Pricing Financial Derivatives Using Stochastic Calculus

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Mathematics

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1. Introduction

A financial derivative is a contract whose value is determined, or “derived”, from the price of a different financial asset, known as the underlying asset of the financial derivative. The underlying asset can be any type of financial asset, but common ones include stocks, indices, commodities, currencies, interest rates, and bonds. Financial derivatives are used by all financial institutions as well as companies that operate in different industries.

Financial derivatives have two major purposes: hedging and speculation. Hedging means minimizing or eliminating the risk of the price movement of an asset. This is done by taking a position in an asset that moves in the opposite direction of the asset being hedged, such as a financial derivative whose value increases when the price of the underlying asset decreases. Investors who are using financial derivatives for hedging purposes are looking to reduce exposure to the underlying asset.

For example, the majority of an airline company’s expense come from fuel costs so as the price of oil rises, the company becomes less profitable. In order to reduce the risk that its costs will increase, the company could purchase a financial derivative that increases in value when the price of oil rises. In the event that the price of oil increases, the airline’s expenses will increase but the losses will be at least partially offset by the increase in the value of the derivative contract.

Speculation, on the other hand, is when an investor wants to increase exposure to an asset. There are many reasons that an investor would choose to gain exposure to an asset by purchasing a financial derivative rather than the underlying asset. The most common reason is that financial derivatives can increase leverage allowing a small move in the price of the underlying asset to provide a much larger movement in the price of the derivative. Another reason is that the upfront cost of a derivative contract is a fraction of the cost of the underlying asset, allowing an investor to gain exposure to an asset without incurring a large cost.

Although financial derivatives are commonly used by businesses in all industries, pricing financial derivatives remains difficult. The pricing of a derivative is not straightforward because its value is a function of the price of the underlying asset, which is inherently random and therefore unpredictable. This randomness creates the need for mathematical modeling of the underlying asset.

The goal of this paper is to show the significance of the mathematics involved in pricing financial derivatives. It is interesting that a product that was originally developed for practical reasons posed such a difficulty in pricing that a variety of branches of mathematics including probability, differential equations, and stochastic calculus are necessary.

This exploration of the pricing of financial derivatives will begin with the stochastic modeling of asset prices, which is used to model the price movement of a financial derivative's underlying asset. Ito's lemma will then be applied to derive the partial differential equation that models the movement of the price of the financial derivative. The analytical solution to the partial differential equation for European call options will be found using a transformation of variables and the solution of the well-studied heat equation. In addition to an analytical solution, two alternative methods for pricing financial derivatives, a binomial tree model and Monte Carlo simulations, are explored. The flexibility, complexity, and accuracy of each of these methods are also analyzed.

2. Stochastic Asset Pricing Model

The price of a financial derivative is dependent on the price of the underlying asset, which is inherently uncertain. If the movement of a financial security could be accurately predicted, it would obtain the fair price immediately, eliminating the possibility of profiting from its movement. Therefore, there must be a random component in the model for the price of a security.

In order to create this model that allows for random movement of asset prices, some simplifying assumptions about the assets must be made. The first assumption is that the returns on an asset are lognormally distributed. The lognormal distribution is chosen because its properties work well with the types of calculations that will be necessary in modeling the price of an asset over time. For example, the lognormal distribution is stable under multiplication. This is important because when calculating cumulative returns, many return factors are multiplied together. Because of this stability under multiplication, the cumulative returns will also follow a lognormal distribution. Therefore, returns will be lognormally distributed over all subdivisions of the time interval T.

Another assumption that we make is that an asset's prices can be modeled using a stochastic process. A stochastic process is a random process indexed by time. Stochastic processes are used to describe the distribution of the possible values that a variable can take on after a certain amount of time. The specific stochastic process that will be used is called geometric Brownian motion, the details of which will be addressed later.

In order to illustrate why a stochastic model is chosen instead of a deterministic model, consider the following example. Suppose the return on an asset after one time period is e^r, where r is a random variable that follows a normal distribution with mean μ and variance σ^2. Therefore, $e^r \sim \ln N(\mu, \sigma^2)$ follows a lognormal distribution with mean μ and variance σ^2. The price of an asset, with a known initial price of S_0, after one period is then

$$S_1 = S_0 e^r$$

And similarly, S_0 can be solved for by dividing by e^r

$$S_0 = \frac{S_1}{e^r}$$

Because we have assumed that $e^r \sim \ln N(\mu, \sigma^2)$, the expected value of S_1 is

$$E(S_1) = \mathrm{E}(S_0 e^r) = S_0 E(e^r) = S_0 e^{\mu + \frac{\sigma^2}{2}}$$

This shows that the rate that an asset is expected to accrue at is $e^{\mu+\frac{\sigma^2}{2}}$. The properties of the lognormal distribution tell us that $\frac{1}{e^r}$ ~ lnN(-μ, σ^2). Then the expected discounted value of S_1 is

$$E(S_0) = E\left(\frac{S_1}{e^r}\right) = S_1 E\left(\frac{1}{e^r}\right) = \frac{S_1}{e^{-\mu+\frac{\sigma^2}{2}}} = S_1 e^{\mu-\frac{\sigma^2}{2}}$$

Which shows that the asset is discounted at a rate of $e^{\mu-\frac{\sigma^2}{2}}$, a different rate than the asset accrues at. This simple example shows the basic reasoning behind the need to model asset prices with stochastic processes rather than deterministic models.

2.1 Asset Price Model

Let the price of an asset be represented by

$$S_t = S_0 e^{\mathrm{r}t+\sigma W_t} \tag{1}$$

where W(t) ~ N(0,t). In this representation, $S_0 e^{rt}$ is the deterministic, or nonrandom, component of the equation. $\sigma W(t)$ is the stochastic component that provides the random movement necessary to account for the difference in the accrual and discounting rate. This is the most general representation of a stochastic process.

To find the distribution of the return on an asset that moves according to (1), the equation can be rearranged to show that.

$$\ln\left(\frac{S_t}{S_0}\right) = \mathrm{r}t + \sigma W_t$$

Because W(t) ~ N(0,t), it follows that $\mathrm{r}t + \sigma W_t \sim \mathrm{N}(\mathrm{r}t, \sigma^2 t)$. Therefore, because $\ln\left(\frac{S_t}{S_0}\right) = \mathrm{r}t + \sigma W_t$, then $\ln\left(\frac{S_t}{S_0}\right) \sim \mathrm{N}(\mathrm{r}t, \sigma^2 t)$. This shows that the continuous interest rate adjusted for time is normally distributed.

Taking the derivative of (1) with respect to time give the differential equation

$$\frac{dS_t}{dt} = S_0 e^{\mathrm{r}t + \sigma W_t}(\mathrm{r} + \sigma \frac{dW_t}{dt})$$

Note that $S_0 e^{\mathrm{r}t + \sigma W_t} = S_t$ so the equation can be rewritten as

$$\frac{dS_t}{dt} = S_t(\mathrm{r} + \sigma \frac{dW_t}{dt})$$

After multiplying by dt and dividing by S_t, the resulting equation is

$$\frac{dS_t}{t} = \mathrm{r}dt + \sigma dW_t \qquad (2)$$

This is the equation for geometric Brownian motion, equation that will be used for the change in the price of an asset.

2.2 History of Brownian Motion in Derivative Pricing Theory

The first research to suggest using stochastic processes to model the price movement of financial securities was done by Bachelier in 1900. In his doctoral thesis, Theory of Speculation, it was proposed that Brownian motion be used to model future asset price movements. While this was the first breakthrough in using rigorous mathematical methods to price financial derivatives, there remained some flaws in the theory. The most glaring being that the arithmetic Brownian motion model allowed securities to take on a negative value, which is not possible in real markets. This ability for prices to be negative is due to the assumption that stock prices follow a normal distribution. Despite this revolutionary suggestion, Bachelier's work was lost in time. It was not cited in any of the later work done by Black, Scholes, and Merton in the 1970s who would eventually be named the founders of derivative pricing. How this great work went unrecognized for so long remains a mystery.

In order to correct for the possibility of negative stock prices, Samuelson used a modified version of Bachelier's model by letting future stock prices follow a lognormal, rather than a normal, distribution. This version of Brownian motion is called geometric Brownian motion. In this model, changes in price are proportional to the assets price rather than using absolute price changes as in Bachelier's model. The proportional rather than absolute change is where the name geometric Brownian motion comes from.

A number of attempts were made to price financial derivatives but all of the methods either required unrealistic assumptions or were dependent on arbitrary parameters. It was not until Black, Scholes, and Merton applied Ito's Lemma, which will be discussed in the next section, that an unbiased derivative pricing method was discovered. This unbiased model is known as the Black-Scholes equation, one of the most famous equations in finance.

The geometric Brownian motion modeling of asset prices is far from a perfect representation of reality. The stochastic component of an asset price follows a normal distribution making large swings in modeled prices very unlikely. However, in real financial markets large swings in price are more common than a normal distribution would predict. Turner and Weigel found that price drops of more than three standard deviations from the mean are three times more likely in financial markets than a normal distribution would predict. Another issue is that the model assumes a constant volatility. This does not necessarily hold true in practical applications. Despite the presence of issues, geometric Brownian motion is the still the most common method used in the industry and it is used in the Black-Scholes equation, which is the foundation for financial derivatives pricing.

3. Ito's Lemma

Although modeling price movements with geometric Brownian motion is the best and most widely accepted practice, its properties make it difficult to analyze. Geometric Brownian motion is continuous but it has unbounded variation. Unbounded variation means that no matter how small an interval is considered the function will not become a smooth curve. Because of this characteristic, Brownian motion cannot be integrated using a Reimann-Stieltjes integral. The notion of unbounded variation is depicted in Figure 1.

