PROPERTIES OF THREE POINTS RELATED BY CONSTRUCTION TO THE BROCARD POINTS OF THE TRIANGLE

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by

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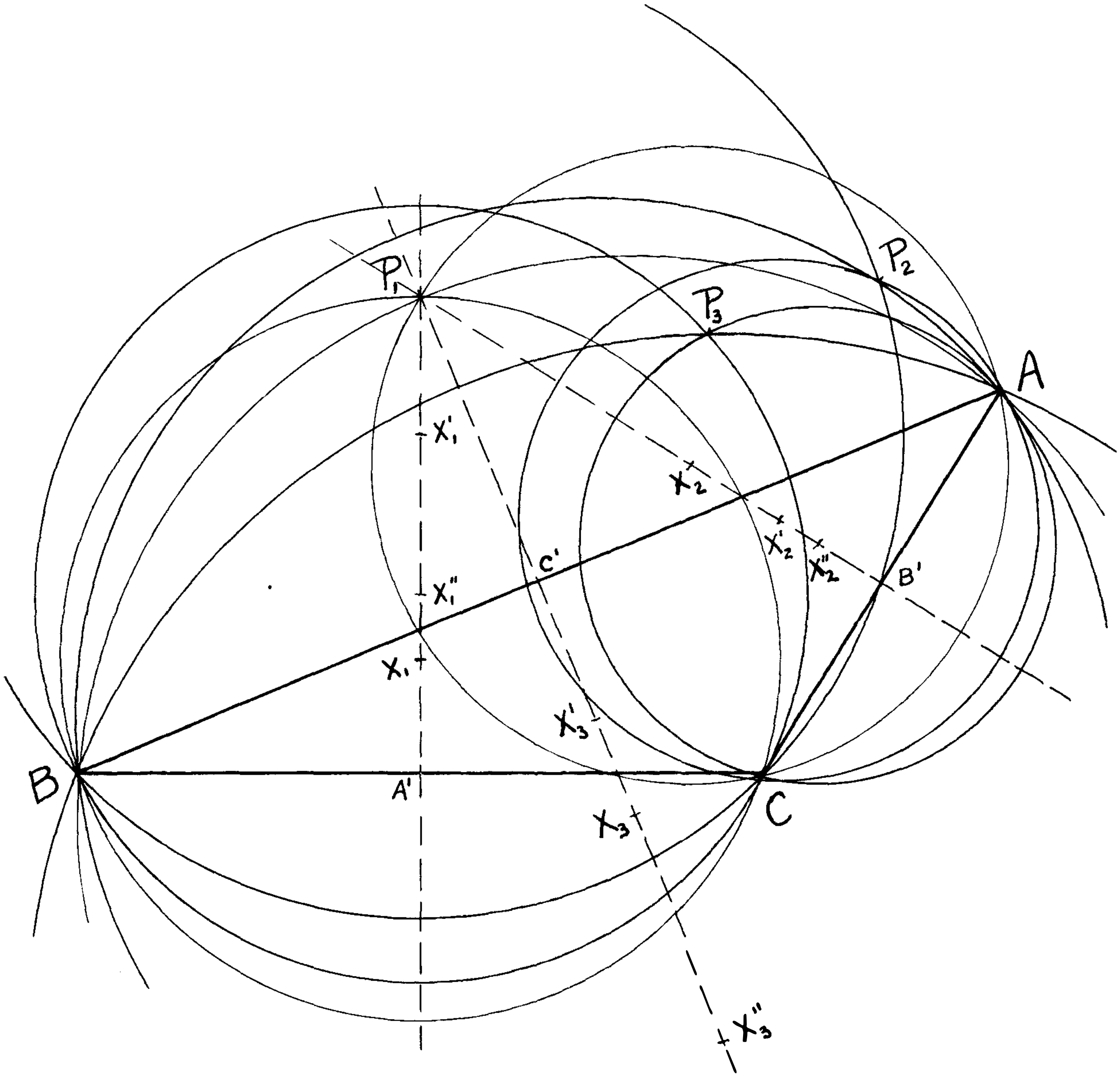

Fig. 1.

Properties of Three Points Related by Construction to the Brocard Points of the Triangle.

Using the three sides AB, BC, CA of any triangle ABC as chords, three systems of circles are constructed so as to be tangent also at A, B, and C to lines which make with AB, BC, and CA exterior angles equal to

(1) 2 C, 2 A, 2 B

(2) 2 A, 2 B, 2 C

(3) 2 B, 2 C, 2 A respectively

retaining throughout the cyclic order (ABC). See figure 1.

