THE MINIMUM ENERGY CRITERIA OF A TRIPLE INTEGRAL PLANT

A Thesis
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By

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INTRODUCTION

In modern control theory, a practical control system represented by a dynamical model has sources whose outputs have a limited range of permissible values. This limitation on the range of source output corresponds to the limited rate of energy conversion which can be achieved in practice and results in restrictions on the amplitude and speed of system variables in the model.

The theory of optimal control is concerned with controlling the system variables of the model, placing restrictions on the source, in such a manner that a scalar quantity, the performance index, achieves an extremal value. In general, the performance index may be written as

$$I = S\,(\underline{x},\underline{u},t), \tag{1-1}$$

where $\underline{x}$ is the state vector and $\underline{u}$ is the control vector.

In a flight control system, for example, this would mean combining into one scalar the effect of error in maintaining the desired trajectory, the amount of fuel expended, the magnitude of the control action, and the time to reach the terminal state.

In economic systems, however, the factors of interest may be economic growth, inflation, and expenditure of resources. In effect, the engineer must select the factor of interest which yields a compromise design to be investigated in order to determine the best optimal system.[1]

In this paper the system model is investigated to discover what variations of source outputs result in a desired change in position for a minimum expenditure of energy.

A. Purpose of Thesis

The problem is to investigate the transfer of a system from an initial state to a final state with a minimum expenditure of control or source energy.

The system is conservative and characterized by a triple integral plant with three poles at zero.

The major portion of this paper is devoted to the linear region of the control; however, the development of the control problem includes saturation, and a brief discussion is presented in Chapter V.

B. Method of Analysis

The performance index that is to be minimized represents the total energy consumed. This performance index is formulated by taking the time integral of the control signal squared. Here, $u(t)$ represents the control signal and the performance index is

$$I = \int_{t_o}^{t_f} u^2(t)\, dt \,. \qquad (1\text{-}2)$$

The problem now becomes one of finding an optimum signal $u^o(t)$, that will force the plant from the initial state $X(t_o)$ to a final state $X(t_f)$ and minimize the performance index.

The method which will be used to determine an optimum control signal, $u^o(t)$, is Pontryagin's Maximum Principle. Briefly stated the

Maximum Principle which is based on the calculus of variations maximizes a scalar H called the Hamiltonian which results in minimizing the scalar performance index I, equation (1-2)[1].

The third order plant considered in this paper is defined via state variable notation -- three first-order differential equations. The performance index will be redefined as a new variable called the Pontryagin function.[2]

The Hamiltonian of the system will be formulated and maximized with respect to the control signal. Thus, in this manner, an optimum control signal will be obtained. Through use of canonical equations, the adjoint system will be defined as function of time and the unknown adjoint initial conditions.

Knowledge of the control signal and adjoint equations determines the state variables as a function of the unknown adjoint initial conditions P_{io} and time. Fixing the final state $X_f(t=T)=0$ establishes the adjoint initial conditions P_{io} in terms of the initial states X_{io} and time T. Here, T is defined as the total time required to move the system state point from an initial position to a terminal location. In addition, an energy equation which is expressed in terms of the initial values of the system variables will be determined and examined for various values of T.

Here, the state variables and control signal are plotted as functions of time, and the corresponding trajectory lengths and consumed energy are also presented.

C. Summary

The design parameter considered in the following chapters is energy; however, other characteristics of system performance will be examined. These include such factors as overshoot, total time required for corrective action, distance traversed, and a realizable controller. First, however, the necessary mathematical background is presented and considered next.

CHAPTER II

DEVELOPMENT OF THE OPTIMAL CONTROL SIGNAL

A. Definition of State Variables

The system to be analyzed is characterized by the third-order differential equation with zero time constants.

$$\frac{d^3x}{dt^3} = u(t) \tag{2-1}$$

Where u(t) is the forcing function and the initial conditions of the equation are

$$x(t_o) = x(0)$$

$$\frac{dx}{dt}(t_o) = \frac{dx}{dt}(0)$$

$$\frac{d^2x}{dt^2}(t_o) = \frac{d^2x}{dt^2}(0)$$

Find the optimum control signal $u^o(t)$ which will transfer the system from initial state $X(0)$ to final state $X(t_f)$ and minimize the energy function

$$I = \int_{t_o}^{t_f} u^2(t)\, dt \tag{2-2}$$

subject to the constraints

$$u_{min} \leq |u| \leq u_{max} \qquad \text{and} \qquad X(t_f) = 0$$

The state variables are defined as

$$\begin{aligned} \text{let} \quad x(t) &= x_1(t) \\ \dot{x}_1(t) &= x_2(t) \\ \dot{x}_2(t) &= x_3(t) \\ \dot{x}_3(t) &= u(t) \end{aligned} \tag{2-3}$$

Where $\dot{x}(t)$ denotes $\frac{dx}{dt}$. Subject to the initial conditions,

$$x_1(t_o) = x_{10}$$
$$x_2(t_o) = x_{20} \qquad (2\text{-}4)$$
$$x_3(t_o) = x_{30}$$

The system can be shown by the simple open-loop process.

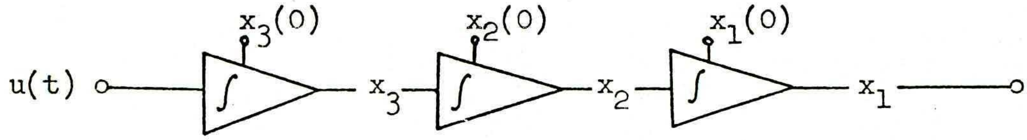

FIGURE 1

Equation (2-3) is written in the following matrix form:

$$\dot{X}(t) = AX(t) + BU(t) \,* \qquad (2\text{-}5)$$

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(t) \qquad (2\text{-}6)$$

A new variable called the Pontryagin function[2] is defined.

$$P = x_{n+1}(t) = x_4(t)$$

*Refer to Appendix A, page 41, for basic definitions.

This variable is set equal to the performance index

$$x_4(t) = \int_{t_o}^{t_f} u^2(t)\, dt \tag{2-7}$$

The energy problem now becomes one of minimizing the new function $x_4(t)$, evaluated at the end of the trajectory with respect to the control signal $u(t)$.

The derivative of equation (2-7) is

$$\dot{x}_4(t) = u^2(t)$$

with $x_4(t_o) = 0$ being the initial condition on the new function.

B. Formation of the Hamiltonian and the Optimum Control Signal

The Hamiltonian* for the system and the energy function is represented by the general equation,[2,4]

$$H = H\left[\underline{x}(t),\ \underline{p}(t),\ u(t),\ t\right]$$

$$H = \underline{X}'(t)\ \underline{A}'\ \underline{p}(t) + u'(t)\ \underline{B}'\ \underline{p}(t) + p_{n+1}(t) \sum_{j=1}^{r} u_j^2(t) \qquad (2\text{-}8)$$

where $\underline{p}(t)$ is a n order vector called the costate.

$\underline{A}'$ and $\underline{B}'$ are the transposed matrices $\underline{A}$ and $\underline{B}$, respectively.

$\underline{X}'(t)$ is the n order transposed state vector.

Substituting the transposed state vector, the costate vector, and the transposed $\underline{A}'$ and $\underline{B}'$ matrices of equation (2-6) into equation (2-8) results in the Hamiltonian of this problem written in matrix form.

In this problem n = 3 and j = r = 1 in equation (2-8).

$$H = \begin{bmatrix} x_1(t) & x_2(t) & x_3(t) \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} p_1(t) \\ p_2(t) \\ p_3(t) \end{bmatrix} + \begin{bmatrix} 0 & 0 & u(t) \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} p_1(t) \\ p_2(t) \\ p_3(t) \end{bmatrix} + p_4\, u^2(t) \qquad (2\text{-}9)$$

By simplifying the matrix the Hamiltonian can be written as

$$H = \begin{bmatrix} 0 & 0 & 0 \\ x_2(t) & 0 & 0 \\ 0 & x_3(t) & 0 \end{bmatrix} \begin{bmatrix} p_1(t) \\ p_2(t) \\ p_3(t) \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & u(t) \end{bmatrix} \begin{bmatrix} p_1(t) \\ p_2(t) \\ p_3(t) \end{bmatrix} + p_4\, u^2(t)$$

*Refer to Appendix A, page 41, for a general development.

$$H = p_1(t)\ x_2(t) + p_2(t)\ x_3(t) + p_3(t)\ u(t) + p_4\ u^2(t) \qquad (2\text{-}10)$$

The Hamiltonian may also be formed by the following expression:

$$H = \sum_{i=1}^{n} p_i f_i = p_1 f_1 + p_2 f_2 + \ . \ . \ . + p_n f_n$$

where

$$\dot{x}_1 = f_1 = x_2$$

$$\dot{x}_2 = f_2 = x_3$$

$$\dot{x}_3 = f_3 = u(t)$$

$$\dot{x}_4 = f_4 = u^2(t)$$

The boundary conditions on the adjoint equation $p_i(t)$ are

$$p_1(t_f) = 0 \quad p_2(t_f) = 0 \quad p_3(t_f) = 0 \quad p_4(t) = -1$$

To summarize, the Hamiltonian function is dependent upon the control $\underline{u}(t)$, the state vector $\underline{x}(t)$, and the adjoint vector $\underline{p}(t)$.

To find the optimal control, Pontryagin's Maximum Principle has to be applied to the Hamiltonian, equation (2-10) which is maximized with respect to the control signal u(t). Equation (2-10) is a quadratic function of u(t) and therefore the optimum control is readily obtained.

Taking the derivative of equation (2-10) with respect to u(t) gives

$$\frac{\partial H}{\partial u(t)} = \frac{2}{\partial u} \left\{ p_1(t)\ x_2(t) + p_2(t)\ x_3(t) + p_3(t)\ u(t) + p_4(t)\ u^2(t) \right\} = 0$$

$$\frac{\partial H}{\partial u(t)} = p_3(t) + 2p_4(t)\ u(t) = 0$$

$$p_3(t) = -2p_4(t)\ u(t)$$

from the boundary conditions set $p_4(t) = -1$ and solve for u(t)

$$u(t) = \tfrac{1}{2}\ p_3(t)$$

subject to the condition stated earlier that

$$|p_3(t)| \leq 2\,U$$

where $U = u_{max}$ the upper bound on $u(t)$.

If $|p_3(t)| \geq 2U$, the optimum control signal that maximizes the Hamiltonian equation (2-9) is

$$u(t) = U \operatorname{sgn} p_3(t)$$

For simplicity the control signal is normalized and U is set equal to 1.

In summary, the optimum control signal is defined as

$$u(t) = \tfrac{1}{2}\, p_3(t) \qquad |p_3(t)| \leq 2\,U \tag{2-11}$$

$$u(t) = U \operatorname{sgn} p_3(t) \qquad |p_3(t)| \geq 2\,U \tag{2-12}$$

C. The Development of the Auxiliary Equations

The canonical equations are defined as[2,3]

$$\frac{\partial H}{\partial p_i} = \dot{x}_i \qquad \text{and}$$

$$\frac{\partial H}{\partial x_i} = -\dot{p}_i$$

First, taking the derivative of equation (2-10) with respect to $p_i(t)$ results in variables in agreement with equation (2-3).

$$\frac{\partial H}{\partial p_1} = \dot{x}_1(t) = x_2(t)$$

$$\frac{\partial H}{\partial p_2} = \dot{x}_2(t) = x_3(t) \tag{2-13}$$

$$\frac{\partial H}{\partial p_3} = \dot{x}_3(t) = u(t)$$

Now, taking the derivative of equation (2-10) with respect to $x_i(t)$ results in the adjoint differential equations.

$$\frac{\partial H}{\partial x_1} = -\dot{p}_1(t) = 0 \tag{2-14}$$

$$\frac{\partial H}{\partial x_2} = -\dot{p}_2(t) = p_1(t)$$

$$\frac{\partial H}{\partial x_3} = -\dot{p}_3(t) = p_2(t)$$

Writing the adjoint differential equations in matrix form gives

$$\begin{bmatrix} \dot{p}_1(t) \\ \dot{p}_2(t) \\ \dot{p}_3(t) \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} p_1(t) \\ p_2(t) \\ p_3(t) \end{bmatrix}$$

Assume the adjoint equations to have the initial conditions

$$p_1(0) = p_{10}$$

$$p_2(0) = p_{20}$$

$$p_3(0) = p_{30}$$

Integrating the adjoint equations (2-14) gives

$$p_1(t) = p_{10} \tag{2-15}$$

$$p_2(t) = -p_{10}t + p_{20}$$

$$p_3(t) = \frac{p_{10}t^2}{2} - p_{20}t + p_{30}$$

D. Conclusion

Note from equation (2-11) that the magnitude of the control signal is less than one and linear; hence, an analytic solution to this problem is possible.

