ON SPLINE FUNCTIONS AND THEIR APPLICATIONS IN INTERPOLATION AND APPROXIMATION THEORY

A Thesis

Presented in Partial Fulfillment of the Requirements
for the Degree Master of Science

by

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1966

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ACKNOWLEDGMENT

Sincere appreciation is expressed to Professor Bojanic for guidance, encouragment and patience in the preparation of this thesis. It was a pleasure to be a student in his series of courses in Real Analysis and in Approximation Theory.

Also, I am indebted to Professor Schoenberg for his visiting lectures at The Ohio State University which provided the initial inspiration for this thesis topic.

To North American Aviation, Inc., Columbus Division, I express gratitude for their employee "Work-Study Program" which encouraged and made financially possible my graduate studies.

Finally, I thank my wife who did not understand the "Greek" I was writing, but did understand me and gave her loving support.

CONTENTS

INTRODUCTION

A spline function f of degree n, or simply an n-spline on $\mathbb{R}$ is any function whose graph consists of arcs of polynomials of degree $\leqq$ n connected so that the (n - 1)st derivative of f is continuous on $\mathbb{R}$. The simplest spline functions are polygonal lines (see Figure 1). As it is well known, any continuous function on $\mathbb{R}$ can be approximated uniformly on $\mathbb{R}$ by a polygonal line.

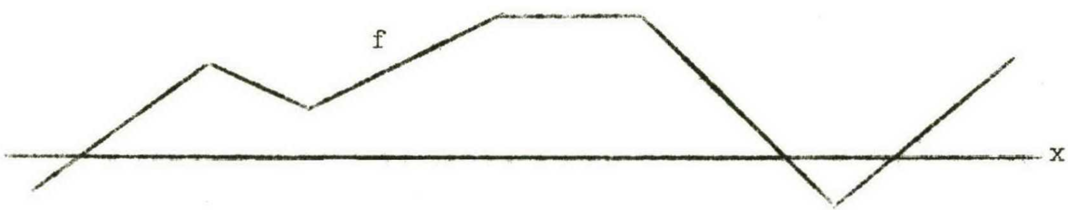


Figure 1. A spline function f of degree 1 on $\mathbb{R}$.

An adjustable curve used by draftsmen is called a "spline". It consists of an elastic rod to which weights are attached to hold the spline in place (see Figure 2). The mechanical spline can be bent to approximate any desired curve, limited only by the elasticity of the material. This instrument gave its name to the class of functions discussed here.

The name <u>spline function</u> was introduced by I. J. Schoenberg in 1944 who systematically studied functions of this type as a tool for the approximation of functions. They were suggested by the work of T. N. E. Greville and other actuarial writers in connection with problems of osculatory interpolation.

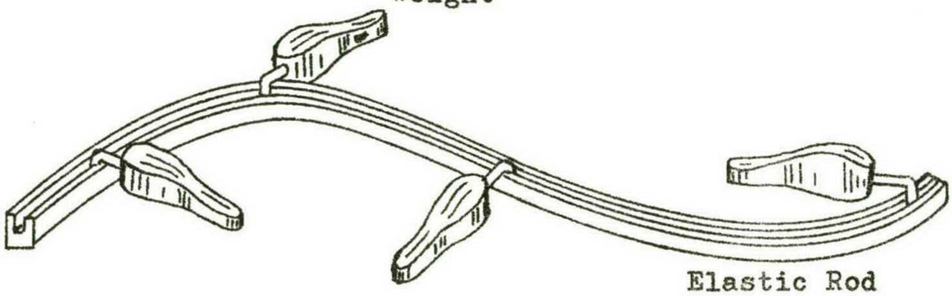


Figure 2. A Mechanical Spline Used by Draftsmen.

For purposes of interpolation the use of spline functions offers substantial advantages. By employing polynomials of relatively low degree one can often avoid the marked undulatory behavior that commonly arises from fitting a single polynomial exactly to a large number of empirical observations [1]. On the other hand, much greater smoothness is obtained by using spline functions instead of the traditional piecewise interpolation procedures, which give rise to discontinuities in the first derivative [1]. A spline function provides continuity of the greatest possible number of derivatives of the interpolating function consistent with the use of polynomials of lower degree than would be required to fit all data points exactly by a single polynomial [1, quoting Schoenberg].

In a series of fundamental papers ([2], [3], [4] and [5]) Schoenberg shows that the fundamental role played by polynomial interpolation in elementary numerical analysis is taken over by spline interpolation, and that the resulting formulae are best in a certain sense.

The procedures for interpolation by means of spline functions are facilitated by Schoenberg's unique representation of an arbitrary spline function as a sum of a linear combination of "elementary spline functions" of particularly simple form [1].

The appearance of spline functions and their use in interpolation and approximation of functions began with Schoenberg's work in 1944 and his published paper in 1946 [2]. Both the theory and applications of spline functions have been developed with increasing intensity in the last four years. The theory now includes the finite and periodic cases; functions of several variables; arbitrarily spaced, multiple (coalescent) abscissae, etc. The role of spline functions of one or several variables in the broad realm of the numerical Analysis of Engineering and mathematical Physics is discussed by Birkoff and deBoor in [6].

This thesis will follow the work of Schoenberg [3], [4] and [5]. We shall consider only spline functions of one real variable with simple, arbitrarily spaced abscissae at which the ordinate is specified.

The first chapter will give definitions and basic properties of the "fundamental spline functions." Its main point will be Schoenberg's representation theorem for arbitrary spline functions by fundamental spline functions.

In the second chapter the existence and uniqueness of the interpolating spline function is shown. This "spline fit" arose from the need in numerical analysis for a method of interpolation which produces derivatives as smooth as possible [10]. Interpolation by spline functions, being essentially the numerical analogue of the draftsman's spline, consists of joining the assigned n points by section of polynomials of degree $\leqq 2m - 1$, requiring that the first $2m - 2$ derivatives be continuous at the junction points.

Letting S be an interpolating function, the quality of approximation is measured by

$$\int_a^b (S^{(m)}(x))^2 \, dx$$

where m is a positive integer. It will be shown in the concluding section that the interpolating spline function minimizes this integral.

We remark here that if the number of points of interpolation is n and $n < m$, then $n \leqq m - 1$ and S is a polynomial of degree $\leqq m - 1$. In this case the integral is zero. Thus, we shall assume that $m \leqq n$.

CHAPTER I

DEFINITIONS AND REPRESENTATION THEOREMS

I.1. Definitions and examples of spline functions

The precise definition of a spline function given by Schoenberg in [3] is as follows:

Definition 1. Let

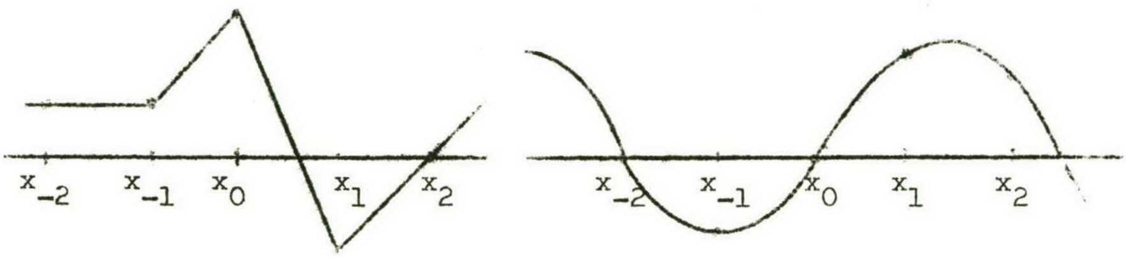
$$(1.1) \qquad \cdots < x_{-2} < x_{-1} < x_0 < x_1 < x_2 < \cdots$$

be a sequence of real numbers, and let n be a natrual number ≥ 1. By a spline function $S(x,n)$, of degree n, having knots (1.1), we mean a function of the class $C^{n-1}(-\infty,\infty)$, such that in each interval $(x_\nu, x_{\nu+1})$ it reduces to a polynomial of degree not exceeding n. $S(x,n)$ will also be referred to as an n-spline.

When the degree of the spline function is clear, we shall use the simpler notation $S(x)$ instead of $S(x,n)$.

Definition 1 shows that an $S(x,1)$ is a continuous broken linear function with possible corners at some or all of the points (1.1). Likewise, an $S(x,2)$ has a graph composed of a sequence of parabolas which join at the knots continuously together with their slopes [4]. These are illustrated in Figure 3.

The simplest n-spline functions are generated by the



(a) A 1-spline S(x,1). (b) A 2-spline S(x,2).
Figure 3. Spline functions.

trincated power function x_+^n defined as follows:

$$x_+^n = \begin{cases} x^n , & 0 \leqq x \\ 0 , & x < 0 \end{cases}$$

where n is a positive integer. The differentiation rule for this function is similar to that for ordinary powers:

$$\frac{d}{dx} x_+^n = (n - 1)x_+^{n-1} .$$

The nth derivative of x_+^n is $n!x_+^0$, where x_+^0 is taken to be the Heaviside function, defined as 1 for positive x and 0 for negative x.

A function of the form $(x - c)_+^n$, where c is a real number, will be called an elementary spline function. The nth derivative of this function has its only discontinuity at x = c, where there is a jump of magnitude n! [1].

The first two special cases of elementary spline functions are shown in Figure 4.

The operation of differentiation (or integration) transforms a spline into another spline of a degree decreased (or increased) by one.

(a) Graph of $(x - c)_+^1$ (b) Graph of $(x - c)_+^2$

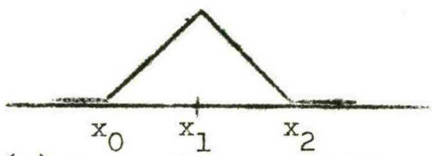
Figure 4. Graphs of the function $(x - c)_+^n$ for n = 1 and 2.

