A STUDY IN HOMOLOGY

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1 Topology: Basic Ideas

Topology is a field of mathematics that studies basic properties of a space or object, such as connectedness or the presence of "holes" in the space, that are preserved under continuous deformation.

In a sense, topology can be thought of as an abstract geometry that has a looser idea of what it means to be the "same." In geometry, two shapes or objects are the "same," or congruent, if one is an exact copy of the other: edges have the same length, meet at the same angles, sides have the same area, and volumes also match. In topology, things are different. In topology, two objects are the "same," or homeomorphic, if one can be continuously deformed into the other. In technical terms, a homeomorphism is a continuous, one-to-one and onto function with a continuous inverse, and if a homeomorphism exists between two spaces, they are homeomorphic. But what exactly does that mean?

Take an object. Any object that you can imagine. Now imagine that that object is made of an incredibly elastic or pliable material. You can fold the object, straighten it, stretch it, compress it,[1] or any such process. The things you are not allowed to do are cut or remove part of the object, as well as gluing anything to the object, or the object to itself. Any object into which you can turn your original object under these rules is homeomorphic to the original object.

There is a classic joke among mathematicians that a topologist cannot tell his doughnut from his coffee mug, because they are homeomorphic. Figure 1 shows the homeomorphism. As another example, triangles are homeomorphic to a circle. In fact, all simple polygons are homeomorphic to a circle: It is easy to see by simply smoothing out every corner and curving every side. That is to say that nearly all objects studied in

[1] It should be noted that you cannot compress any part of the object to an object of lower dimension: No volume to an area, area to a line, or line to a point. This is, in a way, a sort of gluing, which is not allowed. You may compress a volume to something so thin it may seem like an area, but it must still have volume.

basic geometry are topologically equivalent.

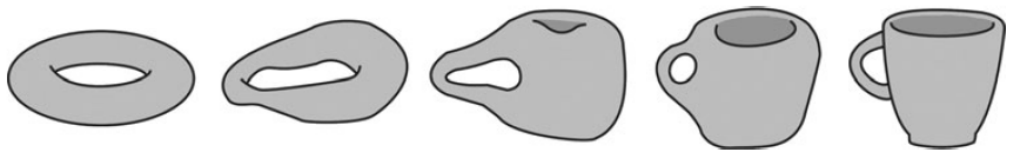

Figure 1: Accessed from rioranchomathcamp.com on February 28, 2013

But then the question is raised: What objects are *not* homeomorphic? For starters, a doughnut (or torus, as it is called) is not homeomorphic to a circle– the volume inside the torus cannot be removed. As another example, an “8” is not homeomorphic to a circle. In order to morph an “8” into a circle, you would need to remove the center point, and then glue the ends together.

It is easy to visualize this morphing process for the objects we have described. But how do we determine with mathematical rigor whether two spaces are homeomorphic? Unfortunately, often the only way to do so is to actually show the homeomorphism– to construct a continuous, one-to-one and onto function with a continuous inverse from one space to the other. For “simple” spaces, this is not necessarily difficult, but what of elaborate spaces? For that matter, what about spaces that exist in 4 or higher dimensions? They would be especially problematic as we could not even visualize the morphing.

Fortunately, while it may be difficult to determine if two spaces are homeomorphic, it is often much easier to determine if two spaces are *not* homeomorphic. This is because of what are known as topological properties (sometimes topological invariants)– properties of a space that are preserved by homeomorphisms. That is to say that if a space has a certain property, then any space homeomorphic to it also has that property. Finding two spaces that share such a property is not enough to conclude they are homeomorphic, because homeomorphic spaces must share *all* topological properties. For example, a fairly intuitive topological property is path-connectedness. We say a space is path-connected if we can take any two points in the space and "draw" a path from one to the other that stays entirely within the space. Both the circle and an "8" are path-connected, but as previously discussed, the circle and "8" are not homeomorphic. But in contrast, if we have two spaces and find a topological property that they do not share, then we can know that they are not homeomorphic. For example, if we take a circle and remove any point from it, the resulting space will still be connected. On the other hand, if we remove the center point of the "8," the resulting space is not connected.

Much of topology is concerned with finding topological invariants and drawing conclusions about a space as a result of that space having or lacking given properties. These conclusions can sometimes inform facts about the real world. For example, the somewhat humorously named "hairy ball theorem" leads us to the conclusion that at any given time, there must be at least one point on the surface of the Earth where the wind is not blowing. There is also the Borsak-Ulam Theorem, which allows us to conclude that at any time, there must be a pair of antipodal points on the surface of the Earth which have the same temperature.

However, rather than deriving any such conclusions, we will be constructing a certain topological invariant: the (simplicial) homology groups.

2 Algebraic Topology and Homology: An Overview

Algebraic Topology was actually begun by Henri Poincarè nearly two decades before general topology (Point-Set Topology) became a field of math on its own; in fact, it was initially called analysis situs [4]. However, it too dealt with underlying structure of spaces which are not dependent on specific distances or angles as in geometry. When it became apparent that the two were studying similar phenomena (properties of spaces preserved by homeomorphisms), they came together under a common name. Poincarè thought the point-set developments to be a “disease” which would later be cured [4], but it would seem that he was incorrect.

The idea that started Algebraic Topology was one that most, if not all mathematicians exploit: to take a new thing that is not well understood, and turn it into a problem from another, well-developed field. Algebra was a field of math fairly well understood by this time, so by creating an algebraic structure on spaces and understanding what that structure means, certain information about the space can be gleaned. Of course, we want this algebraic structure to be topologically invariant, or it would not actually be yielding any information about the topological structure of the space.

There are two main types of groups (the algebraic structure we will be working with; see Section 3 for information on groups) that developed in algebraic topology: homotopy groups and homology groups. The homotopy groups (especially the first homotopy group, otherwise known as the fundamental group) are very interesting, and merit their own discussion. However, to give a proper treatment of even the fundamental group would take too long. Instead, we focus our attention on the homology groups. A fully rigorous development of homology groups (or, to be more precise, simplicial homology groups) will be presented over the course of the remaining sections. For now, we simply give an overview of the basic ideas.

Suppose you have a sphere, and draw a loop on the surface of it. Notice that no matter how you draw the loop, part of the surface of the sphere is “inside” the loop (if you happen to draw an equatorial loop on the sphere, you can still consider one hemisphere to be “inside” while the other is “outside”). Compare this to a torus. Figure 2 shows how things are not quite the same as on the sphere. The loop labelled c is just like all loops on the sphere: it encloses a region of the torus itself. However, the loops a and b are different. Instead of enclosing a part of the torus, they enclose a hole. These loops are more interesting than the loop c, since in a way, c “has” to be there. a and b

only exist because of the specific structure of the torus.

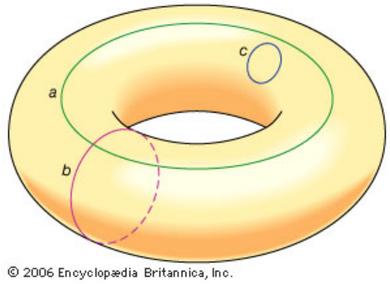

Figure 2: **Accessed from www.britannica.com on April 3rd, 2013**

Now, suppose we had another loop, call it a', which runs parallel to a, but is slightly tighter, by running closer to the center hole of the torus. This loop also encloses a hole in the torus. However, the hole that it encloses is the same hole that a encloses. a' does not give any information about the torus that we do not get from a, and therefore we would like to consider them to be the "same" loop. How do we do that? Notice that if we take the loops a and a' together, they enclose a band on the surface of the torus. We already noted that the loop c was "uninteresting" because it enclosed part of the surface of the torus, and so by extension, we should say that having both of these loops together is "uninteresting." Note that taking the loops a and b together does not enclose a region on the surface of the torus. This is good, because we want these loops to be distinct–they enclose different holes of the torus.

There are obviously more intricacies to homology groups; for example, we can generalize the idea of a loop (or as we will call it, a cycle) to different dimensions and have homology groups in different dimensions. But that is the basic premise: finding "interesting" cycles on a space, and ignoring all uninteresting cycles. Making this mathematically rigorous, however, will require a lot of effort, and will start with something that at first will seem completely unrelated. It will, however, be exactly what we need to define the homology groups. The culmination of this work will be the computation of the homology groups of a class of topological spaces known as surfaces. In addition, we will reverse the process– taking certain groups, and constructing a space that has that as its (first) homology group.

3 Algebraic Background

Before delving into the long construction of homology theory, it is important to discuss some underlying algebraic concepts which, while not a focus, must be known. Since this is not the focus, treatment of this subject will be brief, and no theorems will be presented. All definitions in this section have been taken from Fraleigh [6], with modifications as it is deemed fit.

Definition 1. Let S be a set. A *binary operation* $*$ is a function mapping $S \times S$ into S. For each $(a, b) \in S \times S$, we will denote $*((a, b))$ by $a * b$.

Definition 2. A *group* $(G, *)$ is a set G, closed under a binary operation $*$ such that the following hold:

1. $(G, *)$ is *associative*; that is, for all $a, b, c \in G$, $a * (b * c) = (a * b) * c$.
2. There is an element $e \in G$ called the *identity element* such that for all $a \in G$, $e * a = a * e = a$.
3. For each element $a \in G$, there is an element $a^{-1} \in G$ called the *inverse* of a, such that $a * a^{-1} = a^{-1} * a = e$.

When it is clear from context what the operation of a particular group is, it is usually referred to merely by the set on which it is defined in order to simplify notation. For example, a group that is very frequently used is the integers under the normal addition. We refer to this group as $\mathbb{Z}$ rather than $(\mathbb{Z}, +)$. Technically this is an abuse of notation, but it is standard.

The idea of a group can be difficult to understand at first because it is such an abstract concept. However, groups do appear in the real world. But before these examples can be introduced, a special type of group must first be defined.

Definition 3. Given a set A, a *permutation of* A is a function $\phi : A \to A$ that is both one-to-one and onto.

Definition 4. The *permutation group of* A is the collection of all permutations of A under the operation of composition of functions.

At first these definitions still seem incredibly abstract, but that abstractness is simply mathematical rigor applied to something that can be observed naturally. To help understand, imagine having 4 objects, say, a circle, square, triangle, and an X. They can be arranged in any order. Say to start they are in the order given previously. That is, the circle is the first object, and the X is the last. Now give them a different order. Take the object in, say, the 4th position (in this case, the X), and move it to the first position. Now the X is first, the circle second, the square third, and the triangle fourth.

This rearrangement was a permutation of the positions of the objects. To see how this aligns with the definition given, suppose this permutation is denoted ϕ_1. If $1, 2, 3$ and 4 denote the position of each object. Thus, $\phi_1(1)$ denotes the new position of the object originally in position 1 (the circle) after undergoing the permutation. That is, $\phi_1(1) = 2$. Similarly, $\phi_1(2) = 3$, $\phi_1(3) = 4$ and $\phi_1(4) = 1$.

Now perform a new permutation. Take the square (Currently in position 3), and switch it with the triangle (in position 4). If we call this new permutation ϕ_2, then we have $\phi_2(3) = 4$ and $\phi_2(4) = 3$. Further, the X (position 1) and circle (position 2) did not move. Thus, $\phi_2(1) = 1$ and $\phi_2(2) = 2$.

However, what if we considered this permutation as a permutation of the original positions? We can accomplish this by composing our permutations. Let $\phi = \phi_2 \circ \phi_1$. Originally, the circle was in position 1. After applying ϕ, we see: $\phi(1) = \phi_2(\phi_1(1)) = \phi_2(2) = 2$. As expected, the circle is now in position 2. Similarly, we find that $\phi(2) = 4$,

$\phi(3) = 3$ and $\phi(4) = 1$. Thus, we have taken 2 permutations, composed them, and produced a new permutation. As such, composition induces a binary operation on the collection of permutations. Further, properties of (one to one and onto, in one case) functions guarantee the other properties of groups necessarily hold, so this collection forms a group. Typically, this group in particular is denoted S_4.

For a slightly more complicated example of a real world permutation group, consider the Rubik's Cube. Suppose you have a solved Rubik's Cube, and pick a side to fix. That is, for whichever color is fixed, the center square on the side facing you will always be that color. Imagine assigning to each of the 54 squares a number. Now, suppose you rotate the right side of the cube counter-clockwise by $90°$ (such that squares that were previously on the top of the cube are now facing you). You see that some of the squares have shifted positions. Perhaps square number 3 is now in the position that square number 12 once occupied. Thus, we have permuted the squares of the Rubik's Cube.

