RADIATION OF A DIPOLE ANTENNA IN
A HOMOGENEOUS COLD MAGNETOPLASMA

A Thesis

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INTRODUCTION

The problem of radiation from electrons moving in a magnetoplasma is of interest in astrophysics and radio astronomy. Many authors have treated this problem by taking different kinds of approach. Due to the anisotropic property of the magnetoplasma, formulas for radiation in general are very complicated. The methods that have been used in solving this problem are:

(A) Integral expansion method:

Twiss(1952) developed this method. He and Roberts(1958) used this method to find an asymptotic far field solution for synchrotron radiation in a cold magnetoplasma.

(B) Tensorial Green's function method:

Bunkin(1957) used this method and obtained an asymptotic far field solution for the radiation from a current distribution. And Kuehl (1962) obtained formulas for the radiation from an electric dipole in a cold magnetoplasma.

(C) Fourier transform method:

Sitenko and Kolomenskii(1956) developed this method in solving for Cerenkov radiation. Mansfield(1967) obtained a solution for the radiation from a spiraling electron in a cold magnetoplasma, but his formula was corrected by Melrose(1968). McKenzie(1964) used a different kind of approach of this method and obtained a different solution.

(D) Hamiltonian method:

Ginzburg(1940) first applied this method to a uniaxial anisotropic medium. Kolomenskii(1953) extended this method to solve for the radiation from an oscillating charged particle in a gyrotropic medium, while Eidman(1958) obtained a solution for that from a spiraling electron in a cold magnetoplasma. By taking into account of dispersion, Liemohn(1965) obtained a solution different from that obtained by Eidman.

The Hamiltonian method is a very powerful technique in solving the problem of radiation in anisotropic media as can be seen from those papers cited above in (D). But it has been pointed out by Ko (1969, 1970) that there were errors in both Eidman and Liemohn's papers. These errors have occurred because these authors have failed to take into account the effect of dispersion in a consistent manner. The complete procedures of this method will be presented in a joint paper by Ko, King and Chuang. Applications to other problems will be also discussed in a dissertation by King.

In this thesis we will apply the Hamiltonian method to finding the radiation from an arbitrarily oriented dipole antenna in a homogeneous cold magnetoplasma. First, the Hamiltonian method is partly reviewed. Then the radiation fields are found. And finally, as an example, we will calculate the fields for the case of an infinite magnetostatic field.

HAMILTONIAN METHOD

The discussion of the Hamiltonian method for a free space can be found in (Heitler, 1954). A similar procedure for an anisotropic medium will be taken in this thesis.

In a source-free region Maxwell's equations are

$$\nabla \times \overline{H} - \dot{\overline{D}} = 0 \tag{1}$$

$$\nabla \times \overline{E} + \dot{\overline{B}} = 0 \tag{2}$$

$$\nabla \cdot \overline{B} = 0 \tag{3}$$

$$\nabla \cdot \overline{D} = 0 \tag{4}$$

As usual, we define the vector and scalar potentials as

$$\overline{B} = \nabla \times \overline{A} \tag{5}$$

$$\overline{E} = - \nabla \Phi - \dot{\overline{A}} \tag{6}$$

The coordinate system is oriented such that the uniform magnetostatic field is along the z-axis. Then the homogeneous cold magnetoplasma is characterized by a dielectric complex hermitian tensor $\overline{\overline{\epsilon}}(\omega,\theta)$, where ω is the angular frequency of the wave and θ is the angle measured from the z-axis.

Define $\overline{A}_\omega$ as the Fourier transform of $\overline{A}$,

$$\overline{A}_\omega = \int_0^\infty \overline{A}\, e^{-j\omega t}\, dt \tag{7}$$

where the time t is defined such that for $t \leq 0$, $\overline{A} = \dot{\overline{A}} = 0$.

And the inverse Fourier transform is

$$\overline{A} = \int_{-\infty}^{\infty} \overline{A}_\omega\, e^{j\omega t}\, df \tag{8}$$

where f is the frequency of the wave.

From (5), the magnetic field intensity is

$$\overline{H} = \frac{1}{\mu_0} \nabla \times \overline{A} \tag{9}$$

And since $\overline{D}_\omega = e_0 \overline{\overline{\epsilon}} \cdot (- \nabla \Phi_\omega - j\omega \overline{A}_\omega)$, we have

$$\overline{D} = -\int_{-\infty}^{\infty} e_0 \overline{\overline{e}}\cdot\nabla\Phi_\omega \, e^{j\omega t}\, df - \int_{-\infty}^{\infty} j\omega e_0 \overline{\overline{e}}\cdot\overline{A}_\omega \, e^{j\omega t}\, df \tag{10}$$

By substituting (9) and (10) into (1) and (4), we obtain

$$\nabla x\nabla x\overline{A} + \int_{-\infty}^{\infty} j\omega\mu_0 e_0 \overline{\overline{e}}\cdot\nabla\Phi_\omega \, e^{j\omega t}\, df - \int_{-\infty}^{\infty} \omega^2\mu_0 e_0 \overline{\overline{e}}\cdot\overline{A}_\omega \, e^{j\omega t}\, df = 0 \tag{11}$$

$$\int_{-\infty}^{\infty} \nabla\cdot\overline{\overline{e}}\cdot\nabla\Phi_\omega e^{j\omega t}\, df + \int_{-\infty}^{\infty} j\omega\nabla\cdot\overline{\overline{e}}\cdot\overline{A}_\omega \, e^{j\omega t}\, df = 0 \tag{12}$$

Instead of the Lorentz gauge, the Coulomb gauge will be used,

$$\nabla\cdot\overline{\overline{e}}\cdot\overline{A}_\omega = 0 \tag{13}$$

Under this condition we can set $\Phi_\omega = 0$. Then (11) reduces to

$$\nabla x\nabla x\overline{A} - \int_{-\infty}^{\infty} \omega^2\mu_0 e_0 \overline{\overline{e}}\cdot\overline{A}_\omega \, e^{j\omega t}\, df = 0 \tag{14}$$

