A Thesis

entitled

Hyponormality and Positivity of Toeplitz operators via the Berezin transform

by

Krishna Subedi

Submitted to the Graduate Faculty as partial fulfillment of the requirements for the Doctor of Philosophy Degree in Mathematics

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An Abstract of

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This thesis will present an original work characterizing hyponormality and positivity of Toeplitz operators with bounded symbols on the weighted Bergman spaces of on the disk and whole complex plane.

Cuckovic and Curto [9] have recently obtained a necessary condition for the hyponormality of the Toeplitz operators with a certain harmonic symbol on the disk in $\mathbb{C}$. We further extended the same problem to the weighted Bergman space and obtained the analogous result as in the unweighted Bergman space. This thesis will also survey the positivity of Toeplitz operators with bounded and unbounded symbols on the Bergman, Fock and certain Model spaces in terms of the Berezin transform of the symbol. Inspired by the paper of Zhao and Zheng [10], we have studied positivity of the Toeplitz operators with a bounded symbol on the Model spaces in terms of the Berezin transform. Analogous results have been obtained in the Fock space case as well.

To my mother, father, wife and my children.

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Contents

Chapter 1

Basic Functional Analysis

1.1 Hilbert Spaces[11]

Definition 1. An **inner product** space (also known as a pre-Hilbert space) is a vector space $\mathbb{V}$ over $\mathbb{K} = (\mathbb{R}$ or $\mathbb{C})$ together with a map

$$\langle .,. \rangle : \mathbb{V} \times \mathbb{V} \to \mathbb{K}$$

satisfying (for $x, y, z \in \mathbb{V}$ and $\lambda \in \mathbb{K}$):

(i) $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$

(ii) $\langle x, y \rangle = \overline{\langle y, x \rangle}$

(iii) $\langle \lambda x, y \rangle = \lambda \langle x, y \rangle$

(iv) $\langle x, x \rangle \geq 0$

(v) $\langle x, x \rangle = 0 \implies x = 0$.

Note that it follows from first three properties that:

(vi) $\langle x, y + z \rangle = \langle x, y \rangle + \langle x, z \rangle$

(vii) $\langle x, \lambda y \rangle = \overline{\lambda} \langle x, y \rangle$.

An inner product on $\mathbb{V}$ gives rise to a norm

$$||x|| = \sqrt{\langle x, x \rangle}.$$

If the inner product space is complete in this norm (or in other words, if it is complete in the metric arising from the norm) then we call it a **Hilbert space** and denote it by $\mathcal{H}$.

A sequence of vectors $\{e_n\}$ in a Hilbert space $\mathcal{H}$ is called an **orthonormal basis** for $\mathcal{H}$ if it has the following properties:

(i) The vectors in $\{e_n\}$ are mutually orthogonal;

(ii) Each e_n is a unit vector;

(iii) For every $x \in \mathcal{H}$ we have

$$x = \sum_{n=1}^{\infty} \langle x, e_n \rangle e_n,$$

with the series convergent in the norm topology of $\mathcal{H}$.

Definition 2. Suppose X and Y are normed spaces and $T : X \to Y$ is a linear transformation (also called a linear operator). If there is a constant $C > O$ such that $||Tx|| \leq C||x||$ for all $x \in X$, then we say that T is a **bounded linear operator** from X into Y.

A matrix $(a_{ij})_{1 \leq i,j \leq n}$ gives rise to a bounded linear operator $T : \mathbb{C}^n \to \mathbb{C}^n$ in the natural way: for $x = (x_1, ..., x_n)$, we define as

$$T(x) = \begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} & \dots & a_{1,n} \\ a_{2,1} & a_{2,2} & a_{2,3} & \dots & a_{2,n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & a_{n,2} & n_{n,3} & \dots & a_{n,n} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad for \ \ all \ \ x \in \mathbb{C}^n.$$

Lemma 1. *For $X, Y \in \mathcal{H}$ let $T : X \to Y$ be a linear operator. Then the following are equivalent:*

(i) *T is bounded*

(ii) *T is continuous*

(iii) *T is continuous at 0.*

For the proof, see Conway "A Course in Functional Analysis".

Definition 3. Suppose $T : \mathcal{H}_1 \to \mathcal{H}_2$ is linear operator between the Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$. Then $T^* : \mathcal{H}_2 \to \mathcal{H}_1$ is a linear **adjoint operator** if it satisfy $\langle Tx, y\rangle = \langle x, T^*y\rangle$ for all $x \in \mathcal{H}_1$ and $y \in \mathcal{H}_2$.

A bounded linear operator T on a Hilbert space $\mathcal{H}$ is said to be self-adjoint if $T = T^*$. If T is self-adjoint on $\mathcal{H}$ then it is easy to see that $\langle Tx, x\rangle$ is real for all $x \in \mathcal{H}$. It turns out that the converse of this is also true. Moreover, we have

$$||T|| = \sup\{|\langle Tx, x\rangle| : ||x|| = 1\}$$

if T is self-adjoint.

Lemma 2. *Let $T : H \to K$ be a bounded linear operator between two Hilbert spaces. Then the following statements are true.*

(i) $(T^*)^* = T$

(ii) $||T^*|| = ||T||$

(iii) $ker(T) = ran(T^*)^{\perp}$

(iv) $ker(T^*T) = ker(T)$.

For the proof, see [11].

Let $T = [a_{lj}]_{j,l=1}^{m}$ be $m \times m$ matrix then we say that T is positive (or positive semindefinite) if for any complex numbers $\zeta_1, \zeta_2, ...\zeta_n$ we have

$$\begin{bmatrix} \bar{\zeta}_1 & \bar{\zeta}_2 & \dots & \bar{\zeta}_m \end{bmatrix} T \begin{bmatrix} \zeta_1 \\ \zeta_2 \\ \vdots \\ \zeta_m \end{bmatrix} = \sum_{j,l=1}^{m} a_{j,l} \bar{\zeta}_j \zeta_l \geq 0.$$

Or we say, if the inner product $\langle Tx, x \rangle$ is nonnegative for all $x \in \mathcal{H}$, then T is **positive operator**. In particular, any positive operator is self-adjoint. Just as for a positive number, a positive operator can be raised to any positive power. In particular, if n is a positive integer and T is a positive operator on $\mathcal{H}$, then then there exists a unique positive operator on $\mathcal{H}$, denoted by $T^{\frac{1}{n}}$, such that $(T^{\frac{1}{n}})^n = T$. An easy example of positive operators is T^*T, where T is any bounded linear operator on $\mathcal{H}$. We denote $\sigma(T)$ as a set of complex number, called the spectrum of T, and defined as follows:

$$\sigma(T) = \{\lambda \in \mathbb{C} : \lambda I - T \text{ in not invertible}\},$$

where I is the identity operator on $\mathcal{H}$.

For a bounded linear operator T on $\mathcal{H}$, is invertible if it one-to-one and onto.

Lemma 3. *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be Hilbert spaces. For a bounded linear operator $T : \mathcal{H}_1 \to \mathcal{H}_2$, the following are equivalent:*

(i) T is invertible

(ii) there exists a constant ζ such that $T^*T \geq \zeta I_{\mathcal{H}_1}$ and $TT^* \geq \zeta I_{\mathcal{H}_2}$.

Proposition 1. *Let $\mathcal{H}$ be a Hilbert space, every positive operator $T \in \mathcal{B}(\mathcal{H})$ has non-negative spectrum, i.e. one has the inclusion $\sigma(T) \subset [0, \infty)$.*

Proof. Let $T \in \mathcal{B}(\mathcal{H})$ be a positive operator. We know that positive operator is self-adjoint and has a real spectrum. We wish to prove that this spectrum is positive. We only need to prove that, for every number $a \in (-\infty, 0)$, the operator $A = aI - T$ is invertible.

Since, $A^*A = AA^* = a^2I - 2aT + T^2$ and $-2aT$ and T^2 both are positive so $A^*A \geq a^2I$. Hence, By the above Lemma 3, $A$is invertible so $a \notin \sigma(T)$. Hence the spectrum $\sigma(T)$ is positive. □

Theorem 1. *(The Riesz Representation Theorem):*

If ϕ is a bounded linear functional on a Hilbert space $\mathcal{H}$ then there exists some $g \in \mathcal{H}$ such that for every $f \in \mathcal{H}$, we have $\phi(f) = \langle f, g \rangle$.

A bounded linear operator T on a Hilbert space $\mathcal{H}$ is called **hyponormal operator** if $T^*T \geq TT^*$; **normal operator** if $T^*T = TT^*$. Clearly all self-adjoint and unitary operators are normal.

1.2 Bergman Space [11]

Let $\mathbb{C}$ be the complex plane. The set

$$\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$$

is called the open disk. Let dA denotes area measure on $\mathbb{D}$, normalized so that the area of $\mathbb{D}$ is 1.

$$dA = \frac{1}{\pi} dxdy = \frac{r}{\pi} drd\theta,$$

where

$$z = x + iy = re^{i\theta}.$$

For $p > 0$ and $\zeta > -1$, we define

$$dA_\zeta(z) = (\zeta + 1)(1 - |z|^2)^\zeta dA(z) \text{ and}$$

$$A^p_\zeta(\mathbb{D}) = H(\mathbb{D}) \cap L^p(\mathbb{D}, dA_\zeta),$$

where $H(\mathbb{D})$ is the space of analytic functions in $\mathbb{D}$. These spaces are called **Bergman spaces** or Bergman spaces with standard weights.

By the help mean value theorem, we can prove sup norm of f on $\mathbb{D}$ is dominated by the L^p norm. Hence as a consequence, we can easily see $A^p_\zeta(\mathbb{D})$is closed subspace of $L^p(\mathbb{D}, \mathbb{A}_\zeta)$. As it is well known that $L^2(\mathbb{D}, dA_\zeta)$ is a Hilbert space space so $A^2_\zeta(\mathbb{D})$is also a Hilbert space.

1.2.1 Bergman Kernel

The point evaluation map at $z \in \mathbb{D}$ is a bounded linear functional on the Hilbert space $A^2_\zeta(\mathbb{D})$, the Riesz representation theorem tells that there exists a unique function h_z in $A^2_\zeta(\mathbb{D})$ such that

$$f(z) = \int_{\mathbb{D}} f(w)\overline{h_z(w)} dA_\zeta$$

for all f in $A^2_\zeta(\mathbb{D})$. Let $K_\zeta(z, w)$ denote the function on $\mathbb{D} \times \mathbb{D}$ defined by

$$K_\zeta(z, w) = \overline{h_z(w)}.$$

Then $K_\zeta(z, w)$ is called **the weighted reproducing kernel** of $A^2_\zeta(\mathbb{D})$. When $\zeta = 0$, we call $K(z, w)$ the Bergman kernel of $\mathbb{D}$.

The expression of kernel $K_\zeta(z, w)$ in terms of the orthonormal basis $\{e_n\}$ is

$$K_\zeta(z, w) = \sum_{n=1}^{\infty} e_n(z)\overline{e_n(w)}$$

and the series converges uniformly on the compact subsets of $\mathbb{D} \times \mathbb{D}$. In particular, $K_\zeta(z, w)$ is independent of the choice of the orthonormal basis $\{e_n(z)\}$.

The reproducing kernel of $A^2_\zeta(\mathbb{D})$ is given by

$$K_\zeta(z, w) = \frac{1}{(1 - z\overline{w})^{2+\zeta}}.$$

Since $A^2_\zeta(\mathbb{D})$ is a closed subspace of the Hilbert space $L^2(\mathbb{D}, dA_\zeta)$, there exists an orthogonal projection P_ζ from $L^2(\mathbb{D}, dA_\zeta)$ onto $A^2_\zeta(\mathbb{D})$. Indeed P_ζ is an integral operator because of any $f \in L^2(\mathbb{D}, dA_\zeta)$ we have,

$$P_\zeta f(z) = \int_{\mathbb{D}} K_\zeta(z, w) f(w) dA_\zeta(w).$$

1.2.2 Berezin transform

For any $z \in \mathbb{D}$, $K_\zeta(z, z) > 0$, so we can normalized reproducing kernel to obtain a family of unit vectors k_z, as follows

$$k_z(w) = \frac{K(w, z)}{\sqrt{K(z, z)}}, \qquad w \in \mathbb{D}.$$

Definition 4. Let T be a bounded linear operator on $\mathcal{H}$. We say $\widetilde{T}$ is a Berezin transform of T, as a function on $\mathbb{D}$, is defined as,

$$\widetilde{T}(z) = \langle T k_z, k_z \rangle \text{ for all } z \in \mathbb{D}.$$

In the same way, we define the Berezin transform of the function. We just replace the operator T by a function $f \in L^1(\mathbb{D}, dA_\zeta)$ and interpret $\langle , \rangle$ as an integral pairing rather than an inner product. Thus the Berezin transform of f denoted as the function $\widetilde{f}$

on $\mathbb{D}$, and defined as,

$$\widetilde{f}(z) = \int_{\mathbb{D}} f(w)|k_z(w)|^2 dA_\zeta.$$

By a change of variable, this integral reduces to,

$$\widetilde{f}(z) = \int_{\mathbb{D}} f \circ \varphi_z(w) dA_\zeta(w),$$

where $\varphi_z(w)$ is a Mobius transformation.

Whenever $f \in L^\infty(\mathbb{D})$, the Berezin transform of the function f is same as the Berezin transform of a certain bounded linear operator on $A^2_\zeta(\mathbb{D})$.

Furthermore, the Berezin transform of a operator T is one-to-one. To prove it, one has to show, if $\widetilde{T} = 0$ then $T = 0$.

The fix point characterization of the Berezin transform tells that if $\varphi \in L^1(\mathbb{D}, dA_\zeta)$ is a complex-valued harmonic function, then $\widetilde{\varphi} = \varphi$. The other direction follows if $\varphi \in \mathbb{C}(\overline{\mathbb{D}})$. For proof, see [11].

Another important characterization about the Berezin transform is that if $\varphi \in \mathbb{C}(\overline{\mathbb{D}})$, then $\widetilde{\varphi}$ is in $\mathbb{C}(\overline{\mathbb{D}})$ and $\widetilde{\varphi} = \varphi$ on $\partial\mathbb{D}$.

Main part of the proof is , if $z_0 \in \partial\mathbb{D}$, then $\lim_{z\to z_0} \varphi_z(w) = z_0$.

Let $\varphi \in L^1(\mathbb{D}, dA_\zeta)$ for some ζ. For the sequence of the Berezin transform of φ $\{\widetilde{\varphi}_\beta : \beta \in Z_+\}$ we have,

$$\lim_{\beta\to\infty} \widetilde{\varphi}_\beta = \varphi$$

and the convergence is in the norm topology of $L^1(\mathbb{D}, dA_\zeta)$.

1.2.3 Toeplitz operators on the Bergman space

Given a function $\varphi \in L^\infty(\mathbb{D})$, we define an operator T_φ on $A^2_\zeta(\mathbb{D})$ by

$$T_\varphi f = P_\zeta(\varphi f), \qquad f \in A^2_\zeta(\mathbb{D})$$

where

$$P_\zeta : L^2(\mathbb{D}, dA_\zeta) \to A^2_\zeta(\mathbb{D})$$

is a Bergman projection onto $A^2_\zeta(\mathbb{D})$. The operator T_φ is called the **Toeplitz operator** on the Bergman space $A^2_\zeta(\mathbb{D})$ with symbol φ.

