Number Theoretical and Dynamical Properties of Euclidean Lattices and Their Sublattices

Dissertation

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Abstract

In this dissertation, we study the dynamical and number theoretical behaviors of orbits of lattices in homogeneous spaces. This work is composed of three relatively independent topics.

The first part explores the Diophantine approximation and dynamical properties of the successive minima of Euclidean lattices. We define Diophantine approximation properties and Dani's correspondence with respect to higher successive minima and use a variational principle in parametric geometry of numbers to show that badly approximable matrices of higher orders have full Hausdoff dimension. We also establish estimates of the Haar measure of the sublevel sets of successive minima functions in the space of unimodular lattices. We also prove a few number theoretical and dynamical properties associated to successive minima.

The second part, based on a joint work with Michael Bersudsky, establishes an equidistribution phenomenon of dense orbits in a space of sublattices of rank m in $\mathbb{R}^{m+1}$ under discrete group actions. We study the limiting distribution of dense orbits of a lattice subgroup $\Gamma \leq \mathrm{SL}(m+1, \mathbb{R})$ acting on $H \backslash \mathrm{SL}(m+1, \mathbb{R})$, with respect to a filtration of growing norm balls. One of the main challenges in this work is that the groups H we consider have infinitely many non-trivial connected components. For a specific such H, the homogeneous space $H \backslash G$ identifies with $X_{m,m+1}$, a moduli space of (oriented) rank m-discrete subgroups in $\mathbb{R}^{m+1}$. The proof uses linearization technique and duality principle.

The third part, based on a joint work with Michael Bersudsky and Nimish Shah, studies the equidistribution of definable curves in a polynomially bounded o-minimal structure in homogeneous spaces. For an algebraic subgroup $G \leq \mathrm{SL}(n, \mathbb{R})$ and a lattice $\Gamma \leq G$. We consider definable curves $\{\gamma(t)\} \subset G$ in a polynomially bounded structure that are unipotent upper triangular and show such curves are equidistributed along its orbit closure under homogeneous measure. The proof relies on linearization technique and (C, α)-good properties for certain families of definable functions in a polynomially bounded o-minimal structure.

To my humble self, and everyone with me in this journey

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As an avid book fan of J.K. Rowling's *Harry Potter*, I find myself often draw parallels between Harry's journey and my own and ask this question: what are the most important things that made him into a successful wizard in the magic world? At the end of my journey as a PhD student, I believe they are love, friendship, support and dedication. If there is a reason that I could keep growing my karma towards professional scholarship, it is by such merits of the people who helped me along this journey, for which I am deeply appreciative.

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Publications

Research Publications

Michael Bersudsky and Hao Xing Limiting distribution of dense orbits in a moduli space of rank m discrete subgroups in $(m+1)$-space. *International Mathematics Research Notices (IMRN)*, 2024.

Fields of Study

Major Field: Mathematics

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Throughout this chapter, all norms of vectors in an Euclidean space without specification are assumed to be the maximum Euclidean norm.

1.1 Diophantine approximation and successive minima of lattices

1.1.1 Dirichlet's Theorem and Diophantine Approximation

Diophantine approximation is a branch of number theory studying the approximation of real numbers by rational numbers. The first known result of this kind is due to Adrien-Marie Legendre, which can be proved by the Pigeonhole Principle:

Theorem 1.1.1 (Legendre, 1808)**.** *For any real number x and every $Q > 1$, there exists an integer vector $(p, q) \in \mathbb{Z}^2$ such that*

$$|xq - p| < \frac{1}{Q} \textit{ and } 0 < q < Q.$$

The Classical multidimensional version of approximation theorem, Due to Johann Peter Gustav Lejeune Dirichlet, states that

Theorem 1.1.2 (Dirichlet's approximation theorem [Dir42], 1842)**.** *For every real $m \times n$ matrix A and every $Q > 1$, there exists an integer vector $\boldsymbol{p} \in \mathbb{Z}^m$ and $\boldsymbol{q} \in \mathbb{Z}^n$ such that*

$$\|A\boldsymbol{q} - \boldsymbol{p}\| < \frac{1}{Q^{\frac{n}{m}}} \textit{ and } 0 < \boldsymbol{q} < Q.$$

The modern proof of Dirichlet's approximation theorem often uses a generalized version of "Pigeonhole Principle" for lattices in Euclidean spaces, called the (first) convex body theorem, due to Hermann Minkowski, applied to an appropriately chosen convex set[1]:

Theorem 1.1.3 (Minkowski [Min96], 1889)**.** *Suppose that L is a unimodular lattice in $\mathbb{R}^d$ of covolume 1 and S is a convex subset of $\mathbb{R}^d$ that is symmetric with respect to the origin (namely $x \in S$ if and only if $-x \in S$). If the volume of S is strictly greater than 2^n, then S must contain at least one lattice point other than the origin.*

1.1.2 Matrix diophantine approximation and Dani's correspondence principle

We begin by introducing two classes of matrices that generalizes the Diophantine approximation of real numbers.

