, if and only if*

(i) A is closed and $\overline{\mathcal{D}(A)} = X$;

(ii) the resolvent set $\rho(A)$ of A contains R^+ and for every $\lambda > 0$,

$$||R(\lambda : A)|| \leq \frac{1}{\lambda}. \tag{5.10}$$

As a consequence of this theorem we obtain the following corollary:

Corollary 5.9. *A linear operator A is the infinitesimal generator of a C_0 semigroup satisfying $||T(t)|| \leq e^{wt}$ if and only if*

(i) A is closed and $\overline{\mathcal{D}(A)} = X$;

(ii) the resolvent set $\rho(A)$ of A contains the ray $\{\lambda : Im\lambda = 0, \lambda > w\}$ and for every $\lambda > 0$,

$$||R(\lambda : A)|| \leq \frac{1}{\lambda - w}. \tag{5.11}$$

The Lumer-Phillips theorem and its corollary are used in considering the associated linear problem for the IBVP.

Theorem 5.10. *Let A be a linear operator with dense domain $\mathcal{D}(A)$ in X.*

(i) *If A is dissipative and there is a $\lambda_0 > 0$ such that the range, $R(\lambda_0 I - A)$, of $\lambda_0 I - A$ is X, then A is the infinitesimal generator of a C_0 semigroup of contractions on X.*

(ii) *If A is the infinitesimal generator of a C_0 semigroup of contractions on X then $R(\lambda I - A) = X$ for all $\lambda > 0$ and A is dissipative. Moreover, for every $x \in \mathcal{D}(A)$ and every $x^* \in F(x), Re\left\langle Ax, x^* \right\rangle \leq 0$.*

As a result we have the following corollary:

Corollary 5.11. *Let A be a densely defined closed operator. If both A and A^* are dissipative, then A is the infinitesimal generator of a C_0 semigroup of contractions on X.*

The previous results allows us to prove the following theorem especially useful in the research presented.

Theorem 5.12. *Let A be a closed densely defined linear operator. The initial value problem*

$$\begin{cases} \frac{du(t)}{dt} = Au(t) + f(t), \ t > 0, \\ u(t_0) = x, \ x \in X \end{cases} \tag{5.12}$$

with $f \in C(R^+, X)$ has a unique solution $u(t)$, which is continuously differentiable on $[0, \infty)$, for every initial value $x \in \mathcal{D}(A)$ if and only if A is the infinitesimal generator of a C_0 semigroup $T(t)$.

Bibliography

[1] Benjamin, T. B., Bona, J. L., & Mahony, J. J. *Model equations for long waves in nonlinear dispersive systems*, Proc. R. Soc. Lond. A., **272** (1972), 47–78.

[2] Bona, J. L.& Luo, L.,*Generalized Korteweg-de Vries equation in a quarter plane*, Contemp. Math. **221** (1999), 59–125.

[3] Bona, J. L., Pritchard, W. G. & Scott, L. R., *An evaluation of a model equation for water waves*, Philos. Trans. Roy. Soc. London Ser. A. **302** (1981), 457–510.

[4] Bona, J. L. & Scott, L. R., *Solutions of the Korteweg-de Vries equation in fractional order Sobelev Spaces*, Duke Math Journal, **43** (1976), 87–99.

[5] Bona, J. L. & Smith, R. , *The initial value problem for the Korteweg-de Vries equation*, Philos. Trans. Roy.Soc. London A, **278** (1978), 555–601.

[6] Bona, J. L., Sun, S. M. & Zhang, B.-Y., *A nonhomogeneous boundary- value problem for the Korteweg-de Vries equation in a quarter plane*, Trans. American Math. Soc. **354** (2002), 427–490.

[7] Bona, J. L., Sun, S. M. & Zhang, B.-Y., *Forced Oscillations of a Damped Korteweg-de Vries Equation in a quarter plane*, Comm. in PDEs **5** (2003), 369–400.

[8] Bona, J. L., Sun, S. M. & Zhang, B.-Y., *A nonhomogeneous boundary-value problem for the Korteweg-de Vries Equation on a finite domain*, Comm. in PDEs. **28** (2003), 1391–1436.

[9] Bona, J. L., Sun, S. M. & Zhang, B.-Y., *Conditional and unconditional well-posedness of nonlinear evolution equations*, Adv. Differential Eq. **9** (2004), 241–265.