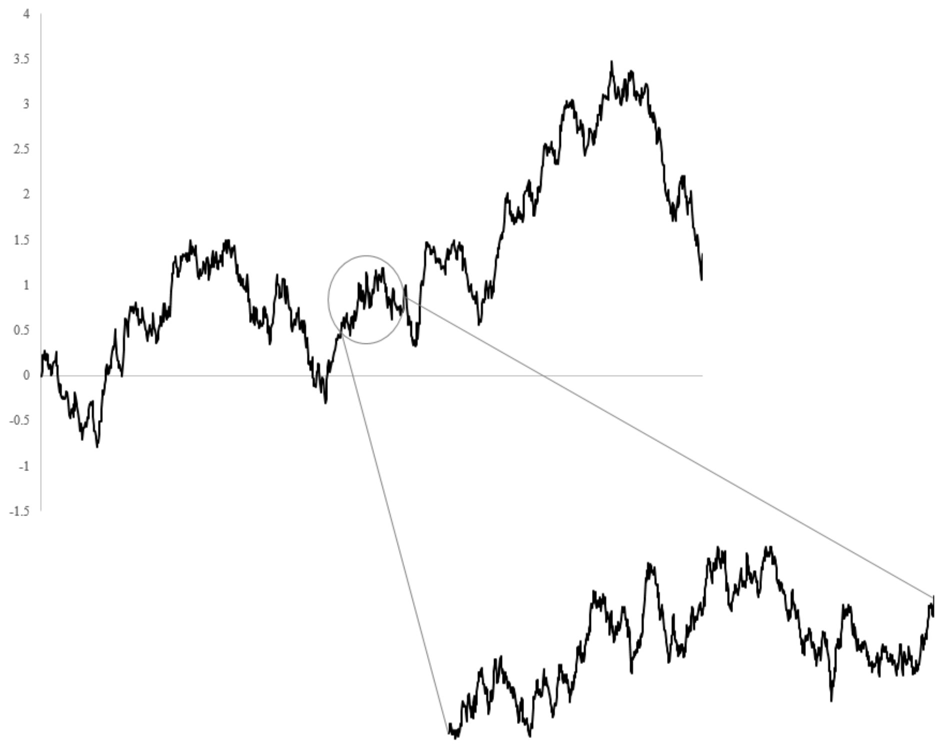


Figure 1: Example of the unbounded variation of fractal motion.

Kiyosi Ito was the first to address the need for a stochastic integral in 1944. Ito derived a method of integrating and differentiating time dependent functions of a stochastic process. The method later became known as Ito's Lemma. The definition of the derivative of a stochastic process was a key component of Black, Scholes, and Merton's ability to solve for the analytical price of a financial derivative.

3.1 Derivation of Ito's Lemma

Ito's Lemma states that for all T > 0

$$f(W(T),T) - f(W(0),0)$$

$$= \int_0^T f_t(W(t),t)dt + \int_0^T f_x(W(t),t)dW(t) + \frac{1}{2}\int_0^t f_{xx}(W(t),t)dt$$

And if we write $f = f(W(t),t)$

$$df = f_t dt + f_x dW + \frac{1}{2} f_{xx} dt.$$

Where $W(t)$ is a standard Brownian motion.

In order to prove this statement, we start by partitioning $f(W(T),T) - f(W(0),0)$ by $\Pi = \{t_0 = 0, t_1, \ldots, t_n = T\}$. We will also assume that the partition is uniform, that is $t_{i+1} - t_i$ is the same for all i. Then it is clear that

$$f(W(T),T) - f(W(0),0) = \textstyle\sum_{i=0}^{n-1} f(W(t_{i+1}),t_{i+1}) - f(W(t_i),t_i).$$

By computing the Taylor expansion of this function the equation becomes

$$f(W(T),T) - f(W(0),0) = \sum_{i=0}^{n-1} f(W(t_{i+1}),t_{i+1}) - f(W(t_i),t_i)$$

$$= \sum_{i=0}^{n-1} f(W(t_i),t_i)\,(t_{i+1} - t_i) + \sum_{i=0}^{n-1} f_x(W(t_i),t_i)\big(W(t_{i+1}) - W(t_i)\big)$$

$$+\frac{1}{2}\sum_{i=0}^{n-1} f_{xx}(W(t_i),t_i)(W(t_{i+1}) - W(t_i))^2 + \sum_{i=0}^{n-1} O(\,(t_{i+1} - t_i)(W(t_{i+1}) - W(t_i)))$$

$$+\sum_{i=0}^{n-1} O((t_{i+1}-t_i)^2) + \sum_{i=0}^{n-1} O((W(t_{i+1})-W(t_i))^3)$$

We then take the limit of the function as $n \to \infty$, which intuitively represents making the partition has infinitely small intervals. We see that the first term converges immediately to a Riemann integral

$$\sum_{i=0}^{n-1} f(W(t_i), t_i)\,(t_{i+1}-t_i) \;\to\; \int_0^T f_t(W(t), t)dt \qquad (3)$$

The second term also converges immediately as $n \to \infty$ because this is the definition of the Ito integral.

$$\sum_{i=0}^{n-1} f_x(W(t_i), t_i)\big(W(t_{i+1})-W(t_i)\big) \;\to\; \int_0^T f_x(W(t), t)dW(t)$$

The convergence of the third term is not immediate. However, it is clear that in order for

$$\frac{1}{2}\sum_{i=0}^{n-1} f_{xx}(W(t_i), t_i)(W(t_{i+1})-W(t_i))^2 \to \int_0^T f_{xx}(W(t), t)(dW(t))^2$$

to be true as it is in Ito's Lemma, it needs to be shown that $dW^2 = dt$. This relationship is intuitive when the properties of $W(t)$, which is a Weiner process, are considered. The standard deviation of $W(t)$ is proportional to $\sqrt{t}$. Therefore it is intuitive that $(dW)^2 = dt$.

It is known that

$$dX = rdt - \sigma dW \approx r(t + \varepsilon - t) + \sigma(W_{t+\varepsilon} - W_t)$$

And

$$(dW)^2 \approx (W_{t+\varepsilon} - W_t)^2 \approx {W_{t+\varepsilon}}^2 - 2W_t W_\varepsilon + {W_t}^2$$

By rewriting the right hand side of this equation as

$${W_{t+\varepsilon}}^2 - 2W_t W_\varepsilon + {W_t}^2 \approx \left({W_{t+\varepsilon}}^2 - {W_t}^2\right) + 2({W_t}^2 - W_{t+\varepsilon}W_t)$$

And because

$$\left({W_{t+\varepsilon}}^2 - {W_t}^2\right) = dW^2$$

Then

$$(dW)^2 \approx dW^2 + 2W_t(W_t - W_{t+\varepsilon})$$

It is clear that

$$(W_t - W_{t+\varepsilon}) = -(W_{t+\varepsilon} - W_t) = -dW$$

Equation for dW^2 then becomes

$$(dW)^2 \approx dW^2 - 2W_t dW$$

According to Ito

$$dW^2 = 2WdW + dt$$

By substituting this value for dW^2 then it is clear that

$$(dW)^2 = dt.$$

Now that it has been established that $dW^2 = dt$ we can conclude that the third term does indeed converge to the equation stated in Ito's Lemma.

$$\frac{1}{2}\sum_{i=0}^{n-1} f_{xx}(W(t_i), t_i)(W(t_{i+1}) - W(t_i))^2 \rightarrow \frac{1}{2}\int_0^T f_{xx}(W(t), t)(dW(t))^2$$

$$= \frac{1}{2}\int_0^T f_{xx}(W(t), t)dt.$$

We have now shown that all of the terms on the right hand side of Ito's Lemma converge from terms of the Taylor expansion. It remains to show that the higher order terms will approach 0 as $n \rightarrow \infty$.

$$\left|\sum_{i=0}^{n-1} O(\,(t_{i+1} - t_i)(W(t_{i+1}) - W(t_i)))\right| < \sup_{0 \le i \le n} |W(t_{i+1}) - W(t_i)| \sum_{i=0}^{n-1} (t_{i+1} - t_i)$$

$$< T \sup_{0 \le i \le n} |W(t_{i+1}) - W(t_i)| \rightarrow T * 0 = 0$$

The same process can be used to show that the last two terms also converge to 0 as $n \rightarrow \infty$.

$$\sum_{i=0}^{n-1} O((t_{i+1} - t_i)^2) < \sup_{0 \le i \le n} |t_{i+1} - t_i| \sum_{i=0}^{n-1} (t_{i+1} - t_i) < T \sup_{0 \le i \le n} |t_{i+1} - t_i| \rightarrow$$

$$T * 0 = 0$$

And

$$\sum_{i=0}^{n-1} O((W(t_{i+1}) - W(t_i))^3) < \sup_{0 \le i \le n} |W(t_{i+1}) - W(t_i)| \sum_{i=0}^{n-1} (W(t_{i+1}) - W(t_i))^2$$

$$\to 0$$

Because the remaining terms all approach 0, we can conclude that

$$f(W(T), T) - f(W(0), 0)$$

$$= \int_0^T f_t(W(t), t)dt + \int_0^T f_x(W(t), t)dW(t) + \frac{1}{2}\int_0^t f_{xx}(W(t), t)dt.$$

3.2 Application of Ito's Lemma

Let F(S_t, t) be the price of a financial derivative of the underlying asset with price S_t. Because F is a function of t and S_t which follows a stochastic process of the form $rdt + \sigma dW_t$ as described in equation (1) Ito's lemma can be used to find the equation for the change in F.

$$dF = \left(\mu S_t \frac{\partial F}{\partial t} + \frac{\partial F}{\partial t} + \frac{1}{2}\sigma^2 {S_t}^2 \frac{\partial^2 F}{\partial {S_t}^2}\right) dt + \left(\sigma S_t \frac{\partial F}{\partial S_t}\right) dW_t \qquad (4)$$

Equation (4) can be used to describe the price movement of any type of financial derivative. However, we will focus on how it is used to price options of stocks.