The n-splines with fixed knots (1.1) form a linear space $\mathcal{S}_n$. Although the knots (1.1) may be points of discontinuities for $S^{(n)}(x,n)$, they need not be. Hence, $\mathcal{S}_n$ contains the family $\mathcal{P}_n$ of polynomials of degree $\leqq n$ of which $\mathcal{S}_n$ is a generalization [4], [5].

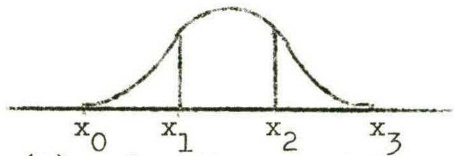
We shall consider now a spline function $S(x,n)$ vanishing outside the range $(x_\nu, x_{\nu+N})$ but not outside any subrange. We say in this case that $S(x,n)$ has the <u>span</u> N [4]. Clearly for n = 1 we have $N \geqq 2$. It will be shown in Theorem 1 that in general $N \geqq n + 1$ for all n.

The spline functions of this type with the smallest possible span, i.e., for which $N = n + 1$, play an important role in the theory of spline functions. We shall see next that these spline functions coincide with the so-called <u>fundamental</u> <u>n-splines</u> [4] (or <u>B-splines</u>, where B stands for "basis" [5]). Figure 5 illustrates spline functions with finite spans.

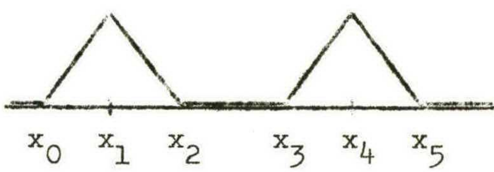
To define the fundamental n-splines we first consider



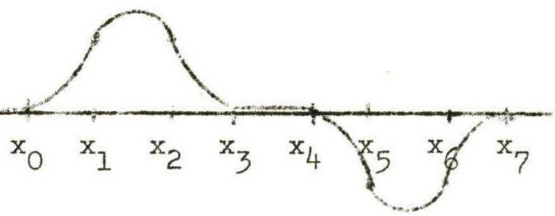
(a) A 1-spline vanishing outside $[x_0, x_2]$, with minimal span $N = 2$.



(b) A 2-spline vanishing outside $[x_0, x_3]$, with minimal span $N = 3$.



(c) A 1-spline vanishing outside $[x_0, x_5]$, with span $N = 5$.



(d) A 2-spline vanishing outside $[x_0, x_7]$, with span $N = 7$.

Figure 5. Spline functions with finite spans.

the elementary spline function M, defined as follows:

Definition 2. The function M is defined by

$$M(x,n;y) = (n + 1)(y - x)_+^n , \tag{1.2}$$

so that M is a polynomial of degree n for $x < y$ and vanishes for all $x \geqq y$, and $M \in C^{n-1}$. Hence, M is an n-spline.

The coefficient $n + 1$ has been employed by Schoenberg [3] to normalize the function for other properties, and will be retained here; although the properties discussed herein do not depend upon this coefficient.

It should be noted that

$$(x_\nu - x)^k = (x_\nu - x)_+^k + (-1)^k (x - x_\nu)_+^k \tag{1.3}$$

for all k = 0, 1, ..., and $\nu = 0, \pm 1, \pm 2, \ldots$. Therefore, we may, without loss of generality, develop the fundamental splines in terms of either $(x_\nu - x)_+$ or $(x - x_\nu)_+$.

Definition 3. The fundamental spline function $M_\nu(x,n;x_\nu,\ldots,x_{\nu+n+1})$ of degree n is defined to be the (n + 1)-st order divided difference of the function M(x,n;y) with respect to the variable y, and based on the points $x_\nu, x_{\nu+1}, \ldots, x_{\nu+n+1}$ [4].

Using the general formula for divided differences in [7], i.e.,

$$\Delta^{n+1}f(y) = f[x_\nu,\ldots,x_{\nu+n+1}] = \sum_{i=\nu}^{\nu+n+1} \frac{f(x_i)}{\omega'(x_i)},$$

where $\omega(x) = (x - x_\nu)\cdots(x - x_{\nu+n+1})$, we see that the fundamental spline $M_\nu(x,n;x_\nu,\ldots,x_{\nu+n+1})$ is defined by

$$(1.4) \qquad M_\nu(x,n;x_\nu,\ldots,x_{\nu+n+1} = \sum_{i=\nu}^{\nu+n+1} \frac{(n+1)(x_i - x)_+^n}{\omega'(x_i)}.$$

The function M_ν is clearly an n-spline function with the n + 2 knots $x_\nu, \ldots, x_{\nu+n+1}$ since it is a linear combination of n-splines, $(x_\nu - x)_+^n, \ldots, (x_{\nu+n+1} - x)_+^n$.

It is also easy to see that M_ν vanishes outside $[x_\nu, x_{\nu+n+1}]$. If $x < x_\nu$, we may omit the subscript "+" on the right hand side of (1.4). Thus, for $x < x_\nu$ we have

$$M_\nu(x,n;x_\nu,\ldots,x_{\nu+n+1}) = \sum_{i=\nu}^{\nu+n+1} \frac{(n+1)(x_i - x)^n}{\omega'(x_i)}$$

$$= \Delta^{n+1}((n+1)(y-x)^n) = 0$$

since the (n + 1)-st divided difference of a polynomial of degree n is zero. On the other hand, if $x > x_{\nu+n+1}$, we have $(x_i - x)_+^n = 0$, $i = \nu, \ldots, \nu + n + 1$, and so

$$M_\nu(x,n;x_\nu,\ldots,x_{\nu+n+1}) = 0.$$

This shows that the span of the fundamental n-spline M_ν is clearly of minimal length n + 1.

The notation will sometimes be simplified by

$$M_\nu(x,n) = M_\nu(x,n;x_\nu,\ldots,x_{\nu+n+1}) .$$

The derivatives of M_ν are given by

$$M_\nu^{(k)}(x,n;x_\nu,\ldots,x_{\nu+n+1}) =$$

$$(n + 1)n\cdots(n + 1 - k)(-1)^k \sum_{i=\nu}^{\nu+n+1} \frac{(x_i - x)_+^{n-k}}{\omega'(x_i)}$$

for $k = 1, \ldots, n$.

Explicit expressions and graphs of the fundamental n-splines will be given for n = 1 and 2, assuming that $x_i = i$, $i = 0, 1, \ldots$.

<u>Example</u>. Let $n = 1$, $x_0 = 0$, $x_1 = 1$, and $x_2 = 2$, then we have

$$M_0(x,1;0,1,2) = \sum_{i=0}^{2} \frac{2(x_i - x)_+}{\omega'(x_i)} ,$$

where $\omega(x) = x(x - 1)(x - 2)$. Thus

$$M_0(x,1;0,1,2) = \begin{cases} 0 & \text{if } x < 0 \\ x & \text{if } 0 \leqq x < 1 \\ -(x - 2) & \text{if } 1 \leqq x < 2 \\ 0 & \text{if } 2 \leqq x \end{cases}$$

The graph of this fundamental 1-spline is shown in Figure 6.

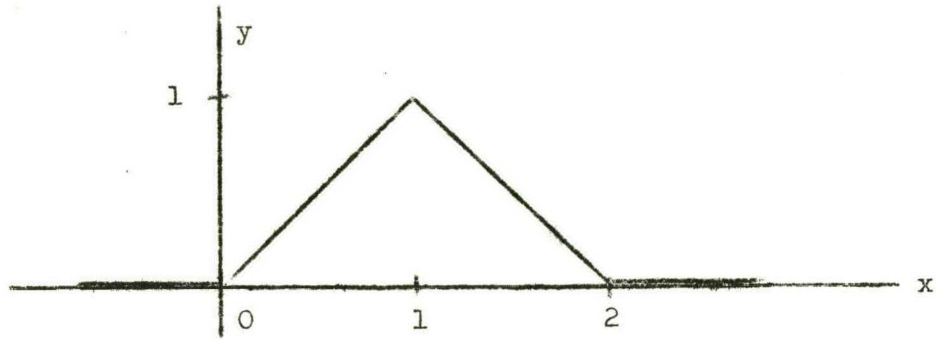


Figure 6. Graph of the fundamental 1-spline $M_0(x,1;0,1,2)$.

Example. Let $n = 2$, and $x_i = i$ for $i = 0, 1, 2, 3$, then we have

$$M_0(x,2;0,1,2,3) = \sum_{i=0}^{3} \frac{3(x_i - x)_+^2}{\omega'(x_i)},$$

where $\omega(x) = x(x - 1)(x - 2)(x - 3)$. Each term in the expansion of this fundamental 2-spline is al a 2-spline as illustrated in Figure 7.

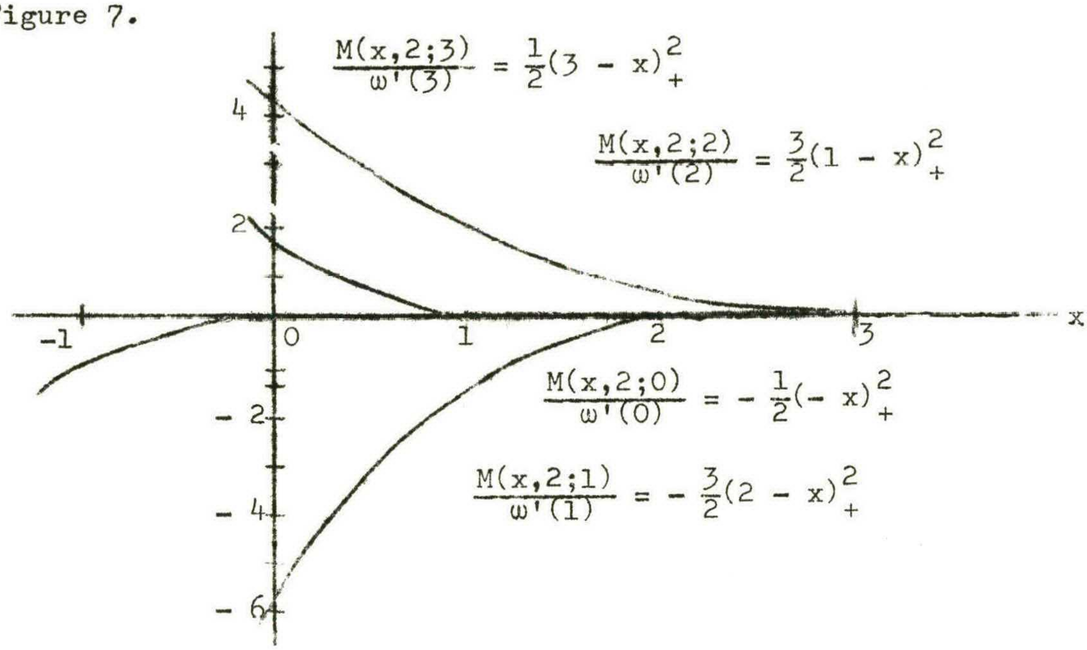


Figure 7. Graph of the 2-splines composing $M_0(x,2;0,1,2,3)$.

