SOME PRACTICAL APPLICATIONS OF LEGENDRE POLYNOMIALS

A Thesis

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by

John Robert Smith, B. A.

The Ohio State University

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Approved by:

Leslie H. Miller

Adviser

TABLE OF CONTENTS

I INTRODUCTION

The Legendre Polynomials of the first kind, defined on the closed interval $[-1,1]$, have important applications in the physical sciences and fields of engineering. The following study of these polynomials shows some of these applications and provides an introduction to the important topic of orthogonal functions.

A survey of the literature brings out the fact that different authors define these polynomials in many different ways. This suggests the question as to whether or not all of these polynomials are identical. To answer this question we shall use Rodrigues' formula as our basic definition, for reasons of convenience, and then show that the same polynomials are obtained if other definitions are used.

To facilitate the equivalence proofs of the other definitions, which we shall prove as theorems, we prove additional theorems on the recurrence relations between the polynomials. In addition to the equivalence proofs we include various theorems to be used later in the representation, expansion and convergence theorems as well as in the practical problems.

A further development of our understanding of these polynomials requires a knowledge of their limitations. To satisfy this need we prove in a simple fashion or quote from other works several theorems on representation, expansion and convergence of these polynomials.

Since our primary purpose is a limited understanding of the applications of mathematics, we take up what seem to be typical problems in which these polynomials may be used. The solution of these problems illustrates the methods of solution which can be applied to more complicated problems than can be included in this paper. Before taking up the actual solution of these problems we discuss the relation between spherical harmonics and Legendre Polynomials.

For the first application we consider some problems of electrostatic potential. The method of solution of these problems may then be applied to other problems of potential at a point, where the vector force field is subject to the inverse square law, or the potential function satisfies Laplace's differential equation ($\nabla^2 V = 0$). We next consider a heat flow problem in which the temperature at any point within a solid hemisphere is computed in terms of the temperature on the surface. This method may be applied to flow problems which satisfy Laplace's differential equation; for example, an irrotational motion of an incompressible fluid. For the last application we take up the Gauss-Legendre method of numerical integration.

In our survey of the literature we make extensive use of the bibliography by Shohat, Hille and Walsh, Bibliography on Orthogonal Polynomials, National Research Council Bulletin #103, 1940.

II DEFINITIONS AND THEIR EQUIVALENCE

2.01 Introduction

A survey of the literature concerned with Legendre Polynomials, shows that many different definitions of these polynomials are given. This suggests that, as introductory material in a study of Legendre Polynomials of the first kind, we might prove that these various definitions are equivalent. It seems convenient to use Rodrigues' formula as our basic definition.

2.02 Definition*

The Legendre Polynomial, $P_n(x)$, is given by the formula

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n}(x^2-1)^n \quad , n = 0, 1, 2, \ldots$$

2.021 Remark

We note that $P_n(x)$ is the n^{th} derivative of a polynomial of degree $2n$ and hence $P_n(x)$ is a polynomial of degree n. Therefore $P_n(x)$ is of the form

$$P_n(x) = a_n x^n + \ldots + a_o, \quad a_n \neq 0$$

The Legendre Polynomials are uniquely determined and may be calculated explicitly by the above definition. The first six are as follows:

$$P_0(x) = 1$$

$$P_1(x) = x$$

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* Jackson, Dunham, Fourier Series and Orthogonal Polynomials, The Carus Mathematical Monographs, Number Six, Published by The Mathematical Association of America, 1941, pages 46 and 50

$$P_2(x) = \tfrac{1}{2}(3x^2-1)$$

$$P_3(x) = \tfrac{1}{2}(5x^3-3x)$$

$$P_4(x) = \tfrac{1}{8}(35x^4-30x^2+3)$$

$$P_5(x) = \tfrac{1}{8}(63x^5-70x^3+15x)$$

2.03 Theorem*

$P_n(x)$ satisfies the formula

$$P_n(x) = \sum_{r=0}^{[\frac{n}{2}]} \frac{(-1)^r (2n-2r)!}{2^n n! (n-2r)!} n_{(r)} x^{n-2r} ,$$

where $n_{(r)}$ is the r^{th} binomial coefficient and $[\frac{n}{2}]$ is the greatest integer in $\frac{n}{2}$.

Proof

From Rodrigues' Formula and the binomial expansion we have

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n}\left\{ \sum_{r=0}^{n} (-1)^r n_{(r)} (x^2)^{n-r} \right\}$$

Taking the n^{th} derivative we get

$$P_n(x) = \frac{1}{2^n n!} \sum_{r=0}^{[\frac{n}{2}]} (-1)^r n_{(r)} (2n-2r)(2n-2r-1) \ldots (n-2r+1) x^{n-2r}$$

$$= \sum_{r=0}^{[\frac{n}{2}]} (-1)^r \frac{(2n-2r)!}{2^n n! (n-2r)!} n_{(r)} x^{n-2r}$$

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* see definition, Helsel, R. G., Mathematics Methods in Science, Vol. I, The Ohio State University, 1948, problem, page 31.

2.04 Theorem*

The Legendre Polynomials satisfy the recurrence relation:

$$P_n'(x) = (2n-1)P_{n-1}(x) + P_{n-2}'(x) \quad , \; n \geq 2$$

Proof

Taking the first derivative with respect to x of $P_n(x)$ we get

$$P_n'(x) = \frac{1}{2^n n!}\frac{d^{n+1}}{dx^{n+1}}(x^2-1)^n$$

$$= \frac{1}{2^n n!}\frac{d^n}{dx^n}\left[\frac{d}{dx}(x^2-1)^n\right]$$

$$= \frac{1}{2^{n-1}(n-1)!}\frac{d^{n-1}}{dx^{n-1}}\left[\frac{d}{dx}\left\{x(x^2-1)^{n-1}\right\}\right]$$

$$= \frac{d^{n-1}}{dx^{n-1}}\left[\frac{(x^2-1)^{n-1}+2(n-1)x^2(x^2-1)^{n-2}}{2^{n-1}(n-1)!}\right]$$

$$= \frac{d^{n-1}}{dx^{n-1}}\left[\left\{(2n-1)(x^2-1) + 2n-2\right\}\frac{(x^2-1)^{n-2}}{2^{n-1}(n-1)!}\right]$$

$$= \frac{d^{n-1}}{dx^{n-1}}\left[\frac{2(n-1)(x^2-1)^{n-1}}{2^{n-1}(n-1)!}\right] + \frac{d^{n-1}}{dx^{n-1}}\left[\frac{(x^2-1)^{n-2}}{2^{n-2}(n-2)!}\right]$$

$$= (2n-1)P_{n-1}(x) + P_{n-2}'(x)$$

2.05 Theorem**

$$P_n'(x) = xP_{n-1}'(x) + nP_{n-1}(x) \quad , \; n \geq 1$$

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* see proof, Hargreaves, Messenger of Mathematics, series 2, Vol. 49 1919, pages 58-62

** see recurrence relation and proof, Hargreaves, op. cit., pages 58-62

Proof

From the proof of Theorem 2.04

$$P_n(x) = \frac{d^n}{dx^n}\left[\frac{x(x^2-1)^{n-1}}{2^{n-1}(n-1)!}\right]$$

Consider $y = f(x_1, x_2)$, then

$$\frac{dy}{dx} = \frac{\partial f}{\partial x_1}\frac{dx_1}{dx} + \frac{\partial f}{\partial x_2}\frac{dx_2}{dx}$$

Now let $y = x(x^2-1)^{n-1}$, where $x_1 = x$ and $x^2 = (x^2-1)^{n-1}$, then

$$\frac{dy}{dx} = (x^2-1)^{n-1} + x\frac{d}{dx}(x^2-1)^{n-1}$$

and

$$\frac{d^2y}{dx^2} = 2\frac{d}{dx}(x^2-1)^{n-1} + x\frac{d^2}{dx^2}(x^2-1)^{n-1}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\frac{d^ny}{dx^n} = (n)\frac{d^{n-1}}{dx^{n-1}}(x^2-1)^{n-1} + x\frac{d^n}{dx^n}(x^2-1)^{n-1}$$

Hence

$$P_n'(x) = n\frac{d^{n-1}}{dx^{n-1}}\left[\frac{(x^2-1)^{n-1}}{2^{n-1}(n-1)!}\right] + x\frac{d^n}{dx^n}\left[\frac{(x^2-1)^{n-1}}{2^{n-1}(n-1)!}\right]$$

Therefore

$$P_n'(x) = nP_{n-1}(x) + xP_{n-1}'(x)$$

2.06 Theorem*

$$nP_n(x) = xP_n'(x) - P_{n-1}'(x) \quad , n \geq 1$$

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* see recurrence relation, Ford, Lester, R., Differential Equations page 197, problem 17, First Edition, McGraw-Hill Book Co., 1933

Proof

Rewriting Theorem 2.04

$$P'_{n+1}(x) = (2n+1)P_n(x) + P'_{n-1}(x)$$

and also rewriting Theorem 2.05

$$P'_{n+1}(x) = xP'_n(x) + (n+1)P_n(x)$$

Subtracting Theorem 2.05 from Theorem 2.04

$$0 = nP_n(x) + P'_{n-1}(x) - xP'_n(x)$$

Therefore

$$nP_n(x) = xP'_n(x) - P'_{n-1}(x)$$

2.07 Theorem*

$$nP_n(x) - (2n-1)xP_{n-1}(x) + (n-1)P_{n-2}(x) = 0 \quad , \; n \geq 2$$

Proof

From Theorem 2.06

$$nP_n(x) = xP'_n(x) - P'_{n-1}(x)$$

But by Theorem 2.04

$$xP'_n(x) = (2n-1)xP_{n-1}(x) + xP'_{n-2}(x)$$

Substituting in Theorem 2.06 we get

$$nP_n(x) = (2n-1)xP_{n-1}(x) + xP'_{n-2}(x) - P'_{n-1}(x)$$

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* see Ford, op. cit., page 197, problem 16