Definition 1.1.4. Let M denote the set of all $m \times n$ matrices with real entries. A matrix $A \in M$ is called *singular* if for all $\epsilon > 0$, there exists Q_ϵ such that for all $Q \geq Q_\epsilon$, there exist integer vectors $\mathbf{p} \in \mathbb{Z}^m$ and $\mathbf{q} \in \mathbb{Z}^n$ such that

$$\|A\mathbf{q} + \mathbf{p}\| \leq \epsilon Q^{-n/m} \text{ and } 0 < \|\mathbf{q}\| < Q \tag{1.1.1}$$

Here $\|\cdot\|$ denotes an arbitrary norm on $\mathbb{R}^m$ or $\mathbb{R}^n$. We denote the set of singular $m \times n$ matrices by $\mathbf{Sing}(m, n)$.

An $m \times n$ matrix A is called *badly approximable* if there exists $c > 0$ such that for all integer vectors $\mathbf{p} \in \mathbb{Z}^m$ and $\mathbf{q} \in \mathbb{Z}^n - \{0\}$ we have

$$\|A\mathbf{q} + \mathbf{p}\| \geq c\|\mathbf{q}\|^{-\frac{n}{m}}.$$

We denote the collection of badly approximable $m \times n$ matrices by $\mathbf{BA}(m, n)$.

[1] For the special case when $m = n = 1$, this convex set can be chosen as

$$S = \left\{(x, y) \in \mathbb{R}^2 : -N - \frac{1}{2} \leq x \leq N + \frac{1}{2}, |\alpha x - y| \leq \frac{1}{N}\right\}.$$

Example 1.1.5. When $m = n = 1$, being singular is the same as being rational. If A is a rational number, then it trivially satisfies (1.1.1). Conversely, by Hurwitz's approximation theorem [BE02], given any irrational number α, there exist infinitely many rational numbers p/q with $(p, q) = 1$ such that

$$|\alpha - \frac{p}{q}| < \frac{1}{\sqrt{5}q^2},$$

and that $\sqrt{5}$ is optimal. So if $\epsilon < \frac{1}{\sqrt{5}}$, then there exist integers p, q with $(p, q) = 1$ such that $|\alpha - \frac{p}{q}| > \frac{\epsilon}{\sqrt{5}}$ which fails (1.1.1).

By Liouville's theorem [EW11], any quadratic irrational (solutions to quadratic equations over $\mathbb{Z}$) is badly approximable, but it is unknown whether all algebraic numbers are badly approximable.

The sets of badly approximable and singular matrices are linked to homogeneous dynamics via the Dani correspondence principle. For each $t \in \mathbb{R}$ and for each matrix A, let $g_t := \begin{bmatrix} e^{t/m} I_m & 0 \\ 0 & e^{-t/n} I_n \end{bmatrix}$ and $u_A := \begin{bmatrix} I_m & A \\ 0 & I_n \end{bmatrix}$, where I_k denotes the k-dimensional identity matrix.

By the Dani's correspondence principle ([Dan85]), the Diophantine properties of A and the dynamical properties of the orbit $(g_t u_A x_0)_{t \geq 0}$ (which consists of unimodular lattices) can be summarized in the following table. Write $x_0 = \mathbb{Z}^{m+n}$,an element in the space of unimodular lattices in $\mathbb{R}^{m+n}$, also identified with the neutral element in $\mathrm{SL}(m+n, \mathbb{R})/\mathrm{SL}(m+n, \mathbb{Z})$:

Theorem 1.1.6. *Let G be a Lie group and Γ be a unimodular Lattice in G. Consider the G-homgeneous space $X := G/\Gamma$, equipped with the G-invariant Borel probability measure μ_X. Then the G-action on X is ergodic.*

Diophantine properties of A	Dynamical properties of $(g_t u_A x_0)_{t\geq 0}$
A is badly approximable	$(g_t u_A x_0)_{t\geq 0}$ is bounded
A is singular	$(g_t u_A x_0)_{t\geq 0}$ is divergent

Table 1.1: Dani's correspondence.

