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Tangent and Cotangent Bundles, Automorphism Groups and Representations of Lie Groups

by

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An Abstract of

Tangent and Cotangent Bundles, Automorphism Groups and Representations of Lie Groups

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We study the tangent TG and cotangent bundles T^*G of a Lie group G which are also Lie groups. Our main results are to show that on TG the canonical Jacobi endomorphism field S is parallel with respect to the canonical Lie group connection Lie group and that dually on the cotangent bundle of G the canonical symplectic form is parallel with respect to the canonical connection.

We next prove some theorems for Lie algebra extensions in which we can obtain a group representation for the extended algebra from the representation of the lower dimensional algebra. We also determine the Lie algebra of the automorphism group of three well known Lie algebras.

Finally we study the Hamilton-Jacobi separability of conformally flat metrics and find a metric, Lagrangian and geodesics for the solvable codimension one nilradical six dimensional Lie Algebras where one exists.

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Chapter 1

TG and T*G as Lie groups

In this chapter we bring together the differential geometric structure of the tangent and cotangent bundles of a smooth manifold and the canonical symmetric connection of any Lie group. More precisely let G be a finite dimensional Lie group and let TG and T^*G denote the tangent and cotangent bundles of G. Then as we explain in Sections 1.4 and 1.5 the bundles TG and T^*G are themselves Lie groups. In Section 1.1 we review the geometric structure enjoyed by tangent and cotangent bundles in general. Section 1.3 reviews the definition and some of the main properties of the canonical symmetric connection of any Lie group. Our main results in sections 1.4 and 1.6 are to show that on TG the canonical Jacobi endomorphism field is parallel with respect to canonical connection and that dually on T^*G the canonical symplectic form is parallel with respect to the canonical connection. Thus T^*G carries a canonical symplectic connection.

It is true that there is some danger of confusion with regard to our lifting formulas for TM and T^*M. However, we have designed this chapter in such a way that we never simultaneously discuss TM and T^*M or TG and T^*G. The summation convention on repeated indices applies throughout.

1.1 Vector fields on Manifolds

Suppose the M is an $m-$manifold and that φ_t is a 1-parameter group of diffeomorphisims of M, that is to say $\varphi : \mathbb{R} \times M \longrightarrow M$ such that $\varphi_s(\varphi_t(x)) = \varphi_{s+t}(x)$ and $\varphi_0(x) = x$ for all $x \in M$. There is a vector field associated to φ which we denote by X_φ and which is characterized by the property that

$$X_\varphi f(x) = \frac{d}{dt}\left(f \circ \varphi_t(x)\right)\Big|_{t=0} \tag{1.1}$$

for all smooth functions f on M. Now we recall that an integral curve $\gamma(t)$ of a vector field X on M is a curve on M such that for all smooth functions f on M with $\gamma(t_0) = x$, we have

$$Xf(x) = \frac{d}{dt}\left(f \circ \gamma(t)\right)\Big|_{t=0}. \tag{1.2}$$

Proposition 1.1.1 *For a fixed $x \in M$, the curves $t \mapsto \varphi_t(x)$ (orbits of x under the action of φ) are integral curves of X_φ.*

Proof: We note that

$$\begin{aligned}
\frac{d}{dt}\left(f\circ\varphi_t(x)\right)\Big|_{t=0} &= \frac{d}{d\tau}\left(f\circ\varphi_\tau\circ\varphi_{t_0}(x)\right)\Big|_{\tau=0} \qquad (\tau=t-t_0)\\
&= \frac{d}{d\tau}\left(f\circ\varphi_\tau(\varphi_{t_0}(x))\right)\Big|_{\tau=0}\\
&= X_\varphi f(\varphi_{t_0}(x)).
\end{aligned}$$

In view of (1.2) this last condition says that $t\mapsto\varphi_t(x)$ is an integral curve of X.

$\square$

Notice that if we think of the vector field X as being specified by an equivalence class of curves $\varphi_t(x)$, gives us a representative from each class at each $x\in M$. Hence we write $X(x)=[\varphi_t(x)]$. Also it is precisely the group property that makes Proposition 1.1.1 work.

Now let $\varphi: M\mapsto N$ be a map of smooth manifolds. We denote the induced bundle map from TM to TN by $T\varphi$. Although there is always such a vector bundle morphism, it does not in general induce a map of vector fields on M to vector fields on N; indeed this situation occurs only in special case where φ is a diffeomorphism. However, it is meaningful to speak of a vector field X on M being $\varphi-\textit{related}$ to a vector field Y on N. In this case

$$\varphi_*X = Y \tag{1.3}$$

at each point in M. We can rewrite this condition more conveniently as

$$X(f \circ \varphi) \quad = \quad (Yf) \circ \varphi \tag{1.4}$$

for all smooth $f : N \mapsto \mathbb{R}$. Here we note that both quantities in (1.4) are regarded as defining functions on M. Using this criterion of φ -relatedness of vector fields, we now show:

Proposition 1.1.2 *The Lie bracket of φ -related vector fields are φ -related.*

Proof: Let X_1, X_2 be φ -related to Y_1, Y_2 respectively. Let $f : N \mapsto \mathbb{R}$. Then

$$X_2(f \circ \varphi) = (Y_2 f) \circ \varphi$$

since X_2, Y_2 are φ -related. But $Y_1 f$ is a function on N, hence

$$X_2(Y_1 f \circ \varphi) = Y_2(Y_1 f) \circ \varphi$$

$$\Rightarrow X_2(X_1(f \circ \varphi)) = Y_2(Y_1 f) \circ \varphi$$

since X_1, Y_1 are φ -related. Hence

$$X_1 X_2(f \circ \varphi) - X_2(X_1(f \circ \varphi)) = Y_1(Y_2 f) \circ \varphi - Y_2(Y_1 f) \circ \varphi$$

that is

$$[X_1, X_2](f \circ \varphi) = ([Y_1, Y_2] f) \circ \varphi$$

and so by (1.4), we have that $[X_1, X_2]$ and $[Y_1, Y_2]$ are φ -related.

$\square$

1.2 Vector Fields on a Lie Group

We specialize next to the case where $M = G$ is a Lie group. Of course the previous developments remains true, but now we have much more structure available. The key to understanding the structure of Lie groups is the following fact [13, 28]:

Theorem 1.2.1 *The following four sets are in one-to-one natural correspondence:*

1. *$T_I G$ the set of tangent vectors to G at the identity I*
2. *The set of left invariant vector fields on G*
3. *The set of right invariant vector fields on G*
4. *The one-parameter subgroups of G.*

Next we define the multiplication on TG and show that TG is a group. In this Section we denote an element in TG by a tilde ($\backsim$), for example $\widetilde{A} \in T_A G$. We also use the maps $L_A, R_A : G \to G$, left and right translation by an element A in the Lie group G, respectively. Those two maps induce another two on TG, namely,

$TL_A, TR_A : TG \to TG$,

Lemma 1.2.1 *The product $\widetilde{A}.\widetilde{B} := L_{A*}\widetilde{B} + R_{B*}\widetilde{A}$ on TG, makes TG a group.*

Proof: We note the following:

- TG is closed under this product: $L_{A*}\widetilde{B}, R_{B*}\widetilde{A} \in T_{AB}G$, and so does their sum.

- We denote $\widetilde{I} = 0_I$ for the identity element of TG, where I is the identity in G. Then

$$\begin{aligned}
\widetilde{A}.\widetilde{I} &= L_{A*}\widetilde{I} + R_{I*}\widetilde{A} \\
&= L_{A*}0_I + \widetilde{A} \\
&= \widetilde{A}.
\end{aligned}$$

  Similarly, we have $\widetilde{I}.\widetilde{A} = \widetilde{A}$.

- Next we show that the inverse element $\widetilde{A}^{-1} = -L_{A^{-1}*}R_{A^{-1}*}\widetilde{A}$. We assume that $\widetilde{B}$ is the inverse of $\widetilde{A}$ in TG, then we have

$$\begin{aligned}
0_I = \widetilde{A}.\widetilde{B} &= L_{A*}\widetilde{B} + R_{B*}\widetilde{A} \\
\Rightarrow L_{A*}\widetilde{B} &= -R_{B*}\widetilde{A} \quad and \quad A.B = I \\
Also \quad 0_I = \widetilde{B}.\widetilde{A} &= L_{B*}\widetilde{A} + R_{A*}\widetilde{B} \\
\Rightarrow L_{B*}\widetilde{A} &= -R_{A*}\widetilde{B} \quad and \quad B.A = I \\
\Rightarrow B &= A^{-1} \quad and \\
\widetilde{B} &= -L_{A^{-1}*}R_{A^{-1}*}\widetilde{A}.
\end{aligned}$$

- Finally we check associativity: we take $\widetilde{A}, \widetilde{B}, \widetilde{C} \in TG$ and we want to show that

$$\left(\widetilde{A}.\widetilde{B}\right).\widetilde{C} = \widetilde{A}.\left(\widetilde{B}.\widetilde{C}\right)$$

so we consider

$$\begin{aligned}
\left(\widetilde{A}.\widetilde{B}\right).\widetilde{C} &= \left(L_{A*}\widetilde{B}+R_{B*}\widetilde{A}\right).\widetilde{C} \\
&= L_{A.B*}\widetilde{C}+R_{C*}\left(L_{A*}\widetilde{B}+R_{B*}\widetilde{A}\right) \\
&= L_{B*}L_{A*}\widetilde{C}+R_{C*}L_{A*}\widetilde{B}+R_{C*}R_{B*}\widetilde{A} \\
&= L_{A*}\left(L_{B*}\widetilde{C}+R_{C*}\widetilde{B}\right)+R_{B.C*}\widetilde{A} \\
&= L_{A*}\left(\widetilde{B}.\widetilde{C}\right)+R_{B.C*}\widetilde{A} \\
&= \widetilde{A}.\left(\widetilde{B}.\widetilde{C}\right).
\end{aligned}$$

□

The following theorems are mentioned in [32] but we do not consider the account given there to be satisfactory because the proof is only given in terms of local coordinates.

Theorem 1.2.2 *If X is a left invariant vector field on G, then its vertical lift X^v is left invariant on TG.*

Proof: If X is left invariant vector field on G, then we have $X(A)=[\varphi_t(A)]$ for all $A \in G$. Also since X is left invariant on G, we have

$$[S.\varphi_t(A)] = [\varphi_t(S.A)]. \qquad (\forall S, A \in G) \tag{1.5}$$

But the equivalence class of curves, $\Phi_t(\widetilde{A})$ associated with X^v on TG is

$$\Phi_t(\widetilde{A}) = \widetilde{A} + t.X(A). \qquad \widetilde{A} \in T_AG \tag{1.6}$$

We note next that $\Phi_0(\widetilde{A}) = \widetilde{A}$ and

$$\begin{aligned}
\Phi_s(\Phi_t(\widetilde{A})) &= \Phi_s(\widetilde{A} + t.X(A)) \\
&= \widetilde{A} + t.X(A) + s.X(A) \\
&= \widetilde{A} + (t+s).X(A) \\
&= \Phi_{s+t}(\widetilde{A}),
\end{aligned}$$

that is to say, that $\Phi_t(\widetilde{A})$ is a 1-parameter group of diffeomorphisms on TG. We would like to show X^v is left invariant, that is, the equivalence class of curves $\left[\Phi_t(\widetilde{A})\right]$ associated with X^v, satisfies (1.5). So we pick any two elements $\widetilde{A}, \widetilde{B}$ in T_AG, T_BG respectively, and consider

$$\begin{aligned}
\widetilde{B}.\Phi_t(\widetilde{A}) &= \widetilde{B}.(\widetilde{A} + t.X(A)) \\
&= L_{B*}\left(\widetilde{A} + t.X(A)\right) + R_{A*}\widetilde{B} \\
&= L_{B*}\widetilde{A} + t.L_{B*}X(A) + R_{A*}\widetilde{B} \\
&= L_{B*}\widetilde{A} + R_{A*}\widetilde{B} + t.X(BA) \qquad (Since \quad X \quad is \quad left-invariant) \\
&= \widetilde{B}.\widetilde{A} + t.X(B.A) \\
&= \widetilde{B.A} + t.X(B.A) \\
&= \Phi_t(\widetilde{B}.\widetilde{A}).
\end{aligned}$$

By applying (1.5) on TG, we may conclude that X^v is left invariant on TG.

$\square$

Theorem 1.2.3 *If X is a left invariant vector field on G, then its complete lift X^c is left invariant on TG.*

Proof: We first note that if $\varphi : G \to G$ is a smooth map, so is $T\varphi : TG \to TG$, in particular $T\varphi_x : T_xG \to T_{\varphi(x)}G$ for each $x \in G$. To see how $T\varphi_x$ is defined, we take an element $\widetilde{A} \in TG$ and a smooth map $F : G \to \mathbb{R}$. Then we have

$$\left(T\varphi_x\widetilde{A}\right) F = \widetilde{A}(F \circ \varphi)(x)$$

Now we take X to be left invariant vector field on G, with equivalence classes $X(A) = [\varphi_t(A)]$ for $A \in G$ as before. Then its complete lift X^c on TG has the equivalence classes $X^c(\widetilde{A}) = [T\varphi_t(\widetilde{A})]$ for $\widetilde{A} \in T_AG$.

Now we want to show that $\widetilde{A}.T\varphi_t(\widetilde{B}) = T\varphi_t(\widetilde{A.B})$, so we take any smooth map $F : TG \to \mathbb{R}$, any $x \in G$, and consider

$$\begin{aligned} \left(\widetilde{A}.T\varphi_t(\widetilde{B})(x)\right) F &= \widetilde{A}.\left(\widetilde{B}.(F \circ \varphi_t)(x)\right) \\ &= \widetilde{A}.\widetilde{B}.\left(F \circ \varphi_t\right)(x) \\ &= \widetilde{AB}.\left(F \circ \varphi_t\right)(x) \\ &= T\varphi_t(\widetilde{A.B})(x)F. \end{aligned}$$

Since x and F were arbitrarily chosen we have $\widetilde{A}.T\varphi_t(\widetilde{B}) = T\varphi_t(\widetilde{A.B})$, and by (1.5) we obtain the result that X^c is left invariant.

$\square$

1.3 The Canonical Connection on a Lie Group

In this Section we shall outline the main properties of the canonical symmetric connection ∇ on a Lie group G [9, 10]. In fact ∇ is defined on left invariant vector fields X and Y by

$$\nabla_X Y = \frac{1}{2}\,[X, Y] \tag{1.7}$$

and then extended to arbitrary vector fields by making ∇ tensorial in the X argument and satisfy the Leibnitz rule in the Y argument. It can be shown that ∇ is symmetric and that the curvature tensor on left invariant vector fields is given by

$$R(X, Y)Z = \frac{1}{4}\,[Z, [X, Y]]. \tag{1.8}$$

Furthermore, G is a symmetric space in the sense that R is a parallel tensor field. It follows from (1.7) that ∇ is flat if and only if the Lie algebra $\mathfrak{g}$ of G is nilpotent of order two. The Ricci tensor R_{ij} of ∇ is symmetric and bi-invariant. In fact, if $\{X_i\}$ is a basis of left invariant vector fields then

$$[X_i, X_j] = C^k_{ij} X_k \tag{1.9}$$

where C^k_{ij} are the structure constants and relative to this basis the Ricci tensor R_{ij} is given by

$$R_{ij} = \frac{1}{4}\,C^l_{jm} C^m_{il} \tag{1.10}$$

from which the symmetry of R_{ij} becomes apparent. Indeed, R_{ij} is obtained by translating to the left or right one quarter of the Killing form. Since R^i_{jkl} is a parallel tensor field and R_{ij} is symmetric, it follows that Ricci gives rise to a quadratic Lagrangian which may, however, not be regular. In fact the Lagrangian is regular if and only if $\mathfrak{g}$ is semi-simple because of Cartan's criterion.