To find a solution for (14), we assume that the whole radiation energy is enclosed in a cube of volume L^3 and the vector potential is required to satisfy some boundary condition, i.e.,

$\overline{A}$ is periodic on the surface of the cube. (15)

Then the general solution of (14) can be represented as a superposition of orthogonal eigenfunctions,

$$\overline{A} = \frac{1}{2}\sum_\lambda\left[q_\lambda(t)\overline{A}_\lambda(\overline{r}) + q_\lambda^*(t)\overline{A}_\lambda^*(\overline{r})\right] \tag{16}$$

or

$$\overline{A} = \mathrm{Re}\sum_\lambda q_\lambda(t)\overline{A}_\lambda(\overline{r}) \tag{17}$$

where $q_\lambda(t)$ and $\overline{A}_\lambda(\overline{r})$ are complex functions and satisfy the following equations

$$\nabla x\nabla x\overline{A}_\lambda(\overline{r}) - \omega_\lambda^2\mu_0 e_0 \overline{\overline{e}}_\lambda\cdot\overline{A}_\lambda(\overline{r}) = 0 \tag{18}$$

$$q_\lambda(t) = b\, e^{j\omega_\lambda t} \tag{19}$$

where $\overline{\overline{e}}_\lambda$ is the dielectric tensor at $\omega = \omega_\lambda$ and b is an arbitrary constant.

From (18), we assume

$$\overline{A}_\lambda(\overline{r}) = \overline{a}_\lambda\sqrt{\frac{4\pi c^2}{L^3}}\, e^{-j\overline{k}_\lambda\cdot\overline{r}} \tag{20}$$

where $\overline{k}_\lambda$ is the propagation vector at $\omega = \omega_\lambda$, $\overline{a}_\lambda$ is the polarization vector associated with $\overline{k}_\lambda$ and c is the velocity of light in free space.

For $\overline{A}_\lambda(\overline{r})$ to satisfy (15), the components of $\overline{k}_\lambda$ are given by $k_{\lambda x} = \frac{2\pi\lambda_x}{L}$, $k_{\lambda y} = \frac{2\pi\lambda_y}{L}$ and $k_{\lambda z} = \frac{2\pi\lambda_z}{L}$, where λ_x, λ_y and λ_z are integers.

The polarization vector is then normalized such that

$$\int_{L^3} dv\ \overline{A}^*_\mu(\overline{r})\cdot\overline{\overline{e}}_\lambda\cdot\overline{A}_\lambda(\overline{r}) = 4\pi c^2 n_\lambda^2\,\delta_{\lambda\mu} \tag{21}$$

where n_λ is the refractive index at $\omega = \omega_\lambda$ along the direction of $\overline{k}_\lambda$ and δ is the Kronecker's delta.

By substituting (20) into (18) and (21), we obtain

$$\overline{\overline{e}}_\lambda\cdot\overline{a}_\lambda = n_\lambda^2\hat{k}_\lambda x(\overline{a}_\lambda x\hat{k}_\lambda) \tag{22}$$

$$\overline{a}^*_\lambda\cdot\overline{\overline{e}}_\lambda\cdot\overline{a}_\lambda = n_\lambda^2 \tag{23}$$

where $\hat{k}_\lambda$ is the unit vector along $\overline{k}_\lambda$.

For (22) and (23) to be consistent with each other, the following equation must be satisfied,

$$|\overline{a}_\lambda|^2 - |\hat{k}_\lambda\cdot\overline{a}_\lambda|^2 = 1 \tag{24}$$

If we let L approach to infinity, then k-space is approximately continuous and the summation over λ can be replaced by integration. The number of modes in the solid angle $d\Omega = \sin\theta\, d\theta\, d\phi$ and the frequency interval between ω_λ and $\omega_\lambda + d\omega_\lambda$ is given by

$$\left(\frac{L}{2\pi c}\right)^3 \omega_\lambda^2 n_\lambda^3\left(1 + \frac{\omega_\lambda}{n_\lambda}\frac{\partial n_\lambda}{\partial\omega_\lambda}\right) d\omega_\lambda\, d\Omega \tag{25}$$

Next we consider the radiation from a current source. Let $\overline{J}$ be the current density. Then the vector potential produced by $\overline{J}$ satisfies

$$\nabla x\nabla x\overline{A} + \int_{-\infty}^{\infty} j\omega\mu_0 e_0\,\overline{\overline{e}}\cdot\nabla\Phi_\omega\, e^{j\omega t}\, df - \int_{-\infty}^{\infty}\omega^2\mu_0 e_0\,\overline{\overline{e}}\cdot\overline{A}_\omega\, e^{j\omega t}\, df = \mu_0\overline{J} \tag{26}$$

Expand $\overline{A}$ as a superposition of the homogeneous eigenfunctions,

$$\overline{A} = \mathrm{Re}\sum_\lambda q_\lambda(t)\overline{A}_\lambda(\overline{r}) \tag{27}$$

Where $q_\lambda(t)$ takes the form of $b_\lambda(t)\, e^{j\omega_\lambda t}$.

For simplicity, from now on we shall delete the word "Re". Then all fields, potentials and currents will be represented by the analytic signals of their real signals (Born and Wolf, 1964).