The scheme of circle construction which is outlined above brings to mind the method of determining the Brocard points of the triangle. The essential differences between the two constructions are that the former places the angle to be inscribed in each circle outside the bounding lines of the triangle while the Brocard construction places the angle within the triangle; also that while Brocard used angles equal to those of the fundamental triangle, the method of construction in the present discussion involves angles equal to twice the size of the angles of the fundamental triangle.

<u>Theorem I.</u> Each of the three systems of circles intersect in a point.

Case 1. The point C is common to the two circles (BC) and (CA) while their second point of intersection may be called P_1 Now $\angle BP_1C = 2A$ and $\angle CP_1A = 2B$. Hence $\angle BP_1A = 2A + 2B = 2\pi - 2C$. This latter is the exact condition for P_1 to lie on the arc of the circle (AB), whose tangent and chord make an angle equal to $2C - \pi$.

Case 2. By the second construction defined above the circles (BC) and (CA) intersect at C and a point designated P_2. Then $\angle BP_2C = 2B$ and $\angle CP_2A = 2C - \pi$. Therefore $\angle BP_2A = 2B + 2C - \pi = \pi - 2A$. Thus P_2 also lies on the circle (AB) whose tangent at A makes an angle equal to 2A with AB.

Case 3. The third construction defined above causes the circles (BC) and (CA) to intersect at C and at P_3. And now $\angle BP_3C = 2C - \pi$ and $\angle CP_3A = 2A$. Then $\angle BP_3A = 2C - \pi + 2A = \pi - 2B$. Thus P_3 is also on the arc of the circle (AB) whose tangent at A makes an angle with the chord AB equal to 2B.

<u>Theorem II.</u> Case 1. If the lines CP_1, BP_2 intersect in a point A_1, and AP_1, CP_2 intersect in a point B_1, and BP_1, AP_2 intersect in C_1, then P_1, P_2, A_1, B_1, C_1 are conclycic. (See fig. 2)

Pass a circle through P_1, P_2, and A_1. Designating $\angle P_1BC$ by d, $\angle P_2BC$ by s, $\angle P_1AC$ by e, and $\angle P_2AC$ by r, then

$$\angle P_1A_1P_2 = 2A + d - s$$

but $$\angle P_1C_1P_2 = \pi - (2A + d - s)$$

Thus, since $\angle P_1A_1P_2$ and $\angle P_1C_1P_2$ are supplementary, C_1 must lie on the circle ($P_1P_2A_1$)

Also
$$\begin{aligned}\angle P_1B_1P_2 &= \pi - \angle P\ B\ C \\ &= \pi - (2\pi - \angle B_1P_1B - \angle P_1BC - \angle BCB_1) \\ &= 2A + d - s\end{aligned}$$

Thus, since $\angle P_1B_1P_2$ and $\angle P_1A_1P_2$ are equal, B_1 must be concyclic with A_1.

Case 2. If the lines CP_1, AP_3 intersect in a point A_1', lines AP_1, BP_3 intersect in B_1', and lines BP_1, CP_3 intersect in C_1', then P_1, P_3, A_1', B_1', C_1' are concyclic.

Pass a circle through P_1, P_3, and C_1'. Then designating $\angle P_3BC$ by x and $\angle P_3AC$ by p,

$$\begin{aligned} \angle P_1C'P_3 &= \pi - d - \angle BCC_1' \\ &= \pi - d - (\pi - 2C + \pi - x) \\ &= 2C - \pi - d + x \end{aligned}$$

but

$$\begin{aligned} \angle P_1A'P_3 &= \pi - (\pi - 2B) - (p - e) \\ &= 2C - \pi - d + x \end{aligned}$$

Thus C_1' and A_1' must be concyclic with P_1 and P_3.

Also

$$\begin{aligned} \angle P_1B_1'P_3 &= 2\pi - x - \angle C - e \\ &= 2\pi - 2C + d - x \\ &= \pi - (2C - \pi - d + x) \end{aligned}$$

Therefore B_1' must be concyclic with A_1' and C_1'.

Case 3. If the lines CP_2, BP_3 intersect in A_1'', and AP_2, CP_3 intersect in B_1'', and BP_2, AP_3 intersect in C_1'', then P_2, P_3, A_1'', B_1'', C_1'' are concyclic.