Since our starting point is the Lie algebra $\mathfrak{g}$ of a Lie group it is of interest to ask how the ideals of $\mathfrak{g}$ are related to ∇. To this end we shall quote the following result [17].

Proposition 1.3.1 *Let ∇ denote a symmetric connection on a smooth manifold M. Necessary and sufficient conditions that there exist a submersion from M to a quotient space Q such that ∇ is projectable to Q are that there exists an integrable distribution D on M that satisfies:*

(i) $\nabla_X Y$ belongs to D whenever Y belongs to D and X is arbitrary.

(ii) $R(Z,X)Y$ belongs to D whenever Z belongs to D and X and Y are arbitrary vector fields on M.

Corollary 1.3.2 *Every ideal $\mathbf{h}$ of $\mathfrak{g}$ gives rise to a quotient space Q consisting of the leaf space of the integrable distribution determined by $\mathbf{h}$ and ∇ on G projects to Q.*

1.4 The tangent bundle of a Lie group TG

On TG the left invariant vector fields are the vertical and complete lifts

$$X_1^v, X_2^v, ..., X_n^v, X_1^c, X_2^c, ..., X_n^c \tag{1.11}$$

of the left invariant vector fields $\{X_i\}$ on G as we proved in the previous section. They determine a Lie algebra $\widetilde{\mathfrak{g}}$. We describe it in terms of the brackets

$$\begin{aligned} \left[X_i^c, X_j^c\right] &= C_{ij}^k X_k^c \\ \left[X_i^v, X_j^v\right] &= 0 \\ \left[X_i^c, X_j^v\right] &= C_{ij}^k X_k^v. \end{aligned} \tag{1.12}$$

Since TG is a Lie group it has a canonical connection $\widetilde{\nabla}$. It is defined on vertical and complete lifts of a left invariant vector fields on G. So we have

$$\begin{aligned} \widetilde{\nabla}_{X_i^v} X_j^v &= \frac{1}{2}\left[X_i^v, X_j^v\right] = 0 \\ \widetilde{\nabla}_{X_i^v} X_j^c &= \frac{1}{2}\left[X_i^v, X_j^c\right] = \frac{1}{2}\left[X_i, X_j\right]^v = \left(\nabla_{X_i} X_j\right)^v \\ \widetilde{\nabla}_{X_i^c} X_j^v &= \frac{1}{2}\left[X_i^c, X_j^v\right] = \frac{1}{2}\left[X_i, X_j\right]^v = \left(\nabla_{X_i} X_j\right)^v \\ \widetilde{\nabla}_{X_i^c} X_j^c &= \frac{1}{2}\left[X_i^c, X_j^c\right] = \frac{1}{2}\left[X_i, X_j\right]^c = \left(\nabla_{X_i} X_j\right)^c. \end{aligned} \tag{1.13}$$

Next we find the nonzero components of the curvature tensor $\widetilde{R}$ on the vector fields

(1.11) of TG, using (1.8), and (1.12). We show here the six possible cases:

$$
\begin{aligned}
\widetilde{R}(X_i^v, X_j^v)X_k^v &= \frac{1}{4}\left[X_k^v, [X_i^v, X_j^v]\right] = 0 \\
\widetilde{R}(X_i^v, X_j^v)X_k^c &= \frac{1}{4}\left[X_k^c, [X_i^v, X_j^v]\right] = 0 \\
\widetilde{R}(X_i^c, X_j^v)X_k^v &= \frac{1}{4}\left[X_k^v, [X_i^c, X_j^v]\right] = [X_k^v, [X_i, X_j]^v] = 0 \\
\widetilde{R}(X_i^c, X_j^v)X_k^c &= \frac{1}{4}\left[X_k^c, [X_i^c, X_j^v]\right] = \frac{1}{4}[X_k^c, [X_i, X_j]^v] \\
&= \frac{1}{4}[X_k, [X_i, X_j]]^v = (R(X_i, X_j)X_k)^v \\
\widetilde{R}(X_i^c, X_j^c)X_k^v &= \frac{1}{4}\left[X_k^v, [X_i^c, X_j^c]\right] = \frac{1}{4}[X_k^v, [X_i, X_j]^c] \\
&= \frac{1}{4}[X_k, [X_i, X_j]]^v = (R(X_i, X_j)X_k)^v \\
\widetilde{R}(X_i^c, X_j^c)X_k^c &= \frac{1}{4}\left[X_k^c, [X_i^c, X_j^c]\right] = \frac{1}{4}[X_k^c, [X_i, X_j]^c] \\
&= \frac{1}{4}[X_k, [X_i, X_j]]^c = (R(X_i, X_j)X_k)^c .
\end{aligned} \tag{1.14}
$$

Remark 1.4.1 *The center of* $\widetilde{\mathfrak{g}}$ *has a basis consisting of vertical and complete lifts in the center of* $\mathfrak{g}$, *that is,* $Z\left(\widetilde{\mathfrak{g}}\right) = \langle Z(\mathfrak{g})^c \cup Z(\mathfrak{g})^v \rangle \equiv \widetilde{Z(\mathfrak{g})}$.

Proof: If $X_i \in Z(\mathfrak{g})$, then $[X_i, X_j] = C_{ij}^k X_k = 0$ for all $j = 1, 2, ..., n$. By (1.12) we have

$$
\begin{aligned}
\left[X_i^c, X_j^c\right] &= [X_i, X_j]^c = 0 \\
\left[X_i^c, X_j^v\right] &= [X_i, X_j]^v = 0 \\
\left[X_i^v, X_j^c\right] &= [X_i, X_j]^v = 0 \\
\left[X_i^v, X_j^v\right] &= 0.
\end{aligned}
$$

So $X_i^c, X_i^v \in Z(\widetilde{\mathfrak{g}})$.

On the other hand, if $X_i^c \in Z(\widetilde{\mathfrak{g}}) \Rightarrow \left[X_i^c, X_j^c\right] = \left[X_i^c, X_j^v\right] = 0$ for all j. But again by (1.12) we have that $[X_i, X_j] = 0$ for all j, that is $X_i \in Z(\mathfrak{g})$. Similarly if $X_i^v \in Z(\widetilde{\mathfrak{g}})$ then $X_i \in Z(\mathfrak{g})$ and hence we obtain the result.

$\square$

Remark 1.4.2 *The lower central series for $\mathfrak{g}$ is defined to be*

$$\begin{aligned} \mathfrak{g}_0 &= \mathfrak{g} \\ \mathfrak{g}_i &= [\mathfrak{g}, \mathfrak{g}_{i-1}] \qquad i = 1, 2, ... \end{aligned}$$

Then by (1.12) and the above definition the lower central series for $\widetilde{\mathfrak{g}}$ will be

$$\begin{aligned} \widetilde{\mathfrak{g}}_0 &= \widetilde{\mathfrak{g}} \\ \widetilde{\mathfrak{g}}_i &= [\widetilde{\mathfrak{g}}, \widetilde{\mathfrak{g}_{i-1}}] \qquad i = 1, 2, ... \end{aligned}$$

and $\widetilde{\mathfrak{g}}_i$ is generated by the vertical and complete lifts of $\mathfrak{g}_i$ to TG. The derived series for $\mathfrak{g}$ is defined to be

$$\begin{aligned} \mathfrak{g}^0 &= \mathfrak{g} \\ \mathfrak{g}^i &= [\mathfrak{g}^{i-1}, \mathfrak{g}^{i-1}] \qquad i = 1, 2, ... \end{aligned}$$

For the same reason we get

$$\begin{aligned} \widetilde{\mathfrak{g}}^0 &= \widetilde{\mathfrak{g}} \\ \widetilde{\mathfrak{g}}^i &= [\widetilde{\mathfrak{g}^{i-1}}, \widetilde{\mathfrak{g}^{i-1}}] \qquad i = 1, 2, ... \end{aligned}$$

and $\widetilde{\mathfrak{g}}^i$ *is generated by the vertical and complete lifts of* $\mathfrak{g}^i$ *to* TG*.*

Theorem 1.4.1 $\widetilde{\mathfrak{g}}$ *is decomposable if and only if* $\mathfrak{g}$ *is decomposable.*

Proof: If $\mathfrak{g}$ is decomposable then $\mathfrak{g}$ is isomorphic to $\mathfrak{g}_1 \oplus \mathfrak{g}_2$. But in that case $\widetilde{\mathfrak{g}}$ is isomorphic to $\widetilde{\mathfrak{g}}_1 \oplus \widetilde{\mathfrak{g}}_2$ and so $\widetilde{\mathfrak{g}}$ is decomposable.

For the converse note that a projection $\pi : \widetilde{\mathfrak{g}} \to \mathfrak{g}$ is defined on the basis of $\widetilde{\mathfrak{g}}$, and can be extended by linearity, as follows

$$\pi(X_i^v) = 0$$

$$\pi(X_i^c) = X_i.$$

Note that π is a Lie algebra homomorphism since

$$\begin{aligned} \pi([X_i^v, X_j^v]) &= 0 \\ &= [\pi(X_i^v), \pi(X_j^v)], \end{aligned}$$

and

$$\begin{aligned}\pi([X_i^c, X_j^v]) &= \pi([X_i, X_j]^v) \\ &= 0.\end{aligned}$$

But

$$\begin{aligned}[\pi(X_i^c), \pi(X_j^v)] &= [\pi(X_i^c), 0] \\ &= 0\end{aligned}$$

Hence $\pi([X_i^c, X_j^v]) = [\pi(X_i^c), \pi(X_j^v)]$,

$$\begin{aligned}\pi([X_i^c, X_j^c]) &= \pi([X_i, X_j]^c) \\ &= [X_i, X_j].\end{aligned}$$

But

$$[\pi(X_i^c), \pi(X_j^c)] = [X_i, X_j].$$

Hence $\pi([X_i^c, X_j^c]) = [\pi(X_i^c), \pi(X_j^c)]$.

Now suppose that $\widetilde{\mathfrak{g}}$ is decomposable so that $\widetilde{\mathfrak{g}}$ is isomorphic to $\mathfrak{h}_1 \oplus \mathfrak{h}_2$. Then since π is surjective $\mathfrak{g}$ is spanned by a basis of $\pi(\mathfrak{h}_1)$ together with a basis of $\pi(\mathfrak{h}_2)$ and $\pi(\mathfrak{h}_1), \pi(\mathfrak{h}_2) \lhd \mathfrak{g}$. Since $[\mathfrak{h}_1, \mathfrak{h}_2] = 0$, then $[\pi(\mathfrak{h}_1), \pi(\mathfrak{h}_2)] = 0$. If there is an element $x \in \pi(\mathfrak{h}_1) \cap \pi(\mathfrak{h}_2)$, then $\pi^{-1}(x) \in h_1 \cap h_2 = 0 \Rightarrow x = 0$. Hence $\mathfrak{g} = \pi(\mathfrak{h}_1) \oplus \pi(\mathfrak{h}_2)$, that is, $\mathfrak{g}$ is decomposable.

$\square$

The tangent bundle TG like any tangent bundle possesses a canonical field of endomorphisms denoted by S, and sometimes known as the Jacobi endomophism [6]. Its local expression is $\frac{\partial}{\partial u^i} \otimes dx^j$. In the next Lemma we quote the main properties enjoyed by S on a general tangent bundle TM. In the Lemma $\mathcal{L}$ denotes the Lie derivative operator.

Lemma 1.4.1 *If X is a vector field on M then:*

(i) $S(X^v) = 0$.

(ii) $S(X^c) = X^v$

(iii) $\mathcal{L}_{X^c} S = 0$

(iv) $\mathcal{L}_{X^v} S = 0$.

We remark that conditions (i) and (ii) of the last Lemma may be taken as the definition of S, so it is characterized by its effect on vertical and complete lifts . For much more thorough accounts we refer to [6, 32].

Theorem 1.4.2 *The canonical field of endomorphisms S is parallel with respect to the canonical connection $\widetilde{\nabla}$ on TG.*

Proof: We need to show that $\widetilde{\nabla}_{X^c} S = \widetilde{\nabla}_{X^v} S = 0$ for all complete and vertical vector fields on TG. We discuss here the four possible cases.

- $$\begin{aligned}\left(\widetilde{\nabla}_{X^v}S\right)Y^v &= \widetilde{\nabla}_{X^v}\left(SY^v\right) - S\left(\widetilde{\nabla}_{X^v}Y^v\right)\\ &= \widetilde{\nabla}_{X^v}\big(0\big) - S\big(0\big)\\ &= 0\end{aligned}$$

- $$\begin{aligned}\left(\widetilde{\nabla}_{X^v}S\right)Y^c &= \widetilde{\nabla}_{X^v}\left(SY^c\right) - S\left(\widetilde{\nabla}_{X^v}Y^c\right)\\ &= \widetilde{\nabla}_{X^v}\big(Y^v\big) - S\big(\nabla_X Y\big)^v\\ &= 0\end{aligned}$$

- $$\begin{aligned}\left(\widetilde{\nabla}_{X^c}S\right)Y^v &= \widetilde{\nabla}_{X^c}\left(SY^v\right) - S\left(\widetilde{\nabla}_{X^c}Y^v\right)\\ &= \widetilde{\nabla}_{X^v}\big(0\big) - S\big(\nabla_X Y\big)^v\\ &= 0\end{aligned}$$

•

$$\begin{aligned}\Big(\widetilde{\nabla}_{X^c}S\Big)Y^c &= \widetilde{\nabla}_{X^c}\Big(SY^c\Big) - S\Big(\widetilde{\nabla}_{X^c}Y^c\Big)\\ &= \widetilde{\nabla}_{X^v}\big(Y^v\big) - S\big(\nabla_X Y\big)^c\\ &= \big(\nabla_X Y\big)^v - \big(\nabla_X Y\big)^v\\ &= 0.\end{aligned}$$

□

1.5 The cotangent bundle of a Lie group T^*G

We will use notation similar to the previous section. We denote elements of T^*G with Greek letters based at the the corresponding English letter, for example $\alpha \in T^*_A G$. We also use the maps $L^*_A, R^*_A : T^*_I G \to T^*_A G$.

Lemma 1.5.1 *The product $\alpha.\beta := L^*_{A^{-1}}\beta + R^*_{B^{-1}}\alpha$ on T^*G, makes it a group.*

Proof: We note the following:

- T^*G is closed under this product, $L^*_{A^{-1}}\beta, R^*_{B^{-1}}\alpha \in T^*_{AB}G$, and so is their sum.

- We denote $\overline{I} = 0^*_I$ for the identity element of T^*G, where I is the identity in G.

$$\begin{aligned} \alpha.\overline{I} &= L^*_{A^{-1}}\overline{I} + R^*_I\alpha \\ &= L^*_{A^{-1}}0^*_I + \alpha \\ &= \alpha \end{aligned}$$

  Similarly, we have $\overline{I}.\alpha = \alpha$.