To find the solution for $q_\lambda(t)$, we multiply (26) by $\int_{L^3} dv\, \overline{A}^*_\lambda(\overline{r})\cdot$ and use (18), (20), (21) and the following relation,

$$\int_{L^3} dv\, \overline{A}^*_\lambda(\overline{r})\cdot\overline{\overline{e}}\cdot\nabla\Phi_\omega = \int_{L^3} dv\, \nabla\Phi_\omega\cdot\overline{\overline{e}}^*\cdot\overline{A}^*_\lambda(\overline{r})$$
$$= \int_{L^3} dv\, [\nabla\cdot(\Phi_\omega\overline{\overline{e}}^*\cdot\overline{A}^*_\lambda(\overline{r})) - \Phi_\omega\nabla\cdot\overline{\overline{e}}^*\cdot\overline{A}^*_\lambda(\overline{r})]$$
$$= -\int_{L^3} dv\, \Phi_\omega\nabla\cdot\overline{\overline{e}}^*\cdot\overline{A}^*_\lambda(\overline{r})$$

to obtain

$$\omega_\lambda^2 n_\lambda^2 q_\lambda(t) - \int_{-\infty}^{\infty} \omega^2\, \overline{a}^*_\lambda\cdot\overline{\overline{e}}\cdot\overline{a}_\lambda q_\lambda(\omega)\, e^{j\omega t}\, df$$
$$= \frac{1}{4\pi}\int_{L^3} dv\, \mu_0\overline{A}^*_\lambda(\overline{r})\cdot\overline{J} + \frac{1}{4\pi}\int_{-\infty}^{\infty} j\omega\mu_0 e_0 [\int_{L^3} dv\, \Phi_\omega\nabla\cdot\overline{\overline{e}}^*\cdot\overline{A}^*_\lambda(\overline{r})]\, e^{j\omega t}\, df \quad (28)$$

where $q_\lambda(\omega)$ is the Fourier transform of $q_\lambda(t)$.

Equation (28) may also be written as

$$(\omega_\lambda^2 n_\lambda^2 - \omega^2\overline{a}^*_\lambda\cdot\overline{\overline{e}}\cdot\overline{a}_\lambda)q_\lambda(\omega) = \frac{1}{4\pi}\int_0^\infty [\int_{L^3} dv\, \mu_0\overline{A}^*_\lambda(\overline{r})\cdot\overline{J}]\, e^{-j\omega t}\, dt$$
$$+ \frac{1}{4\pi} j\omega\mu_0 e_0 \int_{L^3} dv\, \Phi_\omega\nabla\cdot\overline{\overline{e}}^*\cdot\overline{A}^*_\lambda(\overline{r}) \quad (29)$$

From (29), we have

$$q_\lambda(\omega) = \frac{\int_0^\infty [\int_{L^3} dv\, \mu_0\overline{A}^*_\lambda(\overline{r})\cdot\overline{J}]\, e^{-j\omega t}\, dt + j\omega\mu_0 e_0\int_{L^3} dv\, \Phi_\omega\nabla\cdot\overline{\overline{e}}^*\cdot\overline{A}^*_\lambda(\overline{r})}{4\pi(\omega_\lambda^2 n_\lambda^2 - \omega^2\overline{a}^*_\lambda\cdot\overline{\overline{e}}\cdot\overline{a}_\lambda)} \quad (30)$$

Using inverse Fourier transformation, we can find $q_\lambda(t)$. The inverse Fourier transformation may be carried out by residue theorem of complex function. Since the denominator of (30) has a zero at $\omega = \omega_\lambda$, the numerator is evaluated at $\omega = \omega_\lambda$. Then the second integral in the numerator is equal to zero since $\nabla\cdot\overline{\overline{e}}_\lambda\cdot\overline{A}_\lambda(\overline{r}) = 0$. Therefore (30) reduces to

$$q_\lambda(\omega) = \frac{\int_0^\infty [\int_{L^3} dv\, \mu_0\overline{A}^*_\lambda(\overline{r})\cdot\overline{J}]\, e^{-j\omega t}\, dt}{4\pi(\omega_\lambda^2 n_\lambda^2 - \omega^2\overline{a}^*_\lambda\cdot\overline{\overline{e}}\cdot\overline{a}_\lambda)} \quad (31)$$

Then we have

$$q_\lambda(t) = \int_{-\infty}^{\infty} q_\lambda(\omega)\, e^{j\omega t}\, df$$

$$= \int_0^\infty dt_1 \left[\int_{L^3} dv\, \mu_0 \overline{A}_\lambda^*(\overline{r})\cdot\overline{J}(t_1)\right] \int_{-\infty}^{\infty} \frac{e^{j\omega(t - t_1)}}{4\pi(\omega_\lambda^2 n_\lambda^2 - \omega^2 \overline{a}_\lambda^*\cdot\overline{\overline{e}}\cdot\overline{a}_\lambda)}\, df \qquad (32)$$

The pole of (32) is on the real axis of the complex frequency plane. But if we use a complex frequency $(\omega - j\sigma)$ in Fourier transformation, the pole will be off the real axis and on the upper half plane. Then

$$\int_{-\infty}^{\infty} \frac{e^{j\omega(t - t_1)}}{4\pi(\omega_\lambda^2 n_\lambda^2 - \omega^2 \overline{a}_\lambda^*\cdot\overline{\overline{e}}\cdot\overline{a}_\lambda)}\, df = \frac{1}{4\pi j}\, \frac{e^{j\omega_\lambda(t - t_1)}}{2\omega_\lambda n_\lambda^2 + \omega_\lambda^2 \overline{a}_\lambda^*\cdot\frac{\partial\overline{\overline{e}}_\lambda}{\partial\omega_\lambda}\cdot\overline{a}_\lambda} \qquad \text{for } t > t_1$$

$$= 0 \qquad \text{for } t < t_1$$

Since $\overline{a}_\lambda^*\cdot\frac{\partial\overline{\overline{e}}_\lambda}{\partial\omega_\lambda}\cdot\overline{a}_\lambda = \frac{\partial n_\lambda^2}{\partial\omega_\lambda} - \frac{\partial\overline{a}_\lambda^*}{\partial\omega_\lambda}\cdot\overline{\overline{e}}_\lambda\cdot\overline{a}_\lambda - \overline{a}_\lambda^*\cdot\overline{\overline{e}}_\lambda\cdot\frac{\partial\overline{a}_\lambda}{\partial\omega_\lambda}$