[10] Bona, J. L., Sun, S. M. & Zhang, B.-Y., *Boundary Smoothing Properties of the Korteweg-de Vries Equation in a Quarter Plane and Applications*, Dynamics of PDEs. **3** (2006), 1–69.

[11] Bona, J. L., Sun, S. M. and Zhang, B.-Y., Nonhomogeneous problem for the Korteweg-de Vries equation in a bounded domain II, *J. Diff. Eqns*, to appear.

[12] Bona, J. L. & Winther, R. , *The Korteweg-de Vries Equation, Posed in a Quarter-Plane*, SIAM J. Math. Anal **14**(1983), 1056-1106.

[13] Bona, J. L. & Winther, R. , *The Korteweg-de Vries equation in a quarter-plane, continuous dependence results*, Differential and Integral Equations **2**(1989), 228–250.

[14] Bourgain, J., *Fourier transform restriction phenomena for certain lattice subsets and applications to nonlinear evolution equations, part I: Shrödinger equations*, Geom. Funct. Anal. **3**(1993), 107–156.

[15] Bourgain, J., *Fourier transform restriction phenomena for certain lattice subsets and applications to nonlinear evolution equations, part II: the KdV-equation*, Geom. Funct. Anal. **3**(1993), 209–262.

[16] Bourgain, J., *Periodic Korteweg de Vries equation with measures as initial data*, Selecta Math. **3** (1997), 115–159, 1997.

[17] Boussinesq, J., *Théorie de l'intumescence liquide a appelée "onde solitaire" ou "de translation", se propageant dans un canal rectangulaire*, C.R. Acad. Sci. Paris, **72** (1871), 755–759.

[18] Boussinesq, J., *Théorie général de mouvements, qui sont proapgés das un canal rectangulaire horizontal*, C.R. Acad. Sci. Paris, **73** (1871), 256–260.

[19] Boussinesq, J., *Essai sur la théorie des eaux courantes*; Mémoires présentés par divers savants à l'Acad. des Sci. Inst. Nat. France, **23** (1877), 1–680.

[20] Bubnov, B. A., *Generalized boundary value problems for the Korteweg-de Vries equation in bounded domain*, Differential Equations **15**(1979), 17–21.

[21] Bubnov, B. A., *Solvability in the large of nonlinear boundary-value problem for the Korteweg-de Vries equations*,Differential Equations, **16**(1980), 24–30.

[22] Christ, M., Colliander, J., Tao, T., *Asymptotics, frequency modulation, and low regularity ill-posedness for canonical defocusing equations*, Amer. J. Math. **125** (2003), 1235–1293.

[23] Colin, T., Ghidaglia, J.-M., *An initial-boundary-value problem fo the Korteweg-de Vries Equation posed on a finite interval*, Adv. Differential Equations **6** (2001), 1463-1492.

[24] Colliander, J., Keel, M., Staffilani, G., Takaoka, H., Tao, T., *Sharp global well-posedness results for periodic and non-periodic KdV and modified KdV on R and T*, Journ. American Math. Soc. **16**(2003) 705–749.

[25] Colliander, J. E., Kenig, C., *The generalized Korteweg-de Vries equation on the half line*, Comm. Partial Differential Eq. **27** (2002), 2187–2266.

[26] Constantin, P., Saut, J., *Local smoothing properties of dispersive equations*, J. American Math. Soc., **1** (1988), 413 – 446.

[27] de Jager, E. M., *On the Origin of the Korteweg-de Vries Equation*, arXiv:math/0602661v1 [math.HO], 2006

[28] Faminskii, A. V., *The Cauchy problem and the mixed problem in the half strip for equation of Korteweg-de Vries type*, (Russian) Dinamika Sploshn. Sredy **162** (1983), 152–158.

[29] Faminskii, A. V., *A mixed problem in a semistrip for the Korteweg-de Vries equation and its generalizations*, (Russian) Dinamika Sploshn. Sredy **258** (1988), 54–94; English transl. in Trans. Moscow Math. Soc. **51** (1989), 53–91

[30] Faminskii, A. V., *Mixed problms fo teh Korteweg-de Vries equation*, Sbornik: Mathematics **190** (1999), 903–935.