4. Introduction to Options

One of the most common types of financial derivative are options. An option is a contract between two or more parties that gives the buyer the right but not the obligation, or the “option”, to buy or sell the underlying asset to the seller at an agreed upon price, known as the strike price. Each contract comes with an expiration date after which the contract expires and the seller of the option is no longer contractually obliged to buy/sell the underlying asset from/to the buyer of the option. In order to obtain the right to buy/sell the underlying asset from/to the seller of the option, the buyer of the option must pay the seller an upfront fee, known as the option’s premium. This premium will be referred also be referred to as the “price” or “fair value” of an option because it represents the cost of the option to the buyer.

There are two types of basic options, calls and puts. A call is a derivative contract that gives the buyer the right, but not the obligation, to buy the underlying stock at the strike price. This means that the higher the stock price is above the strike price, the move valuable the contract is.

A put is essentially the opposite of a call. It gives the buyer the right, but not the obligation, to sell the underlying stock at the strike price. Similarly to the call option, the put gains value as the price of the stock drops further below the strike price.

There are, in general, two different styles of options: European and American. The difference lies in when the buyer of the option is allowed to exercise the right to buy or sell the underlying security. European options can only be exercised at maturity while American options can be executed at any time before the expiration of the option contract. Because of this difference, American and European options with the same characteristics usually have different prices.

At maturity, the price of American and European options behave in the same manner. For both types of options, if the stock price is above the strike price of a call the buyer of the contract can buy the underlying stock from the seller of the option at the strike price and sell it at the market price. This gives the buyer of the call option a guaranteed profit of S-K, where S is the price of the underlying stock and K is the strike price. Analogously for put options, if the stock price is below the strike price at maturity the buyer of the contract can buy the underlying asset at the market price and sell it to the seller of the option at the strike price for a profit of K-S.

Some terminology that is commonly used in the financial industry when talking about options are “in the money”, “at the money”, and “out of the money”. If an option is in the money, it means exercising the option would result in positive payout. At the money means that the underlying asset is trading at the strike price and exercising the option would cause the buyer to make exactly $0. Out of the money means that exercising the option would cause the buyer to lose money. When an option is out of the money at maturity, the buyer of the option will not exercise the contract because it would cause a loss in addition to the premium they paid for the option.

The profit for calls and puts at expiration with a strike price of $40 are below.

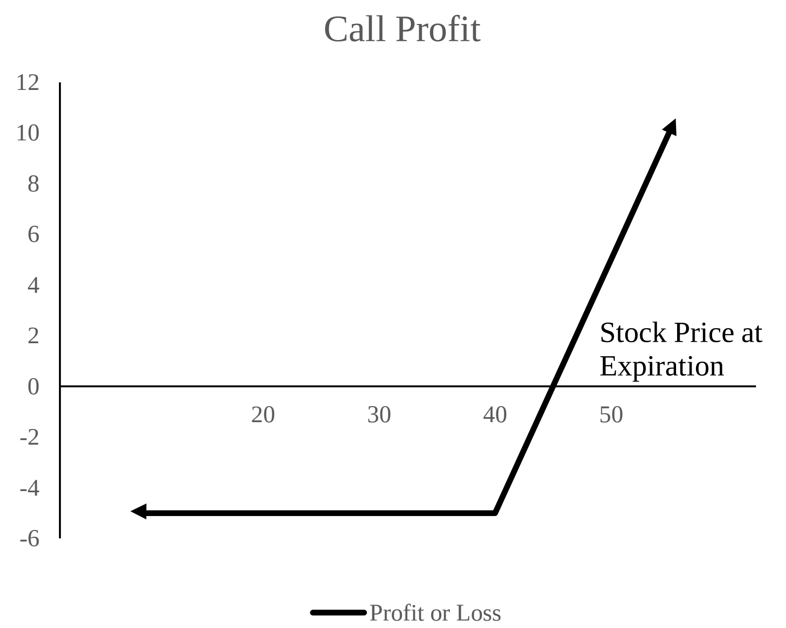


Figure 1: Graph of profit for a put with a strike price of $40.

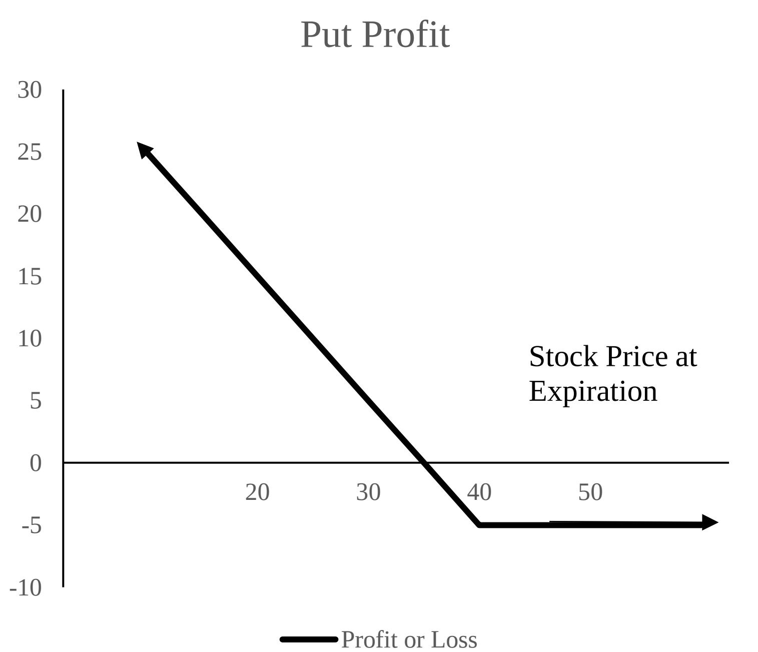


Figure 2: Graph of profit for a call options with a strike price of $40.

While the value of an option at maturity is simple to determine, the value of options before expiration is less obvious due to the uncertainty about where the stock price will be at expiration. There are two components to the price of an option at any given time. The intrinsic value, which is what the option would be worth if it was exercised immediately, and the time value, which is the additional money an investor is willing to pay based on the probability that the value of the option will increase before maturity.

Pricing the intrinsic value of an option is straightforward because both the strike price and market price of the underlying asset are known. The difficulty of pricing options comes from pricing the time value because it must account for the distribution of prices that the underlying asset can have during the time before maturity. The factors that affect the time value of an option are the current stock price, the strike price, the risk free interest rate, the volatility of the underlying asset, and the time until the option's maturity. We will focus more on how to determine the price of an option later in this analysis.

Purchasing a single option can allow the buyer to bet on the price of as stock going either up or down. However, by combining options with different characteristics an investor can create the opportunity to profit from a variety of different outcomes. Some examples include: the price of an asset will stay within a specific range during the life of an option, the price of an asset experience a large movement but the direction of the swing in price is not known, of the price of the asset will move very little by the expiration of the option.

One such construction of a combination of options is referred to as straddle. The payoff diagram is shown below.

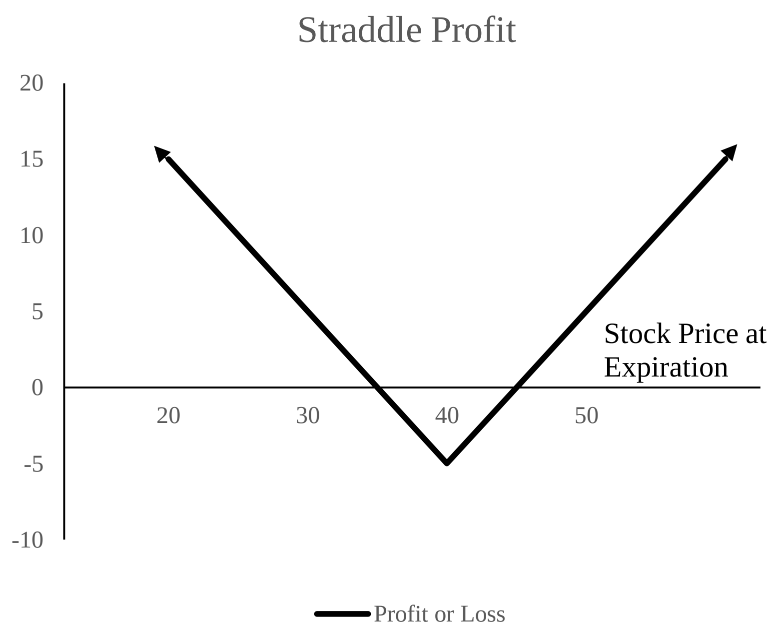


Figure 3: Graph of profit for straddle position constructed from one put option and one call option both with a strike price of $40.

This strategy is created by buying one at the money call and one at the money put. A straddle would be used when an investor believes that the price of the underlying will experience a large swing but does not have an opinion on the direction of the movement.

5. Black-Scholes Model

The first popular use of the Ito integral in pricing financial derivatives was by Fischer Black, Myron Scholes, and Robert Merton. It used the "replicating portfolio" assumption that the price of a derivative can be replicated by some combination borrowing and lending at the risk free rate and owning the underlying asset. The number of shares owned and the amount money borrowed or lent at the risk free rate are constantly adjusted so that the change in the price of the portfolio is equal to the change in the price of the derivative.