and from (22) and (24), it may be shown that

$$\frac{\partial\overline{a}_\lambda^*}{\partial\omega_\lambda}\cdot\overline{\overline{e}}_\lambda\cdot\overline{a}_\lambda + \overline{a}_\lambda^*\cdot\overline{\overline{e}}_\lambda\cdot\frac{\partial\overline{a}_\lambda}{\partial\omega_\lambda} = 0$$

hence (32) reduces to

$$q_\lambda(t) = \int_0^t dt_1 \left[\int_{L^3} dv\, \mu_0 \overline{A}_\lambda^*(\overline{r})\cdot\overline{J}(t_1)\right] \frac{e^{j\omega_\lambda(t - t_1)}}{8\pi j\omega_\lambda n_\lambda^2\left(1 + \frac{\omega_\lambda}{n_\lambda}\frac{\partial n_\lambda}{\partial\omega_\lambda}\right)} \qquad (32)'$$

Under the Coulomb gauge, the scalar potential Φ is a static potential (Heitler, 1954) which doesn't contribute to radiation. Therefore in calculating the electric field intensity only the vector potential will be taken into account, i.e.,

$$\overline{E} = -\dot{\overline{A}}$$

Therefore from (32)' and (25), we obtain, after suppressing the subscript λ, formulas for $\overline{E}$ and $\overline{H}$,

$$\overline{E} = -\iint \left(\frac{L}{2\pi c}\right)^3 \omega^2 n^3 \left(1 + \frac{\omega}{n}\frac{\partial n}{\partial\omega}\right)\dot{q}(t)\overline{A}(\overline{r})\, d\omega\, d\Omega \qquad (33)$$

$$\overline{H} = -\iint \left(\frac{L}{2\pi c}\right)^3 \frac{j\omega^3 n^4}{\mu_0 c} \left(1 + \frac{\omega}{n}\frac{\partial n}{\partial\omega}\right) q(t)\hat{k}\times\overline{A}(\overline{r})\, d\omega\, d\Omega \qquad (34)$$

with

$$q(t) = \int_0^t dt_1 \left[\int_{L^3} dv\, \mu_0 \overline{A}^*(\overline{r})\cdot\overline{J}(t_1)\right] \frac{e^{j\omega(t - t_1)}}{8\pi j\omega n^2 (1 + \frac{\omega}{n}\frac{\partial n}{\partial \omega})} \tag{35}$$

and

$$\overline{A}(\overline{r}) = \overline{a}\sqrt{\frac{4\pi c^2}{L^3}}\, e^{-j\overline{k}\cdot\overline{r}} \tag{36}$$

Now we shall define the polarization coefficients for $\overline{a}$. Let $\hat{k} = (\sin\theta \cos\phi,\ \sin\theta \sin\phi,\ \cos\theta)$, where θ and ϕ are the angles in the conventional spherical coordinate and the components of the vector are expressed in the x, y and z directions. The polarization coefficients associated with $\hat{k}$ are defined as

$$\frac{a_\theta}{a_\phi} = -j\alpha_\theta, \quad \frac{a_K}{a_\phi} = -j\alpha_K$$

where a_θ, a_ϕ and a_K are the spherical components of $\overline{a}$ as shown in Fig. 1, and α_θ and α_K are real functions for a cold magnetoplasma (Ginzburg, 1961). Then

$$\overline{a} = -a_\phi(\sin\phi + j\alpha_\rho\cos\phi,\ -\cos\phi + j\alpha_\rho\sin\phi,\ j\alpha_z) \tag{37}$$

where $\alpha_\rho = \alpha_\theta\cos\theta + \alpha_K\sin\theta$

$$\alpha_z = \alpha_K\cos\theta - \alpha_\theta\sin\theta \tag{38}$$

$$a_\phi^2 = (1 + \alpha_\theta^2)^{-1}$$

Till now we have not specified the source founction $\overline{J}$, hence (33) and (34) are general formulas for the radiation fields in an anisotropic medium.

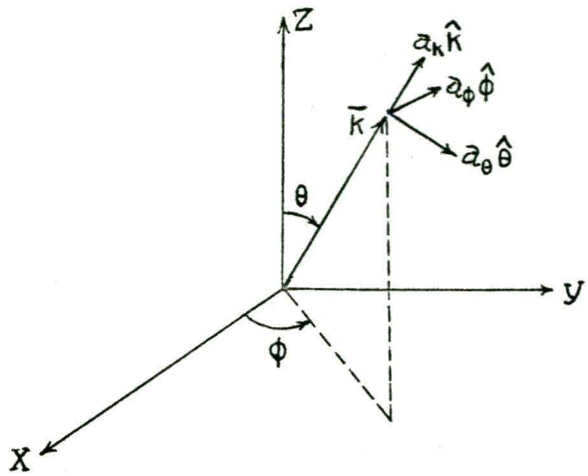

Fig. 1. Illustration of the polarization vector.

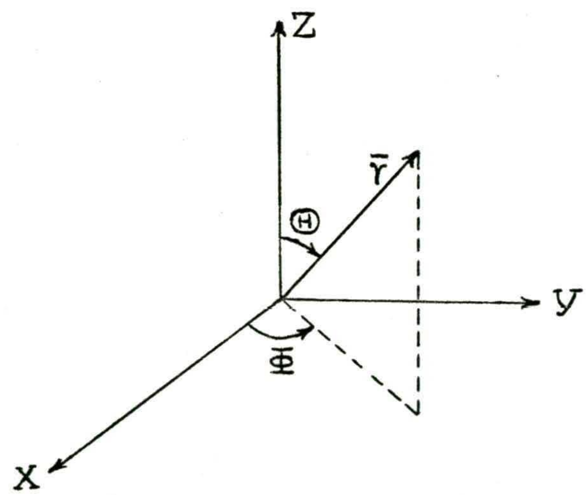

Fig. 2. Illustration of the field point.