- Next we show that the inverse element $\alpha^{-1} = -L^*_A R^*_A \alpha$. We assume that β is the inverse of α in T^*G, then we have

$$\begin{aligned}
0_I^* = \alpha.\beta &= L_{A^{-1}}^*\beta + R_{B^{-1}}^*\alpha \\
\Rightarrow L_{A^{-1}}^*\beta &= -R_{B^{-1}}^*\alpha \quad and \quad A.B = I \\
Also \quad 0_I = \beta.\alpha &= L_{B^{-1}}^*\alpha + R_{A^{-1}}^*\beta \\
\Rightarrow L_{B^{-1}}^*\alpha &= -R_{A^{-1}}^*\beta \quad and \quad B.A = I \\
\Rightarrow B &= A^{-1} \quad and \\
\beta &= -L_A^* R_A^* \alpha.
\end{aligned}$$

- Finally we check the associativity: we take $\alpha, \beta, \gamma \in T_A^*G, T_B^*G, T_C^*G$ respectively, we want to show that

$$(\alpha.\beta).\gamma = \alpha.(\beta.\gamma)$$

so we consider

$$\begin{aligned}
(\alpha.\beta).\gamma &= \left(L_{A^{-1}}^*\beta + R_{B^{-1}}^*\alpha\right).\gamma \\
&= L_{(A.B)^{-1}}^*\gamma + R_{C^{-1}}^*\left(L_{A^{-1}}^*\beta + R_{B^{-1}}^*\alpha\right) \\
&= L_{B^{-1}.A^{-1}}^*\gamma + L_{A^{-1}}^* R_{C^{-1}}^*\beta + R_{C^{-1}}^* R_{B^{-1}}^*\alpha
\end{aligned}$$

$$
\begin{aligned}
&= L^*_{A^{-1}}\left(L^*_{B^{-1}}\gamma + R^*_{C^{-1}}\beta\right) + R^*_{(B.C)^{-1}}\alpha \\
&= L^*_{A^{-1}}\left(\beta.\gamma\right) + R^*_{(B.C)^{-1}}\alpha \\
&= \alpha.\left(\beta.\gamma\right)
\end{aligned}
$$

□

Theorem 1.5.1 *Let X be a left invariant vector field on G, let α be its corresponding Maurer-Cartan left invariant form, then its vertical lift α^v to T^*G is a left invariant vector field on T^*G.*

Proof: Using Similar notations of (1.5), we note that the equivalence class of curves $\Phi_t(\alpha)$ associated with α^v on T^*G is given by:

$$
\Phi_t(\beta) = \beta + t.\alpha(A) \qquad \beta \in T^*_A G \tag{1.15}
$$

We note first that $\Phi_0(\beta) = \beta$, and

$$
\begin{aligned}
\Phi_s(\Phi_t(\beta)) &= \Phi_s(\beta + t.\alpha(A)) \\
&= \beta + t.\alpha(A) + s.\alpha(A) \\
&= \beta + (t+s).\alpha(A) \\
&= \Phi_{s+t}(\beta),
\end{aligned}
$$

that is to say, that $\Phi_t(\beta)$ is a 1-parameter group of diffeomorphisms on T^*G. We would like to show α^v is left invariant, that is the equivalence class of curves $[\Phi_t(\beta)]$

associated with α^v, satisfies (1.5). So we pick elements $\alpha, \beta \in T^*_A G; \gamma \in T^*_C G$, and consider

$$\begin{aligned}
\gamma.\Phi_t(\beta) &= \gamma.(\beta + t.\alpha(A)) \\
&= L^*_{C^{-1}}(\beta + t.\alpha(A)) + R^*_{A^{-1}}\gamma \\
&= L^*_{C^{-1}}\beta + t.L^*_{C^{-1}}\alpha(A) + R^*_{A^{-1}}\gamma \\
&= L^*_{C^{-1}}\beta + R^*_{A^{-1}}\gamma + t.\alpha(C.A) \qquad (Since \quad \alpha \quad is \quad left-invariant) \\
&= \gamma.\beta + t.\alpha(C.A) \\
&= \Phi_t(\gamma.\beta).
\end{aligned}$$

By applying (1.5) on T^*G, we get that α^v is left invariant on T^*G.

□

Theorem 1.5.2 *Let X be a left invariant vector field on G, then its complete lift X^c to T^*G is left invariant on T^*G.*

Proof: We first note that if $\varphi : G \to G$ is a smooth map, then so is $T^*\varphi : T^*G \to T^*G$, in particular $T^*\varphi_A : T^*_A G \to T^*_{\varphi(A)} G$ for each $A \in G$. To see how $T^*\varphi_A$ is defined, we take an element $\alpha \in T^*G$ and vector field $Y \in TG$, then we have

$$(T^*\varphi_A \alpha)\, Y = (\imath_Y \alpha)\varphi(A).$$

Now we take X to be left invariant vector field on G, with equivalence classes $X(A) = [\varphi_t(A)]$ for $A \in G$ as before, then its complete lift X^c on T^*G has the equivalence classes $X^c(\alpha) = [T^*\varphi_t(\alpha)]$ for $\alpha \in T_A^*G$. Now we want to show that $\beta.T^*\varphi_t(\alpha) = T^*\varphi_t(\beta.\alpha)$, so we take any vector field $Y \in TG$, and consider

$$
\begin{aligned}
\Big(\beta.T^*\varphi_t(\alpha)\Big)Y &= \Big(L^*_{B^{-1}}T^*\varphi_t(\alpha) + R^*_{(\varphi_t(A))^{-1}}\beta\Big)Y \\
&= \Big(L^*_{B^{-1}} \circ T^*\varphi_t(\alpha)\Big)Y + \Big(R^*_{(\varphi_t(A))^{-1}}\beta\Big)Y \\
&= \Big(L^*_{B^{-1}}\imath_Y\alpha\Big)\varphi_t(A) + R^*_{(\varphi_t(A))^{-1}}\Big(\imath_Y\beta\Big)(B) \\
&= \imath_Y\Big(L^*_{B^{-1}}\alpha\Big)(B.\varphi_t(A)) + \imath_Y\Big(R^*_{(\varphi_t(A))^{-1}}\beta\Big)(B.\varphi_t(A)) \\
&= \imath_Y\Big(L^*_{B^{-1}}\alpha + R^*_{(\varphi_t(A))^{-1}}\beta\Big)(B.\varphi_t(A)) \\
&= \imath_Y\Big(\beta.\alpha\Big)(\varphi_t(B.A)) \\
&= \Big(T^*\varphi_t(\beta.\alpha)\Big)Y.
\end{aligned}
$$

Since Y was arbitrarily chosen we have $\beta.T^*\varphi_t(\alpha) = T^*\varphi_t(\beta.\alpha)$, and by (1.5) we get the result that X^c is left invariant on T^*G.

$\square$

1.6 The canonical connection ∇^* on T^*G

On T^*G the left invariant vector fields are the complete lifts of left invariant vector fields, and the vertical lifts of the Maurer-Cartan left invariant forms on G, namely:

$$X_1^c, X_2^c, ..., X_n^c, (\alpha^1)^v, (\alpha^2)^v, ..., (\alpha^n)^v. \tag{1.16}$$

They determine a Lie algebra $\bar{\mathfrak{g}}$. We describe it by means of the brackets

$$\begin{aligned} \left[X_i^c, X_j^c\right] &= [X_i, X_j]^c = C_{ij}^k X_k^c \\ \left[(\alpha^i)^v, (\alpha^j)^v\right] &= 0 \\ \left[X_i^c, (\alpha^j)^v\right] &= -C_{ik}^j (\alpha^k)^v. \end{aligned} \tag{1.17}$$

Since T^*G is a Lie group it has a canonical connection that we denote by ∇^*. Its values on complete and vertical lifts are as follows:

$$\begin{aligned} \nabla^*_{(\alpha^i)^v}(\alpha^j)^v &= \frac{1}{2}\left[(\alpha^i)^v, (\alpha^j)^v\right] = 0 \\ \nabla^*_{(\alpha^i)^v} X_j^c &= \frac{1}{2}\left[(\alpha^i)^v, X_j^c\right] = \frac{1}{2} C_{jk}^i (\alpha^k)^v \\ \nabla^*_{X_i^c}(\alpha^j)^v &= \frac{1}{2}\left[X_i^c, (\alpha^j)^v\right] = -\frac{1}{2} C_{ik}^j (\alpha^k)^v \\ \nabla^*_{X_i^c} X_j^c &= \frac{1}{2}\left[X_i^c, X_j^c\right] = \frac{1}{2} C_{ij}^k X_k^c. \end{aligned} \tag{1.18}$$

The following Lemma gives the values of the covariant derivatives of the one-forms $\{\pi^*\alpha^i, d\widehat{X}_i\}$ in T^*G. Here the notation $\widehat{X}$ means that the vector field X on G is regarded as defining a real-valued function on T^*G that is linear in the fiber. In coordinates $\widehat{X} = X^i p_i$.

Lemma 1.6.1

$$\begin{aligned}
\nabla^*_{X^c_k}(\pi^*\alpha^j) &= -\frac{1}{2}C^j_{ki}\pi^*\alpha^i \\
\nabla^*_{(\alpha^k)^v}(\pi^*\alpha^j) &= 0 \\
\nabla^*_{X^c_k}d\widehat{X}_j &= \frac{1}{2}C^i_{kj}d\widehat{X}_i \\
\nabla^*_{(\alpha^k)^v}d\widehat{X}_j &= -\frac{1}{2}C^k_{ij}\pi^*\alpha^i.
\end{aligned} \tag{1.19}$$

Proof: The left invariant vector fields on T^*G are $\{X^c_1, X^c_2, ..., X^c_n, (\alpha^1)^v, (\alpha^2)^v, ..., (\alpha^n)^v\}$ and their dual forms are $\{\pi^*\alpha^1, \pi^*\alpha^2, ..., \pi^*\alpha^n, d\widehat{X}_1, d\widehat{X}_2, ..., d\widehat{X}_n\}$ that is

$$\begin{aligned}
< X^c_i, d\widehat{X}_j > &= 0 \\
< X^c_i, \pi^*\alpha^j > &= \delta^j_i \\
< (\alpha^i)^v, \pi^*\alpha^j > &= 0 \\
< (\alpha^i)^v, d\widehat{X}_j > &= \delta^i_j.
\end{aligned}$$

We use these facts to derive the lemma: thus

- $X^c_k < X^c_i, \pi^*\alpha^j >= 0$

  $< \nabla^*_{X^c_k}X^c_i, \pi^*\alpha^j > + < X^c_i, \nabla^*_{X^c_k}\pi^*\alpha^j >= 0$

  $\frac{1}{2}C^l_{ki} < X^c_l, \pi^*\alpha^j > + < X^c_i, \nabla^*_{X^c_k}\pi^*\alpha^j >= 0$

  $\frac{1}{2}C^j_{ki}X^c_j + < X^c_i, \nabla^*_{X^c_k}\pi^*\alpha^j >= 0$

  $\therefore \nabla^*_{X^c_k}(\pi^*\alpha^j) = -\frac{1}{2}C^j_{ki}\pi^*\alpha^i.$

- $(\alpha^k)^v < X_i^c, \pi^*\alpha^j >= 0$

  $< \nabla^*_{(\alpha^k)^v} X_i^c, \pi^*\alpha^j > + < X_i^c, \nabla^*_{(\alpha^k)^v} \pi^*\alpha^j >= 0$

  $\frac{1}{2} C_{il}^k < (\alpha^l)^v, \pi^*\alpha^j > + < X_i^c, \nabla^*_{(\alpha^k)^v} \pi^*\alpha^j >= 0$

  $< X_i^c, \nabla^*_{(\alpha^k)^v} \pi^*\alpha^j >= 0.$

  Similarly

  $(\alpha^k)^v < (\alpha^i)^v, \pi^*\alpha^j >= 0$

  $< \nabla^*_{(\alpha^k)^v} (\alpha^i)^v, \pi^*\alpha^j > + < (\alpha^i)^v, \nabla^*_{(\alpha^k)^v} \pi^*\alpha^j >= 0$

  $< (\alpha^i)^v, \nabla^*_{(\alpha^k)^v} \pi^*\alpha^j >= 0.$

  Hence we obtain the result that

  $\nabla^*_{(\alpha^k)^v} (\pi^*\alpha^j) = 0.$

- $X_k^c < (\alpha^i)^v, d\widehat{X}_j >= 0$

  $< \nabla^*_{X_k^c} (\alpha^i)^v, d\widehat{X}_j > + < (\alpha^i)^v, \nabla^*_{X_k^c} d\widehat{X}_j >= 0$

  $-\frac{1}{2} C_{kl}^i < (\alpha^k)^l, d\widehat{X}_j > + < (\alpha^i)^v, \nabla^*_{X_k^c} d\widehat{X}_j >= 0$

  $-\frac{1}{2} C_{kj}^i + < (\alpha^i)^v, \nabla^*_{X_k^c} d\widehat{X}_j >= 0$

  $\therefore \nabla^*_{X_k^c} d\widehat{X}_j = \frac{1}{2} C_{kj}^i d\widehat{X}_i.$

- $(\alpha^k)^v < (\alpha^i)^v, d\widehat{X}_k >= 0$

  $< \nabla^*_{(\alpha^k)^v} (\alpha^i)^v, d\widehat{X}_k > + < (\alpha^i)^v, \nabla^*_{(\alpha^k)^v} d\widehat{X}_k >= 0$

  $< (\alpha^i)^v, \nabla^*_{(\alpha^k)^v} d\widehat{X}_k >= 0.$

  Similarly

  $X_k^c < X_i^c, d\widehat{X}_j >= 0$

  $< \nabla^*_{X_k^c} X_i^c, d\widehat{X}_j > + < X_i^c, \nabla^*_{X_k^c} d\widehat{X}_j >= 0$

  $\frac{1}{2} C_{ki}^l < X_l^c, d\widehat{X}_j > + < X_i^c, \nabla^*_{X_k^c} d\widehat{X}_j >= 0$

$$< X_i^c, \nabla^*_{X_k^c} d\widehat{X}_j >= 0$$

$$\therefore \nabla^*_{(\alpha^k)^v} d\widehat{X}_j = -\tfrac{1}{2} C^k_{ij} \pi^* \alpha^i.$$

□

Next we find the nonzero components of the curvature tensor $\overline{R}$ on the vector fields (1.16) of T^*G, using (1.8), and (1.17). We show here the six possible cases:

$$\begin{aligned}
\overline{R}((\alpha^i)^v, (\alpha^j)^v)(\alpha^k)^v &= \frac{1}{4}\left[(\alpha^k)^v, [(\alpha^i)^v, (\alpha^j)^v]\right] = 0 \\
\overline{R}((\alpha^i)^v, (\alpha^j)^v)X_k^c &= \frac{1}{4}\left[X_k^c, [(\alpha^i)^v, (\alpha^j)^v]\right] = 0 \\
\overline{R}(X_i^c, (\alpha^j)^v)(\alpha^k)^v &= \frac{1}{4}\left[(\alpha^k)^v, [X_i^c, (\alpha^j)^v]\right] = -\frac{1}{4}C^j_{il}\left[(\alpha^k)^v, (\alpha^l)^v\right] = 0 \\
\overline{R}(X_i^c, (\alpha^j)^v)X_k^c &= \frac{1}{4}\left[X_k^c, [X_i^c, (\alpha^j)^v]\right] = -\frac{1}{4}C^j_{il}\left[X_k^c, (\alpha^l)^v\right] \qquad (1.20) \\
&= \frac{1}{4}C^j_{il}C^l_{kr}(\alpha^r)^v \\
\overline{R}(X_i^c, X_j^c)(\alpha^k)^v &= \frac{1}{4}\left[(\alpha^k)^v, [X_i^c, X_j^c]\right] = \frac{1}{4}C^l_{ij}\left[(\alpha^k)^v, X_l^c\right] \\
&= \frac{1}{4}C^l_{ij}C^k_{lr}(\alpha^r)^v \\
\overline{R}(X_i^c, X_j^c)X_k^c &= \frac{1}{4}\left[X_k^c, [X_i^c, X_j^c]\right] = \frac{1}{4}\left[X_k^c, [X_i, X_j]^c\right] \\
&= \frac{1}{4}[X_k, [X_i, X_j]]^c = \frac{1}{4}C^l_{ij}C^r_{kl}X_r^c.
\end{aligned}$$

On T^*G we have the canonical one form $\theta = p_i dx^i$, from which we get the canonical two form by taking exterior derivative: $d\theta = dp_i \wedge dx^i$. The following Lemma summarizes all the formulae concerning T^*M that we shall need.