Suppose a and b are complex numbers, φ and ψ are bounded functions on $\mathbb{D}$. Then

(i) $T_{a\varphi+b\psi} = aT_\varphi + T_\psi$

(ii) $T^*_\varphi = T_{\overline{\varphi}}$

Moreover, if $\varphi \in H^\infty$, then

(iii) $T_\psi T_\varphi = T_{\psi\varphi}$

(iv) $T_{\overline{\varphi}} T_\psi = T_{\overline{\varphi}\psi}$.

Chapter 2

A necessary condition on the hyponormality of Toeplitz operators on the weighted Bergman space

2.1 Introduction

Let $\mathbb{D}$ denote the open disc in the complex plane, and dA denotes the normalized Lebesgue area measure on $\mathbb{D}$. For $-1 < \zeta < \infty$, the **weighted Bergman space**, $A^2_\zeta(\mathbb{D})$, is a closed subspace of $L^2(\mathbb{D}, dA_\zeta)$, consisting of all holomorphic square integrable functions on $\mathbb{D}$ with respect to the measure,

$$dA_\zeta(z) = (\zeta + 1)(1 - |z|^2)^\zeta dA(z),$$

If $\zeta = 0$, then $A_0^2(\mathbb{D}) = A^2(\mathbb{D})$ is the Bergman space. The inner product on $L^2(\mathbb{D}, dA_\zeta)$ is defined as

$$\langle f, g\rangle_\zeta = \int_{\mathbb{D}} f(z)\overline{g(z)} dA_\zeta(z) \quad \text{for all} \quad f, g \in L^2(\mathbb{D}, dA_\zeta).$$

For $z^n \in A_\zeta^2(\mathbb{D})$, one can compute that,

$$||z^n||_\zeta^2 = \frac{\Gamma(n+1)\Gamma(\zeta+2)}{\Gamma(n+\zeta+2)}.$$

Hence, the orthonormal basis $\{e_n\}$ is of the form

$$e_n(z) = \sqrt{\frac{\Gamma(n+\zeta+2)}{\Gamma(n+1)\Gamma(\zeta+2)}}\; z^n, \quad \text{for} \quad z \in \mathbb{D}$$

where $\Gamma(s)$ is the usual Gamma function.

The reproducing kernel in $A_\zeta^2(\mathbb{D})$ is defined as

$$K_w^{(\zeta)}(z) = \frac{1}{(1 - z\overline{w})^{2+\zeta}} \;\; \text{for} \;\; z \in \mathbb{D}.$$

For $\varphi \in L^\infty(\mathbb{D})$, the **Berezin transform** of φ on $A_\zeta^2(\mathbb{D})$ is a function $\widetilde{\varphi}_\zeta(z)$ defined by,

$$\widetilde{\varphi_\zeta}(z) := \langle T_\varphi k_z^\zeta, k_z^\zeta\rangle = \int_{\mathbb{D}} \varphi(w)|k_z^\zeta(w)|^2 dA_\zeta(w),$$

For $\varphi \in L^\infty(\mathbb{D})$, the **Toeplitz operator** T_φ on $A_\zeta^2(\mathbb{D})$ is defined by

$$T_\varphi f = P_\zeta(\varphi f) \;\; \text{for} \;\; f \in A_\zeta^2(\mathbb{D}),$$

where P_ζ denotes the orthogonal projection that maps $L^2(\mathbb{D}, dA_\zeta)$ onto $A_\zeta^2(\mathbb{D})$. Similarly, we define the **Hankel operators** by

$$H_\varphi f = (I - P_\zeta)(\varphi f)$$

from $A^2_\zeta(\mathbb{D})$ onto $A^2_\zeta(\mathbb{D})^\perp$ and I is identity operator.

A bounded linear operator acting on $\mathcal{H}$ is said to be **normal** if

$$T^*T = TT^*,$$

and **hyponormal** if

$$T^*T \geq TT^*.$$

This is equivalent to saying

$$||Tx|| \geq ||T^*x|| \text{ for all } x \in \mathcal{H}.$$

2.2 Preliminaries

C. Cowen [2] gave an elegant characterization of the hyponormality of Toeplitz operator on the Hardy Space with a bounded measurable symbol on $\mathbb{T}$, where $\mathbb{T}$ is the unit circle in the complex plane. He proved that for a symbol $\varphi \in L^\infty$,

$$\varphi \equiv f + \overline{g} \quad (f, g \in H^2),$$

the Toeplitz operator T_φ acting on the Hardy space of the unit circle is hyponormal if and only if

$$f = c + T_{\overline{h}} g, \quad \text{for some} \quad c \in \mathbb{C}, \quad h \in H^\infty, \quad ||h||_\infty \leq 1.$$

It is natural for us to consider the same problem on the Bergman space. However, C. Cowen's proof does not adapt to the Bergman space, since the multiplication operator M_z is no longer an isometry.

Then, for the case that φ is a continuous function, Yufeng Lu and Chaomei Liu [7] found a sufficient condition for the hyponormality of T_φ by using the Mellin transform on weighted Bergman space.

H. Sadraoui [8] gave a necessary and sufficient condition on the Bergman space for $\varphi = f + \overline{g}$ with f, g bounded and analytic, which is also true on the weighted Bergman space.

Theorem 2. *(H. Sadraoui): If $\varphi \equiv f + \bar{g}$, then following are equivalent*

(i) T_φ is hyponormal in $A^2(\mathbb{D})$;

(ii) $H_{\bar{g}}^* H_{\bar{g}} \leq H_{\bar{f}}^* H_{\bar{f}}$;

(iii) $H_{\bar{g}} = C H_{\bar{f}}$, where C is a contraction on $A^2(\mathbb{D})$.

Later, P. Ahern and Z. Cuckovic [1] generalized H. Sadraouis result by using a mean value inequality and the Berezin transform.

Theorem 3. *(P. Ahern and Z. Cuckovic):*

If T_φ is hyponormal with the symbol $\varphi \equiv \overline{g} + f \in L^\infty(\mathbb{D})$, then $\widetilde{u} \geq u$, and $u := |f|^2 - |g|^2$.

Hwang and Lee [5], and Hwang, Lee and Park [6] gave some necessary and sufficient conditions for the hyponormality of Toeplitz operators on weighted Bergman space with the class of functions $\varphi \equiv \overline{f} + g \ \ (f, g \in H^2)$.

Recently, Cuckovic and Curto [9] gave a necessary condition for the hyponormality of the Toeplitz operators on the Bergman space which stated as follows.

Theorem 4. *Assume that T_φ is hyponormal in $A^2(\mathbb{D})$, with symbols of the form*

$$\varphi \equiv \alpha z^n + \beta z^m + \gamma \bar{z}^p + \delta \bar{z}^q$$

where $\alpha, \beta, \gamma, \delta \in \mathbb{C}$, $m, n, p, q \in \mathbb{Z}_+, m < n$ *and* $p < q$, *and* $n - m = q - p$. *Then*

$$|\alpha|^2 n^2 + |\beta|^2 m^2 - |\gamma|^2 p^2 - |\delta|^2 q^2 \geq 2|\bar{\alpha}\beta mn - \bar{\gamma}\delta pq|.$$

A natural question arises, do we get the same necessary condition for the weighted Bergman space with weight $(1 - |z|^2)^\zeta dA$ where $\zeta \in \mathbb{Z}_+$?

We show the same result holds as in non-weighted case. In fact, we let T_φ act on the vectors of the form

$$z^k + cz^l + dz^r \quad (k < l < r),$$

and then we study the asymptotic behavior of a suitable matrix of inner products, as $k \to \infty$. As a result, we obtain the same result.

Lemma 4. *For* $u, v \geq 0$, *we have*

$$P_\zeta(\overline{z}^u z^v) = \begin{cases} 0, \ \textit{if} \ v < u \\ \frac{\Gamma(v-u+\zeta+2)\Gamma(v+1)}{\Gamma(v-u+1)\Gamma(v+\zeta+2)} z^{v-u}, \ \textit{if} \ v \geq u. \end{cases}$$

Proof.

$$\begin{aligned} P(\overline{z}^u z^v) &= \sum_{n=0}^{\infty} \left\langle \overline{z}^u z^v, \frac{z^n}{||z^n||} \right\rangle \frac{z^n}{||z^n||} \\ &= \sum_{n=0}^{\infty} \frac{\langle \overline{z}^u z^v, z^n \rangle z^n}{||z^n||^2} \\ &= \sum_{n=0}^{\infty} \frac{\Gamma(n+\zeta+2)}{\Gamma(n+1)\Gamma(\zeta+2)} \langle z^v, z^{n+u} \rangle z^n \\ &= \begin{cases} 0 & \text{if } v < u \\ \frac{\Gamma(v-u+\zeta+2)\Gamma(v+1)}{\Gamma(v-u+1)\Gamma(v+\zeta+2)} z^{v-u} & \text{if } \ v \geq u. \end{cases} \end{aligned} \tag{2.1}$$

□

Corollary 1. *For $v \geq u$ and $t \geq w$, we have the statement,*

$$\begin{aligned}
\langle P_\zeta(\overline{z}^u z^v), P_\zeta(\overline{z}^w z^t)\rangle =& \left\langle \frac{\Gamma(v-u+\zeta+2)\Gamma(v+1)}{\Gamma(v-u+1)\Gamma(v+\zeta+2)} z^{v-u}, \frac{\Gamma(t-w+\zeta+2)\Gamma(t+1)}{\Gamma(t-w+1)\Gamma(t+\zeta+2)} z^{t-w} \right\rangle \\
=& \frac{\Gamma(v-u+\zeta+2)\Gamma(v+1)}{\Gamma(v-u+1)\Gamma(v+\zeta+2)} \frac{\Gamma(t-w+\zeta+2)\Gamma(t+1)}{\Gamma(t-w+1)\Gamma(t+\zeta+2)} \langle z^{v-u}, z^{t-w}\rangle \\
=& \frac{\Gamma(t-w+\zeta+2)^2\Gamma(v+1)\Gamma(t+1)}{\Gamma(t-w+1)^2\Gamma(v+\zeta+2)\Gamma(t+\zeta+2)} \frac{\Gamma(t-w+1)\Gamma(\zeta+2)}{\Gamma(t-w+\zeta+2)} \delta_{u+t,v+w} \\
=& \frac{\Gamma(t-w+\zeta+2)\Gamma(\zeta+2)\Gamma(v+1)\Gamma(t+1)}{\Gamma(t-w+1)\Gamma(v+\zeta+2)\Gamma(t+\zeta+2)} \delta_{u+t,v+w}. \qquad (2.2)
\end{aligned}$$

Other useful results:

For any

$$f \in A^2_\zeta(\mathbb{D}), f = \sum_{n=0}^{\infty} b_n z^n \quad \text{where } b_n \in \mathbb{C}$$

1. $||z^k f||^2 = \sum\limits_{n=0}^{\infty} |b_n|^2 \frac{\Gamma(k+n+1)\Gamma(\zeta+2)}{\Gamma(k+n+\zeta+2)}$

$$\begin{aligned}
||z^k f||^2 &= \langle z^k f, z^k f, \rangle \\
&= \left\langle z^k \sum_{n=0}^{\infty} b_n z^n, z^k \sum_{m=0}^{\infty} b_m z^m \right\rangle \\
&= \sum_{n=0}^{\infty} |b_n|^2 \langle z^{k+n}, z^{k+n}\rangle \\
&= \sum_{n=0}^{\infty} |b_n|^2 \frac{\Gamma(k+n+1)\Gamma(\zeta+2)}{\Gamma(k+n+\zeta+2)}. \qquad (2.3)
\end{aligned}$$

2. Norm value of $P(\overline{z}^k f)$

Since we have, $||P(\overline{z}^k f)||^2 = \langle P(\overline{z}^k f), P(\overline{z}^k f)\rangle$ so we compute the value of $P(\overline{z}^k f)$ first,

(2.4)

$$
\begin{aligned}
P(\overline{z}^k f) &= \sum_{n=0}^{\infty} \langle P(\overline{z}^k f), e_n \rangle e_n \\
&= \sum_{n=0}^{\infty} \langle \overline{z}^k f, e_n \rangle e_n \\
&= \sum_{n=0}^{\infty} \frac{\Gamma(n+\zeta+2)}{\Gamma(n+1)\Gamma(\zeta+2)} \ \langle f, z^{k+n} \rangle z^n \\
&= \sum_{n=0}^{\infty} \frac{\Gamma(n+\zeta+2)}{\Gamma(n+1)\Gamma(\zeta+2)} \left\langle \sum_{m=0}^{\infty} b_m z^m, z^{k+n} \right\rangle z^n \\
&= \sum_{n=0}^{\infty} \frac{\Gamma(n+\zeta+2)}{\Gamma(n+1)\Gamma(\zeta+2)} \langle b_{k+n} z^{k+n}, z^{k+n} \rangle z^n \\
&= \sum_{n=0}^{\infty} \frac{\Gamma(n+\zeta+2)}{\Gamma(n+1)\Gamma(\zeta+2)} b_{k+n} \frac{\Gamma(k+n+1)\Gamma(\zeta+2)}{\Gamma(k+n+\zeta+2)} z^n \\
&= \sum_{n=0}^{\infty} \ b_{k+n} \frac{\Gamma(n+\zeta+2) \ \Gamma(k+n+1)}{\Gamma(n+1) \ \Gamma(k+n+\zeta+2])} z^n. \qquad (2.5)
\end{aligned}
$$

So, (2.4) turns out to be

$$
\begin{aligned}
||P(\overline{z}^k f)||^2 &= \left\langle \sum_{n=0}^{\infty} \ b_{k+n} \frac{\Gamma(n+\zeta+2) \ \Gamma(k+n+1)}{\Gamma(n+1) \ \Gamma(k+n+\zeta+2])} z^n, \sum_{n=0}^{\infty} \ b_{k+n} \frac{\Gamma(n+\zeta+2) \ \Gamma(k+n+1)}{\Gamma(n+1) \ \Gamma(k+n+\zeta+2])} z^n \right\rangle \\
&= \sum_{n=0}^{\infty} \ |b_{k+n}|^2 \frac{\Gamma(n+\zeta+2)^2 \ \Gamma(k+n+1)^2}{\Gamma(n+1)^2 \ \Gamma(k+n+\zeta+2)^2} \ \langle z^n, z^n \rangle \\
&= \sum_{n=0}^{\infty} \ |b_{k+n}|^2 \frac{\Gamma(n+\zeta+2)^2 \ \Gamma(k+n+1)^2}{\Gamma(n+1)^2 \ \Gamma(k+n+\zeta+2)^2} \frac{\Gamma(n+1)\Gamma(\zeta+2)}{\Gamma(n+\zeta+2)} \\
&= \sum_{n=0}^{\infty} \ |b_{k+n}|^2 \frac{\Gamma(\zeta+2)\Gamma(n+\zeta+2) \ \Gamma(k+n+1)^2}{\Gamma(n+1) \ \Gamma(k+n+\zeta+2)^2}. \qquad (2.6)
\end{aligned}
$$

In particular for $k=1$,

$$
\begin{aligned}
||P(\overline{z} f)||^2 &= \sum_{n=0}^{\infty} \ |b_{n+1}|^2 \frac{\Gamma(\zeta+2)\Gamma(n+\zeta+2) \ \Gamma(n+2)^2}{\Gamma(n+1)\Gamma(n+\zeta+3)^2} \\
&= \sum_{n=0}^{\infty} \ |b_{n+1}| \frac{\Gamma(\zeta+2) \ (n+1)\Gamma(n+2)}{(n+\zeta+2)\Gamma(n+\zeta+3)} \\
&= \sum_{n=1}^{\infty} \ |b_n|^2 \frac{(n)\Gamma(\zeta+2) \ \Gamma(n+1)}{(n+\zeta+1)\Gamma(n+\zeta+2)} \qquad (2.7)
\end{aligned}
$$

and for $k = 2$,

$$
\begin{aligned}
||P(\overline{z}^2 f)||^2 &= \sum_{n=0}^{\infty} |b_{n+2}|^2 \frac{\Gamma(\zeta+2)\Gamma(n+\zeta+2)\ \Gamma(n+3)^2}{\Gamma(n+1)\ \Gamma(n+\zeta+4)^2} \\
&= \sum_{n=0}^{\infty} |b_{n+2}|^2 \frac{(n+2)(n+1)\Gamma(\zeta+2)\ \Gamma(n+3)}{(n+\zeta+3)(n+\zeta+2)\Gamma(n+\zeta+4)} \\
&= \sum_{n=2}^{\infty} |b_n|^2 \frac{(n)(n-1)\Gamma(\zeta+2)\ \Gamma(n+1)}{(n+\zeta+1)^2(n+\zeta)^2\Gamma(n+\zeta)} \qquad (2.8)
\end{aligned}
$$

Proposition 2. *(D.Farenick and W.Y. Lee [3]) Let φ be a trigonometric polynomial of the form*

$$\varphi(z) = \sum_{n=-m}^{N} a_n z^n. \qquad (2.9)$$

If T_φ is a hyponormal operator then $m \leq N$ and $|a_m| \leq |a_N|$.