RADIATION FROM A DIPOLE ANTENNA

The dipole antenna is placed at the origin of the coordinate system and is oriented in the yz-plane making an angle α with the z-axis. Then the source function $\bar{J}$ takes the form of

$$\bar{J} = Il\, e^{j\omega_0 t}\,\delta(r)(0, \quad \sin\alpha, \quad \cos\alpha) \tag{39}$$

where Il is the dipole moment of the antenna.

The solution for q(t) may be found from (35), (36) and (37). It is found that

$$q(t) = \sqrt{\frac{4\pi c^2}{L^3}}\,\frac{\mu_0 Il a_\phi}{4\pi j\omega n^2\left(1 + \frac{\omega}{n}\frac{\partial n}{\partial\omega}\right)}\,(\sin\alpha\cos\phi + j\alpha_\rho\sin\alpha\sin\phi + j\alpha_z\cos\alpha)$$
$$\cdot\frac{\sin\left(\frac{\omega-\omega_0}{2}\right)t}{\omega-\omega_0}\,e^{j\left(\frac{\omega+\omega_0}{2}\right)t}$$

In the limit $t\to\infty$, the following equation is valid in the sense of distribution function (Papoulis, 1962),

$$\lim_{t\to\infty}\frac{\sin\omega t}{\omega} = \pi\delta(\omega)$$

Then

$$q(t) = \sqrt{\frac{4\pi c^2}{L^3}}\,\frac{\mu_0 Il a_\phi}{4j\omega n^2\left(1 + \frac{\omega}{n}\frac{\partial n}{\partial\omega}\right)}\,(\sin\alpha\cos\phi + j\alpha_\rho\sin\alpha\sin\phi + j\alpha_z\cos\alpha)$$
$$\cdot\delta(\omega-\omega_0)\,e^{j\omega t} \tag{40}$$

The field point is chosen at $\bar{r} = r(\sin\Theta\cos\Phi, \quad \sin\Theta\sin\Phi, \quad \cos\Theta)$, where r, Θ and Φ are the spherical coordinates of the field point as shown in Fig. 2. The fields at this point may be found by substituting (40) into (33) and (34). After carrying out the integration over ω, the fields take the forms of

$$\bar{E} = \int d\Omega\,\frac{\omega^3 n\mu_0 Il}{8\pi^2 c(1+\alpha_\theta^2)}\,(\sin\alpha\cos\phi + j\alpha_\rho\sin\alpha\sin\phi + j\alpha_z\cos\alpha)$$
$$\cdot e^{j\omega t - j\frac{\omega n}{c}r[\sin\Theta\sin\theta\cos(\phi-\Phi) + \cos\Theta\cos\theta]}$$

$$\cdot(\sin\phi + j\alpha_\rho\cos\phi,\quad -\cos\phi + j\alpha_\rho\sin\phi,\quad j\alpha_z) \tag{41}$$

$$\bar{H} = \int d\Omega \frac{\omega^2 n^2 Il}{8\pi^2 c^2(1 + \alpha_\theta^2)} (\sin\alpha\cos\phi + j\alpha_\rho\sin\alpha\sin\phi + j\alpha_z\cos\alpha)$$

$$\cdot e^{j\omega t - j\frac{\omega n}{c}r[\sin\Theta\sin\theta\cos(\phi - \Phi) + \cos\Theta\cos\theta]}$$

$$\cdot(\cos\theta\cos\phi - j\alpha_\theta\sin\phi,\quad \cos\theta\sin\phi + j\alpha_\theta\cos\phi,\quad -\sin\theta) \tag{42}$$

It should be noted that in (41) and (42) the frequency ω is evaluated at the frequency of the source and the integration is with respect to θ and ϕ.

The integration over ϕ is easy to be carried out by

$$\int_0^{2\pi} e^{-jX\cos(\phi - \Phi) + js\phi}\, d\phi = 2\pi j^{-s} e^{js\Phi} J_s(X)$$

where $X = \frac{\omega n}{c} r\sin\Theta\sin\theta$ and J_s is a Bessel function of order s.

Then

$$\bar{E} = \int_0^\pi d\theta\,\sin\theta\,\frac{\omega^2 n I l \mu_0}{4\pi c(1 + \alpha_\theta^2)} \exp(j\omega t - j\frac{\omega n}{c}r\cos\Theta\cos\theta)$$

$$\begin{bmatrix} j\alpha_\rho\sin\alpha\, J_0 + \alpha_z\cos\alpha\,(\sin\Phi + j\alpha_\rho\cos\Phi)\, J_1 \\ -\frac{1}{2}(1 - \alpha_\rho^2)\sin\alpha\sin 2\Phi\, J_2\ , \\ -\frac{1}{2}(1 + \alpha_\rho^2)\sin\alpha\, J_0 + \alpha_z\cos\alpha\,(j\alpha_\rho\sin\Phi - \cos\Phi)\, J_1 \\ +\frac{1}{2}(1 - \alpha_\rho^2)\sin\alpha\cos 2\Phi\, J_2\ , \\ -\alpha_z^2\cos\alpha\, J_0 + \alpha_z\sin\alpha\,(\cos\Phi + j\alpha_\rho\sin\Phi)\, J_1 \end{bmatrix} \tag{43}$$

$$\bar{H} = \int_0^\pi d\theta\,\sin\theta\,\frac{\omega^2 n^2 Il}{4\pi c^2(1 + \alpha_\theta^2)}\exp(j\omega t - j\frac{\omega n}{c}r\cos\Theta\cos\theta)$$