Pass a circle through P_2, P_3, and C_1'', then

$$\begin{aligned} \angle P_2C_1''P_3 &= 2\pi - s - \angle C - p \\ &= \pi - (2B + s - x) \end{aligned}$$

but

$$\angle P_2A_1''P_3 = 2B + s - x$$

and

$$\begin{aligned} \angle P_2B''P_3 &= \pi - r - \angle ACB_1'' \\ &= 2B + s - x \end{aligned}$$

Thus C_1'', A_1'', and B_1'' must be concyclic with P_2 and P_3.

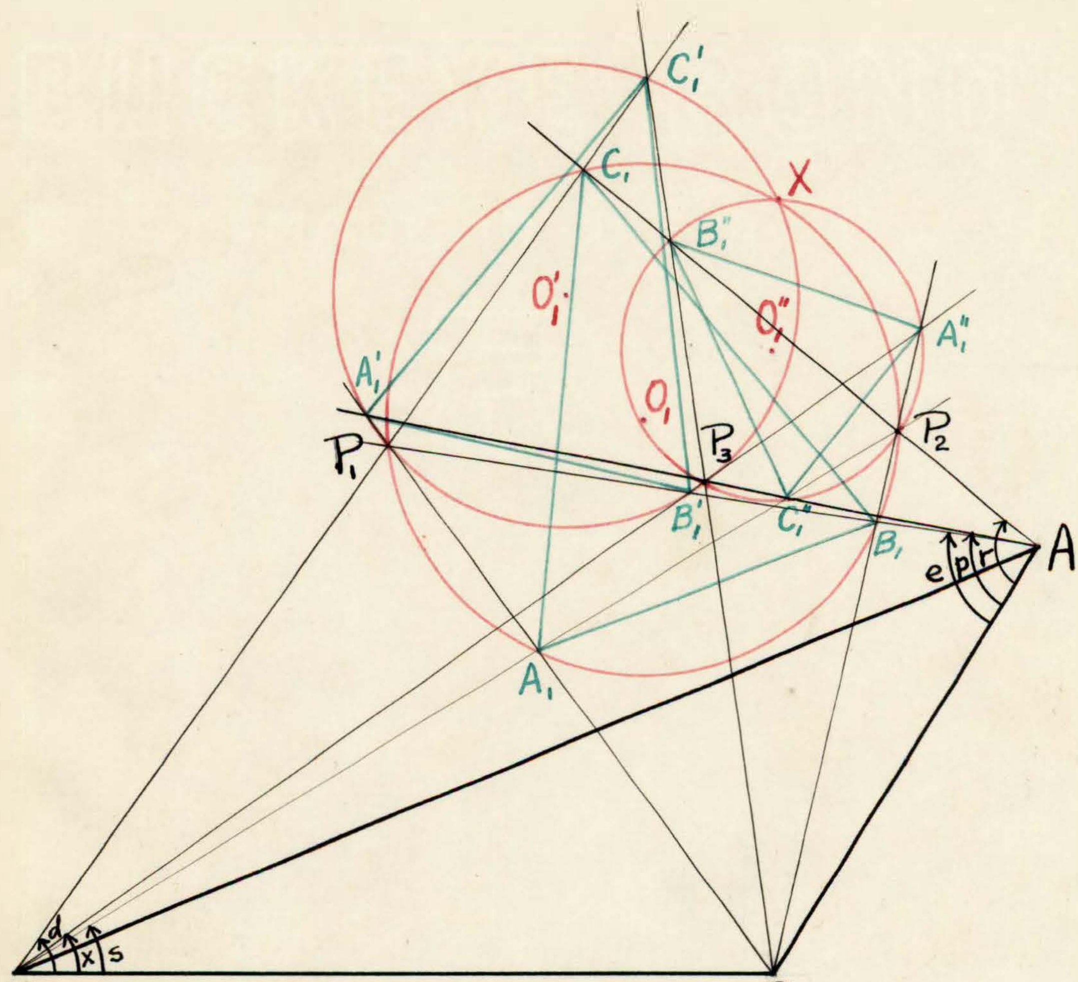

Fig. 2.

<u>Theorem III.</u> The three circles ($P_1 P_2 A_1 B_1 C_1$), ($P_1 P_3 A'_1 B'_1 C'_1$), and ($P_2 P_3 A''_1 B''_1 C''_1$) intersect in a point.

The two circles ($P_1 P_3$) and ($P_2 P_3$) have the point P_3 in common and also intersect in another point X. Now as shown in Case 2, Theo. II $\angle P_1 X P_3 = 2C - \pi - d + x$ and by Case 3, Theo. II $\angle P_3 X P_2 = 2B + s - x$

Hence $\angle P_1 X P_2 = (2C - \pi - d + x) + (2B + s - x)$

$$= \pi - (2A + d - s)$$

Thus by Case 1, Theo. II, the point X also lies on the arc of the circle ($P_1 P_2$) .