Lemma 1.6.2 *For the canonical one form θ on T^*G we have the following:*

(i) $\theta\big(\alpha^v\big) = 0$

(ii) $\theta\big(X^c\big) = \widehat{X}$

(iii) $Y^c\big(\widehat{X}\big) = \widehat{[Y,X]}$

(iv) $d\theta\big(\alpha^v, \beta^v\big) = 0$

(v) $d\theta\big(\alpha^v, X^c\big) = \pi^* < \alpha, X >$

(vi) $d\theta\big(X^c, Y^c\big) = \widehat{[X,Y]}$

(vii) $\mathcal{L}_{\alpha^v}\theta = \pi^*\alpha$

(viii) $\mathcal{L}_{X^c}\theta = 0$

(ix) $\mathcal{L}_{\alpha^v}\widehat{X} = \pi^* < X, \alpha >$

(x) $\mathcal{L}_{X^c}\widehat{Y} = \frac{1}{2}\widehat{[X,Y]}$.

Lemma 1.6.3 *For the canonical 1-form θ on T^*G we have the following:*

(i) $\nabla^*_{X^c}\theta = 0$

(ii) $\nabla^*_{\alpha^v}\theta = \pi^*\alpha$.

Proof: We discuss the four possible cases:

(i)

$$\begin{aligned}
\Big(\nabla^*_{X^c}\theta\Big)Y^c &= \nabla^*_{X^c}(\theta Y^c) - \theta(\nabla^*_{X^c}Y^c) \\
&= \nabla^*_{X^c}\widehat{Y} - \frac{1}{2}\theta\Big([X,Y]^c\Big) \\
&= \frac{1}{2}\widehat{[X,Y]} - \frac{1}{2}\widehat{[X,Y]} \\
&= 0.
\end{aligned}$$

(ii)

$$\begin{aligned}\left(\nabla^*_{X^c}\theta\right)\alpha^v &= \nabla^*_{X^c}(\theta\alpha^v) - \theta(\nabla^*_{X^c}\alpha^v)\\ &= 0.\end{aligned}$$

(iii)

$$\begin{aligned}\left(\nabla^*_{(\alpha^i)^v}\theta\right)X^c_j &= \nabla^*_{(\alpha^i)^v}(\theta X^c_j) - \theta(\nabla^*_{(\alpha^i)^v}X^c_j)\\ &= \nabla^*_{(\alpha^i)^v}\widehat{X_j} - \frac{1}{2}C^i_{jk}\theta((\alpha^k)^v)\\ &= \pi^* < X_j, \alpha^i > -0\\ &= \delta^i_j.\end{aligned}$$

(iv)

$$\begin{aligned}\left(\nabla^*_{(\alpha^i)^v}\theta\right)(\alpha^j)^v &= \nabla^*_{(\alpha^i)^v}(\theta(\alpha^j)^v) - \theta(\nabla^*_{(\alpha^i)^v}(\alpha^j)^v)\\ &= 0.\end{aligned}$$

□

Theorem 1.6.1 *The canonical two-form $d\theta$ on T^*G is parallel with respect to the canonical connection on T^*G.*

Proof: We show the six possible cases here.

(i)

$$\begin{aligned}
\Big(\nabla^*_{(\alpha^k)^v} d\theta\Big)(X_i^c, X_j^c) &= \nabla^*_{(\alpha^k)^v}\Big(d\theta(X_i^c, X_j^c)\Big) - d\theta\Big(\nabla^*_{(\alpha^k)^v} X_i^c, X_j^c\Big) \\
&\quad - d\theta\Big(X_i^c, \nabla^*_{(\alpha^k)^v} X_j^c\Big) \\
&= \nabla^*_{(\alpha^k)^v}[\widehat{X_i^c, X_j^c}] - \frac{1}{2} C_{il}^k d\theta\Big((\alpha^l)^v, X_j^c\Big) \\
&\quad - \frac{1}{2} C_{jl}^k d\theta\Big(X_i^c, (\alpha^l)^v\Big) \\
&= C_{ij}^l \mathcal{L}_{(\alpha^k)^v} \widehat{X}_l - \frac{1}{2} C_{il}^k \pi^* < \alpha^l, X_j > + \frac{1}{2} C_{jl}^k \pi^* < X_i, \alpha^l > \\
&= C_{ij}^l \delta_l^k - \frac{1}{2} C_{il}^k \delta_j^l + \frac{1}{2} C_{jl}^k \delta_i^l \\
&= C_{ij}^k - \frac{1}{2} C_{ij}^k + \frac{1}{2} C_{ji}^k \\
&= 0.
\end{aligned}$$

(ii)

$$\begin{aligned}
\Big(\nabla^*_{X_k^c} d\theta\Big)(X_i^c, X_j^c) &= \nabla^*_{X_k^c}\Big(d\theta(X_i^c, X_j^c)\Big) - d\theta\Big(\nabla^*_{X_k^c} X_i^c, X_j^c\Big) - d\theta\Big(X_i^c, \nabla^*_{X_k^c} X_j^c\Big) \\
&= \nabla^*_{X_k^c}[\widehat{X_i^c, X_j^c}] - \frac{1}{2} d\theta\Big([X_k, X_i]^c, X_j^c\Big) - \frac{1}{2} d\theta\Big(X_i^c, [X_k, X_j]^c\Big) \\
&= \frac{1}{2}\big[X_k, \widehat{[X_i^c, X_j^c]}\big] - \frac{1}{2}\big[\widehat{[X_k, X_i]}, X_j^c\big] - \frac{1}{2}\big[X_i, \widehat{[X_k, X_j]}\big] \\
&= 0.
\end{aligned}$$

(iii)

$$\begin{aligned}\Big(\nabla^*_{(\alpha^k)^v} d\theta\Big)(X_i^c, (\alpha^j)^v) &= \nabla^*_{(\alpha^k)^v}\Big(d\theta(X_i^c, (\alpha^j)^v)\Big) - d\theta\Big(\nabla^*_{(\alpha^k)^v} X_i^c, (\alpha^j)^v\Big) \\ &\quad -d\theta\Big(X_i^c, \nabla^*_{(\alpha^k)^v}(\alpha^j)^v\Big) \\ &= -\nabla^*_{(\alpha^k)^v}\delta_i^j - \frac{1}{2}C_{il}^k d\theta\Big((\alpha^l)^v, (\alpha^j)^v\Big) \\ &= 0.\end{aligned}$$

(iv)

$$\begin{aligned}\Big(\nabla^*_{X_k^c} d\theta\Big)(X_i^c, (\alpha^j)^v) &= \nabla^*_{X_k^c}\Big(d\theta(X_i^c, (\alpha^j)^v)\Big) - d\theta\Big(\nabla^*_{X_k^c} X_i^c, (\alpha^j)^v\Big) \\ &\quad -d\theta\Big(X_i^c, \nabla^*_{X_k^c}(\alpha^j)^v\Big) \\ &= -\nabla^*_{X_k^c}\delta_i^j - \frac{1}{2}C_{ki}^l d\theta\Big(X_l^c, (\alpha^j)^v\Big) + \frac{1}{2}C_{kl}^j d\theta\Big(X_i^c, (\alpha^l)^v\Big) \\ &= \frac{1}{2}C_{ki}^l\delta_l^j - \frac{1}{2}C_{kl}^j\delta_i^l \\ &= 0.\end{aligned}$$

(v)

$$\begin{aligned}\Big(\nabla^*_{(\alpha^k)^v} d\theta\Big)((\alpha^i)^v, (\alpha^j)^v) &= \nabla^*_{(\alpha^k)^v}\Big(d\theta((\alpha^i)^v, (\alpha^j)^v)\Big) - d\theta\Big(\nabla^*_{(\alpha^k)^v}(\alpha^i)^v, (\alpha^j)^v\Big) \\ &\quad -d\theta\Big((\alpha^i)^v, \nabla^*_{(\alpha^k)^v}(\alpha^j)^v\Big) \\ &= 0.\end{aligned}$$

(vi)

$$\begin{aligned}\left(\nabla^*_{X_k^c} d\theta\right)((\alpha^i)^v, (\alpha^j)^v) &= \nabla^*_{X_k^c}\left(d\theta((\alpha^i)^v, (\alpha^j)^v)\right) - d\theta\left(\nabla^*_{X_k^c}(\alpha^i)^v, (\alpha^j)^v\right) \\ &\quad -d\theta\left(X_k^c, \nabla^*_{(\alpha^k)^v}(\alpha^j)^v\right) \\ &= 0.\end{aligned}$$

It follows that ∇^* on T^*G is an example of a symplectic connection.

$\square$

For more information about symplectic connections we refer the reader to [2, 3].

Chapter 2

Representation theorems for Lie algebras and Lie groups

If a Lie algebra has a trivial center then the adjoint representation is faithful. Then by exponentiating the adjoint matrices with parameters and multiplying all the exponentiated adjoint matrices together we can obtain a representation for the corresponding matrix Lie group. In fact if we have a faithful representation of a Lie algebra $\mathfrak{g}$ as a subalgebra of $gl(p, \mathbb{R})$ for some p that is not necessarily the dimension of $\mathfrak{g}$, the same method works to produce a subgroup of $GL(p, \mathbb{R})$ whose Lie algebra is isomorphic to $\mathfrak{g}$.

However, when there is a non-trivial center we have to use some other techniques to get a matrix representation for $\mathfrak{g}$. The fact such a representation always exists follows from Ado's theorem [13, 31] The classification of Lie algebras involves families depending on parameters. Typically a given Lie algebra belongs to a family of Lie algebras which depends on several parameters. There are a few Lie algebras which do not belong to such families and which may or may not have a trivial center. For generic values the Lie algebra may have a trivial center whereas typically for certain non-generic values there may be a non-trivial center. It seems that there is no way to take a limit when the parameters take special values and the algebra has a non-trivial center.

If we *start* from a linear Lie group G we can get a representation for the corresponding Lie algebra $\mathfrak{g}$ in two very different ways. Usually in this dissertation we denote a typical element in such a group by S. The element S depends on the coordinates on G. A Lie algebra *matrix* representation can be found by differentiating S with respect to the coordinates involved and evaluating at the identity.

The other way to obtain a representation is via the Maurer-Cartan form. Let dS be the differential of the matrix S. Then by calculating $S^{-1}dS$ and dSS^{-1} we obtain left and right invariant Maurer-Cartan forms (see A.3) which are $\mathfrak{g}$-valued. We can choose a basis for left or right invariant forms from the entries of $S^{-1}dS$ or dSS^{-1} and by dualizing this basis we obtain a basis for the left or right invariant vector fields (see A.5) and we can check whether we have obtained the required Lie algebra.

The process of finding representations for Lie algebras that have a non-trivial center is difficult. Part of the goal of the program of my adviser Professor G. Thompson is to find representations for all known Lie algebras and to formulate a number of Theorems that give the representations in various cases [10, 11, 14, 26]. In the Theorems below we consider the very special cases of such representations involving Heisenberg algebras.

2.1 Heisenberg extensions

We start with an n-dimensional Lie algebra $\mathfrak{g}$ with structure equations $[e_i, e_j] = C_{ij}^k e_k$, so $1 \leq k \leq n$. We assume only that one vector in $\mathfrak{g}$, which without loss of generality we take to be e_n, is not in the derived algebra $[\mathfrak{g}, \mathfrak{g}]$ of $\mathfrak{g}$. We introduce two more vectors $\{e_{n+1}, e_{n+2}\}$ satisfying: $[e_{n+1}, e_i] = [e_{n+2}, e_i] = 0 \quad (1 \leq i \leq n-1)$, the Heisenberg bracket $[e_{n+1}, e_n] = e_{n+2}$ and $[e_{n+1}, e_{n+2}] = [e_{n+2}, e_n] = 0$. In other words we want to extend the algebra $\mathfrak{g}$ by two dimensions in such a way that there is only one new non-zero bracket, a Heisenberg bracket.

We claim that the extended $(n+2)$-dimensional space is a Lie algebra. As such we must check the following triples, to ensure that the Jacobi identity is satisfied. For $1 \leq i, j, k \leq n$ we have:

$$[[e_i, e_j], e_k] + [[e_k, e_i], e_j] + [[e_j, e_k], e_i] = 0$$

$$[[e_i, e_j], e_{n+1}] + [[e_{n+1}, e_i], e_j] + [[e_j, e_{n+1}], e_i] = 0$$

$$[[e_i, e_j], e_{n+2}] + [[e_{n+2}, e_i], e_j] + [[e_j, e_{n+2}], e_i] = 0$$

$$[[e_i, e_{n+1}], e_{n+2}] + [[e_{n+2}, e_i], e_{n+1}] + [[e_{n+1}, e_{n+2}], e_i] = 0.$$

Of these four conditions the first follows from the fact that $\mathfrak{g}$ is a Lie algebra and the third and fourth because e_{n+2} commutes with every vector. In the second identity, the expression on the left hand side reduces to $C_{ij}^k [e_k, e_{n+1}]$: however, this term is also zero because of the assumption that e_n is not in $[\mathfrak{g}, \mathfrak{g}]$. We obtain a Lie algebra extension $\mathfrak{g}'$ of $\mathfrak{g}$. In fact $\langle e_{n+1}, e_{n+2} \rangle$ forms a two-dimensional ideal and $\mathfrak{g}'$ splits over this ideal.

The next issue that we address is that if we know a matrix representation for $\mathfrak{g}$, can we find a representation for $\mathfrak{g}'$? We shall denote the matrix Lie groups associated to $\mathfrak{g}$ and $\mathfrak{g}'$ by G and G', respectively.

Theorem 2.1.1 *Let S be the matrix of a representation for G in $GL(p,\mathbb{R})$. Then S' is the matrix representation of G' in $GL(p+2,\mathbb{R})$ where S' is given by:*

$$S' = \left(\begin{array}{cc|ccc} 1 & w & 0 & 0 & y \\ 0 & 1 & 0 & 0 & z \\ \hline 0 & 0 & & & \\ 0 & 0 & & S & \\ 0 & 0 & & & \end{array}\right)$$

Proof: We give two proofs here. The first one is based on Maurer-Cartan forms and the second one on the matrix Lie algebra associated with the extended representation.

(i) Note that

$$dS' = \left(\begin{array}{cc|ccc} 0 & dw & 0 & 0 & dy \\ 0 & 0 & 0 & 0 & dz \\ \hline 0 & 0 & & & \\ 0 & 0 & & dS & \\ 0 & 0 & & & \end{array}\right)$$

and

$$(S^{'})^{-1} = \left(\begin{array}{cc|ccc} 1 & -w & 0 & 0 & wz-y \\ 0 & 1 & 0 & 0 & -z \\ \hline 0 & 0 & & & \\ 0 & 0 & & S^{-1} & \\ 0 & 0 & & & \end{array}\right)$$

so that

$$dS^{'}.(S^{'})^{-1} = \left(\begin{array}{cc|ccc} 0 & dw & 0 & 0 & -zdw+dy \\ 0 & 0 & 0 & 0 & dz \\ \hline 0 & 0 & & & \\ 0 & 0 & & dS.S^{-1} & \\ 0 & 0 & & & \end{array}\right).$$

So the new independent Maurer-Cartan forms are dz and $dy-zdw$. But $d(dz) = 0 \Longrightarrow C_{ij}^{n+1} = 0$. Similarly $d(dy-zdw) = -dz.dw \Longrightarrow C_{n,n+1}^{n+2} = 1$ and thus we obtain the result.