$$\begin{bmatrix} \frac{1}{2}(\cos\theta + \alpha_\theta\alpha_\rho)\sin\alpha\, J_0 + \alpha_z(\cos\theta\cos\Phi - j\alpha_\theta\sin\Phi)\cos\alpha\, J_1 \\ +\frac{1}{2}(\alpha_\theta\alpha_\rho\cos 2\Phi - \cos\theta\cos 2\Phi + j\alpha_\theta\sin 2\Phi - j\alpha_\rho\cos\theta\sin 2\Phi)\sin\alpha\, J_2\ , \\ \frac{1}{2}j(\alpha_\theta + \alpha_\rho\cos\theta)\sin\alpha\, J_0 + \alpha_z(\cos\theta\sin\Phi + j\alpha_\theta\cos\Phi)\cos\alpha\, J_1 \\ +\frac{1}{2}(\alpha_\theta\alpha_\rho\sin 2\Phi - \cos\theta\sin 2\Phi + j\alpha_\rho\cos\theta\cos 2\Phi - j\alpha_\theta\cos 2\Phi)\sin\alpha\, J_2\ , \\ -j\alpha_z\sin\theta\cos\alpha\, J_0 + (j\cos\Phi - \alpha_\rho\sin\Phi)\sin\theta\sin\alpha\, J_1 \end{bmatrix}$$

(44)

In (43) and (44), replace $J_s(X)$ by $\frac{1}{2}(H_s^{(1)}(X) + H_s^{(2)}(X))$. And as r approaches to infinity, the asymptotic forms of Hankel functions of large argument may be used. Then we perform the stationary phase method to carry out the integration over θ (Copson, 1965). It is found that

$$\int_{-\pi}^{\pi} H_s^{(2)}(X) \exp(- j\tfrac{\omega n}{c}r \cos\Theta \cos\theta)\, d\theta$$

$$= \int_{-\pi}^{\pi} j^s \left(\frac{2c}{\pi\omega nr \sin\Theta \sin\theta}\right)^{\frac{1}{2}} \exp(j\tfrac{\pi}{4} - j\tfrac{\omega n}{c}r \cos(\Theta - \theta))\, d\theta$$

$$= j^s \left(\frac{2c}{\pi\omega nr \sin\Theta \sin\theta}\right)^{\frac{1}{2}} \exp(j\tfrac{\pi}{4} - j\tfrac{\omega n}{c}r \cos\eta)$$

$$\cdot \int_{-\infty}^{\infty} \exp\left[\tfrac{1}{2}j\tfrac{\omega}{c}r\left((n - n'') \cos\eta - 2n' \sin\eta\right)\theta^2\right] d\theta$$

$$= j^{s+1}\left(\frac{2c}{\omega r}\right)\exp(- j\tfrac{\omega n}{c}r \cos\eta)\left[n \sin\Theta \sin\theta \left((n - n'') \cos\eta - 2n' \sin\eta\right)\right]^{-\frac{1}{2}} \tag{45}$$

where $\eta = \Theta - \theta$, and θ is the angle where the stationary phase occurs and satisfies the following equations,

$$n' \cos(\Theta - \theta) + n \sin(\Theta - \theta) = 0 \tag{46}$$

$$\cos(\Theta - \theta) > 0 \tag{47}$$

$$-\pi \leqslant \theta \leqslant \pi$$

where the primes denote differentiation with respect to θ and equation (47) is due to the radiation condition at infinity.

The range of integration from $-\pi$ to 0 in (45) is due to the first kind Hankel function according to the following transformation,

$$H_s^{(1)}(- X) = (- 1)^{s+1}\, H_s^{(2)}(X)$$

If there are more than one stationary phase point in the range, $-\pi \leqslant \theta \leqslant \pi$, the right hand side of (45) will be replaced by a summation over each stationary phase point.

By using (46), we rewrite (45) as

$$\int_{-\pi}^{\pi} H_s^{(2)}(X) \exp(-\, j\tfrac{\omega n}{c} r \cos\Theta \cos\theta)\, d\theta$$

$$= j^{s\,+\,1}\left(\frac{2c}{\omega r}\right) \exp(-\, j\tfrac{\omega n}{c} r \cos\eta)\left[\sin\Theta \sin\theta \cos\eta\ (n^2 - nn'' + 2n'^2)\right]^{-\frac{1}{2}}$$

After carrying out the integration over θ, we find that the fields are

$$\overline{E} = \frac{j\omega\mu_0 n Il \sin\theta \exp(j\omega t - j\frac{\omega n}{c} r \cos\eta)}{4\pi(1 + \alpha_\theta^2)[\sin\Theta \sin\theta \cos\eta\ (n^2 - nn'' + 2n'^2)]^{\frac{1}{2}}\, r}$$

$$\begin{bmatrix} j\alpha_\rho \sin\alpha + \alpha_z \cos\alpha\ (j \sin\Phi - \alpha_\rho \cos\Phi) + \frac{1}{2}(1 - \alpha_\rho^2) \sin\alpha \sin 2\Phi, \\ -\frac{1}{2}(1 + \alpha_\rho^2) \sin\alpha - \alpha_z \cos\alpha\ (\alpha_\rho \sin\Phi + j \cos\Phi) - \frac{1}{2}(1 - \alpha_\rho^2) \\ \cdot \sin\alpha \cos 2\Phi, \\ -\alpha_z^2 \cos\alpha + j\alpha_z \sin\alpha \cos\Phi - \alpha_z\alpha_\rho \sin\alpha \sin\Phi \end{bmatrix} \quad (48)$$

$$\overline{H} = \frac{j\ n\ Il\ \sin 0\ \exp(j\ \ t - j\ \ r \cos\ \ )}{4\pi c(1 + \alpha_\theta^2)[\sin\Theta \sin\theta \cos\eta\ (n^2 - nn'' + 2n'^2)]^{1/2}\, r}$$