By using these assumptions, a partial differential equation (4) for the change in the price of a financial derivative is created. While the PDE is for the change in the price of any financial derivative, solutions do not exist for all possible financial derivatives. Black, Scholes, and Merton leveraged other branches of mathematics in order to use this PDE to solve for the price of a call option, for which Scholes and Merton were awarded a Nobel Prize in economics. This equation, now known as the Black-Scholes call price equation is the most famous result of financial derivative pricing research.

5.1 Derivation of Black-Scholes Partial Differential Equation

Based on the Black-Scholes assumption of a replicating portfolio, the price of a derivative can be expressed as some weightings of the underlying asset, w_S, and the risk free asset, w_A. This can be expressed as

$$F = w_S S + \ w_A A$$

Where F is the value of the replicating portfolio. From this equation it can be seen that

$$w_A = \frac{F - \ w_A S}{A}.$$

The change in the price of the portfolio can be written as

$$dF = \; w_S dS + \; w_A dA \qquad (5)$$

Let A_t represent the price of a risk free asset at time *t*. Then we can use the representation of the price of an asset that was determined equation (1) to calculate the price of the risk free asset as

$$A_t = \; A_0 e^{\mu t + \, \sigma W(t)}$$

However, because this the risk free asset, there is no volatility in the return. This means that σ = 0 so the resulting equation for the price of the risk free asset is

$$A_t = A_0 \, e^{\mu t}$$

Because this is a special asset, we give this rate a new name, r, which represents the risk free rate. The price of the risk free asset at time t is then

$$A_t = A_0 \, e^{rt}$$

The differential of which is

$$dA_t = \; A_t r dt$$

After substituting w_A and dA_t into equation (5) the equation is now

$$dF = \; w_S dS + (F - \; w_S S) r dt$$

Using the equation for dS from (2) the resulting equation is

$$dF = w_S(\mu S dt + \; \sigma S dW_t) + (F - w_S S) dt$$

By rearranging the equation, the result is

$$dF = (w_S S(\mu - r) + rF)dt + (w_S \sigma S)dW_t$$

Now we have two representations of dF. We know from the properties of an Ito process that the dt and dW_t terms of the two equations must be equal. By setting corresponding terms equal to each other and simplifying we find

$$w_S = \frac{\delta F}{\delta S}$$

And

$$\frac{\partial F}{\partial S} S(\mu - r) + rF = \mu S \frac{\partial F}{\partial P} + \frac{\partial F}{\partial t} + \frac{1}{2}\sigma^2 \frac{\partial^2 F}{\partial S}$$

By expanding the left hand side of the equation, the equation becomes

$$\mu S \frac{\partial F}{\partial S} - rS \frac{\partial F}{\partial S} + rF = \mu S \frac{\partial F}{\partial P} + \frac{\partial F}{\partial t} + \frac{1}{2}\sigma^2 \frac{\partial^2 F}{\partial S}$$

The term $\mu S \frac{\delta F}{\delta S}$ appears on both sides of the equation so it can be cancelled out. This is an important step because μ, the return on the underlying asset, drops out of the equation. Without μ in the equation, all of the variables can be observed in the market, eliminating the need for arbitrary parameters and estimations of average asset returns. The fully simplified version of the equation, known as the Black-Scholes partial differential equation, is:

$$rS \frac{\partial F}{\partial S} + \frac{\partial F}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 F}{\partial S^2} - rF = 0.$$

The step where μ drops out of the equation shows one of the most important breakthroughs that Black and Scholes made. Prior to their work, all research done on options pricing resulted in equations that depended on some arbitrary factor such as return on the underlying asset, discount factors based on the risk of the underlying, or the shape of a utility function. In addition to its dependence on only observable variables, there are no restriction on the structure of F. Because of this flexibility, the Black-Scholes partial differential equation can be used for any type of derivative.

5.2 Derivation of Call Option Price from Black-Scholes PDE

While the Black-Scholes PDE is able to model the price of any financial derivative, solutions do not exist for all types, the PDE has a solution for the price of a European call option. In order to find the equation for the price of the call option, the Black-Scholes PDE must be transformed. By using changes of variables, the PDE can be transformed into the form of what is known as the heat equation. The heat equation has been studied extensively and under the correct conditions it is both solvable and stable. European call options have initial and boundary conditions for which the heat equation is solvable.

In order to find the solution to the Black-Scholes PDE, it must first be transformed from its original form

$$rS\frac{\partial F}{\partial S} + \frac{\partial F}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 F}{\partial S^2} - rF = 0 \tag{6}$$

Into a version of the heat equation. A number of change of variables will be used to transform the PDE into

$$\frac{\partial u}{\partial \tau} = \frac{1}{2}\sigma^2 \frac{\partial^2 u}{\partial x^2}$$

The first change of variables that will be made is

$$F = u(S,t)e^{rt}$$

The goal of this transformation is to eliminate the rF term in equation (6). From the equation of F and using the product rule

$$\frac{\partial F}{\partial t} = \frac{\partial u}{\partial t}e^{rt} + rue^{rt} = \frac{\partial u}{\partial t}e^{rt} + rF$$

By rearranging this equation we find that

$$\frac{\partial u}{\partial t}e^{rt} = \frac{\partial F}{\partial t} - rF$$

The terms on the right hand side are present in equation (6) so the equation becomes

$$\frac{\partial u}{\partial t}e^{rt} + rS\frac{\partial F}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 F}{\partial S^2} = 0 \tag{7}$$

Successfully eliminating the rF term.

The next change of variables that will be made is

$$S = e^y$$

This change of variables combined with the previous allow us to solve for the remaining terms in equation (7) in terms of u and y. First, it is clear that

$$\frac{\partial F}{\partial S} = \frac{\partial u}{\partial S} e^{rt} \tag{8}$$

However this form of the differential does not eliminate any variables when substituted back into equation (7). We will multiply the right hand side by 1 in order to make the substitution more useful.

$$\frac{\partial F}{\partial S} = \frac{\partial u}{\partial S} e^{rt} \left(\frac{\partial S}{\partial y}\right)\left(\frac{\partial y}{\partial S}\right)$$

From the definition of S we know that

$$\frac{\partial S}{\partial y} = S \text{ or equivalently } \frac{\partial y}{\partial S} = \frac{1}{S}$$

By substituting this new value into equation (8) and noting that

$$\left(\frac{\partial S}{\partial y}\right)\left(\frac{\partial u}{\partial S}\right) = \frac{\partial u}{\partial y}$$

Equation (8) becomes

$$\frac{\partial F}{\partial S} = \frac{\partial u}{\partial y} e^{rt} \frac{1}{S}$$

From this equation we can solve for the second partial derivative of F with respect to S by using the product rule.

$$\frac{\partial^2 F}{\partial S^2} = \frac{\partial}{\partial S}\left(\frac{\partial F}{\partial S}\right) = -\frac{1}{S^2}\frac{\partial u}{dy}e^{rt} + \frac{1}{S}\frac{\partial^2 u}{\partial u \partial S}$$

By using the same method as the previous equation, multiplying by 1 on the right hand side and noting that $\frac{\partial y}{\partial S} = \frac{1}{S}$

$$\frac{\partial^2 F}{\partial S^2} = -\frac{1}{S^2}\frac{\partial u}{dy}e^{rt} + \frac{1}{S}\frac{\partial^2 u}{\partial y \partial S}\left(\frac{\partial S}{\partial y}\right)\left(\frac{\partial y}{\partial S}\right) = -\frac{1}{S^2}\frac{\partial u}{dy}e^{rt} + \frac{1}{S^2}\frac{\partial^2 u}{\partial y^2}\,e^{rt}$$

Substituting these new values into equation (7) the PDE becomes

$$\frac{\partial u}{\partial t}e^{rt} + r\frac{\partial u}{\partial y}e^{rt} + \frac{1}{2}\sigma^2 e^{rt}\left(-\frac{\partial u}{dy} + \frac{\partial^2 u}{\partial y^2}\right) = 0$$

After dividing both sides of the equation by e^{rt} and collecting like terms the equation becomes

$$\frac{\partial u}{\partial t} + \left(r - \frac{1}{2}\sigma^2\right)\frac{\partial u}{\partial y} + \frac{1}{2}\sigma^2\frac{\partial^2 u}{\partial y^2} = 0$$

We now have a constant coefficient equation. In order to make the initial condition into a terminal condition, we will make the substitution

$$\tau = T - t$$

From this equation it is clear that

$$d\tau = -dt \text{ or equivalently } \frac{\partial t}{\partial \tau} = -1$$

This allows the substitution

$$\frac{\partial u}{\partial t} = -\frac{\partial u}{\partial \tau}$$

The Black-Scholes PDE is now

$$-\frac{\partial u}{\partial \tau} + \left(r - \frac{1}{2}\sigma^2\right)\frac{\partial u}{\partial y} + \frac{1}{2}\sigma^2\frac{\partial^2 u}{\partial y^2} = 0$$

The final step that is needed in order to finish the transformation into the heat equation is to eliminate the linear term. The substitution that will allow this is

$$y = x - \left(r - \frac{1}{2}\sigma^2\right)\tau$$

The first derivative of y in terms of x show

$$\frac{\partial y}{\partial x} = 1$$

And therefore

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial x}$$

Because both y and x are functions of τ, x can replace y in the PDE

$$-\frac{\partial u}{\partial \tau} + \left(r - \frac{1}{2}\sigma^2\right)\frac{\partial u}{\partial y} + \frac{1}{2}\sigma^2\frac{\partial^2 u}{\partial x^2} = 0$$