Proof. Let μ_G be a left Haar measure on G. Let $f \in L^2(X, \mu_X)$ be such that f is invariant under G-action, namely for any $g \in G$

$$f(g.x) = f(x), \text{for } \mu_X\text{-almost every } x.$$

We will show that f is a constant almost everywhere with respect to μ_X.

Consider the product space $G \times X$. We will apply Fubini-Tonelli Theorem to the function $|f(g.x) - f(x)|$ on $G \times X$:

$$0 = \int_G \int_X |f(g.x) - f(x)| d\mu_X d\mu_G = \int_X \int_G |f(g.x) - f(x)| d\mu_G d\mu_X.$$

Hence, $0 = \int_G |f(g.x) - f(x)| d\mu_G(g)$ for μ_X-almost every $x \in X$. Therefore for almost all $x \in X$ (and in particular there exists $x \in X$), there exists $U \subset G$ with $\mu_G(U) = 1$ such that for every $g \in U$,

$$f(g.x) = f(x).$$

That f is μ_X-almost everywhere constant follows from the claim below.

Claim: $\mu_X(U.x) = 1$ for any $x \in X$.

To show this claim, we use the quotient integral formula (See for example Theorem 1.5.3 in [DE14] or Theorem 2.51 in [Fol15]):

Take $h = 1_{(G-U)g_0}$, noticing that $\mu((G - U)g_0) = 0$ because of the unimodularity. Now let μ_Γ be a Haar measure on Γ, by the the quotient integral formula and Fubini's theorem we have:

$$0 = \int_G h d\mu_G = \int_{G/\Gamma} \int_\Gamma h(g\gamma) d\mu_\Gamma \ d\mu_X = \int_\Gamma \int_{G/\Gamma} h(g\gamma) d\mu_X \ d\mu_\Gamma.$$

Since Γ is countable, the Haar measure on it must be a scalar multiple of counting measure. So we must have for every $\gamma \in \Gamma$, in particular for $\gamma = e$,

$$0 = \int_{G/\Gamma} h(g) d\mu_X = \mu_X((G-U)g_0\Gamma).$$

Since $g_0 \in G$ is arbitrary, and by the transitivity of the action, $G/\Gamma - Ux$ is contained in $(G-U).x$, we are done.

$\square$

Theorem 1.1.7 (See Chapter III Corollary 2.2 of [BM00])**.** *If a simple Lie group with finite center acts ergodically on a probability space X, then every subgroup of G with a non-compact closure is strongly mixing, and thus ergodic on X.*

It follows from the ergodicity of (g_t)-action on $\mathrm{SL}(m+n,\mathbb{R})/\mathrm{SL}(m+n,\mathbb{Z})$ and the equidistribution of orbits under ergodic actions that $\mathbf{BA}(m,n), \mathbf{Sing}(m,n)$ and $\mathbf{VSing}(m,n)$ all have Lebesgue measure zero.

To further investigate the sets with various Diophantine properties, we will look into their fractional dimensions. In order to compute the Hausdorff dimension of badly approximable numbers and matrices, Schmidt invented the following topological game [Sch66] [Sch69] called Schmidt Games:

Choose two parameters $0 < \alpha < 1$ and $0 < \beta < 1$. Two players, called Alice and Bob, will play the following game:

- First Bob choose a closed ball B_1 in $\mathbb{R}^d$;
- Then Alice choose closed ball $A_1 \subset B_1$ in $\mathbb{R}^d$ whose radius is α times the radius of B_1

- Next Bob chooses a closed ball $B_2 \subset A_1$ whose radius is β times the radius of A_1
- Then Alice chooses a closed ball $A_2 \subset B_2$ whose radius is α times the radius of B_2
- $\cdots\cdots$

We call a sequence of choices by Alice (resp. Bob) depending on the choices of Bob (resp. Alice) a *strategy*. A set $S \subset \mathbb{R}^d$ is called *(Alice)-winning* if Alice has a strategy to make sure (no matter how Bob chooses his strategy), we have

$$\cap_{k=1}^{\infty} A_k \subset S.$$

Schmidt proved the following

Theorem 1.1.8 (Theorem 2, [Sch69])**.** *The set of badly approximable matrices in $\mathbb{R}^{m\times n}$ is a winning set.*

Theorem 1.1.9 (Corollary 2 to Theorem 6, [Sch66])**.** *Any winning set in an Euclidean space is of full Hausdorff dimension.*

By introducing a modified version of Schmidt's game, Kleinbock and Weiss [KW10] proved that the set of weighted badly approximable matrices is also winning and thus of full Hausdorff dimension. Specifically,