[31] Faminskii, A. V., *An initial boundary-value problem in a half-strip for the Korteweg-de Vries equation in fractional order Sobelev Spaces*, Comm. Partial Differential Eq. **29** (2004), 1653–1695.

[32] Fermi, E., Pasta, J., and Ulam, S. "Studies in Nonlinear Problems, I." Los Alamos report LA 1940, 1955.

[33] Fokas, A. S. & Its, A. R., *Soliton generation for initial-boundary value problems*, Phys. Rev. Lett. **68** (1992), 3117–3120.

[34] Fokas, A. S. & Its, A. R., *An initial-boundary value problem for the Korteweg-de Vries equation*, Proceedings of the conference: Solitons, nonlinear wave equations and computation (New Brunswick, NJ, 1992), Math. Comput. Simulation **37** (1994), 293–321.

[35] Fokas, A. S. & Its, A. R., *Integrable equations on the half-infinite line. Solitons in science and engineering: theory and applications*, Chaos Solitons Fractals **5** (1995), 2367–2376.

[36] Fokas, A. S. & Pelloni, B., *The solution of certain initial boundary-value problems for the linearized Korteweg-de Vries equation*, Proc. Royal. Soc. London Series A **454** (1998), 645–657.

[37] Gardner, C., Greene, J., Kruskal, M., and Miura, R., *Method for Solving the Korteweg-deVries Equation*, Phys. Rev. Lett. **19** (1967), 1095–1097.

[38] Ghidaglia, J.-M., *Weakly damped forced Korteweg-de Vries equations behave as a finite-dimensional dynamical system in the long time*, J. Differential Eqns. **74** (1988), 369–390.

[39] Ghidaglia, J.-M., *A note on the strong convergence towards attractors of damped forced KdV equations*, J. Differential Eqns. **110**(1994), 356–359.

[40] Hammack, J. L., *A note on tsunamis: their generation and porpagation in an ocen of uniform depth*, J. Fluid Mech. **60** (1973), 769–799.

[41] Hammack, J. L., Segur, H., *The Korteweg-de Vries equation and water waves, Part 2. Comparison Experiments*, J. Fluid Mech. **65** (1974), 237–246.

[42] Hille, E. & Phillips, R. S., *Functional analysis and semigroups*, Amer. Math. Soc. Coll. Publ., Vol. 31, 1957.

[43] Holmer, J., *The Initial-Boundary Value Problem for the Korteweg-de Vries Equation*, Comm. Partial Differential Eq. **31**(2006), 1151 - 1190.

[44] Kappeler, T., Topalov, P.,*Global well-posedness of KdV in* $H^{-1}(T,R)$, Duke Math. J., **135** (2006), 327–360.

[45] Kato, T., *On the Korteweg-de Vries Equation*, manuscripta mathematica, **28** (1979), 89–99.

[46] Kato, T., *On the Cauchy problem for the (generalized) Korteweg-de Vries equations*, Advances in Mathematics Supplementary Studies, Studies in Applied Math. **8**(1983), 93–128.

[47] Kenig, C.,Ponce, G. & Vega, L., *On the (generalized) Korteweg-de Vries equation*, Duke Math. J., **59** (1989), 585–610.

[48] Kenig, C.,Ponce, G. & Vega, L., *Oscillatory integrals and regularity of dispersive equations*, Indiana Univ. Math. J. **40** (1991),33–69.

[49] Kenig, C.,Ponce, G. & Vega, L., *Well-Posedness of the Initial Value Problem for the Korteweg-de Vries Equation*, J. Amer. Math. Soc. **4**(1991), 323–347.

[50] Kenig, C.,Ponce, G. & Vega, L., *The Cauchy problem for the Korteweg-de Vries equation in Sobolev spaces of negative indices*, Duke Math. J., **71** (1993), 1–21.

[51] Kenig, C.,Ponce, G. & Vega, L., *Well-Posedness and scattering results for teh generalized Korteweg-de Vries equations via the contraction principle*, Comm. Pure Appl. Math. **46** (1993), 527–620

[52] Kenig, C.,Ponce, G. & Vega, L., *A Bilinear Estimate with Applicatios to the KdV Equation*, J. Amer. Math. Soc., **9** (1996), 573–603.

[53] Komornik, V., Russell, D. L. & Zhang, B.-Y., *Stabilization de l'equation de Korteweg-de Vries*, C. R. Acad. Sci. Paris **312** (1991), 841–843.