<u>Definition:</u> Let the centers of the circles through $P_1 BC$, $P_1 AC$, $P_1 AB$ be x_1, x_2, x_3 respectively; the centers of the circles through $P_2 BC$, $P_2 CA$, $P_2 AB$ be x'_1, x'_2, x'_3 respectively; and the centers of the circles through $P_3 BC$, $P_3 CA$, $P_3 AB$ be x''_1, x''_2, x''_3 respectively. (See fig. 3)

<u>Theorem IV.</u> The triangles $x_1 x_2 x_3$, $x'_1 x'_2 x'_3$, $x''_1 x''_2 x''_3$ and the triangles $A_1 B_1 C_1$, $A'_1 B'_1 C'_1$, $A''_1 B''_1 C''_1$ are all similar.

As a line of centers to a common chord $x_1 x_3$ is perpendicular to BP_1 and $x_2 x_3$ perpendicular to AP_1. Therefore $\angle x_1 x_3 x_2$ is supplementary to angle $AP_1 B$ and thus equal to $2C - \pi$. Also $x_1 x_2$ is perpendicular to CP_1, so $\angle x_1 x_2 x_3$ is directly equal to $\angle AP_1 C$ or 2B, and $\angle x_2 x_1 x_3$ is equal to $\angle BP_1 C$ or 2A.

Also $x'_1 x'_2$, $x'_1 x'_3$, $x'_2 x'_3$ are perpendicular to CP_2, BP_2, and AP_2 respectively, so that $\angle x'_1 x'_3 x'_2$ is eequal to 2A, $\angle x'_1 x'_2 x'_3$ is equal to $2C - \pi$, and $\angle x'_2 x'_1 x'_3$ is equal to 2B.

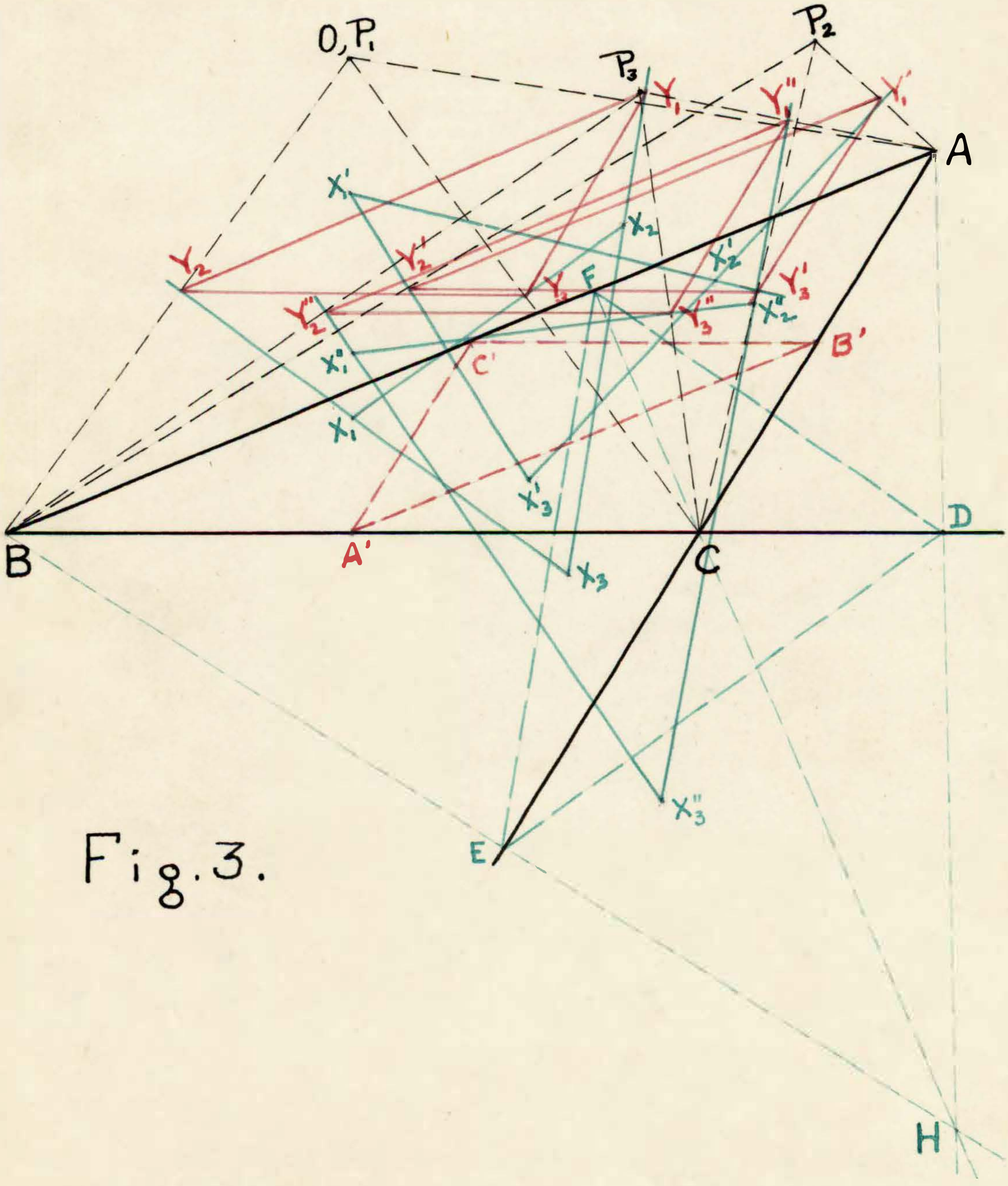
O, P₁
P₂
P₃
Y₁
Y''₁
Y'₁
A
X'₁
X₂
Y₂
Y'₂
X'₂
Y₃
F
Y'₃
Y''₂
Y''₃
X''₂
X''₁
C'
B'
X₁
X'₃
B
A'
C
D
X₃
X''₃
E
H