(ii) Let x_i be the variables "corresponding" to e_i $(1 \leq i \leq n-1)$ and w, z, y "correspond" to e_n, e_{n+1}, e_{n+2}, respectively. Let $M_i^{'} = \frac{\partial}{\partial x_i}\Big|_I S^{'}$, so we have

$$M_i^{'} = \left(\begin{array}{cc|ccc} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & & & \\ 0 & 0 & & M_i & \\ 0 & 0 & & & \end{array}\right), M_w^{'} = \left(\begin{array}{cc|ccc} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & & & \\ 0 & 0 & & M_w & \\ 0 & 0 & & & \end{array}\right)$$

$$M_y^{'} = \left(\begin{array}{cc|ccc} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & & & \\ 0 & 0 & & 0 & \\ 0 & 0 & & & \end{array}\right), M_z^{'} = \left(\begin{array}{cc|ccc} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ \hline 0 & 0 & & & \\ 0 & 0 & & 0 & \\ 0 & 0 & & & \end{array}\right)$$

Note that $[M_i^{'}, M_j^{'}] = C_{ij}^k M_i^{'}; 1 \leq i < j \leq n.$

$[M_y^{'}, M_i^{'}] = 0; 1 \leq i \leq n-1.$

$[M_i^{'}, M_z^{'}] = 0; 1 \leq i \leq n-1.$

$[M_w^{'}, M_z^{'}] = M_y^{'}.$

$[M_y^{'}, M_z^{'}] = 0.$

Thus our proof is finished.

$\square$

Remark 2.1.1 *It is a bit difficult to find suitable terminology that describes the construction of the previous Theorem. In this dissertation it will be referred to simply as a "Heisenberg extension".*

2.2 One-dimensional extension to a three-dimensional Heisenberg nilradical

Again let $\mathfrak{g}$ be a Lie algebra. A *nil ideal* of $\mathfrak{g}$ is an ideal $\mathfrak{m}$ of $\mathfrak{g}$ such that ad $\boldsymbol{X}$ is nilpotent for each $\boldsymbol{X} \in \mathfrak{m}$. An ideal $\mathfrak{m}$ of $\mathfrak{g}$ is a nil ideal if and only if $\mathfrak{m}$, as an algebra, is nilpotent. Any Lie algebra $\mathfrak{g}$ has a unique maximal nil ideal which contains every nil ideal. We call it the *nilradical* of $\mathfrak{g}$. The Lie algebra $\mathfrak{g}$ is nilpotent if and only if $\mathfrak{g}$ is equal to its own nilradical.

Among the simplest of all Lie algebras are algebras that have a codimension one abelian nilradical. We now want to begin to study the next simplest kind of algebra, namely, in which the codimension one nilradical is isomorphic to a direct sum of a three-dimensional Heisenberg and an abelian Lie algebra. Of the low-dimensional Lie algebras, algebras 7-11 in dimension four [25], algebras 19-29 in dimension five [25] and algebras 13-38 in dimension six [22] are of this type. Ultimately our goal is to find a representation for this kind of algebra in terms of an algebra with a codimension one abelian nilradical. If we quotient by the one-dimensional ideal that is the center of the nilradical we obtain an algebra with a codimension one abelian nilradical or, equivalently, we have a non-split one-dimensional extension of the codimension one abelian nilradical algebra.

Let $\mathfrak{g}$ be a Lie algebra of dimension $n-1$ that has an abelian nilradical ideal of codimension one, that is, of dimension $n-2$. We suppose that a basis for $\mathfrak{g}$ is given by $\{e_2, e_3, ..., e_n\}$ where e_n is not in the nilradical. Then $\mathfrak{g}$ has only the following non-zero brackets and the non-zero structure constants are contained in the matrix

C_i^j:

$$[e_i, e_n] = C_i^j e_j. \quad (2 \leq i, j \leq n)$$

We introduce a new vector e_1 that gives an extended algebra $\mathfrak{g}'$ that will have the following extra brackets:

- $[e_\alpha, e_\beta] = e_1$ for fixed $\alpha \neq \beta$ between 2 and $n-1$ inclusive
- $[e_1, e_i] = 0; 2 \leq i \leq n-1$
- $[e_1, e_{2n}] = \lambda^i e_i; 1 \leq i \leq n-1.$

Next we check the Jacobi identity on this algebra and find the sufficient conditions for the extension.

- $[[e_i, e_j], e_1] + [[e_1, e_i], e_j] + [[e_j, e_1], e_i] = 0$
- $[[e_i, e_\alpha], e_1] + [[e_1, e_i], e_\alpha] + [[e_\alpha, e_1], e_i] = 0$
- $[[e_\beta, e_\alpha], e_1] + [[e_1, e_\beta], e_\alpha] + [[e_\alpha, e_1], e_\beta] = 0$
- $[[e_n, e_\alpha], e_1] + [[e_1, e_n], e_\alpha] + [[e_\alpha, e_1], e_n] = \lambda^\beta e_\beta \Rightarrow \lambda^\beta = 0$
- $[[e_n, e_\beta], e_1] + [[e_1, e_n], e_\beta] + [[e_\beta, e_1], e_n] = \lambda^\alpha e_\alpha \Rightarrow \lambda^\alpha = 0$
- $[[e_\beta, e_\alpha], e_n] + [[e_n, e_\beta], e_\alpha] + [[e_\alpha, e_n], e_\beta] = 0 \Rightarrow \lambda^k e_k - C_\beta^\beta e_1 - C_\alpha^\alpha e_1 = 0$ with a sum over k. So for $i \neq 1; \lambda^i = 0$ and $\lambda = \lambda^1 = C_\alpha^\alpha + C_\beta^\beta$. Hence we can assume that $[e_1, e_n] = \lambda e_1; \lambda = C_\alpha^\alpha + C_\beta^\beta$.
- $[[e_n, e_\alpha], e_i] + [[e_i, e_n], e_\alpha] + [[e_\alpha, e_i], e_n] = C_i^k [e_k, e_\alpha] = 0$. So for $i \neq \alpha, 2 \leq i \leq n$, we have: $C_i^\beta = 0$. Similarly we get:

- $[[e_n, e_\beta], e_i] + [[e_i, e_n], e_\beta] + [[e_\beta, e_i], e_n] = C_i^k[e_k, e_\beta] = 0$ so for $i \neq \beta, 2 \leq i \leq n$, we have, $C_i^\alpha = 0$.

This shows that $ad(e_n)$ must have the form

$$-ad(e_n) = \left(\begin{array}{ccccccccc} C_\alpha^\alpha + C_\beta^\beta & * & * & * & * & * & * & * & * \\ 0 & & & & & & & & * \\ 0 & & & & & & & & * \\ 0 & 0 & 0 & C_\alpha^\alpha & 0 & 0 & C_\beta^\alpha & 0 & 0 \\ 0 & & & & & & & & * \\ 0 & & & & & & & & * \\ 0 & 0 & 0 & C_\alpha^\beta & 0 & 0 & C_\beta^\beta & 0 & 0 \\ 0 & & & & & & & & * \\ 0 & & & & & & & & *. \end{array} \right)$$

Since $ad^*(e_n) = -\big(ad(e_n)\big)^t$, it follows that e_α, e_β must be chosen so that e^α, e^β span an $ad^*(e_n)$-invariant subspace. We summarize the preceding discussion by means of the following Theorem.

Theorem 2.2.1 *Let $\mathfrak{g}$ be a Lie algebra of dimension $n-1$ that has an abelian nilradical ideal of codimension one and such that $ad^*(e_n)$ has a two-dimensional invariant subspace where e_n is not in the nilradical. Then $\mathfrak{g}$ has a non-split one-dimensional extension to an algebra with a codimension one nilradical isomorphic to a direct sum of a three-dimensional Heisenberg and an abelian Lie algebra.*

2.3 Absorption Lemma

Let us apply the theory of the previous section starting with a three-dimensional Lie algebra so as to obtain a four-dimensional Lie algebra that has a three-dimensional Heisenberg nilradical. The extended algebra will have the following brackets and note that e_1 spans the center of the nilradical of the extended algebra, which is an ideal:

$$\begin{aligned}
[e_2, e_3] &= e_1 \\
[e_1, e_4] &= \lambda e_1 \\
[e_2, e_4] &= \alpha_1 e_1 + \beta_1 e_2 + \gamma_1 e_3 \\
[e_3, e_4] &= \alpha_2 e_1 + \beta_2 e_2 + \gamma_2 e_3.
\end{aligned}$$

In this extension process we refer to the following Lemma that has been mentioned in [21] in a much more general form: unfortunately no proof is given and we hope later on to prove the Lemma in that more general form.

Lemma 2.3.1 *Without loss of generality we may assume that* $\alpha_1, \alpha_2 = 0$

Proof: To see that we introduce the change of basis

$$\overline{e_1} = e_1, \overline{e_2} = e_2, \overline{e_3} = e_3, \overline{e_4} = \alpha_2 e_2 - \alpha_1 e_3 + e_4.$$

So the new brackets will be

$$
\begin{aligned}
[\overline{e_2}, \overline{e_3}] &= \overline{e_1} \\
[\overline{e_1}, \overline{e_4}] &= [e_1, \alpha_2 e_2 - \alpha_1 e_3 + e_4] \\
&= \alpha_2 [e_1, e_2] - \alpha_1 [e_1, e_3] + [e_1, e_4] \\
&= \lambda \overline{e_1} \\
[\overline{e_2}, \overline{e_4}] &= [e_2, \alpha_2 e_2 - \alpha_1 e_3 + e_4] \\
&= \alpha_2 [e_2, e_2] - \alpha_1 [e_2, e_3] + [e_2, e_4] \\
&= -\alpha_1 e_1 + \alpha_1 e_1 + \beta_1 e_2 + \gamma_1 e_3 \\
&= \beta_1 \overline{e_2} + \gamma_1 \overline{e_3} \\
[\overline{e_3}, \overline{e_4}] &= [e_3, \alpha_2 e_2 - \alpha_1 e_3 + e_4] \\
&= \alpha_2 [e_3, e_2] - \alpha_1 [e_3, e_3] + [e_3, e_4] \\
&= -\alpha_2 e_1 + \alpha_2 e_1 + \beta_2 e_2 + \gamma_2 e_3 \\
&= \beta_2 \overline{e_2} + \gamma_2 \overline{e_3}
\end{aligned}
$$

$\square$

Remark 2.3.1 *The Lemma clearly has an obvious generalization as follows. Let* $\mathfrak{g}$ *be an algebra of dimension* $2n+2$ *with a basis* $\{e_0, e_i, e_{n+i}, e_{2n+1}\}$ *where* $1 \leq i \leq n$ *and non-zero brackets:*

$[e_i, e_{n+i}] = e_0 \quad (1 \leq i \leq n)$

$[e_i, e_{2n+1}] = C_i^a e_a + C_i e_0 \quad (1 \leq a \leq 2n)$

$[e_{n+i}, e_{2n+1}] = D_i^a e_a + D_i e_0 \quad (1 \leq a \leq 2n).$

The change of basis in which only e_{2n+1} *is modified, being replaced by* $e_{2n+1} + \sum_{i=1}^{n}(D_i e_i - C_i e_{n+i})$*, allows us to assume without loss of generality that* $C_i = D_i = 0$*.*

The Lemma allows us to establish a one-one correspondence between the generic three-dimensional solvable Lie algebras, they have two-dimensional abelian nilradicals, and the algebras of dimension four that have a Heisenberg nilradical. Again the numbering refers to [25].

3D co-dimension one abelian nilradical	4D co-dimension one Heisenberg nilradical
$\underline{A_{3.2}}$ $[e_1, e_3] = e_1$ $[e_2, e_3] = e_1 + e_2$	$\underline{A_{4.7}}$ $[e_2, e_3] = e_1$ $[e_1, e_4] = 2e_1$ $[e_2, e_4] = e_2$ $[e_3, e_4] = e_2 + e_3$
$\underline{A_{3.3}}$ $[e_1, e_3] = e_1$ $[e_2, e_3] = e_2$	$\underline{A_{4.9}; (b = 1)}$ $[e_2, e_3] = e_1$ $[e_1, e_4] = 2e_1$ $[e_2, e_4] = e_2$ $[e_3, e_4] = e_3$
$\underline{A_{3.4}}$ $[e_1, e_3] = e_1$ $[e_2, e_3] = -e_2$	$\underline{A_{4.8}}$ $[e_2, e_3] = e_1$ $[e_2, e_4] = e_2$ $[e_3, e_4] = -e_3$
$\underline{A_{3.5}^a}$ $[e_1, e_3] = e_1$ $[e_2, e_3] = ae_2$	$\underline{A_{4.9}^b}$ $[e_2, e_3] = e_1$ $[e_1, e_4] = (1 + b)e_1$ $[e_2, e_4] = e_2$ $[e_3, e_4] = be_3$

3D co-dimension one abelian nilradical	4D co-dimension one Heisenberg nilradical
$\underline{A_{3.6}}$ $[e_1, e_3] = -e_2$ $[e_2, e_3] = e_1$	$\underline{A_{4.10}}$ $[e_2, e_3] = e_1$ $[e_2, e_4] = -e_3$ $[e_3, e_4] = e_2$
$\underline{A_{3.7}^a}$ $[e_1, e_3] = ae_1 - e_2$ $[e_2, e_3] = e_1 + ae_2$	$\underline{A_{4.11}^a}$ $[e_2, e_3] = e_1$ $[e_1, e_4] = 2ae_1$ $[e_2, e_4] = ae_2 - e_3$ $[e_3, e_4] = e_2 + ae_3$

Remark 2.3.2 *We note that there is only one choice for the Heisenberg bracket when we extend from three to four dimensions. We might have more than one choice in higher dimensions (three choices if we want to extend four to five dimensions). Our choice is subject to the co-adjoint condition discussed in Theorem 2.2.1. Thus the simple pattern exhibited above in extending from three to four dimensions will not be valid in higher dimensions. Nonetheless we hope eventually to develop an inductive that will enable us to analyze algebras that have a co-dimension one nilradical that is isomorphic to the direct sum of a three-dimensional Heisenberg and an abelian Lie algebra.*

Chapter 3

Automorphism group of special Lie algebras

For any Lie algebra $\mathfrak{g}$ its group of automorphisms itself is a Lie group. Its Lie algebra is of the space of derivations of $\mathfrak{g}$ under the operation of commutator and consists of inner automorphisms, namely, essentially the adjoint matrices, and outer automorphisms. We determine the Lie algebra of the automorphism group of certain well known Lie algebras. In this Chapter we have made extensive use of Maple to find the dimension and pattern and for those algebras, even though in the end it plays no direct role in the proofs.

3.1 Automorphism group of the 2n+1 dimensional Heisenberg algebra

Let $\mathcal{H}$ be the $2n+1$ dimensional Heisenberg Lie algebra with basis $\{e_0, e_1, e_2, ..., e_n, e_{n+1}, e_{n+2}, ..., e_{2n}\}$ and nonzero brackets

$$[e_i, e_{i+n}] = e_0, \qquad 1 \leq i \leq n$$

that is, the non-zero structure constants are $C^0_{i,i+n} = 1$ for $1 \leq i \leq n$.