Theorem 1.1.10. *For $r_i, s_j, 1 \leq i \leq m, 1 \leq j \leq n$ with $\sum_{i=1}^m r_i = 1 = \sum_{j=1}^n s_j$, let $\boldsymbol{r} = (r_1, \ldots, r_m)$ and $\boldsymbol{s} = (s_1, \ldots, s_n)$. Then the set of badly approximable matrices with weight $(\boldsymbol{r}, \boldsymbol{s})$, denoted*

$$\mathbf{BA}^{\mathbf{r},\mathbf{s}}(m,n) := \{A \in \mathbb{R}^{m\times n} : \inf_{\mathbf{p}\in\mathbb{Z}^m, \mathbf{q}\in\mathbb{Z}^n - \{0\}} \|Aq - p\|_{\mathbf{r}} \cdot \|q\|_{\mathbf{s}} > 0\},$$

where the notation $\|x\|_{\boldsymbol{r}} := \max\{|x_1|^{\frac{1}{r_1}}, \cdots, |x_m|^{\frac{1}{r_m}}\}$ for any $x \in \mathbb{R}^m$, is a winning set of modified Schmidt game and thus of full Hausdorff dimension mn.

However, it is a major challenge in Diophantine approximations to compute the Hausdorff dimension of the set $\mathbf{Sing}(m,n)$ of singular matrices. The first breakthrough was made in 2011 by Cheung [Che11], who proved that the Hausdorff dimension of $\mathbf{Sing}(2,1)$ is $4/3$; this was extended in 2016 by Cheung and Chevallier [CC16], who proved that the Hausdorff dimension of $\mathbf{Sing}(m,1)$ is $m^2/(m+1)$ for all $m \geq 2$; while Kadyrov, Kleinbock, Lindenstrauss, and Margulis [KKLM17] proved that the Hausdorff dimension of $\mathbf{Sing}(m,n)$ is at most $\delta_{m,n} := mn(1 - \frac{1}{m+n})$. Most recently, Das, Fishman, Simmons and Urbański [DFSU20] proved that this upper bounded is sharp. Their proof is based on a generalized variational principle and is independent of the previous results.

Theorem 1.1.11 ([DFSU20])**.** *For all $(m,n) \neq (1,1)$, we have*

$$\dim_H(\mathbf{Sing}(m,n)) = \dim_P(\mathbf{Sing}(m,n)) = \delta_{m,n},$$

where $\dim_H(S)$ and $\dim_P(S)$ denote the Hausdorff and packing dimensions of a set S, respectively.

Remark 1.1.12. When $m = n = 1, \dim_H(\mathbf{Sing}(m,n)) = \dim_P(\mathbf{Sing}(m,n)) = 0$, since in this case $\mathbf{Sing}(m,n)$ is simply the set of rational numbers.

1.1.3 Successive minima functions and matrices with Diophantine Approximation properties of higher orders

Definition 1.1.13. Let $d = m + n$, and for each $j = 1, \ldots, d$, let $\lambda_j(\Lambda)$ denote the j-th minimum of a lattice $\Lambda \subset \mathbb{R}^d$ (with respect to the l^2 norm on $\mathbb{R}^d$ [2]), i.e. the infimum of λ such that the set $\{r \in \Lambda : \|r\|_2 \leq \lambda\}$ contains j linearly independent vectors.

[2]Note that $\|\cdot\|_\infty \leq \|\cdot\|_2 \leq \sqrt{d}\|\cdot\|_\infty$. So if we use the maximum norm $\|\cdot\| := \|\cdot\|_\infty$ to define λ_i, the resulting λ_i^∞ is equivalent to λ_i up to a multiple constant depending on d, which doesn't change any results below. We use l^2 norm here since it is most common in literature.

Diophantine properties of A	Dynamical properties of $(g_t u_A x_0)_{t\geq 0}$
A is badly approximable	$\sup_{t\geq 0} -\mathbf{h}_{A,1(t)} < \infty$
A is singular	$\lim_{t\to\infty} -\mathbf{h}_{A,1(t)} = \infty$

Table 1.2: Dani's correspondence with successive minima function.