But in Theorem 2.05

$$P'_{n-1}(x) = xP'_{n-2}(x) + (n-1)P_{n-2}(x)$$

and on substitution we get

$$nP_n(x) = (2n-1)xP_{n-1}(x) + xP'_{n-2}(x) - \left\{xP'_{n-2}(x) + (n-1)P_{n-2}(x)\right\}$$

Therefore

$$nP_n(x) - (2n-1)xP_{n-1}(x) + (n-1)P_{n-2}(x) = 0$$

2.08 Theorem*

$$(x^2-1)P'_n(x) = nxP_n(x) - nP_{n-1}(x) \quad , \quad n \geq 1$$

Proof

Multiply Theorem 2.06 by x:

$$nxP_n(x) = x^2P'_n(x) - xP'_{n-1}(x)$$

then

$$nxP_n(x) - nP_{n-1}(x) = x^2P'_n(x) - xP'_{n-1}(x) - nP_{n-1}(x)$$

From Theorem 2.05 we have

$$-xP'_{n-1}(x) = nP_{n-1}(x) - P'_n(x)$$

and on substitution we get

$$nxP_n(x) - nP_{n-1}(x) = x^2P'_n(x) + nP_{n-1}(x) - P'_n(x) - nP_{n-1}(x)$$

Therefore

$$(x^2-1)P'_n(x) = nxP_n(x) - nP_{n-1}(x)$$

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* Ford, op. cit. page 197, problem #19

2.09 Theorem

$P_n(x)$ satisfies the recurrence relation:

$$nP_n(x) = (x^2-1)P'_{n-1}(x) + nxP_{n-1}(x)$$

Proof

Rewrite Theorem 2.08 as follows

$$-(x^2-1)P'_{n-1}(x) = -(n-1)xP_{n-1}(x) + (n-1)P_{n-2}(x)$$

and add to Theorem 2.07, i.e.

$$nP_n(x) = (2n-1)xP_{n-1}(x) - (n-1)P_{n-2}(x)$$

and get

$$nP_n(x) = (x^2-1)P'_{n-1}(x) + nxP_{n-1}(x)$$

2.10 Theorem*

$$\int_{-1}^{1} P_n(x)P_{n-j}(x)dx = 0 \quad , \quad j \geqslant 1 \text{ and } -1 \leqslant x \geqslant 1$$

$$P_n(1) = 1 \quad , \quad n \geqslant 0$$

Proof

Substituting for $P_n(x)$ and $P_{n-j}(x)$ from the basic definition we get

$$\int_{-1}^{1} P_n(x)P_{n-j}(x)dx = \frac{1}{2^{2n-j}\,n!\,(n-j)!}\int_{-1}^{1} \frac{d^n}{dx^n}(x^2-1)^n \; \frac{d^{n-j}}{dx^{n-j}}(x^2-1)^{n-j}dx$$

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* see definition, Jackson, op. cit., pages 46 and 50

Integrating by parts n times we get

$$= \frac{1}{2^{2n-j}\, n!\,(n-j)!} \int_{-1}^{1} \frac{d^{n-n}}{dx^{n-n}}(x^2-1)^n \frac{d^{n-j}}{dx^{n-j}}(x^2-1)^{n-j} dx = 0$$

Since

$$\frac{d^{2n-j}}{dx^{2n-j}}(x^2-1)^{n-j} = 0$$

and no other terms appear due to the following:

$$\int_{-1}^{1} \frac{d^{n-j}}{dx^{n-j}}(x^2-1)^{n-j} \frac{d^{n}}{dx^{n}}(x^2-1)^{n} dx = \frac{d^{n-j}}{dx^{n-j}}(x^2-1)^{n-j} \frac{d^{n-1}}{dx^{n-1}}(x^2-1)^{n}\Big]_{-1}^{1}$$
$$- \int_{-1}^{1} \frac{d^{n-j+1}}{dx^{n-j+1}}(x^2-1)^{n-j} \frac{d^{n-1}}{dx^{n-1}}(x^2-1)^{n} dx$$

Now $\frac{d^{n-1}}{dx^{n-1}}(x^2-1)^n$ contains (x^2-1) as a factor and on evaluation at the limits -1 and 1 equals zero. By Remark 2.021 and using mathematical induction

$$P_0(1) = 1$$

$$P_1(1) = 1$$

$$P_2(1) = 1 \quad \text{etc.}$$

Assume $P_{n-1}(1) = 1$ and $P_{n-2}(1) = 1$, $n \geq 2$. Then by Theorem 2.07

$$nP_n(1) = (2n-1)P_{n-1}(1) - (n-1)P_{n-2}(1) = 2n-1-n+1 = n$$

Hence

$$P_n(1) = 1$$

2.11 <u>Theorem</u>*

$$P_n(-1) = (-1)^n$$

<u>Proof</u>

Utilizing the recurrence relation Theorem 2.07, i.e.

$$nP_n(x) - (2n-1)xP_{n-1}(x) + (n-1)P_{n-2}(x) = 0$$

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* Jackson, <u>op. cit</u>, page 46.

and Remark 2.021 and evaluating

$$P_0(-1) = 1 \quad ; \quad P_1(-1) = -1 \quad ; \quad P_2(-1) = \tfrac{1}{2}(3-1) = 1$$

Assuming

$$P_{n-1}(-1) = (-1)^{n-1}$$

and

$$P_{n-2}(-1) = (-1)^{n-2}$$

consider the following:

a. Let n be even, then by Theorem 2.07 we have

$$nP_n(-1) = (2n-1)(-1)(-1) - (n-1)(1) = n$$

Hence

$$P_n(-1) = 1$$

b. Let n be odd, then by Theorem 2.07 we have

$$nP_n(-1) = (2n-1)(-1)(1) - (n-1)(-1) = -n$$

Hence

$$P_n(-1) = (-1)$$

Therefore we may conclude that

$$P_n(-1) = (-1)^n$$

Note This may also be proved by proving

$$P_n(x) = (-1)^n P_n(-x)*$$

2.12 Theorem**

On the closed interval $[-1,1]$

$$\int_{-1}^{1} P_n(x)P_n(x)\,dx = \frac{2}{2n+1}$$

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* see Thesis, Thomas, R. E., Some Elementary Aspects of Legendre Polynomials, The Ohio State University, page 14

** Jackson, op. cit., page 52

Proof

From Definition 2.02

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n}(x^2-1)^n$$

Substituting in the above theorem we get

$$\frac{1}{2^{2n}(n!)^2}\int_{-1}^{1} \frac{d^n}{dx^n}(x^2-1)^n \frac{d^n}{dx^n}(x^2-1)^n dx$$

After n integrations by parts this becomes

$$= \frac{1}{2^{2n}(n!)^2}(2n)!(-1)^n \int_{-1}^{1}(x^2-1)^n dx$$

Since this is an even function we may change the limits of integration from 0 to 1 and multiply by 2 as follows:

$$= \frac{2(2n)!}{2^{2n}(n!)^2}\int_{0}^{1}(1-x^2)^n dx$$

Integrating this by the trigonometric substitution of $x = \sin\theta$, we get

$$= \frac{2(2n)!}{2^{2n}(n!)^2}\int_{0}^{\frac{\pi}{2}}\cos^{2n+1}\theta d\theta = \frac{2(2n)!}{2^{2n}(n!)^2}\left\{\frac{1}{2n+1}\cos^{2n}\theta\sin\theta\Big]_{0}^{\frac{\pi}{2}} + \frac{2n}{2n+1}\int_{0}^{\frac{\pi}{2}}\cos^{2n-1}\theta d\theta\right\}$$

Since the first term of each integration contains a cosine term until the last, the integral becomes

$$= \frac{2(2n)!(2^n n!)^2}{2^{2n}(n!)^2(2n+1)!} = \frac{2}{2n+1}$$

Hence

$$\int_{-1}^{1}P_n(x)P_n(x)dx = \frac{2}{2n+1}$$

2.13 Theorem*

On the closed interval $[-1,1]$

$$\int_{-1}^{1} xP_{n-1}(x)P_n(x)\,dx = \frac{2n}{4n^2-1}$$

Proof

Substituting

$$xP_{n-1}(x) = \frac{nP_n(x) + (n-1)P_{n-2}(x)}{2n-1}$$

from Theorem 2.07, the integral becomes

$$\int_{-1}^{1} P_n(x)\left[\frac{nP_n(x) + (n-1)P_{n-2}(x)}{2n-1}\right]dx = \frac{n}{2n-1}\int_{-1}^{1}\left[P_n(x)\right]^2 dx + \frac{n-1}{2n-1}\int_{-1}^{1} P_n(x)P_{n-2}(x)\,dx$$

Evaluating this by Theorem 2.10, we have the last integral as zero and by the previous theorem the first integral becomes

$$\frac{n}{2n-1}\,\frac{2}{2n+1} = \frac{2n}{4n^2-1}$$

Hence our theorem is true.