And also into the solution to $\frac{\partial S}{\partial \tau}$

$$\frac{\partial S}{\partial \tau} = e^y\frac{\partial y}{\partial \tau} = S\frac{\partial y}{\partial \tau} = S\left(\frac{\partial x}{\partial \tau} - \left(r - \frac{1}{2}\sigma^2\right)\right)$$

Multiplying both sides of the equation by $\left(\frac{\partial x}{\partial S}\right)\left(\frac{\partial u}{\partial x}\right)$ gives

$$\frac{\partial S}{\partial \tau} = \frac{\partial S}{\partial \tau}\left(\frac{\partial x}{\partial S}\right)\left(\frac{\partial u}{\partial x}\right) = \frac{\partial u}{\partial \tau}$$

$$S\left(\frac{\partial x}{\partial \tau} - \left(r - \frac{1}{2}\sigma^2\right)\right)\left(\frac{\partial x}{\partial S}\right)\left(\frac{\partial u}{\partial x}\right) = S\left(\frac{\partial u}{\partial \tau} - \left(r - \frac{1}{2}\sigma^2\right)\frac{\partial u}{\partial x}\right)\left(\frac{1}{S}\right)$$

$$= \frac{\partial u}{\partial \tau} - \left(r - \frac{1}{2}\sigma^2\right)\frac{\partial u}{\partial x}$$

Showing that

$$\frac{\partial u}{\partial \tau} = \frac{\partial u}{\partial \tau} - \left(r - \frac{1}{2}\sigma^2\right)\frac{\partial u}{\partial x}$$

By making this final substitution into the PDE, the linear term in eliminated.

$$-\frac{\partial u}{\partial \tau} + \frac{1}{2}\sigma^2\frac{\partial^2 u}{\partial x^2} = 0$$

Which is exactly the heat equation that the transformations were intended to find

$$\frac{\partial u}{\partial \tau} = \frac{1}{2}\sigma^2\frac{\partial^2 u}{\partial x^2}$$

For a European call conditions after the transformation of variables are

$$u(x,\tau) = \max(e^{x-\left(r-\frac{\sigma^2}{2}\right)\tau} - K)e^{-r\tau}$$

$$-\infty < x < \infty, \qquad 0 \leq \tau < T$$

The well-known solution to the heat equation is

$$u(x,\tau) = \frac{1}{\sqrt{4\pi\left(\frac{1}{2}\sigma^2\right)\tau}}\int_{-\infty}^{\infty} \exp\left(-\frac{(x-c)^2}{4\left(\frac{1}{2}\sigma^2\right)\tau}\right) g(c)dc$$

Where $g(c)$ is the terminal condition of the financial derivative. The condition for the European call option can be written as

$$u(x,T) = \begin{cases} e^{x-rT+\frac{\sigma^2}{2}\tau} - Ke^{-r(T-\tau)}, & \text{x} > \ln(K) + \left(r - \frac{\sigma^2}{2}\right)\tau \\ 0, & \text{x} \leq 0 \end{cases}$$

By substituting this condition into the solution to the heat equation

$$u(x,\tau) = \frac{1}{\sqrt{2\pi\sigma^2\tau}}\int_{\ln(K)+\left(r-\frac{\sigma^2}{2}\right)\tau}^{\infty} \exp\left(-\frac{(x-c)^2}{2\sigma^2\tau}\right) e^{\left(x-rT+\frac{\sigma^2}{2}\tau\right)} dc -$$

$$\frac{Ke^{-r(T-\tau)}}{\sqrt{2\pi\sigma^2\tau}}\int_{\ln(K)+\left(r-\frac{\sigma^2}{2}\right)\tau}^{\infty} \exp\left(-\frac{(x-c)^2}{2\sigma^2\tau}\right) dc$$

This expression can be simplified and the resulting solution to the partial differential equation is precisely the Black-Scholes equation for the price of a call option.

$$F(S,t) = SN\left(\frac{\ln\left(\frac{S}{K}\right) + (r + \frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}}\right) - Ke^{-rT}N\left(\frac{\ln\left(\frac{S}{K}\right) + \left(r - \frac{\sigma^2}{2}\right)\tau}{\sigma\sqrt{\tau}}\right).$$

6. Alternative Methods for Pricing Derivatives

While there is a closed form solution to the Black-Scholes equation for the price of a European call option, most financial derivatives do not have conditions that allow the existence of a solution. This lack of solutions for leaves a need for alternative methods of pricing more complex financial derivatives. The most common method used in practice is a Monte Carlo simulation. This method avoids the need for analytical solution by leveraging the processing power of modern computers. Alternatively, binomial trees can be used to price financial derivatives. These trees allow for an analytical solution but are computationally expensive.

6.1 Convergence of Binomial Tree To Black-Scholes Equation

One way to determine the fair value of one a more complex option is by using a binomial tree to model the movement of the price of the underlying asset. The binomial tree allows for conditions that are more complex than simply execution at maturity to be placed on the derivatives. This is important because most derivatives have more restrictions than European options and cannot be priced by using the Black-Scholes PDE.

Although this is a technique that is typically used for more complex derivatives, the Black-Scholes call price can be derived from the binomial tree. In addition to being accurate, the binomial tree method is generally considered to be easier than using the Black-Scholes equation. We will now show the derivation of a call price from a binomial tree.

The binomial tree for a call option that matures in T units of time will have n steps each with a length of T/n. We will define the change in price after each step as either an upward movement, u, or a downward movement, d. We will let these movements have magnitudes such that from a beginning price one upward movement followed by one downward or one downward movement followed by an upward movement will both result in the beginning price. These values are

$$u = e^{\sigma\sqrt{\frac{T}{n}}}, \qquad d = e^{-\sigma\sqrt{\frac{T}{n}}}$$

But we will continue to refer to them as u and d for simplicity.

After n steps, the price of the underlying security will be $S_0u^jd^{n-j}$ where j is the number of upward movements. The payoff of the call option with strike K will be $\max(S_0u^jd^{n-j} - K, 0)$.

From the definition of a binomial distribution, we know that the probability of j upward movements and $n - j$ downward movements is

$$\frac{n!}{(n-j)!\,j!}p^j(1-p)^{n-1}$$

Where p is the probability of an upward movement.

Because the probability of having j upward movements is known, we can calculate the expected value of $\max(S_0u^jd^{n-j} - K, 0)$.

$$E\big(\max(S_0u^jd^{n-j} - K, 0)\big) = \sum_{j=0}^{n}\frac{n!}{(n-j)!\,j!}p^j(1-p)^{n-1}\max(S_0u^jd^{n-j} - K, 0)$$

The expected value of the call option represents a future cash flow so it must be discounted back to the present value. We will use the risk free rate r for this discounting. The fair value of a call price is then

$$c = e^{-rT} \sum_{j=0}^{n} \frac{n!}{(n-j)!\,j!} p^{j} (1-p)^{n-1} \max\left(S_0 u^j d^{n-j} - K, 0\right).$$

We know that the payoff of a call option is nonzero when the price of the underlying security is greater than the strike price at maturity. This occurs when

$$S_0 u^j d^{n-j} > K$$

By taking the natural log of both sides the relationship becomes

$$\ln\left(\frac{S_0}{K}\right) > -j \ln(u) - (n-j) \ln(d). \tag{9}$$

Because we have already defined u and d, the values can be substituted into the (9) resulting in

$$\ln\left(\frac{S_0}{K}\right) > -j\sigma \frac{T}{n} - (n-j)\left(-\sigma \frac{T}{n}\right).$$

After simplification, this becomes

$$\ln\left(\frac{S_0}{K}\right) > -2j\sigma \frac{T}{n} + n\sigma \sqrt{\frac{T}{n}}.$$

The goal of this relationship is to find how many upward movements j are necessary for the call option to end in the money at maturity. Solving the inequality for j will give this number

$$j > \frac{n}{2} - \frac{\ln\left(\frac{S_0}{K}\right)}{2\sigma\sqrt{\frac{T}{n}}}.$$

Let this quantity be denoted by α. Then the price of a call option can be written as

$$c = e^{-rT} \sum_{j>\alpha} \frac{n!}{(n-j)!\,j!} p^j (1-p)^{n-1} \left(S_0 u^j d^{n-j} - K\right)$$

$$= e^{-rT} \left[S_0 \sum_{j>\alpha} \frac{n!}{(n-j)!\,j!} p^j (1-p)^{n-1} u^j d^{n-j} - K \sum_{j>\alpha} \frac{n!}{(n-j)!\,j!} p^j (1-p)^{n-1} \right]$$

We know from statistics that as the number of binomial trials approaches infinity, the number of successes approaches a normal distribution with mean np and standard deviation $\sqrt{np(1-p)}$. The second term in the equation is the probability that the number of successes will be greater than α. Then as the number of trails approaches infinity, or in this case the number of steps in the binomial tree approaches infinity, the value of the second term is

$$\sum_{j>\alpha} \frac{n!}{(n-j)!\,j!} p^j (1-p)^{n-1} = N\left(\frac{np-\alpha}{\sqrt{np(1-p)}}\right)$$

where N is the cumulative probability distribution function for the standard normal distribution. By replacing α, the equation becomes

$$\sum_{j>\alpha} \frac{n!}{(n-j)!\,j!} p^j (1-p)^{n-1} = N\left(\frac{\ln\left(\frac{S_0}{K}\right)}{2\sigma\sqrt{T}\sqrt{p(1-p)}} + \frac{\sqrt{n}\left(p-\frac{1}{2}\right)}{\sqrt{p(1-p)}} \right)$$

In "Option pricing: A simplified approach", Cox, Ross, and Rubinstein define p as

$$p = \frac{e^{\frac{rT}{n}} - e^{-\sigma\sqrt{T/n}}}{e^{\sigma\sqrt{T/n}} - e^{-\sigma\sqrt{T/n}}}.$$

This value of p is chosen because Cox, Ross and Rubinstein assumed that stock operated in a risk neutral world. In this world, an investor is indifferent between an investment in a risk free asset and a risky asset because both assets have the same expected return. This value of p makes the return on a stock equal to the risk free rate, matching the assumptions of a risk neutral world.