Fig.3.

Likewise $x_1''x_2''$, $x_1''x_3''$, $x_2''x_3''$ are perpendicular to CP_3 , BP_3 , AP_3 respectively and $\angle x_1''x_3''x_2''$ is equal to 2B, $\angle x_1''x_2''x_3''$ is equal to 2A, and $\angle x_2''x_1''x_3''$ is equal to $2C - \pi$.

Now in the triangle $A_1B_1C_1$, $\angle A_1B_1C_1$ is supplementary to $\angle A_1P_1C_1$ and thus equal to $\angle BP_1C$ or 2A. Also $\angle A_1C_1B_1$ is equal to $\angle CP_1A$ or 2B, and $\angle B_1A_1C_1$ is equal to $\angle AP_2C$ or $2C - \pi$.

Similarly in triangle $A_1'B_1'C_1'$, $\angle A_1'B_1'C_1'$ is equal to $\angle BP_1C$ or 2A, $\angle A_1'C_1'B_1'$ is supplementary to $\angle A_1'P_1B_1'$ or equal to 2B, and $\angle B_1'A_1'C_1'$ is supplementary to $\angle B_1'P_3C_1'$ or equal to $2C - \pi$.

In the triangle $A_1''B_1''C_1''$, $\angle A_1''B_1''C_1''$ is equal to $\angle BP_2C$ or 2B, $\angle A_1''C_1''B_1''$ is equal to $\angle BP_3C$ or to $2C - \pi$, and $\angle B_1''A_1''C_1''$ is equal to $\angle AP_3C$ or to 2A.

Hence the triangles $x_1x_2x_3$, $x_1'x_2'x_3'$, $x_1''x_2''x_3''$, $A_1B_1C_1$, $A_1'B_1'C_1'$, $A_1''B_1''C_1''$ are similar.

<u>Theorem V.</u> The point P_1 coincides with O, the circumcenter of ABC, and therefore takes on the properties of that point.

$\angle BOC = 2A = \angle BP_1C$ by Casel, Theo. I

hence O must lie on the circle through B, P_1, and C.

$\angle COA = 2B = \angle CP_1A$ by Case 1, Theo. I

thus O must lie on the circle through A, P_1 , and C.

also $\angle AOB = 2\pi - 2C = \angle AP_1B$ by Case 1, Theo. I

therefore O must be concyclic with A, P_1 , and B. But if O is to be on all three circles, it must be coincident with their common point of intersection which was shown to be P_1 in Theo. I

<u>Definition:</u> Two triangles are said to be orthologic if perpendiculars dropped from the vertices of one to the corresponding sides of the of the other are concurrent.

<u>Theorem VI.</u> Case 1. The two triangles ABC and $x_1'x_2'x_3'$ are orthologic, the points of concurrency being P_2 and O; triangles ABC and $x_1''x_2''x_3''$ are orthologic, the concurrency points being P_3 and O.