Notice that we could have rotated the left side of the cube. Or even the top, bottom, front, or back. We also could have rotated the right two sides of the cube, but this would have been equivalent to a rotation of the left side of the cube. We also could have rotated the right side by $180°$ or $270°$, but that would be equivalent to two or three $90°$ rotations. Similarly, we could have rotated the right side clockwise by $90°$, but this is equivalent to a $270°$ counter-clockwise rotation (which, as previously discussed, is three $90°$ counter-clockwise rotations). Thus we see that any permutation on the Rubik's Cube can be generated by (often very long) combinations of the six basic permutations: $90°$ counter-clockwise rotations of the right, left, top, bottom, front, and back. It should be noted that solution algorithms for the Rubik's Cube are based on this and its remaining underlying group structure.

Something should be observed about both S_4 and the Rubik's Cube's group. In the latter, for example, notice that performing a right rotation followed by a top rotation will yield a different permutation than first performing a top rotation followed by a right rotation. A similar phenomenon can be observed in S_4. The reason this occurs is because (almost) all permutation groups lack a certain property: their operation is not commutative.

Definition 5. Let $(G, *)$ be a group and $a, b \in G$. If $a * b = b * a$, then it is said that a *and* b *commute*. If for all $a, b \in G$, a and b commute, it is said that G is an *Abelian Group*.

As homology groups will necessarily be Abelian, it will be assumed at any groups discussed are Abelian, unless it is specifically mentioned that we are dealing with a non-Abelian group (such as S_4 or the Rubik's Cube group). Further, it is common when dealing with Abelian groups to let "+" denote the group operation.

There are still are concepts of importance left to be addressed. First, consider S_4 once more. Suppose we only performed permutations that left the X in the fourth position. Then combining any two of these permutations will result in another permutation that leaves the X alone. Further, these elements are still associative, have an identity (the permutation that moves no object does not move the X in particular), and each have an inverse which leaves the X alone. Thus we see that there is a group "within" S_4 such that combining any two elements within it produces another element within it. This is what is known as a subgroup.

Definition 6. Let $(G, *)$ be a group and $H \subset G$. If $*$ induces a closed operation on H (that is, for all if $a, b \in H$, $a * b \in H$), and H along with the operation induced by $*$ forms a group itself, then H is a *subgroup of* G, and we write $H < G$.

It should be noted that G is a subgroup of itself. This will be important later.

Now suppose in addition to our four objects in S_4, we had another three objects. But instead of considering permutations of all seven objects together, we only considered permutations which permute the first four objects with each other, and the new three object with each other, but never permute the first set with the second set. What we essentially get is a new group that has two subgroups which do not interact with each other. That is the idea of the direct sum of groups.

Definition 7. Let $G_1, G_2, \ldots, G_n$ be abelian groups. The group $G_1 \oplus G_2 \oplus \ldots \oplus G_n = G$, called the *direct sum of the groups* G_i, has elements of the form $(g_1, g_2, \ldots, g_n)$, where $g_i \in G_i$. Further, the operation is defined such: if $(a_1, a_2, \ldots, a_n), (b_1, b_2, \ldots, b_n) \in G$, then $(a_1, a_2, \ldots, a_n) + (b_1, b_2, \ldots, b_n) = (a_1 + b_1, a_2 + b_2, \ldots, a_n + b_n)$.[2]

Again consider S_4. Suppose that instead of a circle, square, triangle, and X, we had an oval, rectangle, pentagon, and Z. Would the resulting permutation groups be any different? We should hope not, or there would be an incredibly large number of groups possessing identical structure. Thus, we introduce the concept of an isomorphism.

Definition 8. Let $(G, *)$ and $(H, \times)$ be groups. We call a function $\psi \colon G \to H$ an *isomorphism* if ψ is one to one, onto, and for all $a, b \in G$, $\psi(a * b) = \psi(a) \times \psi(b)$. If there exists an isomorphism between G and H, we say G and H are *isomorphic* and write $G \cong H$.

Thus we have that S_4 as we originally defined it and this "new" S_4 are isomorphic. It should be noted that though in this example these two representations of S_4 "looked" very similar, it is not always the case that isomorphic groups "look" the same. For example, $(\mathbb{R}, +)$, the real numbers under addition, is isomorphic to $(\mathbb{R}^+, \times)$, the positive real numbers under multiplication, via the isomorphism $\psi(x) = e^x$. Lastly note that as

[2] Slightly more generality can be gained by allowing the groups to be non-Abelian, but because a different term is used for this, and we will not be working with non-Abelian groups, this is not addressed.

isomorphisms form an equivalence relation, people will often refer to two isomorphic groups as being the same group: "This group *is* that group" rather than "this group *is isomorphic to* that group." While technically this is an abuse of terminology, it is a widespread practice and little care will be given to ensure this distinction is made.

Sometimes there are maps that, like isomorphisms, preserve structure, but do not preserve the "size" of the groups. Still, these maps can provide useful information, and will be quite essential for developing homology groups.

Definition 9. Let G, H be groups, with operations notated by juxtaposing elements. A map $\varphi : G \to H$ is a *homomorphism* if for all $a, b \in G$, $\varphi(ab) = \varphi(a)\varphi(b)$.

There are two very important subgroups associated with homomorphisms.

Definition 10. Let $\varphi : G \to H$ be a homomorphism. The *kernel* of φ, ker φ = $\{g \in G \mid \varphi(g) = e\}$, where e is the identity in H. The *image* of φ, im $\varphi = \{h \in H \mid \exists g \in G$ s.t. $\varphi(g) = h\}$. It is known that ker $\varphi < G$ and im $\varphi < H$.

There are two more important ideas left to discuss, but both of them go together. To begin, fix a non-zero positive integer n. If we divide any integer by n, then there are a limited number of possibilities for the remainder. In fact, if we let $r = m \mod n$, where $m \mod n$ denotes the remainder of $\frac{m}{n}$, then $r \in \{0, 1, \dots, n-1\}$. We can also see that if $r_1 = m_1 \mod n$ and $r_2 = m_2 \mod n$, then $(m_1 + m_2) \mod n = (r_1 + r_2) \mod n$. For example, if $n = 7$, $m_1 = 10$ and $m_2 = 6$, then $m_1 + m_2 = 16$, and $16 \mod 7 = 2$. But $r_1 = 3$ and $r_2 = 6$, so $r_1 + r_2 = 9$, and $9 \mod 7 = 2$.

Thus we suspect that there is some group structure here. That is in fact the case. But before we can define the group structure, another object must be defined.

Definition 11. Let G be a group and $H < G$. If $a \in G$, then we define $a + H = \{g \in G \mid \exists h \in H$ s.t. $g = a + h\}$ as *the coset of H containing a*.

Definition 12. Let G be a group and $H < G$. We denote by G/H the *factor group* of G by H, whose elements are the cosets of H. The operation on G/H is defined as such: if $a + H, b + H \in G/H$, then $(a + H) + (b + H) = (a + b) + H$.[3] We also say that this is the group obtained by *modding H out of G*

It might be difficult to see at first, but the above example describes a factor group. In this case, G will be $\mathbb{Z}$ and H is $n\mathbb{Z}$, the multiples of n. Note that if $m_1, m_2 \in \mathbb{Z}$, and if $n \mid (m_2 - m_1)$, then m_1 and m_2 are in the same coset of $n\mathbb{Z}$. For if $m_2 - m_1 = nk_1$ and $m_1 = a + nk_2$, then $m_2 = nk_1 + (a + nk_2) = a + n(k_1 + k_2)$. Thus we see that the only unique cosets are those of the form $r\mathbb{Z}$ where $r \in \{0, 1, \ldots, n-1\}$. It is no coincidence that this set coincides with the set of possible remainders.

4 Geometric Independence and Simplices

Constructing the simplicial homology groups takes some work. To start, we have a definition:

Definition 13. A set $A = \{x_0, x_1, \ldots, x_k\}$ of $k + 1$ points in $\mathbb{R}^n$ is *geometrically independent* if no $k - 1$ dimensional hyperplane contains all of them [4]. Otherwise, if all points of A do lie on a $k - 1$ dimensional hyperplane, we say A is *geometrically dependent*.

Definition 13 essentially says that if A is geometrically independent, each of the points of A are in their own dimension. For an example, see Figure 3. In (a), the set $\{a_0, a_1, a_2\}$ is geometrically independent. All points lie on the same plane, but not the same line, which is a 1-dimensional hyperplane. By contrast, in (b), the set $\{b_0, b_1, b_2\}$ is geometrically dependent, because all points are collinear.

[3] Usually, in order for this operation to be well-defined, the subgroup H need have another property known as normality. But in Abelian groups, all subgroups are normal, so this is ignored.

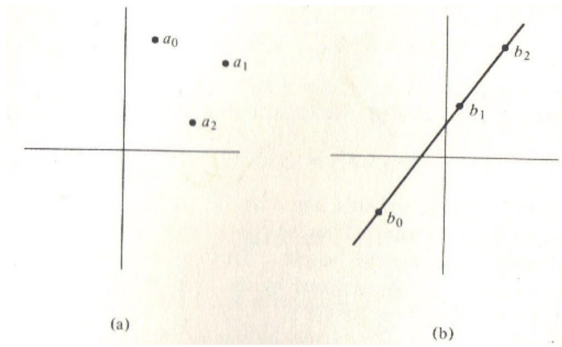

Figure 3: Scanned from Croom, page 8 [4]

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There are a couple of equivalent definitions of geometric independence which may prove useful later.

Proposition 1. *A set $A = \{x_0, x_1, \ldots, x_k\}$ of $k+1$ points is geometrically independent if and only if for $p \leq k$ no $p+1$ of the points lie on a hyperplane of dimension less than or equal to $p-1$.*

Proof. Note that one direction of this proof is trivial. If we know that no $p+1$ of the points lie on a hyperplane of dimension less than or equal to $p-1$, then in particular we know that all $k+1$ of the points do not lie on a hyperplane of dimension $k-1$.

To prove the other implication, suppose the converse. Suppose there exists $A_p \subset A$ such that $|A_p| = p+1$ and all points of A_p lie on a hyperplane of dimension less than or equal to $p-1$, call it H^*. Without loss of generality, let $\dim H = p-1$ and let $A_p = \{x_0, x_1, \ldots, x_p\}$. Lastly, let H^* have basis $B^* = \{b_1, b_2, \ldots, b_{p-1}\}$ and translation vector v.

Now, for all $x_j \in A \backslash A_p$, there exists $b_j \in \mathbb{R}^n$ such that $x_j = h_j + b_j$ for some $h_j \in H^*$. Define a new hyperplane, H, with basis $B = B^* \cup \{b_{p+1}, b_{p+2}, \ldots, b_k\}$ and translation vector v. Note that $|B| = k-1$.

Clearly $A \subset H$, as for $x_j \in A_p$, $H^* \subset H$, and for $x_j \in A \backslash A_p$:

$$x_j = h_j + b_j = v + \sum_{i=1}^{p-1} a_{i,j} b_i + b_j \text{ for } a_{i,j} \in \mathbb{R}$$

which is in H. Further, $\dim H = k - 1$. Thus, A is geometrically dependent. $\square$

In a sense, Proposition 1 says that a set is geometrically independent if and only if any subset of it is geometrically independent. This is, however, a self-referential statement, so it does not make a good definition. The next proposition provides a more useful defintion.

Proposition 2. *A set $A = \{x_0, x_1, \ldots, x_k\}$ of points in $\mathbb{R}^n$ is is geometrically independent if and only if the set of vectors $\{x_1 - x_0, x_2 - x_0, \ldots, x_k - x_0\}$ is linearly independent.*

The proof of Proposition 2 involves a lot of linear algebra and provides little new insight, so it is skipped. The statement is more useful, though, because it lends itself readily to the well-developed theory of linear algebra.

Definition 14. Let $A = \{x_0, x_1, \ldots, x_k\}$ be a set of points. The *k-simplex*, σ^k, spanned by A is the set of all $x \in \mathbb{R}^n$ for which there exist nonnegative real numbers $\lambda_0, \lambda_1, \ldots, \lambda_k$ such that

$$x = \sum_{i=0}^{k} \lambda_i x_i \quad \text{and} \quad \sum_{i=0}^{k} \lambda_i = 1.$$

The numbers λ_i are the *barycentric coordinates* of the point x. The points a_i are the *vertices* of σ^k [4].

We see that a 0-simplex is a single point, a 1-simplex is a line segment, a 2-simplex is a triangle with interior, and a 3-simplex is a tetrahedron with interior. Higher dimensional simplices cannot be easily visualized, but for the results presented here, only 0-, 1-, and 2-simplices will be needed. Note, 1-simplices will sometimes be referred to as edges.