Combining the functions shown in Figure 7 gives

$$M_0(x,2;0,1,2,3) = \begin{cases} 0 & \text{if } x < 0 \\ \frac{1}{2}x^2 & \text{if } 0 \leqq x < 1 \\ -x^2 + 3x - \frac{3}{2} & \text{if } 1 \leqq x < 2 \\ \frac{1}{2}x^2 - 3x + \frac{9}{2} & \text{if } 2 \leqq x < 3 \\ 0 & \text{if } 3 \leqq x \end{cases}$$

The graph of this fundamental 2-spline is shown in Figure 8.

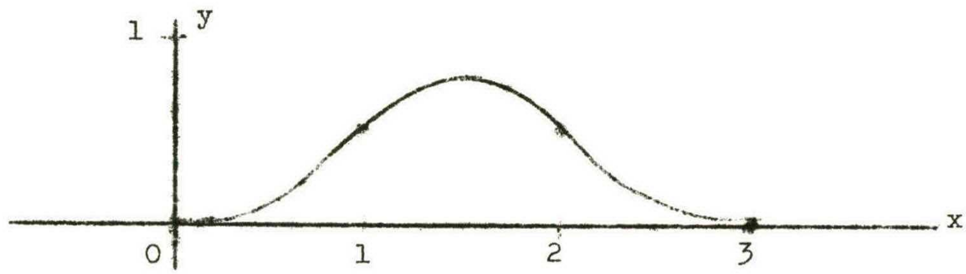


Figure 8. Graph of the fundamental 2-spline $M_0(x,2;0,1,2,3,)$.

The fundamental spline functions of higher orders will be similar to the illustrated fundamental 2-spline, except of course the polynomial arcs between nodes will be of correspondingly higher degree.

It is useful to observe that the fundamental splines on simple, equidistant knots over $[a,b]$ is symmetric with respect to the bisector $x = \frac{a+b}{2}$.

For the discussion of spline interpolation we shall need to consider the set of all spline functions of degree 2 - 1 having the knots $x_0 < \dots < x_n$. This set will be denoted by $\mathscr{S}_{2m-1}(x_0,\dots,x_n)$. We shall also need the following

Definition 4. A function S is called a natural spline function of degree $2m - 1$ provided that

(a) $S(x,2m-1) \in \mathscr{S}_{2m-1}(x_0,\ldots,x_n)$ and

(b) $S(x,2m-1) \in \mathscr{P}_{m-1}$ in $(-\infty,x_0) \cup (x_n,+\infty)$ [5].

The class of such natural spline functions is denoted by $\mathscr{S}^*_{2m-1}(x_0,\ldots,x_n)$, and clearly $\mathscr{S}^*_{2m-1}(x_0,\ldots,x_n) \subset \mathscr{S}_{2m-1}(x_0,\ldots,x_n)$.

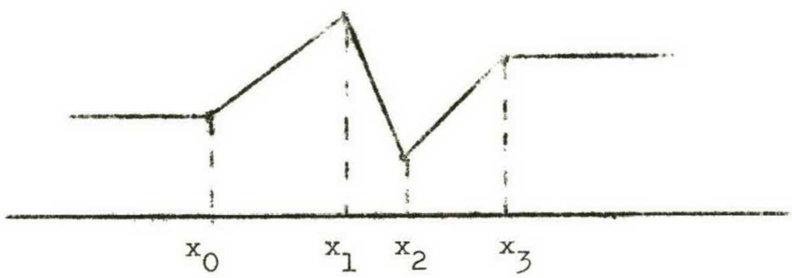


Figure 9. Graph of a natural spline function $S(x,1) \in \mathscr{S}^*_1(x_0,\ldots,x_3)$.

I.2. Representation of arbitrary spline functions by fundamental spline functions

The fundamental n-splines will be shown to form a basis for all spline functions of degree n and given knots [3]. Thus, the functions do indeed merit their name from this fundamental role. The discussion will follow Schoenberg's paper [3].

The proof of the representation theorem is based on the following elementary lemma.

Lemma 1. If P is a polynomial of degree $\leqq$ n, and

$$P^{(k)}(c) = 0, \quad k = 0, 1, \ldots, n-1$$

then

$$P(x) = A(x-c)^n .$$

Proof. Since $P^{(k)}(c) = 0$, $k = 0, 1, \ldots, n-1$, the polynomial has a zero of degree $m \geqq n$ at c. Thus

$$P(x) = (x - c)^m Q(x)$$

where Q is a polynomial such that

$$m + \deg Q \leqq n \ .$$

Since $m \geqq n$, it follows that $\deg Q = 0$, i.e.

$$Q(x) = A \quad \text{for all } x \in \mathbb{R},$$

and the lemma is proved.

For simplicity, the representation of arbitrary spline functions by fundamental spline functions is discussed in four theorems, each increasing the span of the spline functions of the preceding case.

Theorem 1. If $0 < N \leqq n$ and if S is an n-spline having the knots $x_0 < x_1 < \ldots < x_N$, and such that

$$S(x,n) = 0 \quad \text{everywhere outside the interval } (x_0,x_N),$$

then

$$S(x,n) = 0 \quad \text{for all } x.$$

Proof. On each of the subintervals $[x_0,x_1), \ldots, [x_{N-1},x_N)$ the graph of S is a polynomial of degree $\leqq n$ and $S \in C^{n-1}(-\infty,\infty)$. By hypothesis, the n-spline S vanishes for all $x \leqq x_0$, i.e.,

$$S(x) = 0, \quad x \leqq x_0 \ .$$

Thus

$$S^{(k)}(x) = 0, \quad \text{for all } x < x_0,\ k = 0, 1, 2, \ldots .$$

The continuity condition, $S \in C^{n-1}(-\infty,\infty)$, requires that

$$S^{(k)}(x_0) = 0, \quad k = 0, 1, \ldots, n-1. \qquad (1.5)$$

Since on $[x_0, x_1]$ S coincides with a polynomial of degree $\leqq n$, then from (1.5) it follows, by Lemma 1, that S is of the form

$$S(x) = a_0(x - x_0)^n , \quad x \in [x_0, x_1],$$

where a_0 is a constant.

Proceeding to the next interval $[x_1, x_2]$, the continuity condition at x_1 requires that

$$S(x_1) = a_0(x_1 - x_0)^n,$$

and

$$S^{(k)}(x_1) = a_0(n)\cdots(n + 1 - k)(x_1 - x_0)^{n-k},$$

$k = 1, 2, \ldots, n - 1$. Let

$$P(x) = S(x) - a_0(x - x_0)^n, \quad x \in [x_1, x_2].$$

We have then, deg $P \leqq n$ and

$$P^{(k)}(x_1) = 0, \quad k = 0, 1, \ldots, n - 1.$$

Thus, by Lemma 1, we have

$$P(x) = a_1(x - x_1)^n,$$

i.e.,

$$S(x) - a_0(x - x_0)^n = a_1(x - x_1)^n,$$

and so

$$S(x) = a_0(x - x_0)^n + a_1(x - x_1)^n, \quad x \in [x_1, x_2].$$

Continuing this process for succeeding intervals gives,

$$(1.6) \qquad S(x) = \sum_{\nu=0}^{k} a_\nu (x - x_\nu)^n , \quad x \in [x_k, x_{k+1}]$$

for $k = 0, 1, \ldots, N - 1$.

In particular, for $x \in [x_{N-1}, x_N]$ we have

$$S(x) = \sum_{\nu=0}^{N-1} a_\nu (x - x_\nu)^n.$$

By hypothesis, $S(x) = 0$ for $x \geqq x_N$, so the continuity condition requires that

$$S(x_N) = \sum_{\nu=0}^{N-1} a_\nu (x_N - x_\nu)^n = 0,$$

and

$$S^{(k)}(x_N) = (n)\cdots(n + 1 - k) \sum_{\nu=0}^{N-1} a_\nu (x_N - x_\nu)^{n-k} = 0,$$

for $k = 1, 2, \ldots, n - 1$. It follows that the numbers $a_0, \ldots, a_{N-1}$ are solutions of the following system:

$$\begin{aligned}
a_0(x_N - x_0)^n + \ldots + a_{N-1}(x_N - x_{N-1})^n &= 0 \\
a_0(x_N - x_0)^{n-1} + \ldots + a_{N-1}(x_N - x_{N-1})^{n-1} &= 0 \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & \\
a_0(x_N - x_0)^{n-k} + \ldots + a_{N-1}(x_N - x_{N-1})^{n-k} &= 0 \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & \\
a_0(x_N - x_0) + \ldots + a_{N-1}(x_N - x_0) &= 0
\end{aligned}$$

We have here n equations in N unknowns, and $N \leqq n$.