The Lie algebra of the group of automorphisms consist of derivations D that satisfy the Leibnitz rule which can be written in the following three ways:

$$\begin{aligned} D[e_i, e_j] &= [De_i, e_j] + [e_i, De_j] \\ C^k_{ij} D^m_k e_m &= C^m_{kj} D^k_i e_m + C^m_{ik} D^j_k e_m \\ C^k_{ij} D^m_k &= C^m_{kj} D^k_i + C^m_{ik} D^j_k. \end{aligned} \tag{3.1}$$

Now we discuss all the possible cases to determine the pattern of D.

- $m \neq 0$, (3.1) becomes

  $C^0_{ij} D^m_0 = 0 \quad \forall i, j \Rightarrow D^m_0 = 0 \quad \forall m \neq 0$

- $1 \leq i, j \leq n$, (3.1) becomes

  $0 = -D^{j+n}_i + D^{i+n}_j$

  $D^{j+n}_i = D^{i+n}_j$

- $n < i, j \leq 2n$, (3.1) becomes

  $0 = D_i^{j-n} - D_j^{i-n}$

  $0 = D_i^{j-n} = D_j^{i-n}$

- $1 \leq i \leq n < j \leq 2n, j - i \neq n$, (3.1) becomes

  $0 = D_i^{j-n} + D_j^{i+n}$

  $D_i^{j-n} = -D_j^{i+n}$

- $j = i + n$, (3.1) becomes

  $D_0^0 = D_i^i + D_{i+n}^{i+n} \quad 1 < i \leq n.$

Hence the D matrix has the form:

$$\left(\begin{array}{c|c|c}
\lambda & D_2^1 \quad \cdots & \cdots \quad D_{2n+1}^1 \\
\hline
0 & & \\
\vdots & -C^{tr} + \lambda I & B \\
& & \\
\hline
& & \\
\vdots & A & C \\
0 & &
\end{array}\right)$$

where A, B are symmetric $n \times n$ matrices, and C is an arbitrary $n \times n$ matrix.

An example below when $n = 3$, see A.6

$$\left[\begin{array}{c|ccc|ccc}
d_{4,4}+d_{7,7} & d_{1,2} & d_{1,3} & d_{1,4} & d_{1,5} & d_{1,6} & d_{1,7} \\
\hline
0 & d_{4,4}+d_{7,7}-d_{5,5} & -d_{6,5} & -d_{7,5} & d_{2,5} & d_{3,5} & d_{4,5} \\
0 & -d_{5,6} & d_{4,4}+d_{7,7}-d_{6,6} & -d_{7,6} & d_{3,5} & d_{3,6} & d_{4,6} \\
0 & -d_{5,7} & -d_{6,7} & d_{4,4} & d_{4,5} & d_{4,6} & d_{4,7} \\
\hline
0 & d_{5,2} & d_{6,2} & d_{7,2} & d_{5,5} & d_{5,6} & d_{5,7} \\
0 & d_{6,2} & d_{6,3} & d_{7,3} & d_{6,5} & d_{6,6} & d_{6,7} \\
0 & d_{7,2} & d_{7,3} & d_{7,4} & d_{7,5} & d_{7,6} & d_{7,7}
\end{array}\right].$$

Remark 3.1.1 *In the Propositions below we have used 7×7 matrices, but the reason is to demonstrate the pattern when $n = 3$. In fact it can be extended to arbitrary n.*

Proposition 3.1.1 *The dimension of $Aut(\mathcal{H})$=$(n+1)(2n+1)$. In the Lie algebra of $Aut(\mathcal{H})$ there are $2n$ inner automorphisms, namely the $ad(e_i), 1 \leq i \leq 2n$.*

$$\left\{ \begin{bmatrix} 0&1&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0 \end{bmatrix}, \begin{bmatrix} 0&0&1&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0 \end{bmatrix}, \ldots, \begin{bmatrix} 0&0&0&0&0&0&1\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0 \end{bmatrix} \right\}$$

There are $n+1$ semisimple outer automorphisms for which a basis consists of:

$$\left\{ \left[\begin{array}{c|ccc|ccc} 1&0&0&0&0&0&0\\ \hline 0&1&0&0&0&0&0\\0&0&\ddots&0&0&0&0\\0&0&0&1&0&0&0\\ \hline 0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0 \end{array} \right], \left[\begin{array}{c|ccc|ccc} 0&0&0&0&0&0&0\\ \hline 0&1&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\ \hline 0&0&0&0&-1&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0 \end{array} \right], \right.$$

$$\left[\begin{array}{c|ccc|ccc} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right], \ldots, \left.\left[\begin{array}{c|ccc|ccc} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{array}\right]\right\}.$$

The derivations that are neither semisimple, nor inner are of dimension $2n + 4\binom{n}{2}$. *They are of two kinds each of dimensions* $n, 2\binom{n}{2}$, *respectively, and the others come from their transposes:*

- *Tho ones that come from every Heisenberg bracket* $[e_i, e_{i+n}] = e_0, 1 \leq i \leq n$. *Each matrix has a one in the* i^{th} *row and* $i + n^{th}$ *column, and zeros elsewhere, that is, their rank is one.*

$$\left\{\left[\begin{array}{c|ccc|ccc} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right], \ldots, \left[\begin{array}{c|ccc|ccc} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]\right\}$$

- *The other kind are matrices of rank* 2*, they come from a pair of distinct Heisenberg brackets* $[e_i, e_{i+n}] = e_0, [e_j, e_{j+n}] = e_0, 1 \leq i < j \leq n$*. They split into two kinds:*

    1. *The first has* $D^i_j = -1, D^{j+n}_{i+n} = 1$ *and zeros elsewhere:*

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

    2. *The second has* $D^i_{j+n} = 1, D^j_{i+n} = 1$ *and zeros elsewhere:*

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

3.2 Automorphism group of the standard n-dimensional Filiform Lie algebra

Let $\mathfrak{g}$ be the standard n-dimensional filiform Lie algebra, $n \geq 4$, with basis $\{e_1, e_2, ..., e_n\}$ and the following nonzero brackets:

$$[e_1, e_2] = e_3, [e_1, e_3] = e_4, ..., [e_1, e_{n-1}] = e_n.$$

We apply (3.1) to obtain the following conditions:

$$\begin{aligned} D_i^1 &= 0, \quad 2 \leq i \leq n & (3.2) \\ D_{i+1}^{i+1} &= D_1^1 + D_i^i, \quad 2 \leq i \leq n-1 & (3.3) \\ D_{i+1}^m &= D_i^{m-1}, \quad 2 \leq i \leq n-1, 2 \leq m \leq n, m \neq i+1. & (3.4) \end{aligned}$$

Proposition 3.2.1 *There are $n-1$ inner automorphisms, namely $ad(e_i), 1 \leq i \leq n-1$.*

For example when $n = 5$ we have

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Proposition 3.2.2 *There are n outer automorphisms, namely,*

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} D_1^1 & & & & \\ & D_2^2 & & & \\ & & D_1^1 + D_2^2 & & \\ & & & 2D_1^1 + D_2^2 & \\ & & & & 3D_1^1 + D_2^2 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}.$$

Theorem 3.2.1 *The dimension of the Lie algebra of* $Aut(\mathfrak{g})$ *is* $2n-1$*. Moreover the most general matrix of a derivation of* $\mathfrak{g}$ *is of the following form:*

$$\begin{bmatrix} D_1^1 & 0 & 0 & 0 & 0 & 0 \\ D_1^2 & D_2^2 & 0 & 0 & 0 & 0 \\ D_1^3 & D_{n-1}^n & D_1^1 + D_2^2 & 0 & 0 & 0 \\ \vdots & D_{n-2}^n & \ddots & 2D_1^1 + D_2^2 & 0 & 0 \\ D_1^{n-1} & D_3^n & \ddots & D_{n-2}^n & \ddots & 0 \\ D_1^n & D_2^n & D_3^n & D_{n-2}^n & D_{n-1}^n & (n-2)D_1^1 + D_2^2 \end{bmatrix}$$

3.3 Automorphism group of the n-dimensional standard solvable Lie algebra

Let $\mathfrak{g}$ be an n-dimensional solvable algebra, $n \geq 4$, with basis $e_1, e_2, ..., e_n$ and the following nonzero brackets:

$$[e_1, e_2] = e_2 + e_3, [e_1, e_3] = e_3 + e_4, ..., [e_1, e_{n-2}] = e_{n-2} + e_{n-1}, [e_1, e_{n-1}] = e_n.$$

By analogy with the standard filiform algebra of the previous Section we refer to this algebra as the "standard solvable algebra" although our terminology is not standard! We apply (3.1) to get the following conditions:

$$\begin{aligned}
D_i^1 &= 0, \quad 1 \leq i \leq n \\
D_{n-1}^{n-1} &= D_n^n \\
D_i^i &= D_{n-2}^n + D_{n-1}^n + D_n^n, \quad 2 \leq i \leq n-2 \\
D_i^{n-1} &= D_i^n + D_{i+1}^n, \quad 2 \leq i \leq n-2 \\
D_i^{n-2} &= D_i^n + 2D_{i+1}^n + D_{i+2}^n, \quad 2 \leq i \leq n-3 \\
D_{i+1}^{m+1} &= D_i^m, \quad 3 \leq m \leq n-3, 2 \leq i \leq n-3, i \neq m.
\end{aligned}$$

We have solved the previous system of equations and we able to formulate the results in terms of the following Theorem.

Theorem 3.3.1 *The dimension of the Lie algebra of* $Aut(\mathfrak{g}) = 2n - 2$. *Moreover the most general matrix of a derivation of* $\mathfrak{g}$ *is of the following form:*

$$
\begin{pmatrix}
0 & 0 & 0 & & \cdots & & & 0 \\
D_1^2 & A_2 & 0 & 0 & \cdots & & & 0 \\
D_1^3 & A_3 & A_2 & 0 & & & & \\
 & & \ddots & \ddots & \ddots & & & \vdots \\
\vdots & \vdots & \ddots & \ddots & \ddots & 0 & & \\
D_1^{n-2} & A_{n-2} & A_{n-3} & \cdots & A_3 & A_2 & 0 & 0 \\
D_1^{n-1} & B_2 & B_3 & \cdots & B_{n-3} & B_{n-2} & D_n^n & 0 \\
D_1^n & D_2^n & D_3^n & \cdots & D_{n-3}^n & D_{n-2}^n & D_{n-1}^n & D_n^n
\end{pmatrix}
$$

$$
\begin{aligned}
A_2 &= D_{n2}^n + D_{n-1}^n + D_n^n \\
A_2 &= D_{n-3}^n + 2D_{n-2}^n + D_{n-1}^n \\
&\vdots \\
A_{n-2} &= D_2^n + 2D_3^n + D_4^n \\
B_2 &= D_2^n + D_3^n \\
B_3 &= D_3^n + D_4^n \\
&\vdots \\
B_{n-2} &= D_{n-2}^n + D_{n-1}^n.
\end{aligned}
$$

Remark 3.3.1 *In the examples above the derivations are either nilpotent or semi-simple. We are not sure if this property holds in general and we intend to study this question in the future.*

Chapter 4

Hamilton-Jacobi separability and Lie group Lagrangian systems

This Chapter begins with some results about Hamilton-Jacobi separability. It has to be admitted that this topic takes us in a different direction from most of the other material in this dissertation. In the subsequent Section we sketch the main ideas from the geometric formulation of Euler-Lagrange dynamical systems. We also consider the inverse problem for Lagrangian dynamical systems and the very special case of the geodesic equations of the canonical symmetric connection belonging to any Lie group G that we studied in Chapter 1.

4.1 Separability of conformally flat metrics

For a Riemannian manifold, the Levi-Civita criterion for the separability of the Hamilton-Jacobi equation for the Hamiltonian $H(x_i, p_j)$ is that *in the separation coordinates* (x_i, p_i)

$$\frac{\partial H}{\partial p_i}\frac{\partial H}{\partial p_j}\frac{\partial^2 H}{\partial x_i \partial x_j} - \frac{\partial H}{\partial p_i}\frac{\partial H}{\partial x_j}\frac{\partial^2 H}{\partial x_i \partial p_j} - \frac{\partial H}{\partial x_i}\frac{\partial H}{\partial p_j}\frac{\partial^2 H}{\partial p_i \partial x_j} + \frac{\partial H}{\partial x_i}\frac{\partial H}{\partial x_j}\frac{\partial^2 H}{\partial p_i \partial p_j} = 0 \tag{4.1}$$

for $i \neq j$, $i, j = 1, 2, ..., n$ (see for example [16] and we note that (4.1) is symmetric in i and j.

The geometric meaning, if any, of Levi-Civita's criterion is difficult to find. Ideally one would like to have a coordinate-free way to characterize separable systems but we seem to be very far from achieving it.

We will use the the Levi-Civita criterion to prove the following theorem.

Theorem 4.1.1 *The only separable conformally flat metrics are Liouville metrics.*

Proof: The Hamiltonian for a conformally flat metric takes the form

$$H = e^{2f}(p_1^2 + ... + p_n^2) \tag{4.2}$$

where $f = f(x_1, ..., x_n)$. The proof is a brute calculation, showing that (4.1) gives exactly Liouville's conditions on f. We will use the comma, in $f_{,i}$ to denote $\frac{\partial f}{\partial x_i}$. We note that

$\frac{\partial H}{\partial x_i} = 2f_{,i}H = 2f_{,i}e^{2f}(p_1^2 + ... + p_n^2)$

$\frac{\partial^2 H}{\partial x_i \partial x_j} = 2f_{,ji}e^{2f}(p_1^2 + ... + p_n^2) + 4f_{,i}f_{,j}e^{2f}(p_1^2 + ... + p_n^2) = 2e^{2f}(p_1^2 + ... + p_n^2)(f_{,ij} + 2f_{,i}f_{,j})$

$\frac{\partial H}{\partial p_i} = 2e^{2f}p_i$

$\frac{\partial^2 H}{\partial x_i \partial p_j} = 4f_{,i}e^{2f}p_j$ and $\frac{\partial^2 H}{\partial p_j \partial p_i} = 0.$

Substituting these quantities in (4.1) will give

$$\begin{aligned}4e^{4f}p_ip_j(2f_{,ij}e^{2f}(p_1^2+...+p_n^2)+4f_{,i}f_{,j}e^{2f}(p_1^2+...+p_n^2))\\ -16e^{4f}p_if_{,j}(p_1^2+...+p_n^2)f_{,i}e^{2f}p_j-16e^{6f}f_{,i}p_j(p_1^2+...+p_n^2)p_if_{,j} &= 0.\end{aligned}$$

So

$$8e^{6f}p_ip_j(f_{,ij}-2f_{,i}f_{,j})=0$$

and hence we obtain the following condition, which is just what is needed to say that the metric is of Liouville type:

$$f_{,ij}=2f_{,i}f_{,j}.$$

□

Let (M,g) be a pseudo-Riemannian manifold of dimension n. We shall use $|$ to denote covariant derivatives rather a semi-colon. A symmetric tensor L_{ij} is said to be a *special conformal Killing tensor* or SCKT if it satisfies the condition, see for example [7],

$$L_{ij|k}=\frac{1}{2}(f_{|i}g_{jk}+f_{|j}g_{ik}). \tag{4.3}$$

Theorem 4.1.2 *Let K be a Killing vector field for a metric and let α be its dual one-form. Then α^2 satisfies the SCKT equation if and only if α is parallel.*

Proof: It is clear that if α is parallel then α^2 has constant trace f. Indeed $f=g(\alpha,\alpha)$ and so if X is an arbitrary vector field then $Xf=0$. Hence each side of 4.3 zero.