Also note that equation (2-12) the magnitude of the control signal is greater than one and non-linear. An analytic solution is extremely difficult to obtain and one must usually rely on geometrical techniques to overcome this difficulty.

CHAPTER III

DERIVATION OF THE SYSTEM EQUATIONS

A. The Control Law, Function of Known Variables

The linear optimum control signal given by equation (2-11) is

$$u^o(t) = \tfrac{1}{2}\, p_3(t) \qquad |p_3(t)| \leq 2U$$

Upon substitution of equation (2-15) into the above expression yields

$$u^o(t) = \frac{p_{10}t^2}{4} - \frac{p_{20}t}{2} + \frac{p_{30}}{2} \tag{3-1}$$

This equation is a function of the adjoint system initial conditions and is irrelevant unless there is a method to solve for p_{10}, p_{20}, and p_{30}. This is accomplished by substituting equation (3-1) into equations (2-13)

$$\dot{x}_3(t) = u(t) = u^o(t) = \frac{p_{10}t^2}{4} - \frac{p_{20}t}{2} + \frac{p_{30}}{2}$$

integrating

$$x_3(t) = \frac{p_{10}t^3}{12} - \frac{p_{20}t^2}{4} + \frac{p_{30}t}{2} + x_{30} \tag{3-2}$$

$$\dot{x}_2(t) = x_3(t)$$

$$x_2(t) = \frac{p_{10}t^4}{48} - \frac{p_{20}t^3}{12} + \frac{p_{30}t^2}{4} + x_{30}t + x_{20} \tag{3-3}$$

$$\dot{x}_1(t) = x_2(t)$$

$$x_1(t) = \frac{p_{10}t^5}{240} - \frac{p_{20}t^4}{48} + \frac{p_{30}t^3}{12} + \frac{x_{30}t^2}{2} + x_{20}t + x_{10} \tag{3-4}$$

p_{10}, p_{20}, and p_{30} can now be determined as functions of final time ($t_f = T$) and the initial conditions of the state vectors x_{10}, x_{20}, and x_{30}.

Substituting the final condition equation, $X\ (t_f = T) = 0$ into equations (3-2) through (3-4) results in

$$0 = \frac{p_{10}T^5}{240} - \frac{p_{20}T^4}{48} + \frac{p_{30}T^3}{12} + \frac{x_{30}T^2}{2} + x_{20}T + x_{10} \tag{3-5}$$

$$0 = \frac{p_{10}T^4}{48} - \frac{p_{20}T^3}{12} + \frac{p_{30}T^2}{4} + x_{30}T + x_{20} \tag{3-6}$$

$$0 = \frac{p_{10}T^3}{12} - \frac{p_{20}T^2}{4} + \frac{p_{30}T}{2} + x_{30} \tag{3-7}$$

Clearing the above equations of fractions and multiplying equation (3-6) and (3-7) by T and T^2 respectively gives

$$0 = p_{10}T^5 - 5p_{20}T^4 + 20p_{30}T^3 + 120x_{30}T^2 + 240x_{20}T + 240x_{10} \tag{3-8}$$

$$0 = p_{10}T^5 - 4p_{20}T^4 + 12p_{30}T^3 + 48x_{30}T^2 + 48x_{20}T \tag{3-9}$$

$$0 = p_{10}T^5 - 3p_{20}T^4 + 6p_{30}T^3 + 12x_{30}T^2 \tag{3-10}$$

Subtract equation (3-10) from equation (3-9).

$$0 = -p_{20}T^4 + 6p_{30}T^3 + 36x_{30}T^2 + 48x_{20}T \tag{3-11}$$

Subtract equation (3-9) from equation (3-8).

$$0 = -p_{20}T^4 + 8p_{30}T^3 + 72x_{30}T^2 + 192x_{20}T + 240x_{10} \tag{3-12}$$

Subtract equation (3-11) from equation (3-12).

$$0 = 2p_{30}T^3 + 36x_{30}T^2 + 144x_{20}T + 240x_{10}$$

Solving for $p_{30} = -\frac{1}{T^3}\left\{18x_{30}T^2 + 72x_{20}T + 120x_{10}\right\}$ (3-13)

Substituting this value of p_{30} into equation (3-11),

$$p_{20} = -\frac{1}{T^4}\left\{72x_{30}T^2 + 384x_{20}T + 720x_{10}\right\} \tag{3-14}$$

Substituting this value of p_{20} into equation (3-8) gives

$$p_{10} = \frac{-1}{T^5} \left\{ 120x_{30}T^2 + 720x_{20}T + 144x_{10} \right\} \tag{3-15}$$

Further substitution of equations (3-13), (3-14), and (3-15) into equation (3-1) results in

$$\begin{aligned} u^o(t) = & \frac{-1}{T^5} \left\{ 30x_{30}T^2 + 180x_{20}T + 360x_{10} \right\} t^2 \\ & + \frac{1}{T^4} \left\{ 36x_{30}T^2 + 192x_{20}T + 360x_{10} \right\} t \\ & - \frac{1}{T^3} \left\{ 9x_{30}T^2 + 36x_{20}T + 60x_{10} \right\} \end{aligned} \tag{3-16}$$

The control law satisfies the necessary conditions and is a unique function of the initial and final conditions and time.

B. The Performance Index, Function of Known Variables

So far the minimum energy problem has been treated from a general point of view. The mathematical analysis has progressed through the necessary steps needed to obtain the optimum control law $u^o(t)$, equation (3-16). With the aid of the optimum law, the initial conditions x_{10}, x_{20}, x_{30}, and the knowledge that at t_f=T, x(T)=0; the position vectors $x_1(t)$, $x_2(t)$, and $x_3(t)$ were obtained. By using the optimum control law, equation (3-16) and the energy criterion function, equation (2-7), an expression for energy can be obtained.

Equation (2-7)

$$E(t) = \int_{t_o}^{t_f} u^2(t) = \int_{t_o=0}^{t_f=T} \{u^o(t)\}^2 dt \tag{3-17}$$

From equation (3-1)

$$\{u^o(t)\}^2 = \left\{\frac{p_{10}t^2}{4} - \frac{p_{20}t}{2} + \frac{p_{30}}{2}\right\}^2 \qquad \text{or}$$

$$\{u^o(t)\}^2 = \frac{p_{10}{}^2 t^4}{16} - \frac{p_{10}p_{20}}{4} t^3 + \frac{p_{20}{}^2 + p_{10}p_{30}}{4} t^2$$

$$-\frac{p_{20}p_{30}t}{2} + \frac{p_{30}{}^2}{4}$$

Substituting the above expression into equation (3-17) and integrating with respect to time gives

$$E(t) = \int_{t_o=0}^{t_f=T} \left\{\frac{p_{10}{}^2}{16} t^4 - \frac{p_{10}p_{20}}{4} t^3 + \frac{p_{20}{}^2 + p_{10}p_{30}}{4} t^2 \right.$$

$$\left. -\frac{p_{20}p_{30}}{2} t + \frac{p_{30}{}^2}{4} \right\} dt$$

$$= \left[\frac{p_{10}{}^2 t^5}{80} - \frac{p_{10}p_{20}}{16} t^4 + \frac{p_{20}{}^2 + p_{10}p_{30}}{12} t^3 - \frac{p_{20}p_{30}}{4} t^2 \right.$$

$$\left. + \frac{p_{30}{}^2}{4} t \right]_{t_o=0}^{t_f=T}$$

$$E(t) = \frac{p_{10}{}^2}{80} T^5 - \frac{p_{10}p_{20}}{16} T^4 + \frac{p_{20}{}^2 + p_{10}p_{30}}{12} T^3$$

$$-\frac{p_{20}p_{30}}{4} T^2 + \frac{p_{30}{}^2}{4} T \qquad (3\text{-}18)$$

C. Conclusion

The system equations restated are as follows:

a) The state variables.

$$x_1(t) = \frac{p_{10}}{240} t^5 - \frac{p_{20}}{48} t^4 + \frac{p_{30}}{12} t^3 + \frac{x_{30}}{2} t^2 + x_{20} t + x_{10}$$

$$x_2(t) = \frac{p_{10}}{48} t^4 - \frac{p_{20}}{12} t^3 + \frac{p_{30}}{4} t^2 + x_{30}t + x_{20}$$

$$x_3(t) = \frac{p_{10}}{12} t^3 - \frac{p_{20}}{4} t^2 + \frac{p_{30}}{2} t + x_{30}$$

b) The adjoint initial condition equations.

$$p_{10} = -\frac{1}{T^5}\left\{120x_{30}T^2 + 720x_{20}T + 144x_{10}\right\}$$

$$p_{20} = -\frac{1}{T^4}\left\{72x_{30}T^2 + 384x_{20}T + 720x_{10}\right\}$$

$$p_{30} = -\frac{1}{T^3}\left\{18x_{30}T^2 + 72x_{20}T + 120x_{10}\right\}$$

c) The optimum control law.

$$u^o(t) = -\frac{1}{T^5}\left\{30x_{30}T^2 + 180x_{20}T + 360x_{10}\right\} t^2$$

$$+ \frac{1}{T^4}\left\{36x_{30}T^2 + 192x_{20}T + 360x_{10}\right\} t$$

$$- \frac{1}{T^3}\left\{9x_{30}T^2 + 36x_{20}T + 60x_{10}\right\}$$

d) The energy equation.

$$E(T) = \frac{p_{10}^2}{80} T^5 - \frac{p_{10}p_{20}}{16} T^4 + \frac{p_{20}^2 + p_{10}p_{30}}{12} T^3 - \frac{p_{20}p_{30}T^2}{4} + \frac{p_{30}^2}{4} T$$

CHAPTER IV

EXAMPLE OF THE APPLICATION OF NUMERICAL VALUES TO SYSTEM EQUATIONS

A. Introduction

The minimization technique will be best displayed if numerical values are substituted into the system equations. By allowing the time of movement, $(T = t_f - t_o)$, to assume the values, $T = 1.0$, $T = 2.0$, ..., $T = 40.0$ seconds and letting the initial and final states be

$$x_1(0) = x_{10} = 1 \qquad x_1(T) = 0$$

$$x_2(0) = x_{20} = 1 \qquad x_2(T) = 0$$

$$x_3(0) = x_{30} = 1 \qquad x_3(T) = 0$$

then the initial conditions of the adjoint system can be calculated as function of x_{io} and final time T.

There is no particular reason for choosing the initial conditions to be these specific values, but the initial point was placed in the lower right hand octant for the purpose of plotting.

Small t is given increments of value $t = 0, 0.5, \ldots, T$ for each $T = 1, 2, \ldots, 40$ seconds.

Establishing the initial states of the adjoint system allows the calculation of:

a) The state variables $x_1(t)$, $x_2(t)$, $x_3(t)$, and the control function $u^o(t, p_{io})$.

b) The performance index or energy function $E\ (T, x_{io})$ as function of total time.

Because the calculations of the state variables, control function, length of the trajectories, and energy are repetitious and tedious, the computer was used. The data compiled are voluminous if all values of $T = 1.0$ second through $T = 40.0$ seconds are considered. For the sake of brevity, it was necessary to scrutinize the data for only pertinent information. Appendix B consists of these data.

The purpose of this paper is to find an optimum control function that minimizes the energy of the system as the state space is traversed. This can be accomplished only by plotting the data and comparing the curves. Before a specific control function is chosen, such parameters as trajectory overshoot, time of movement, and energy expended are all taken into account. In addition, the control function must be realizable.

The lengths of the trajectories (S) are calculated by using the well known relationship

$$\Delta S = \sqrt{(x_1 - x_{10})^2 + (x_2 - x_{20})^2 + x_3 - x_{30})^2} \qquad (4\text{-}1)$$

for each value of time of movement ($T = t_f - t_o$), $T = 1.0$ through $T = 40.0$ seconds.

For example, let $T = 1.0$ second in the expression of the position vectors x_1, x_2, and x_3, equations (3-2), (3-3), and (3-4) respectively. Let the instantaneous time t span the range $0 \leqslant t \leqslant T$ with increments of $\Delta t = 1.0$ second. For each increment of time Δt an increment of the trajectory (ΔS) is calculated by substituting the value of the position vector and its initial condition into the above equation (4-1). Each previous value of the position vector serves as the initial condition for the next calculation. Initially the position vectors x_{10}, x_{20}, and x_{30} take on the arbitrary values of 1, 1, and -1 respectively. Finally, adding all of the increments ΔS results in the length of the trajectory S.

The calculations of the trajectory lengths (S) for each value of time of movement T are in Appendix B, page 46. The plot of the above calculations is in Figure 11.

B. Discussion of Data

The graphs of Figure 2 and Figure 3, showing energy versus time of movement, represent the minimum energy expended by the optimal control for changes in final time.