$$\begin{bmatrix} \frac{1}{2}(\cos\theta + \alpha_\theta\alpha_\rho) \sin\alpha + \alpha_z \cos\alpha\ (j \cos\theta \cos\Phi + \alpha_\theta \sin\Phi) \\ + \frac{1}{2}(\cos\theta - \alpha_\theta\alpha_\rho) \sin\alpha \cos 2\Phi + \frac{1}{2}j(\alpha_\rho \cos\theta - \alpha_\theta) \sin\alpha \sin 2\Phi, \\ \frac{1}{2}j(\alpha_\theta + \alpha_\rho \cos\theta) \sin\alpha + \alpha_z \cos\alpha\ (j \cos\theta \sin\Phi - \alpha_\theta \cos\Phi) \\ + \frac{1}{2}(\cos\theta - \alpha_\theta\alpha_\rho) \sin\alpha \sin 2\Phi + \frac{1}{2}j(\alpha_\theta - \alpha_\rho \cos\theta) \sin\alpha \cos 2\Phi, \\ -\, j\alpha_z \sin\theta \cos\alpha - \sin\theta \sin\alpha \cos\Phi - j\alpha_\rho \sin\theta \sin\alpha \sin\Phi \end{bmatrix} \quad (49)$$

In (48) and (49) the components of the vectors are expressed in the rectangular coordinate. To transform to the spherical coordinate we need the following relations,

$$\begin{aligned} \hat{x} &= \sin\Theta \cos\Phi\ \hat{r} + \cos\Theta \cos\Phi\ \hat{\Theta} - \sin\Phi\ \hat{\Phi} \\ \hat{y} &= \sin\Theta \sin\Phi\ \hat{r} + \cos\Theta \sin\Phi\ \hat{\Theta} + \cos\Phi\ \hat{\Phi} \\ \hat{z} &= \cos\Theta\ \hat{r} - \sin\Theta\ \hat{\Theta} \end{aligned} \quad (50)$$

When expressed in the spherical coordinate, (48) and (49) reduce to

$$\overline{E} = -\frac{j\omega\mu_0 nIl \sin\theta \left[\alpha_z\cos\alpha + (\alpha_\rho\sin\Phi - j\cos\Phi)\sin\alpha\right]}{4\pi(1+\alpha_\theta^2)\left[\sin\theta \sin\Theta \cos\eta \, (n^2 - nn'' + 2n'^2)\right]^{1/2}}$$

$$\cdot\frac{\exp(j\omega t - j\frac{\omega n}{c}r\cos\eta)}{r}(\alpha_\theta\sin\eta + \alpha_K\cos\eta, \ \ \alpha_\theta\cos\eta - \alpha_K\sin\eta, \ \ j) \tag{51}$$

$$\overline{H} = \frac{j\omega n^2 Il \sin\theta \left[\alpha_z\cos\alpha + (\alpha_\rho\sin\Phi - j\cos\Phi)\sin\alpha\right]}{4\pi c(1+\alpha_\theta^2)\left[\sin\theta \sin\Theta \cos\eta \, (n^2 - nn'' + 2n'^2)\right]^{1/2}}$$

$$\cdot\frac{\exp(j\omega t - j\frac{\omega n}{c}r\cos\eta)}{r}(j\sin\eta, \ \ j\cos\eta, \ \ -\alpha_\theta) \tag{52}$$

The average Poynting vector is defined as $\frac{1}{2}\mathrm{Re}(\overline{E}\times\overline{H}^*)$. From (51) and (52) we see that the average Poynting vector has a radial component and a Θ-component. The radial component has a factor

$$\left[(1+\alpha_\theta^2)\cos\eta - \alpha_\theta\alpha_K\sin\eta\right],$$

which, by using (46), may be rewritten as

$$\frac{1}{n}\left[n(1+\alpha_\theta^2) + n'\alpha_\theta\alpha_K\right]\cos\eta .$$

And the Θ-component has a factor

$$\left[(1+\alpha_\theta^2)\sin\eta + \alpha_\theta\alpha_K\cos\eta ,\right.$$

which may also be rewritten as

$$\frac{1}{n}\left[-n'(1+\alpha_\theta^2) + n\alpha_\theta\alpha_K\right]\cos\eta .$$

It will be shown in the next section that

$$n'(1+\alpha_\theta^2) = n\alpha_\theta\alpha_K \tag{53}$$

Therefore the Θ-component of the average Poynting vector is zero. Then the average Poynting vector is in the radial direction. The radiation intensity is found to be

$$\frac{dP}{d\Omega} = \frac{\omega^2\mu_0 nI^2l^2 \sin\theta\,(n^2+n'^2)\left[(\alpha_z\cos\alpha + \alpha_\rho\sin\alpha\sin\Phi)^2 + \sin^2\alpha\cos^2\Phi\right]}{32\pi^2 c\sin\Theta\,|n^2 - nn'' + 2n'^2|\,(1+\alpha_\theta^2)} \tag{54}$$

where $d\Omega = \sin\Theta \, d\Theta \, d\Phi$ and the parameters (n, α_θ, α_k, etc.) in the equation refer to the appropriate polarization.

INFINITE MAGNETOSTATIC FIELD

The magnetoplasma is characterized by two parameters, the plasma frequency $\omega_p = (Ne/me_o)^{\frac{1}{2}}$ and the gyro-frequency $\omega_B = |e|B/m$, where N is the density of electrons in the magnetoplasma, B is the magnetostatic field and m and e are the mass and the charge of an electron, respectively.

Define

$$X = (\omega_p/\omega)^2$$

and

$$Y = (\omega_B/\omega).$$

Then the refractive index and the polarization coefficients are given by (Ginzburg, 1961)

$$n_\pm^2 = 1 - \frac{2X(1 - X)}{D_\pm}$$

$$D_\pm = 2(1 - X) - Y^2 \sin^2\theta \pm \left[Y^4 \sin^4\theta + 4Y^2(1 - X)^2 \cos^2\theta\right]^{\frac{1}{2}}$$

$$\alpha_{\theta\pm} = - \frac{(n_\pm^2 - 1)Y \cos\theta}{n_\pm^2 - (1 - X)} \tag{55}$$

$$\alpha_{K\pm} = \frac{(n_\pm^2 - 1)Y \sin\theta}{1 - X}$$

In (55) the upper (+) sign denotes the ordinary mode and the lower (-) sign denotes the extraordinary mode of polarization.