First, assume $N = n$. Since

$$\det \begin{vmatrix} (x_N - x_0)^n & \ldots & (x_N - x_{N-1})^n \\ \cdot & \cdots & \cdot \\ (x_N - x_0) & \ldots & (x_N - x_{N-1}) \end{vmatrix} \neq 0 ,$$

it follows that the only solutions of the preceding system are the trivial solutions, $a_0 = 0, \ldots, a_{N-1} = 0$. Thus, if $N = n$, then

$$S(x) = 0 \quad \text{for all } x.$$

Next, assume $N < n$. Without loss of generality, we can add simple knots between x_0 and x_N so that there are exactly $n + 1$ knots. This results in precisely the previous case, proving the theorem.

The following corollary is a useful restatement of Theorem 1.

Corollary. A non-zero n-spline S has at least $n + 2$ knots, i.e.,

$$\deg S \leqq \operatorname{span} S + 1.$$

Theorem 2. If $N \geqq n + 1$ and if S is an n-spline having the knots $x_0 < x_1 < \dots < x_N$ and such that

$$S(x,n) = 0 \quad \text{everywhere outside the interval } (x_0, x_N), \tag{1.7}$$

then S can be uniquely represented in the form

$$S(x,n) = \sum_{\nu=0}^{N-n-1} c_\nu M_\nu(x,n;x_\nu,\dots,x_{\nu+n+1}) , \tag{1.8}$$

where $M_0, \dots, M_{N-n-1}$ are fundamental n-splines.

Proof. Assuming that (1.7) holds, and using (1.6), an n-spline S may be written as

$$S(x,n) = \begin{cases} 0 & x \leqq x_0 \\ a_0(x - x_0)^n & x_0 \leqq x \leqq x_1 \\ \quad \cdot \quad \cdot \quad \cdot & \\ \sum_{i=0}^{k} a_i(x - x_i)^n & x_k \leqq x \leqq x_{k+1} \\ \quad \cdot \quad \cdot \quad \cdot & \\ \sum_{i=0}^{N-1} a_i(x - x_i)^n , & x_{N-1} \leqq x \leqq x_N \\ 0 , & x_N \leqq x \end{cases} \tag{1.9}$$

Figure 10 illustrates S for $N \geqq n + 1$.

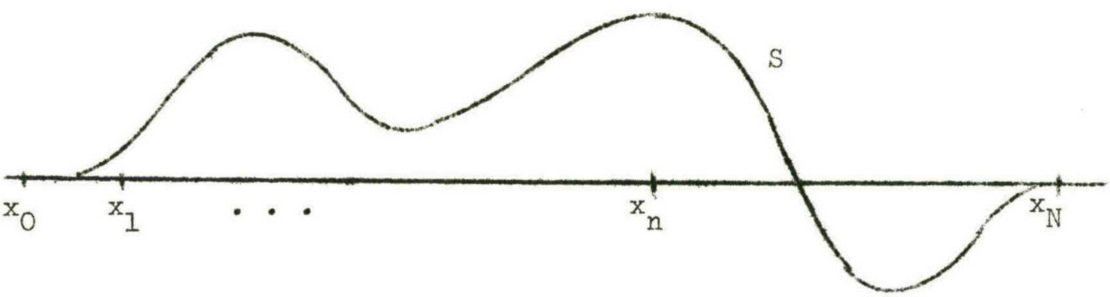


Figure 10. An n-spline on (x_0, x_N) for $N \geqq n + 1$.

By (1.3) and (1.4), the fundamental n-splines may also be written in the form (1.9). Thus, for suitable constants b_{ij}, we have

$$M_0(x,n) = \begin{cases} 0 & , \quad x \leqq x_0 \\ \sum_{j=0}^{k} b_{0j}(x - x_j)^n & , \quad x_k \leqq x \leqq x_{k+1}, \\ & \quad k = 0, \ldots, n \\ 0 & , \quad x_{n+1} \leqq x \end{cases}$$

$$M_1(x,n) = \begin{cases} 0 & , \quad x \leqq x_1 \\ \sum_{j=1}^{k} b_{1j}(x - x_j)^n & , \quad x_k \leqq x \leqq x_{k+1}, \\ & \quad k = 1, \ldots, n + 1 \\ 0 & , \quad x_{n+2} \leqq x \end{cases}$$

and in general, for $\nu = 0, 1, \ldots, N - n - 1$

$$(1.10) \qquad M_\nu(x,n) = \begin{cases} 0 & , \quad x \leqq x_\nu \\ \sum_{j=\nu}^{k} b_{\nu j}(x - x_j)^n & , \quad x \in [x_k, x_{k+1}] \\ & \quad k = \nu, \ldots, \nu + n \\ 0 & , \quad x_{\nu+n+1} \leqq x \end{cases}$$

The following figure shows the fundamental n-splines M_0, M_1, ..., M_{N-n-1} on $[x_0,x_N]$ for $N \geqq n + 1$.

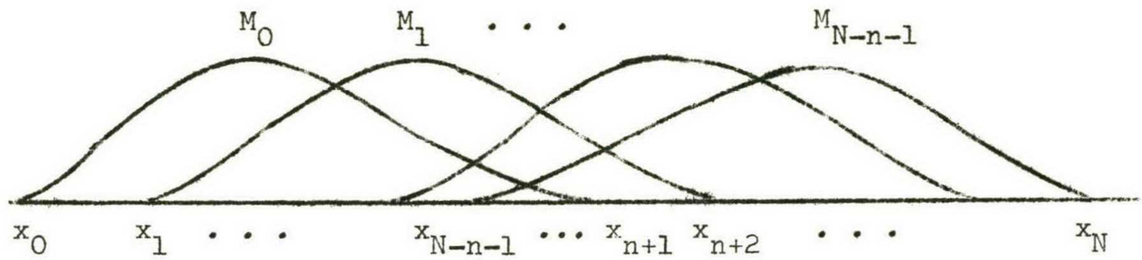


Figure 11. The fundamental n-splines on $[x_0,x_N]$, $N \geqq n + 1$.

It is required to show for an arbitrary n-spline S vanishing outside (x_0,x_N), $N \geqq n + 1$ that

$$(1.8) \qquad S(x,n) = \sum_{\nu=0}^{N-n-1} c_\nu M_\nu(x,n) , \quad \text{for all } x,$$

where the c_ν's are uniquely determined. From (1.9) and (1.10) it follows that on $[x_0,x_1]$ we have

$$S(x,n) = a_0(x - x_0)^n$$

$$M_0(x,n) = b_{00}(x - x_0)^n , \quad \text{where } b_{00} \neq 0, \text{ since } M_0 \text{ is a fundamental n-spline}$$

$$M_\nu(x,n) = 0 , \quad \nu = 1, \ldots, N - n - 1.$$

If we define S_0 by

$$S_0(x,n) = S(x,n) - \sum_{\nu=0}^{N-n-1} c_\nu M_\nu(x,n) ,$$

we have on $[x_0,x_1]$

$$S_0(x,n) = a_0(x - x_0)^n - c_0 b_{00}(x - x_0)^n$$

$$= (a_0 - c_0 b_{00})(x - x_0)^n$$

Then $S_0(x,n) = 0$ for all $x \in [x_0, x_1]$ if and only if $c_0 = \dfrac{a_0}{b_{00}}$.

Next, let $x \in [x_1, x_2]$, then by the same procedure

$$S(x,n) = a_0(x - x_0)^n + a_1(x - x_1)^n,$$

$$M_0(x,n) = b_{00}(x - x_0)^n + b_{01}(x - x_1)^n,$$

$$M_1(x,n) = b_{11}(x - x_1)^n, \quad b_{11} \neq 0,$$

$$M_\nu(x,n) = 0, \quad \nu = 2, \ldots, N - n - 1.$$

Define S_1 by

$$S_1(x,n) = S(x,n) - \sum_{\nu=0}^{N-n-1} c_\nu M_\nu(x,n) .$$

Then on $[x_1, x_2]$ we have

$$S_1(x,n) = (a_0 - c_0 b_{00})(x - x_0)^n +$$

$$(a_1 - c_0 b_{01} - c_1 b_{11})(x - x_1)^n$$

i.e.
$$S_1(x,n) = (a_1 - c_0 b_{01} - c_1 b_{11})(x - x_1)^n$$

Thus, $S_1(x,n) = 0$ for all $x \in [x_1, x_2]$ if and only if

$$c_1 = \frac{a_1 - c_0 b_{01}}{b_{11}} .$$

Continuing to the next interval, $x \in [x_2, x_3]$, we have

$$S(x,n) = a_0(x - x_0)^n + a_1(x - x_1)^n + a_2(x - x_2)^n,$$

$$M_0(x,n) = b_{00}(x - x_0)^n + b_{01}(x - x_1)^n + b_{02}(x - x_2)^n,$$

$$M_1(x,n) = b_{11}(x - x_1)^n + b_{12}(x - x_2)^n,$$

$$M_2(x,n) = b_{22}(x - x_2)^n, \quad b_{22} \neq 0,$$

$$M_\nu(x,n) = 0 \ , \quad \nu = 3, \ldots, N - n - 1.$$

The difference S_2 defined by

$$S_2(x,n) = S(x,n) - \sum_{\nu=0}^{N-n-1} c_\nu M_\nu(x,n)$$

is then

$$\begin{aligned} S_2(x,n) = {} & (a_0 - c_0 b_{00})(x - x_0)^n + \\ & (a_1 - c_0 b_{01} - c_1 b_{11})(x - x_1)^n + \\ & (a_2 - c_0 b_{02} - c_1 b_{12} - c_2 b_{22})(x - x_2)^n, \end{aligned}$$

i.e., $$S_2(x,n) = (a_2 - c_0 b_{02} - c_1 b_{12} - c_2 b_{22})(x - x_2)^n.$$

Then $S_2(x,n) = 0$ for all $x \in [x_2, x_3]$ if and only if

$$c_2 = \frac{a_2 - c_0 b_{02} - c_1 b_{12}}{b_{22}} \ .$$

In general, for $x \in [x_k, x_{k+1}]$, $k = 0, \ldots, N - n - 1$ we have the recursion relation

$$S_0(x,n) = (a_0 - c_0 b_{00})(x - x_0)^n, \text{ and}$$

$$S_k(x,n) = S_{k-1}(x,n) + [a_k - \sum_{\nu=0}^{k} c_\nu b_{\nu,k}](x - x_k)^n,$$

$$k = 1, \ldots, N - n.$$

Thus,

$$S_k(x,n) = S(x,n) - \sum_{\nu=0}^{N-n-1} c_\nu M_\nu(x,n) = 0$$

for all $x \in [x_k, x_{k+1}]$, $k = 0, \ldots, N - n - 1$ if and only if

$$c_0 = \frac{a_0}{b_{00}} \ , \text{ and}$$

$$c_k = \frac{a_k - \sum_{\nu=0}^{k-1} c_\nu b_{\nu,k-1}}{b_{k,k}}, \quad k = 1, \ldots, N - n - 1.$$