For the symbol,

$$\varphi = \overline{z}^2 + 2z, \qquad (2.10)$$

T_φ is not hyponormal on the Hardy space $H^2(\mathbb{T})$, because $m = 2, N = 1$, and $m > N$ (see proposition 2). However, Cuckovic and Curto [9] proved that T_φ is hyponormal in $A^2(\mathbb{D})$.

Following the similar idea, we show that T_φ is hyponormal in the weighted Bergman space as well.

Theorem 5. *For $\zeta > 0$ and $\varphi = \overline{z}^2 + \beta z$, show that T_φ is hyponormal iff*

$$|\beta|^2||zf||^2 + ||P(\overline{z}^2 f)||^2 \geq ||\beta||^2||P(\overline{z}f)||^2 + ||z^2 f||^2$$

iff

$$|\beta| \geq 2.$$

Proof. Since we know,

$$
\begin{aligned}
[T_\varphi^*, T_\varphi] &= T_{\overline{\varphi}} T_\varphi - T_\varphi T_{\overline{\varphi}} \\
&= (T_{z^2} + \overline{\beta} T_{\overline{z}})(T_{\overline{z}^2} + \beta T_z) - (T_{\overline{z}^2} + \beta T_z)(T_{z^2} + \overline{\beta} T_{\overline{z}})
\end{aligned}
$$

Toeplitz operators with analytic or co-analytic symbols commute so only the remaining terms are,

$$
= T_{z^2} T_{\overline{z}^2} - T_{\overline{z}^2} T_{z^2} + |\beta|^2 T_{\overline{z}} T_z - |\beta|^2 T_z T_{\overline{z}}. \tag{2.11}
$$

We know that T_φ is hyponormal iff

$$
\begin{aligned}
&\langle [T_\varphi^*, T_\varphi] f, f \rangle \geq 0 \\
&\langle T_{z^2} T_{\overline{z}^2} f, f \rangle - \langle T_{\overline{z}^2} T_{z^2} f, f \rangle + |\beta|^2 \langle T_{\overline{z}} T_z f, f \rangle - |\beta|^2 \langle T_z T_{\overline{z}} f, f \rangle \geq 0 \\
&||P(\overline{z}^2 f)||^2 + |\beta|^2 ||zf||^2 \geq ||z^2 f||^2 + |\beta|^2 ||P(\overline{z} f)||^2
\end{aligned} \tag{2.12}
$$

This proves the first part of the theorem.

Next we prove second part of the theorem, which states that,

$$
|\beta|^2 ||zf||^2 + ||P(\overline{z}^2 f)||^2 \geq |\beta|^2 ||P(\overline{z} f)||^2 + ||z^2 f||^2 \tag{2.13}
$$

if and only if

$$
|\beta| \geq 2.
$$

We have already computed the values of $||P(\overline{z}^2 f)||^2, ||zf||^2,\ ||z^2 f||^2$ and $||P(\overline{z} f)||^2$ so

(2.13) becomes

$$\Longrightarrow \sum_{n=2}^{\infty} |b_n|^2 \frac{(n)(n-1)\Gamma(\zeta+2)\ \Gamma(n+1)}{(n+\zeta+1)^2(n+\zeta)^2\Gamma(n+\zeta)} + |\beta|^2 \sum_{n=0}^{\infty} |b_n|^2 \frac{\Gamma(n+2)\Gamma(\zeta+2)}{\Gamma(n+\zeta+3)} \geq$$
$$\sum_{n=0}^{\infty} |b_n|^2 \frac{\Gamma(n+3)\Gamma(\zeta+2)}{\Gamma(n+\zeta+4)} + |\beta|^2 \sum_{n=1}^{\infty} |b_n|^2 \frac{(n)\Gamma(\zeta+2)\ \Gamma(n+1)}{(n+\zeta+1)\Gamma(n+\zeta+2)}$$
$$\Longrightarrow \sum_{n=2}^{\infty} |b_n|^2 \frac{(n)(n-1)\ \Gamma(n+1)}{(n+\zeta+1)^2(n+\zeta)^2\Gamma(n+\zeta)} + |\beta|^2 \sum_{n=0}^{\infty} |b_n|^2 \frac{\Gamma(n+2)}{\Gamma(n+\zeta+3)} \geq$$
$$\sum_{n=0}^{\infty} |b_n|^2 \frac{\Gamma(n+3)}{\Gamma(n+\zeta+4)} + |\beta|^2 \sum_{n=1}^{\infty} |b_n|^2 \frac{(n)\ \Gamma(n+1)}{(n+\zeta+1)\Gamma(n+\zeta+2)}. \tag{2.14}$$

Equation (2.14) must hold for every sequences (b_n) of coefficients of f. Consider first sequence (b_n) with $b_0 := 1$, and $b_n ;= 0$ for all $n \geq 1$.

For this sequence (2.14) yields,

$$|\beta|^2 \frac{\Gamma(2)}{\Gamma(\zeta+3)} \geq \frac{\Gamma(3)}{\Gamma(\zeta+4)}$$
$$|\beta|^2 \geq \frac{2\Gamma(\zeta+3)}{\Gamma(\zeta+4)}$$
$$|\beta|^2 \geq \frac{2}{\zeta+3}. \tag{2.15}$$

Next, we take $b_0 := 0, b_1 = 1$, and $b_n := 0$ for all $n \geq 2$, then (2.14) yields,

$$|\beta|^2 \frac{\Gamma(3)}{\Gamma(\zeta+4)} \geq \frac{\Gamma(4)}{\Gamma(\zeta+5)} + |\beta|^2 \frac{1}{(\zeta+2)(\Gamma(\zeta+3))}$$
$$|\beta|^2 \frac{2}{(\zeta+3)} \geq \frac{6}{(\zeta+4)(\zeta+3)} + |\beta|^2 \frac{1}{\zeta+2}$$
$$|\beta|^2 \geq \frac{6.(\zeta+2)}{(\zeta+1)(\zeta+4)}. \tag{2.16}$$

Finally, we use sequence $b_0 := 0, b_1 := 0, ..., b_{k-1} = 0, b_k := 1,$ and $b_n = 0$ for all $n > k$. Then (2.14) gives us,

$$\Longrightarrow \frac{(k)(k-1)\ \Gamma(k+1)}{(k+\zeta+1)^2(k+\zeta)^2\Gamma(k+\zeta)} + |\beta|^2\frac{\Gamma(k+2)}{\Gamma(k+\zeta+3)} \geq \frac{\Gamma(k+3)}{\Gamma(k+\zeta+4)} + |\beta|^2\ \frac{(k)\ \Gamma(k+1)}{(k+\zeta+1)\Gamma(k+\zeta+2)}$$

$$\Longrightarrow \frac{(k)(k-1)\ \Gamma(k+1)}{(k+\zeta+1)^2(k+\zeta)^2\Gamma(k+\zeta)} + |\beta|^2\frac{(k+1)\Gamma(k+1)}{(k+\zeta+2)(k+\zeta+1)(k+\zeta)\Gamma(k+\zeta)} \geq \frac{(k+2)(k+1)\Gamma(k+1)}{(k+\zeta+3)(k+\zeta+2)(k+\zeta+1)(k+\zeta)\Gamma(k+\zeta)} + |\beta|^2\ \frac{(k)\ \Gamma(k+1)}{(k+\zeta+1)(k+\zeta+1)(k+\zeta)\Gamma(k+\zeta)}$$

$$\Longrightarrow \frac{(k)(k-1)}{(k+\zeta+1)(k+\zeta)} + |\beta|^2\frac{(k+1)}{(k+\zeta+2)} \geq \frac{(k+2)(k+1)}{(k+\zeta+3)(k+\zeta+2)} + |\beta|^2\ \frac{(k)}{(k+\zeta+1)}$$

$$\Longrightarrow |\beta|^2\frac{(k+1)(k+\zeta+1)-k(k+\zeta+2)}{(k+\zeta+1)(k+\zeta+2)} \geq \frac{(k+\zeta)(k+\zeta+1)(k^2+3k+2)-(k^2-k)(k+\zeta+2)(k+\zeta+3)}{(k+\zeta)(k+\zeta+1)(k+\zeta+2(k+\zeta+3))}$$

$$\Longrightarrow |\beta|^2(\zeta+1) \geq \frac{4k^2+4k\zeta^2+4k^2+12k\zeta+2\zeta^2+8k+2\zeta}{(k+\zeta)(k+\zeta+3)}$$

$$\Longrightarrow |\beta|^2(\zeta+1) \geq \frac{4k^2(\zeta+1)+2\zeta(\zeta+1)+4k(\zeta+1)(\zeta+2)}{(k+\zeta)(k+\zeta+3)}$$

$$\Longrightarrow |\beta|^2 \geq \frac{4k^2+2\zeta+4k(\zeta+2)}{(k+\zeta)(k+\zeta+3)} \tag{2.17}$$

By the help of Wolfram alpha, (See the appendix for details).

$$\max\left\{\frac{4k^2+2\zeta+4k(\zeta+2)}{(k+\zeta)(k+\zeta+3)} : k \geq 1, \zeta > 0\right\} = \frac{4(k+2)}{k+3}\ \ k \geq 1.$$

Since $\frac{4(k+2)}{k+3}$ is increasing in k so max attains at the infinity. One can show the value of $\frac{4(k+2)}{k+3}$ at infinity is 4.

Thus, T_φ is hyponormal if and only if

$$|\beta| \geq 2.$$

$\square$

Thus the hyponormality in the Hardy space does not imply the hyponormality in weighted Bergman space.

Next, we introduce facts about the asymptotic expansion of ratios of Gamma functions. For $a, b \in \mathbb{R}$,

$$\frac{\Gamma(k+a)}{\Gamma(k+b)} \sim k^{a-b} \sum_{i=0}^{\infty} \frac{G_i(a,b)}{z^i} \tag{2.18}$$

where,

$$\begin{aligned}
G_0(a,b) &= 1 \\
G_1(a,b) &= \frac{1}{2}(a-b)(a+b-1) \\
G_2(a,b) &= \frac{1}{12}\binom{a-b}{2}\{3(a+b-1)^2 - (a-b+1)\} \\
&\vdots
\end{aligned}$$

We use this facts to compute following ratios of Gamma functions,

$$\frac{\Gamma(k+1)}{\Gamma(k+\zeta+2)} \sim k^{-\zeta-1}\left\{1 + \frac{1}{2k}(-\zeta-1)(\zeta+2) + \frac{1}{12k^2}\binom{-\zeta-1}{2}\big(3(\zeta+2)^2 - (-\zeta)\big)\right\} + \mathcal{O}(k^{-3}). \tag{2.19}$$

$$\frac{\Gamma(k+n+1)}{\Gamma(k+n+\zeta+2)} \sim k^{-\zeta-1}\Bigg\{1-\frac{1}{2k}(\zeta+1)(2n+\zeta+2)+\frac{1}{12k^2}\binom{-\zeta-1}{2}$$
$$\big(3(2n+\zeta+2)^2+\zeta\big)\Bigg\}+\mathcal{O}(k^{-3}). \tag{2.20}$$

$$\frac{\Gamma(k+\zeta+2)}{\Gamma(k+1)} \sim k^{\zeta+1}\Bigg\{1+\frac{1}{2k}(\zeta+1)(\zeta+2)+\frac{1}{12k^2}\binom{\zeta+1}{2}\big(3(\zeta+2)^2-(\zeta+2)\big)\Bigg\}+\mathcal{O}(k^{-3}). \tag{2.21}$$

$$\frac{\Gamma(k-n+\zeta+2)}{\Gamma(k-n+1)} \sim k^{\zeta+1}\Bigg\{1+\frac{1}{2k}(\zeta+1)(-2n+\zeta+2)+\frac{1}{12k^2}\binom{\zeta+1}{2}$$
$$\big(3(-2n+\zeta+2)^2-(\zeta+2)\big)\Bigg\}+\mathcal{O}(k^{-3}). \tag{2.22}$$

2.3 Main Results

We study the necessary condition for the hyponormality of the Toeplitz operators on the weighted Bergman space of the disc with symbol of type

$$\varphi \equiv \alpha z^n + \beta z^m + \gamma \bar{z}^p + \delta \bar{z}^q$$

where $\alpha, \beta, \gamma, \delta \in \mathbb{C}$ and $m, n, p, q \in \mathbb{Z}_+, m < n$ and $p < q$ with $n - m = q - p$. We show if T_φ is hyponormal then

$$|\alpha|^2 n^2 + |\beta|^2 m^2 - |\gamma|^2 p^2 - |\delta|^2 q^2 \geq (\bar{\alpha}\beta mn - \bar{\gamma}\delta pq).$$

Proof. Here $C := [T_\varphi^*, T_\varphi]$ denotes the self-commutator of T_φ. We study necessary conditions on the symbol φ, assuming positivity on C.

We consider the expression $\langle Cf, f\rangle$, given by

$$\left\langle [(T_{\alpha z^n+\beta z^m+\gamma\bar{z}^p+\delta\bar{z}^q})^*, T_{\alpha z^n+\beta z^m+\gamma\bar{z}^p+\delta\bar{z}^q}](z^k+cz^l+dz^r), z^k+cz^l+dz^r\right\rangle,$$

for large values of k (and consequently large value of l and r). And it is easy to see $\langle Cf, f\rangle$ is a quadratic form in c and d, that is,

$$\langle Cf, f\rangle \equiv A_{00} + 2\mathrm{Re}(A_{10}c) + 2\mathrm{Re}(A_{01}d) + A_{20}c\bar{c} + 2\mathrm{Re}(A_{11}\bar{c}d) + A_{02}d\bar{d}, \qquad (2.23)$$

where $A_{00} = \langle Cz^k, z^k\rangle$, $A_{10} = \langle Cz^l, z^k\rangle$, $A_{01} = \langle Cz^r, z^k\rangle$, $A_{20} = \langle Cz^l, z^l\rangle$, $A_{11} = \langle Cz^r, z^l\rangle$ and $A_{02} = \langle Cz^r, z^r\rangle$.