The graphs are plots of equation (3-18). They illustrate well the importance of choosing time of movement ($T = t_f - t_o$). Since it is physically not practical to operate in the high energy region, $T \leq 6.0$ seconds, the region of interest to be considered is for $T > 6.0$ seconds.

A careful examination of Figure 3 discloses that for $8.0 \leq T \leq 11.0$ seconds the energy curve has an inflection point. This point occurs at T = 10.0 seconds. For values of T greater than 12.0 seconds, the minimum energy required decreases at a slow rate.

If the system is to be moved from the initial state to the final state by using relatively small amounts of energy, and if there is no restriction on time, then T = 8.0 seconds and upward would fulfill the requirement.

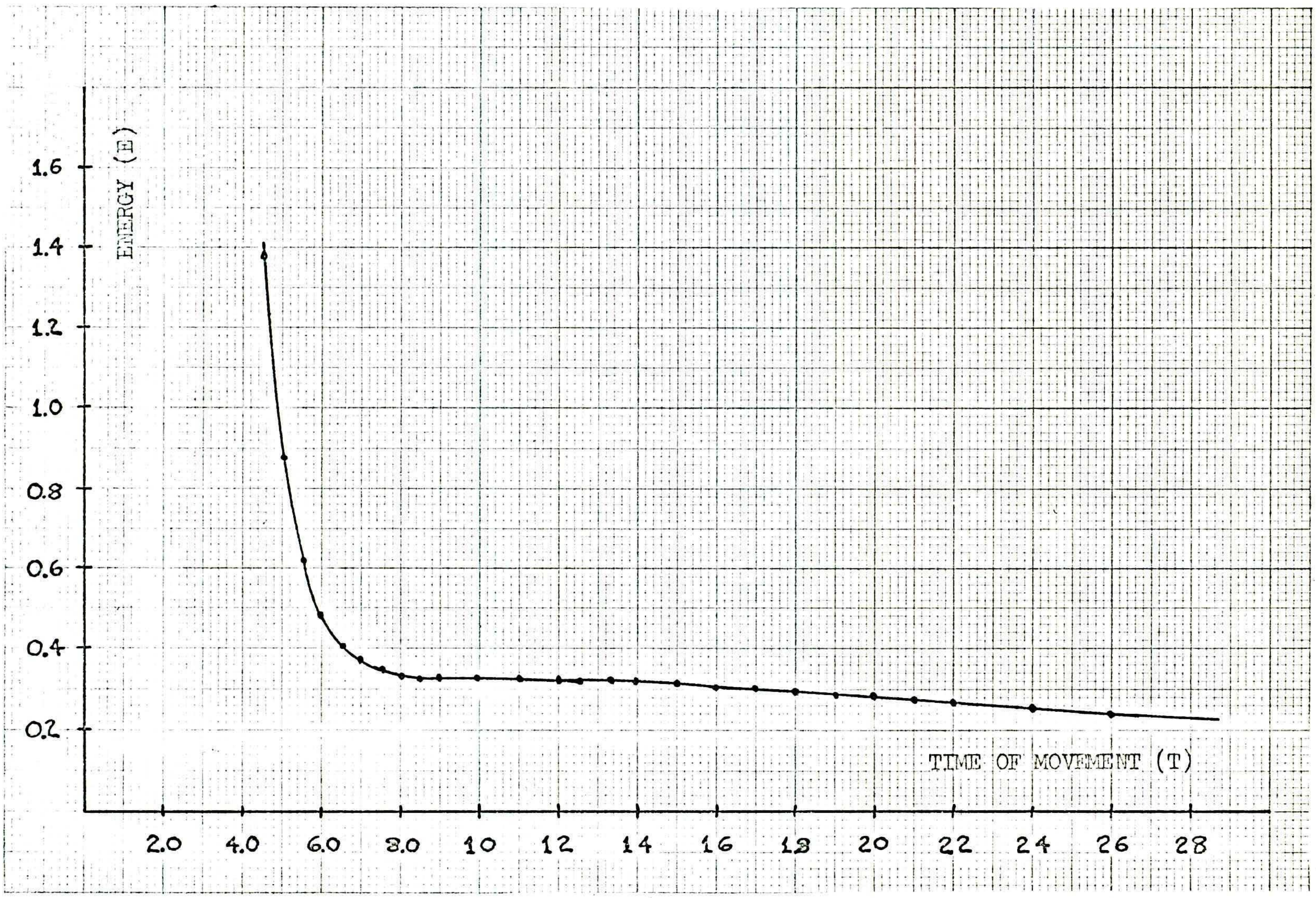

FIGURE 2 - ENERGY VERSUS TIME OF MOVEMENT (T)

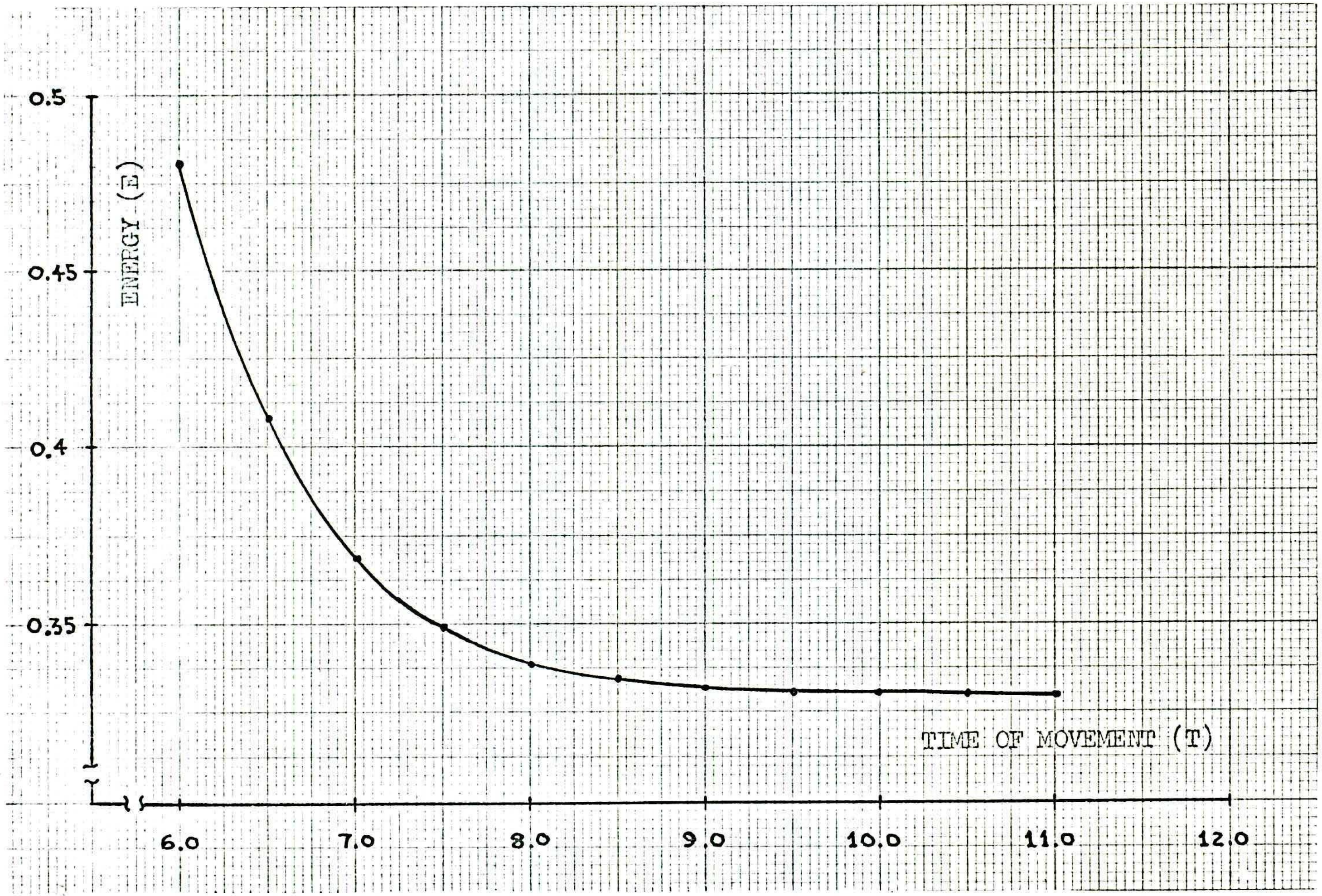

FIGURE 3 - ENERGY VERSUS TIME OF MOVEMENT (T). EXPANDED SCALE.

The next point to be studied is the system behavior of the position vectors x_1, x_2, and x_3. Figure 5 through 10 are plots of the optimal trajectories for different values of time of movement ($T = t_f - t_o$). These figures are the plots of equation (3-2) through (3-4).

A point of great interest is the severe changes that occur in the trajectories as T increases. From Figure 2 and Figure 3 it is known that the energy differential for T = 5.0 and T = 15.0 seconds is very small; however, the trajectories are radically different with large overshoots occurring when T is both small and large in value. Figure 4 depicts a three-dimensional state space $x_1 x_2 x_3$ of the trajectories; it illustrates best the dissimilarity of the paths of T = 5.0 seconds and T = 15.0 seconds.

In the same figure for T = 8.0 seconds, the trajectory displays less overshoot than T = 5.0 and T = 15.0 seconds; however, T = 10.0 seconds has even better characteristics than T = 8.0 seconds.

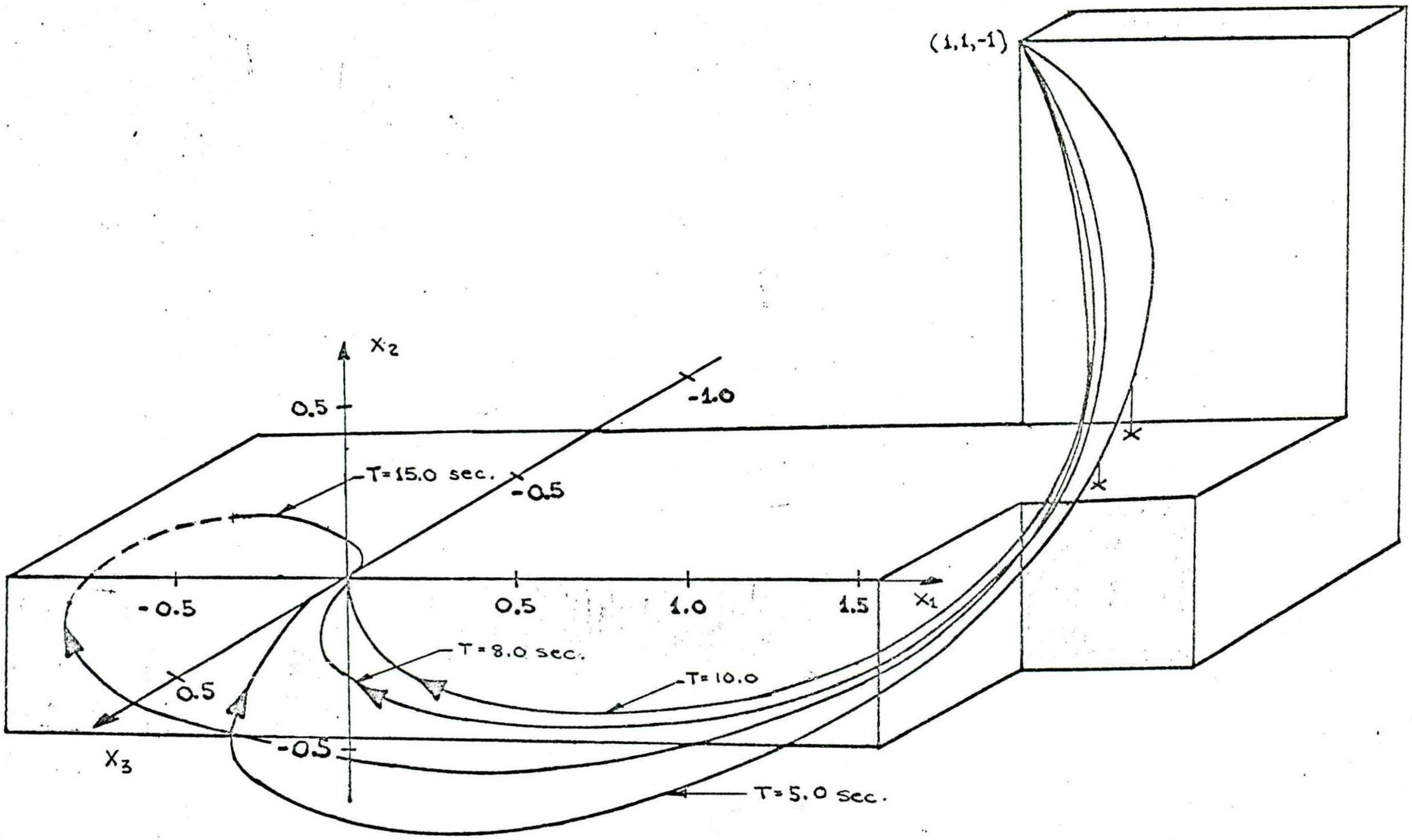

FIGURE 4 - SYSTEM TRAJECTORIES IN THREE DIMENSIONAL SPACE

Figure 5 is the plot of the position vectors x_1, x_2, and x_3 for $T = 1.0$ second. The curves of Figure 5 disclose that x_1 and x_2 exhibit satisfactory responses, but x_3 is very irregular and not acceptable. Figures 6 through 10 are all plots of the position vectors for $T = 6.0$, 8.0, 9.0, 10.0, and 15.0 seconds respectively.