For this value of p, as n approaches infinity $p(1-p)$ approaches $\frac{1}{4}$ and $\sqrt{n}\left(p-\frac{1}{2}\right)$ approaches

$$\frac{(r-\frac{\sigma^2}{2})\sqrt{T}}{2\sigma}.$$

These can limits be shown by using the Taylor expansion

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

By substituting these values into the equation, the second term in the price of a call option simplifies to

$$\sum_{j>\alpha} \frac{n!}{(n-j)!\,j!} p^j (1-p)^{n-1} = N\left(\frac{\ln\left(\frac{S_0}{K}\right) + \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right).$$

In order to solve for

$$\sum_{j>\alpha} \frac{n!}{(n-j)!\,j!} p^j (1-p)^{n-1} u^j d^{n-j}$$

We will define

$$p^* = \frac{pu}{pu + (1-p)d}$$

And then

$$1 - p^* = \frac{pd}{pu + (1-p)d}.$$

By substituting these values into the first term of the equation for the price of a call option

$$\sum_{j>\alpha} \frac{n!}{(n-j)!\,j!} p^j (1-p)^{n-1} u^j d^{n-j}$$

$$= (pu + (1-p)d)^n \sum_{j>\alpha} \frac{n!}{(n-j)!\,j!} (p^*)^j (1-p^*)^{n-j}$$

The expected return on a stock for one unit of time is $pu + (1-p)d$. As mentioned previously, in the Cox-Ross-Rubenstein paper it is assumed that stocks prices move in a risk-neutral world. Because of this assumption, it follows that the expected return on a stock is equal to the risk free rate so one can write

$$e^{\frac{rT}{n}} = pu + (1-p)d.$$

Then this can be substituted into the equation

$$\sum_{j>\alpha} \frac{n!}{(n-j)!\,j!} p^j (1-p)^{n-1} u^j d^{n-j} = e^{rT} \sum_{j>\alpha} \frac{n!}{(n-j)!\,j!} (p^*)^j (1-p^*)^{n-j}$$

This equation is now another binomial distribution but in this case, the probability of an upward movement is p^*. We will again use the fact that a binomial distribution approaches a normal distribution as the number of trials goes to infinity to conclude that

$$\sum_{j>\alpha} \frac{n!}{(n-j)!\,j!} p^j (1-p)^{n-1} u^j d^{n-j} = e^{rT} N\left(\frac{np^* - \alpha}{\sqrt{np^*(1-p^*)}}\right)$$

Replacing α with its value, the expression becomes

$$e^{rT} N\left(\frac{\ln\left(\frac{S_0}{k}\right)}{2\sigma\sqrt{T}\sqrt{p^*(1-p^*)}} + \frac{\sqrt{n}\left(p^* - \frac{1}{2}\right)}{\sqrt{p^*(1-p^*)}}\right)$$

The final step is to replace the p^*in the term. In order to make this substitution, we will use the fact that as n approaches infinity, $p^*(1-p^*)$ approaches $\frac{1}{4}$ and $\sqrt{n}\left(p^* - \frac{1}{2}\right)$ approaches

$$\frac{(r + \frac{\sigma^2}{2})\sqrt{T}}{2\sigma}.$$

These limits are solved for using the same Taylor series expansion method as used in simplifying the expressions in the second term. By substituting these values into the expression, we find that

$$e^{rT}N\left(\frac{\ln\left(\frac{S_0}{k}\right)+\ (r+\frac{\sigma^2}{2})T}{\sigma\sqrt{T}}\right)$$

Now that we have all of the pieces of the equation simplified, each term can be put together to get the Black-Scholes equation

$$c = S_0N\left(\frac{\ln\left(\frac{S_0}{k}\right)+\ (r+\frac{\sigma^2}{2})T}{\sigma\sqrt{T}}\right) - Ke^{-rT}N\left(\frac{\ln\left(\frac{S_0}{K}\right)+\left(r-\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right).$$

6.1.2 Examples of Pricing Options with a Binomial Tree

The most basic option to price with a binomial tree is a European call option. Below is a simple binomial tree with four steps for visualization. This tree models a European call with the parameters specified in Figure 5.

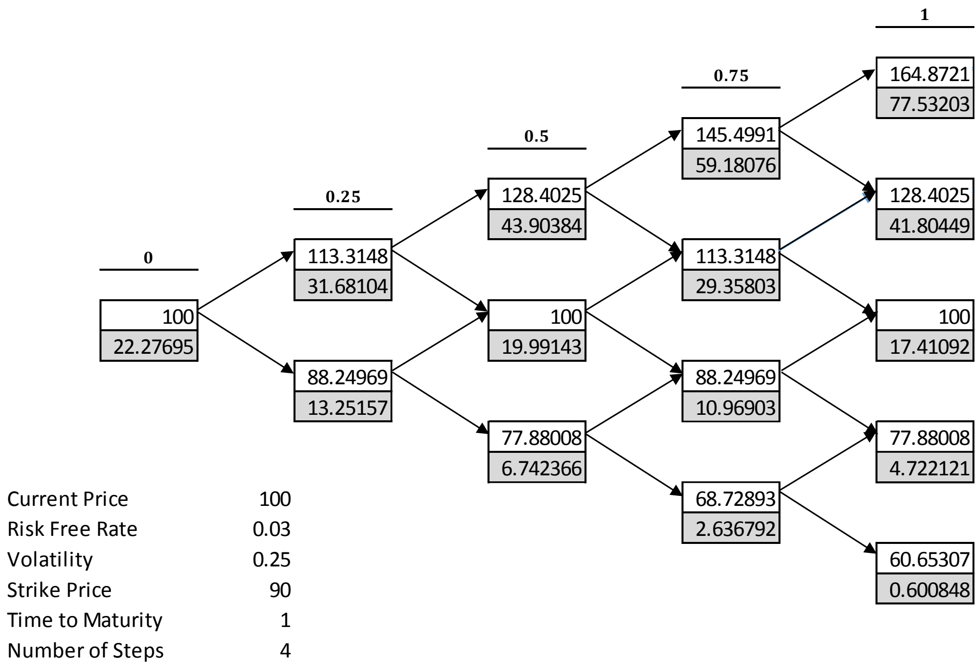


Figure 5: Example of binomial tree to price a European option.

In the illustration, the price of the underlying stock is shown in the top box. In each step the price either increases by u or decreases by d which are both defined as in the Cox, Ross, and Rubenstein research. The shaded box underneath the stock price is the present value of the call option at the corresponding step.

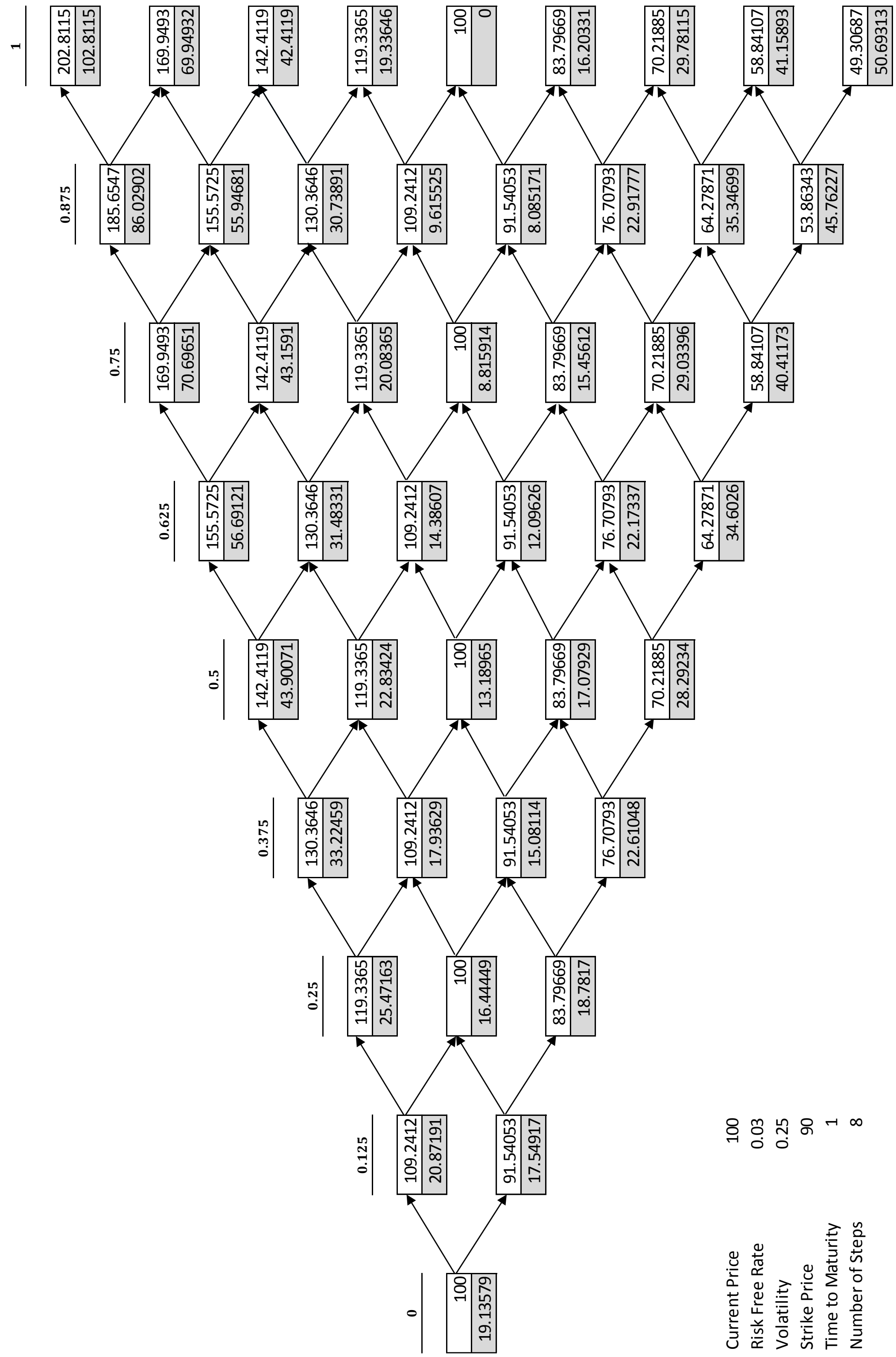
Rather than drawing binomial trees by hand, the trees are modeled using a computer program. An example of a program that computes the value of a European option with a binomial tree is shown below in Figure 6.

```
Function myCall(S, r, sigma, X, t, n)

     u = Exp(sigma * (t / n) ^ (1 / 2))
     d = 1 / u
     pu = (Exp(r * t / n) - d) / (u - d)
     pd = 1 – pu

     For i = 0 To n
          mySum = mySum + Exp(-r * t) *
                 Application.WorksheetFunction.Max(S*u^(n-i)
                 * d^i - X, 0)*
                 Application.WorksheetFunction.Combin(n,i)
                 *pu^(n-i) * pd^i
     Next

     myCall = mySum

End Function
```