2.14 Theorem**

$P_n(x)$ is a particular solution of Legendre's differential equation

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0$$

Proof

Rewriting Legendre's differential equation and substituting $P_n(x)$ for y as follows:

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* Ford, op. cit., page 197, problem #5

** see previous reference, page 190.

$$\frac{d}{dx}\left[(1-x^2)P_n(x)\right] + n(n+1)P_n(x) = 0$$

By use of Theorem 2.08 we get

$$= \frac{d}{dx}\left[nP_{n-1}(x) - nxP_n(x)\right] + n(n+1)P_n(x)$$

Performing the indicated derivation we get

$$= nP'_{n-1}(x) - nxP'_n(x) - nP_n(x) + n^2P_n(x) + nP_n(x).$$

But Theorem 2.06 states

$$n^2P_n(x) = nxP'_n(x) - nP'_{n-1}(x).$$

Hence on substitution in the above equation for $nP'_{n-1}(x) - nxP'_n(x)$ we get

$$= -n^2P_n(x) + n^2P_n(x) = 0$$

Therefore $P_n(x)$ is a particular solution of Legendre's differential equation.

2.15 Theorem*

$$P_n(x) = \frac{1}{n!}\begin{vmatrix} x & 1 & 0 & 0 & \cdots & 0 \\ 1 & 3x & 2 & 0 & \cdots & 0 \\ 0 & 2 & 5x & 3 & \cdots & 0 \\ \vdots & & & & & \vdots \\ 0 & 0 & 0 \cdots & & (n-1) & (n-1)(2n-1)x \end{vmatrix}, \quad n \geq 1$$

* see exercise #2, Whittaker and Robinson, The Calculus of Observation, Blackie and Son, Ltd., 2nd Edition, 1937, pages 74,75

Proof

Evaluating

$$P_1(x) = x$$

$$P_2(x) = \tfrac{1}{2}(3x^2-1)$$

$$P_3(x) = \tfrac{1}{3!}\left[x(15x^2-4) + (0-5x)\right] = \tfrac{1}{3!}\left[15x^3-9x\right] = \tfrac{1}{2}(5x^2-3x)$$

The proof will be completed by showing that $P_n(x)$ as given above satisfies the recurrence relation of Theorem 2.07, i.e.

$$nP_n(x) = (2n-1)xP_{n-1}(x) - (n-1)P_{n-2}(x)$$

Evaluating the determinant by expanding according to the n^{th} column we get, after multiplication by n!

$$n!P_n(x) = (2n-1)x(n-1)!P_{n-1}(x) - (n-1)^2(n-2)!P_{n-2}(x).$$

Dividing by (n-1)!, we get

$$nP_n(x) = (2n-1)xP_{n-1}(x) - (n-1)P_{n-2}(x)$$

Hence the determinant form of $P_n(x)$ is equivalent to Rodrigues' formula.

2.16 Theorem*

$P_n(x)$ is given by the generating function

$$(1-2xy+y^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} P_n(x)y^n$$

Proof

Consider

$$F(x,y) = (1-2xy+y^2)^{-\frac{1}{2}}$$

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* see definition and proof, Margenau and Murphy, The Mathematics of Physics and Chemistry, pages 94-109, and Helsel, op. cit. page 31.

Evaluating at $y = 0$

$$F(x,y)\Big|_{y=0} = 1$$

Taking the partial derivative with respect to y and evaluating at $y = 0$ we get

$$\frac{\partial}{\partial y}F(x,y)\Big|_{y=0} = (x-y)(1-2xy+y^2)^{-\frac{3}{2}}\Big|_{y=0} = 1!P(x)$$

Similarly we see

$$\frac{\partial^2}{\partial y^2}F(x,y)\Big|_{y=0} = -(1-2xy+y^2)^{-\frac{3}{2}} + 3(x-y)^2(1-2xy+y^2)^{-\frac{5}{2}}\Big|_{y=0} = 3x^2-1 = 2!P_2(x)$$

and

$$\frac{\partial^3}{\partial y^3}F(x,y)\Big|_{y=0} = -9(x-y)(1-2xy+y^2)^{-\frac{5}{2}} + 3\cdot5(x-y)^3(1-2xy+y^2)^{-\frac{7}{2}}\Big|_{y=0} = 3!P_3(x)$$

. . . .

$$\frac{\partial^n}{\partial y^n}F(x,y)\Big|_{y=0} = n!P_n(x)$$

Expanding in a MacLaurin series about $F(x,o)$ we get

$$F(x,y) = F(x,o) + y\,\frac{\partial F}{\partial y}\Big|_{y=0} + \frac{y^2}{2!}\frac{\partial^2 F}{\partial y^2}\Big|_{y=0} + \dots + \frac{y^n}{n!}\frac{\partial^n F}{\partial y^n}\Big|_{y=0} + \dots$$

$$= P_0(x) + yP_1(x) + y^2P_2(x) + \dots + y^nP_n(x) + \dots$$

Hence

$$F(x,y) = \sum_{n=0}^{\infty} P_n(x)y^n$$

To show its equivalence to Rodrigues' formula we derive the recurrence relation Theorem 2.07. Taking partial derivatives of both sides with respect to y, we get

$$(x-y)(1-2xy+y^2)^{-\frac{1}{2}}(1-2xy+y^2)^{-1} = \sum_{n=0}^{\infty} nP_n(x)y^{n-1}$$

Rewriting and substituting for $(1-2xy+y^2)^{-\frac{1}{2}}$

$$(1-2xy+y^2)\sum_{n=0}^{\infty} nP_n(x)y^{n-1} = (x-y)\sum_{n=0}^{\infty} P_n(x)y^n$$

Equating the coefficients of y^{n-1} we get

$$xP_{n-1}(x) - P_{n-2}(x) = nP_n(x) - 2x\left\{P_{n-1}(x)\right\}(n-1) + (n-2)P_{n-2}(x)$$

Combining terms we have

$$nP_n(x) - (2n-1)xP_{n-1}(x) + (n-1)P_{n-2}(x) = 0$$

which is Theorem 2.07.

Note! For $x = 1$

$$(1-2y+y^2)^{-\frac{1}{2}} = \frac{1}{1-y} = 1 + y + y^2 + \dots + y^n + \dots = \sum_{n=0}^{\infty} y^n$$

This implies an equation of Theorem 2.10, i.e.

$$P_n(1) = 1$$

2.17 Theorem*

$$P_n(\cos\theta) = \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots(2n)}\left\{2\cos n\theta + 2\frac{1\cdot n}{1\cdot(2n-1)}\cos(n-2)\theta + \right.$$

$$\left. 2\frac{1\cdot 3\cdot n(n-1)}{1\cdot 2(2n-1)(2n-3)}\cos(n-4)\theta + 2\frac{1\cdot 3\cdot 5\cdot n(n-1)(n-2)}{1\cdot 2\cdot 3(2n-1)(2n-3)(2n-5)}\cos(n-6)\theta + \dots\right\}$$

Proof

Let

$$x = \cos\theta = \tfrac{1}{2}(e^{i\theta} + e^{-i\theta}),$$

and we see that

$$(1-2xy+y^2)^{-\frac{1}{2}} = (1-ye^{i\theta})^{-\frac{1}{2}}(1-ye^{-i\theta})^{-\frac{1}{2}}$$

If $|y| < 1$, we have, by the binomial theorem

$$(1-ye^{i\theta})^{-\frac{1}{2}} = 1 + \tfrac{1}{2}ye^{i\theta} + \frac{1\cdot 3}{2\cdot 4}y^2e^{2i\theta} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}y^3e^{3i\theta} + \dots \qquad (1)$$

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* see definition and proof, Prasad, G., A Treatise on Spherical Harmonics and Functions of Bessel and Lame, Part 1, page 31, The Benares Mathematical Society, 1930.

and

$$(1-ye^{-i\theta})^{-\frac{1}{2}} = 1 + \tfrac{1}{2}ye^{-i\theta} + \tfrac{1\cdot 3}{2\cdot 4}\, y^2 e^{-2i\theta} + \tfrac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\, y^3 e^{-3i\theta} + \dots \qquad (2)$$

Hence their product must be equal to

$$\sum_{n=0}^{\infty} P_n(\cos\theta)y^n$$

by Theorem 2.16. Therefore the coefficient of y^n must be equal to

$$P_n(\cos\theta)$$

Taking the product of equations (1) and (2) for the coefficient of y^n we get

$$\frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots(2n)}\left\{e^{in\theta} + e^{-in\theta} + \frac{1\cdot 2\cdot n}{2\cdot(2n-1)}\left[e^{i(n-2)\theta} + e^{-(n-2)\theta}\right]\right.$$
$$\left. + \frac{1\cdot 3\cdot(2n)(2n-2)}{2\cdot 4(2n-1)(2n-3)}\left[e^{i(n-4)\theta} + e^{-i(n-4)\theta}\right] + \dots\right\}$$

which is Theorem 2.17.

2.18 Theorem*

$$P_n(x) = \frac{1}{\pi}\int_0^{\pi}(x + \sqrt{x^2-1}\cos\theta)^n d\theta$$

Proof

Evaluating

$$P_0(x) = \frac{1}{\pi}\int_0^{\pi} d\theta = 1 \qquad ;$$

$$P_1(x) = \frac{1}{\pi}\int_0^{\pi}(x + \sqrt{x^2-1}\cos\theta)d\theta = \frac{1}{\pi}\Big[x\theta + \sqrt{x^2-1}\sin\theta\Big]_0^{\pi} = x$$

The proof will be completed by showing that it satisfies the recurrence relation Theorem 2.07, i. e.