The graphs of $T = 6.0$ seconds through $T = 10.0$ seconds show that as T increases the overshoot of x_1 increases while that of x_2 and x_3 decrease. However, for values of T greater than 10.0 seconds, particularly $T = 15.0$ seconds, damped oscillations are introduced into the system. The graph of $T = 15.0$ seconds is shown in Figure 10.

If a choice were to be made of the time of movement $(T = t_f - t_o)$, $T = 10.0$ seconds displays the most satisfactory system response.

The data for the position vectors x_i with their respective time of movement are shown in Appendix B, page 47.

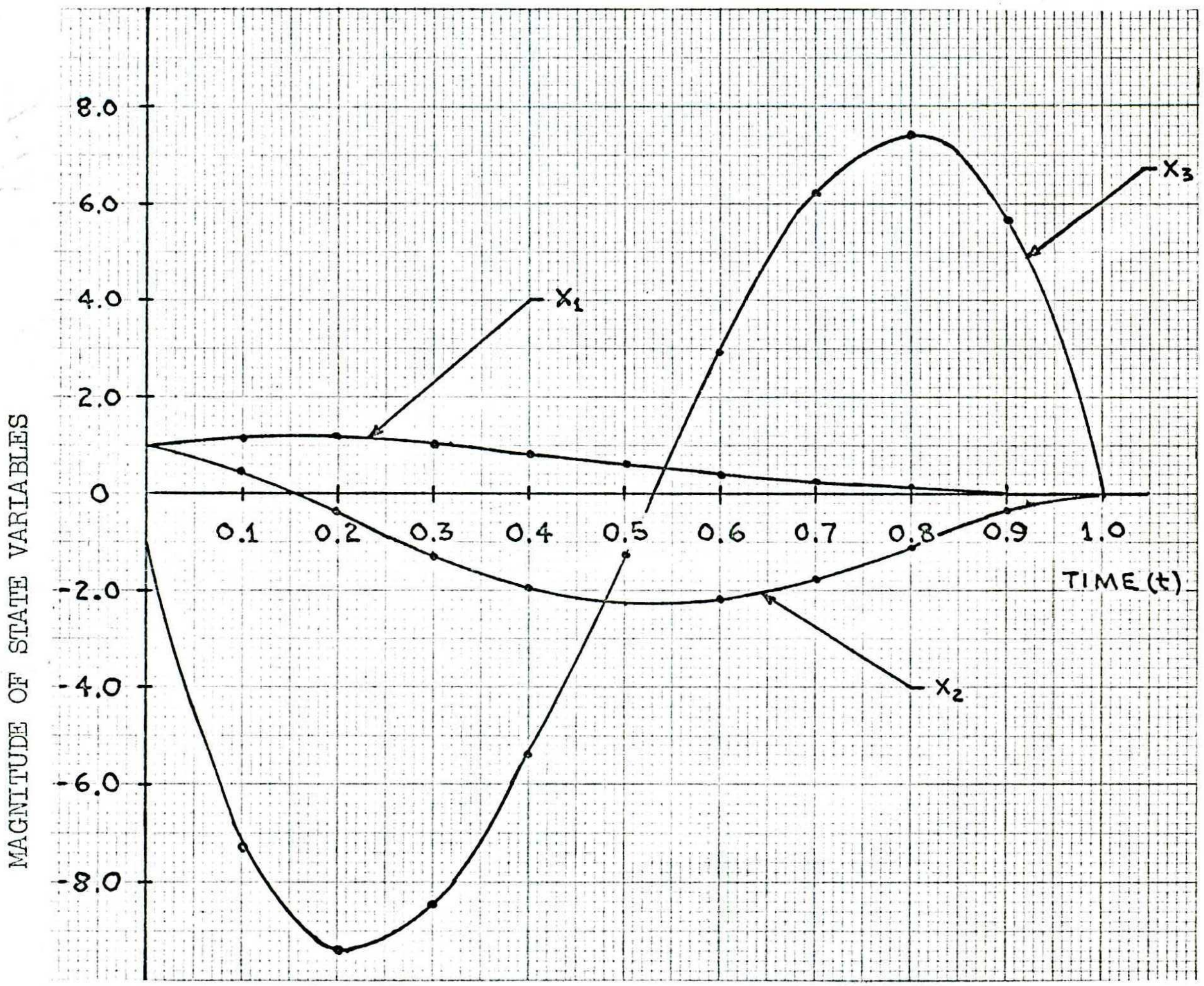

FIGURE 5 - STATE VARIABLES x_1, x_2, AND x_3 VERSUS TIME t.

TIME OF MOVEMENT T = 1.0 SECOND.

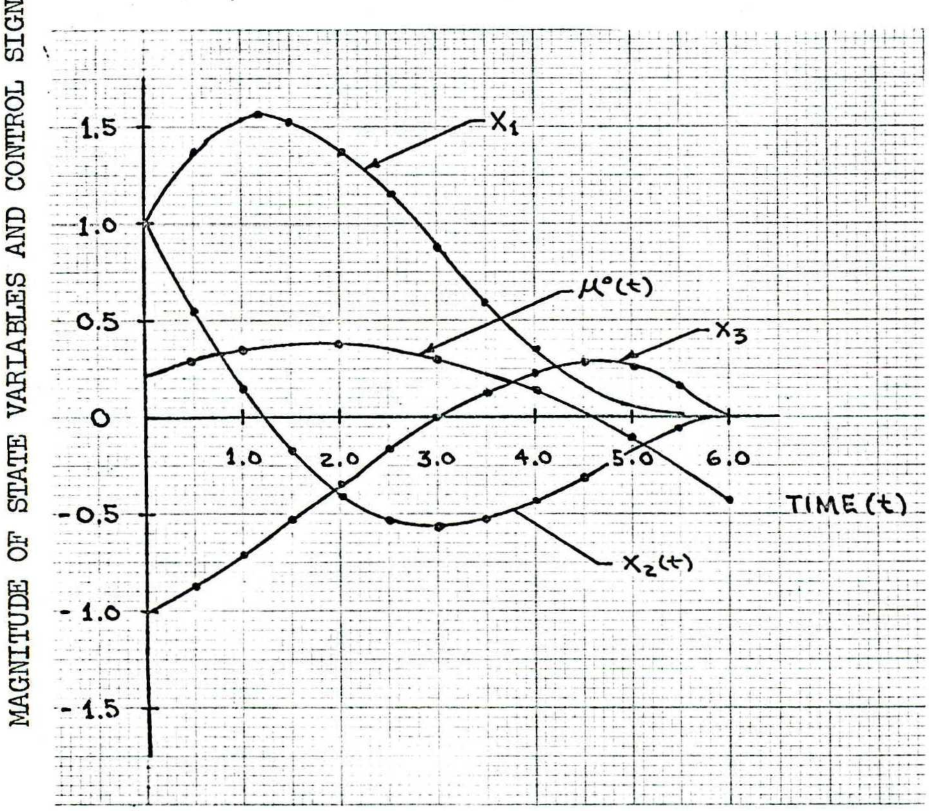

FIGURE 6 - STATE VARIABLES x_1, x_2, x_3 AND OPTIMUM CONTROL SIGNAL VERSUS TIME t. TIME OF MOVEMENT T = 6.0 SECONDS.

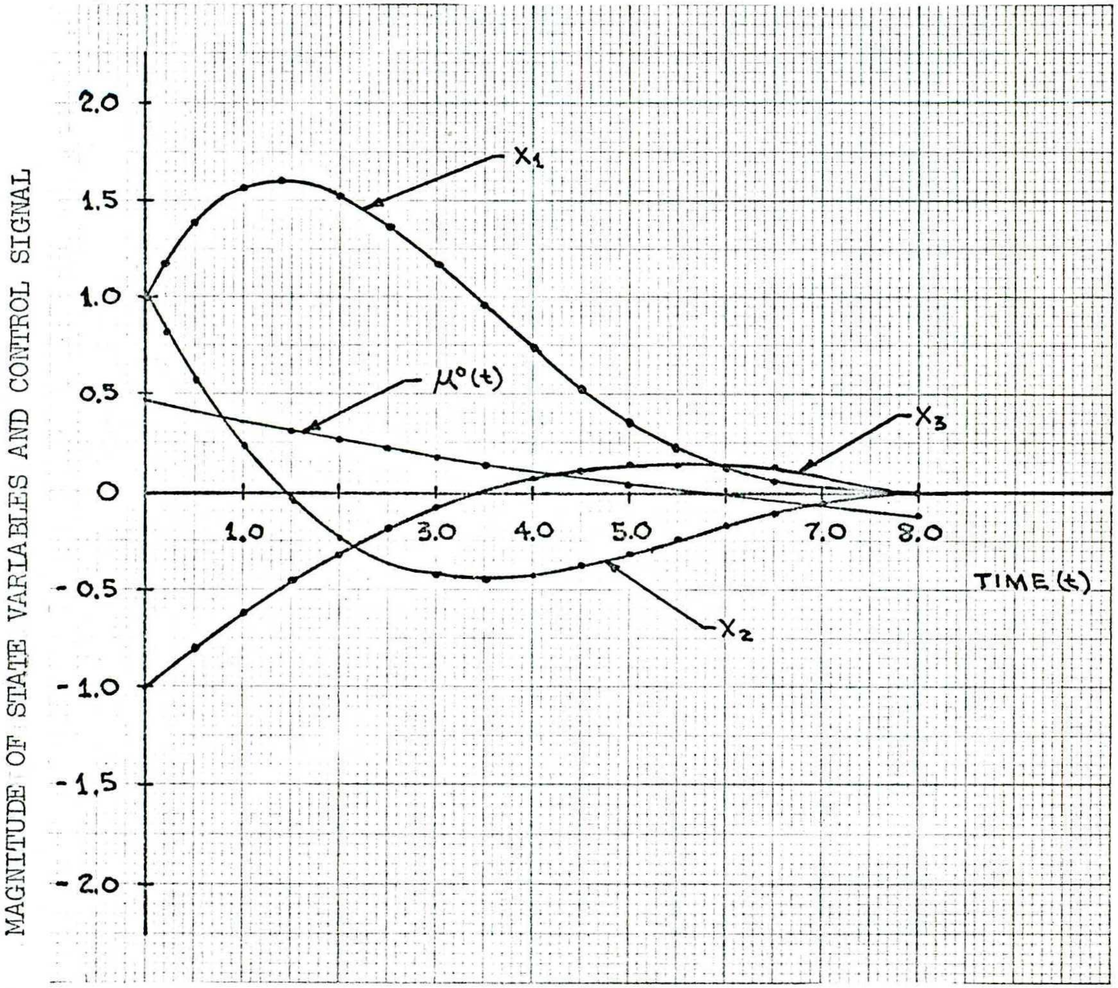

FIGURE 7 - STATE VARIABLES x_1, x_2, x_3 AND OPTIMUM CONTROL SIGNAL $u^o(t)$ VERSUS TIME t. TIME OF MOVEMENT T = 8.0 SECONDS.