Proposition 3. *Let $A = \{x_0, x_1, \ldots, x_k\}$ be vertices for a k-simplex σ^k. Then for each $x \in \sigma^k$, $\lambda_0, \lambda_1, \ldots, \lambda_k$, the barycentric coordinates of x, are unique.*

Proof. Suppose x has another set of barycentric coordinates, $\lambda'_0, \lambda'_1, \ldots, \lambda'_k$. Consider

$$\sum_{i=0}^{k} (\lambda_i - \lambda'_i)(x_i - x_0) = \sum_{i=0}^{k} (\lambda_i x_i - \lambda'_i x_i + \lambda_i x_0 - \lambda'_i x_0)$$

Now, by definition, we have that the first two terms must sum to x. Further, because barycentric coordinates must sum to 1, the last two terms must sum to x_0. Thus $\sum_{i=0}^{k} (\lambda_i - \lambda'_i)(x_i - x_0) = x - x + x_0 - x_0 = 0$ And since the set $\{x_1 - x_0, x_2 - x_0, \ldots, x_k - x_0\}$ is linearly independent by Proposition 2, we must have that for each i, $\lambda_i - \lambda'_i = 0$, or $\lambda_i = \lambda'_i$. □

Notice that if we have a k-simplex, σ^k and for a given vertex x_i, if we consider only points of σ^k whose ith coordinate is 0, those points themselves form a simplex. The idea is addressed below.

Definition 15. A simplex σ^n is a *face* of a simplex σ^k, $n \leq k$ if every vertex of σ^n is also a vertex of σ^k [4].

We will shortly prove a property of simplices, but first, a definition.

Definition 16. We say a set X is *convex* if for any two points $a, b \in X$, the line segment connecting a and b is contained in X. That is, if $x = ta + (1 - t)b$, where $0 \leq t \leq 1$, $x \in X$. If $A \subset \mathbb{R}^n$, we say X is the *convex hull of* A if it is the smallest convex set such that $A \subset X$.

Proposition 4. *A k-simplex σ^k is the convex hull of its set of vertices.*

Proof. We must first prove σ^k is convex. To do so, take $a, b \in \sigma^k$. Let $a = \sum_{i=0}^{k} \lambda_i x_i$ and $b = \sum_{i=0}^{k} \lambda_i' x_i$. If $x = ta + (1-t)b$, $0 \leq t \leq 1$, then

$$x = \sum_{i=0}^{k} t\lambda_i x_i + \sum_{i=0}^{k} (1-t)\lambda_i' x_i$$

or

$$x = \sum_{i=0}^{k} (t\lambda_i + (1-t)\lambda_i')x_i$$

As t, $(1-t)$, λ_i, and λ_i' are all nonnegative, $\lambda_i'' = (t\lambda_i + (1-t)\lambda_i')$ is nonnegative. Further, since λ_i and λ_i' sum to 1 and $t + (1-t) = 1$, λ_i'' are barycentric coordinates for x. Thus $x \in \sigma^k$

Now, let X_α br an arbitrary convex set containing the vertices of σ^k and let $X = \bigcap X_\alpha$. We wish to show $\sigma^k = X$. That $X \subset \sigma^k$ is trivial, as X is the intersection of all convex sets containing the vertices of σ^k, and σ^k is merely one such convex set.

To prove $\sigma^k \subset X$, we will prove that for any arbitrary X_α, $\sigma^k \subset X_\alpha$. The idea is to show that all of the n-faces of $\sigma^k \subset X_\alpha$ by induction on n. Clearly the 0-faces are in X_α as they are the vertices. Further, all 1-faces are in X_α, as a 1-face is a line segment connecting two vertices, and σ^k is convex.

Let σ^2 be a 2-face of σ^k. Without loss of generality, suppose σ^2's vertices are x_0, x_1, x_2. Let $x \in \sigma^2$ and let $x = \lambda_0 x_0 + \lambda_1 x_1 + \lambda_2 x_2$. Now fix λ_0 and let λ_1, λ_2 vary between 0 and $1 - \lambda_0$ (such that $\lambda_0 + \lambda_1 + \lambda_2 = 1$. See Figure 4). Let x'' be the point where $\lambda_1 = 0$ and x' be the point where $\lambda_2 = 0$.

Clearly x', x'' are on (distinct) 1-faces of σ^k and thus $x', x'' \in X_\alpha$. But X_α is convex, and x lies on the line segment connecting x', x'', so $x \in X_\alpha$.

For higher dimensional faces, the proof is similar. Fix all $k-2$ of the coordinates of the point, and let the last two vary. Proceed as before. $\square$

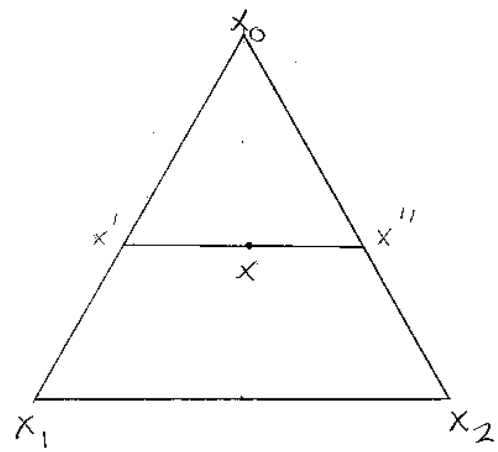

Figure 4

Note that an individual simplex can only produce a few spaces, even under homeomorphisms. Therefore, we next look to joining many simplices together.

Definition 17. Two simplices, σ^n and σ^m are *properly joined* provided that either $\sigma^m \cap \sigma^n = \emptyset$ or $\sigma^m \cap \sigma^n$ is a face of both σ^n and σ^m [4].

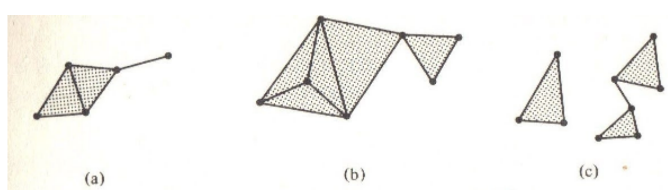

Figure 5: Scanned from Croom, page 9 [4]

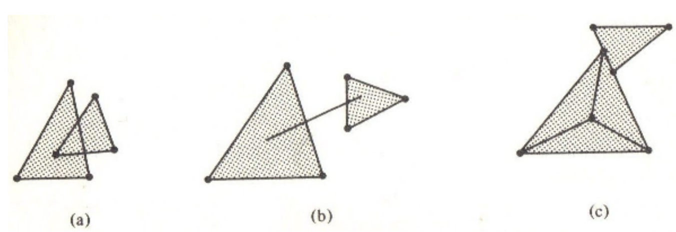

Figure 6: Scanned from Croom, page 9 [4]

Figure 5 shows examples of simplices that are properly joined, while Figure 6 gives examples that are not.

Definition 18. A (*geometric or simplicial*) *complex* is a finite family K of simplices which are properly joined and have the property that each face of a member of K is also a member of K. The *dimension* of K is the largest integer r such that K has an r-simplex. The union of K with the Euclidean subspace topology is denoted by $|K|$ and is called the *geometric carrier of* K or the *polyhedron associated with* K, or sometimes the *underlying space of* K [4].

Polyhedra are relatively simple to visualize, and most common topological spaces are polyhedra, so homology theory initially developed through them [4]. It is this classic path of development that we are traveling.

Before moving on, one more notion needs to be made precise.

Definition 19. Let X be a topological space. If there is a geometric complex K such that $|K|$ is homeomorphic to X, then X is a *triangulable space*, and K is a *triangulation of* X [4].

We will only be working with triangulable spaces, though we will take spaces known to be triangulable, and then triangulate them.

5 Oriented Complexes

Definition 20. An *oriented n-simplex*, $n \geq 1$ is obtained from an n-simplex, $\sigma^n = \langle x_o \dots x_n \rangle$ by choosing an ordering for its vertices. The equivalence class of even permuations of the chosen ordering determines the *positively oriented simplex* $+\sigma^n$ while the equivalence class of odd permutations determines the *negatively oriented simplex* $-\sigma^n$. An *oriented complex* is obtained from a complex by assigning an orientation to each of its simplices [4].

Orienting simplices in a sense gives us a direction to move around the vertices.

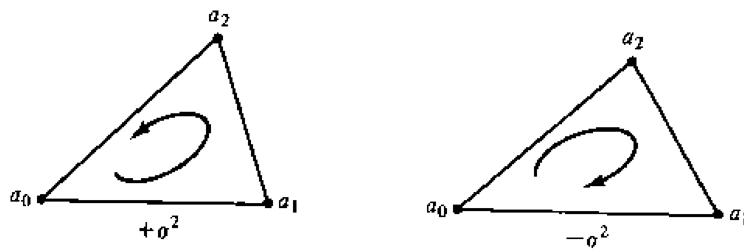

Figure 7: Scanned from Croom, page 12 [4]

For example, if we had an oriented 1-simplex, $+\sigma^1 = \langle a_0 a_1 \rangle$, then $\langle a_0 a_1 \rangle$ is moving "forward" through the simplex, while $\langle a_1 a_0 \rangle$ would be moving "backward." Similarly for an oriented 2-simplex, $\sigma^2 = \langle a_0 a_1 a_2 \rangle$, $\langle a_0 a_1 a_2 \rangle$ is moving forward while $\langle a_2 a_1 a_0 \rangle$ is moving backward. Note that in this case, these are not the only representations of $\pm\sigma^2$. For example, we also have $+\sigma^2 = \langle a_1 a_2 a_0 \rangle$ and $-\sigma^2 = \langle a_0 a_2 a_1 \rangle$. In fact, as we see in Figure 7 that any ordering that goes through the vertices counter-clockwise will be $+\sigma^2$ and any going clockwise will be $-\sigma^2$. For higher dimensional simplices, it is difficult to use geometric intuition to distinguish orientations, but we will not be dealing with such simplices.

Definition 21. Let K be an oriented complex with simplices σ^{p+1} and σ^p whose dimensions differ by 1. We associate to each pair (σ^{p+1}, σ^p) an *incidence number* $[\sigma^{p+1}, \sigma^p]$, which is defined as follows: If σ_p is not a face of σ_{p+1}, then $[\sigma^{p+1}, \sigma^p] = 0$. Otherwise, label the vertices of σ^p such that $+\sigma^p = \langle x_0 \dots x_p \rangle$. Let v be the vertex of σ^{p+1} not in σ^p. Thus we must have that $+\sigma^{p+1} = \pm\langle v x_0 \dots x_p \rangle$. If $+\sigma^{p+1} = +\langle v x_0 \dots x_p \rangle$, then $[\sigma^{p+1}, \sigma^p] = 1$. If $+\sigma^{p+1} = -\langle v x_0 \dots x_p \rangle$, then $[\sigma^{p+1}, \sigma^p] = -1$ [4].

We need incidence numbers to prove the next theorem, which will be of critical importance to defining the homology groups in the next section.

Theorem 5.1. *Let K be an oriented complex, σ^p an oriented p-simplex of K and σ^{p-2} a $(p-2)$-face of σ^p. Then*

$$\sum_{\sigma^{p-1}\in K}[\sigma^p,\sigma^{p-1}][\sigma^{p-1},\sigma^{p-2}]=0.$$

Proof. The following proof is taken from Croom ([4]), though there are 2 cases at the end of the proof which he leaves as an exercise.

Label the vertices of σ^{p-2} as $x_0,\dots x_{p-2}$ such that $+\sigma^{p-2}=\langle x_0\dots x_{p-2}\rangle$. Since σ^{p-2} is a face of σ^p, there are two additional vertices, a,b, and we may assume without loss of generality that $+\sigma^p=\langle abx_0\dots x_{p-2}\rangle$.

Nonzero terms in the sum will only occur when both $[\sigma^p,\sigma^{p-1}]$ and $[\sigma^{p-1},\sigma^{p-2}]$ are nonzero. That is, when σ^{p-1} is a face of σ^p and σ^{p-2} is a face of σ^{p-1}. This only happens for two $(p-1)$-simplices, namely:

$$\sigma_1^{p-1}=\langle ax_0\dots x_{p-2}\rangle,\quad \sigma_2^{p-1}=\langle bx_0\dots x_{p-2}\rangle.$$

The proof now results in 4 cases, determined by the orientations of σ_1^{p-1} and σ_2^{p-1}.

Case 1. Suppose

$$+\sigma_1^{p-1}=+\langle ax_0\dots x_{p-2}\rangle,\quad +\sigma_2^{p-1}=+\langle bx_0\dots x_{p-2}\rangle.$$

Then

$$\begin{aligned}[\sigma^p,\sigma_1^{p-1}]=-1,\quad [\sigma_1^{p-1},\sigma^{p-2}]=+1,\\ [\sigma^p,\sigma_2^{p-1}]=+1,\quad [\sigma_2^{p-1},\sigma^{p-2}]=+1,\end{aligned}$$

so the sum of the products as indicated in the theorem is 0, as desired.