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* see definition and proof, Jackson, op. cit., pages 59-60

$$(n+1)P_{n+1}(x) - (2n+1)xP_n(x) + nP_{n-1}(x) = 0$$

By direct substitution into Theorem 2.07 we get

$$\frac{1}{\pi}\int_0^\pi \left\{x + \sqrt{x^2-1}\cos\theta\right\}^{n-1}\left\{(n+1)(x + \sqrt{x^2-1}\cos\theta)^2 - (2n+1)x(x+\sqrt{x^2-1}\cos\theta) + n\right\}d\theta$$

$$= \frac{1}{\pi}\int_0^\pi \left\{x + \sqrt{x^2-1}\cos\theta\right\}^{n-1}\left\{-nx^2 + n(x^2-1)\cos^2\theta + (x^2-1)\cos^2\theta + x\sqrt{x^2-1}\cos\theta + n\right\}d\theta$$

$$= \frac{1}{\pi}\int_0^\pi \left\{x + \sqrt{x^2-1}\cos\theta\right\}^{n-1}\left\{-n(x^2-1)\sin^2\theta + (x + \sqrt{x^2-1}\cos\theta)(\sqrt{x^2-1}\cos\theta)\right\}d\theta$$

Then to show

$$\frac{1}{\pi}\int_0^\pi \left\{x + \sqrt{x^2-1}\cos\theta\right\}^{n-1} n(x^2-1)\sin^2\theta\, d\theta = \frac{1}{\pi}\int_0^\pi \left\{x + \sqrt{x^2-1}\cos\theta\right\}^{n}\left\{\sqrt{x^2-1}\cos\theta\right\}d\theta \qquad (1)$$

Integrating the right side by parts we get

$$\frac{1}{\pi}\int_0^\pi \left\{x + \sqrt{x^2-1}\cos\theta\right\}^{n}\left\{\sqrt{x^2-1}\cos\theta\right\}d\theta = \underbrace{\frac{1}{\pi}\left\{x + \sqrt{x^2-1}\cos\theta\right\}^{n}\sqrt{x^2-1}\sin\theta\Big]_0^\pi}_{=0} + \frac{1}{\pi}\int_0^\pi \left\{x + \sqrt{x^2-1}\cos\theta\right\}^{n-1} n(x^2-1)\sin^2\theta\, d\theta$$

Hence equation (1) is true and theorem 2.16 is equivalent to Rodrigues' formula.

2.181 <u>Corollary</u>

$$P_n(x) = \frac{1}{\pi}\int_0^\pi (x + i\sqrt{1-x^2}\cos\theta)^n d\theta$$

This is a restatement of 2.15.

2.19 Theorem*

$$P_n(\cos\theta) = \frac{(-1)^n}{n!} r^{n+1} \frac{\partial^n}{\partial x^n}\left(\frac{1}{r}\right)$$

where $r = (x^2+y^2+z^2)^{\frac{1}{2}}$ and $\cos\theta = \frac{x}{r}$

Proof

Evaluating

$P_0(\cos\theta) = 1 \cdot r\,\frac{1}{r} = 1$;

$P_1(\cos\theta) = \frac{(-1)}{1} r^2 \frac{\partial}{\partial x}\left(\frac{1}{r}\right) = (-1)r^2 \frac{-x}{r^3} = \frac{x}{r} = \cos\theta$;

$P_2(\cos\theta) = \frac{1}{2!} r^3 \frac{\partial^2}{\partial x^2}\left(\frac{1}{r}\right) = \frac{r^3}{2}\frac{\partial}{\partial x}\left(\frac{-x}{r^3}\right) = \frac{r^3}{2}\left(\frac{3x^2}{r^5} - \frac{1}{r^3}\right) = \frac{1}{2}\left[3\left(\frac{x}{r}\right)^2 - 1\right] = \frac{1}{2}(3\cos^2\theta - 1)$

Let $y = \cos\theta$. Assume $P_{n-1}(y)$ and $P'_{n-1}(y) = \frac{d}{dy}P_{n-1}(y)$ as known to be equivalent to the first $P_{n-1}(y)$'s of Rodrigues' formula, then we compute $P_n(y)$ in terms of $P_{n-1}(y)$ and $P'_{n-1}(y)$. We know

$$\frac{\partial y}{\partial x}\frac{df}{dy} = \frac{\partial f}{\partial x} \quad , \; f = P_{n-1}(y)$$

Now

$$\frac{\partial y}{\partial x} = \frac{\partial}{\partial x}\left(\frac{x}{r}\right) = \frac{1}{r^3}(r^2 - x^2)$$

and

$$\frac{\partial}{\partial x} P_{n-1}(y) = P'_{n-1}(y)\frac{\partial y}{\partial x}$$

Then taking the partial of $P_{n-1}(y)$ with respect to x we get

$$\frac{\partial}{\partial x} P_{n-1}(y) = \frac{(-1)^{n-1}}{(n-1)!}(n)r^{n-1}\frac{x}{r}\frac{\partial^{n-1}}{\partial x^{n-1}}\left(\frac{1}{r}\right) + \frac{(-1)^{n-1}}{(n-1)!} r^n \frac{\partial^n}{\partial x^n}\left(\frac{1}{r}\right)$$

Substituting from above

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* see definition, Byerly, William E., Fourier Series and Spherical Harmonics, page 165, Ginn and Co., copyright 1893.

$$\frac{(r^2-x^2)}{r^3} P'_{n-1}(y) = \frac{ny}{r} P_{n-1}(y) - \frac{n}{r} P_n(y)$$

Multiplying through by r we get

$$nP_n(y) = (y^2-1)P'_{n-1}(y) + nyP_{n-1}(y)$$

Hence by Theorem 2.09 the $P_n(\cos\theta)$ of Theorem 2.19 is equivalent to Rodrigues' formula with $y = \cos\theta$ substituted for x.

2.20 Remarks

Other definitions of Legendre Polynomials may be found in the references previously cited.

3.01 Introduction

Orthogonal functions are extensively used in a series expansion of a given function. With a given set of orthogonal functions two questions of interest are:

(1) How do you expand a function in a series using the given set of orthogonal functions?

(2) Under what conditions will the series converge to the given function?

In this chapter we shall show how one may expand a function in a series of Legendre Polynomials. We shall also state theorems concerning the convergence of the Legendre expansion to a given function. We shall see that under certain conditions*, continuous, or finitely discontinuous functions, may be represented by a convergent series of Legendre Polynomials. If the series is terminated, after n+1 terms, we shall see that the finite sum of the Legendre expansion is the best n^{th} degree polynomial approximation to the given function in the sense of least squares.

If a function $f(x)$ can be represented by a series of Legendre Polynomials, i.e.

$$f(x) = \sum_{i=0}^{\infty} a_i P_i(x),$$

then formal multiplication termwise by $P_i(x)$ and integration gives

$$a_i = \frac{2i+1}{2}\int_{-1}^{1} f(x)P_i(x)\,dx$$

* Prasad, op. cit., pages 73 and 74

3.02 Definition

If $f(x)$ is any integrable function we define the series

$\sum_{i=0}^{\infty} a_i P_i(x)$, where a_i is given by

$$a_i = \frac{2i+1}{2}\int_{-1}^{1} f(x)P_i(x)dx \quad ,$$

to be the Legendre series associated with $f(x)$.

3.03 Theorem

Any polynomial of degree no greater than n is equal to a linear combination of the first n+1 Legendre Polynomials.

Proof

Consider the polynomial $F(x) = x^k$, where k is any fixed integer. We wish to find constants a_i such that

$$x^k = a_k P_k(x) + a_{k-1}P_{k-1}(x) + \dots + a_1 P_1(x) + a_o \qquad (1)$$

By Remark 2.021 $P_n(x)$ is a polynomial of n^{th} degree. Let

$$P_n(x) = b_{nn}x^n + b_{nn-1}x^{n-1} + \dots + b_{n1}x + b_{no} \quad ,$$

$$n = 0,1,2\dots,k \text{ and } b_{nn} \neq 0.$$

On substituting $P_n(x)$ in equation (1) we have:

$$\begin{aligned} x^k = {} & a_k \left[b_{kk}x^k + b_{kk-1}x^{k-1} + \dots + b_{k1}x + b_{ko} \right] \\ & + a_{k-1}\left[b_{k-1\ k-1}x^{k-1} + b_{k-1\ k-2}x^{k-2} + \dots + b_{k-11}x + b_{k-10} \right] \\ & + \dots + a_2\left[\tfrac{3}{2}x^2 - \tfrac{1}{2} \right] + a_1 x + a_o. \end{aligned} \qquad (2)$$

Equating powers of x and solving for the a_i's, we get, for the coefficient of x^k

$$1 = a_k b_{kk} \qquad \text{or} \qquad a_k = \frac{1}{b_{kk}}$$

Substituting $\frac{1}{b_{kk}}$ for a_k in equation (2) and equating the coefficients of x^{k-1} we have

$$0 = \frac{1}{b_{kk}}\left[b_{kk-1}\right] + a_{k-1}\left[b_{k-1\ k-1}\right]$$

or

$$a_{k-1} = -\frac{b_{kk-1}}{b_{kk}b_{k-1\ k-1}} \quad \text{etc.}$$

Since the leading coefficients of each of the Legendre Polynomials, $P_n(x)$, is different from zero the a_i's will be defined. Hence we may evaluate any polynomial as a linear combination of Legendre Polynomials and determine the coefficients in this manner.