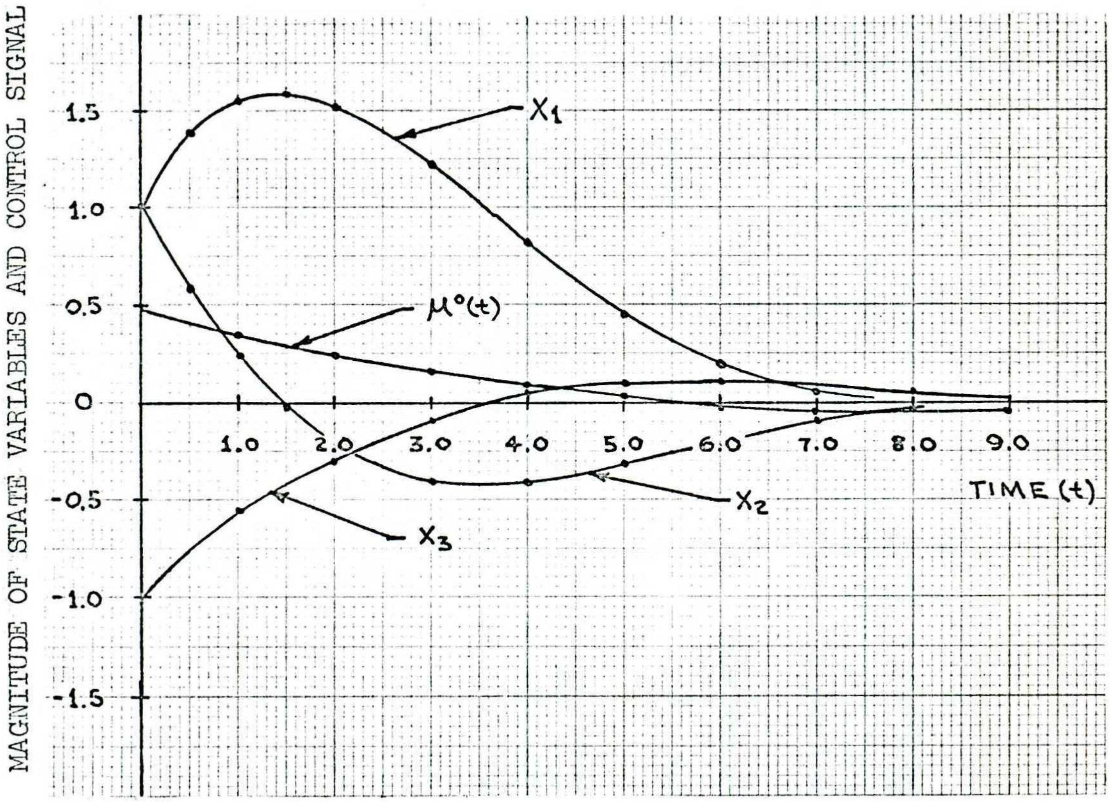

FIGURE 8 - STATE VARIABLES x_1, x_2, x_3 AND OPTIMUM SIGNAL $u^o(t)$ VERSUS TIME t. TIME OF MOVEMENT T = 9.0 SECONDS.

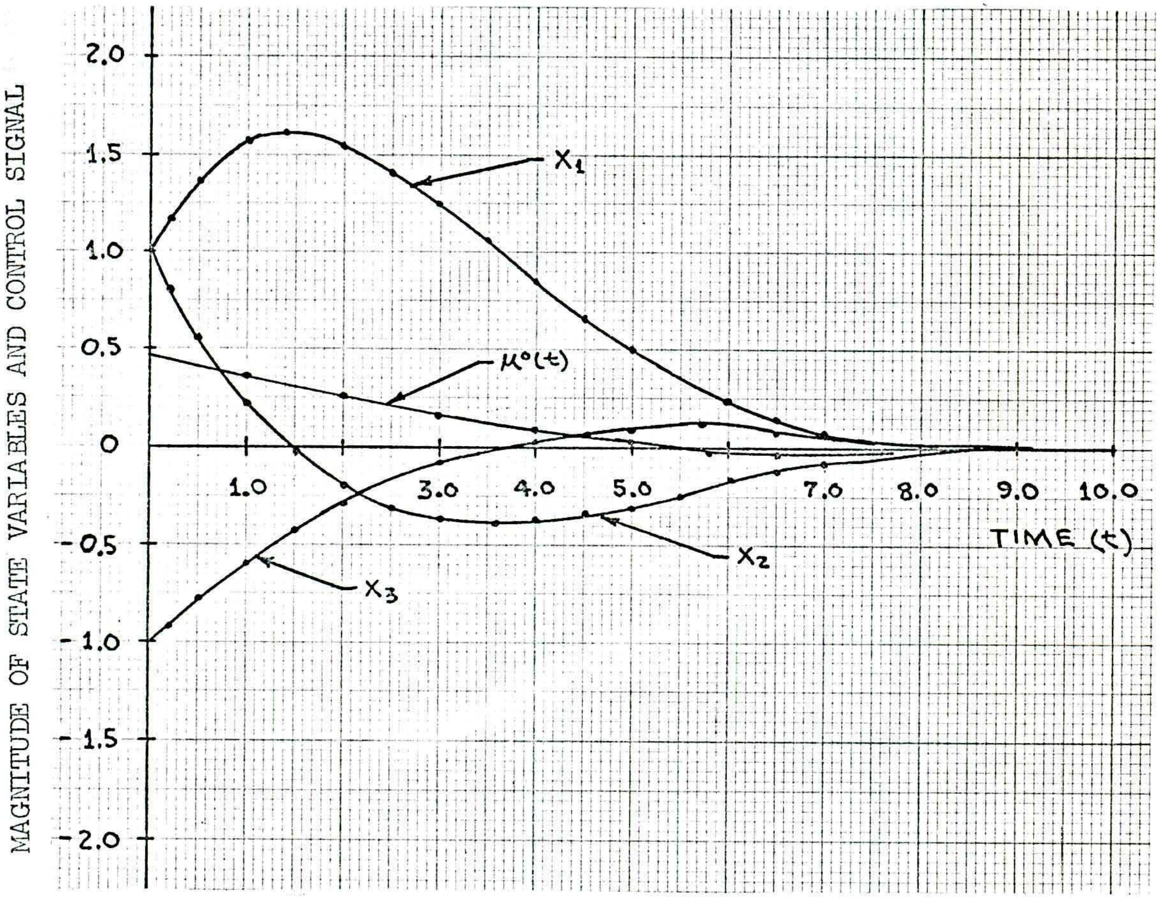

FIGURE 9 - STATE VARIABLES x_1, x_2, x_3 AND OPTIMUM CONTROL SIGNAL $u^o(t)$ VERSUS TIME.t. TIME OF MOVEMENT T = 10.0 SECONDS.

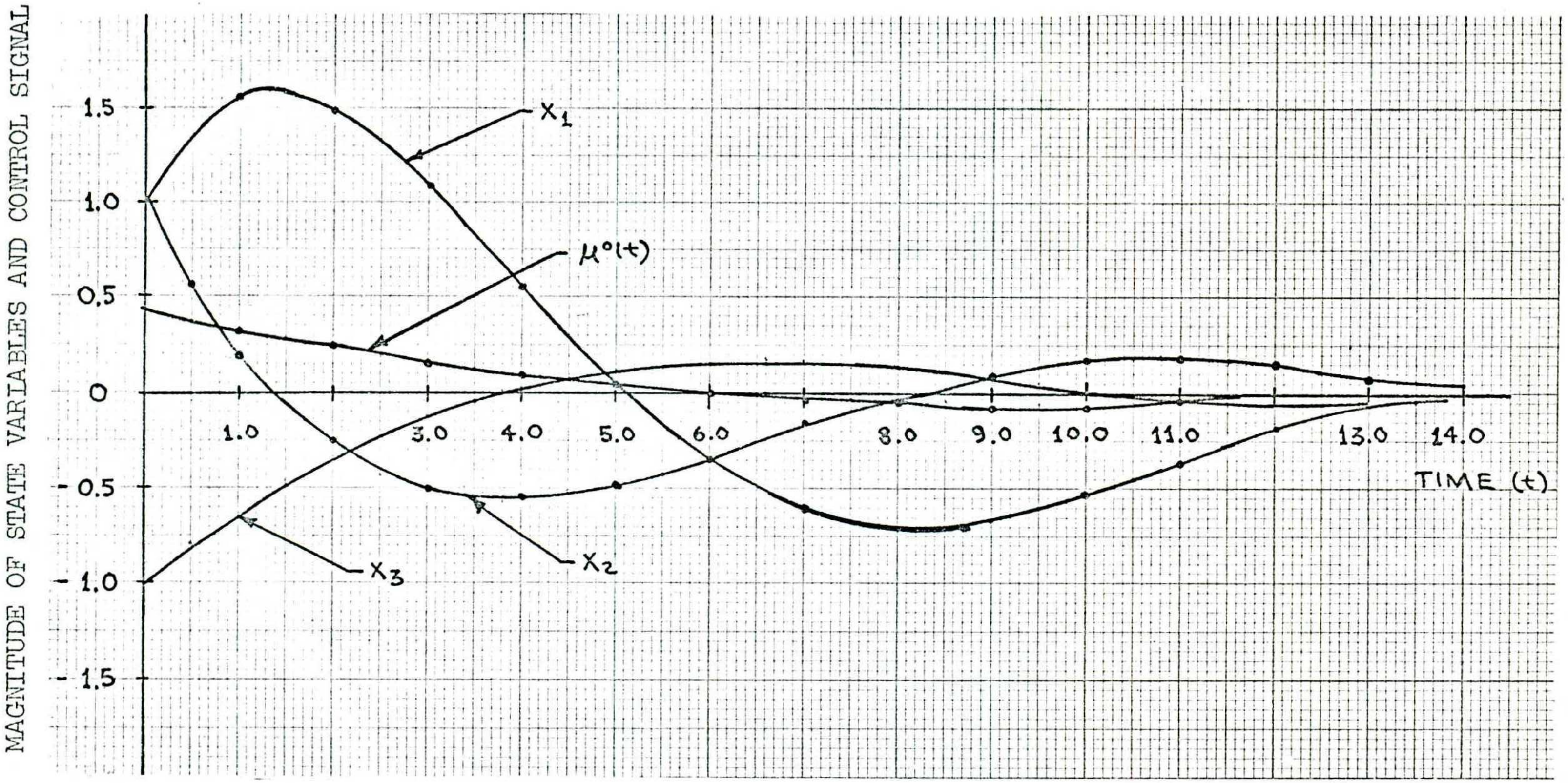

FIGURE 10 - STATE VARIABLES x_1, x_2, x_3 AND OPTIMUM CONTROL SIGNAL $u^o(t)$ VERSUS TIME t. TIME OF MOVEMENT T = 15.0 SECONDS.

Since the region of minimum overshoot has been determined to be approximately 10.0 seconds, the data pertaining to position vectors x_i reveal that the trajectory lengths vary greatly, depending on the time of movement chosen.

Figure 11 and Figure 12 are plots of trajectory lengths versus time. The trajectory lengths should be of some importance because at some point there must exist a minimum path. The minimum trajectory length or optimum path is found to occur when time of movement T is 10.0 seconds.

The computed lengths with their respective time of movement are shown in Appendix B, page 46.

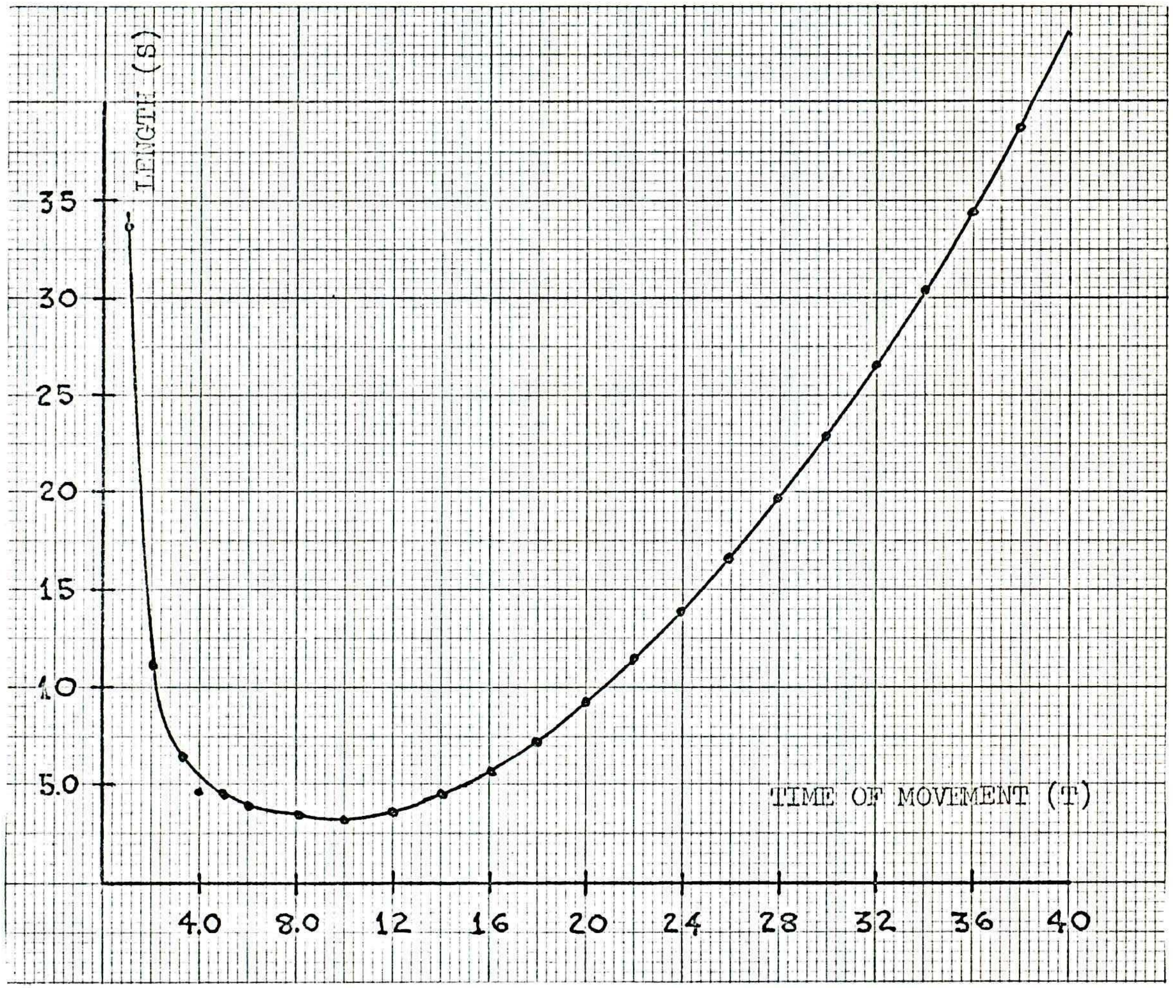

FIGURE 11 - TRAJECTORY LENGTH (S) VERSUS TIME OF MOVEMENT (T)