Case 2. Suppose

$$+\sigma_1^{p-1}=+\langle ax_0\dots x_{p-2}\rangle,\quad +\sigma_2^{p-1}=-\langle bx_0\dots x_{p-2}\rangle.$$

Then

$$[\sigma^p, \sigma_1^{p-1}] = -1, \quad [\sigma_1^{p-1}, \sigma^{p-2}] = +1,$$
$$[\sigma^p, \sigma_2^{p-1}] = -1, \quad [\sigma_2^{p-1}, \sigma^{p-2}] = -1,$$

so the desired sum is 0.

Case 3. Suppose

$$+\sigma_1^{p-1} = -\langle a x_0 \dots x_{p-2} \rangle, \quad +\sigma_2^{p-1} = +\langle b x_0 \dots x_{p-2} \rangle.$$

Then

$$[\sigma^p, \sigma_1^{p-1}] = +1, \quad [\sigma_1^{p-1}, \sigma^{p-2}] = -1,$$
$$[\sigma^p, \sigma_2^{p-1}] = +1, \quad [\sigma_2^{p-1}, \sigma^{p-2}] = +1,$$

so once more, the sum we are seeking is 0.

Case 4. Suppose

$$+\sigma_1^{p-1} = -\langle a x_0 \dots x_{p-2} \rangle, \quad +\sigma_2^{p-1} = -\langle b x_0 \dots x_{p-2} \rangle.$$

Then

$$[\sigma^p, \sigma_1^{p-1}] = +1, \quad [\sigma_1^{p-1}, \sigma^{p-2}] = -1,$$
$$[\sigma^p, \sigma_2^{p-1}] = -1, \quad [\sigma_2^{p-1}, \sigma^{p-2}] = -1,$$

so in this last case, the desired sum is 0. □

As previously stated, Theorem 5.1 will be key to defining homology groups, which we will be doing shortly.

6 Defining Homology Groups

We are very near to defining the homology groups. To make the final push to this definition, we must first define another class of groups.

Definition 22. Let K be an oriented complex. A *p-chain* on K is a function c from the family of oriented p-simplices of K to the integers, such that $c(-\sigma^p) = -c(\sigma^p)$. Under the operation of addition of their values, the p-chains form a group, $C_p(K)$, called the *group of p-chains* of K. For a given p-simplex σ^p, the *elementary chain* c corresponding to σ^p is the function defined such that $c(+\sigma^p) = 1$ and $c(\tau^p) = 0$ for all other oriented simplices τ^p [11].

Usually, σ^p is used to denote its corresponding elementary chain, as otherwise notation can become cluttered. Though this is an abuse, we will adopt it. It should also be noted that for $p < 0$ and $p > \dim K$, $C_p(K)$ is the trivial group. Thus, we are only concerned with p-chains if K has p-simplices.

Chain groups can loosely be thought of as linear combinations of simplices. With this in mind, it is easy to see that if K has n oriented p-simplices, then

$$C_p(K) \cong \underbrace{\mathbb{Z} \oplus \ldots \oplus \mathbb{Z}}_{\text{n times}}.$$

However, triangulations of a space are never unique, and so chain groups cannot be topologically invariant. In order to get groups that are topologically invariant, we need one more thing.

Definition 23. If σ^p is an elementary p-chain, $p \geq 1$, the *boundary* of σ^p, denoted $\partial(\sigma^p)$ is defined by

$$\partial(\sigma^p) = \sum_{\sigma^{p-1} \in K} [\sigma^p, \sigma^{p-1}]\sigma^{p-1}.$$

The boundary operator can easily be extended to a homomorphism $\partial : C_p(K) \to C_{p-1}(K)$ by linearity. The boundary of any 0-chain is defined to be 0.

Since there is a boundary operator for each p, we should note the boundary operator ∂_p, but since it is usually easy to know which operator should be used by context, we omit it for the sake of convenience.

There is a seemingly alternative but quickly equivalent definition of the boundary that will generally be more useful for computing boundaries. If $\sigma^k = \langle x_0 \ldots x_k \rangle$ is an oriented k-simplex, then $\langle x_0 \ldots \hat{x}_i \ldots x_k \rangle$ denotes the $k-1$ face of σ^k obtained by removing the ith face. Then

$$\partial(\sigma^k) = \sum_{i=0}^{k} (-1)^i \langle x_0 \ldots \hat{x}_i \ldots x_k \rangle. \tag{1}$$

Thus, for a 1-simplex σ^1, we see that $\partial(\sigma^1) = \langle x_1 \rangle - \langle x_0 \rangle$ and for a 2-simplex σ^2, $\partial(\sigma^2) = \langle x_1 x_2 \rangle - \langle x_0 x_2 \rangle + \langle x_0 x_1 \rangle$. This latter case especially helps clarify why it is called the boundary operator. Using Figure 8, we see that the boundary operator lets us "travel" around the boundary of σ^2. Through $\langle a_1 a_2 \rangle$, then backwards through $\langle a_0 a_2 \rangle$, and finally through $\langle a_0 a_1 \rangle$.

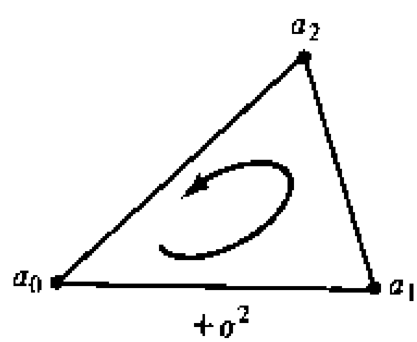

Figure 8: Scanned from Croom, page 12 [4]

We now proceed to a key theorem that will allow homology groups to exist.

Theorem 6.1. *Let K be an oriented complex and $p \geq 2$. Then for any p-chain c_p, $\partial \circ \partial(c_p) = 0$. To be more precise, $\partial\partial : C_p(K) \to C_{p-2}(K)$, diagramed by*

$$C_p(K) \xrightarrow{\partial} C_{p-1}(K) \xrightarrow{\partial} C_{p-2}(K)$$

is the trivial homomorphism [4].

Proof. This proof is taken from Croom [4]. Since any p-chain is a linear combination of elementary p-chains, it suffices to prove that for each elementary p-chain, σ^p, $\partial\partial(\sigma^p) = 0$. Notice that

$$\begin{aligned}
\partial\partial(\sigma^p) &= \partial \left(\sum_{\sigma_i^{p-1} \in K} [\sigma^p, \sigma_i^{p-1}] \sigma_i^{p-1} \right) = \sum_{\sigma_i^{p-1} \in K} \partial([\sigma^p, \sigma_i^{p-1}] \sigma_i^{p-1}) \\
&= \sum_{\sigma_i^{p-1}} \sum_{\sigma_j^{p-2}} [\sigma^p, \sigma_i^{p-1}][\sigma_i^{p-1}, \sigma_j^{p-2}] \sigma_j^{p-2}.
\end{aligned}$$

If we reverse the order of summatation and pull σ_j^{p-2} out one summation, we obtain

$$\partial\partial(\sigma^p) = \sum_{\sigma_j^{p-2} \in K} \left(\sigma_j^{p-2} \sum_{\sigma_i^{p-1} \in K} [\sigma^p, \sigma_i^{p-1}][\sigma_i^{p-1}, \sigma_j^{p-2}] \right).$$

But Theorem 5.1 guarantees that $\sum_{\sigma_i^{p-1} \in K} [\sigma^p, \sigma_i^{p-1}][\sigma_i^{p-1}, \sigma_j^{p-2}] = 0$ for each σ_j^{p-2}, so $\partial\partial(\sigma^p) = \sum_{\sigma_j^{p-2} \in K} (0 \cdot \sigma_j^{p-2}) = 0$. □

Corollary 6.2. *The image of ∂_{p+1} is contained in the kernel of ∂_p. That is, since both are subgroups of $C_p(K)$,* im $\partial_{p+1} <$ ker ∂_p

Proof. Let $c_p \in$ im $\partial_{p+1} < C_p(K)$. Thus $\exists c_{p+1}$ such that $\partial(c_{p+1}) = c_p$. But then $0 = \partial(\partial(c_{p+1})) = \partial(c_p)$. Thus $\partial(c_p) = 0$, so $c_p \in$ ker ∂_p. □

With this in mind, we are finally ready to define the homology groups.

Definition 24. Let K be an oriented complex. For each p, we define the *group of p-cycles*, $Z_p(K)$ to be the kernel of ∂_p. That is, $Z_p(K) = \text{ker } \partial_p$. The *group of p-boundaries*, $B_p(K)$ to be the image of ∂_{p+1}. That is, $B_p(K) = \text{im } \partial_{p+1}$. Corollary 6.2 guarantees $B_p(K) < Z_p(K)$, so we define the pth homology group, $H_p(K)$ to be the factor group of $Z_p(K)$ by $B_p(K)$. That is,

$$H_p(K) = Z_p(K)/B_p(K).$$

After so much construction, it would be good to now take a step back and consider what we have accomplished. For instance, what exactly are cycles? They are easiest to describe in the one-dimensional case, because despite all of the mathematical rigor necessary to make all definitions precise, it turns out that 1-cycles are just loops.[4] While there are different ways to think of loops, one is that they are one-dimensional objects (in a topological sense) which enclose a two-dimensional space, or an area. With this way of thinking, we then expand it to see that 2-cycles are two-dimensional objects which enclose a three-dimensional space, or a volume. In the zero-dimensional case it is more difficult to see, but because "enclosing" a one-dimensional space simply entails being the endpoints of that space, and any given point can be both "endpoints" of a loop, every 0-chain is a 0-cycle.

However, that these cycles enclose a higher dimensional space is not enough. Consider a sphere. Any 1-cycle on a sphere will enclose an area that is also on the sphere (this fact, though presented differently, is actually the result of a theorem known as the Jordan Separation Theorem). Compare this with the torus, seen in Figure 9. Notice the cycles labeled a and b. Unlike any 1-cycle on a sphere, these cycles enclose space that is not a part of the torus itself. These cycles are more interesting. The cycles on the sphere, in a sense "need" to be there, but these ones on the torus exist because of the specific

[4]Strictly speaking, they are the image of loops

structure of the torus. That is why we have the boundaries. Boundaries are essentially those cycles which are on the surface of the sphere. They enclose an area that is a part of the space, and because they are not interesting, we mod them out.

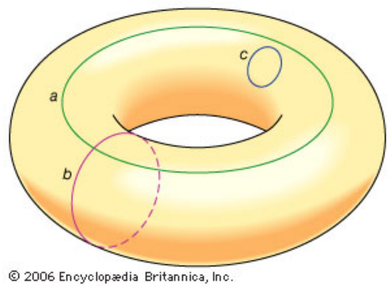

Figure 9: **Accessed from www.britannica.com on April 3rd, 2013**

Boundaries also serve another purpose. Suppose that there was another cycle that, like the cycle b in Figure 9 encloses part of the visible hole of the torus. Should this cycle be considered distinct from b? This cycle does not detail any new information about the torus, so it would not make sense to consider it as distinct. How then can we make this distinguishment more quantifiable? Notice however that together, these cycles enclose a "band" of the torus. In fact, the group structure detects this. The fact that they enclose a two-dimensional space which is a part of the torus means that together they form a boundary.

Definition 25. Let K be an oriented complex. We call the elements of $H_p(K)$ (the cosets of $B_p(K)$) the *homology classes* of cycles. If $z_1, z_2 \in Z_p(K)$, we say that z_1 and z_2 are *homologous* if they belong to the same homology class. That is, if $z_1 - z_2 \in B_p(K)$. We use the notation $z_1 \simeq z_2$ to show z_1 and z_2 are homologous.

It should be noted that we can and will say chains are homologous, even though we technically only defined cycles as being homologous.

7 Examples of Homology Groups and Computation Shortcuts

Now that we have defined homology groups, it would help to give some examples of spaces, and the computations of their homology groups. To start, consider the complex K in Figure 10 whose underlying space is a triangle. This space, of course, consists of only a single 2-simplex.