Figure 6: Sample code for option pricing using a binomial tree.

The binomial tree method can be applied to other types of options as well. The only piece of the program or tree that needs to be modified is the terminal condition. For call options this condition is $\max(S - K, 0)$. The example below provides a visualization of how to price a straddle position with the binomial method. The terminal condition in this tree is $\max(S - K, K - S)$. This binomial tree in particular has 8 steps and is used to price a 1 year straddle position constructed by buying one call and one put each with a strike of 100 on a security with a current price of 100, volatility of 25% and a risk free rate of 3%.

In the nonterminal nodes, the present value of the option is calculated as the present value of the expected value of the option in the next step based on p_u and p_d which are also calculated according to the Cox, Ross, Rubenstein method. The possible values of the option is calculated by going through the tree backwards until time 0 is reached and the fair value of the option is found.



0
100
19.13579
0.125
109.2412
20.87191
91.54053
17.54917
0.25
119.3365
25.47163
100
16.44449
83.79669
18.7817
0.375
130.3646
33.22459
109.2412
17.93629
91.54053
15.08114
76.70793
22.61048
0.5
142.4119
43.90071
119.3365
22.83424
100
13.18965
83.79669
17.07929
70.21885
28.29234
0.625
155.5725
56.69121
130.3646
31.48331
109.2412
14.38607
91.54053
12.09626
76.70793
22.17337
64.27871
34.6026
0.75
169.9493
70.69651
142.4119
43.1591
119.3365
20.08365
100
8.815914
83.79669
15.45612
70.21885
29.03396
58.84107
40.41173
0.875
185.6547
86.02902
155.5725
55.94681
130.3646
30.73891
109.2412
9.615525
91.54053
8.085171
76.70793
22.91777
64.27871
35.34699
53.86343
45.76227
1
202.8115
102.8115
169.9493
69.94932
142.4119
42.4119
119.3365
19.33646
100
0
83.79669
16.20331
70.21885
29.78115
58.84107
41.15893
49.30687
50.69313
Current Price 100
Risk Free Rate 0.03
Volatility 0.25
Strike Price 90
Time to Maturity 1
Number of Steps 8

The accuracy of the calculated value can be showing that it is equal to the value the equation

$$c = \sum_{i=0}^{n} \max\left(S_0 u^{n-i} d^i - K, K - S_0 u^{n-i} d^i\right) {p_u}^{n-i} {p_d}^{i} \binom{n}{i} \tag{10}$$

The value for this calculation can be done by creating a computer program. The program follows the same process as the one used to model a European call with the binomial tree but with a different terminal condition. Figure 6 shows an example of VBA code that calculates the value for c in Equation (10). For the parameters used in the visualized example of a straddle position, the program returns a price of $17.2064 matching the value calculated by iterating steps in the binomial tree.

```
Function myStraddle(S, r, sigma, X, t, n)


     u = Exp(sigma * (t / n) ^ (1 / 2))
     d = 1 / u
     pu = (Exp(r * t / n) - d) / (u - d)
     pd = 1 - pu

     For i = 0 To n
          mySum = mySum + Exp(-r * t) *
                 Application.WorksheetFunction.Max(S*u^(n-i)
                 * d^i - X, X - S*u^(n-i)* d^i)*
                 Application.WorksheetFunction.Combin(n,i)
                 *pu^(n-i) * pd^i
     Next

     myStraddle = mySum

End Function
```

Figure 7: Sample code for pricing a straddle position with a binomial tree.

The convergence of the binomial tree model to the Black-Scholes equation can be shown empirically as well. Below is a graph illustrating this convergence as the number of nodes in the tree increases.

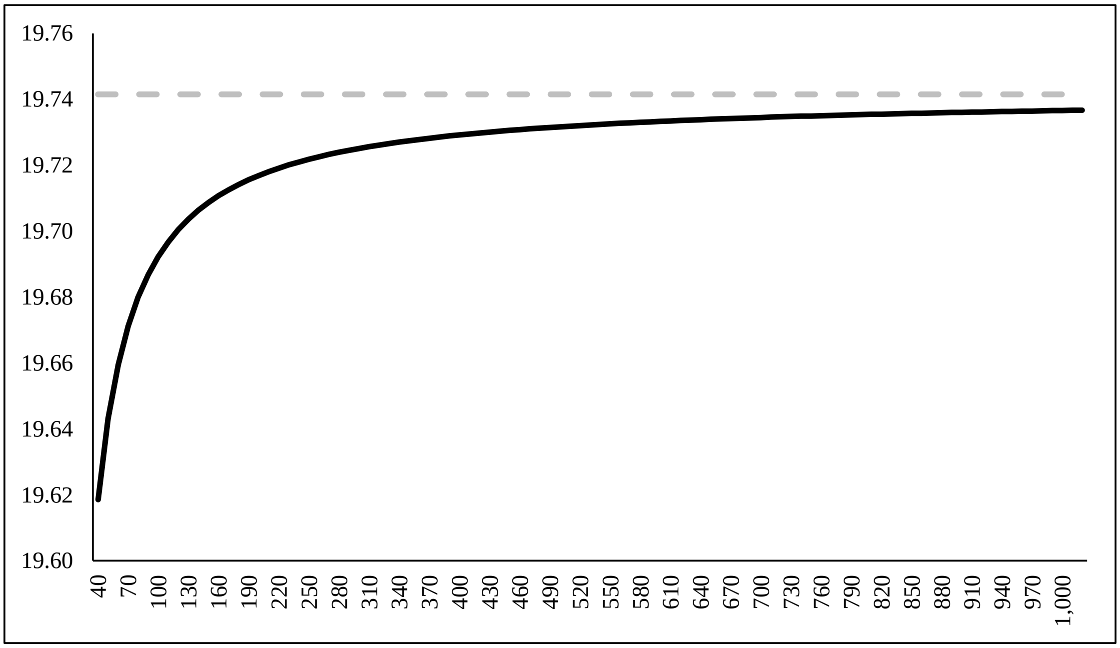


Figure 8: Convergence of binomial model to Black-Scholes price.

The same structure of the binomial tree can be used to price a variety of options. For European options or positions constructed from a combination of European options, the only change that needs to be made to the binomial tree is the equation used to price the value of the option at the terminal nodes of the tree.

This shows how easily the binomial tree can be modified to price a variety of options. However, the flexibility of using a binomial tree to price derivatives cannot fully shown by using only positions created from only European options. Because

European options can only be executed at maturity, the advantages of the iterative calculation of the derivative price is not fully leveraged. Additionally, there is no advantage to using a binomial tree to price a European option because the Black-Scholes equation is computationally faster than the binomial tree. This computational complexity is due to the many combinatorial calculations that are made with very large numbers.

The true value of the ability to price derivatives using a binomial tree comes from its flexibility. Because the price of the derivative is calculated at each node in the tree, there is the ability to make a decision at each point in time. By allowing for the ability to make decisions to be made at each node, it is able to price derivatives whose value depends on the path of the price of the underlying security.

American options are the classic example price path dependent derivatives. Because American options can be executed before expiration, the value of the option at each nonterminal node is the maximum of the payoff from early exercise, $S_i - K$, and the present value of the option at that node if early exercise were not allowed.

Figure 9 illustrates how the ability to execute before maturity can be worked into a binomial tree valuation of options. The binomial tree on the top is being used to price an American put option. The value of the put option at the terminal nodes is $\max(K - S_T, 0)$. Working backwards through the tree the value of an option at time 0.5 is the maximum of the present value of the expected value of the option at that time or the payoff for executing the put. Executing the option would result in a payoff of $5.595496 while the option would have been worth $0.870155 so it is more profitable to execute than to hold it until expiration.