Now consider the n-spline S in the interval $[x_{N-n}, x_N]$. Define S* by

$$(1.11) \qquad S^*(x,n) = \begin{cases} S(x,n) - \sum_{\nu=0}^{N-n-1} c_\nu M_\nu(x,n), & x \in [x_0, x_N] \\ 0, & x \notin (x_0, x_N) \end{cases}$$

Since S* is a linear combination of n-splines, it is also an n-spline, with knots $x_0 < x_1 < \ldots < x_N$. By the previous arguments

$$(1.12) \qquad S^*(x,n) = 0 \quad \text{for } x \in [x_0, x_{N-n}].$$

Hence, S* degenerates into an n-spline with only n + 1 knots, so by Theorem 1

$$(1.13) \qquad S^*(x,n) = 0 \quad \text{for } x \in [x_{N-n}, x_N].$$

Finally, combining (1.11), (1.12) and (1.13)

$$S^*(x,n) = 0 \quad \text{for all } x,$$

which implies that

$$S(x,n) = \sum_{\nu=0}^{N-n-1} c_\nu M_\nu(x,n) \quad \text{for all } x,$$

completing the proof for Theorem 2.

<u>Theorem 3</u>. An n-spline S vanishing if $x < x_0$ can be uniquely represented in the form

$$(1.14) \qquad S(x,n) = \sum_{\nu=0}^{\infty} c_\nu M_\nu(x,n),$$

and conversely any such series represents an n-spline vanishing for $x < x_0$.

Proof. Figure 12 illustrates this case.

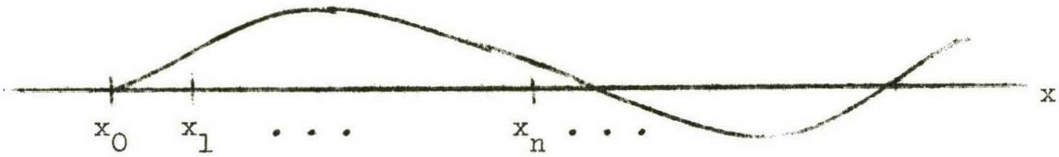


Figure 12. An n-spline vanishing for $x < x_0$.

The proof of this theorem is the same as for Theorem 2, except the sum continues indefinitely, resulting in

$$S(x,n) = \sum_{\nu=0}^{\infty} c_\nu M_\nu(x,n) \quad . \tag{1.14}$$

Theorem 2 is clearly a special case of this where $c_\nu = 0$ for $\nu > N - n - 1$, N finite.

We can now establish the following property of the fundamental splines which will be needed for the next theorem.

Lemma 2. The n + 1 fundamental n-splines

$$M_{-n},\ M_{-n+1},\ \ldots,\ M_{-1},\ M_0$$

are linearly independent in the interval (x_0,x_1) and therefore form in this interval a basis for $\mathcal{P}_n$.

Proof. Figure 13 illustrates the n + 1 fundamental n-splines over (x_{-n},x_{n+1}).

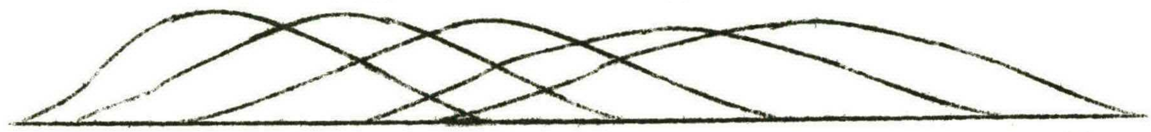


Figure 13. The fundamental n-splines over (x_{-n},x_{n+1}).

Each fundamental n-spline has exactly n + 2 knots so that exactly 2n + 2 distinct knots are required for these n + 1 functions.

Assume these n-splines are dependent in (x_0,x_1), $x_0 \neq x_1$, then there exists constants c_{-n}, ..., c_0, not all zero, such that

$$S(x,n) = \sum_{j=-n}^{0} c_j M_j(x,n) = 0 , \quad x \in (x_0,x_1) .$$

We may represent S by

$$S(x,n) = \begin{cases} 0 , & -\infty < x < x_{-n} \\ S_L(x) , & x_{-n} \leqq x < x_0 \\ S_0(x) , & x_0 \leqq x < x_1 \\ S_R(x) , & x_1 \leqq x < x_{n+1} \\ 0 , & x_{n+1} \leqq x < \infty \end{cases}$$

which is illustrated in Figure 14.

S is an n-spline vanishing for all $x \in (-\infty,x_{-n}) \cup [x_0,x_1) \cup [x_{n+1},\infty)$, with the usual continuity conditions at x_{-n}, x_0, x_1 and x_{n+1}. Let

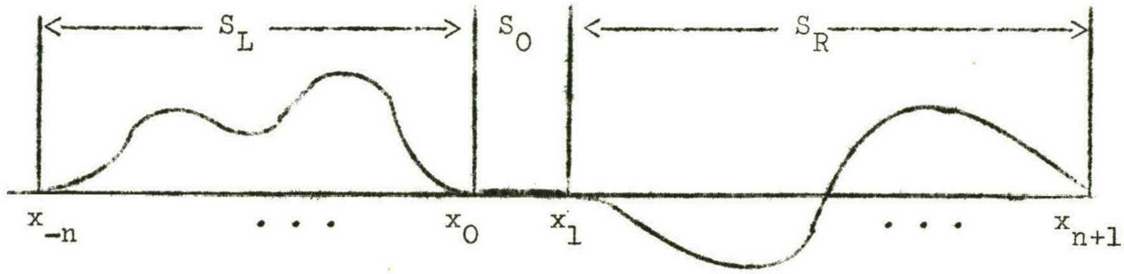


Figure 14. An n-spline on (x_{-n}, x_{n+1}) vanishing in (x_0, x_1).

$$S_R^*(x,n) = \begin{cases} S_R(x,n) \;, & x_1 \leqq x < x_{n+1} \\ 0 \;, & \text{elsewhere} \end{cases}$$

Since S_R is a segment of an n-spline and the continuity conditions at its endpoints are satisfied, then S_R^* is an n-spline with at most n + 1 knots, namely $x_1, \ldots, x_{n+1}$. Hence, by Theorem 1

$$S_R^*(x,n) = 0 \quad \text{for all } x,$$

so that

$$S_R(x,n) = 0 \;, \quad x_1 < x < x_{n+1}$$

and thus

$$S(x,n) = 0 \;, \quad x_1 < x < x_{n+1} \;.$$

We can now represent S as an n-spline with at most n + 1 knots, namely $x_{-n}, \ldots, x_0$. Again applying Theorem 1, this implies

$$S(x,n) = 0 \quad \text{for all } x.$$

From the uniqueness property in Theorem 2, it follows that $c_j = 0$, $j = -n, \ldots, 0$, contradicting the assumption of linear dependence. This completes a proof of Lemma 2.

Theorem 4. Every n-spline S can be uniquely represented in the form

(1.15) $$S(x,n) = \sum_{\nu=-\infty}^{+\infty} c_\nu M_\nu(x,n) ,$$

where the c_ν are constants and the M_ν are fundamental n-splines. Conversely, any such series represents an n-spline.

Proof. Let S be an arbitrary n-spline, and let

$$S(x,n) = P(x) , \quad x_0 < x < x_1 , \quad (P \in \mathcal{P}_n) .$$

By Lemma 2, we can write

(1.16) $$P(x) = \sum_{j=-n}^{0} c_j M_j(x,n) , \quad x \in (x_0,x_1) .$$

Define S* by

(1.17) $$S^*(x,n) = S(x,n) - P(x)$$

which is an n-spline vanishing in the interval (x_0,x_1). We may therefore write

(1.18) $$S^*(x,n) = S_0(x,n) + S_1(x,n) ,$$

where S_0 and S_1 are n-splines vanishing in the intervals $(x_0,+\infty)$ and $(-\infty,x_1)$, respectively. By Theorem 3 we may therefore write uniquely

(1.19) $$S_0(x,n) = \sum_{j=-\infty}^{-n-1} c_j M_j(x,n) ,$$
$$S_1(x,n) = \sum_{j=1}^{\infty} c_j M_j(x,n) ,$$

for all x.

Combining (1.16) through (1.19) and solving for S gives

$$S(x,n) = S^*(x,n) + P(x)$$
$$= S_0(x,n) + P(x) + S_1(x,n) ,$$

i.e.,

$$(1.15) \qquad S(x,n) = \sum_{j=-\infty}^{\infty} c_j M_j(x,n) ,$$

which is the desired representation. This completes the proof of the representation theorems.

We observe that on any finite interval the series (1.15) always reduces to a finite sum [4].

I.3. Examples

This chapter will be concluded with some examples to illustrate the representation theorems.