[54] Korteweg, D. J., & Vries, G., de ,*On the change of form of long waves advancing in a rectangular canal, and on a new type of long stationary waves*, Phil. Mag., **39** (5), 1895.

[55] Lax, P.,*Integrals of Nonlinear Equations of Evolution and Solitary Waves*, Comm. on Pure and Applied Math., **21** (1968), 467–490.

[56] Miura, R., *The Korteweg-de Vries equation: A survey of results*, SIAM Review,**18**(1976), 412 – 459.

[57] Palais, R. *The Symmetries of Solitions*, Bulliten of the American Mathematical Society **34**: 339–403, 1997.

[58] Pazy, A., *Semigroups of linear operators and applications to partial differential equations*, Applied Mathematical Sciences, **44**, Springer-Verlag, New York, 1983.

[59] Rosier, L., *Exact boundary controllability for the Korteweg-de Vries equation on a bounded domain.*, ESAIM: COCV, www.emath.fr/cocv/, **2**(1997), 33–55.

[60] Rosier, L. and Zhang, B.-Y., *Control and stabilization of the Korteweg-de Vries equation: recent progresses*, Journal of System Science and Complexity, to appear.

[61] Russell, D. L. & Zhang, B.-Y.,*Controllability and stabilizability of the third-order linear dispersion equation on a periodic domain*, SIAM J. Control Optim. **31**(1993), 659–676.

[62] Russell, D. L. & Zhang, B.-Y., *Smoothing and decay properties of solutions of the Korteweg-de Vries equation on a periodic domain with point dissipation*, J. Math. Anal. Appl., **190** (1995), 449–488.

[63] Russell, D. L. & Zhang, B.-Y., *Stabilization of the Korteweg-de Vries equation on a periodic domain*, Control and optimal design of distributed parameter systems (Minneapolis, MN, 1992), 195–211, IMA Vol. Math. Appl., 70, Springer, New York, 1995.

[64] Russell, J. S., *Report on waves*, Rept. 14th Meeting of the British Association for the Advancement of Science, John Murray, London, 311–390, 1844.

[65] Saut, J. C. & Temam, R., *Remarks on the Korteweg-de Vries equation*, Israel J. Math. **24** (1976), 78–87.

[66] Sjöberg, A., *On the Korteweg-de Vries equation*, J. Math. Anal. Appl., **29** (1970), 569-579.

[67] Sun, S., *The Korteweg-de Vries equation on a periodic domain with singular-point dissipation*, SIAM J. Control Optim., **34** (1996), 892 – 912.

[68] Temam, R., *Sur un problme non linaire*, J. Math. Pures Appl., **48** (1969), 159-172.

[69] Ton, B. A., *Initial boundary value problems for the Korteweg-de Vries equation*, J. Differential Eq. **25** (1977), 288–309

[70] Zabusky, N. J., Galvin, C. J., *Shallow-water waves,the Korteweg-de Vries equation and solitons*, J. Fluid Mech., **47**(1971), 811 – 824.

[71] Zabusky, N. J. & Kruskal, M. D., *Interaction of "solitons " in a collisionless plasma and the recurrence of initial states*, Phys. Rev. Letters, **15** (1965), 240–243.

[72] Zhang, B.-Y., *Boundary stabilization of the Korteweg-de Vries equations*, Proc. of International Conference on Control and Estimation of Distributed Parameter Systems: Nonlinear Phenomena, Vorau (Styria, Austria), July 18-24, 1993, International Series of Numerical Mathematics, **118**, 371 – 389, 1994.

[73] Zhang, B.-Y., *A remark on the Cauchy problem for the Korteweg de-Vries equation on a periodic domain*, Diff. and Integral Eq., **8** (1995), 1191–1204.

[74] Zhang, B.-Y., *Analyticity of solutions for the generalized Korteweg de-Vries equation with respect to their initial datum* , SIAM J. Math. Anal., **26** (1995), 1488–1513.

[75] Zhang, B.-Y., *Taylor series expansion for solutions of the Korteweg-de Vries equation with respect to their initial values*, J. Func. Anal.,**129** (1995), 293–324.

[76] Zhang, B.-Y., Exact boundary controllability of the Korteweg-de Vries equation, *SIAM J. Cont. Optim.*, 37 (1999), 543–565.