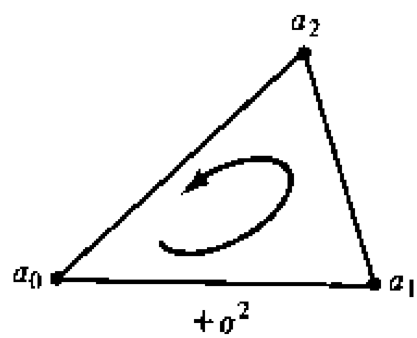

Figure 10: Scanned from Croom, page 12 [4]

Calculating $Z_0(K)$ is easy. Every elementary 0-chain is a cycle, so $Z_0(K) \cong C_0(K) \cong \mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z}$. Let $c_0 = h_0a_0+h_1a_1+h_2a_2 \in Z_0(K)$, but instead of doing anything with it now, we calculate $Z_1(K)$. Suppose we have a 1-chain, $c_1 = g_1e_1 + g_2e_2 + g_3e_3$. Then $\partial(c_1) = g_1(a_1 - a_0) + g_2(a_2 - a_1) + g_3(a_0 - a_2)$. Assuming we want c_1 to be a cycle, we can say that $0 = (g_3 - g_1)a_0 + (g_1 - g_2)a_1 + (g_2 - g_3)a_2$. Thus we see this will hold if $g_1 = g_2 = g_3$. So letting $g_1 = g$ will determine a cycle, and $Z_1(K) \cong \mathbb{Z}$.

Now instead of supposing $\partial(c_1) = 0$, suppose $\partial(c_1) = h_0a_0 + h_1a_1 + h_2a_2$. That is $h_0 = g_3 - g_1, h_1 = g_1 - g_2, h_2 = g_2 - g_3$. Thus, we are allowed to select any values for any 2 of the gs, and we will still arrive at the desired equality. Thus, $B_0(K) \cong \mathbb{Z} \oplus \mathbb{Z}$, meaning $H_0(K) \cong \mathbb{Z}$.

Now suppose we take a 2-chain, c_2. Because there is only one 2-simplex, $c_2 = n\sigma$.

Now, $\partial(n\sigma) = ne_1 + ne_2 + ne_3$. If we suppose c_2 is a cycle, then we set $\partial(c_2) = 0$ and conclude this can only happen if $n = 0$. Thus, $Z_2(K) \cong \{0\}$ and subsequently, $H_2(K) \cong \{0\}$.

If we instead consider $\partial(c_2) = g_1e_1 + g_2e_2 + g_3e_3$, we see this occurs when $g_1 = g_2 = g_3 = n$, so $B_1(K) \cong \mathbb{Z}$, meaning that $H_1(K) \cong \{0\}$.

Before going on to other examples, we notice that this was a lot of work for computing the homology groups of such a simple space. Doing these computations directly by calculating the cycle and boundary groups involves increasingly more work as the space becomes more complicated. Fortunately, there are some shortcuts.

First, for any of the spaces we will be working with, the dimension of the associated complex will be at most two. As a result, there are no 3-chains for 2-cycles to bound, and $B_2(K)$ will necessarily be trivial, meaning $H_2(K) \cong Z_2(K)$.

Next, there is a very simple geometric interpretation of $H_0(K)$ that makes computing it completely trivial. First, a definition:

Definition 26. Let X be a space. Let "$\sim$" define an equivalence relation of points in X where for $x, y \in X, x \sim y$ if and only if x and y can be connected by a path. That is, if K is a triangulation of X with x, y vertices of K, then there is a sequence $\sigma_1, \sigma_2, \ldots, \sigma_p$ of 1-simplices such that x is a vertex of σ_1, y is a vertex of σ_p, and for all i, σ_i and σ_{i+1} share a vertex. The equivalence classes of points of X under $\sim$ are known as the *path components* of X.

Theorem 7.1. *Let K be a complex whose underlying space has n path components. Then $H_0(K)$ is isomorphic to the direct sum of n copies of $\mathbb{Z}$.*

Proof. This proof is mostly taken from Croom [4], with some simplification near the end. Choose a path component of K and pick a vertex of K in that component, call it $\langle a' \rangle$. For any other vertex of K in the same path component, call it $\langle b \rangle$, there is a sequence of 1-simplices of the form

$$\langle ba_0\rangle, \langle a_0a_1\rangle, \ldots, \langle a_v a'\rangle.$$

Let c_1' be a 1-chain that has a value of either g or $-g$. It is easy to see that $\partial(c_1') = g\dot{(}\langle b\rangle + \langle a'\rangle)$ or $g\dot{(}\langle b\rangle - \langle a'\rangle)$. Thus, any elementary 0-chain $g\dot{\langle}b\rangle$ in the path component that contains a' is homologous to one of $\pm g\dot{\langle}a'\rangle$. Hence, any 0-chain on this component is homologous to $h\dot{\langle}a'\rangle$ for some integer h.

We can repeat this process on every path component. If we index the n path components of K, and we let a^i be an elementary 0-chain in the ith path component, then for any 0-cycle, c_0 on K, there are integers $h_1, h_2, \ldots, h_n$ such that

$$c_0 = \sum_{i=0}^{n} h_i \dot{\langle}a^i\rangle.$$

There might be some concern of uniqueness of representation of elements. Suppose we have 0-chains, $g = \sum g_i\dot{\langle}a^i\rangle$ and $h = \sum h_i\dot{\langle}a^i\rangle$ that represent the same homology class. But then for some 1-chain, c_1

$$\sum (g_i - h_i)\langle a^i\rangle = \partial(c_1),$$

but since for $i \neq j$, a^i and a^j are in different path components, this can only hold if $g_i = h_i$. Thus, g and h are the same chain. Thus we have that $H_0(K) \cong \underbrace{\mathbb{Z} \oplus \ldots \oplus \mathbb{Z}}_{\text{n times}}$ via the isomorphism

$$\sum_{i=0}^{n} h_i \dot{\langle}a^i\rangle \to (h_1, h_2, \ldots, h_n).$$

□

Thus Theorem 7.1 allows us to immediately determine $H_0(K)$ if we know how many path components K has.

The last shortcut is not to compute $Z_p(K)$ and $B_p(K)$ separately. Instead, we compute $H_p(K)$ directly and use some intuition to otherwise skip steps. The idea is to first "reduce" the complex by passing to things homologous to find the most essential form of a cycle, and then determine under what conditions, if any, that cycle may bound.

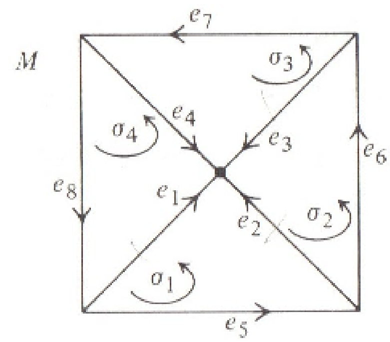

Figure 11: Scanned from Munkres, page 32 [11]

.

To help understand how we do this, consider the complex M in Figure 11 whose underlying space is a square. To start, M has only 1 path component, so $H_0(M) \cong \mathbb{Z}$. Next, we compute $H_2(M)$: Suppose we have a 2-cycle, z_2. Suppose z_2 has a value of a on σ_1. We notice that 1-simplex e_5 is a face of σ_1 and no other 2-simplex. Thus, when we compute $\partial(z_2)$, we find that it will have a value of a on e_5, because nothing can "cancel" the value of e_5 picked up by σ_1. Thus $a = 0$. Using similar reasoning, we find that z_2 must have a value of 0 on every 2-simplex. Thus, $H_2(M) \cong \{0\}$.

It is for $H_1(M)$ that we truly exploit our shortcut. The following, including this technique, is presented in Munkres [11]. Suppose we have a 1-chain, c on M and let c have a value of a on e_1. Computation will show that the chain

$$c_1 = c + \partial(a\sigma_1)$$

has a value of 0 on e_1. Clearly c_1, c are homologous because $c_1 - c = \partial(a\sigma_1)$. Next, if

c_1 has a value of b on e_2, then computations show that

$$c_2 = c_1 + \partial(b\sigma_2)$$

has a value of 0 e_2. Further, since $\partial(b\sigma_2)$ has a value of 0 on σ_1, c_2 has a value of 0 on e_1 as well. Lastly, if c_2 has a value of d on e_3. Again, computations show that

$$c_3 = c_2 + \partial(d\sigma_3)$$

has a value of 0 on e_1, e_2, e_3. Further, $c \simeq c_3$, as $c_3 - c = \partial(d\sigma_3) + \partial(b\sigma_2) + \partial(a\sigma_1)$. Thus, any 1-chain c is homologous to a 1-chain c_3 which can take nonzero values only on the subcomplex M' in Figure 12. Note, c_3 can still have a value of 0 on M', but on any 1-simplex not in M', c_3 necessarily has a value of 0. This concept will be useful in the future, and so we give it a formal name.

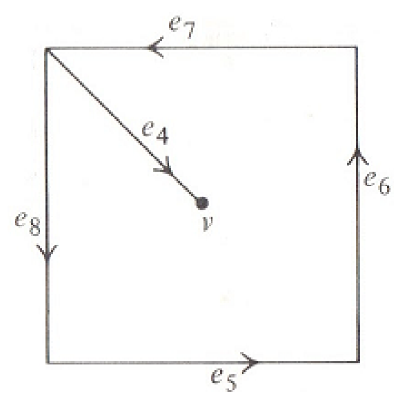

Figure 12: Scanned from Munkres, page 32 [11]

.

Definition 27. Let K be a complex and K' be a subcomplex of K. We say that a chain $c \in C_p(K)$ is *carried by* K' if $c \in C_p(K')$. That is, c takes nonzero values only on K'.

Now, we consider a cycle, z_1. Since a cycle is still a chain, z_1 is homologous to a cycle z_1' which is carried by M'. But then z_1' must also have a value of 0 on e_4, as otherwise $\partial(z_1)$ will have a nonzero value on v. Thus, any 1-cycle on M must be homologous to one of the form $g(e_5+e_6+e_7+e_8)$. But $g(e_5+e_6+e_7+e_8) = \partial(g\sum\sigma_i)$. Thus we conclude that $H_1(M) \cong \{0\}$

It should be observed that the complexes K and M have isomorphic homology groups in each dimension. This is no mistake, as $|K|$ and $|M|$ are homeomorphic. This was our goal, after all: finding groups which are topologically invariant.

Theorem 7.2. *Let K and L be complexes such that $|K|$ and $|L|$ are homeomorphic. Then $H_p(K) \cong H_p(L)$ for each p.*

Proving Theorem 7.2 would take too long and involve a lot of discussion of topics which ultimately serve no other purpose than proving this very theorem, but it is a essential result nonetheless.

It should be noted that the converse of Theorem 7.2 does not hold. That is, if you have two complexes that have isomorphic homology groups in each dimension, it is not necessarily true that they are homeomorphic. The following examples will show such a situation.

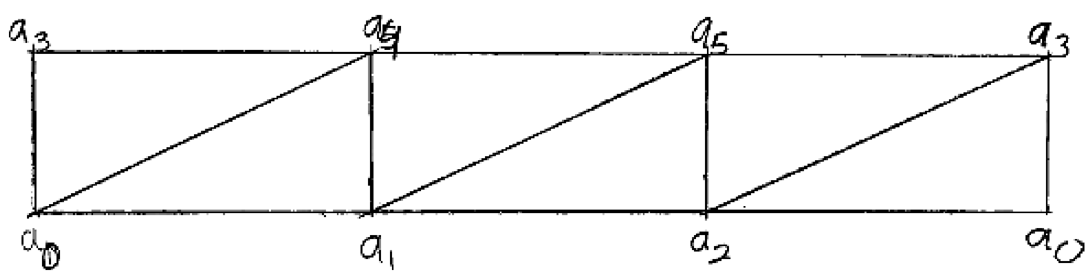

Figure 13

Figure 13 shows a plane diagram. Plane diagrams are something topologists use to display a three-dimensional space in two dimensions. Notice in the diagram how on we have two vertices that are labelled x_0, and two that are labeled x_3. Thus they are in fact the same point, and the line segment between them is the same 1-simplex on that space.

A good way to think about it is that you take the rectangle, and "glue" the edges together that are indicated in the diagram. So the plane diagram in Figure 13 shows a "ring" or what we call an annulus.

The annulus (call it A) has 1 path component, so $H_0(A) \cong \mathbb{Z}$. We can use a similar argument as we used for L to see that $H_2(A) \cong \{0\}$. For $H_1(A)$, we can use similar methods as in the previous example (which Munkres describes as "pushing off" the 1-simplices) to show that any 1-chain is homologous to one carried by the subcomplex A' in Figure 14.