Alternatively, the matricial form of 2.23 is

$$\left\langle \begin{pmatrix} A_{00} & A_{10} & A_{01} \\ \bar{A}_{10} & A_{20} & A_{11} \\ \bar{A}_{01} & \bar{A}_{11} & A_{02} \end{pmatrix} \begin{pmatrix} 1 \\ c \\ d \end{pmatrix}, \begin{pmatrix} 1 \\ c \\ d \end{pmatrix} \right\rangle. \qquad (2.24)$$

Now, we compute the values of coefficient A_{00} which is just action of T_φ on monomial z^k.

$$\begin{aligned} A_{00} =& \langle Cz^k, z^k\rangle \equiv \left\langle [(T_{\alpha z^n+\beta z^m+\gamma\bar{z}^p+\delta\bar{z}^q})^*, T_{\alpha z^n+\beta z^m+\gamma\bar{z}^p+\delta\bar{z}^q}]z^k, z^k\right\rangle \\ &= \langle (T_{\bar{\alpha}\bar{z}^n+\bar{\beta}\bar{z}^m+\bar{\gamma}z^p+\bar{\delta}z^q}).(T_{\alpha z^n+\beta z^m+\gamma\bar{z}^p+\delta\bar{z}^q}) - (T_{\alpha z^n+\beta z^m+\gamma\bar{z}^p+\delta\bar{z}^q}).(T_{\bar{\alpha}\bar{z}^n+\bar{\beta}\bar{z}^m+\bar{\gamma}z^p+\bar{\delta}z^q})z^k, z^k\rangle \\ &= \langle (T_{\bar{\alpha}\bar{z}^n+\bar{\beta}\bar{z}^m+\bar{\gamma}z^p+\bar{\delta}z^q}).(T_{\alpha z^n+\beta z^m+\gamma\bar{z}^p+\delta\bar{z}^q})z^k, z^k\rangle - \\ &\quad \langle (T_{\alpha z^n+\beta z^m+\gamma\bar{z}^p+\delta\bar{z}^q}).(T_{\bar{\alpha}\bar{z}^n+\bar{\beta}\bar{z}^m+\bar{\gamma}z^p+\bar{\delta}z^q})z^k, z^k\rangle. \end{aligned} \qquad (2.25)$$

To compute A_{00} and other coefficients explicitly, we use the facts that Toeplitz operators with analytic symbols is commutative, two monomials z^u and z^v are orthogonal whenever $u \neq v$. Only the remaining terms are,

$$\begin{aligned}A_{00} = &|\alpha|^2(\langle z^{k+n}, z^{k+n}\rangle - \langle P(\bar{z}^n z^k), P(\bar{z}^n z^k)\rangle) \\ &+ |\beta|^2(\langle z^{m+k}, z^{m+k}\rangle - \langle P(\bar{z}^m z^k), P(\bar{z}^m z^k)\rangle) \\ &- |\gamma|^2(\langle z^{p+k}, z^{p+k}\rangle - \langle P(\bar{z}^p z^k), P(\bar{z}^p z^k)\rangle) \\ &- |\delta|^2(\langle z^{q+k}, z^{q+k}\rangle - \langle P(\bar{z}^q z^k), P(\bar{z}^q z^k)\rangle). \end{aligned} \tag{2.26}$$

We compute the first two term of A_{00}; the rest will follow similarly.

$$\begin{aligned}|\alpha|^2(||z^{k+n}||_\zeta^2 - ||P(\bar{z}^n z^k)||_\zeta^2) = |\alpha|^2 &\left(\frac{\Gamma(k+n+1)\Gamma(\zeta+2)}{\Gamma(k-n+\zeta+2)} - \frac{\Gamma(k+n+\zeta+2)\Gamma(k+1)^2\Gamma(\zeta+2)}{\Gamma(k-n+1)\Gamma(k+\zeta+2)^2}\right) \\ = \frac{|\alpha|^2\Gamma(\zeta+2)\Gamma(k+1)}{\Gamma(k+\zeta+2)} &\left(\frac{\Gamma(k+n+1)\Gamma(k+\zeta+2)}{\Gamma(k+n+\zeta+2)\Gamma(k+1)} - \frac{\Gamma(k-n+\zeta+2)\Gamma(k+1)}{\Gamma(k-n+1)\Gamma(k+\zeta+2)}\right)\end{aligned} \tag{2.27}$$

To compute the difference in (2.27), we multiply (2.20) & (2.21), and (2.19) & (2.22) and take their difference. First we multiply (2.20) & (2.21).

$$\begin{aligned}\frac{\Gamma(k+n+1)}{\Gamma(k+n+\zeta+2)}\frac{\Gamma(k+\zeta+2)}{\Gamma(k+1)} = &1 + \frac{1}{2k}(\zeta+1)(\zeta+2) + \frac{1}{12k^2}\binom{\zeta+1}{2}\Big((2+\zeta)(3\zeta+5)\Big) - \\ &\frac{1}{2k}(\zeta+1)(2n+\zeta+2) - \frac{1}{4k^2}(\zeta+1)^2(2+\zeta)(2n+\zeta+2) + \\ &\frac{1}{12k^2}\binom{-\zeta-1}{2}\Big(3(2n+\zeta+2)^2+\zeta\Big) + \mathcal{O}(k^{-3}).\end{aligned} \tag{2.28}$$

Similarly we multiply (2.19) & (2.22),

$$\frac{\Gamma(k+1)}{\Gamma(k+\zeta+2)}\frac{\Gamma(k-n+\zeta+2)}{\Gamma(k-n+1)} = 1+\frac{1}{2k}(\zeta+1)(-2n+\zeta+2)$$
$$+\frac{1}{12k^2}\binom{\zeta+1}{2}\left(3(-2n+\zeta+2)^2-(\zeta+2)\right)$$
$$-\frac{1}{2k}(\zeta+1)(2+\zeta)-\frac{1}{4k^2}(\zeta+1)^2(2+\zeta)(-2n+\zeta+2)$$
$$+\frac{1}{12k^2}\binom{-\zeta-1}{2}\left(3(\zeta+2)^2+\zeta\right)+\mathcal{O}(k^{-3}). \tag{2.29}$$

Now we subtract (2.28)−(2.29) to get the difference (2.27)

$= \frac{1}{2k}(\zeta+1)(2+\zeta+2n-\zeta-2)+\frac{1}{12k^2}\binom{\zeta+1}{2}\{3(2+\zeta)^2-(2+\zeta)-3(-2n+\zeta+2)^2+(2+\zeta)\}-$
$\frac{1}{2k}(\zeta+1)(2n+\zeta+2-2-\zeta)-\frac{1}{4k^2}(\zeta+1)^2(2+\zeta)(2n+\zeta+2-2n-\zeta-2)+$
$\frac{1}{12k^2}\binom{-\zeta-1}{2}\{3(2n+\zeta+2)^2+\zeta)-3(\zeta+2)^2-\zeta\}+\mathcal{O}(k^{-3})$

$= \frac{1}{12k^2}\binom{\zeta+1}{2}\{3(2+\zeta)^2-3(2+\zeta)^2+12n(\zeta+2)-12n^2\}-\frac{1}{4k^2}4n(\zeta+1)^2(2+\zeta)+$
$\frac{1}{4k^2}\binom{-\zeta-1}{2}\{3(\zeta+2)^{+}12n(\zeta+2)+12n^2-3(\zeta+2)^2\}+\mathcal{O}(k^{-3})$

$= \frac{n}{k^2}\Big\{\binom{\zeta+1}{2}(\zeta+2-n)-\frac{1}{4k^2}4n(\zeta+1)^2(2+\zeta)+\frac{n}{k^2}\binom{-\zeta-1}{2}(\zeta+2-n)\Big\}+\mathcal{O}(k^{-3})$

$= \frac{n}{k^2}\Big\{\frac{(\zeta+1).\zeta}{2}(\zeta+2-n)-(\zeta+1)^2(2+\zeta)+\frac{(\zeta+1)(2+\zeta)}{2}.(\zeta+2+n)\Big\}+\mathcal{O}(k^{-3})$

$= \frac{n}{k^2}.\frac{(\zeta+1)}{2}\Big\{\zeta(\zeta+2-n)-2(\zeta+1)(2+\zeta)+(2+\zeta)(\zeta+2+n)\Big\}+\mathcal{O}(k^{-3})$

$= \frac{n}{k^2}.\frac{(\zeta+1)}{2}\Big\{\zeta(\zeta+2)-n\zeta-2(\zeta+1)(2+\zeta)+(2+\zeta)^2+2n+n\zeta\Big\}+\mathcal{O}(k^{-3})$

$= \frac{n^2}{k^2}(\zeta+1)+\mathcal{O}(k^{-3}).$

Now (2.27) becomes,

$$= \frac{|\alpha|^2\Gamma(\zeta+2)\Gamma(k+1)}{\Gamma(k+\zeta+2)}\Big\{\frac{n^2}{k^2}(\zeta+1)+\mathcal{O}(k^{-3})\Big\}$$

using (A), we get

$$= |\alpha|^2\Gamma(\zeta+2)k^{-\zeta-1}\Big\{1+\mathcal{O}(k^{-1})\Big\}\Big\{\frac{n^2}{k^2}(\zeta+1)+\mathcal{O}(k^{-3})\Big\}$$

$$= |\alpha|^2\Gamma(\zeta+2)k^{-\zeta-1}\frac{n^2}{k^2}(\zeta+1)+\mathcal{O}(k^{-4-\zeta})$$

$$= |\alpha|^2\Gamma(\zeta+2)k^{-\zeta-3}n^2(\zeta+1)+\mathcal{O}(k^{-4-\zeta})$$

Hence the first two terms of A_{00};

$$|\alpha|^2(||z^{k+n}||^2_\zeta - ||P(\bar{z}^n z^k)||^2_\zeta) = k^{-\zeta-3}|\alpha|^2\Gamma(\zeta+2)n^2(\zeta+1)+\mathcal{O}(k^{-1})$$

Following the similar pattern, we obtain remaining terms of A_{00} as below.

$|\beta|^2(||z^{m+k}||^2_\zeta - ||P(\bar{z}^m z^k)||^2_\zeta) = k^{-\zeta-3}|\beta|^2\Gamma(\zeta+2)m^2(\zeta+1)+\mathcal{O}(k^{-1})$

$|\gamma|^2(||z^{p+k}||^2_\zeta - ||P(\bar{z}^p z^k)||^2_\zeta) = k^{-\zeta-3}|\gamma|^2\Gamma(\zeta+2)p^2(\zeta+1)+\mathcal{O}(k^{-1})$

$|\delta|^2(||z^{q+k}||^2_\zeta - ||P(\bar{z}^q z^k)||^2_\zeta) = k^{-\zeta-3}|\delta|^2\Gamma(\zeta+2)q^2(\zeta+1)+\mathcal{O}(k^{-1})$

Finally we take get,

$$A_{00} = k^{-\zeta-3}(\zeta+1)\Gamma(\zeta+2)\Big\{n^2|\alpha|^2+m|\beta|^2-p^2|\gamma|^2-q^2|\delta|^2\Big\}+\mathcal{O}(k^{-1}).$$

Now we compute A_{10},

$$\begin{aligned} A_{10} =& \langle Cz^l, z^k\rangle \equiv \Big\langle[(T_{\alpha z^n+\beta z^m+\gamma\bar{z}^p+\delta\bar{z}^q})^*, T_{\alpha z^n+\beta z^m+\gamma\bar{z}^p+\delta\bar{z}^q}]z^l, z^k\Big\rangle \\ &= \langle(T_{\bar{\alpha}\bar{z}^n+\bar{\beta}\bar{z}^m+\bar{\gamma}z^p+\bar{\delta}z^q}).(T_{\alpha z^n+\beta z^m+\gamma\bar{z}^p+\delta\bar{z}^q}) - (T_{\alpha z^n+\beta z^m+\gamma\bar{z}^p+\delta\bar{z}^q}).(T_{\bar{\alpha}\bar{z}^n+\bar{\beta}\bar{z}^m+\bar{\gamma}z^p+\bar{\delta}z^q})z^l, z^k\rangle \end{aligned} \tag{2.30}$$

$$= \langle (T_{\bar{\alpha}\bar{z}^n+\bar{\beta}\bar{z}^m+\bar{\gamma}z^p+\bar{\delta}z^q}).(T_{\alpha z^n+\beta z^m+\gamma\bar{z}^p+\delta\bar{z}^q})z^l, z^k\rangle -$$

$$\langle (T_{\alpha z^n+\beta z^m+\gamma\bar{z}^p+\delta\bar{z}^q}).(T_{\bar{\alpha}\bar{z}^n+\bar{\beta}\bar{z}^m+\bar{\gamma}z^p+\bar{\delta}z^q})z^l, z^k\rangle. \tag{2.31}$$

We again use the facts that Toeplitz operators with analytic symbols are commutative, two monomials z^u and z^v are orthogonal whenever $u \neq v$. Only the remaining terms of A_{10} are,

$$\begin{aligned} A_{10} = {} & \bar{\alpha}\beta(\langle T_{\bar{z}^n}T_{z^m}z^l, z^k\rangle - \langle T_{z^m}T_{\bar{z}^n}z^l, z^k\rangle)\delta_{k+n,m+l} \\ & +\bar{\beta}\alpha(\langle T_{\bar{z}^m}T_{z^n}z^l, z^k\rangle - \langle T_{z^n}T_{\bar{z}^m}z^l, z^k\rangle)\delta_{m+k,n+l} \\ & +\bar{\gamma}\delta(\langle T_{z^p}T_{\bar{z}^q}z^l, z^k\rangle - \langle T_{\bar{z}^q}T_{z^p}z^l, z^k\rangle)\delta_{q+k,p+l} \\ & +\bar{\delta}\gamma(\langle T_{z^q}T_{\bar{z}^p}z^l, z^k\rangle - \langle T_{\bar{z}^p}T_{z^q}z^l, z^k\rangle)\delta_{p+k,q+l}. \end{aligned} \tag{2.32}$$

Since we have $m < n$ and $k < l$, so that $m + k < n + l$, and therefore $\delta_{m+k,n+l} = 0$. Also, $p < q$ implies $p + k < q + l$, so that $\delta_{p+k,q+l} = 0$. As a consequence,

$$\begin{aligned} A_{10} = {} & \bar{\alpha}\beta\Big(\langle z^{m+l}, z^{k+n}\rangle - \langle P(\bar{z}^n z^l), P(\bar{z}^m z^k\rangle\Big)\delta_{k+n,m+l} \\ & -\bar{\gamma}\delta\Big(\langle z^{p+l}, z^{q+k}\rangle - \langle P(\bar{z}^q z^l), P(\bar{z}^p z^k\rangle\Big)\delta_{q+k,p+l}. \end{aligned} \tag{2.33}$$

We compute the first two terms of A_{10}; From the corollary (2.2), we have

$$\bar{\alpha}\beta\Big(\langle z^{m+l}, z^{k+n}\rangle - \langle P(\bar{z}^n z^l), P(\bar{z}^m z^k\rangle\Big)\delta_{k+n,m+l} \tag{2.34}$$

$$
\begin{aligned}
=&\bar{\alpha}\beta\left(\frac{\Gamma(k+n+1)\Gamma(\zeta+2)}{\Gamma(k+n+\zeta+2)}-\frac{\Gamma(k-m+\zeta+2)\Gamma(\zeta+2)\Gamma(l+1)\Gamma(k+1)}{\Gamma(k-m+1)\Gamma(l+\zeta+2)\Gamma(k+\zeta+2)}\right)\\
=&\bar{\alpha}\beta\Gamma(\zeta+2)\left(\frac{\Gamma(k+n+1)}{\Gamma(k+n+\zeta+2)}-\frac{\Gamma(k-m+\zeta+2)\Gamma(l+1)\Gamma(k+1)}{\Gamma(k-m+1)\Gamma(l+\zeta+2)\Gamma(k+\zeta+2)}\right)\\
=&\bar{\alpha}\beta\Gamma(\zeta+2)\frac{\Gamma(k+1)}{\Gamma(k+\zeta+2)}\left\{\frac{\Gamma(k+\zeta+2)}{\Gamma(k+1)}\frac{\Gamma(k+n+1)}{\Gamma(k+n+\zeta+2)}-\right.\\
&\left.\frac{\Gamma(k-m+\zeta+2)}{\Gamma(k-m+1)}\frac{\Gamma(k+n-m+1)}{\Gamma(k+n-m+\zeta+2)}\right\}\quad \text{(because } l=k+n-m\text{).} \qquad (2.35)
\end{aligned}
$$