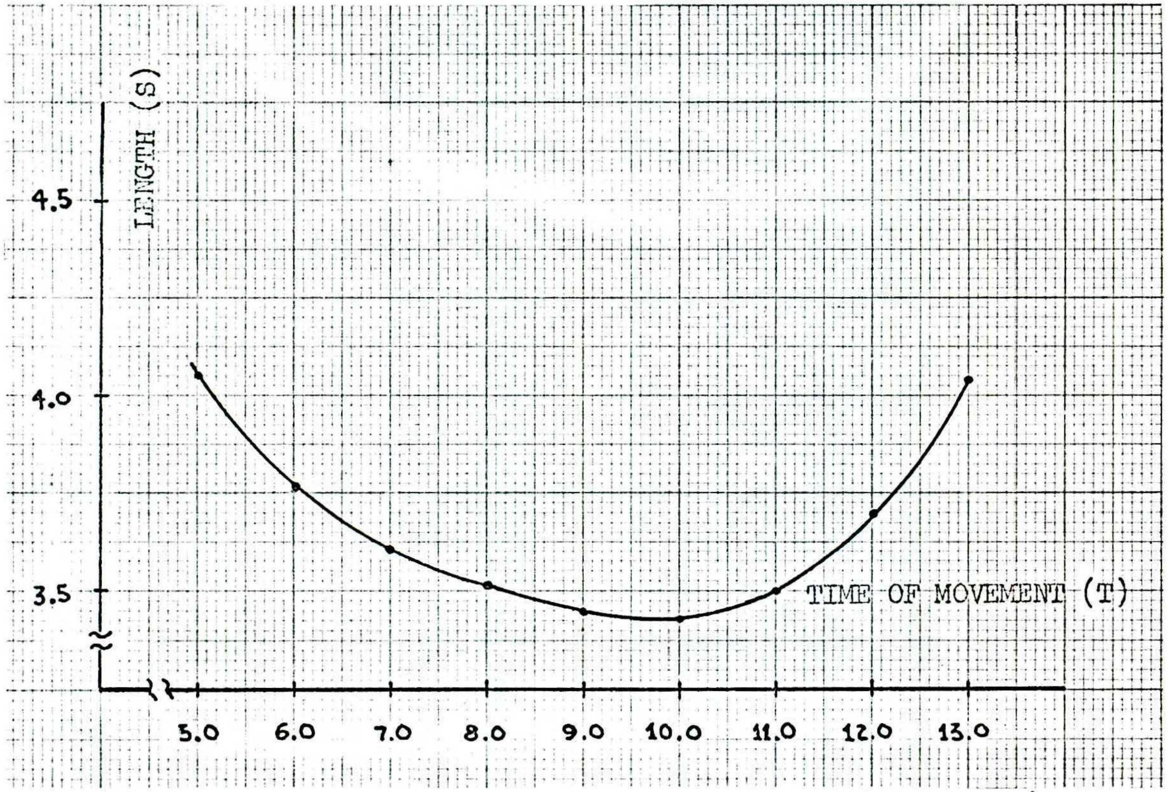

FIGURE 12 - TRAJECTORY LENGTH (S) VERSUS TIME OF MOVEMENT (T).
EXPANDED SCALE.

As the reader will note, Figures 6 through 10 consist of not only the position vector x_1, x_2, and x_3 but also the optimal control function $u^o(t)$. Because of the large magnitude of the optimum control signal for T = 1.0 second the function had to be plotted separately and not on the same graph as the position vectors. Figure 13 is this graph.

The plot of the control function $u^o(t)$ for T = 1.0 second (Figure 13) appears to be realizable having the shape of half a sine wave; however, the magnitude of the wave is very large implying the use of enormous amount of energy by the system. This, of course, could have been predicted by re-examining the energy curve of Figure 2.

As T increases, the control function deviates greatly from the shape of a sine wave, becoming linear at approximately T = 8.0 seconds and thereafter evolving into a concave upward curve.

The data to plot the control function $u^o(t)$ are in Appendix B, page 47.

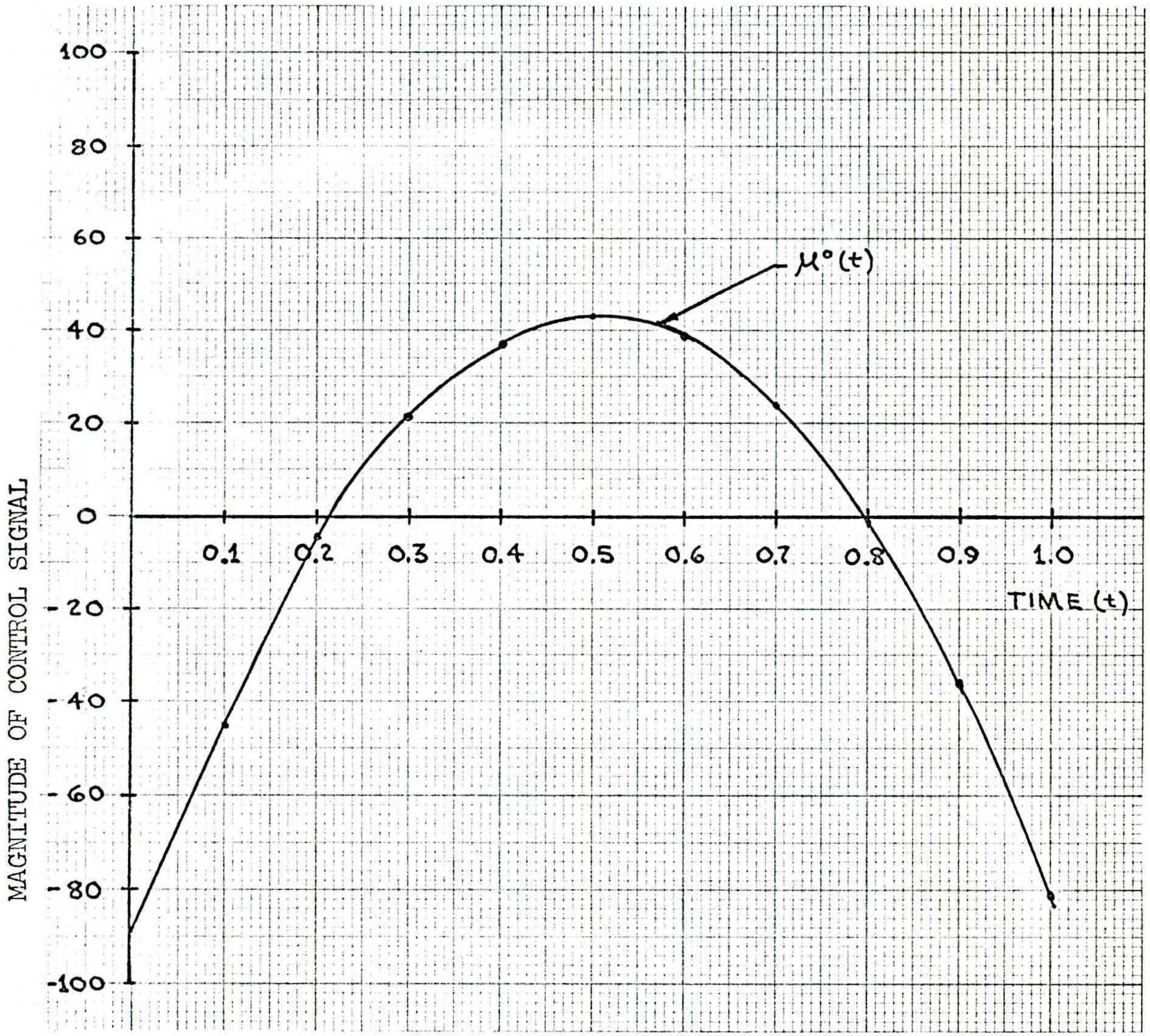

FIGURE 13 - OPTIMUM CONTROL SIGNAL $u^{o}(t)$ VERSUS TIME t.

TIME OF MOVEMENT T = 1.0 SECOND.

C. Conclusion

On page 20, it was stated that if time was not a critical factor, then the system state could be changed with any desired amount of energy. Now that the behavior of the system variables has been investigated, it has been found that if the proper time T is not chosen, large overshoot and/or damped oscillations will be introduced into the system. By choosing the time of movement T = 10.0 seconds a relatively small amount of energy will be used by the system and also the optimum path will be utilized.

CHAPTER V

DISCUSSION OF THE NON-LINEAR CONTROL SIGNAL

A. Introduction

When a constraint is put on the magnitude of the control signal, the optimal control law is given by equation (2-12),

$$u(t) = U \text{ sgn } p_3(t)$$

where $p_3(t)$ is given by equation (2-15),

$$p_3(t) = \frac{p_{10}t^2}{2} - p_{20}t + p_{30}$$

This equation is a function of the unknown adjoint initial conditions. The value of p_{10}, p_{20}, and p_{30} cannot be chosen freely since they must be a function of the initial state X_{io} and the final state X_f.

Substituting equation (2-15) into equation (2-12) results in

$$u(t) = U \text{ sgn } \left[\frac{p_{10}t^2}{2} - p_{20}t + p_{30}\right] \qquad (5\text{-}1)$$

The arbitrary constants p_{10}, p_{20}, and p_{30} in equation (5-1) must be chosen such that given $x_1(0)$, $x_2(0)$, and $x_3(0)$, then $x_1(T) = x_2(T) = x_3(T) = 0$. The relationship between p_{io}, X_{io}, and X_f is non-linear and unknown. Nevertheless, an important result can be deduced immediately. Since p_{10}, p_{20}, and p_{30} are real constants (some of which may be zero) the function $\left(\frac{p_{10}t^2}{2} - p_{20}t + p_{30}\right)$ has 0, 1, or 2 real roots.

Therefore, $u(t) = U \operatorname{sgn} (\frac{1}{2} p_{10}t^2 - p_{20}t + p_{30})$ implies that optimal control consists of $u = \pm U$ with 0, 1, or 2 switchings. A further consideration reveals that, only for certain initial values of x_1, x_2, and x_3 can zero or one switch be employed. Generally, two switchings are required to bring to rest this third-order system. U(t) in the literature is referred to as bang-bang control and in general (n-1) switchings are required to bring all the n state variables to zero.

B. Conclusion

The difficulty in solving eq. (5-1) can be overcome by using geometric techniques to construct switching boundaries. However, this technique becomes extremely difficult when the order of the system is greater than two. In this problem the linear switching surface is a plane in three dimensions. Results for the triple integral plant with optimal non-linear switching have been given by Grensted and Fuller (1965).[6]

CHAPTER VI

SUMMARY AND CONCLUSION

When minimizing the energy criterion function $I = \int_{t_o}^{t_f} u^2(t)\, dt$, a system can be made to change states in any specified time. The curve on page 21 gives the energy movement time relationship. Note from the figure that the system can change states in zero time, however, this movement would require an infinite amount of energy. On the other hand, it is possible for the system to change states using no energy, but in this case the movement time would be infinite.

There exists a minimum trajectory or optimum path such that the overshoot of the system variables x_1, x_2, and x_3 is minimized. The time and energy associated with this optimum path is not a minimum, but the magnitude of the energy required for the system movement is relatively small and the movement time is not excessive.

If minimum time and energy were required for the system, then another criterion function must be minimized. This performance index would be some combination of time and energy.

The preceding discussion brings the sensitivity of optimal control problems into focus. A small change in one parameter such as time can cause either large or small variations in minimum energy, and of course energy means cost. A good engineering design must take all such effects into account.