By using (55), we may prove (53).

Next, it is interesting to examine the radiation fields at the approximation, $\omega \ll \omega_B$ and $\omega_p \ll \omega_B$, because it is possible to obtain explicit equations for the radiation fields under this assumption.

When ω_B approaches to infinity, (55) becomes

$$D_- = - 2Y^2 \sin^2\theta$$

$$n_-^2 = 1$$

$$\alpha_{\theta -} = 0$$

$$\alpha_{K-} = 0 \qquad (56)$$

and

$$D_+ = 2(1 - X)[(1 - X)\cot^2\theta + 1]$$

$$n_+^2 = 1 - \frac{X}{(1 - X)\cot^2\theta + 1}$$

$$\alpha_{\theta +} = \frac{Y\cos\theta}{(1 - X)\cot^2\theta} \rightarrow \infty$$

$$\alpha_{K+} = - \frac{XY\sin\theta}{(1 - X)[(1 - X)\cot^2\theta + 1]} \rightarrow \infty \qquad (56)$$

Then from (46), we have

$$\theta_- = \Theta$$

and

$$\cot\theta_+ = \frac{\cot\Theta}{1 - X} \qquad (57)$$

Combine (56) and (57) to obtain

$$n_+^2 = \frac{(1 - X)^2 \sin^2\Theta + \cos^2\Theta}{1 - X\sin^2\Theta} \qquad (58)$$

We see that the extraordinary mode can radiate in all directions, but the ordinary mode can only radiate in the directions where $X\sin^2\Theta \lessgtr 1$.

From (51) and (52), we find the radiation fields as follows:

$$\bar{E}_- = - \frac{j\omega\mu_0 Il\sin\alpha\cos\Phi}{4\pi r} \exp(j\omega t - j\tfrac{\omega}{c}r)\,\hat{\Phi}$$

$$\bar{H}_- = \frac{j\omega Il\sin\alpha\cos\Phi}{4\pi cr} \exp(j\omega t - j\tfrac{\omega}{c}r)\,\hat{\Theta} \qquad (59)$$

and

$$\overline{E}_+ = \frac{j\omega\mu_0 Il(1 - X)(\sin\Theta \cos\alpha - \cos\Theta \sin\Phi \sin\alpha)}{4\pi r(1 - X \sin^2\Theta)^{3/2}}$$

$$\cdot\exp\left[j\omega t - j\frac{\omega}{c}r(1 - X \sin^2\Theta)^{\frac{1}{2}}\right] \hat{\Theta}$$

$$\overline{H}_+ = \frac{j\omega Il(1 - X)(\sin\Theta \cos\alpha - \cos\Theta \sin\Phi \sin\alpha)}{4\pi cr(1 - X \sin^2\Theta)}$$

$$\cdot\exp(j\omega t - j\frac{\omega}{c}r(1 - X \sin^2\Theta)^{\frac{1}{2}}) \hat{\Phi} \qquad (59)$$

And the radiation intensity is

$$\frac{dP_-}{d\Omega} = \frac{\omega^2\mu_0 I^2 l^2 \sin^2\alpha \cos^2\Phi}{32\pi^2 c} \qquad (60)$$

and

$$\frac{dP_+}{d\Omega} = \frac{\omega^2\mu_0 I^2 l^2 (1 - X)^2(\sin\Theta \cos\alpha - \cos\Theta \sin\Phi \sin\alpha)^2}{32\pi^2 c(1 - X \sin^2\Theta)^{5/2}} \qquad (60)$$

By integrating (60), it is found that the power radiated by the extraordinary mode is

$$P_- = \frac{\omega^2\mu_0 I^2 l^2 \sin^2\alpha}{16\pi c} \qquad (61)$$

and that radiated by the ordinary mode is

$$P_+ = \int \frac{\omega^2\mu_0 I^2 l^2 (1 - X)^2 \sin\Theta[2 \cos^2\alpha + \cos^2\Theta\ (1 - 3 \cos^2\alpha)]}{32\pi c(1 - X \sin^2\Theta)^{5/2}} d\Theta \qquad (62)$$

The range of integration in (62) depends on whether X is greater or less than 1. It is from 0 to π for X less than 1 and from 0 to Θ_0 and $\pi - \Theta_0$ to π for X greater than 1, where Θ_0 satisfies

$X \sin^2\Theta_0 = 1.$

For X less than 1, the integration in (62) may be carried out by making the following substitutions of variables,

$u = \cos\Theta$

and

$$\tan v = \sqrt{\frac{X}{1 - X}}\, u$$

It is found that

$$P_+ = \frac{\omega^2 \mu_0 I^2 l^2}{48\pi c}\left[(1 - X) + (3 + X)\cos^2\alpha\right] \tag{63}$$

Then the total power radiated by the dipole antenna is

$$P = P_+ + P_-$$

$$= \frac{\omega^2 \mu_0 I^2 l^2 (4 - X \sin^2\alpha)}{48\pi c} \tag{64}$$

where X is less than 1.

Thus for X less than 1, the radiation resistance of the dipole antenna placed in a homogeneous cold magnetoplasma is

$$R = \frac{2P}{I^2} = \frac{\omega^2 \mu_0 l^2 (4 - X \sin^2\alpha)}{24\pi c} \tag{65}$$

This result is in agreement with that obtained by Kuehl(1962).

From (59), we see that for a longitudinally oriented dipole antenna, the extraordinary mode can not be excited, since $\alpha = 0$, and the ordinary mode reduces to the fields in free space obtained by the conventional method, if we let X equal to zero. This argument gives another check of the equations obtained in this thesis.

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