Before computing the difference in (2.35), we calculate the product of ratios first. To do that we repalce n by $n-m$ in (2.20) to get,

$$
\begin{aligned}
\frac{\Gamma(k+n-m+1)}{\Gamma(k+n-m+\zeta+2)}\sim k^{-\zeta-1}\Bigg(&1-\frac{1}{2k}(\zeta+1)(2n-2m+\zeta+2)+\frac{1}{12k^2}\binom{-\zeta-1}{2}\\
&\Big(3(2n-2m+\zeta+2)^2+(\zeta)\Big)\Bigg)+\mathcal{O}(k^{-3}) \qquad (2.36)
\end{aligned}
$$

Assuming $a=-m+\zeta+2$ and $b=-m+1$ in (2.18) we get,

$$
\begin{aligned}
\frac{\Gamma(k-m+\zeta+2)}{\Gamma(k-m+1)}\sim k^{\zeta+1}\Bigg(&1+\frac{1}{2k}(\zeta+1)(-2m+\zeta+2)+\frac{1}{12k^2}\binom{\zeta+1}{2}\\
&\Big(3(-2m+\zeta+2)^2-(\zeta+2)\Big)\Bigg) \qquad (2.37)
\end{aligned}
$$

The first product inside the bracket of (2.35) has been obtained in (2.28). And, we multiply (2.36) and (2.37) to get the second product.

$$
\begin{aligned}
&\frac{\Gamma(k+n-m+1)}{\Gamma(k+n-m+\zeta+2)}\frac{\Gamma(k-m+\zeta+2)}{\Gamma(k-m+1)}=1+\frac{1}{2k}(\zeta+1)(-2m+\zeta+2)\\
&+\frac{1}{12k^2}\binom{\zeta+1}{2}\Big(3(-2m+\zeta+2)^2-(\zeta+2)\Big)-\frac{1}{2k}(\zeta+1)(2n-2m+\zeta+2)\\
&-\frac{1}{4k^2}(\zeta+1)^2(2n-2m+\zeta+2)(-2m+\zeta+2)+\frac{1}{12k^2}\binom{-\zeta-1}{2}\Big(3(2n-2m+\zeta+2)^2+\zeta\Big)\\
&+\mathcal{O}(k^{-3}) \qquad (2.38)
\end{aligned}
$$

Subtracting (2.28)−(2.38), we get

$= -\frac{1}{2k}(\zeta+1)(2n+\zeta+2-2m+\zeta+2)+\frac{1}{12k^2}\binom{-\zeta-1}{2}\Big\{3(2n+\zeta+2)^2+\zeta-3(2n-2m+\zeta+2)^2-\zeta\Big\}+\frac{1}{2k}(\zeta+1)(2+\zeta+2n-2m+\zeta+2)-\frac{1}{4k^2}(\zeta+1)^2\Big\{(2+\zeta)(2n+\zeta+2)-(2n-2m+\zeta+2)(-2m+\zeta+2)\Big\}+\frac{1}{12k^2}\Big\{3(\zeta+2)^2-(\zeta+2)-3(-2m+\zeta+2)^2+(\zeta+2)\Big\}+\mathcal{O}(k^{-3})$

$= \frac{1}{12k^2}\binom{-\zeta-1}{2}\Big\{3(\zeta+2)^2+12n(\zeta+2)+12n^2-3(\zeta+2)^2-12(n-m)^2\Big\}-\frac{1}{4k^2}(\zeta+1)^2\Big\{2n(2+\zeta)+(2+\zeta)^2+4m(n-m)-2(n-m)(\zeta+2)+2m(\zeta+2)-(\zeta+2)^2\Big\}+\frac{1}{12k^2}\binom{\zeta+1}{2}\Big\{3(\zeta)^2-3(\zeta+2)^2+12m(\zeta+2)-12m^2\Big\}+\mathcal{O}(k^{-3})$

$= \frac{1}{24k^2}(\zeta+1)(2+\zeta)\Big\{12n(\zeta+2)+12n^2-12n(\zeta+2)+12m(\zeta+2)-12n^2+24mn-12m^2\Big\}-\frac{1}{4k^2}(\zeta+1)^2\Big\{2n(2+\zeta)+4m(n-m)-2n(\zeta+2)+2m(\zeta+2)+2m(\zeta+2)+\frac{1}{24k^2}\zeta(\zeta+1)\Big\{12m(\zeta+2)-12m^2\Big\}+\mathcal{O}(k^{-3})$

$= \frac{m}{2k^2}(\zeta+1)(2+\zeta)(\zeta+2+2n-m)-\frac{m}{2k^2}(\zeta+1)^2(2n-2m+2\zeta+4)+\frac{m}{2k^2}\zeta(\zeta+1)(\zeta+2-m)+\mathcal{O}(k^{-3})$

$= \frac{m}{2k^2}(\zeta+1)\Big\{(2+\zeta)(\zeta+2+2n-m)-(\zeta+1)(2n-2m+2\zeta+4)+\zeta(\zeta+2-m)\Big\}+\mathcal{O}(k^{-3})$

$= \frac{m}{2k^2}(\zeta+1)\Big\{\zeta(\zeta+2+2n-m-2n+2m_2\zeta-4+\zeta+2-m)+(2\zeta+4+4n-2m-2n+2m-2\zeta-4)\Big\}+\mathcal{O}(k^{-3})$

$= \frac{m}{2k^2}(\zeta+1)2n+\mathcal{O}(k^{-3})$

$= \frac{mn(\zeta+1)}{k^2}+\mathcal{O}(k^{-3})$

Therefore (2.35) becomes,

$\bar{\alpha}\beta\Gamma(\zeta+2)\frac{\Gamma(k+1)}{\Gamma(k+\zeta+2)}\Big\{\frac{mn(\zeta+1)}{k^2}+\mathcal{O}(k^{-3})\Big\}$

$$=\bar{\alpha}\beta\Gamma(\zeta+2)(1+\mathcal{O}(k^{-1}))\Big\{\tfrac{mn(\zeta+1)}{k^2}+\mathcal{O}(k^{-3})\Big\}$$
$$=k^{-\zeta-3}\bar{\alpha}\beta mn\Gamma(\zeta+2)(\zeta+1)+\mathcal{O}(k^{-4-\zeta}).$$

Other terms of A_{10} follow similarly, and we get,

$$A_{10}=k^{-\zeta-3}\bar{\alpha}\beta mn\Gamma(\zeta+2)(\zeta+1)+k^{-\zeta-3}\bar{\gamma}\delta pq\Gamma(\zeta+2)(\zeta+1)+\mathcal{O}(k^{-4-\zeta})$$
$$=k^{-\zeta-3}\Gamma(\zeta+2)(\zeta+1)\{\bar{\alpha}\beta mn-\bar{\gamma}\delta pq\}+\mathcal{O}(k^{-4-\zeta}).$$

Next we compute $A_{01}=\langle Cz^r,z^k\rangle$. We will imitate the calculation for A_{10}. Observe that $k<r$, the remaining terms of A_{01} will be same as (2.32),

$$\begin{aligned}A_{01}=\ &\bar{\alpha}\beta(\langle T_{\bar{z}^n}T_{z^m}z^r,z^k\rangle-\langle T_{z^m}T_{\bar{z}^n}z^r,z^k\rangle)\delta_{k+n,m+r}\\ &+\bar{\beta}\alpha(\langle T_{\bar{z}^m}T_{z^n}z^r,z^k\rangle-\langle T_{z^n}T_{\bar{z}^m}z^r,z^k\rangle)\delta_{m+k,n+r}\\ &+\bar{\gamma}\delta(\langle T_{z^p}T_{\bar{z}^q}z^r,z^k\rangle-\langle T_{\bar{z}^q}T_{z^p}z^r,z^k\rangle)\delta_{q+k,p+r}\\ &+\bar{\delta}\gamma(\langle T_{z^q}T_{\bar{z}^p}z^r,z^k\rangle-\langle T_{\bar{z}^p}T_{z^q}z^r,z^k\rangle)\delta_{p+k,q+r}.\end{aligned}\tag{2.39}$$

Since we have $m<n$ and $k<r$, so that $m+k<n+r$, and therefore $\delta_{m+k,n+r}=0$. Also, $p<q$ implies $p+k<q+r$, so that $\delta_{p+k,q+r}=0$. As a consequence,

$$\begin{aligned}A_{01}=&\bar{\alpha}\beta\Big(\langle z^{m+r},z^{k+n}\rangle-\langle P(\bar{z}^nz^r),P(\bar{z}^mz^k\rangle\Big)\delta_{k+n,m+r}\\ &-\bar{\gamma}\delta\Big(\langle z^{p+r},z^{q+k}\rangle-\langle P(\bar{z}^qz^r),P(\bar{z}^pz^k\rangle\Big)\delta_{q+k,p+r}.\end{aligned}\tag{2.40}$$

We let $l:=n+k-m$ and $r:=l+q-p$. It follows that $n+k=m+l<m+r$ and $q+k<q+l=p+r$. Therefore, both Kronecker deltas appearing in A_{01} are zero, and thus $A_{01}=0$.

We compute $A_{11}=\langle Cz^r,z^l\rangle$ where $l<r$. We replace l with r and k with l in the

calculation of A_{10}.

$$A_{11} = \bar{\alpha}\beta\Big(\langle z^{m+r}, z^{l+n}\rangle - \langle P(\bar{z}^n z^r), P(\bar{z}^m z^l\rangle\Big)\delta_{l+n,m+r}$$
$$-\bar{\gamma}\delta\Big(\langle z^{p+r}, z^{q+l}\rangle - \langle P(\bar{z}^q z^r), P(\bar{z}^p z^l\rangle\Big)\delta_{q+l,p+r}. \tag{2.41}$$

The calculation of $A_{20} = \langle Cz^l, z^l\rangle$ and $A_{02} = \langle Cz^r, z^r\rangle$ are similar to A_{00}. We replace k by l and r separately to get as follows,

$$\begin{aligned} A_{20} = {} & |\alpha|^2(\langle z^{l+n}, z^{l+n}\rangle - \langle P(\bar{z}^n z^l), P(\bar{z}^n z^l)\rangle) \\ & + |\beta|^2(\langle z^{m+l}, z^{m+l}\rangle - \langle P(\bar{z}^m z^l), P(\bar{z}^m z^l)\rangle) \\ & - |\gamma|^2(\langle z^{p+l}, z^{p+l}\rangle - \langle P(\bar{z}^p z^l), P(\bar{z}^p z^l)\rangle) \\ & - |\delta|^2(\langle z^{q+l}, z^{q+l}\rangle - \langle P(\bar{z}^q z^l), P(\bar{z}^q z^l)\rangle). \end{aligned} \tag{2.42}$$

and,

$$\begin{aligned} A_{02} = {} & |\alpha|^2(\langle z^{r+n}, z^{r+n}\rangle - \langle P(\bar{z}^n z^r), P(\bar{z}^n z^r)\rangle) \\ & + |\beta|^2(\langle z^{m+r}, z^{m+r}\rangle - \langle P(\bar{z}^m z^r), P(\bar{z}^m z^r)\rangle) \\ & - |\gamma|^2(\langle z^{p+r}, z^{p+r}\rangle - \langle P(\bar{z}^p z^r), P(\bar{z}^p z^r)\rangle) \\ & - |\delta|^2(\langle z^{q+r}, z^{q+r}\rangle - \langle P(\bar{z}^q z^r), P(\bar{z}^q z^r)\rangle). \end{aligned} \tag{2.43}$$

In the calculation of A_{11}, A_{20} and A_{02}, we use the assumption $n-m = q-p$, and let $g := n-m = q-p$. It follows that $l = k+g$ and $r = l+g = k+2g$. Then we imitate calculation of A_{10} to obtain A_{11}, and A_{00} to obtain A_{02} and A_{02}. It follows,

$$A_{11} = k^{-\zeta-3}\Gamma(\zeta+2)(\zeta+1)\{\bar{\alpha}\beta mn - \bar{\gamma}\delta pq\} + \mathcal{O}(k^{-4-\zeta}),$$

$$A_{20} = k^{-\zeta-3}\Gamma(\zeta+2)(\zeta+1)\{|\alpha|^2 n^2 + |\beta|^2 m^2 - |\gamma|^2 p^2 - |\delta|^2 q^2\} + \mathcal{O}(k^{-4-\zeta}),$$

$$A_{02} = k^{-\zeta-3}\Gamma(\zeta+2)(\zeta+1)\{|\alpha|^2n^2 + |\beta|^2m^2 - |\gamma|^2p^2 - |\delta|^2q^2\} + \mathcal{O}(k^{-4-\zeta})$$

The associated 3×3 matrix of C become,

$$M := \begin{pmatrix} A_{00} & A_{10} & 0 \\ \bar{A}_{10} & A_{20} & A_{11} \\ 0 & \bar{A}_{11} & A_{02} \end{pmatrix}.$$

We study the asymptotic behavior of $k^{\zeta+3}M$ as $k \to \infty$. We get,

$$\lim_{k\to\infty} k^{\zeta+3}A_{00} = \Gamma(\zeta+2)(\zeta+1)\{|\alpha|^2n^2 + |\beta|^2m^2 - |\gamma|^2p^2 - |\delta|^2q^2\} = a \tag{2.44}$$

$$\lim_{k\to\infty} k^{\zeta+3}A_{10} = \Gamma(\zeta+2)(\zeta+1)\{\bar{\alpha}\beta mn - \bar{\gamma}\delta pq\} = \rho. \tag{2.45}$$

Similarly asymptotic expansion of the other entries are,

$$\lim_{k\to\infty} k^{\zeta+3}A_{20} = \lim_{k\to\infty} k^{\zeta+3}A_{02} = a$$

and,

$$\lim_{k\to\infty} k^{\zeta+3}A_{11} = \rho.$$

It follows that asymptotic expansion of $k^{\zeta+3}M$ as $k \to \infty$ will be a tridiagonal matrix,

$$\begin{pmatrix} a & \rho & 0 \\ \bar{\rho} & a & \rho \\ 0 & \bar{\rho} & a \end{pmatrix}.$$

Now, if instead of using a vector of the form

$$f : z^k + cz^l + dz^r \quad (k < l < r)$$

with $l = k+g$ and $r = l+g = k+2g$ (that is, a vector of the form

$$f : z^k + cz^{k+g} + dz^{k+2g},$$

we were to use a longer vector with similar power structure,

$$f : z^k + c_1 z^{k+g} + c_2 z^{k+2g} + ... + c_N z^{k+Ng},$$

The associated matrix of C with respect to new f will be of $N \times N$ order.