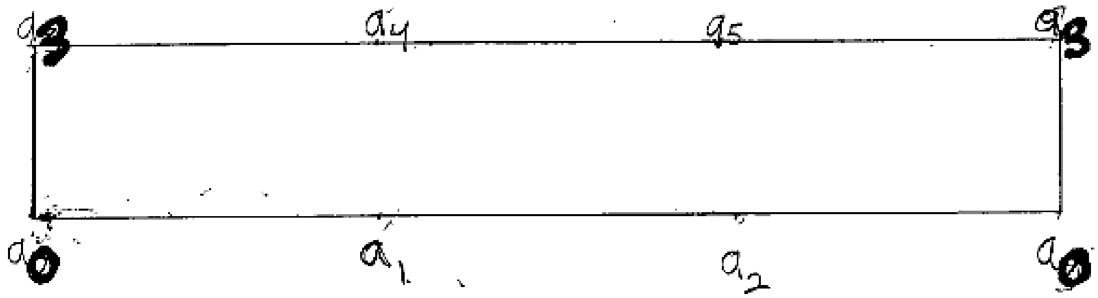

Figure 14

Now we notice two possible cycles: one which runs along the top of the annulus once (call it z_1), and one which runs along the bottom once. However, they are clearly homologous, as they are the boundary of the 2-chain which takes equal value on every 2-simplex. Thus, we only need to consider the 1-cycle which runs along the top of the annulus (in which case, the cycle must take a value of 0 on $\langle x_0 x_3 \rangle$). So any 1-cycle on A is homologous to one of the form $g\dot{z}_1$ for an integer g and $H_1(A) \cong \mathbb{Z}$.

Now, consider the Möbius band, M, in Figure 15. This plane diagram looks very similar to the one for the annulus. The difference is in the gluing. Essentially, unlike with the annulus, you create a half twist before gluing the ends together. Using our previous work, we immediately get that $H_0(M) \cong \mathbb{Z}$ and $H_2(M) \cong \{0\}$.

Using the pushing off technique, we get that any 1-chain on M is homologous to

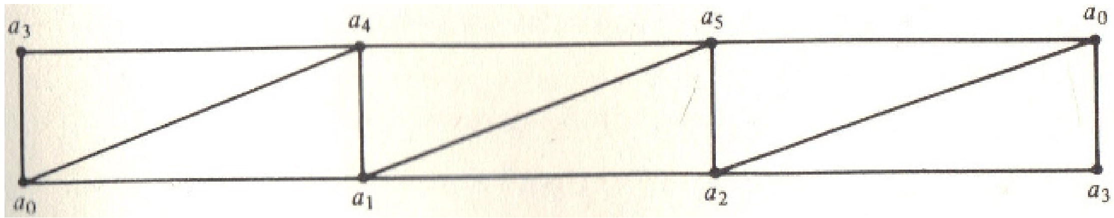

Figure 15: Scanned from Croom, page 20 [4]

.

one carried by the subcomplex M' in Figure 16. Notice that $z_1' = \sum_{i=1}^{6} e_i$ is a cycle. There are also cycles which run along the top or bottom, and then goes through e_7, but these are homologous to multiples of z_1'. So any 1-cycle on M is homologous to gz_1' and $H_1(M) \cong \mathbb{Z}$.

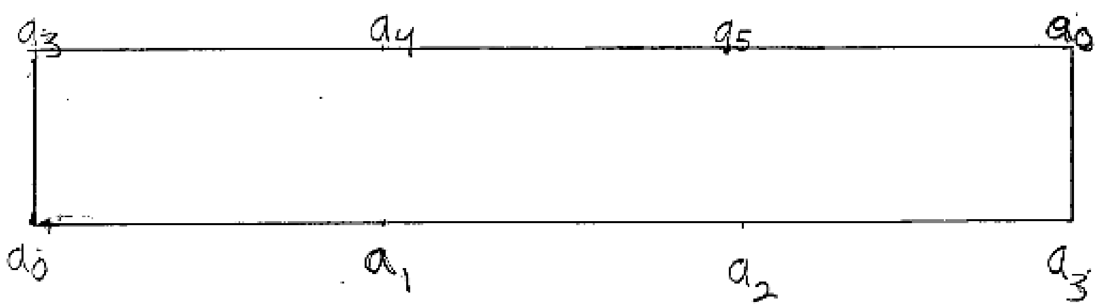

Figure 16

.

So, the homology groups of the annulus and Möbius band are isomorphic. However, it is known that A and M are not homeomorphic. The annulus is orientable, but the Möbius band is not. However, despite this deficiency, homology groups are still quite useful. Even if they cannot show that two spaces are homeomorphic, knowing that they are not is still very useful.

8 Homology Groups of Surfaces, Part I

We now begin the main goal: homology groups of surfaces. Surfaces (technically closed surfaces) are a special class of spaces. The main feature of surfaces is that they are locally homeomorphic to the plane, $\mathbb{R}^2$. Essentially this means that for every point of a surface, when it is looked at closely, appears flat, like the plane, and there are no edges, or boundaries. This latter condition excludes such spaces as the annulus or Möbius band. It should be noted that all surfaces are path-connected, so we will pay no attention to H_0 as it always be isomorphic to $\mathbb{Z}$.

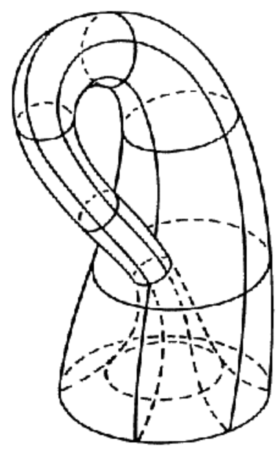

Figure 17: Accessed from www.math.osu.edu on March 13, 2013

.

Some examples of surfaces are the sphere and torus, which have already been seen. Another example is the Klein bottle, seen in Figure 17. One can think of a Klein bottle by taking a cylinder, and instead of gluing the circular edges directly, as we would do to create a torus, we glue them in the "reverse" direction (this is analogous to the half-twist in creating a Möbius band). One may object calling this a surface, because there are apparently points where the Klein bottle intersects itself. The plane never self-instersects, so how is it locally homeomorphic to the plane at these points? The answer is that the Klein bottle is actually a subset of $\mathbb{R}^4$, but we are viewing it in $\mathbb{R}^3$. In $\mathbb{R}^3$, it cannot exist without this self-intersection. We must blame human beings' inability to perceive four spatial dimensions for this problem.

At first, we will only work with surfaces which can have a plane diagram made from a square or rectangle, with appropriate gluings. In this case, "appropriate" means that edges are glued in pairs. We should also have that no 2-simplex is glued to itself, nor should any 2-simplex be glued to another 2-simplex along more than one edge. To help facilitate these computations, we have a lemma, which comes from Munkres [11]. Before we state the lemma, we will note that from this point forward, we assume that every 2-simplex is oriented counter-clockwise, and the orientation of 1-simplices is determined arbitrarily. We can have arbitrary orientations for 1-simplices because 0-simplices (which arise in calculations of the boundary of 1-chains) have no orientation.

Lemma 8.1. *Let L be the complex in Figure 18, whose underlying space is a rectangle. Let* Bd L *denote the complex whose underlying space is the boundary of the rectangle. Then*

1. *Every 1-cycle of L is homologous to a 1-cycle carried by* Bd *L.*
2. *If d is a 2-chain of L and if ∂d is carried by* Bd L*, then d is a multiple of the chain $\sum \sigma_i$ [11].*

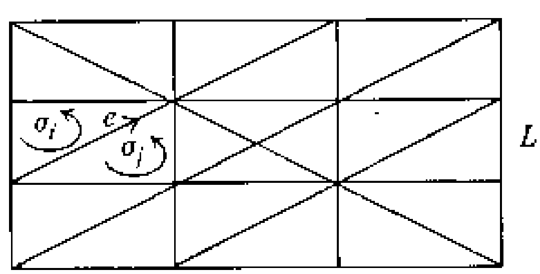

Figure 18: Scanned from Munkres, page 34 [11]

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Proof. The proof of 2 is easy. For any interior edge e, there are exactly two 2-simplices, σ_i, σ_j that have e has an edge. Since ∂d must have a value of 0 on e (as ∂d is carried by Bd L), we see that d must have the same value on σ_i as σ_j (having both of the 2-simplices are oriented counter-clockwise, one of σ_i, σ_j will go through e in its positive direction, and the other will go through it in its negative direction). Extending this on every interior edge e, we get that d must have the same value on every 2-simplex, and d should be a multiple of $\sum \sigma_i$.

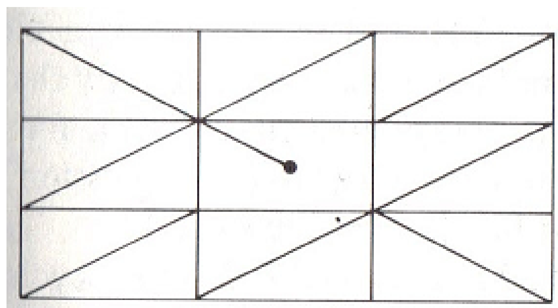

Figure 19: Scanned from Munkres, page 35 [11]

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The proof of 1 requires the use of the "pushing off" technique used in our previous example. For any 1-chain c, we first we "push off" the 1-simplices in the center, meaning that c is homologous to a 1-chain c_1, carried by the complex in Figure 19. We then push off the remaining interior edges, meaning c_1 is homologous to a 1-chain c_2, carried by the complex in Figure 20. Now, if z_2 is a cycle, we must have that z_2 is carried by Bd L, else $\partial(z_2)$ has non-zero value on $v_1, \ldots, v_5$.

$\square$

With this lemma in hand, we begin our computations. One last note on our notation for these computations. While all 1-simplices still have an orientation, the fact that we have chosen their orientation arbitrarily means we cannot use our old notation, as this would suggest a particular orientation. We introduce a modified notation. If σ^1 is a

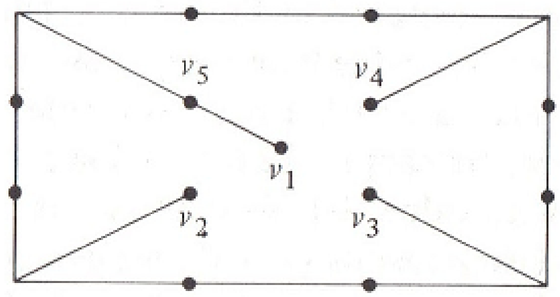

Figure 20: Scanned from Munkres, page 35 [11]

.

1-simplex with a, b as vertices, then we will use the notation $[a, b]$ to be the orientation of σ^1 that starts at a and ends at b.

In this first section, all proofs come from Munkres [11]

Theorem 8.2. *Let T denote the complex represented by the labelled rectangle L in Figure 21. $|T|$ is the torus. Then*

$$H_1(T) \cong \mathbb{Z} \oplus \mathbb{Z} \quad \text{and} \quad H_2(T) \cong \mathbb{Z}.$$

Further, if we let $\gamma = \sum \sigma_i$, where σ_i denotes a 2-simplex, and let

$$w_1 = [a, b] + [b, c] + [c, a],$$

$$z_1 = [a, d] + [d, e] + [e, a].$$

Then γ generates $H_2(T)$ and w_1, z_1 are generators for $H_1(T)$.

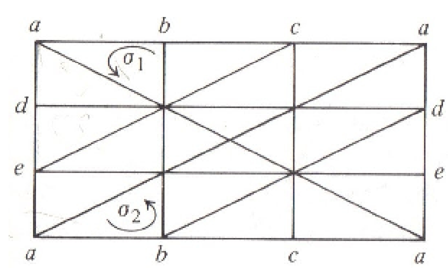

Figure 21: Scanned from Munkres, page 35 [11]

.

Proof. Let $g : |L| \to |T|$ be the gluing map, and let $A = g(|\text{Bd } L|)$. Then A is homeomorphic to a wedge of two circles (the union of 2 circles with a vertex in common), seen in Figure 22. Because g only glues edges in Bd L, Lemma 8.1 immediately gives

1. Every 1-cycle of L is homologous to a 1-cycle carried by A.
2. If d is a 2-chain of T and if ∂d is carried by A, then d is a multiple of the chain γ.

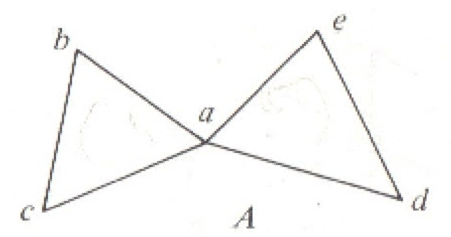

Figure 22: Scanned from Munkres, page 36 [11]

.

However, in T we also get that

3. If c is a 1-cycle of T carried by A, then c is of the form $mw_1 + nz_1$.
4. $\partial\gamma = 0$.

3 follows because A is just the space in Figure 22. 4 is also direct. $\partial\gamma$ definitely has a value of 0 on every 1-simplex not in A, simply by computing boundaries of the 2-simplices with that 1-simplex as a face. For the 1-simplices in A, direct computations also show that each of these 1-simplices takes a value of 0. For example, $[a, b]$ is a face of σ_1 and σ_2 (see Figure 21). $\partial\sigma_1$ has a value of -1 on $[a, b]$, and $\partial\sigma_2$ has a value of $+1$ on $[a, b]$.