APPENDIX A

STATEMENT OF BASIC DEFINITIONS AND TERMINOLOGY

The following is taken from M. Athanassiades article, "Optimal Control for Linear Time-Invariant Plants with Time, Fuel, and Energy Constraints".[4]

A linear time-invariant system may be described by the matrix differential equation

$$\dot{\underline{x}}(t) = \underline{Ax}(t) + \underline{Bu}(t) \tag{1}$$

where

x(t) is an n-vector called the state of the plant,

u(t) is an r-vector called the control function,

A is a constant nxn matrix,

B is a constant nxr matrix, and

n is the order of the plant.

The performance index is defined by

$$I = \int_{t_o}^{t_f} L\,(\underline{x},\ \underline{u}^o, t)\ dt \tag{2}$$

where u^o denotes the optimal control function.

The Hamiltonian is defined using eq. (1) and (2).

$$H(\underline{x},\underline{p},\underline{u},t) = L(\underline{x},\underline{u},t) + \langle \dot{\underline{x}},\underline{p} \rangle \tag{3}$$

Where $\langle \underline{x},\underline{p} \rangle$ indicates the scalar product of vectors $\underline{x}$ and $\underline{p}$, the $\underline{p}$ is an n-vector called the co-state. Substituting equation (1) into (3) gives

$$H(\underline{x},\underline{p},\underline{u},t) = L(\underline{x},\underline{u},t) + \langle \underline{Ax},\underline{p} \rangle + \langle \underline{Bu},\underline{p} \rangle \tag{4}$$

or expressed in another form gives

$$H = L(\underline{x},\underline{u},t) + \langle \underline{x},\underline{A}'\underline{p} \rangle + \langle \underline{u},\underline{B}'\underline{p} \rangle \qquad (5)$$

where $\underline{A}'$ and $\underline{B}'$ are the transposed matrices $\underline{A}$ and $\underline{B}$, respectively.

The canonical equations are defined as

$$\frac{\partial H}{\partial x_i} = -\dot{p}_i \qquad i = 1,2,\ldots,n \qquad (6)$$

or

$$\frac{\partial H}{\partial \underline{x}} = -\dot{\underline{p}}$$

and

$$\frac{\partial H}{\partial p_i} = \dot{x}_i \qquad i = 1,2,\ldots,n \qquad (7)$$

or

$$\frac{\partial H}{\partial \underline{p}} = \dot{\underline{x}}$$

Equation (6) and (7) represent a total of 2n equations where n is the order of the plant equation (1).

Equation (6) defines mathematically the co-state vector $\underline{p}(t)$. The vector $\underline{p}(t)$, which is the solution of equation (6), depends:

a) On the equation of the plant due to the presence of $\underline{A}'$;

b) On the performance function L.

The above theory briefly describes the definitions and equations used in this paper.

APPENDIX B

A. Numerical Data

The following pages 44-62 consist of data compiled by a computer and used in plotting the graphs of Chapter IV.

UUTPUT VECTOR ***(T,E)***

0.50	0.348660000E 05
1.00	0.144900000E 04
1.50	0.232370377E 03
2.00	0.630000000E 02
2.50	0.224928017E 02
3.00	0.951852179E 01
3.50	0.454203200E 01
4.00	0.239062500E 01
4.50	0.138058189E 01
5.00	0.878399938E 00
5.50	0.618847452E 00
6.00	0.481481437E-00
6.50	0.408049710E-00
7.00	0.368953399E-00
7.50	0.348562934E-00
8.00	0.338378906E-00
8.50	0.333680093E-00
9.00	0.331809223E-00
9.50	0.331263781E-00
10.00	0.331199944E-00
10.50	0.331153989E-00
11.00	0.330882728E-00
11.50	0.330272317E-00
12.00	0.329282403E-00
12.50	0.327914447E-00
13.00	0.326192409E-00
13.50	0.324151367E-00
14.00	0.321830064E-00
14.50	0.319268346E-00
15.00	0.316503674E-00
15.50	0.313571036E-00
16.00	0.310501099E-00
16.50	0.307322681E-00
17.00	0.304059565E-00
17.50	0.300733566E-00
18.00	0.297363281E-00
18.50	0.293964535E-00
19.00	0.290550798E-00
19.50	0.287134320E-00
20.00	0.283724815E-00
20.50	0.280330837E-00
21.00	0.276958942E-00

21.50	0.273615539E-00
22.00	0.270305604E-00
22.50	0.267032713E-00
23.00	0.263800979E-00
23.50	0.260612130E-00
24.00	0.257468969E-00
24.50	0.254372597E-00
25.00	0.251324803E-00
25.50	0.248326391E-00
26.00	0.245377660E-00
26.50	0.242478997E-00
27.00	0.239630759E-00
27.50	0.236832917E-00
28.00	0.234084815E-00
28.50	0.231386721E-00
29.00	0.228737623E-00
29.50	0.226137340E-00
30.00	0.223585054E-00
30.50	0.221080184E-00
31.00	0.218622237E-00
31.50	0.216209918E-00
32.00	0.213842869E-00
32.50	0.211519942E-00
33.00	0.209240451E-00
33.50	0.207003862E-00
34.00	0.204808667E-00
34.50	0.202654362E-00
35.00	0.200540051E-00
35.50	0.198465019E-00
36.00	0.196428105E-00
36.50	0.194428787E-00
37.00	0.192465976E-00
37.50	0.190539077E-00
38.00	0.188647077E-00
38.50	0.186789244E-00
39.00	0.184965029E-00
39.50	0.183173224E-00
40.00	0.181413278E-00

OUTPUT VECTOR ***(T,S)***

1.00	0.338753562E 02
2.00	0.112030919E 02
3.00	0.638097447E 01
4.00	0.471163034E 01
5.00	0.405416465E 01
6.00	0.375720006E 01
7.00	0.360049379E 01
8.00	0.350828055E 01
9.00	0.345406929E 01
10.00	0.343331569E 01
11.00	0.349629107E 01
12.00	0.370126912E 01
13.00	0.404761040E 01
14.00	0.451474118E 01
15.00	0.508493978E 01
16.00	0.574658138E 01
17.00	0.649234319E 01
18.00	0.731755370E 01
19.00	0.821915495E 01
20.00	0.919508100E 01
21.00	0.102439023E 02
22.00	0.113645985E 02
23.00	0.125564440E 02
24.00	0.138183986E 02
25.00	0.151515522E 02
26.00	0.165541003E 02
27.00	0.180262797E 02
28.00	0.195679638E 02
29.00	0.211789665E 02
30.00	0.228591921E 02
31.00	0.246085563E 02
32.00	0.264269991E 02
33.00	0.283144426E 02
34.00	0.302708712E 02
35.00	0.322962332E 02
36.00	0.343905029E 02
37.00	0.365536618E 02
38.00	0.387857299E 02
39.00	0.410866194E 02
40.00	0.434564300E 02

OUTPUT VECTOR ***(T1,X1,X2,X3,X4)***

0.00	0.100000000E 01	0.100000000E 01	-0.100000000E 01	-0.870000000E 02
0.10	0.108256499E 01	0.546750009E 00	-0.728999996E 01	-0.405000005E 02
0.20	0.109568000E 01	-0.319999978E-00	-0.944000006E 01	-0.420000076E 01
0.30	0.101699500E 01	-0.123724997E 01	-0.847000027E 01	0.218999987E 02
0.40	0.855360024E 00	-0.194400007E 01	-0.540000033E 01	0.377999992E 02
0.50	0.640625015E 00	-0.228125009E 01	-0.125000191E 01	0.434999962E 02
0.60	0.413440041E-00	-0.219200021E 01	0.295999908E 01	0.390000000E 02
0.70	0.215055011E-00	-0.172124997E 01	0.620999956E 01	0.243000050E 02
0.80	0.771200433E-01	-0.101600075E 01	0.748000050E 01	-0.599990845E 00
0.90	0.114850923E-01	-0.325249538E-00	0.575000191E 01	-0.356999855E 02
1.00	-0.894069672E-07	0.372529030E-07	0.000000000E-39	-0.809999847E 02
0.00	0.100000000E 01	0.100000000E 01	-0.100000000E 01	-0.120000000E 02
0.10	0.109315312E 01	0.846093752E 00	-0.201874998E 01	-0.843750000E 01
0.20	0.116639999E 01	0.607500002E 00	-0.269999999E 01	-0.525000006E 01
0.30	0.121289687E 01	0.316093765E-00	-0.308125001E 01	-0.243750012E 01
0.40	0.122880000E 01	0.223517418E-07	-0.319999996E 01	-0.238418579E-06
0.50	0.121289063E 01	-0.316406220E-00	-0.309375009E 01	0.206249976E 01
0.60	0.116620001E 01	-0.612499967E 00	-0.280000013E 01	0.374999964E 01
0.70	0.109163439E 01	-0.871406227E 00	-0.235625011E 01	0.506249952E 01
0.80	0.993600041E 00	-0.107999998E 01	-0.180000019E 01	0.599999976E 01
0.90	0.877628170E 00	-0.122890612E 01	-0.116875029E 01	0.656249976E 01
1.00	0.750000067E 00	-0.131249994E 01	-0.500000238E 00	0.675000000E 01
1.10	0.617371932E 00	-0.132890639E 01	0.168749928E-00	0.656250024E 01
1.20	0.486400038E-00	-0.128000003E 01	0.799999595E 00	0.600000048E 01
1.30	0.363365665E-00	-0.117140624E 01	0.135624981E 01	0.506250072E 01
1.40	0.253800109E-00	-0.101250011E 01	0.179999995E 01	0.375000095E 01
1.50	0.162109420E-00	-0.816406533E 00	0.209375024E 01	0.206250143E 01