$$A = \begin{pmatrix} \langle Cz^k, z^k\rangle & \langle Cz^{k+g}, z^k\rangle & \cdots & \langle Cz^{k+Ng}, z^k\rangle \\ \overline{\langle Cz^{k+g}, z^k\rangle} & \langle Cz^{k+g}, z^{k+g}\rangle & \cdots & \langle Cz^{k+Ng}, z^{k+g}\rangle \\ \vdots & \vdots & \ddots & \\ \overline{\langle Cz^{k+Ng}, z^k\rangle} & \overline{\langle Cz^{k+Ng}, z^{k+g}\rangle} & \cdots & \langle Cz^{k+Ng}, z^{k+Ng}\rangle \end{pmatrix}.$$

We imitate the calculation of A_{00} to calculate asymptotic expansion of diagonal entries; that is we replace k by $k+Ng$, we get

$\lim\limits_{k\to\infty} k^{\zeta+3}\langle Cz^{k+Ng}, z^{k+Ng}\rangle = (\zeta+1)\Gamma(\zeta+2)\Big\{n^2|\alpha|^2+m|\beta|^2-p^2|\gamma|^2-q^2|\delta|^2\Big\} = a. \quad \forall N$

Similarly we imitate the calculation of A_{10} to obtain upper diagonal; we replace l by $k+Ng$ and k by $l+(N-1)g$, we get,

$\lim\limits_{k\to\infty} k^{\zeta+3}\langle Cz^{k+Ng}, z^{k+(N-1)g}\rangle = \Gamma(\zeta+2)(\zeta+1)\{\bar{\alpha}\beta mn - \bar{\gamma}\delta pq\} = \rho \quad \forall\ N\ .$

By our assumption we have,

$$m + l = k + n$$

$$k + g + m = k + n$$

$$k + Ng + g + m = k + n + Ng$$

$$k + Ng + m = k + n + (N-1)g \ \neq k + n + (N-t)g \quad \forall\ t \geq 2$$

Similarly,

$$p + k + Ng \neq n + q + (N - t)g \quad \forall\, t \geq 2$$

As in A_{10}, the surviving terms of $\langle Cz^{k+Ng}, z^{k+(N-t)g}\rangle$, $N \geq 1$ & $t \geq 2$ are,

$$=\bar{\alpha}\beta\Big(\langle\, .\,,\,.\,\rangle - \langle\, .\,,\,.\,\rangle\Big)\delta_{k+n+(N-t)g,k+Ng+m}$$
$$+\,\bar{\gamma}\delta\Big(\langle\, .\,,\,.\,\rangle - \langle\, .\,,\,.\,\rangle\Big)\delta_{k+(N-t)g+q,k+Ng+p}.$$

Therefore, $\langle Cz^{k+Ng}, z^{k+(N-t)g}\rangle = 0$, $N \geq 1$ & $t \geq 2$.

It follows that asymptotic expansion of $k^{\zeta+3}A$ as $k \to \infty$ will be still a tridiagonal matrix,

$$\begin{pmatrix} a & \rho & 0 & 0 & \cdots & 0 \\ \overline{\rho} & a & \rho & 0 & \cdots & 0 \\ 0 & \overline{\rho} & a & \rho & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & & \rho \\ 0 & 0 & 0 & \cdots & \overline{\rho} & a \end{pmatrix}.$$

Since this must be true for all $N \geq 1$, It follows that asymptotic expansion of $k^{\zeta+3}A$ as $k \to \infty$ is infinite tridiagonal matrix,

$$B = \begin{pmatrix} a & \rho & 0 & 0 & \cdots \\ \overline{\rho} & a & \rho & 0 & \cdots \\ 0 & \overline{\rho} & a & \rho & \cdots \\ \vdots & \vdots & \vdots & \ddots & \end{pmatrix}.$$

The hyponormality of T_φ, detected by the positivity of the self-commutator C, leads to the positive semi-definiteness of associated matrix. It follows that a necessary condition for the hyponormality of T_φ is the positive semidefiniteness of the infinite tridiagonal matrix B.

We now consider the spectral behavior of B as an operator on $l^2(Z_+)$.

Lemma 5. *For $a \in \mathbb{R}$ and $\rho \in \mathbb{C}$, the spectrum of the infinite tridiagonal matrix B is $[a-2|\rho|, a+2|\rho|]$.*

Proof. The proof is already known. Here B represents the Toeplitz opertator on $H^2(\mathbb{T})$ with symbol $\varphi(z) = a + 2Re(\bar{\rho}z)$. As symbol is harmonic, the spectrum of $T_\varphi = aI + T_{\bar{\rho}z+\rho\bar{z}}$ is the set $a + 2Re(\{\bar{\rho}z : z \in \mathbb{D}\}) = a + 2[-|\rho|, |\rho|]$. □

As a consequences, if B is positive (as an operator on $l^2(\mathbb{Z}_+)$), then

$$a \geq 2|\rho|.$$

The values of a and b from (2.44) and (2.45) gives,

$$\Gamma(\zeta+2)(\zeta+a)\{|\alpha|^2n^2 + |\beta|^2m^2 - |\gamma|^2p^2 - |\delta|^2q^2\} \geq \Gamma(\zeta+2)(\zeta+a)\{(\bar{\alpha}\beta mn - \bar{\gamma}\delta pq)\}$$

$$|\alpha|^2n^2 + |\beta|^2m^2 - |\gamma|^2p^2 - |\delta|^2q^2\} \geq (\bar{\alpha}\beta mn - \bar{\gamma}\delta pq)\}.$$

□

Chapter 3

Positivity of Toeplitz operator via Berezin transform on Model spaces

3.1 Introduction

Denote by $\mathbb{T}$ the unit circle in $\mathbb{C}$, and let ds represent the normalized arc length measure on $\mathbb{T}$. The **Hardy Space** of the unit disc $H^2(\mathbb{D})$ is the set of all functions f which are holomorphic on $\mathbb{D}$ and satisfy the condition

$$||f||^2 := \sup_{0<r<1} \int_{\mathbb{T}} |f(rz)|^2 ds(z) < \infty.$$

It is well known that for $f \in H^2(\mathbb{D})$ the integral means

$$r \to \int_{\mathbb{T}} |f(rz)|^2 ds(z)$$

are increasing in r for $r \in (0, 1)$, and so we may conclude that

$$\sup_{0<r<1} \int_{\mathbb{T}} |f(rz)|^2 ds(z) = \lim_{r \to 1^-} \int_{\mathbb{T}} |f(rz)|^2 ds(z).$$

It is also well known for $f \in H^2(\mathbb{D})$ that the radial limits

$$f(z) := \lim_{r \to 1^-} f(rz)$$

exists almost everywhere on $\mathbb{T}$ and that the H^2 norm of f coincides with the L^2 norm of the boundary function of f on $\mathbb{T}$. In this way, we regard H^2 as a closed subspace of $L^2(\mathbb{T})$, so that it is indeed a Hilbert space.

Definition 5. We say that a function u is an **inner function** if u is a bounded holomorphic function on $\mathbb{D}$ such that $|u(z)| = 1$ for almost every $z \in \mathbb{T}$.

Examples of inner functions includes: $z^n, n \in \mathbb{N}$ and Blaschke products

$$\beta \prod_{i=1}^{k} \frac{z - a_i}{1 - \bar{a}_i z},$$

where $|\beta| = 1$, $a_i \in \mathbb{D}$ for $1 \leq i \leq k$.

Definition 6. The **unilateral shift operator** $S : H^2 \to H^2$ is defined by

$$Sf(z) := zf(z),$$

for $z \in \mathbb{D}$. The unilateral shift operator S is unitarily equivalent to $S : l^2 \to l^2$ as an operator defined on sequences of Fourier coefficients of functions in H^2 by

$$S(a_0, a_1, a_2, ...) := (0, a_0, a_1, a_2, ...).$$

In 1949 Beurling characterized all the S-invariant subspaces in H^2.

Theorem 6. *Beurling's Theorem: The nontrivial S-invariant subspaces of H^2 are precisely the subspaces of the form*

$$uH^2 = \{uf : f \in H^2\},$$

where u is an inner function.

Definition 7. Suppose u is an inner function. Then the **model space** $\mathcal{K}_u$ is given by $\mathcal{K}_u = (uH^2)^\perp = H^2 \ominus uH^2$.

In other words, model spaces are the orthogonal complements of the nontrivial invariant subspaces for the unilateral shift $Sf(z) := zf(z)$ on H^2. Hence, $\mathcal{K}_u$ are precisely the nontrivial invariant subspaces of S^*.

Hence we conclude, model spaces are invariant subspaces for the backward shift operator,

$$S^* f(z) = \frac{f(z) - f(0)}{z} \qquad z \in \mathbb{D}.$$

The following known proposition describes Model spaces in a different way.

Proposition 3. *If u is an inner function, then*

$$\mathcal{K}_u = H^2 \cap u\overline{zH^2},$$

where we regard the right hand side as a set of functions on $\mathbb{T}$.

Since $\mathcal{K}_u$ are closed subspaces of the Hardy space, they are Hilbert spaces with reproducing kernels, which we will now identify. The reproducing kernels for the $H^2(\mathbb{D})$, known as the Szegö kernels, are given by

$$S_\lambda(z) = \frac{1}{1 - \overline{\lambda} z},$$

for $\lambda \in \mathbb{D}$ and $z \in \mathbb{T}$. From here, one finds that the reproducing kernels K_λ^u for $\mathcal{K}_u$ are given by

$$K_\lambda^u(z) = \frac{1 - \overline{u(\lambda)} u(z)}{1 - \overline{\lambda} z},$$

for $\lambda \in \mathbb{D}$ and $z \in \mathbb{T}$.

Since $\mathcal{K}_u$ is a closed subspace of the Hilbert space H^2, there must be an orthogonal projection $P_u : H^2 \to \mathcal{K}_u$. This projection is an integral operator given by

$$P_u f(\lambda) = \langle f, K_\lambda^u \rangle = \int_{\mathbb{T}} f(z) \frac{1 - u(\lambda)\overline{u(z)}}{1 - \lambda \overline{z}} ds(z),$$

for $\lambda \in \mathbb{D}$. If u is an inner function and $\varphi \in L^\infty(\mathbb{T})$, we define the truncated Toeplitz operator $T_\varphi^u : \mathcal{K}_u \to \mathcal{K}_u$ by

$$T_\varphi^u f := P_u(\varphi f).$$

From the integral representation of P_u, we may write T_φ^u as an integral operator,

$$T_\varphi^u f(\lambda) = \int_{\mathbb{T}} f(z)\varphi(z) \frac{1 - u(\lambda)\overline{u(z)}}{1 - \lambda \overline{z}} ds(z),$$

for $\lambda \in \mathbb{D}$. For $\varphi \in L^\infty(\mathbb{T})$, let $\widetilde{\varphi}$ represent the Berezin transform of φ on K_u, given by

$$\widetilde{\varphi}_u(\lambda) := \langle T_\varphi^u k_\lambda^u, k_\lambda^u \rangle = \int_{\mathbb{T}} \varphi(e^{i\theta}) |k_\lambda^u(e^{i\theta})|^2 d\theta,$$

where $k_\lambda^u(z)$ represents the normalized kernel of $\mathcal{K}_u$.

We will concern ourselves with Model Spaces generated by the inner functions $u = z^n$. It is well known that the matrices of truncated Toeplitz operators with bounded, real valued symbols on $\mathcal{K}_{z^n}$ are simply $n \times n$ Toeplitz matrices of the form

$$\begin{bmatrix} \widehat{\varphi_0} & \overline{\widehat{\varphi_1}} & \cdots & \overline{\widehat{\varphi_n}} \\ \widehat{\varphi_1} & \widehat{\varphi_0} & \cdots & \overline{\widehat{\varphi_{n-1}}} \\ \vdots & \vdots & \ddots & \vdots \\ \widehat{\varphi_n} & \widehat{\varphi_{n-1}} & \cdots & \widehat{\varphi_0} \end{bmatrix},$$

where

$$\widehat{\varphi_n} = \int_0^{2\pi} \varphi(e^{i\theta})e^{-in\theta}d\theta,$$

is the n^{th} Fourier coefficient of φ. If $\varphi \in L^\infty(\mathbb{D})$, we say $T_\varphi^{z^n}$ is a **positive operator** if it's associated matrix T_φ^n is positive definite, or, in other words, if the determinant of every principal minor is positive. For more introductory information on Model spaces, see [4].

3.2 Preliminaries

In the recent years, the Berezin transform has been a useful tool for characterizing certain properties of Toeplitz operators. Here, we give a brief overview of some major previous results concerning the characterization of the positivity of Toeplitz operators in terms of the Berezin transforms of their symbols.

Recently, Zhao and Zheng [10] proved that the positivity of a Toeplitz operator on the Bergman space is not completely determined by the positivity of the Berezin transform of its symbol. In fact, they showed that even if the Berezin transform of a quadratic polynomial in $|z|$ on the unit disk is bounded from below by a positive number, the Toeplitz operator associated with that symbol may not be positive. We start with their result in the positive direction.

Theorem 7. *[10] Let $\varphi = |z|^2 + a|z| + b$, where a, b are real. Suppose $a \in \mathbb{R}\backslash(-2, -\frac{5}{2})$, then T_φ is positive if and only if $\widetilde{\varphi}(z)$ is a nonnegative function on $\mathbb{D}$.*

On the other hand, they gave a counterexample to show that the positivity of the Berezin transform of such a symbol is not sufficient to prove the positivity of its associated Toeplitz operator.

Theorem 8. *[10] Let $\varphi(z) = |z|^2 + a|z| + b$, where a, b are real. For each $a \in$*

$(-\frac{14}{9}, -\frac{5}{4}) \subset (-2, -\frac{5}{4})$, *there exist* $b \in \mathbb{R}$ *and* $\delta > 0$ *such that*

$$\widetilde{\varphi}(z) \geq \delta,$$

for all $z \in \mathbb{D}$, *but* T_φ *is not positive.*

Motivated by this paper, we study the relationship between the positivity of truncated Toeplitz operators on finite dimensional model spaces with corresponding inner functions $u = z^n$ and the positivity of their Berezin transforms. We ask the following question:

3.3 Main Results

Question 1. *For the inner functions* z^n, $n \geq 2$, *if* $\varphi \in L^\infty(\mathbb{D})$ *and* $\widetilde{\varphi_{z^n}}(\lambda) \geq 0$ *for all* $\lambda \in \mathbb{D}$, *is* $T_\varphi^{z^n}$ *positive on* $\mathcal{K}_\lambda^{z^n}$?

We answer this question affirmatively in the case that φ is real valued and $n = 2$, and negatively in the case that φ is real valued and $n \geq 3$.