3.04 Theorem*

The Legendre Polynomial, $P_n(x)$, is bounded as follows:

$$\left|P_n(x)\right| < 1 \text{ on the closed interval } [-1,1]$$

$$\left|P_n(x)\right| < \sqrt{\frac{\pi}{2n(1-x^2)}} \quad \text{on the open interval } (-1,1)$$

3.05 Definition**

An orthogonal set of functions is said to be complete with respect to a class of continuous functions, C, if whenever f, belonging to C, is orthogonal to every member of the orthogonal set, then

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* see proof, Jackson, op. cit., pages 61-63.
** see Thomas for definition, following theorem and corollary, op. cit., pages 31-34

f is identically zero.

3.06 Theorem

The set of Legendre Polynomials is complete with respect to the class of continuous functions defined on the closed interval $[-1,1]$.

3.061 Corollary

The Legendre coefficients of a continuous function defined on the closed interval $[-1,1]$ are all zero if and only if the function is identically zero.

3.07 Theorem*

If the sum of the absolute values of the Legendre coefficients of a continuous function forms a convergent series, then the Legendre expansion is absolutely and uniformly convergent, and converges to the function.

3.08 Theorem**

If $f'(x)$ exists and is finite on the open interval $(-1,1)$, and has only a finite number of discontinuities and is monotone in each of a finite number of parts of $(-1,1)$, then the Legendre series associated with $f(x)$ converges to $f(x)$.

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* see previous reference, pages 37 and 38.
** Prasad, op. cit., pages 68-73

3.09 Theorem*

If f(x) is finite on the closed interval $[-1,1]$ and is of limited total fluctuation in the interval; then at every point x interior to the interval, the Legendre series

$$\sum_{i=0}^{\infty} a_i P_i(x)$$

converges to

$$\frac{1}{2}\left\{f(x+0) + f(x-0)\right\}$$

if this expression exists; at x 1 and at x -1, the series converges to f(1-0) and f(-1+0) respectively when these limits exist.

3.10 Theorem**

If $f(x) = \sum_{n=0}^{\infty} a_n P_n(x)$ and $R_m(x)$ is any polynomial of degree $\leq m$, then

$$\int_{-1}^{1}\left[f(x) - \sum_{n=0}^{m} a_n P_n(x)\right]^2 dx \leq \int_{-1}^{1}\left[f(x) - R_m(x)\right]^2 dx$$

We say that $\sum_{n=0}^{m} a_n P_n(x)$ is the best approximation to f(x) in the sense of least squares.

Proof

Let $R_m(x) = \sum_{n=0}^{m} \alpha_n P_n(x)$. To complete the proof we must determine the coefficients α_n such that

$$\int_{-1}^{1}\left[f(x) - R_m(x)\right]^2 dx$$

* Prasad, op. cit., Part II, (Advanced) 1932, page 94.
** Helsel, op. cit., pages 33-34

is a minimum. Taking the minimum of

$$\int_{-1}^{1}\left[f(x) - R_m(x)\right]^2 dx$$

as follows: Let

$$\frac{\partial}{\partial\alpha_r}\left\{ \int_{-1}^{1}\left[f(x) - \sum_{n=0}^{m}\alpha_n P_n(x)\right]^2 dx\right\} = 0 \quad ,$$

where α_r is any α_n. Taking the indicated partial derivative we get:

$$= -2\int_{-1}^{1}\left[f(x) - \sum_{n=0}^{m}\alpha_n P_n(x)\right] P_r(x)dx = 0$$

Hence

$$\int_{-1}^{1} f(x)P_r(x)dx = \int_{-1}^{1}\sum_{n=0}^{m}\alpha_n P_n(x)P_r(x)dx = \alpha_r \frac{2}{2r+1}$$

Therefore we have

$$\alpha_r = \frac{2r+1}{2}\int_{-1}^{1} f(x)P_r(x)dx$$

From this we may conclude the minimum exists and is obtained with every α_r equal to the corresponding a_r.

3.11 <u>Definition*</u>

A sequence of functions, $f_n(x)$, defined on the closed interval $[a,b]$ is said to converge in the mean to the limit function f(x), provided

$$\lim_{n\to\infty}\int_{a}^{b}\left[f_n(x) - f(x)\right]^2 dx = 0$$

3.12 <u>Theorem**</u>

Let f(x) be a bounded Riemann integrable function defined on

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* Helsel, <u>op. cit.</u>, page 71.
** Thomas, <u>op. cit.</u> page 55.

$[-1,1]$. The necessary and sufficient conditions for the Legendre expansion of f(x) to converge in the mean to f(x) is that

$$\sum_{n=0}^{\infty} a_n^2 \frac{2}{2n+1} = \int_{-1}^{1} \left[f(x)\right]^2 dx$$

is satisfied.

IV APPLICATIONS

4.01 Introduction

We illustrate the uses of Legendre Polynomials by solving some typical problems in electrostatic potential, heat transfer and numerical integration. Before taking up the problems some remarks will be made concerning the relation of spherical harmonics and Legendre Polynomials.

4.02 Remarks*

In these remarks we will define a solid spherical harmonic and discuss the differential equation method of obtaining the spherical harmonics.

4.021 Definition

A solid spherical harmonic is a homogeneous function of x,y,z, satisfying Laplace's differential equation, i.e. $\nabla^2 V = 0$, where the degree of V is called the degree of the harmonic and may be any constant, real or complex, integral or fractional, positive or negative.

If in spherical coordinates $V = r^n Y_n(\theta,\phi)$, then $Y_n(\theta,\phi)$ is called a surface spherical harmonic of degree n

4.022 The Differential Equation Method

Laplace's equation in spherical coordinates becomes

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* Prasad, op. cit., pages 1-4.

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 V}{\partial\phi^2} = 0 \qquad (1)$$

Let $V = r^n f(\theta)\cos m\phi$. Then $r^2\frac{\partial V}{\partial r} = nr^{n+1}f(\theta)\cos m\phi$ and

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) = n(n+1)r^n f(\theta)\cos m\phi\frac{df(\theta)}{d\theta}$$

also

$$\sin\theta\frac{\partial V}{\partial\theta} = \sin\theta\ r^n\cos m\phi$$

and

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right) = \left\{\frac{\cos\theta}{\sin\theta}\frac{df(\theta)}{d\theta} + \frac{d^2f(\theta)}{d\theta^2}\right\}r^n\cos m\phi$$

and finally

$$\frac{\partial V}{\partial\phi} = -r^n f(\theta)m\sin m\phi \quad\text{and}\quad \frac{1}{\sin^2\theta}\frac{\partial^2 V}{\partial\phi^2} = -\frac{m^2}{\sin^2\theta}f(\theta)r^n\cos m\phi$$

Substituting the above in equation (1) and combining terms we get

$$\frac{d^2f(\theta)}{d\theta^2} + \frac{\cos\theta}{\sin\theta}\frac{df(\theta)}{d\theta} + \left\{n(n+1) - \frac{m^2}{\sin^2\theta}\right\}f(\theta) = 0 \qquad (2)$$

Making a change of variables, let $y = \cos\theta$. Then

$$\frac{df(\theta)}{d\theta} = \frac{df(\theta)}{dy}\frac{dy}{d\theta} = -\sin\theta\frac{df(\theta)}{dy}$$

and

$$\frac{d^2f(\theta)}{d\theta^2} = -\cos\theta\frac{df(\theta)}{dy} + \sin^2\theta\frac{d^2f(\theta)}{dy^2}$$

Substituting these in equation (2) we get

$$(1-y^2)\frac{d^2f(\theta)}{dy^2} - 2y\frac{df(\theta)}{dy} + \left\{n(n+1) - \frac{m^2}{1-y^2}\right\}f(\theta) = 0 \qquad (3)$$

Now let $f(\theta) = (1-y^2)^{\frac{m}{2}}W$ and substituting in equation (3) as follows:

$$\frac{df(\theta)}{dy} = -m(1-y^2)^{\frac{m}{2}-1}y\,W + (1-y^2)^{\frac{m}{2}}\frac{dW}{dy}$$

and

$$\frac{d^2f(\theta)}{dy^2} = \left\{\frac{m}{2}\left(\frac{m}{2}-1\right)(1-y^2)^{\frac{m}{2}-2}(2y)^2 - m(1-y^2)^{\frac{m}{2}-1}\right\}W - 2my(1-y^2)^{\frac{m}{2}-1}\frac{dW}{dy} + (1-y^2)^{\frac{m}{2}}\frac{d^2W}{dy^2}$$

and we get

$$(1-y^2)\frac{d^2W}{dy^2} - 2(m+1)y\frac{dW}{dy} + (n-m)(n+m+1)W = 0 \qquad (4)$$

Consider now Legendre's differential equation i.e.

$$(1-y^2)\frac{d^2s}{dy^2} - 2y\frac{ds}{dy} + n(n+1)s = 0 \qquad (5)$$

Let P_n and Q_n denote two independent particular solutions of Legendre's differential equation, where $P_n(1) = 1$ and P_n is of degree n. Differentiating equation (5) m times with respect to y we get:

$$(1-y^2)\frac{d^{m+2}s}{dy^{m+2}} - 2(m+1)y\frac{d^{m+1}s}{dy^{m+1}} + (n-m)(n+m+1)\frac{d^m s}{dy^m} = 0 \qquad (6)$$

Hence $\frac{d^m P_n}{dy^m}$ and $\frac{d^m Q_n}{dy^m}$, which shall be denoted by $\frac{P_n^m}{(1-y^2)^{\frac{m}{2}}}$ and $\frac{Q_n^m}{(1-y^2)^{\frac{m}{2}}}$, are independent particular solutions of equation (4).