Example. The fundamental spline functions of degree 1 and 2, being easily computed, are frequently used to illustrate the theory. Their explicit expressions, when the integers serve as knots, are given below:

$$M_\nu(x,1;\nu,\nu+1,\nu+2) = \begin{cases} 0 & , \quad -\infty < x < \nu \\ (x-\nu) & , \quad \nu \leqq x < \nu+1 \\ -(x-\nu-2) & , \quad \nu+1 \leqq x < \nu+2 \\ 0 & , \quad \nu+2 \leqq x < +\infty \end{cases}$$

$$M_\nu(x,2;\nu,\nu+1,\nu+2) = \begin{cases} 0 & , \quad -\infty < x < \nu \\ \frac{1}{2}(x-\nu)^2 & , \quad \nu \leqq x < \nu+1 \\ -(x-\nu-\frac{3}{2})^2 + \frac{3}{4} & , \quad \nu+1 \leqq x < \nu+2 \\ \frac{1}{2}(x-\nu-3)^2 & , \quad \nu+2 \leqq x < \nu+3 \\ 0 & , \quad \nu+3 \leqq x < +\infty \end{cases}$$

for any $\nu = 0, \pm 1, \ldots$.

<u>Example</u>. To obtain the constant function 1 we may use the so-called "roof function", or fundamental 1-splines, since

$$\sum_{\nu=-\infty}^{\infty} M_\nu(x,1;\nu,\nu+1,\nu+2) = 1 \ , \quad \text{for all } x.$$

This representation is obvious in view of Figure 15.

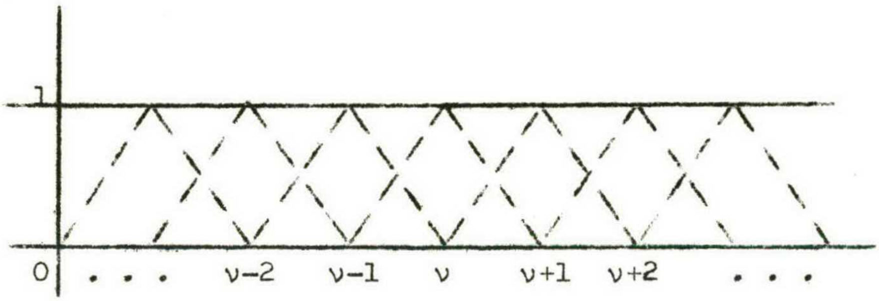


Figure 15. The constant function 1 as a linear combination of fundamental 1-splines.

<u>Example</u>. As illustrations of Theorems 3 and 4, Schoenberg [3] gives the following identities:

let $x_\nu = \nu$ for all integers ν, then

$$x_+^k = \sum_{\nu=0}^{\infty} (\nu+1)(\nu+2)\cdots(\nu+k)M_\nu(x,k,\nu,\ldots,\nu+k+1),$$

and

$$x^k = \sum_{\nu=-\infty}^{+\infty} (\nu + 1)(\nu + 2)\cdots(\nu + k)M_\nu(x,k;\nu,\ldots,\nu+k+1).$$

The first four special cases are:

$$1 = \sum_{-\infty}^{\infty} M_\nu(x,k;\nu,\ldots,\nu+k+1) , \quad \text{for any } k \geqq 0$$

$$x = \sum_{-\infty}^{\infty} (\nu + 1)M_\nu(x,1;\nu,\nu+1,\nu+2)$$

$$x^2 = \sum_{-\infty}^{\infty} (\nu + 1)(\nu + 2)M_\nu(x,2;\nu,\nu+1,\nu+2,\nu+3)$$

$$x^3 = \sum_{-\infty}^{\infty} (\nu + 1)(\nu + 2)(\nu + 3)M_\nu(x,3;\nu,\nu+1,\nu+2,\nu+3,\nu+4).$$

Example. Define a 1-spline function S by

$$S(x,1) = \begin{cases} 0 & , \quad x < 0 \\ x & , \quad 0 \leqq x < 1 \\ -\frac{1}{2}x + \frac{3}{2} & , \quad 1 \leqq x < 2 \\ \frac{3}{2}x - \frac{5}{2} & , \quad 2 \leqq x < 3 \\ -2x + 8 & , \quad 3 \leqq x < 4 \\ 0 & , \quad 4 \leqq x \end{cases}$$

and find the representation of S as a linear combination of fundamental 1-splines.

Solution. Figure 16 shows the graph of S with the solid lines, and the graph of the fundamental 1-splines with dotted lines. We have as the degree of the spline $n = 1$ and the span is $N = 4$, so that $n + 1 \leqq N < \infty$, and thus Theorem 2 applies. Hence, we are to find c_0, c_1 and c_2 such that

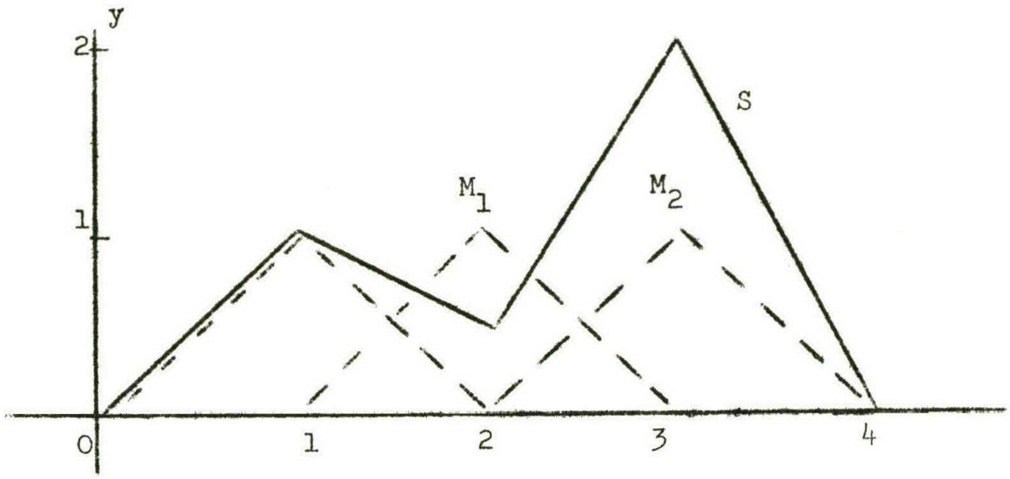


Figure 16. An example for Theorem 2.

$$S(x,1) = \sum_{\nu=0}^{2} c_\nu M_\nu(x,1;\nu,\nu+1,\nu+2) , \quad \text{for all } x.$$

The required fundamental 1-splines are given in Table 1.

TABLE 1

FUNDAMENTAL 1-SPLINES

Interval	$M_0(x,1;0,1,2)$	$M_1(x,1;1,2,3)$	$M_2(x,1;2,3,4)$
$-\infty < x < 0$	0	0	0
$0 \leqq x < 1$	x	0	0
$1 \leqq x < 2$	$-(x - 2)$	$(x - 1)$	0
$2 \leqq x < 3$	0	$-(x - 3)$	$(x - 2)$
$3 \leqq x < 4$	0	0	$-(x - 4)$
$4 \leqq x < \infty$	0	0	0

To evaluate c_0, let $x \in [0,1)$, then

$$S(x,1) = x,$$

and

$$\sum_{\nu=0}^{2} c_\nu M_\nu(x,1) = c_0 x \ .$$

Thus, $c_0 = 1.$

Now let $x \in [1,2)$, then

$$S(x,1) = -\frac{1}{2}x + \frac{3}{2} \ ,$$

and

$$\sum_{\nu=0}^{2} c_\nu M_\nu(x,1) = -(x-2) + c_1(x-1)$$

$$= (c_1 - 1)x + (2 - c_1) \ .$$

Thus, $c_1 = \frac{1}{2}$.

Finally let $x \in [2,3)$, then

$$S(x,1) = \frac{3}{2}x - \frac{5}{2} \ ,$$

and

$$\sum_{\nu=0}^{2} c_\nu M_\nu(x,1) = -\frac{1}{2}(x-3) + c_2(x-2)$$

$$= (c_2 - \tfrac{1}{2})x + (\tfrac{3}{2} - 2c_2) \ .$$

Thus, $c_2 = 2$.

The desired representation is then

$$S(x,1) = M_0(x,1;0,1,2) + \frac{1}{2} M_1(x,1;1,2,3) +$$

$$2M_2(x,1;2,3,4) \ , \quad \text{for all } x.$$

Example. An additional property of the function M given by Definition 2, i.e.

$$M(x,k;y) = (k+1)(y-x)_+^k \ ,$$

is that $M(x,k;y) \geqq 0$ for all x. Moreover its $(k+1)$st divided difference is also non-negative, i.e.

$$M_\nu(x,k;x_\nu,\ldots,x_{\nu+k+1}) > 0 \quad \text{in } (x_\nu,x_{\nu+k+1})$$

and

$$M_\nu(x,k) = 0 \quad \text{outside } (x_\nu,x_{\nu+k+1}) \text{ .}$$

As shown by Schoenberg [5]

$$\int_{-\infty}^{\infty} M_\nu(x,k;x_\nu,\ldots,x_{\nu+k+1})\, dx = 1 \text{ ,} \quad \text{for any } \nu \text{ ,}$$

follows from Peano's Theorem. Hence, we may think of the fundamental spline functions as frequency functions [3], [5].

CHAPTER II

INTERPOLATION

II.1. Interpolation by spline functions

It will be shown that spline functions can be used with great advantages as interpolating functions. The usual questions of existence, uniqueness, characterization, and best approximation will be discussed.

First, we consider three lemmas needed for the existence and uniqueness theorem proof.

Lemma 3 [4]. If $f \in C^{k-1}[x_0,x_k]$ while $f^{(k-1)}$ is absolutely continuous, then its divided difference of order k may be expressed as

$$f[x_0,\ldots,x_k] = \frac{1}{k!}\int_{x_0}^{x_k} M_0(x,k-1;x_0,\ldots,x_k) f^{(k)}(x)\,dx \; . \tag{2.1}$$

Proof. This lemma follows from Peano's general theorem ([8], p. 69ff) concerning linear functionals which vanish for polynomials of degree at most k − 1 [4].