Now, 1 and 3 give us that every 1-cycle of T is homologous to one of the form $z = mw_1 + nz_1$. We see that z will bound if and only if it is trivial, because if $z = \partial d$, then d is a multiple of γ by 2. So $d = p\gamma$ and $c = \partial d = \partial p\gamma = p\partial\gamma = 0$. Thus

$$H_1(T) \cong \mathbb{Z} \oplus \mathbb{Z}$$

with w_1, z_1 generating the group.

For $H_2(T)$, we note that for a 2-cycle d, ∂d is carried by A, since ∂d is trivial. Thus, 2 says that d is of the form $p\gamma$. Further, 4 tells us that any such 2-chain of that form is in fact a cycle. Since there are no 3-chains to bound, we conclude that

$$H_2(T) \cong \mathbb{Z}$$

with γ as a generator. □

Theorem 8.3. *Let K denote the complex represented by the labelled rectangle in Figure 23. $|K|$ is the Klein bottle. Then*

$$H_1(K) \cong \mathbb{Z} \oplus \mathbb{Z}_2 \quad \textit{and} \quad H_2(K) \cong \{0\}$$

Further, if

$$w_1 = [a, b] + [b, c] + [c, a],$$

$$z_1 = [a, d] + [d, e] + [e, a],$$

then z_1 generates the torsion of $H_1(K)$ ($\mathbb{Z}_2$), and w_1 generates $H_1(K)/\mathbb{Z}_2$.

Proof. This proof begins the same as for the torus in Theorem 8.2. We let g be the corresponding gluing map, and let $A = g(|\text{Bd } L|)$. Again, A is the wedge of two circles. Again, properties 1 and 2 hold. 3 is also the same. But this time, $\partial\gamma = 2z_1$. This fact is simple to establish. Take $[a, b]$ for example. As before, $[a, b]$ is a face of σ_1 and σ_2, and when computing the boundary, we get a negative value from σ_1 and a positive value from σ_2. However, consider $[a, d]$, which is a face of σ_3 and σ_4. Computing boundaries, we

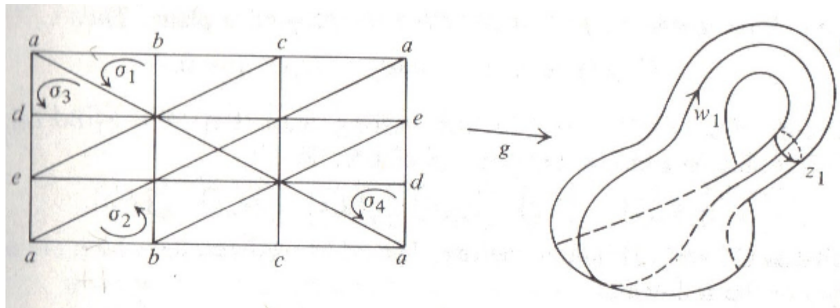

Figure 23: Scanned from Munkres, page 37 [11]

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get a positive value for $[a, d]$ from both σ_3 and σ_4. This also happens for each 1-simplex in z_1, so $\partial\gamma = 2z_1$.

Now, every 1-cycle on K is of the form $z = mw_1 + nz_1$. To see when z bounds, we suppose $z = \partial d$ for some 2-chain d. Since z is carried by A, 2 tells us that $d = p\gamma$. Thus, $\partial d = 2pz_1$, and we see that z bounds if and only if n is even and m is 0. We thus obtain

$$H_1(K) \cong \mathbb{Z} \oplus \mathbb{Z}_2$$

with z_1 generating $\mathbb{Z}_2$, and w_1 generating $\mathbb{Z}$.

For $H_2(K)$, we once again get that any 2-cycle d must be of the form $p\gamma$. However, this time $p\gamma$ is a cycle if and only if $p = 0$, since $\partial\gamma = 2z_1$. Hence,

$$H_2(K) \cong \{0\}.$$

□

The Klein bottle shows some interesting things. Remembering that the second homology group indicates an enclosed volume, $H_2(K)$ being trivial means that the Klein bottle does not enclose a volume. Said another way, the Klein bottle has no definite inside. The other interesting feature is the presence of $\mathbb{Z}_2$ in the first homology group.

Understanding what this means can be difficult, especially given our original interpretation of what a non-trivial cycle in H_1 means. How can this cycle not enclose a part of the Klein bottle when we "go through it" once, but it does when we go through it a second time?

The twisting nature (non-orientability) of the Klein bottle is what allows this to happen. Notice that if we pick a point on the Klein bottle, and start on its "outside" (as previously discussed, the Klein bottle has no definite inside or outside, but we appeal to the intuitive sense of an outside here). We are able to trace a path along the Klein bottle such that we will eventually reach our initial point, but now we are on the "inside" of the bottle. Thus, if we think of one pass through of z_1 to be on the "outside" and one to be on the "inside" we see that we could "unravel" the bottle, with half of the bottle "within" the two copies of z_1 and other half "outside" of them.

Next we have another surface, known as the projective plane. The projective plane is a very strange space, and incredibly difficult to visualize. One way the projective plane can be defined is by taking a circle (with interior), and gluing every point on the boundary of the circle with its antipode (the point directly opposite it on the boundary). It should be obvious that if embedded in $\mathbb{R}^3$, the projective plane will have self-intersections. Like the Klein bottle, the projective plane properly exists in $\mathbb{R}^4$.

Theorem 8.4. *Let P denote the complex represented by the labelled rectangle in Figure 24. $|P|$ is the projective plane. Then*

$$H_1(P) \cong \mathbb{Z}_2 \quad \textit{and} \quad H_2(P) \cong \{0\}.$$

Further, the cycle

$$z_1 = [a, b] + [b, c] + [c, d] + [d, e] + [e, f] + [f, a]$$

will generate $H_1(P)$.

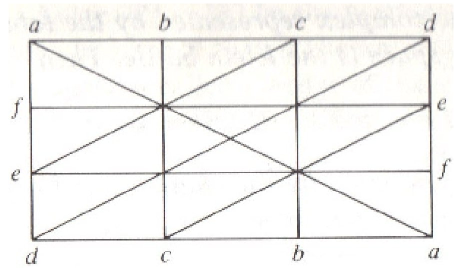

Figure 24: Scanned from Munkres, page 38 [11]

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Proof. Again, let g be the gluing map, and let $A = g(|\text{Bd } L|)$. This time, A is homeomorphic to a circle. The results 1 and 2 are the same as before. But now

3. Every 1-cycle carried by A is a multiple of z_1.
4. $\partial\gamma = 2z_1$.

Using similar reasoning to what we did for the Klein bottle, we see that

$$H_1(P) \cong \mathbb{Z}_2 \quad \text{and} \quad H_2(P) \cong \{0\}.$$

□

Before moving on to the next example, we need another definition, which essentially allows us to build new surfaces out of simpler ones.

Definition 28. Let K and L be surfaces. The *connected sum* of K and L, denoted $K\#L$ is created by deleting a small open disk from both K and L, and gluing them along their boundaries. If K and L are homeomorphic, we sometimes notate $K\#L$ as $2K$ (or $2L$).

Theorem 8.5. *Let* $2P$ *be the connected sum of projective planes. Then*

$$H_1(2P) \cong \mathbb{Z} \oplus \mathbb{Z}_2 \quad \textit{and} \quad H_2(2P) \cong \{0\}$$

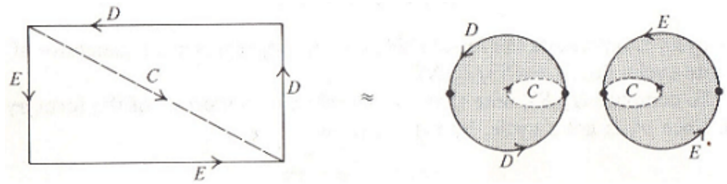


Figure 25: Scanned from Munkres, page 38 [11]

.

Proof. We still use the rectangle L, with identifications as indicated in Figure 25 (here, the arrows denote the direction of gluing). Again, we let g be the gluing map, with $A = g(|\text{Bd } L|)$, which is a wedge of two circles. We let w_1 be the cycle which runs across the top right edges (and through the diagonal), and z_1 being the cycle which runs across the bottom left edges (and across the diagonal). 1 and 2 from Theorem 8.2 hold, with 3 and 4 being

3. Every 1 cycle carried by A is of the form $mw_1 + nz_1$.

4. $\partial\gamma = 2w_1 + 2z_1$.

The same logic as presented in the computations for K and P will show that $H_2(2P) \cong \{0\}$. H_1 is a different matter. The key is to use w_1 and z_1 to create more "useful" generating cycles. To this end, we suggest $\{w_1, w_1 + z_1\}$. This works, because z_1 is a linear combination of these cycles ($z_1 = -(w_1) + (w_1 + z_1)$). With this in mind, and following the logic of the other computations, we see that

$$H_1(2P) \cong \mathbb{Z} \oplus \mathbb{Z}_2$$

with $w_1 + z_1$ generating $\mathbb{Z}_2$ and w_1 generating $\mathbb{Z}$. $\square$

It should be noted that $H_n(2P) \cong H_n(K)$ for all n. This is no mistake. While our comments at the end of Section 7 tell us it should not necessarily be the case, it is known that the Klein bottle is homeomorphic to the connected sum of two projective planes.

9 Homology Groups of Surfaces, Part II

There are many surfaces left to consider. In fact, there are infinitely many. There is a known classification theorem for surfaces.

Theorem 9.1. *Let X be a surface. Then X homeomorphic to the sphere, S^2, the connected sum of n tori, nT, or the connected sum of m projective planes, mP.*

How, then, can we proceed with these calculations? The lemma we used in Section 8 does not directly apply, because none of the remaining surfaces can be created from it with appropriate gluings. However, looking back at the proof of Lemma 8.1, there is nothing particularly important about that particular triangulation of the rectangle, or the fact that it even if a rectangle. Any n-gon with any given triangulation will yield the same conclusions: every 1-cycle will be homologous to one carried by the boundary of the n-gon, and any 2-chain whose boundary is carried by the boundary of the n-gon will be a multiple of $\gamma = \sum \sigma_i$.

With that in mind, we start with the sphere, S^2. The sphere can be constructed from the plane diagram of a rectangle, by gluing the top and left sides together, as well as the bottom and right sides. This gluing, under the triangulation of the rectangle in Lemma 8.1 would glue two 2-simplices along more than one face in the top left and bottom right corners. To get around this, we need only modify the triangulation. If we replace the rectangle in the top left by the one in Figure 26, and a rotation of that rectangle to replace the bottom right, we have an acceptable triangulation.

Theorem 9.2. *Let S^2 denote the complex represented rectangle discussed above. $|S^2|$ is the sphere. Then*

$$H_1(S^2) \cong \{0\} \quad \textit{and} \quad H_2(S^2) \cong \mathbb{Z}.$$

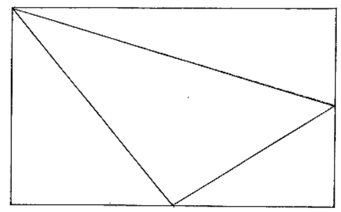
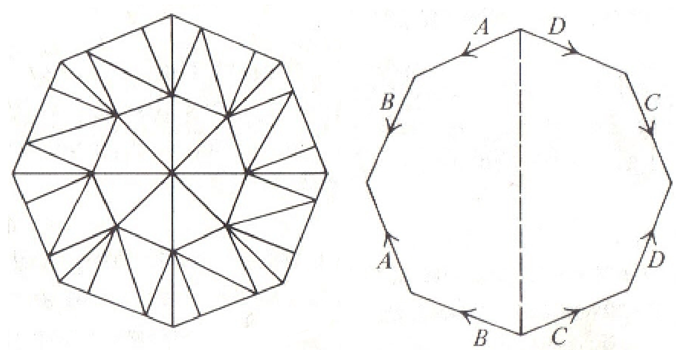


Figure 26



(a) (b)

Figure 27: Scanned from Munkres, page 40 [11]

Proof. Let g by the gluing map, and $A = g(|\text{Bd } L'|)$. Then A is a line segment. Since every cycle is homologous to one carried by A, and the only cycles on a line segment are trivial, we immediately get that $H_1(S^2) \cong \{0\}$. Further, the same logic as in Theorem 8.2 will show that $H_2(S^2) \cong \mathbb{Z}$. □

This result should be no surprise, as we already discussed that every cycle on a sphere bounds, and the sphere clearly encloses a volume.