Therefore $V = r^n P_n^m \cos m\phi$ and $V = r^n Q_n^m \cos m\phi$ are solutions of Laplace's equation (1). It will be noted that equation (3) is not altered by replacing n by -n-1. Hence $V = r^{-n-1}P_{-n-1}^m \cos m\phi$ and $V = r^{-n-1}Q_{-n-1}^m \cos m\phi$ are also solutions of equation (1). Similarly we may show for $V = r^n f(\theta)\sin m\phi$, we get like results involving the $\sin m\phi$ instead of $\cos m\phi$. These solutions of Laplace's equation give the following:

i. Zonal harmonics, when $m = 0$.

ii. Tesseral harmonics, when $m \neq 0$, and $m \neq n$

iii. Sectorial harmonics, when $m = n$

We note that for $m = 0$, the zonal harmonic $V = r^n P_n(y)$ and $V = \frac{1}{r^{n+1}} P_n(y)$ contain the Legendre Polynomial, $P_n(y)$, as given by

Theorem 2.19. Therefore the Legendre Polynomials are a special case of spherical harmonics.

4.03 Electrostatic Potential

4.031 Definition*

Potential V at a point in an electrostatic field is the work necessary to bring a unit positive charge from infinity; i.e. from outside the field up to the point in question.

4.032 Remarks**

Let us calculate the potential V at P (see figure 1) due to a point charge q placed at the origin O. The electric intensity at Q_2 is $E = \frac{q}{r}$ (Coulomb's Law) in the direction of the radius vector, r, and the work done against the repulsion of q in moving a unit positive charge the distance dl from Q_1 to Q_2 is $dV = \frac{q}{r^2} dl \cos\alpha$. But $dl \cos\alpha = - dr$. Hence $dV = -\frac{q}{r^2} dr$. Integrating from ∞ to R the distance from P to O, we get:

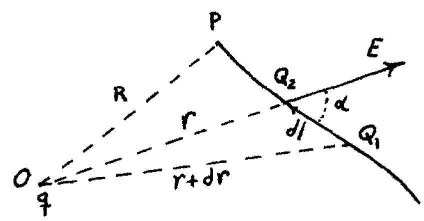

figure 1.

$$V = -q\int_{\infty}^{R} \frac{dr}{r^2} = \frac{q}{R},$$

where the potential is independent of the path.

In case of a continuous distribution of electricity consisting of ρ units of charge per unit volume occupying a volume τ and σ

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* Page and Adams, Principles of Electricity, University Physics Series, Van Nostrands Co., 11th printing, page 15.

** see previous reference, pages 15-18.

units of charge per unit area distributed over a surface, S, then the formula for the potential is as follows:

$$V = \int_{\tau} \frac{\rho\, d\tau}{r} + \int_{S} \frac{\sigma\, ds}{r} \quad ,$$

where r is the distance from P to the elements, $d\tau$ and ds. We see from these expressions that V, the electrostatic potential in all cases, is a function only of the coordinates of the point P at which it is evaluated, and is independent of the path along which the unit positive charge is carried to P. In a region in which there are no free charges, the potential in an isotropic medium satisfies Laplace's differential equation $\nabla^2 V = 0$.*

4.033 <u>Problem</u>**

To determine the potential at a point P (see figure 2) due to two charges, -q and +q, located a distance 2d apart.

<u>Solution</u>

From Remark 4.032 we see that the potential at P is $V = \frac{q}{r_1} - \frac{q}{r_2}$. Now by the law of cosines, $r_1^2 = R^2+d^2-2dR\cos\gamma$ and $r_2^2 = R^2+d^2+2dR\cos\gamma$. We need to consider two cases:

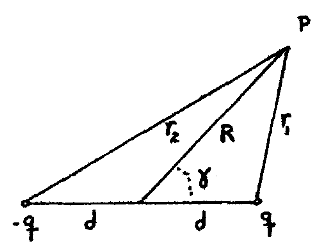

figure 2.

case i.

For $\frac{d}{R} \leq 1$, then $V = \frac{q}{R}(1-2\frac{d}{R}\cos\gamma + \frac{d^2}{R^2})^{-\frac{1}{2}} - \frac{q}{R}(1+2\frac{d}{R}\cos\gamma + \frac{d^2}{R^2})^{-\frac{1}{2}}$

Hence by Theorem 2.16

$$V = \frac{q}{R}\sum_{n=0}^{\infty} P_n(\cos\gamma)(\frac{d}{R})^n - \frac{q}{R}\sum_{n=0}^{\infty} P_n(-\cos\gamma)(\frac{d}{R})^n$$

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* Page and Adams, <u>op. cit.</u>, pages 83-85.
** see previous reference, pages 34-35, and Helsel, <u>op. cit.</u>, pages 29-30.

By note of Theorem 2.11; i.e. $P_n(x) = (-1)^n P_n(-x)$, this becomes

$$= \frac{2q}{R}\sum_{n=0}^{\infty} P_{2n+1}(\cos\gamma)\left(\frac{d}{R}\right)^{2n+1}$$

since even terms cancel.

case ii.

For $\frac{R}{d} \leq 1$, then $V = \frac{q}{d}\left(1-2\frac{R}{d}\cos\gamma + \frac{R^2}{d^2}\right)^{-\frac{1}{2}} - \frac{q}{d}\left(1+2\frac{R}{d}\cos\gamma + \frac{R^2}{d^2}\right)^{-\frac{1}{2}}$

and similarly to case i,

$$V = \frac{2q}{d}\sum_{n=0}^{\infty} P_{2n+1}(\cos\gamma)\left(\frac{R}{d}\right)^{2n+1}$$

4.034 Problem*

To determine the potential at points A and P (see figure 3) due to a circular wire ring of small cross section of uniform charge density ρ per unit length.

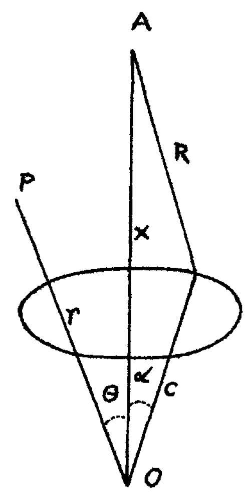

figure 3.

Solution

i) Since A is located on the axis of the ring at a distance x from the origin and is equidistant from all positions of the ring, its potential $V_a = \frac{Q}{R}$, where Q is the total charge on the ring and $R = (c^2+x^2-2cx\cos\alpha)^{\frac{1}{2}}$. Hence as in Problem 4.033

$$V_a = \frac{Q}{c}\sum_{n=0}^{\infty} P_n(\cos\alpha)\left(\frac{x}{c}\right)^n \quad \text{, where } \frac{x}{c} \leq 1 \text{ and}$$

$$= \frac{Q}{c}\sum_{n=0}^{\infty} P_n(\cos\alpha)\left(\frac{c}{x}\right)^{n+1} \quad \text{, where } \frac{c}{x} \leq 1.$$

ii) To obtain the value of the potential function at any point in

*Smythe, Static and Dynamic Electricity, pages 137-138.

space we must satisfy Laplace's equation, which in spherical coordinates is as follows: *

$$\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial V}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 V}{\partial\phi^2} = 0 \qquad (1)$$

subject to the conditions that $V_p = V_a$ for $\theta = 0$.

Let the polar axis be the axis of the ring and hence, due to symmetry, the potential V_p is independent of 0 and Laplace's equation reduces to

$$\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial V}{\partial\theta}\right) = 0 \qquad (2)$$

By 4.032 we know that

$$V_p = \sum_{n=0}^{\infty} A_n r^n P_n(\cos\theta)$$

and

$$V_p = \sum_{n=0}^{\infty} A_n \frac{1}{r^{n+1}} P_n(\cos\theta)$$

are solutions of equation (2).

Applying the boundry conditions we have

$$V_p = \sum_{n=0}^{\infty} A_n r^n P_n(\cos\theta) = \frac{Q}{c}\sum_{n=0}^{\infty} P_n(\cos\alpha)\left(\frac{r}{c}\right)^n \quad , \text{ for } \theta = 0 \text{ or } \cos\theta = 1.$$

But by Theorem 2.10 $P_n(1) = 1$, hence

$$\sum_{n=0}^{\infty} A_n r^n = \frac{Q}{c}\sum_{n=0}^{\infty} P_n(\cos\alpha)\left(\frac{r}{c}\right)^n$$

Therefore we may conclude that

$$V_p = \frac{Q}{c}\sum_{n=0}^{\infty} P_n(\cos\alpha)\left(\frac{r}{c}\right)^n P_n(\cos\theta) \quad , \text{ for } \frac{r}{c} < 1 \text{ or } \theta \neq \alpha, \frac{r}{c} = 1$$

Similarly

$$V_p = \frac{Q}{c}\sum_{n=0}^{\infty} P_n(\cos\alpha)\left(\frac{c}{r}\right)^{n+1} P_n(\cos\theta) \quad , \text{ for } \frac{c}{r} < 1 \text{ or } \theta \neq \alpha, \frac{c}{r} = 1$$

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* Byerly, op. cit., pages 8-12, 152-158.

iii) Remark

As is stated in previous references, this problem is an example of a type and is stated as follows:

Whenever, in a problem involving the solving of the special form of Laplace's equation (equation (2) of ii), the value of V is given or can be found for all points on the axis of X and this value can be expressed as sum or a series involving only whole powers positive or negative of the radius vector of the point, the solution for a point not on the axis can be obtained by multiplying each term by the appropriate zonal harmonic, subject only to the condition that the result, if a series, must be convergent.