<u>Case 1 for $n = 2$</u>

The Fourier coeficients of φ with respect to orthonormal basis $\{e_0 = 1, e_1 = z\}$ are

$\hat{\varphi}_0 = \langle T_\varphi^{z^2} e_0, e_0 \rangle = \frac{1}{2\pi} \int\limits_0^{2\pi} \varphi(e^{i\theta}) d\theta$

$\hat{\varphi}_1 = \langle T_\varphi^{z^2} 1, z \rangle = \frac{1}{2\pi} \int\limits_0^{2\pi} \varphi(e^{i\theta}) e^{-i\theta} d\theta$

$\hat{\varphi}_{-1} = \langle T_\varphi^{z^2} z, 1 \rangle = \frac{1}{2\pi} \int\limits_0^{2\pi} \varphi(e^{i\theta}) e^{i\theta} d\theta$

On $\mathcal{K}_\lambda^{z^2}$, we compute the kernel and normalized reproducing kernel,

$$K_\lambda^{z^2}(z) = 1 + \bar{\lambda} z, \quad \text{for } \lambda \in \mathbb{D} \text{ and } z \in \mathbb{T}$$

and,

$$k_\lambda^{z^2}(z) = \frac{1+\bar{\lambda}z}{\sqrt{1+|\lambda|^2}}, \qquad \text{for } \lambda \in \mathbb{D} \text{ and } z \in \mathbb{T}.$$

The truncated Toeplitz operator on the model space $\mathcal{K}_{z^2}$ is,

$$T_\varphi^{z^2} = \begin{bmatrix} \hat{\varphi}_0 & \hat{\varphi}_{-1} \\ \hat{\varphi}_1 & \hat{\varphi}_0 \end{bmatrix} = \begin{bmatrix} \hat{\varphi}_0 & \overline{\hat{\varphi}_1} \\ \hat{\varphi}_1 & \hat{\varphi}_0 \end{bmatrix}.$$

Then $T_\varphi^{z^2}$ is positive iff $\hat{\varphi}_0 > 0$ and $\hat{\varphi}_0^2 - |\hat{\varphi}_1|^2 > 0$. (A)

Similarly the Berezin transform of φ is defined as: $\widetilde{\varphi}_{z^2}(\lambda) = \langle T_\varphi^{z^2} k_\lambda^{z^2}, k_\lambda^{z^2}\rangle$ for all $\lambda \in \mathbb{D}$.

Lemma 6. *For $\varphi \in L^\infty(\partial\mathbb{D})$, we show $\widetilde{\varphi}(\lambda) = \hat{\varphi}_0 + \frac{2Re(\lambda\hat{\varphi}_1)}{1+|\lambda|^2}$ for all $\lambda \in \mathbb{D}$.*

Proof.

$$\begin{aligned}
\widetilde{\varphi}_{z^2}(\lambda) &= \langle T_\varphi^{z^2} k_\lambda^{z^2}, k_\lambda^{z^2}\rangle \\
&= \frac{1}{2\pi}\int_0^{2\pi} \varphi(e^{i\theta}) k_\lambda^{z^2}(e^{i\theta})\overline{k_\lambda^{z^2}(e^{i\theta})}\,d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} \varphi(e^{i\theta}) \frac{1+\overline{\lambda}e^{i\theta}}{\sqrt{1+|\lambda|^2}}\frac{1+\lambda e^{-i\theta}}{\sqrt{1+|\lambda|^2}}\,d\theta \\
&= \frac{1}{2\pi(1+|\lambda|^2)}\int_0^{2\pi} \varphi(e^{i\theta})(1+|\lambda|^2+\overline{\lambda}e^{i\theta}+\lambda e^{-i\theta}) \\
&= \hat{\varphi}_0 + \frac{\overline{\lambda}\hat{\varphi}_{-1}+\lambda\hat{\varphi}_1}{1+|\lambda|^2} \\
&= \hat{\varphi}_0 + \frac{\overline{\lambda\hat{\varphi}_1}+\lambda\hat{\varphi}_1}{1+|\lambda|^2} \\
&= \hat{\varphi}_0 + \frac{2Re(\lambda\hat{\varphi}_1)}{1+|\lambda|^2}.
\end{aligned}$$

□

Now we show positivity of the Berezin transform of the real valued symbol implies the positivity of truncated Toeplitz operator corresponding to inner function z^2.

Theorem 9. *Let $\varphi \in L^\infty(\mathbb{T})$ be a real valued function. Then $\widetilde{\varphi}_{z^2} > 0$ on $\mathbb{D}$ if and only if $T_\varphi^{z^2}$ is positive on $\mathcal{K}_{z^2}$.*

Proof. From Lemma 6, we know that

$$\widetilde{\varphi}_{z^2}(\lambda) = \widehat{\varphi}_0 + \frac{2Re(\lambda\widehat{\varphi}_1)}{1+|\lambda|^2} \text{ for all } \lambda \ \in \mathbb{D}$$

Thus,

$$\begin{aligned}
\widetilde{\varphi}_{z^2}(\lambda) \geq 0 \iff& \widehat{\varphi}_0 + \frac{2Re(\lambda\widehat{\varphi}_1)}{1+|\lambda|^2} \geq 0 \\
\iff& \widehat{\varphi}_0 \geq -\frac{2Re(\lambda\widehat{\varphi}_1)}{1+|\lambda|^2} \\
\iff& \widehat{\varphi}_0 \geq \sup_{\lambda\in\mathbb{D}} \frac{-2Re(\lambda\widehat{\varphi}_1)}{1+|\lambda|^2} \\
\text{choose } \lambda = -\overline{\widehat{\varphi}}_1 x& \text{ for } 0 \leq x \leq \frac{1}{|\hat{\varphi}_1|} \\
\iff& \widehat{\varphi}_0 \geq \sup_{0\leq x\leq \frac{1}{|\widehat{\varphi}_1|}} \frac{2|\widehat{\varphi}_1|^2 x}{1+|\widehat{\varphi}_1|^2x^2} \qquad (3.1)
\end{aligned}$$

Let

$$f(x) = \frac{|\widehat{\varphi}_1|^2 x}{1+|\widehat{\varphi}_1|^2x^2}.$$

Then

$$f'(x) = \frac{|\widehat{\varphi}_1|^2 - x^2|\widehat{\varphi}_1|^4}{(1+|\widehat{\varphi}_1|^2x^2)^2},$$

which shows us that f has a critical point at $x = \frac{1}{|\widehat{\varphi}_1|}$. Continuing,

$$f''(x) = \frac{-2x|\widehat{\varphi}_1|^4(1+|\widehat{\varphi}_1|^2x^2)^2 - 4x|\widehat{\varphi_1}|^2(1+|\widehat{\varphi_1}|^2x^2)(|\widehat{\varphi_1}|^2 - x^2|\widehat{\varphi_1}|^2)}{(1+|\widehat{\varphi}_1|^2x^2)^4}.$$

Note that $f''(\frac{1}{|\widehat{\varphi}_1|}) < 0$. At this value of λ, (3.1) becomes

$$\widehat{\varphi}_0 \geq |\widehat{\varphi_1}|.$$

Squaring both sides, we have

$$\widehat{\varphi}_0^2 \geq |\widehat{\varphi}_1|^2$$

By (A), $T_\varphi^{z^2}$ is positive. $\square$

<u>Case 2 for $n \geq 3$</u>

Next, we prove that the positivity of the Berezin transform of real valued symbol is not enough to prove the positivity of the truncated Toeplitz operator corresponding to inner function $n \geq 3$.

Theorem 10. *Let φ be the real valued, bounded function given by $\varphi(\theta) = 1.1 + 3\cos(2\theta)$ for $\theta \in [0, 2\pi)$. Then $\widetilde{\varphi}_{z^3}(\lambda) > 0$ for all $\lambda \in \mathbb{D}$, but $T_\varphi^{z^3}$ is not positive on $\mathcal{K}_{z^3}$.*

Proof. One can compute, $K_\lambda(z) = 1 + \overline{\lambda}z + \overline{\lambda}^2 z^2$ and normalized kernel: $k_\lambda(z) = \frac{1+\overline{\lambda}z+\overline{\lambda}^2z^2}{\sqrt{1+|\lambda|^2+|\lambda|^4}}$ for all $\lambda \in \mathbb{D}$. For $\varphi \in L^\infty(\partial\mathbb{D})$, one can compute,

$$\begin{aligned}
\widetilde{\varphi}_{z^3}(\lambda) &= \frac{1}{2\pi}\int_0^{2\pi} \varphi(e^{i\theta})|k_\lambda(e^{i\theta})|^2 d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} \varphi(e^{i\theta})\frac{|1+\overline{\lambda}e^{i\theta}+\overline{\lambda}^2e^{2i\theta}|^2}{1+|\lambda|^2+|\lambda|^4}d\theta \\
&= \frac{1}{2\pi(1+|\lambda|^2+|\lambda|^4)}\int_0^{2\pi} \varphi(e^{i\theta})(1+|\lambda|^2+|\lambda|^4+(\lambda+\overline{\lambda}\lambda^2)e^{-i\theta}+(\overline{\lambda}+\overline{\lambda}^2\lambda)e^{i\theta} \\
&\quad + \lambda^2e^{-2i\theta}+\overline{\lambda}^2e^{2i\theta})d\theta \\
&= \hat{\varphi}_0 + \frac{\lambda+\overline{\lambda}\lambda^2}{1+|\lambda|^2+|\lambda|^4}\hat{\varphi}_1 + \frac{\overline{\lambda}+\overline{\lambda}^2\lambda}{1+|\lambda|^2+|\lambda|^4}\overline{\hat{\varphi}_1} + \frac{\lambda^2}{1+|\lambda|^2+|\lambda|^4}\hat{\varphi}_2 + \frac{\overline{\lambda}^2}{1+|\lambda|^2+|\lambda|^4}\overline{\hat{\varphi}_2} \\
&= \hat{\varphi}_0 + \frac{2}{1+|\lambda|^2+|\lambda|^4}Re\{(\lambda+\overline{\lambda}\lambda^2)\hat{\varphi}_1\} + \frac{2}{1+|\lambda|^2+|\lambda|^4}Re\{(\lambda^2)\hat{\varphi}_2\}
\end{aligned}$$

$$= \hat{\varphi}_0 + \frac{2(1+|\lambda|^2)}{1+|\lambda|^2+|\lambda|^4} Re(\lambda\hat{\varphi}_1) + \frac{2}{1+|\lambda|^2+|\lambda|^4} Re\{(\lambda^2)\hat{\varphi}_2\}$$

Since $\varphi(\theta) = 1.1 + 3\cos 2(\theta) = 1.1 + 3\frac{e^{2i\theta}+e^{-2i\theta}}{2}$,

so, the calculations yield,

$$\widehat{\varphi}_0 = 1.1$$

$$\widehat{\varphi}_1 = 0$$

$$\widehat{\varphi}_2 = \frac{3}{2}.$$

$\square$

Hence,

$$\begin{aligned}
\widetilde{\varphi}_{z^3}(\lambda) &= 1.1 + \frac{2}{1+|\lambda|^2+|\lambda|^4} Re\{\frac{3}{2}(\lambda^2)\} \\
&\geq 1.1 + \inf_{\lambda \in \mathbb{D}} \frac{1}{1+|\lambda|^2+|\lambda|^4} Re\{(3\lambda^2) \\
&= 1.1 + 3(-\frac{1}{3}) \\
&= .1 > 0
\end{aligned}$$

However,

$$det \begin{bmatrix} \widehat{\varphi}_0 & \widehat{\varphi}_{-1} & \widehat{\varphi}_{-2} \\ \widehat{\varphi}_1 & \widehat{\varphi}_0 & \widehat{\varphi}_{-1} \\ \widehat{\varphi}_2 & \widehat{\varphi}_1 & \widehat{\varphi}_0 \end{bmatrix} = det \begin{bmatrix} 1.1 & 0 & \frac{3}{2} \\ 0 & 1.1 & 0 \\ \frac{3}{2} & 0 & 1.1 \end{bmatrix} = 1.331 - \frac{9.9}{4} = \frac{-4.576}{4} < 0.$$

We have provided an example of a bounded, real-valued symbol φ satisfying $\widetilde{\varphi}_{z^3} > 0$ on $\mathbb{D}$ whose associate truncated Toeplitz operator $T_\varphi^{z^3}$ is not positive on $\mathcal{K}_{z^3}$. One can use the same φ to prove $\widetilde{\varphi}_{z^n} > 0$ on $\mathbb{D}$ whose corresponding truncated Toeplitz operator $T_\varphi^{z^n}$ is not positive on $\mathcal{K}_{z^n}$ for all $n \geq 4$.

Next, we observe that the positivity of a truncated Toeplitz operator does not guarantee the positivity of its symbol.

Theorem 11. *Let $\varphi = \sin\theta + a$, for $a \in \mathbb{R}$. We show for certain value of $a \in \mathbb{R}$, $T_\varphi^{z^2}$ is positive but φ is negative.*

Proof. We first compute Fourier coefficients of $T_\varphi^{z^2}$:

$$\widehat{\varphi}_0 = \frac{1}{2\pi}\int_0^{2\pi}(\sin\theta + a)d\theta = a$$

$$\widehat{\varphi}_1 = \frac{1}{2\pi}\int_0^{2\pi}(\sin\theta + a)e^{-i\theta}d\theta = \frac{1}{2\pi}\int_0^{2\pi}(\sin\theta e^{-i\theta} + ae^{-i\theta})d\theta = -\frac{i}{2}$$

We know that the operator $T_\varphi^{z^2}$ is positive if and only if $\widehat{\varphi}_0 \geq 0$ and $\widehat{\varphi}_0^2 - |\widehat{\varphi}_1|^2 \geq 0$. From the above, we have

$$\widehat{\varphi}_0 = a$$

$$\widehat{\varphi}_0^2 - |\widehat{\varphi}_1|^2 = a^2 - \frac{1}{4}.$$

Thus, $T_\varphi^{z^2}$ is positive if and only if $a \geq \frac{1}{2}$. However, if we let $a = \frac{3}{4}$, we still while $T_\varphi^{z^2}$ is positive, the symbol $\varphi(e^{i\theta}) = \sin\theta + \frac{3}{4}$ is negative for certain values of θ. $\square$

Chapter 4

Positivity of Toeplitz operator via Berezin transform on the Fock space.

4.1 Introduction

For fixed α, consider the Gaussian measure, $d\lambda_\alpha = \frac{\alpha}{\pi} e^{-\alpha|z|^2} dA(z)$ on $\mathbb{C}$ so that $\int_{\mathbb{C}} d\lambda_\alpha = 1$, where $dA(z) = dxdy$ is ordinary area measure. We define **Fock space**, denoted by $F^2(\mathbb{C})$, is a set of all entire functions f with the property that the function $f(z)e^{-\frac{\alpha}{2}|z|^2}$ is in $L^2(\mathbb{C}, d\lambda)$. $F^2(\mathbb{C})$ is closed subspace of $L^2(\mathbb{C})$ with norm

$$||f||_2 = \left(\frac{\alpha}{\pi} \int_{\mathbb{C}} |f(z)e^{-\frac{\alpha}{2}|z|^2}|^2 dA \right)^{\frac{1}{2}}$$

and inner product on $F^2(\mathbb{C})$ is defined as,

$$\langle f, g \rangle = \int_{\mathbb{C}} f(z)\overline{g(z)} d\lambda_\alpha(z) \text{ for all } f, g \in F^2(\mathbb{C}).$$

We study positivity on the Fock space corresponding to $\alpha = 1$.

Here $K_z(w) = e^{\bar{z}w}$ is a reproducing kernel of $F^2(\mathbb{C})$ and $k_z(w) = \frac{K_z(w)}{||K_z||} = e^{\bar{z}w - \frac{|z|^2}{2}}$ is a normalized kernel. For $\varphi(z) = \varphi(|z|)$ we define the Berezin transform as $\widetilde{\varphi}(z) = \langle T_\varphi k_z, k_z \rangle$ that maps $\mathbb{C} \to \mathbb{R}$.

Let $P : L^2(\mathbb{C}) \to F^2(\mathbb{C})$ is the orthogonal projection onto the Fock space. For a radial symbol $\varphi(z) = \varphi(|z|)$ on $\mathbb{C}$, we define the Toeplitz operator T_φ as

$$T_\varphi f(z) = P(\varphi f)(z) = \int_{\mathbb{C}} \varphi(w) f(w) \overline{k_z(w)} d\lambda(w)$$

for all holomorphic polynomials $f \in F^2(\mathbb{C})$.