Lemma 4. Suppose that $\varphi[x_0,\ldots,x_m] = 0$, then there exists a unique polynomial Q of degree $\leqq m-1$ such that

$$\varphi(x_\nu) = Q(x_\nu) \; , \quad \nu = 0, \ldots, m \; .$$

Proof. The interpolating polynomial Q of degree $\leqq$ m may be uniquely represented by a finite Newton series (see [8], p. 39ff) such that

$$\varphi(x_\nu) = Q(x_\nu) \ , \quad \nu = 0, \ldots, m \ .$$

The Newton representation is defined by

$$Q(x) = \varphi[x_0] + (x - x_0)\varphi[x_0,x_1] + (x - x_0)(x - x_1)\varphi[x_0,x_1,x_2] + \ldots + (x - x_0)\cdots(x - x_{m-1})\varphi[x_0,\ldots,x_m] \ .$$

Since $\varphi[x_0,\ldots,x_m] = 0$, we have clearly deg $Q \leqq m - 1$.

Lemma 5. If $1 \leqq m \leqq n$ and

$$\varphi[x_i,\ldots,x_{i+m}] = 0 \ , \quad i = 0, \ldots, n - m \ ,$$

then there exists a unique polynomial P of degree $\leqq m - 1$ such that

$$P(x_\nu) = \varphi(x_\nu) \ , \quad \nu = 0, \ldots, n \ .$$

Proof. By Lemma 4, there exists a unique polynomial P_i of degree $\leqq m - 1$ such that for each fixed i ($i = 0, \ldots, n - m$)

$$P_i(x_\nu) = \varphi(x_\nu) \ , \quad \nu = i, \ldots, i + m \ .$$

But this implies that for each i ($i = 0, \ldots, n - m$)

$$P_i(x_\nu) = P_{i+1}(x_\nu) \ , \quad \text{for } \nu = i + 1, \ldots, i + m \ .$$

Since two polynomials of degree $\leqq m - 1$ coinciding at m points are identical, it follows that

$$P_0 = P_1 = \ldots = P_{n-m} = P,$$

and the lemma is proved.

Let m be a natural integer and define a class of functions F_m in a given finite interval $[a,b]$ as follows:

(2.2) $$f \in F_m[a,b] \Leftrightarrow f \in C^{m-1}[a,b],\ f^{(m-1)} \text{ absolutely continuous on } [a,b] \text{ and } f^{(m)} \in L^2(a,b)\ .$$

Let the integer $n > 0$ be given and suppose that we are also given $n + 1$ abscissae

$$x_0 < x_1 < \dots < x_n \quad \text{in } [a,b]\ .$$

We now choose arbitrary but fixed reals x_ν for $\nu > n$ and $\nu < 0$ so as to obtain a sequence of knots

(2.3) $$\dots < x_{-1} < x_0 < x_1 < \dots < x_{n-1} < x_n < x_{n+1} < \dots\ .$$

All of the spline functions in the following discussion may be considered to be defined on the knots (2.3). However, only the $n + 1$ knots $x_0, \dots, x_n$ will be of direct concern.

The following theorem gives the existence and uniqueness of interpolating spline functions.

<u>Theorem 5</u> [4]. Let there be given in $[a,b]$ a function f in the class $F_m[a,b]$ defined by (2.2) and a set of $n + 1$ points

(2.4) $$a \leqq x_0 < x_1 < \dots < x_n \leqq b$$

with $1 \leqq m \leqq n$. Then there is a uniquely determined natural spline function S in $\mathscr{S}^*_{2m-1}(x_0,\dots,x_n)$ which interpolates the function f at the points (2.4), i.e.,

$$S(x_\nu) = f(x_\nu)\ , \quad \nu = 0, \dots, n\ .$$

<u>Proof</u>. Consider the $(m - 1)$-spline function $\mathfrak{s}$ represented by

$$\mathfrak{s}(x,m-1) = \sum_{\nu=0}^{n-m} c_\nu M_\nu(x,m-1;x_\nu,\dots,x_{\nu+m}) ,$$

where the c_ν are arbitrary constants and the M_ν are fundamental $(m-1)$-splines. We have then

$$\mathfrak{s}(x,m-1) = 0 \quad \text{if } x < x_0 \text{ or } x > x_n .$$

The numbers $c_0, \dots, c_{n-m}$ are determined from the following equations

$$\int_a^b \Big(\sum_{\nu=0}^{n-m} c_\nu M_\nu(x,m-1) - f^{(m)}(x) \Big) M_j(x,m-1)\, dx = 0 , \tag{2.5}$$

$$j = 0, \dots, n-m ,$$

i.e.

$$c_0\int_a^b M_0^2 dx + c_1\int_a^b M_1 M_0 dx + \dots + c_{n-m}\int_a^b M_{n-m} M_0 dx = \int_a^b f^{(m)} M_0 dx$$

$$c_0\int_a^b M_0 M_1 dx + c_1\int_a^b M_1^2 dx + \dots + c_{n-m}\int_a^b M_{n-m} M_1 dx = \int_a^b f^{(m)} M_1 dx$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$c_0\int_a^b M_0 M_{n-m} dx + c_1\int_a^b M_1 M_{n-m} dx + \dots + c_{n-m}\int_a^b M_{n-m}^2 dx = \int_a^b f^{(m)} M_{n-m} dx.$$

The determinant of this linear system in the unknowns c_ν is the Gramian of the functions $M_\nu(x,m-1)$, $(\nu = 0, \dots, n-m)$. These being linearly independent in $[a,b]$, by Lemma 2, we conclude that the problem (2.5) has a unique solution $\bar{\mathfrak{s}}$.

Assuming now that $\bar{\mathfrak{s}}$ has been uniquely determined by the above procedure, we integrate this $(m-1)$-spline function m times, obtaining a $(2m-1)$-spline, denoted by $\bar{S}$:

$$(2.6)\qquad \bar{S}(x,2m-1) = \frac{1}{(m-1)!}\int_{x_0}^{x}(x-t)^{m-1}\bar{s}(t,m-1)\,dt\ .$$

Clearly $\bar{S} \in \mathscr{S}^*_{2m-1}(x_0,\ldots,x_n)$.

Define the function φ by

$$\varphi(x) = f(x) - \bar{S}(x,2m-1)\ .$$

Then

$$\varphi^{(m)}(x) = f^{(m)}(x) - \bar{S}^{(m)}(x,2m-1)\ ,$$

or

$$(2.7)\qquad \varphi^{(m)}(x) = f^{(m)}(x) - \bar{s}(x,m-1)\ .$$

Substituting (2.7) into (2.5) gives

$$(2.8)\qquad \int_a^b \varphi^{(m)}(x)M_j(x,m-1)\,dx = 0\ ,\quad j = 0,\ \ldots,\ n-m\ .$$

By Lemma 3 we have for each i ($i = 0,\ \ldots,\ n-m$)

$$(2.9)\qquad \varphi[x_i,\ldots x_{i+m}] = \frac{1}{m!}\int_{x_i}^{x_{i+m}} \varphi^{(m)}(x)M_i(x,m-1;x_i,\ldots,x_{i+m})dx,$$

and so, from (2.8) and (2.9)

$$(2.10)\qquad \varphi[x_i,\ldots,x_{i+m}] = 0\ ,\quad i = 0,\ \ldots,\ n-m\ .$$

Applying Lemma 5 to (2.10) there exists a unique polynomial P of degree $\leqq m-1$ such that

$$P(x_\nu) = \varphi(x_\nu)\ ,\quad \nu = 0,\ \ldots,\ n\ ,$$

i.e.

$$P(x_\nu) = f(x_\nu) - \bar{S}(x_\nu,2m-1)\ ,$$

or

$$\bar{S}(x_\nu,2m-1) + P(x_\nu) = f(x_\nu)\ .$$

we may therefore define S by

$$S(x,2m-1) = \bar{S}(x,2m-1) + P(x) ,$$

and thus obtaining uniquely $S \in \mathcal{S}^*_{2m-1}(x_0,\dots,x_n)$ such that

$$S(x_\nu) = f(x_\nu) , \quad \nu = 0, \dots, n ,$$

proving Theorem 5.

Remark. The natural spline function S in Theorem 5 is also characterized by the following property:

$$(2.11) \qquad \int_a^b (S^{(m)}(x,2m-1) - f^{(m)}(x))^2 dx =$$

$$\inf_{g \in \mathcal{S}^*_{2m-1}(x_0,\dots,x_n)} \int_a^b (g^{(m)}(x,2m-1) - f^{(m)}(x))^2 dx .$$

Corollary [4]. Given n + 1 points in the plane

$$(x_\nu, y_\nu) , \quad (\nu = 0, \dots, n;\ x_0 < x_1 < \dots < x_n),$$

and an integer m, $1 \leqq m \leqq n$, then there exists a unique spline $S \in \mathcal{S}^*_{2m-1}(x_0,\dots,x_n)$ such that

$$S(x_\nu) = y_\nu , \quad (\nu = 0, \dots, n) .$$

Example. Given $f(x) = (x - 1)^2$ on $[0,3]$, and let $m = 2$, $n = 3$, and the knots be $x_\nu = \nu$, $\nu = 0, 1, 2, 3$. Construct a unique 3-spline function S with the given knots and such that

a) $S''(x,3) = 0$ if $x < 0$ or $x > 3$, and

b) $S(x_\nu,3) = f(x_\nu)$, $\nu = 0, 1, 2, 3$.