4.035 Problem*

To determine the potential at a point P (see figure 4) due to a thin spherical shell of given surface charge density.

Solution

The potential due to all the elements of the surface ds is by Remarks 4.022

$$V = \int_S \frac{\sigma\, ds}{R} ,$$

where σ is the surface charge density. To find ds we let the sphere be centered at O with coordinates of the point Q on the surface being (a, θ', ϕ') and the coor-

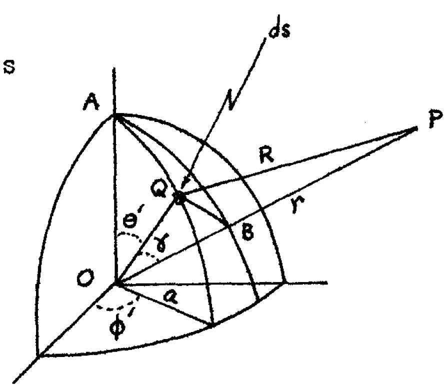

figure 4.

* MacRoberts, T. M., Spherical Harmonics, Methuen and Co., Ltd., London, 2nd Edition, revised 1947, page 158.

dinates of the point P being (r,θ,ϕ). Then $ds = a^2 \sin\theta' d\theta' d\phi'$ $= a^2 dy' d\phi'$, where $-y = \cos\theta'$. As in problem 4.033

$$R = (r^2 + a^2 - 2ar\cos\gamma)^{\frac{1}{2}}$$

where $\cos\gamma = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi-\phi')$ which arises on application of the law of cosines for spherical triangles to triangle ABQ in figure 4. Hence

$$V = \int_S \frac{\sigma\, ds}{(r^2+a^2-2ar\cos\gamma)^{\frac{1}{2}}} = \int_0^{2\pi}\int_{-1}^{1} \frac{\sigma\, a^2 dy' d\phi'}{(r^2+a^2-2ar\cos\gamma)^{\frac{1}{2}}}$$

and therefore

$$V = \int_0^{2\pi}\int_{-1}^{1} \frac{\sigma a^2}{r}\left\{\sum_{n=0}^{\infty} P_n(\cos\gamma)\left(\frac{a}{r}\right)^n\right\} dy' d\phi' \quad , \text{ for } \frac{a}{r} < 1$$

$$= \int_0^{2\pi}\int_{-1}^{1} \sigma \left\{\sum_{n=0}^{\infty} \frac{a^{n+2}}{r^{n+1}} P_n(\cos\gamma)\right\} dy' d\phi' = \sigma \sum_{n=0}^{\infty} \frac{a^{n+2}}{r^{n+1}} \int_0^{2\pi}\int_{-1}^{1} P_n(\cos\gamma)\, dy' d\phi'$$

for $\frac{a}{r} < 1$

and similarly

$$V = \sigma \sum_{n=0}^{\infty} \frac{r^n}{a^{n-1}} \int_0^{2\pi}\int_{-1}^{1} P_n(\cos\gamma)\, dy' d\phi' \quad , \text{ for } \frac{r}{a} < 1$$

4.0351 Remark

Since this problem may be solved much more easily in the standard manner, we use this problem as an illustration of the method of applying Legendre Polynomials which for a more complicated problem may be the best method of solution. The standard solution to problem 4.035 follows:*

Let V_i and V_o be the potentials at points P and Q inside and

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* Page and Adams, op. cit., pages 19-32 and Problem # 9, page 25.

outside the charged spherical shell respectively. By Gauss' law, i.e. $N = 4\pi\int_S \sigma\, da$, where N is the electric flux, the electric intensity field of a uniformly charged spherical shell outside is the same as if the total charge were concentrated at a point and therefore $V_o = \frac{Q}{r}$, where Q is the total charge. Another result of Gauss' law is that there is no field interior to this charged sphere and hence the potential is the same at all interior points which requires that the potential $V_i = \frac{Q}{a}$.

4.04 Heat Transfer Problem*

If the convex surface of a solid hemisphere (figure 5) of radius a is kept at the constant temperature unity and the base at the constant temperature zero show that after the permanent state of temperature is set up, the temperature of any internal point is U, where

$$U = \frac{3}{2}\left(\frac{r}{a}\right)P_1(y) - \frac{7}{2}\cdot\frac{1}{2}\left(\frac{r}{a}\right)^3 P_3(y) + \frac{11}{6}\cdot\frac{1\cdot 3}{2\cdot 4}\left(\frac{r}{a}\right)^5 P_5(y) - \ldots$$

Solution**

When there are no sources or sinks in a uniform solid body it must satisfy Fourier's heat equation, i.e.

$$\nabla^2 U = \frac{c\rho}{K}\frac{\partial u}{\partial t}$$

since our problem calls for the steady state solution. The temperature at any

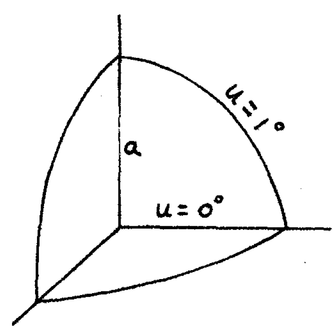

figure 5.

* Byerly, op. cit., page 176, Problem # 8.
** Sokolinkoff, I. S. and E. S., Higher Mathematics for Engineers and Physicists, Second Edition, McGraw-Hill, 1941, pages 425-428 and 382-385.

point must satisfy Laplace's equation $\nabla^2 U = 0$. Since the temperature distribution will be symmetrical and therefore independent of ϕ, Laplace's equation becomes

$$\frac{\partial}{\partial r}\left(r^2 \frac{\partial u}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial u}{\partial\theta}\right) = 0,$$

in spherical coordinates and particular solutions, as in 4.032, are

$$U_a = \sum_{n=0}^{\infty} A_n r^n P_n(y)$$

and

$$U_b = \sum_{n=0}^{\infty} B_n \frac{1}{r^{n+1}} P_n(y)$$

where r is the radius vector of the point in question.

From the statement of the problem the boundry conditions are:

i) $u = f(\theta) = 1$, for $0 \leq \theta < \frac{\pi}{2}$ and $\frac{r}{a} = 1$.

ii) $u = f(\theta) = 0$, for $\theta = \frac{\pi}{2}$ and $\frac{r}{a} = 1$.

iii) $r = 0$, $u = 0$.

Condition iii) implies $U_b = \infty$ at $r = 0$, hence we consider only U_a in the solution of the problem. Thus we use the boundry conditions to evaluate the constants in the series expansion of U_a. Since the boundry conditions are stated at $r = a$, we let

$$U(y) = \sum_{n=0}^{\infty} A_n'\left(\frac{r}{a}\right)^n P_n(y) \qquad (1)$$

instead of U_a as defined previously. Then when $r = a$ this becomes

$$U = \sum_{n=0}^{\infty} A_n' P_n(y) \qquad (2)$$

and if we may determine the constants A_n' such that the equation satisfies the boundry conditions, then we shall have the desired solution to the problem. Let

$$F(y) = \sum_{n=0}^{\infty} A_n' P_n(y).$$

In order to expand F(y) in a series of Legendre Polynomials we define it to be an odd function such that

a) $F(y) = 1$, for $0 \leq \theta < \frac{\pi}{2}$

b) $F(y) = 0$, for $\frac{\pi}{2} \leq \theta < \pi$

Hence by Theorem 3.02

$$A_n' = (2n+1) \int_0^1 F(y) P_n(y)\,dy \quad , \text{ where } F(y) = 1 \text{ on the range 0 to 1.}$$

Evaluating this equation for various values of n we get:

$$A_0' = \int_0^1 P_0(y)\,dy = 1 \quad ; \quad A_1' = 3\int_0^1 1\cdot P_1(y)\,dy = 3\int_0^1 y\,dy = 3\left.\frac{y^2}{2}\right]_0^1 = \frac{3}{2}$$

$$A_2' = 5\int_0^1 P_2(y)\,dy = \frac{5}{2}\int_0^1 (3y^2-1)\,dy = \frac{5}{2}\left[3y^3 - y\right]_0^1 = 0$$

$$A_3' = \frac{7}{2}\int_0^1 (5y^3-3y)\,dy = \frac{7}{2}\left[\frac{5}{4}y^4 - \frac{3}{2}y^2\right]_0^1 = \frac{7}{2}\left(\frac{5-6}{4}\right) = -\frac{1\cdot 7}{2\cdot 4}$$

$$A_5' = \frac{11}{8}\int_0^1 (63y^5-70y^3+15y)\,dy = \frac{11}{8}\left[\frac{63}{6}y^6 - \frac{70}{4}y^4 + \frac{15}{2}y^2\right]_0^1 = \frac{11}{8}\left(\frac{21+15-35}{2}\right)$$

$$= \frac{11}{8}\cdot\frac{1}{2} = \frac{1\cdot 3}{2\cdot 4}\cdot\frac{11}{6}.$$

Substituting these results into equation (1) we get

$$U(y) = P_0(y) + \frac{3}{2}\left(\frac{r}{a}\right)P_1(y) - \frac{1\cdot 7}{2\cdot 4}\left(\frac{r}{a}\right)^3 P_3(y) + \frac{1\cdot 3\cdot 11}{2\cdot 4\cdot 6}\left(\frac{r}{a}\right)^5 P_5(y) - \ldots$$