Conversely, we assume now that $L_{ij}=\alpha_i\alpha_j$ satisfies the SCKT equation. Since α is a

Killing one-form we have

$$\alpha_{i|j} + \alpha_{j|i} = 0. \tag{4.4}$$

Taking the covariant derivative of L_{ij} we get

$$L_{ij|k} = \alpha_{i|k}\alpha_j + \alpha_i\alpha_{j|k}.$$

We let f be the trace of L_{ij}, that is,

$$f = g^{mn}L_{mn} = g^{mn}\alpha_m\alpha_m.$$

But L_{ij} satisfies (4.3) so we have

$$2(\alpha_{i|k}\alpha_j + \alpha_i\alpha_{j|k}) = g_{ik}(g^{mn}(\alpha_{m|j}\alpha_n + \alpha_m\alpha_{n|j}) + g_{jk}(g^{mn}(\alpha_{m|i}\alpha_n + \alpha_m\alpha_{n|i})$$

and since we are summing over m and n this equation reduces to

$$\alpha_{i|k}\alpha_j + \alpha_i\alpha_{j|k} = g_{ik}g^{mn}\alpha_{m|j}\alpha_n + g_{jk}g^{mn}\alpha_{m|i}\alpha_n$$

or

$$\alpha_{i|k}\alpha_j + \alpha_i\alpha_{j|k} = g_{ik}\alpha^n_{|j}\alpha_n + g_{jk}\alpha^n_{|i}\alpha_n. \tag{4.5}$$

Permuting i, j and k in (4.5) we obtain

$$\begin{aligned} \alpha_{k|i}\alpha_j + \alpha_k\alpha_{j|i} &= g_{ij}\alpha^n_{|k}\alpha_n + g_{ki}\alpha^n_{|j}\alpha_n \qquad (4.6)\\ \alpha_{i|j}\alpha_k + \alpha_i\alpha_{k|j} &= g_{kj}\alpha^n_{|i}\alpha_n + g_{ij}\alpha^n_{|k}\alpha_n. \qquad (4.7) \end{aligned}$$

Adding (4.5), (4.6) and (4.7), and by using (4.4), we obtain

$$\begin{aligned} 0 &= 2(g_{ik}\alpha^n_{|j}\alpha_n + g_{kj}\alpha^n_{|i}\alpha_n + g_{ij}\alpha^n_{|k}\alpha_n) \\ &= g_{ik}(\alpha^n\alpha_n)_{|j} + g_{kj}(\alpha^n\alpha_n)_{|i} + g_{ij}(\alpha^n\alpha_n)_{|k} \\ &= g_{ik}f_{,j} + g_{kj}f_{,i} + g_{ij}f_{,k}. \end{aligned}$$

Finally we take a trace by multiplying through by g^{ik} to obtain

$$(n+2)f_j = 0$$

and hence

$$f_j = 0.$$

Thus f is constant and hence $L_{ij|k} = 0$. Hence for an arbitrary vector field X, $\nabla_X\alpha^2 = 0$, or

$$\alpha.\nabla_X\alpha = 0.$$

Thus, without loss of generality,

$$\nabla_X\alpha = 0,$$

that is, α is parallel.

$\square$

4.2 Elements of Lagrangian systems

In this section we are going to briefly formulate the theory of Lagrangian systems on a general tangent bundle TM that is not necessarily a Lie group. We use the canonical Jacobi endomorphism S that we introduced in Chapter 1 and the Liouville vector field $\Delta = u^i \frac{\partial}{\partial u^i}$. A Lagrangian $L : TM \to \mathbb{R}$. There is an exact two-form associated to L defined by

$$\omega = d\big(dL \circ S\big).$$

The two-form ω is known as the Cartan two-form of the Euler-Lagrange system. It is a symplectic form if L is non-degenerate. The Cartan two-form determines a vector field Γ, that is called the Euler-Lagrange vector field that is defined by [5, 18, 12]

$$\iota_\Gamma \omega = -d(\Delta L - L).$$

The energy associated with a Lagrangian is defined by

$$E = \Delta L - L$$

where $\Delta = u^i \frac{\partial}{\partial u^i}$.

Another way to look at Lagrangian theory is as the pullback under the inverse of the Legendre transformation in which the canonical two-form on T^*M pulls back to ω on TM.

The inverse problem of Lagrangian dynamics consists of finding necessary and sufficient conditions for a system of second order ordinary differential equations to be the Euler-Lagrange equations of a regular Lagrangian function and in case they are, to describe all possible such Lagrangians. Work had begun on the problem even at the end of the nineteenth century but by far the most important contribution is the 1941 article of Douglas [8].

One aspect of the inverse problem which until recently was little explored is the very special case of the geodesic equations of the canonical symmetric connection belonging to any Lie group G. Thompson has investigated the situation for Lie groups of dimension two and three [29], and together with Ghanam and Miller of dimension four [10]. It was found in [29] that in all these cases the geodesics were the Euler -Lagrange equations of a suitable Lagrangian defined on an open subset of the tangent bundle TG. In the next Section we examine the inverse problem for the canonical Lie group connection focussing on the existence of a metric Lagrangian and the non-uniqueness aspect of the inverse problem.

4.3 Bi-Lagrangian systems

Let us consider the case where $M = G$, a Lie group. The Lie group G acts on itself by left and right translations $L_A, R_A : G \to G$. These actions induce actions on the tangent bundle of the Lie group TG, namely $TL_A, TR_A : TG \to TG$. A Lagrangian $L : TG \to \mathbb{R}$ is said to be right invariant if

$$L \circ TR_A = L \quad \forall A \in G$$

and similarly left invariant if we use left translations. It is called bi-invariant if it is both right and left invariant.

Proposition 4.3.1 *The connection ∇ on G is bi-invariant.*

Proof: Let X, Y be any left-invariant vector fields on G. Consider then by definition:

$$\begin{aligned}
L_*(\nabla_X Y) &= (L_*\nabla)_{L_*X} L_*Y \\
(L_*\nabla)_X Y &= L_*\Big(\frac{1}{2}[X,Y]\Big) \\
&= \frac{1}{2}[L_*X, L_*Y] \\
&= \frac{1}{2}\,[X,Y]
\end{aligned}$$

where the penultimate equality follows from Proposition 1.1.2, which implies that $(L_*\nabla) = \nabla$, since ∇ is unique. Hence ∇ is left-invariant. Similarly the connection ∇ is right-invariant, hence we have the result.

□

Remark 4.3.1 *Notice that the factor of $\frac{1}{2}$ could be replaced by any other constant but only the value of $\frac{1}{2}$ gives a connection with zero torsion. The fact that the connection is unique* *follows from a series of difficult exercises in [13].*

Corollary 4.3.2 *If φ is a left or right translation then the geodesic spray Γ is preserved under $T\varphi$, that is*

$$(T\varphi)_*\Gamma = \Gamma$$

and $(T\varphi^{-1})^*L$ (and for that matter $(T\varphi)^*L$) is also a Lagrangian for Γ.

□

Considering $(T\varphi)^*L$ as a Lagrangian for ∇, we have four possibilities:

- L is left invariant; that is $(T\varphi)^*L = L$ for all left translations.
- L is right invariant; that is $(T\varphi)^*L = L$ for all right translations.
- L is bi-invariant; that is $(T\varphi)^*L = L$ for all right and left translations.
- $(T\varphi)^*L \neq L$ which is also interesting because then we will have an alternative Lagrangian for Γ.

In other words the group G acts on the left and right on the space of Lagrangians for Γ. If the fourth possibility occurs then Γ will be an example of a bi-Lagrangian system or perhaps, more accurately, a multi-Lagrangian system. Such a situation represents rather extreme phenomena within in the context of the inverse problem.

We take up next the question of when ∇ is the Levi-Civita connection of a metric. It is known that in the case where the metric is Riemannian the necessary and sufficient conditions for a group G to admit a metric is that G should be the product of a compact and an abelian group [20]. More generally one can pose the question of whether a given connection, not necessarily the canonical connection, is the Levi-Civita connection of some metric. The answer is provided by the following Theorem [9, 29].

Theorem 4.3.1 *The necessary and sufficient conditions for a connection to be a Levi-Civita connection are that the following system of linear equations for unknown g stabilize and that it admit a non-singular solution, R denoting the curvature tensor of the connection:*

$$gR + (gR)^t = 0 \tag{4.8}$$

$$g\nabla R + (g\nabla R)^t = 0 \tag{4.9}$$

$$g\nabla^2 R + (g\nabla^2 R)^t = 0. \tag{4.10}$$

$$\cdots$$

□

In Theorem 4.3.1 the symbol gR would be written in coordinates as $g_{im}R^m_{jkl}$ and ∇R would be denoted $R^m_{jkl|n}$. If the hypotheses of the theorem hold one obtains the family of all possible compatible metrics by integrating a Frobenius-integrable distribution. For the canonical Lie group connection only the conditions from the curvature itself are important because the covariant derivatives are zero. We used MAPLE to implement these conditions for a class of six-dimensional Lie algebras described below. Notice that the every curvature matrix in the Lie algebra of the holonomy group must have even rank if there is going to be a metric associated to a particular algebra. All the examples we are aware of of bi-invariant Lagrangians are metric Lagrangians [23].

An example that can be found in [10] is the four-dimensional algebra $A_{4,8}$ for which the non-zero brackets are:

$$[e_2, e_3] = e_1, \quad [e_2, e_4] = e_2, \quad [e_3, e_4] = -e_3.$$

In this example and the others that follow we use a matrix S to denote the group representation:

$$S = \begin{bmatrix} 1 & 0 & xe^{w} & y \\ 0 & e^{-w} & 0 & x \\ 0 & 0 & e^{w} & z \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

The right invariant forms are given by $dw, dx + xdw, dz - zdw$ and $dy - zdx - xzdw$ and the corresponding right invariant vector fields are given by

$$W = \frac{\partial}{\partial w} - x\frac{\partial}{\partial x} + z\frac{\partial}{\partial z}, \quad X = \frac{\partial}{\partial x} + z\frac{\partial}{\partial y}, \quad Y = \frac{\partial}{\partial y}, \quad Z = \frac{\partial}{\partial z}.$$

The geodesic equations are

$$\ddot{w} = 0, \quad \ddot{x} = -\dot{w}\dot{x}, \quad \ddot{y} = \dot{z}(\dot{x} + x\dot{w}), \quad \ddot{z} = \dot{z}\dot{w}.$$

It turns out that a metric is given by

$$g = xdwdz - dydw + dxdz.$$

Notice, however, that g can be re-written in either of the following two equivalent forms using left and right invariant forms, respectively:

$$g = dw(xdz - dy) + (e^{w}dx)(e^{-w}dz)$$

$$g = (dz - zdw)(xdw + dx) - dw(dy - zdx - xzdw).$$

Even if a metric exists for the canonical connection on a Lie group it may not be bi-invariant. In fact if the metric is bi-invariant then each left or right invariant vector field is a Killing vector field. We have examined Mubarakzyanov's list of 99 classes of solvable codimension one nilradical six dimensional Lie algebras [22]. Examining all 99 cases required considerable effort and was complicated by the fact that the algebras depend on parameters that have to be handled interactively. Their group representations are given in [14]. We found only five algebras that have a metric, namely, $A_{6.82}, A_{6.89}, A_{6.90}, A_{6.91}, A_{6.93}$ with the value $\alpha = 0$. Here they are discussed in more detail.

$\underline{\mathbf{A}_{\mathbf{6.82}}^{\alpha,\lambda,\lambda_1}(\alpha = \mathbf{0})}$: The nonzero brackets are, following Mubarakzyanov's notation here and in the other cases: $[e_2, e_4] = e_1, [e_3, e_5] = e_1, [e_2, e_6] = \lambda e_2, [e_3, e_6] = \lambda_1 e_3, [e_4, e_6] = \lambda e_4, [e_5, e_6] = \lambda_1 e_5$ and

$$S = \begin{bmatrix} 1 & e^{\lambda w} z & e^{\lambda_1 w} q & 0 & 0 & p \\ 0 & e^{\lambda w} & 0 & 0 & 0 & x \\ 0 & 0 & e^{\lambda_1 w} & 0 & 0 & y \\ 0 & 0 & 0 & e^{-\lambda w} & 0 & z \\ 0 & 0 & 0 & 0 & e^{\lambda_1 w} & q \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

The right-invariant vector fields are: $\{D_p, D_x, D_y, D_z + xD_p, D_q + yD_p, -\lambda_1 q D_q + \lambda x D_x + \lambda_1 y D_y - \lambda z D_z + D_w\}$. The metric as a matrix is given by

$$g = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & -a\lambda_1 \\ 0 & 0 & 0 & \frac{a\lambda_1}{\lambda} & 0 & 0 \\ 0 & 0 & 0 & 0 & a & 0 \\ 0 & \frac{a\lambda_1}{\lambda} & 0 & 0 & 0 & 0 \\ 0 & 0 & a & 0 & 0 & 0 \\ -a\lambda_1 & 0 & 0 & 0 & 0 & b \end{bmatrix}$$

where a, b are arbitrary constants. The Lagrangian is

$$L = a\dot{q}\dot{y} - a\lambda_1\dot{p}\dot{w} + a\lambda_1 q\dot{y}\dot{w} + \frac{a\lambda_1}{\lambda}\dot{x}\dot{z} + a\lambda_1 z\dot{x}\dot{w} + b\dot{w}^2$$

and the geodesics are

$$\ddot{p} = \dot{q}\dot{y} + \dot{x}\dot{z} + (\lambda(z\dot{x} + x\dot{z}) + \lambda_1(y\dot{q} + q\dot{y}))\dot{w}, \ddot{x} = \lambda\dot{x}\dot{w}, \ddot{y} = \lambda_1\dot{y}\dot{w}, \ddot{z} = -\lambda\dot{z}\dot{w}, \ddot{q} = -\lambda_1\dot{q}\dot{w}, \ddot{w} = 0.$$

Remark 4.3.2 *The metric g is not bi-invariant since*

$$\mathcal{L}_{-xD_p + D_z} g = a\lambda_1 dx dw \neq 0.$$

$\underline{\mathbf{A_{6.89}^{\alpha s\nu_0}}(\alpha=\mathbf{0})}$: The nonzero brackets are $[e_2,e_4]=e_1,[e_3,e_5]=e_1,[e_2,e_6]=se_2,[e_3,e_6]=\nu_0 e_5,[e_4,e_6]=-se_4,[e_5,e_6]=-\nu_0 e_3$. The group representation S is given by

$$S=\begin{bmatrix} 1 & 0 & xe^{sw} & -\cos(w\nu_0)\,q-p\sin(w\nu_0) & -\sin(w\nu_0)\,q+p\cos(w\nu_0) & y \\ 0 & e^{-sw} & 0 & 0 & 0 & x \\ 0 & 0 & e^{sw} & 0 & 0 & z \\ 0 & 0 & 0 & \cos(w\nu_0) & \sin(w\nu_0) & p \\ 0 & 0 & 0 & -\sin(w\nu_0) & \cos(w\nu_0) & q \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

The right-invariant vector fields are: $\{2D_y,2D_z,-D_p-qD_y,D_x+zD_y,D_q-pD_y,\nu_0 qD_p-\nu_0 pD_q-sxD_x+szD_z+D_w\}$.