The tree on the bottom shows how to derive the price for a European put with the same inputs. The difference between the two trees is highlighted in red. The difference in this node causes the American option to be worth much more than the European option. This again highlights the importance of the flexibility of the binomial tree model because a small change in the structure of the derivative can cause a large change in its fair value.

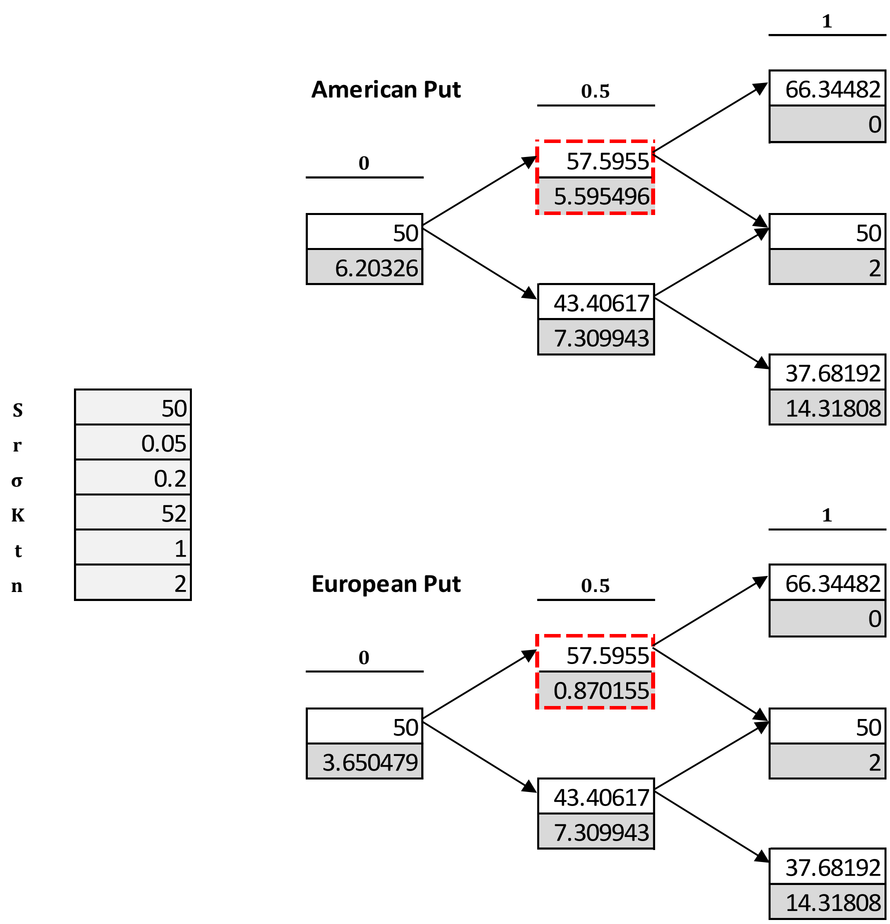


Figure 9: Comparison of binomial trees for American and European put options.

6.2 Monte Carlo Simulation to Price Options

Another alternative way to price derivatives is to use a Monte Carlo simulation. Monte Carlo simulations are computer programs that provide a large number of different outcomes for a given scenario. These outcomes are called sample paths. The results of these sample paths can be used in a few different ways. The most common is to make a calculation based on the outcome of each sample path. These calculations are then averaged to find the expected value of the calculation based on the assumed probability distribution of all possible outcomes. This type of analysis is useful for applications where there is uncertainty in the outcome of an event and a decision needs to be made depending on which outcome occurs. In the case of financial derivatives, the uncertain event is the price of the underlying security at the time of maturity and the decision to be made is the price an investor is willing to pay for the derivative.

In a Monte Carlo simulation used to price a derivative, the simulation first generates a large number of possible price paths for the underlying security. This is done by using the assumption the price of an asset can be modeled by a geometric Brownian motion process and that returns follow a lognormal distribution. It is necessary to use a large number of sample paths, typically at least 1000, to ensure that each path has only a small influence on the average in order to prevent a few extreme sample paths from skewing the results. For pricing options, these sample paths are created by generating a random number and using it in the stochastic portion of the equation for geometric Brownian motion to determine the price of the underlying security at expiration.

For each path, the final price of the underlying security is used to calculate the price of the option. The present value of that financial derivative is then calculated by discounting the value of the option by the risk free rate. These present values represent the fair price of the option for each path. The average of option's price in each sample path is the fair price for that option. Below is an example of a MATLAB script that can be used to price a straddle position. As the number of sample paths increases, the fair value calculated by the program approaches the price that the Black-Scholes equation gives.

```
nSim = 1000000;
r = .03;
sigma = .25;
strike = 100;
initS = 100;
t = 1;

randNums= normrnd(0,1, [1 nSim]);

for i = 1:nSim
      prices(1,i)  = initS * exp((r - 0.5*sigma^2)*t+
                        randNums(1, i)*sigma*sqrt(t));

      value(1,i) = max(strike - prices(1,i),
                    prices(1,i) - strike);

     optionPrice(1,i) = value(1,i) / exp(-r*t);
end

averagePrice = mean(optionPrice)
```

Figure 10: Sample code for pricing an option with a Monte Carlo simulation.

The Monte Carlo simulation method for pricing financial derivatives is commonly used in the industry. Similarly to the binomial tree method, it allows for a large amount of flexibility. However, the binomial tree is only able to handle the addition of decisions over the life of a derivative because all of the other variables, such as the volatility of the underlying asset and the risk free rate, are static over the life of the derivative. In addition to the capabilities of a binomial tree, the Monte Carlo simulation is able to allow these variables to change over time.

6.3 Comparing of Methods of Pricing Derivatives

Each method of pricing financial derivatives has its advantages and disadvantages. The most important characteristics to consider are each method's computational complexity, accuracy, and flexibility. Depending on what the use of the calculated price will be, one method may be preferred over another due.

A method's computational complexity is an important characteristic to consider when choosing which method to use. The Black-Scholes equation is the simplest computation. It requires only the ability to compute exponentials and use the standard normal cumulative distribution function. The Monte Carlo simulation follows the Black-Scholes equation in terms of computational complexity. It does not involve any computationally difficult functions but it requires a large number, up to millions, of iterations. The binomial model is the most computationally demanding because it requires the calculation of combinations. For binomial trees with more than approximately 1000 nodes, the combinatorial calculations involved are too large for even modern computers to compute.

The most accurate methods for pricing options is the Black-Scholes equation because this is the analytical price of an option. The other method's accuracies are compared to the result that the Black-Scholes equation gives. The Monte Carlo simulation is the next most accurate followed by the binomial model. Below is a chart showing the convergence of the models to the price given by the Black-Scholes equation. Note that there are not values for the binomial model with more than 1000 nodes. The computational complexity is too great for the computer to handle as mentioned above. However, we have shown that the binomial tree method does converge to the Black-Scholes equation price as the number of nodes approaches infinity. Therefore, in the event that a computer has the ability to handle the computation, the binomial price model would be more accurate than a Monte Carlo simulation.

n	**Binomial Model Price**	**Monte Carlo Price**
2	17.4960	47.4629
10	19.2550	26.5176
100	19.6922	21.2230
1000	19.7366	19.3600
10000	-	19.7625
100000	-	19.7423

BS Model Price	19.7415

Figure 11: Speed of convergence of pricing methods to Black-Scholes price.

A method's flexibility, or its ability to handle different types of financial derivatives, is an important characteristic. While the Black-Scholes equation is the most accurate and least computationally demanding for simple European options, it is not able to handle the conditions of more complicated financial derivatives. For example, it cannot account for early exercise in American option or a variable risk free interest rate. The binomial model is more flexible than the Black-Scholes equation because it allows for decisions at each node. However, the Monte Carlo simulation is the most flexible. It is able to allow for both decisions at intermediate steps as well as variable values for the parameters.

7 Conclusion

Financial derivatives are an integral part of both business activities and the financial markets. As the largest financial security market, estimated to be more than $1.2 quadrillion, the ability to price financial derivatives is a necessity. However, pricing financial derivatives is not a straightforward task because the value of the derivatives is dependent on the movement of an underlying security which is inherently random.

As it has been shown, for even the simplest financial derivatives, European options, a significant amount of sophisticated mathematical molding is necessary to find an analytical solution for its fair value. First, the price of the underlying asset is modeled using a stochastic process, geometric Brownian motion. Because geometric Brownian motion has unbounded variation, Reimann-Stieltjes integral cannot be used.

Instead, Ito's lemma, the foundation of stochastic calculus, must be used to find the equation that models the underlying asset's movement.

Based on the model for the movement of the underlying asset's price, one can find a partial differential equation describing the fair value of a financial derivative. After a transformation of variables, this partial differential equation can be represented in the form of the extensively studied heat equation. The heat equation has a well-known solution that is used to find the equation for the fair price of a call option, known as the Black-Scholes equation.

While the call option has a closed solution, not all financial derivatives have conditions that allow for the existence of a solution. This creates a need for alternative methods of pricing financial derivatives. Binomial trees and Monte Carlo simulations are two other methods that can be used. For European call options, both methods converge to the price given by the Black-Scholes equation. Both of these methods are more flexible in handling conditions than the Black-Scholes equation. The flexibility is not leveraged when using these methods to price European options. However, the ability to handle complex conditions is extremely important for pricing other, less straightforward types of financial derivatives.

New types of financial derivatives are always being created. This creates a constant need to modify existing pricing techniques and to develop new methods for calculating the fair value of new financial derivatives. Further work can be done on the pricing of novel financial derivatives by expanding on the basic foundations presented here.

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