1.60	0.912001282E-01	-0.600000158E 00	0.220000005E 01	0.190734863E-05
1.70	0.421032459E-01	-0.383906707E-00	0.208125067E 01	-0.243749762E 01
1.80	0.136000067E-01	-0.192499831E-00	0.170000052E 01	-0.524999714E 01
1.90	0.184698403E-02	-0.539064854E-01	0.101875114E 01	-0.843749619E 01
2.00	-0.149011612E-07	-0.134110451E-06	0.476837158E-06	-0.119999952E 02
0.00	0.100000000E 01	0.100000000E 01	-0.100000000E 01	-0.322222218E 01
0.10	0.109449400E 01	0.885126539E 00	-0.128530863E 01	-0.249259254E 01
0.20	0.117619358E 01	0.745283954E 00	-0.150024690E 01	-0.181481479E 01
0.30	0.124294499E 01	0.587250009E 00	-0.164999998E 01	-0.118888891E 01
0.40	0.129324642E 01	0.417283967E-00	-0.173975305E 01	-0.614814818E 00
0.50	0.132619599E 01	0.241126567E-00	-0.177469133E 01	-0.925925970E-01
0.60	0.134143999E 01	0.640000328E-01	-0.176000001E 01	0.377777725E-00
0.70	0.133912104E 01	-0.109391928E-00	-0.170086420E 01	0.796296209E 00
0.80	0.131982619E 01	-0.274364152E-00	-0.160246915E 01	0.116296288E 01
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1.10	0.117153420E 01	-0.689688221E 00	-0.112358034E 01	0.195185176E 01
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1.50	0.828125104E 00	-0.968749955E 00	-0.250000119E-00	0.227777782E 01
1.60	0.730378374E 00	-0.982419655E 00	-0.241976380E-01	0.222962973E 01
1.70	0.632383510E 00	-0.973836407E 00	0.194197357E-00	0.212962982E 01
1.80	0.536320105E 00	-0.943999961E 00	0.399999857E-00	0.197777811E 01
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2.00	0.358024851E-00	-0.827160493E 00	0.753086269E 00	0.151851895E 01
2.10	0.279315174E-00	-0.744750112E 00	0.889999926E 00	0.121111169E 01
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2.30	0.149543971E-00	-0.547317892E 00	0.105864203E 01	0.440741569E-00
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2.50	0.615354776E-01	-0.332947463E-00	0.105246949E 01	-0.537036032E 00
2.60	0.333907604E-01	-0.231308639E-00	0.970864534E 00	-0.110370263E 01
2.70	0.149051845E-01	-0.140749961E-00	0.830000639E 00	-0.172222075E 01
2.80	0.466582179E-02	-0.674567223E-01	0.624692082E 00	-0.239259133E 01
2.90	0.615686178E-03	-0.181327760E-01	0.349754214E-00	-0.311481318E 01
3.00	0.238418579E-06	0.298023224E-06	0.834465027E-06	-0.388888726E 01
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0.30	0.125148524E 01	0.667120121E 00	-0.118949218E 01	-0.343359388E-00
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0.60	0.139713436E 01	0.302546896E-00	-0.121656249E 01	0.145312458E-00
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1.80	0.102736570E 01	-0.735453114E 00	-0.333437622E-00	0.104531249E 01
1.90	0.952327535E 00	-0.763567403E 00	-0.228867307E-00	0.104414064E 01
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2.40	0.563200086E 00	-0.752000093E 00	0.259999901E-00	0.862500072E 00
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2.80	0.291600103E-00	-0.587250024E 00	0.539999902E 00	0.506250203E 00
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3.80	0.208413601E-02	-0.304531455E-01	0.288437963E-00	-0.120468700E 01
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0.40	0.131940894E 01	0.597219847E 00	-0.997721598E 00	0.122687993E-00
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0.70	0.145440564E 01	0.305750661E-00	-0.938019201E 00	0.270431984E-00
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1.50	0.142859504E 01	-0.325849950E-00	-0.607600056E 00	0.520799965E 00
1.60	0.139305957E 01	-0.383976921E-00	-0.554662459E 00	0.537407957E 00
1.70	0.135197873E 01	-0.436732531E-00	-0.500227250E 00	0.550751969E 00
1.80	0.130589661E 01	-0.483983323E-00	-0.444620878E-00	0.560831979E 00
1.90	0.125536598E 01	-0.525628537E 00	-0.388169676E-00	0.567647971E 00
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2.20	0.108277810E 01	-0.616412148E 00	-0.217011288E-00	0.568511993E 00
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2.40	0.955909520E 00	-0.648527384E 00	-0.104665704E-00	0.552767992E 00
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2.80	0.693874896E 00	-0.647708237E 00	0.104051113E-00	0.482112039E-00
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3.40	0.339817345E-00	-0.508969113E 00	0.338022336E-00	0.278208081E-00
3.50	0.290655106E-00	-0.473850161E-00	0.363599911E-00	0.232800102E-00
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3.70	0.203435153E-00	-0.397123128E-00	0.400316745E-00	0.132192111E-00
3.80	0.165744245E-00	-0.356521189E-00	0.410803169E-00	0.769921308E-01
3.90	0.132156640E-00	-0.315151989E-00	0.415606320E-00	0.185281402E-01
4.00	0.102720022E-00	-0.273600221E-00	0.414399907E-00	-0.431998307E-01
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4.30	0.388731957E-01	-0.154185593E-00	0.371459261E-00	-0.247967742E-00
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4.50	0.150849223E-01	-0.858502686E-01	0.306800142E-00	-0.400799654E-00
4.60	0.796377659E-02	-0.573083758E-01	0.262681767E-00	-0.482111596E-00
4.70	0.346255302E-02	-0.335903168E-01	0.210269049E-00	-0.566687576E 00
4.80	0.105673075E-02	-0.155419707E-01	0.149235532E-00	-0.654527508E 00
4.90	0.135540962E-03	-0.404155254E-02	0.792547613E-01	-0.745631479E 00
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0.30	0.125605437E 01	0.710718751E 00	-0.926250003E 00	0.268055551E-00
0.40	0.132254024E 01	0.619456798E 00	-0.898765430E 00	0.281481478E-00
0.50	0.138003953E 01	0.531008877E 00	-0.869984567E 00	0.293981478E-00
0.60	0.142884000E 01	0.445500016E-00	-0.840000004E 00	0.305555549E-00
0.70	0.146924137E 01	0.363045931E-00	-0.808904327E 00	0.316203699E-00
0.80	0.150155456E 01	0.283753105E-00	-0.776790135E 00	0.325925920E-00
0.90	0.152610062E 01	0.207718775E-00	-0.743750013E 00	0.334722213E-00
1.00	0.154320987E 01	0.135030888E-00	-0.709876560E 00	0.342592586E-00
1.10	0.155322099E 01	0.657681674E-01	-0.675262369E 00	0.349537030E-00
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1.40	0.154415753E 01	-0.120820954E-00	-0.567901261E 00	0.364814807E-00
1.50	0.152929689E 01	-0.175781205E-00	-0.531250030E 00	0.368055549E-00
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1.70	0.148400800E 01	-0.274639234E-00	-0.457206830E-00	0.371759255E-00
1.80	0.145432003E 01	-0.318499967E-00	-0.420000047E-00	0.372222219E-00
1.90	0.142043208E 01	-0.358639240E-00	-0.382793255E-00	0.371759255E-00
2.00	0.138271609E 01	-0.395061702E-00	-0.345679067E-00	0.370370369E-00
2.10	0.134154320E 01	-0.427781224E-00	-0.308750056E-00	0.368055556E-00
2.20	0.129728253E 01	-0.456820980E-00	-0.272098824E-00	0.364814814E-00
2.30	0.125030059E 01	-0.482213333E-00	-0.235817961E-00	0.360648148E-00

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2.60	0.109663022E 01	-0.536969140E 00	-0.130123518E-00	0.342592604E-00
2.70	0.104233944E 01	-0.548281267E 00	-0.962500647E-01	0.334722236E-00
2.80	0.987085521E 00	-0.556246936E 00	-0.632099360E-01	0.325925939E-00
2.90	0.931198716E 00	-0.560954109E 00	-0.310957432E-01	0.316203721E-00
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3.30	0.707585722E 00	-0.549281314E 00	0.862499326E-01	0.268055584E-00
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3.60	0.547840059E 00	-0.512000084E 00	0.159999937E-00	0.222222257E-00
3.70	0.497476429E-00	-0.494917154E-00	0.181373388E-00	0.205092628E-00
3.80	0.448925018E-00	-0.475784063E-00	0.200987592E-00	0.187037077E-00
3.90	0.402381927E-00	-0.454781353E-00	0.218749940E-00	0.168055605E-00
4.00	0.358024776E-00	-0.432098866E-00	0.234567836E-00	0.148148190E-00
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4.20	0.276480079E-00	-0.382500112E-00	0.259999961E-00	0.105555628E-00
4.30	0.239546716E-00	-0.356009781E-00	0.269428968E-00	0.828704499E-01
4.40	0.205305696E-00	-0.328691512E-00	0.276543170E-00	0.592593513E-01
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4.80	0.963199139E-01	-0.216000199E-00	0.280000016E-00	-0.444442816E-01
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5.00	0.586419702E-01	-0.161265671E-00	0.265432119E-00	-0.101851661E-00
5.10	0.438243151E-01	-0.135281444E-00	0.253750056E-00	-0.131944243E-00
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5.60	0.433373451E-02	-0.315063596E-01	0.147654563E-00	-0.296296004E-00
5.70	0.187045336E-02	-0.182814598E-01	0.116250321E-00	-0.331944127E-00
5.80	0.566720963E-03	-0.837665796E-02	0.812349170E-01	-0.368518170E-00
5.90	0.723004341E-04	-0.215786696E-02	0.425158143E-01	-0.406018171E-00
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0.30	0.125668362E 01	0.716805726E 00	-0.888376869E 00	0.367793176E-00
0.40	0.132398348E 01	0.629801869E 00	-0.851752251E 00	0.364669479E-00
0.50	0.138276555E 01	0.546444565E 00	-0.815448925E 00	0.361367282E-00
0.60	0.143339284E 01	0.466700792E-00	-0.779484741E 00	0.357886590E-00
0.70	0.147622499E 01	0.390535727E-00	-0.743877560E 00	0.354227401E-00
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3.70	0.952841610E 00	-0.406978935E-00	0.158598572E-01	0.110638645E-00
3.80	0.912241101E 00	-0.404851288E-00	0.265772343E-01	0.103734210E-00
3.90	0.871905953E 00	-0.401686251E-00	0.366117656E-01	0.969821923E-01
4.00	0.831936240E 00	-0.397551209E-00	0.459787697E-01	0.903825946E-01
4.10	0.792425871E 00	-0.392512202E-00	0.546933860E-01	0.839354061E-01
4.20	0.753461778E 00	-0.386633754E-00	0.627708882E-01	0.776406303E-01
4.30	0.715124965E 00	-0.379978865E-00	0.702265650E-01	0.714982748E-01
4.40	0.677489936E 00	-0.372608662E-00	0.770756304E-01	0.655083358E-01
4.50	0.640625119E 00	-0.364583403E-00	0.833333284E-01	0.596708171E-01
4.60	0.604593158E 00	-0.355961263E-00	0.890148580E-01	0.539857075E-01
4.70	0.569450796E 00	-0.346799105E-00	0.941355526E-01	0.484530143E-01
4.80	0.535249412E 00	-0.337152362E-00	0.987105519E-01	0.430727378E-01
4.90	0.502034783E 00	-0.327074647E-00	0.102755189E-00	0.378448814E-01
5.00	0.469847262E-00	-0.316618472E-00	0.106284618E-00	0.327694379E-01
5.10	0.438722074E-00	-0.305834442E-00	0.109314114E-00	0.278464034E-01
5.20	0.408689559E-00	-0.294771761E-00	0.111858964E-00	0.230757892E-01
5.30	0.379775345E-00	-0.283478320E-00	0.113934368E-00	0.184575990E-01
5.40	0.351999998E-00	-0.272000134E-00	0.115555555E-00	0.139918141E-01
5.50	0.325379908E-00	-0.260381818E-00	0.116737813E-00	0.967844576E-02
5.60	0.299926996E-00	-0.248666763E-00	0.117496327E-00	0.551749393E-02
5.70	0.275648415E-00	-0.236896217E-00	0.117846355E-00	0.150895491E-02
5.80	0.252548456E-00	-0.225110412E-00	0.117803201E-00	-0.234715641E-02
5.90	0.230625927E-00	-0.213348210E-00	0.117382020E-00	-0.605086610E-02
6.00	0.209876776E-00	-0.201646209E-00	0.116598129E-00	-0.960215181E-02
6.10	0.190293252E-00	-0.190040171E-00	0.115466669E-00	-0.130010285E-01
6.20	0.171864271E-00	-0.178563952E-00	0.114003003E-00	-0.162474774E-01
6.30	0.154575169E-00	-0.167250097E-00	0.112222269E-00	-0.193415210E-01
6.40	0.138407946E-00	-0.156129658E-00	0.110139757E-00	-0.222831480E-01

6.50	0.123341858E-00	-0.145231783E-00	0.107770696E-00	-0.250723586E-01
6.60	0.109353066E-00	-0.134584367E-00	0.105130404E-00	-0.277091451E-01
6.70	0.964154601E-01	-0.124214053E-00	0.102233976E-00	-0.301935263E-01
6.80	0.845003128E-01	-0.114145637E-00	0.990967751E-01	-0.325254910E-01
6.90	0.735756755E-01	-0.104402185E-00	0.957339406E-01	-0.347050391E-01
7.00	0.636082888E-01	-0.950059391E-01	0.921608210E-01	-0.367321707E-01
7.10	0.545620322E-01	-0.859766006E-01	0.883926153E-01	-0.386068784E-01
7.20	0.464000702E-01	-0.773333311E-01	0.844445229E-01	-0.403291844E-01
7.30	0.390821099E-01	-0.690932870E-01	0.803318322E-01	-0.418990664E-01
7.40	0.325673819E-01	-0.612719655E-01	0.760698020E-01	-0.433165394E-01
7.50	0.268129706E-01	-0.538838506E-01	0.716736317E-01	-0.445815884E-01
7.60	0.217759609E-01	-0.469411612E-01	0.671585798E-01	-0.456942283E-01
7.70	0.174100995E-01	-0.404555202E-01	0.625398755E-01	-0.466544442E-01
7.80	0.136688948E-01	-0.344361663E-01	0.578327477E-01	-0.474622585E-01
7.90	0.105065703E-01	-0.288913846E-01	0.530525446E-01	-0.481176414E-01
8.00	0.787472725E-02	-0.238276720E-01	0.482144058E-01	-0.486206077E-01
8.10	0.572514534E-02	-0.192499161E-01	0.433335006E-01	-0.489711650E-01
8.20	0.400829315E-02	-0.151617527E-01	0.384252071E-01	-0.491693206E-01
8.30	0.267601013E-02	-0.115654469E-01	0.335046947E-01	-0.492150523E-01
8.40	0.167894363E-02	-0.846076012E-02	0.285872817E-01	-0.491083674E-01
8.50	0.967741013E-03	-0.584745407E-02	0.236880481E-01	-0.488492586E-01
8.60	0.493764877E-03	-0.372231007E-02	0.188224614E-01	-0.484377407E-01
8.70	0.207424164E-03	-0.208091736E-02	0.140056312E-01	-0.478738137E-01
8.80	0.612735748E-04	-0.918865204E-03	0.925281644E-02	-0.471574478E-01
8.90	0.810623169E-05	-0.228166580E-03	0.457921624E-02	-0.462886803E-01
9.00	0.476837158E-06	-0.000000000E-39	0.208616257E-06	-0.452675037E-01

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