The metric is given by

$$g=\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & -as \\ 0 & 0 & 0 & a & 0 & 0 \\ 0 & 0 & \frac{as}{\nu_0} & 0 & 0 & 0 \\ 0 & a & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{as}{\nu_0} & 0 \\ -as & 0 & 0 & 0 & 0 & b \end{bmatrix}.$$

The Lagrangian is

$$L=\frac{as}{\nu_0}\dot{p}^2-asq\dot{p}\dot{w}+\frac{as}{\nu_0}\dot{q}^2+asp\dot{q}\dot{w}+a\dot{x}\dot{z}-as\dot{y}\dot{w}+asx\dot{z}\dot{w}+b\dot{w}^2$$

and the geodesics are

$$\ddot{p} = \nu_0 \dot{q}\dot{w}, \ddot{q} = -\nu_0 \dot{p}\dot{w}, \ddot{x} = -s\dot{x}\dot{w}, \ddot{y} = -\nu_0 p\dot{p}\dot{w} - \nu_0 q\dot{q}\dot{w} + \dot{x}\dot{z} + sx\dot{z}\dot{w}, \ddot{z} = s\dot{z}\dot{w}, \ddot{w} = 0.$$

Remark 4.3.3 *The metric g is not bi-invariant since*

$$\mathcal{L}_{D_x + zD_y} g = adqdz \neq 0.$$

$\underline{\mathbf{A}^{\alpha\nu_0}_{\mathbf{6.90}}(\alpha = \mathbf{0}, \nu_\mathbf{0} \neq \mathbf{1}):}$ The nonzero brackets are: $[e_2, e_4] = e_1, [e_3, e_5] = e_1, [e_2, e_6] = e_4, [e_3, e_6] = \nu_0 e_5, [e_4, e_6] = e_2, [e_5, e_6] = -\nu_0 e_3$.

The group representation S is given by

$$S = \begin{bmatrix} 1 & 0 & e^w x & -\cos(w\nu_0)\, q - p\sin(w\nu_0) & -\sin(w\nu_0)\, q + p\cos(w\nu_0) & y \\ 0 & e^{-w} & 0 & 0 & 0 & x \\ 0 & 0 & e^w & 0 & 0 & z \\ 0 & 0 & 0 & \cos(w\nu_0) & \sin(w\nu_0) & p \\ 0 & 0 & 0 & -\sin(w\nu_0) & \cos(w\nu_0) & q \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

The right-invariant vector fields are: $\{2D_y, -2^{\frac{1}{2}}D_x - 2^{-\frac{1}{2}}xD_y + 2^{-\frac{1}{2}}D_z, D_q - 2pD_y, 2^{\frac{1}{2}}D_x - 2^{-\frac{1}{2}}xD_y + 2^{-\frac{1}{2}}D_z, D_p, \nu_0 qD_p - \nu_0 pD_q - xD_x + (\nu_0 p^2 - \nu_0 q^2)D_y + zD_z + D_w\}$. The metric is given by

$$g = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & -a\nu_0 \\ 0 & -a\nu_0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a & 0 & 0 & 0 \\ 0 & 0 & 0 & a\nu_0 & 0 & 0 \\ 0 & 0 & 0 & 0 & a & 0 \\ -a\nu_0 & 0 & 0 & 0 & 0 & b \end{bmatrix}.$$

The Lagrangian is

$$L = a\dot{p}^2 - 2a\nu_0 q\dot{p}\dot{w} + a\dot{q}^2 + a\nu_0\dot{x}\dot{z} - a\nu_0 z\dot{x}\dot{w} - a\nu_0\dot{y}\dot{w} + b\dot{w}^2$$

and the geodesics are

$$\ddot{p} = \nu_0\dot{q}\dot{w}, \ddot{q} = -\nu_0\dot{p}\dot{w}, \ddot{x} = -\dot{x}\dot{w}, \ddot{y} = -2\dot{p}\dot{q} - 2\nu_0 q\dot{q}\dot{w} - \dot{x}\dot{z} + z\dot{x}\dot{w}, \ddot{z} = \dot{z}\dot{w}, \ddot{w} = 0.$$

Remark 4.3.4 *The metric g is not bi-invariant since*

$$\mathcal{L}_{D_q - 2pD_y} g = -2a\nu_0(dpdy + dydp) \neq 0.$$

$\underline{\mathbf{A_{6.91}}}$: The nonzero brackets are $[e_2, e_4] = e_1, [e_3, e_5] = e_1, [e_2, e_6] = e_4, [e_3, e_6] = e_5, [e_4, e_6] = e_2, [e_5, e_6] = -e_3$. The group representation S is given by

$$S = \begin{bmatrix} 1 & y & -x & q & -z & p \\ 0 & e^w & 0 & 0 & 0 & x \\ 0 & 0 & e^{-w} & 0 & 0 & y \\ 0 & 0 & 0 & \cos(w) & -\sin(w) & z \\ 0 & 0 & 0 & \sin(w) & \cos(w) & q \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

The right-invariant vector fields: $\{4D_p, -2xD_p + D_x - D_y, 2^{\frac{1}{2}}D_q, 2xD_p + D_x + D_y, 2^{\frac{3}{2}}qD_p - 2^{\frac{1}{2}}D_z, (-z^2 + q^2)D_p + zD_q + xD_x - yD_y - qD_z + D_w\}$. The metric is given by

$$g = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & -a \\ 0 & -a & 0 & 0 & 0 & 0 \\ 0 & 0 & a & 0 & 0 & 0 \\ 0 & 0 & 0 & a & 0 & 0 \\ 0 & 0 & 0 & 0 & a & 0 \\ -a & 0 & 0 & 0 & 0 & b \end{bmatrix}$$

The Lagrangian is

$$L = -a\dot{p}\dot{w} + a\dot{q}^2 - 2az\dot{q}\dot{w} + 2ay\dot{x}\dot{w} + 2a\dot{x}\dot{y} + b\dot{z}^2 + b\dot{w}^2$$

and the geodesics are

$$\ddot{p} = -2\dot{z}\dot{y} + 2\dot{x}\dot{y} + 2y\dot{x}\dot{w} - 2z\dot{z}\dot{w}, \ddot{q} = \dot{z}\dot{w}, \ddot{x} = \dot{x}\dot{w}, \ddot{y} = -\dot{y}\dot{w}, \ddot{z} = -\dot{q}\dot{w}, \ddot{w} = 0.$$

Remark 4.3.5 *The metric g is not bi-invariant since*

$$\mathcal{L}_{-2xD_p+D_x-D_y} g = -2adxdw \neq 0.$$

$\underline{\mathbf{A_{6.93}}(\alpha = 0) :}$ The nonzero brackets are $[e_2, e_4] = e_1, [e_3, e_5] = e_1, [e_2, e_6] = e_4 + \nu_0 e_5, [e_3, e_6] = \nu_0 e_4, [e_4, e_6] = e_2 - \nu_0 e_3, [e_5, e_6] = -\nu_0 e_2$.

We showed the existence of a metric g using only the Lie algebra. Unfortunately we do not have a group or even a vector field representation for the algebra [14], so we are unable to compute the metric concretely in coordinates.

Remark 4.3.6 *The algebra $A_{6.91}(\alpha = 0)$ can be thought of a special case of the algebra $A_{6.90}(\alpha = 0)$ if we let $\nu_0 = 1$*

Remark 4.3.7 *The algebra $A_{6.90}(\alpha = 0)$ is isomorphic the algebra $A_{6.89}(\alpha = 0)$ if we use the change of basis*

$$\overline{e_2} = \frac{1}{\sqrt{2}}\Big(e_2 + e_4\Big), \overline{e_4} = \frac{1}{\sqrt{2}}\Big(-e_2 + e_4\Big)$$

Suppose that ∇ is the Levi-Civita connection of a metric that is not bi-invariant. Then clearly it is possible to apply a Legendre transformation. As such one would obtain a multi-Hamiltonian description of the corresponding Hamiltonian system. It remains also to study the complete integrability properties of these systems.

We conclude with the example of the Heisenberg group

$$S = \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix}.$$

The geodesic equations are easily found to be

$$\ddot{x} = 0, \qquad \ddot{y} = 0, \qquad \ddot{z} = \dot{x}\dot{y}.$$

The connection is flat as can easily be seen by defining $\bar{z} = z - \frac{xy}{2}$. As such a metric is given by

$$g = dx^2 + dy^2 + (dz - \frac{y}{2}dx - \frac{x}{2}dy)^2$$

and the corresponding Lagrangian is given by $L = \dot{x}^2 + \dot{y}^2 + (\dot{z} - \frac{y}{2}\dot{x} - \frac{x}{2}\dot{y})^2$. On the other hand if we act on the right by an element of the group

$$\begin{bmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix}$$

we find the following transformed but equivalent Lagrangian:

$$L = \dot{x}^2 + \dot{y}^2 + (\dot{z} + b\dot{x} - \frac{y+b}{2}\dot{x} - \frac{x+a}{2}\dot{y})^2.$$

where a, b and c are constants.

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Appendix A

Maple Routines

We are listing some Maple procedures that were written during the course of research.

A.1 Levi-Civita

A procedure to implement the Levi-Civita separability criterion for a Hamiltonian function H. The inputs are H and the dimension of the space n.

```
> LeviCivita:=proc(H,n) local A,alg_cond,i,j;
> alg_cond:=[ ];
> for i from 1 to n do
> for j from i+1 to n do
> A:=simplify(diff(H,p||i)*diff(H,p||j)*diff(H,x||i,x||j)
>  -diff(H,p||i)*diff(H,x||j)*diff(H,x||i,p||j)
>  -diff(H,x||i)*diff(H,p||j)*diff(H,p||i,x||j)
>  +diff(H,x||i)*diff(H,x||j)*diff(H,p||i,p||j));
> if not(A  in alg_cond) then <br> alg_cond:=[op(alg_cond), A];
> fi;
> od;
> od;
> map(factor,alg_cond);
> end:
```

A.2 Euler Lagrange Equation

This is a procedure to find the Euler-Lagrange equations

$$\frac{d}{dt}\Big(\frac{\partial L}{\partial u^i}\Big) = \frac{\partial L}{\partial x^i}$$

for a given Lagrangian function $L(x^i, u^i)$, where $u^i = \dot{x}^i$, and $i = 1, 2, ..., n$. The inputs are $L(x^i, u^i), n$ and the output consists of the geodesics in the form $f^i = \ddot{x}^i$

```
> EL:= proc(L,n) local i,Z,eq,soln:
> for i from 1 to n do
> Z[i]:=add(u||m*diff(L,x||m,u||i)+
>  f||m*diff(L,u||m,u||i),m=1..n)-diff(L,x||i)
> od:
> eq:=convert(Z,set):
> soln:=solve(eq,seq(f||i,i=1..n)):
> assign(soln):
> end:
```

A.3 L_forms

This procedure produces a basis for the right invariant Maurer-Cartan forms, for a Lie group with matrix group representation S. The input for this procedure is the matrix $dS.S^{-1}$. This is important because later we want to use the forms to get their dual vector fields and hence a Lie algebra vector field representation.

```
> L_forms:=proc(V) local n,Z,T,V_to_vector,k,i,j,vector_to_coefficient,cond,C,C1;
> n:=frameBaseDimension();
> V_to_vector:=Matrix(n^2,1):
> k:=0:
> for i from 1 to n do
> for j from 1 to n do
> k:=k+1;
> V_to_vector[k,1]:=V[i,j]:
> od:
> od:
> vector_to_coefficient:=array(1..(n^2),1..n):
> for i from 1 to (n^2) do
> for j from 1 to n do
> vector_to_coefficient[i,j]:=coeff_list(V_to_vector[i,1],[[j]]);
> od:
> od:
> cond:=[]:
> for i from 1 to (n^2) do
> C:=convert((linalg[row](vector_to_coefficient,i)),'list'):
> Z:=convert(Vector[row](n),'list');
> C1:=v_zip( C,frameBaseForms(),plus):
```

```
> if ((C <>Z) and (not(C1 in cond)))  then
> if nops(cond)=0 then
> cond:=[op(cond), C1];
> elif
> linear_combo(C1,cond)=[] then
> cond:=[op(cond), C1];
> fi:
> fi:
> od:
> convert(cond,'list');
> end:
```

A.4 dual_vectors

This procedures takes a list of vector fields in Vessiot format and gives the dual 1-forms.

```
> dual_vectors:=proc(V) local
> n,i,j,vector_to_coefficient,G,New_form;
> n:=frameBaseDimension();
> vector_to_coefficient:=array(1..n,1..n);
> for i from 1 to n do
> for j from 1 to n do
> vector_to_coefficient[j,i]:=coeff_list(V[i],[[j]]):
> od:
> od:
> G:=convert(inverse(vector_to_coefficient),Matrix):
> for i from 1 to n do
> New_form[i]:=v_zip( convert(LinearAlgebra[Row](G,i),'list'),frameBaseForms(),plus)
> od:
> convert(New_form,'list');
> end:
```

A.5 L_vectors

This procedure takes a list of one-forms in Vessiot format and gives a list of the dual vector fields.

```
> L_vectors:=proc(F) local n,i,j,vector_to_coefficient,G,Vec;
> n:=frameBaseDimension();
> vector_to_coefficient:=array(1..n,1..n);
> for i from 1 to n do
> for j from 1 to n do
> vector_to_coefficient[i,j]:=coeff_list(F[i],[[j]]):
> od:
> od:
> G:=convert(inverse(vector_to_coefficient),'Matrix'):
> for i from 1 to n do
> Vec[i]:=evalV(add(LinearAlgebra[Column](G,i)[k]
> *frameBaseVectors()[k],k=1..n));
> od:
> convert(Vec,'list');
> end:
```

A.6 Heisenberg automorphism group

In this section we outline a routine written to implement the algebraic conditions of equation (3.1). In this example we have the 7 dimensional Heisenberg algebra but in fact we can specify the choice of the structure constants C^i_{jk} and the dimension of the algebra n.

```
> restart;
> n:=7:
> d:=matrix(n,n):
> C:=array(1..n,1..n,1..n):
> for i from 1 to n do
> for j from 1 to n do
> for k from 1 to n do C[i,j,k]:=0:  od:
> od:
> od:
> C[2,5,1]:=1:C[5,2,1]:=-1:C[3,6,1]:=1:
> C[6,3,1]:=-1:C[4,7,1]:=1:C[7,4,1]:=-1:
> eq:=:
> for i from 1 to n do
> for j from 1 to n do
> for m from 1 to n do
> eq1:=add(C[i,j,k]*d[m,k],k=1..n)=add(C[k,j,m]*d[k,i],k=1..n)
>       +add(C[i,k,m]*d[k,j],k=1..n);
> eq:=op(eq),eq1;
> od;
> od;
> od;
> D_var:=:
> for i from 1 to n do
> for j from 1 to n do <br>D_var:=op(D_var),d[i,j];
> od;
> od;
> solve(eq,D_var):
> assign(op(%));
> print(d);
```

A.7 Forms to Geodesics

This procedure takes the input from L_forms routine or you can input your list of right invariant forms in Vessiot format as a list. We call it "T" The output is a list of geodesics where u_i is the time derivative of the corresponding Frame Independent Variable specified in *coord_in_it* at the beginning of your maple worksheet.

```
> Forms_to_Geo:=proc(T) local n,forms_to_coefficient,i,j,u,F,G,eq;
> n:=frameBaseDimension();
> forms_to_coefficient:=array(1..n,1..n);
> for i from 1 to n do
> for j from 1 to n do
> forms_to_coefficient[i,j]:=coeff_list(T[i],[[j]]):
> od:
> od:
> u:=[u||1,u||2,u||3,u||4,u||5,u||6]:
> F:=evalm(forms_to_coefficient&*u);
> for i from 1 to n do
> G[i]:=simplify((add(u||m*diff(F[i],frameIndependentVariables()[m]),m=1
..n))+(add(f||m*diff(F[i],u||m),m=1..n)));
> od:
> eq:=convert(G,set);
> solve(eq,{seq(f||i,i=1..n)}):
> end:
```