Theorem 9.3. *Let nT be a space homeomorphic to the connected sum of n tori. Then*

$$H_1(nT) \cong \underbrace{\mathbb{Z} \oplus \ldots \oplus \mathbb{Z}}_{2n \text{ times}} \quad \textit{and} \quad H_2(nT) \cong \mathbb{Z}.$$

Proof. We will prove the result for $2T$, and the remaining cases will follow similar arguments. Let Figure 27a be a triangulation of $2T$ (call the octagon M) with gluings indicated in Figure 27b. Let g be the gluing map, with $X = g(|\text{Bd } M|)$. X is the wedge of four circles. Then

1. Every 1-cycle of of $2T$ is homologous to a 1-cycle carried by X.
2. If d is a 2-chain of $2T$ and if ∂d is carried by X, then d is a multiple of γ
3. If z is a 1-cycle of $2T$ carried by X, then z is of the form $mA + nB + sC + tD$
4. $\partial\gamma = 0$

3 follows because X is a wedge of four circles, and 4 follows from computations of boundaries: for a 1-simplex in the boundary of M, e_i, the two 2-simplices that have e_i as a face go through e_i in opposite directions, so they cancel each other out.

Now, every 1-cycle of T is homologous to one of the form $z = mA+nB+sC+tD$. A cycle z_0 will bound if $z_0 = \partial d$. But then ∂d is carried by A, so $d = p\gamma$. Then $z_0 = \partial d = p\partial\gamma = 0$. Thus, z_0 bounds only if z_0 is trivial. Thus

$$H_1(2T) \cong \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}.$$

Now if d is a 2-cycle of $2T$, then ∂d is carried by A and $d = p\gamma$. Since $\partial\gamma = 0$, d is a cycle for any p. Thus

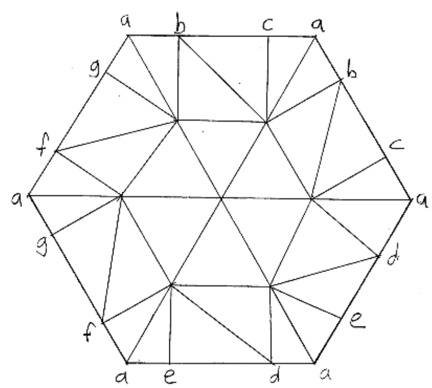
$$H_2(2T) \cong \mathbb{Z}.$$

For $n > 2$, the calculations go exactly the same. We take a $4n$-gon, with sides glued in pairs, with one side between the two glued sides (notice in Figure 27b how B is between the two sides marked A, an A between the Bs, a C between the Ds and a D between the Cs). We will get $2n$ cycles which will never bound. And because $\partial\gamma = 0$, $H_2(nT) \cong \mathbb{Z}$. $\square$

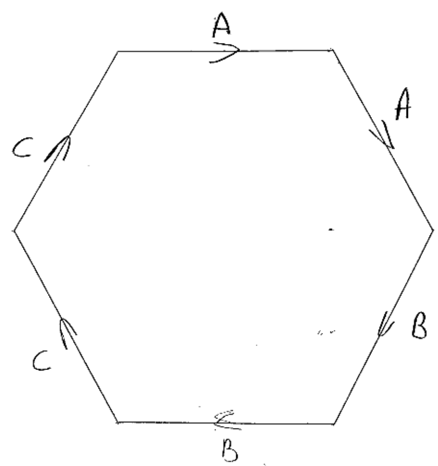
The process for proving Theorem 9.3 is extremely similar to that of Theorem 8.2. This is not surprising, because they are similar spaces.

Theorem 9.4. *Let mP be a space homeomorphic to the connected sum of m projective planes. Then*

$$H_1(mP) \cong \underbrace{\mathbb{Z} \oplus \ldots \oplus \mathbb{Z}}_{m-1 \text{ times}} \oplus \mathbb{Z}_2 \quad \textit{and} \quad H_2(mP) \cong \{0\}.$$



(a)



(b)

Figure 28

Proof. We will prove the result in the case of $3P$, and the remaining cases will follow similar arguments. Let Figure 28a be a triangulation of $3P$ (call the hexagon M) with gluings indicated in Figure 28b. Let g be the gluing map, with $X = g(|\text{Bd } M|)$. X is a wedge of three circles. Then

1. Every 1-cycle of of $3P$ is homologous to a 1-cycle carried by X.
2. If d is a 2-chain of $3P$ and if ∂d is carried by X, then d is a multiple of γ
3. If z is a 1-cycle of $3P$ carried by X, then z is of the form $mA + nB + sC$

4. $\partial\gamma = 2(A + B + C)$

3 follows because X is a wedge of three circles, and 4 follows from computations of boundaries: every 2-simplex goes through 1-simplices in X in the same direction, and since every such 1-simplex is a face of two 2-simplices, we get two copies of each 1-cycle, A, B, C.

Like we did for Theorem 8.5, we create a new set of generating cycles to aid our computations. We choose $\{A, B, A+B+C\}$. Note that $C = -A + -B + (A+B+C)$. Now, if z is a 1-cycle of $3P$, it is homologous to one of the form $z = mA + nB + sC = (m-s)A + (n-s)B + s(A+B+C)$. Then z bounds if and only if $m = n = s$ is even. Thus

$$H_1(3P) \cong \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}_2.$$

For H_2, if d is a 2-cycle, then ∂d is carried by X, and $d = p\gamma$. But since $\partial\gamma = 2(A + B + C)$, then d is a cycle if and only if $p = 0$. Thus

$$H_2(3P) \cong \{0\}.$$

For $m > 3$, we take a $2m$-gon, gluing adjacent sides in pairs in "opposite" directions, creating new generating cycles in the same way. Doing so gives $m - 1$ non-bounding cycles and 1 cycle which bounds with every other copy of itself. □

As we see, all of these surfaces have non-isomorphic homology groups, so they are, in fact, distinct. This is not enough to conclude that these are the only surfaces, unfortunately, though it is the case. This might seem suspect. The connected sum of a torus and projective plane should certainly be a surface, after all. As it turns out, $T\#P$ is homeomorphic to $3P$. While we will not show this fully, we will show that they have isomorphic homology groups.

Theorem 9.5. *Let L denote the complex represented by the labelled hexagon in Figure 29. $|L|$ is the connected sum of a torus and a projective plane. Then*

$$H_1(L) \cong \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}_2 \quad \textit{and} \quad H_2(L) \cong \{0\}.$$

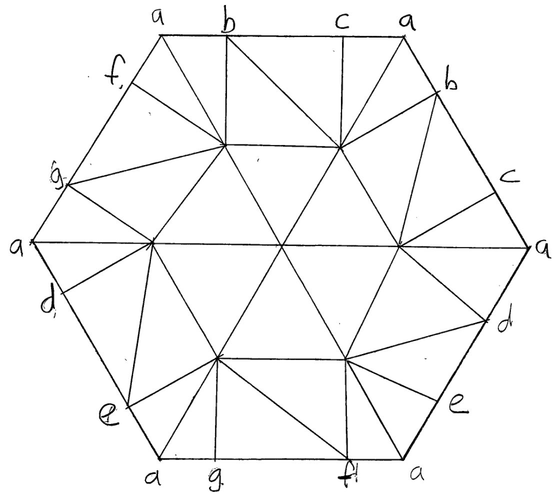


Figure 29

Proof. Let g be the gluing map, and let $X = g(|\text{Bd } M|)$. X is a wedge of three circles. Then

1. Every 1-cycle of of L is homologous to a 1-cycle carried by X.

2. If d is a 2-chain of L and if ∂d is carried by X, then d is a multiple of γ

3. If z is a 1-cycle of L carried by X, then z is of the form $mA + nB + sC$

4. $\partial\gamma = 2C$

3 follows from X being a wedge of three circles, and 4 follows from boundary computations: 2-simplices which have a 1-simplex e_i in A or B as a face will go through e_i in opposite directions, whereas 2-simplices which have a 1-simplex e_j in C as a face will go through e_j in the same direction. Then following the same logic that we have

now used many times, we quickly arrive at the conclusions. This time we do not even need to create new generating cycles. □

10 Finding Spaces With Certain Homology Groups

After finishing all of these computations of homology groups of surfaces, we notice that there has not been much variety in the homology groups thus far. All second homology groups have been trivial or isomorphic to $\mathbb{Z}$. The lack of 3-simplices to bound prevents anything like $\mathbb{Z}_2$, and the requirement of connectedness and being locally homeomorphic to the plane restricts it to only $\mathbb{Z}$. H_1 is also similarly restricted. For surfaces, we cannot even have something as simple $\mathbb{Z}$, let alone $\mathbb{Z}_3$.

It is then clear that to get some such groups, we cannot have a surface. After all, with a full classification of surfaces known, we know that we cannot find some strange surface with such groups. While another space may not be as "nice" as a surface, we will no longer bound by restrictions such as being locally homeomorphic to a plane. In particular, this means we no longer need to glue edges in pairs. This will give us all the freedom we need to get other finite groups such as $\mathbb{Z}_3$.

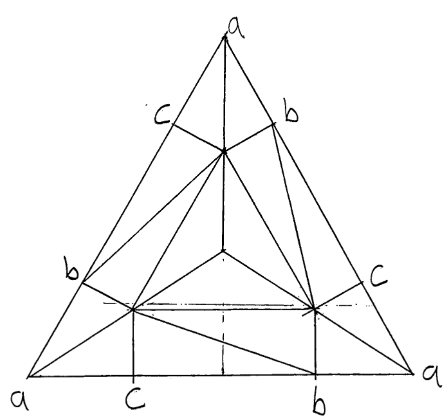


Figure 30

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To get a space whose first homology group is isomorphic to $\mathbb{Z}_3$, we use the space L in Figure 30, with gluing indicated by the labels. The side of the triangle, which will denote by A, will be a cycle. In fact, since we are gluing every side together, "any" side is a cycle, and they are all the same side. Following our usual process from all of the previous computations, we get that any 1-cycle carried by the boundary is a multiple of A, and $\partial\gamma = 3A$ (as every 2-simplex on the boundary of L will go through the edges on the boundary in the same direction, and each one is the fact of three such 2-simplices). Thus, every three copies of A will bound, and $H_1(L) \cong \mathbb{Z}_3$. If using an n-gon with analogous gluings, we will get a first homology group of $\mathbb{Z}_n$.

Using these spaces, we can build anything else that we want. Suppose, for example, we wanted a space with a first homology group of $\mathbb{Z}_3 \oplus \mathbb{Z}_3$, we need only glue a vertex of one to a vertex of the other, making sure to not glue them in a "loop," but rather in a "chain" (that is to say, if we were gluing four of these things together, we glue one to another, then a third to one of them, and a fourth to the third, but then not gluing the fourth to the first). If we want a copy of $\mathbb{Z}$, we glue on a circle. If we want to change the second homology group on top of it, we can paste sphere onto it. Since the sphere has trivial first homology group, this will not alter the first homology group at all. Of course, we still do not have any 3-simplices to bound, so we still cannot have anything like $\mathbb{Z}_2$ in our second homology group.[5]

There might be some concern that doing this gluing might create new, unwanted cycles. The simple explanation for why this does not happen is that a point is not a 2-dimensional space, and so no new area is enclosed. More mathematically, if a new cycle were created by this process, it would necessarily have to pass through the vertex where the 2 spaces were glued. But then the only way to get back to that vertex is to travel through cycles which were already present. This is not completely rigorous, but

[5] There is a method for getting such groups in the second homology group, known as suspension, but it is outside of the scope of this work.

it is easy to convince oneself of it. This is, however, why we require that the objects be glued in "chains," as if 3 spheres were glued in a "loop," a new cycle would be created by going through each one.

11 Closing Comments

Arriving at the end of this development of simplicial homology theory, we look back and take note of how much effort was needed to simply define the homology groups, let alone compute theme for a special class of spaces. Indeed, simplicial homology theory mostly dead-ends after a few more developments past what was discussed here. A different homology theory, known as singular homology, began to prevail and still prevails as a topic of research.

Still, studying this original homology is important, as singular homology (as well as other homology theories) is much less intuitive. Often it is easier to reach some conclusions in singular homology than their simplicial counterpart. For example, Theorem 7.2 in the singular case becomes almost a triviality, while in the simplicial case, several other concepts need to be developed, which eventually dead-end along with the rest of simplicial homology. But the proofs of these results give no insight into what is happening. For that matter, the groups are defined in such an abstract way that figuring out what they mean is a challenge. Thus, starting with simplicial homology is necessary to have the intuition to truly understand what the singular groups actually mean.

Thus, the work presented here has provided a good starting point for the study of homology theory. From here, one can go on to study the generalized homology theory with the Eilenberg-Steenrod Axioms, as well as singular homology theory, and eventually to cohomology theories. Those are, however, topics for another work.

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