It may easily be shown from the fundamental construction that the perpendicular dropped from x_1' to BC passes through O, while the perpendicular from C to $x_1'x_2'$ passes through P_2, etc.

Case 2. The two triangles ABC and $x_1x_2x_3$ are orthologic and have a common point of concurrency at O.

The proof of this also follows from the fundamental construction as in Case 1 of this theorem and also from the fact of Theorem V.

<u>Corollary 1.</u> The triangles $x_1x_2x_3$, $x_1'x_2'x_3'$, $x_1''x_2''x_3''$, and A'B'C' (the pedal triangle of the circumcenter) are perspective, the center of perspection being O.

This follows from the theorem since the vertices x_1, x_1', x_1'' lie on the perpendicular bisector of BC which also determines A'. Likewise B' and C' are determined by the lines containing the other two sets of vertices. And these perpendicular bisectors are concurrent at O.

<u>Corollary 2.</u> The triangles $x_1x_2x_3$ and DEF (the pedal triangle of the orthocenter) are homothetic.

By the fundamental construction as illustrated in Theo. VI the side x_1x_2 is perpendicular to the circumradius OC. But the radius of the circumcircle passing through a vertex of the triangle is perpendicular to the corresponding side of the orthic triangle DEF. (See Court's College Geometry #144.)

Thus OC is perpendicular to DE and DE is therefore parallel to x_1x_2. Similarly EF and x_2x_3, FD and x_1x_3 are parallel.

Moreover the homothetic center, being collinear with O and H (the orthocenter), is on the Euler line of the triangle.

Corollary 3. The triangles $x_1'x_2'x_3'$, $x_1''x_2''x_3''$, $A_1B_1C_1$, $A_1'B_1'C_1'$, and $A_1''B_1''C_1''$ are all similar to DEF.

Theo. IV stated that $x_1'x_2'x_3'$, $x_1''x_2''x_3''$, $A_1B_1C_1$, $A_1'B_1'C_1'$, and $A_1''B_1''C_1''$ were all similar triangles to $x_1x_2x_3$ which from the preceeding corollary is homothetic with DEF.

Construction: Let Y_1, Y_2, Y_3 be the feet of the perpendiculars dropped from A, B, C respectively on x_2x_3, x_1x_3, x_1x_2; also Y_1', Y_2', Y_3' the feet of the perpendiculars from A, B, C to $x_2'x_3'$, $x_1'x_3'$, $x_1'x_2'$ respectively; and Y_1'', Y_2'', Y_3'' the feet of the perpendiculars from A, B, C respectively to $x_2''x_3''$, $x_1''x_3''$, $x_1''x_2''$

Theorem VII. The triangles $Y_1Y_2Y_3$ and ABC are homothetic with their homothetic center at O and their homothetic ratio equal to one-half; the triangles $Y_1'Y_2'Y_3'$ and ABC have their homothetic center at P_2 and their homothetic ratio equal to one-half; also $Y_1''Y_2''Y_3''$ and ABC are homothetic having their homothetic center at P_3 and their homothetic ratio equal to one-half.

That the triangles $Y_1Y_2Y_3$, $Y_1'Y_2'Y_3'$, $Y_1''Y_2''Y_3''$ are each homothetic with ABC rests essentially upon the fact that they are respectively the pedal triangles of A, B, C on the triangles $x_1x_2x_3$, $x_1'x_2'x_3'$, $x_1''x_2''x_3''$ which are orthologic with ABC.

To show that the ratio of similitude is equal to one-half, it is only necessary to recall that OB and x_1x_3 are perpendicular to each other at Y_2; also that x_1 was determined on the perpendicular bisector of OB. Thus Y_2x_1 is the perpendicular bisector of OB and Y_2 is its midpoint. Similarly Y_1 may be shown to be the midpoint of OA and Y_3 the midpoint of OC.

Likewise by construction Y'_1 is the midpoint of P_2A, Y'_2 the midpoint of P_2B, and Y'_3 the midpoint of P_2C. Also Y''_1, Y''_2, Y''_3 are respectively the midpoints of P_3A, P_3B, P_3C.

Corollary 1. The triangles $Y_1Y_2Y_3$, $Y'_1Y'_2Y'_3$, $Y''_1Y''_2Y''_3$, and $A'B'C'$ are congruent and homothetic with each other.

All four triangles are homothetic with ABC and bear the same homothetic ratio with respect to it.

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