But the condition $U = 0$ at $r = 0$ implies that the first term of the expansion is not present. Hence the solution is:

$$U = \frac{3}{2}\left(\frac{r}{a}\right)P_1(y) - \frac{1\cdot 7}{2\cdot 4}\left(\frac{r}{a}\right)^3 P_3(y) + \frac{1\cdot 3\cdot 11}{2\cdot 4\cdot 6}\left(\frac{r}{a}\right)^5 P_5(y) - \dots$$

Note that this could have been arrived at by using the extension of an odd function, i.e.

$$A_{2n+1} = \left[2(2n+1) + 1\right] \int_0^1 U(y) P_{2n+1}(y)\,dy$$

4.05 Numerical Integration*

This method of numerical integration is sometimes called the Gauss-Legendre method. Gauss has shown that by the proper choice of the interpolation points $x_0, x_1, x_2, \dots, x_n$, we can obtain an approximation to the given integral equivalent to the approximation obtained by replacing $f(x)$ by a polynomial of degree $2n+1$ or less.

To determine the Gauss-Legendre numerical integration method we let $P_{n+1}(x)$ have its $n+1$ roots at $x_0, x_1, x_2, \dots, x_n$ in order between -1 and 1. Also let $f(x)$ be a function with known values at each x_i , $i = 0, 1, 2, \dots, n$ and are equal to $f(x_0), f(x_1), f(x_2), \dots, f(x_n)$.

From Lagrange's interpolation formula form a polynomial of degree n as follows:

$$F_n(x) = f(x_0)\,\frac{(x-x_1)(x-x_2)\cdots(x-x_n)}{(x_0-x_1)(x_0-x_2)\cdots(x_0-x_n)} + f(x_1)\,\frac{(x-x_0)(x-x_2)(x-x_3)\cdots(x-x_n)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)\cdots(x_1-x_n)}$$

$$+ f(x_2)\,\frac{(x-x_0)(x-x_1)(x-x_3)\cdots(x-x_n)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)\cdots(x_2-x_n)} + \dots + f(x_n)\,\frac{(x-x_0)(x-x_1)(x-x_2)\cdots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)(x_n-x_2)\cdots(x_n-x_{n-1})} \quad (1)$$

We note that $F_n(x_i) = f(x_i)$ for $i = 0, 1, 2, \dots, n$. Integrating equation (1) we have

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* Helsel, op. cit., pages 35-36. Also L. M. Milne-Thompson, Calculus of Finite Differences, pages 173-177, Macmillan and Co., Ltd. 1933

$$\int_{-1}^{1} F_n(x)\,dx = H_0 f(x_0) + H_1 f(x_1) + H_2 f(x_2) + \dots + H_n f(x_n) \quad ,$$

where

$$H_i = \int_{-1}^{1} \frac{(x-x_0)(x-x_1)\cdots(x-x_{i-1})(x-x_{i+1})\cdots(x-x_n)}{(x_i-x_0)(x_i-x_1)\cdots(x_i-x_{i-1})(x_i-x_{i+1})\cdots(x_i-x_n)}\,dx \quad , \; i = 0,1,2,\dots,n \qquad (2)$$

Proof of Gauss' assertion:

If $f(x)$ is a polynomial of degree $\leq 2n+1$, then

$$\frac{f(x)}{P_{n+1}(x)} = Q_n(x) + \frac{R_n(x)}{P_{n+1}(x)} \quad ,$$

where $Q_n(x)$ and $R_n(x)$ are of degree $\leq n$. Or

$$f(x) = Q_n(x)P_{n+1}(x) + R_n(x) \qquad (3)$$

Let $x = x_0$ and substitute in equation (3) and get

$$f(x_0) = 0 \cdot Q_n(x_0) + R_n(x_0)$$

Then let $x = x_1$ in equation (3) and get

$$f(x_1) = R_n(x_1) \quad \text{etc.}$$

Hence we have that

$$f(x_i) = R_n(x_i) \quad \text{for } i = 0,1,2,\dots,n \qquad (4)$$

Now

$$\int_{-1}^{1} f(x)\,dx = \int_{-1}^{1} P_{n+1}(x)Q_n(x)\,dx + \int_{-1}^{1} R_n(x)\,dx$$

This implies that

$$\int_{-1}^{1} f(x)\,dx = \int_{-1}^{1} R_n(x)\,dx \quad ,$$

since

$$\int_{-1}^{1} P_{n+1}(x)Q_n(x)\,dx = 0$$

by the orthogonality property of Theorem 2.10. We know

$$\int_{-1}^{1} R_n(x)\,dx = H_0R_n(x_0) + H_1R_n(x_1) + \dots + H_nR_n(x_n) \qquad (5)$$

Therefore from equations (4) and (5) we may conclude that

$$\int_{-1}^{1} f(x)\,dx = H_0 f(x_0) + H_1 f(x_1) + \dots + H_n f(x_n)$$

exactly.

4.051 Solution of a Problem by Simpson's Method

$\int_{1.0}^{2.0} \frac{dx}{x}$ for $n = 4$, where n is the number of intervals in $[a,b]$.

Solution

$$\int_{1.0}^{2.0} \frac{dx}{x} = \frac{1}{12}\left[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + f(x_4)\right]$$

$$= \frac{1}{12}\left[1 + 4\frac{1}{1.25} + 2\frac{1}{1.5} + 4\frac{1}{1.75} + \frac{1}{2}\right] = 0.693254$$

For $n = 8$ this becomes 0.693154

4.052 Solution of the Problem of 4.051 by the Gauss-Legendre Method

$$\int_{1.0}^{2.0} \frac{dx}{x} \qquad \text{for } n = 4$$

Solution

Since the Legendre Polynomials are symmetric in the interval $[-1,1]$ the zeros of $P_n(\alpha)$ may be arranged in such a way that

$$\alpha_s = -\alpha_{n-s+1}$$

for n even and if n is odd the middle one is zero. This implies that

$$H_s = H_{n-s+1}$$

To change the limits from $[a,b]$ to $[-1,1]$ we make the substitution

$$x = \frac{a+b}{2} + \frac{b-a}{2}\alpha$$

Hence with $a = 1$ and $b = 2$ we get

$$x = \frac{3+\alpha}{2}$$

and

$$dx = \tfrac{1}{2}d\alpha$$

For $n = 4$ the roots of $P_4(\alpha)$ are the α_i's as follows:*

$$-\alpha_0 = \alpha_3 = 0.861136311$$

and

$$-\alpha_1 = \alpha_2 = 0.339981043$$

and

$$H_i = \frac{A_i}{2} \text{ , where } A_i = \int_{-1}^{1} \frac{(\alpha-\alpha_0)(\alpha-\alpha_1)\cdots(\alpha-\alpha_{i-1})(\alpha-\alpha_{i+1})\cdots(\alpha-\alpha_n)}{(\alpha_i-\alpha_0)(\alpha_i-\alpha_1)\cdots(\alpha_i-\alpha_{i-1})(\alpha_i-\alpha_{i+1})\cdots(\alpha_i-\alpha_n)}\, d\alpha$$

For this problem the $\frac{A_i}{2}$'s are as follows:

$$\frac{A_0}{2} = \frac{A_3}{2} = 0.173927422$$

and

$$\frac{A_1}{2} = \frac{A_2}{2} = 0.3260725774$$

Using the above data to evaluate $\int_{1.0}^{2.0} \frac{dx}{x}$ we get:

$$\int_{1.0}^{2.0} \frac{dx}{x} = \sum_{i=0}^{3} \frac{A_i}{2}\left(\frac{2}{3+\alpha_i}\right) = \frac{A_0}{2}\left(\frac{2}{3+\alpha_3} + \frac{2}{3-\alpha_3}\right) + \frac{A_1}{2}\left(\frac{2}{3+\alpha_2} + \frac{2}{3-\alpha_2}\right)$$

$$= 0.17392742(0.13507595 + 0.51798223) + 0.326072577(0.75187434 + 0.5988058)$$

$$= \underline{\underline{0.693146}}$$

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* Hobson, E. W., Spherical and Ellipsoidal Harmonics, pages 80-81, Cambridge at the University Press, 1931; also, Margenau and Murphy, op. cit., pages 462-464.

4.053 The Exact Value of Problem in 5.051

$$\int_{1.0}^{2.0} \frac{dx}{x} = \ln 2.0 = 0.693147 \quad *$$

A comparison of the results indicates that the Gauss-Legendre method is exact to the 6th decimal place for $n = 4$ while Simpson's method for $n = 8$ is only exact up to the 5th decimal place.

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* Peirce, B. O., A Short Table of Integrals, Third Revised Edition, Ginn and Company, page 109.