Solution. Following the proof of Theorem 5, we first consider the 1-spline $\mathfrak{S}$ represented by

$$\mathfrak{s}(x,1) = \sum_{\nu=0}^{1} c_\nu M_\nu(x,1;\nu,\nu+1,\nu+2) ,$$

where

$$M_0(x,1;0,1,2) = \begin{cases} x & , \quad 0 \leqq x < 1 \\ -(x-2) & , \quad 1 \leqq x < 2 \\ 0 & , \quad \text{elsewhere} \end{cases}$$

$$M_1(x,1;1,2,3) = \begin{cases} (x-1) & , \quad 1 \leqq x < 2 \\ -(x-3) & , \quad 2 \leqq x < 3 \\ 0 & , \quad \text{elsewhere} \end{cases}$$

Using (2.5) we may evaluate c_0 and c_1. We have $f''(x) = 2$, so the linear system becomes

$$\begin{cases} c_0 \int_0^3 M_0^2(x,1)dx + c_1 \int_0^3 M_0(x,1)M_1(x,1)dx = 2 \int_0^3 M_0(x,1)dx \\ c_0 \int_0^3 M_0(x,1)dx + c_1 \int_0^3 M_1^2(x,1)dx = 2 \int_0^3 M_1(x,1)dx \end{cases}$$

Performing the simple integration results in

$$\begin{cases} \frac{2}{3} c_0 + \frac{1}{6} c_1 = 2 \\ \frac{1}{6} c_0 + \frac{2}{3} c_1 = 2 \end{cases}$$

with unique solution $c_0 = c_1 = 12/5$.

The 1-spline $\bar{\mathfrak{s}}$ is now uniquely determined, i.e.,

$$\bar{\mathfrak{s}}(x,1) = \begin{cases} \frac{12}{5} x & , \quad 0 \leqq x < 1 \\ \frac{12}{5} & , \quad 1 \leqq x < 2 \\ -\frac{12}{5}(x-3) & , \quad 2 \leqq x < 3 \\ 0 & , \quad \text{elsewhere} \end{cases}$$

Integrating $\bar{\mathfrak{s}}$ twice we obtain the 3-spline $\bar{S}$. Finally, we can find α and β such that the function

$$S(x) = \bar{S}(x) + \alpha x + \beta$$

satisfies the conditions

$$S(x_\nu) = f(x_\nu) \;, \quad \nu = 0, 1, 2, 3 \;.$$

A simple calculation shows that

$$S(x,3) = \begin{cases} -\frac{7}{5}x + 1 & , \quad x < 0 \\ \frac{2}{5}x^3 - \frac{7}{5}x + 1 & , \quad 0 \leqq x < 1 \\ \frac{6}{5}(x-2)^2 + \frac{11}{5}x - \frac{17}{5} & , \quad 1 \leqq x < 2 \\ -\frac{2}{5}(x-3)^3 + \frac{17}{5}x - \frac{31}{5} & , \quad 2 \leqq x < 3 \\ \frac{17}{5}x - \frac{31}{5} & , \quad 3 \leqq x \end{cases}$$

Figure 2.1 illustrates the problem.

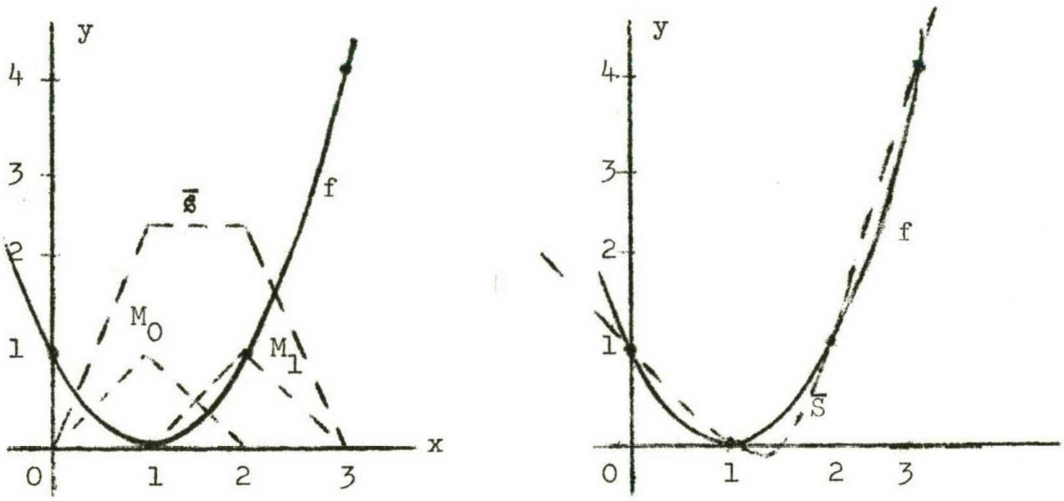


Figure 17. An interpolation problem.

II.2. Minimal property of interpolating spline functions

As an application of this section, Holladay [11] uses the problem of bending a stick a little bit. "A bent stick assumes a position which, subject to the bending constraints, will minimize its potential energy. Assuming the stick to be originally straight and uniform, Hooke's Law implies that its potential energy will be proportional to the integral along its arc length of the square of its curvature. If the problem is two dimensional, and if the bending is sufficiently slight [so] that the arc length may be considered as being practically proportional to some coordinate axis, then we get the problem of minimizing

$$\int (f''(t))^2 dt."$$

After describing the class of functions which interpolates a given set of points, we then seek to minimize the amount of "twisting", and whenever twisting is necessary, we wish to "spread it out."

The minimal property of the interpolating spline functions is given in the following.

Theorem 6 (Minimal Property) [4]. Let f be any function such that $f^{(m)} \in C[a,b]$, and such that

(2.12) $$f(x_i) = y_i \;, \quad i = 0, \ldots, n \;,$$

where

$$a \leqq x_0 < \ldots < x_n \leqq b \;.$$

Then

$$(2.13) \qquad \int_a^b (f^{(m)}(x))^2 dx \geqq \int_a^b (S^{(m)}(x,2m-1))^2 dx ,$$

where $S \in \mathscr{S}^*_{2m-1}(x_0,\ldots,x_n)$ and

$$(2.14) \qquad S(x_i) = y_i , \quad i = 0, \ldots, n .$$

Proof. Let f be any function such that (2.12) holds and $f^{(m)} \in L_2[a,b]$. Let $S \in \mathscr{S}^*_{2m-1}(x_0,\ldots,x_n)$ be such that (2.14) holds. Following [4] we shall show that

$$(2.15) \qquad \int_a^b (f^{(m)}(x))^2 dx = \int_a^b (f^{(m)}(x) - S^{(m)}(x,2m-1))^2 dx + \int_a^b (S^{(m)}(x,2m-1))^2 dx ,$$

from which the minimal property (2.13) of the spline interpolation immediately follows.

Since

$$\int_a^b (f^{(m)}(x))^2 dx = \int_a^b (f^{(m)}(x) - S^{(m)}(x,2m-1))^2 dx + \int_a^b S^{(m)}(x,2m-1)(f^{(m)}(x) - S^{(m)}(x,2m-1)) dx + \int_a^b (S^{(m)}(x,2m-1))^2 dx ,$$

we have only to show that the middle term on the right side vanishes.

Letting $\varphi(x) = f(x) - S(x,2m-1)$, we have from (2.12) and (2.14) that

$$(2.16) \qquad \varphi(x_i) = 0 , \quad i = 0, \ldots, n .$$

Also, since $S \in \mathcal{P}_{m-1}$ outside (x_0,x_n), we have

$$(2.17)\qquad S^{m+j)}(x,2m-1) = 0\ , \quad j = 0,\ 1,\ \ldots$$

for $x \in (-\infty, x_0] \cup [x_n, +\infty)$.

Integrating by parts, we get

$$\int_a^b S^{(m)}(x,2m-1)\varphi^{(m)}(x)\,dx =$$

$$= [S^{(m)}(x,2m-1)\varphi^{(m-1)}(x)]_a^b - \int_a^b S^{(m+1)}(x,2m-1)\varphi^{(m-1)}(x)dx$$

$$= [S^{(m)}\varphi^{(m-1)}]_a^b - [S^{(m+1)}\varphi^{(m-2)}]_a^b + \int_a^b S^{(m+2)}\varphi^{(m-2)}\,dx$$

$$= \ldots =$$

$$= \sum_{\nu=0}^{k} (-1)^\nu [S^{(m+\nu)}(x,2m-1)\varphi^{(m-1-\nu)}(x)]_a^b +$$

$$(-1)^{k+1} \int_a^b S^{(m+1+k)}(x,2m-1)\varphi^{(m-1-k)}(x)\,dx\ .$$

Finally, letting $k = m - 2$ gives

$$(2.18)\qquad \int_a^b S^{(m)}(x,2m-1)\varphi^{(m)}(x)\,dx =$$

$$\sum_{\nu=0}^{m-2} (-1)^\nu [S^{(m+\nu)}(x,2m-1)\varphi^{(m-1-\nu)}(x)]_a^b +$$

$$(-1)^{m-1} \int_a^b S^{(2m-1)}(x,2m-1)\varphi'(x)\,dx\ .$$

But, since $a \leqq x_0$ and $x_n \leqq b$, and by (2.17), the finite sum on the right side of (2.18) vanishes. Then since $S \in \mathscr{S}_{2m-1}$, we have that $S^{(2m-1)}$ is constant on each subinterval, i.e.,

$$S^{(2m-1)}(x,2m-1) = \alpha_\nu\ , \quad x \in [x_\nu, x_{\nu+1}),\ \nu = 0,\ \ldots,\ n-1.$$

Thus

$$\int_a^b S^{(m)}(x,2m-1)\varphi^{(m)}(x)\,dx = (-1)^{m-1}\int_{x_0}^{x_n} S^{(2m-1)}(x,2m-1)\varphi'(x)\,dx$$

$$= (-1)^{m-1}\sum_{\nu=0}^{n-1}\alpha_\nu\int_{x_\nu}^{x_{\nu+1}}\varphi'(x)\,dx$$

$$= (-1)^{m-1}\sum_{\nu=0}^{n-1}\Big[\alpha_\nu\varphi(x)\Big]_{x_\nu}^{x_{\nu+1}} = 0\ ,$$

where the last step follows from (2.16). This establishes (2.15), and hence proves the theorem.

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