We say T_φ is positive on $F^2(\mathbb{C})$ iff $\langle T_\varphi f, f \rangle \geq 0$ for all $f \in F^2(\mathbb{C})$. To show the positivity of T_φ, it suffices to prove $\langle T_\varphi e_n, e_n \rangle \geq 0$ for all $n \geq 0$ where $e_n(z) = \frac{1}{\sqrt{n!}} z^n$ is the orthonormal basis for $F^2(\mathbb{C})$.

4.2 Lemmas and Theorems

In this section, we study the relation between the Berezin transforms of the monomial symbols and the positivity of the Toeplitz operators with that symbols. First, we will show that the positivity of monomial symbol to any integer power is enough to prove the positivity of the Toeplitz operator. Second, we will show the positivity of the Toeplitz operator is not enough to prove the positivity of the associated symbol.

Lemma 7. *For any symbol $\varphi(z)$, show that,*

$$\widetilde{\varphi}(z) = \frac{1}{\pi} \int_{\mathbb{C}} \varphi(w) e^{-|z-w|^2} dA(w).$$

Proof. We have,

$$\widetilde{\varphi}(z) = \langle T_\varphi k_z, k_z \rangle$$

$$
\begin{aligned}
&= \int_{\mathbb{C}} \varphi(w) k_z(w) \overline{k_z(w)} d\lambda(w) && (4.1)\\
&= \frac{1}{\pi} \int_{\mathbb{C}} \varphi(w) \frac{K_z(w)}{||K_z||} \frac{K_w(z)}{||K_z||} e^{-|w|^2} dA(w)\\
&= \frac{1}{\pi} \int_{\mathbb{C}} \varphi(w) \frac{K_z(w)}{||K_z||} \frac{K_w(z)}{||K_z||} e^{-|w|^2} dA(w)\\
&= \frac{1}{\pi} \int_{\mathbb{C}} \varphi(w) \frac{e^{\overline{z}w}}{e^{\frac{|z|^2}{2}}} \frac{e^{\overline{w}z}}{e^{\frac{|z|^2}{2}}} e^{-|w|^2} dA(w)\\
&= \frac{1}{\pi} \int_{\mathbb{C}} \varphi(w) e^{\overline{z}w-|z|^2+z\overline{w}-|w|^2} dA(w)\\
&= \frac{1}{\pi} \int_{\mathbb{C}} \varphi(w) e^{-(z-w)(\overline{z}-\overline{w})} dA(w)\\
&= \frac{1}{\pi} \int_{\mathbb{C}} \varphi(w) e^{-|z-w|^2} dA(w). && (4.2)
\end{aligned}
$$

□

We will show that limit of the Berezin transform at the infinity of a monomial symbol is infinity.

Lemma 8. *For symbol $\varphi(z) = |z|$, show that,*

$$\lim_{z\to\infty} \widetilde{|z|} = \lim_{z\to\infty} \int_{\mathbb{C}} |w| e^{-|z-w|^2} dA(w) = \infty.$$

Proof. Since,

$$\int_{\mathbb{C}} |w| e^{-|z-w|^2} dA(w) \geq \int_{\{w:|z-w|\leq 1\}} |w| e^{-|z-w|^2} dA(w)$$

and,

$$|z-w| \geq |z| - |w|$$

$$|w| \geq |z| - |z-w|$$

$$|w| \geq |z| - 1$$

also,

$$|z-w| \leq 1$$

$$|z-w|^2 \leq 1$$

so,

$$e^{-|z-w|^2} \geq e^{-1}.$$

Therefore,

$$\begin{aligned}\int_{\mathbb{C}} |w| e^{-|z-w|^2} dA(w) \geq & \int_{\{w:|z-w|\leq 1\}} (|z|-1) e^{-1} dA(w) \\ =& (|z|-1) \int_{\{w:|z-w|\leq 1\}} e^{-1} dA(w) \\ =& (|z|-1) e^{-1} \pi\end{aligned}$$

Hence,

$$\lim_{z\to\infty} \int_{\mathbb{C}} |w| e^{-|z-w|^2} dA(w) = \infty$$

i.e

$$\lim_{z\to\infty} \widetilde{\varphi}(z) = \lim_{z\to\infty} \widetilde{|z|} = \infty$$

□

The following theorem shows that if a symbol of the Toeplitz operator is harmonic with certain property then the symbol and its Berezin transform are same.

Theorem 12. *Suppose φ is harmonic function on $\mathbb{C}$ satisfying $\varphi \circ t_a \in L^1(\mathbb{C}, d\lambda), a \in \mathbb{C}$, where $t_a(z) = z + a$. Then $\widetilde{\varphi} = \varphi$. Consequently, $\widetilde{\varphi}(z) \geq 0 \iff T_\varphi \geq 0$*

Proof. For fixed z, t_z is holomorphic so $\varphi \circ t_z$ is harmonic. By the mean-value theorem,

$$\begin{aligned}
&\varphi \circ t_z(0) = \int_{\mathbb{C}} \varphi \circ t_z(w) d\lambda(w)\\
&\varphi(0+z) = \int_{\mathbb{C}} \varphi(w+z)\frac{1}{\pi}e^{-|w|^2}dA(w)\\
&\varphi(z) = \frac{1}{\pi}\int_{\mathbb{C}} \varphi(w+z)e^{-|w|^2}dA(w)\\
&let \;\; w+z=\zeta \;\; so \;\; that \;\; w=\zeta - z\\
&\varphi(z) = \frac{1}{\pi}\int_{\mathbb{C}} \varphi(\zeta)e^{-|z-\zeta|^2}dA(\zeta)\\
&\varphi(z) = \widetilde{\varphi}(z)
\end{aligned}$$

□

Next, we show the relation between positivity of Toeplitz operators and the Berezin transforms of monomial symbols.

Lemma 9. *For $\varphi(z) = |z|^m, m \in \mathbb{Z}_+$ prove that,*

$$\langle T_{|z|^m} e_n, e_n\rangle = \frac{1}{n!}\Gamma(n + \frac{m}{2} + 1)$$

Proof.

$$\begin{aligned}
\langle T_{|z|^m} e_n, e_n\rangle &= \int_{\mathbb{C}} \varphi(w)|e_n(w)|^2 d\lambda\\
&= \int_{\mathbb{C}} |w|^m \frac{1}{n!}|w|^{2n}\frac{1}{\pi}e^{-|w|^2}dA(w)\\
&= \frac{1}{\pi n!}\int_0^\infty \int_0^{2\pi} r^{m+2n}e^{-r^2} r dr d\theta\\
&= \frac{1}{\pi n!}\int_0^\infty \int_0^{2\pi} r^{m+2n+1}e^{-r^2} dr d\theta\\
&= \frac{1}{n!}\int_0^\infty r^{\frac{m}{2}+n}e^{-r}dr \quad (substitution)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{n!}\int_0^\infty r^{\frac{m}{2}+n+1-1}e^{-r}dr \qquad (4.3)\\
&= \frac{1}{n!}\Gamma(\frac{m}{2}+n+1).
\end{aligned}
$$

□

Next, we compute the Berezin transforms of the monomial symbols.

Lemma 10. *For the symbol $\varphi(z) = |z|^m$, show that $\widetilde{\varphi}(z) = e^{-R^2}\sum_{k=0}^{\infty}\frac{R^{2k}}{(k!)^2}\Gamma(k+\frac{m}{2}+1)$ where $R \in \mathbb{R}$ and $z = Re^{i\alpha}$.*

Proof.

$$
\begin{aligned}
\widetilde{\varphi}(z) =& \frac{1}{||K_z||^2}\langle T_\varphi K_z, K_z\rangle \\
=& \frac{1}{||K_z||^2}\left\langle \sum_{k=0}^{\infty}\langle T_\varphi K_z, e_k\rangle e_k, \sum_{k=0}^{\infty}\langle K_z, e_k\rangle e_k\right\rangle \\
=& \frac{1}{||K_z||^2}\sum_{k=0}^{\infty}\langle T_\varphi K_z, e_k\rangle\overline{\langle K_z, e_k\rangle} \\
=& \frac{1}{||K_z||^2}\sum_{k=0}^{\infty}\langle K_z, T_\varphi^* e_k\rangle\langle e_k, K_z\rangle \\
=& \frac{1}{||K_z||^2}\sum_{k=0}^{\infty}\langle K_z, \overline{\lambda_k}e_k\rangle\langle e_k, K_z\rangle \text{ where } \lambda_k = \langle T_{|z|^m}e_k, e_k\rangle \\
=& \frac{1}{||K_z||^2}\sum_{k=0}^{\infty}\lambda_k\langle K_z, e_k\rangle e_k(z) \\
=& \frac{1}{||K_z||^2}\sum_{k=0}^{\infty}\lambda_k\overline{\langle e_k, K_z\rangle}e_k(z) \\
=& \frac{1}{||K_z||^2}\sum_{k=0}^{\infty}\lambda_k\overline{e_k(z)}e_k(z) \\
=& \frac{1}{||K_z||^2}\sum_{k=0}^{\infty}\lambda_k|e_k(z)|^2
\end{aligned}
$$

we use by lemma 9 for λ_k, and we get,

$$=\frac{1}{e^{|z|^2}}\sum_{k=0}^{\infty}\frac{1}{k!}\Gamma(k+\frac{m}{2}+1)\frac{1}{k!}|z|^{2k} \tag{4.4}$$

for $z = Re^{i\alpha}$ we get,

$$=\frac{1}{e^{R^2}}\sum_{k=0}^{\infty}\frac{R^{2k}}{(k!)^2}\Gamma(k+\frac{m}{2}+1)$$

□

Now, we prove the positivity of the Berezin transform of the monomial radial symbol implies the positivity of the Toeplitz operator associated with that symbol.

Theorem 13. *For the symbols $\varphi(z) = |z|^m + a,\ m \in \mathbb{Z}_+, a \in \mathbb{R}$, show that $\widetilde{\varphi}(z) \geq 0 \iff T_\varphi \geq 0$.*

Proof. Assume that $\widetilde{\varphi}(z) \geq 0$. From Lemma 10 we get,

$$\widetilde{\varphi}(z) = e^{-R^2}\sum_{k=0}^{\infty}\frac{R^{2k}}{(k!)^2}\Gamma(k+\frac{m}{2}+1)+a, \quad \text{where } z = Re^{i\alpha}.$$

The Berezin transform at the origin,

$\widetilde{\varphi}(0) = \Gamma(\frac{m}{2}+1) + a \geq 0$

By Lemma 9 we get,

$$\begin{aligned}
\langle T_{|z|^m+a}e_n, e_n\rangle = \ & \frac{1}{n!}\Gamma(n+\frac{m}{2}+1)+a \ \ \forall n \geq 0 \\
\iff & \frac{1}{n!}\Gamma(\frac{m}{2}+1+n)+a \ \ \forall n \geq 0 \\
\iff & \frac{1}{n!}\Big\{(\frac{m}{2}+1+n-1)(\frac{m}{2}+1+n-2)(\frac{m}{2}+1+n-3)...(\frac{m}{2}+1+2) \\
& (\frac{m}{2}+1+1)(\frac{m}{2}+1)\Gamma(\frac{m}{2}+1)\Big\}+a
\end{aligned}$$

$$\begin{aligned}
&\geq \frac{1}{n!}\{n(n-1)(n-2)(n-3)...3.2.1\Gamma(\frac{m}{2}+1)\}+a \\
&= \Gamma(\frac{m}{2}+1)+a \\
&= \widetilde{\varphi}(0) \geq 0
\end{aligned}$$

Thus, $\widetilde{\varphi}(z) \geq 0 \implies \langle T_\varphi e_n, e_n\rangle \geq 0$ for all $n \geq 0 \implies T_\varphi \geq 0$. □

Another way to prove Theorem 13 is to show $\frac{1}{n!}\Gamma(n+\frac{m}{2}+1)$ is increasing in n for fixed m. For that we prove,

$$\frac{(n+1)^{th} term}{n^{th} term} > 1$$

Here, n^{th} term= $\frac{(n+\frac{m}{2})!}{n!}$ and $(n+1)^{th}$ term $= \frac{(n+\frac{m}{2}+1)!}{(n+1)!}$

So that,

$$\frac{(n+1)^{th} term}{n^{th} term} = \frac{(n+\frac{m}{2}+1)!}{(n+1)!}.\frac{n!}{(n+\frac{m}{2})!} = \frac{n+\frac{m}{2}+1}{n+1} = 1+\frac{\frac{m}{2}}{n+1} > 1 \ \forall\ n.$$

Since $\langle T_{|z|^m+a} e_0, e_0\rangle = \widetilde{\varphi}(0) \geq 0$ is a first term of $\langle T_{|z|^m+a} e_n, e_n\rangle$ and we showed $\langle T_{|z|^m+a} e_n, e_n\rangle$ is increasing in n so that $\langle T_{|z|^m+a} e_n, e_n\rangle \geq 0$.

Now, we prove that positivity of the Toeplitz operators on the Fock space is not enough to prove the positivity of associated symbols.

Theorem 14. *For symbol, $\varphi(z) = |z| - a, a \in \mathbb{R}$, the following are equivalent:*

(a) $T_\varphi \geq 0$ on $F^2(\mathbb{C})$

(b) $a \leq \Gamma(\frac{3}{2})$

(c) $\widetilde{\varphi}(z) \geq 0$ on $\mathbb{C}$

Proof. (a) $\iff$ (b)

By the Lemma 9 we have,

$\langle T_\varphi e_n, e_n\rangle = \frac{1}{n!}\Gamma(n+\frac{3}{2}) - a$

$$\begin{aligned} T_\varphi \geq 0 &\iff \frac{1}{n!}\Gamma(n+\frac{3}{2}) - a \geq 0 \ \ \forall n \geq 0 \\ &\iff a \leq \frac{1}{n!}\Gamma(n+\frac{3}{2}) \ \ \forall n \geq 0 \\ &\iff a \leq \Gamma(\frac{3}{2}) \text{ (It is true because } \frac{1}{n!}\Gamma(n+\frac{3}{2}) \text{ increasing in } n). \end{aligned}$$

(a) $\Longrightarrow$ (c) is obvious.

(c) $\Longrightarrow$ (b)

Let $\widetilde{\varphi}(z) \geq 0$ for all $z \in \mathbb{C}$

so, $\widetilde{\varphi}(0) \geq 0$

by Lemma 10 we get,

$\Gamma(\frac{3}{2}) - a \geq 0$

Hence, $a \leq \Gamma(\frac{3}{2})$.

Thus, for $0 < a \leq \Gamma(\frac{3}{2})$, we have $T_{\varphi(z)} \geq 0$ but $\varphi(0) < 0$. $\square$

4.3 Appendix

maximize[{4k^2+2m+4k(m+2)}/{(k+m)(k+m+3)}, m>0, k>=1}]

An attempt was made to fix mismatched parentheses, brackets, or braces.

Input interpretation:

maximize	function	$\frac{4k^2+2m+4k(m+2)}{(k+m)(k+m+3)}$
	domain	$m>0 \wedge k \geq 1$

Open code

$e_1 \wedge e_2 \wedge \ldots$ is the logical AND function

Global maximum:

$$\max\left\{\frac{4k^2+2m+4k(m+2)}{(k+m)(k+m+3)} \,\middle|\, m>0 \wedge k\geq 1\right\} = \begin{cases} \frac{4(k+2)}{k+3} & k \geq 1 \\ -\infty & \text{(otherwise)} \end{cases}$$

at m = indeterminate

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