For a $m \times n$ matrix A, the *successive minima function* of the matrix A, denoted $\mathbf{h} = \mathbf{h}_A = (h_1, \ldots, h_d) : [0, \infty) \to \mathbb{R}^d$ is defined by the formula

$$h_i(t) := \log \lambda_i(g_t u_A \mathbb{Z}^d). \tag{1.1.2}$$

Then the Dani's correspondence principle can be translated into the language of successive minima function as follows:

In light of successive minima functions, we can generalize the notion of badly approximable matrices, singular matrices as follows:

Definition 1.1.14. For $r = 1, 2, \ldots, d = m + n$, A matrix $A \in M(m \times n, \mathbb{R})$ is called **badly approximable of order** r if

$$\sup_{t\geq 0} -\mathbf{h}_{A,r}(t) < \infty.$$

$A \in M(m \times n, \mathbb{R})$ is called **singular of order** r if

$$\lim_{t\to\infty} -\log \mathbf{h}_{A,r}(t) = \infty.$$

Let $\mathbf{BA}_r(m, n)$ (resp. $\mathbf{Sing}_r(m, n)$) denote the set of badly approximable (resp. singular) $m \times n$ matrices. $\mathbf{BA}_r(m, n)$ (resp. $\mathbf{Sing}_r(m, n)$) form an ascending (descending) sequence of sets in r.

1.1.4 Dani's correspondence for Diophantine Approximation properties of higher orders

For $r = 1, 2, ..., d$ and a lattice Λ, let $I^r(\Lambda)$ denote the set of all r-tuples of linearly independent vectors $(v_1, \ldots, v_r)$.

Theorem 1.1.15. *A matrix $A \in \mathbb{R}^{m\times n}$ is badly approximable of order r if and only if there exists $c > 0$ such that for all linearly independent r vectors $(\boldsymbol{p}_1, \boldsymbol{q}_1), \ldots, (\boldsymbol{p}_r, \boldsymbol{q}_r) \in \mathbb{Z}^m \times (\mathbb{Z}^n - \{0\})$, there exists $1 \le i \le r$ satisfying*

$$\|A\boldsymbol{q}_i - \boldsymbol{p}_i\| \ge \frac{c}{\|\boldsymbol{q}_i\|^{\frac{n}{m}}}. \tag{1.1.3}$$

Proof. For the forward implication, notice that the definition of badly approximable of order r is equivalent to saying that there exists $\delta > 0$ such that

$$\lambda_r(g_t u_A \mathbb{Z}^d) \ge \delta, \forall t > 0.$$

Using our I^r notation, this is the same as saying there exists $\delta > 0$ such that

$$I^r(g_t u_A \mathbb{Z}^d) \cap B_\delta^r = \varnothing, \forall t \ge 0. \tag{1.1.4}$$

where B_δ denotes the (open) ball in $\mathbb{R}^d$ centered at the origin with radius δ and B_δ^r means its r-fold Cartesian product.

But

$$g_t u_A \begin{bmatrix} \mathbf{p} \\ \mathbf{q} \end{bmatrix} = \begin{bmatrix} e^{\frac{t}{m}} I_m & \\ & e^{-\frac{t}{n}} I_n \end{bmatrix} \begin{bmatrix} I_m & A \\ & I_n \end{bmatrix} \begin{bmatrix} \mathbf{p} \\ \mathbf{q} \end{bmatrix} = \begin{bmatrix} e^{\frac{t}{m}}(\mathbf{p} + A\mathbf{q}) \\ e^{-\frac{t}{n}} \mathbf{q} \end{bmatrix}. \tag{1.1.5}$$

So the above equation 1.1.4 implies (from here we change the 2-norm to ∞-norm) that there exists $\delta > 0$ such that for all linearly independent r vectors $(\mathbf{p}_1, \mathbf{q}_1), \ldots, (\mathbf{p}_r, \mathbf{q}_r) \in \mathbb{Z}^m \times (\mathbb{Z}^n - \{0\}) = \mathbb{Z}^d$, there exists $1 \le i \le r$ satisfying

$$\|e^{\frac{t}{m}}(A\mathbf{q}_i + \mathbf{p}_i)\| \ge \delta \text{ or}$$

$$\|e^{-\frac{t}{n}}\mathbf{q}_i\| \ge \delta$$

for all $t \geq 0$. Note that since $\mathbf{q}_i \neq 0$, we can choose $t = t(\delta, q_i)$ so that $\|e^{-\frac{t}{n}}\mathbf{q}_i\| = \frac{\delta}{2} < \delta$, then the second possibility is blocked and

$$\|A\mathbf{q}_i + \mathbf{p}_i\| \geq e^{-\frac{t}{m}}\delta = (e^{-\frac{t}{n}})^{\frac{n}{m}}\delta = \left(\frac{\delta/2}{\|\mathbf{q}_i\|}\right)^{\frac{n}{m}} \delta =: \frac{c}{\|\mathbf{q}_i\|^{\frac{n}{m}}}.$$

For the backward implication, there exists $c > 0$ such that for all linearly independent r vectors $(\mathbf{p}_1, \mathbf{q}_1), \ldots, (\mathbf{p}_r, \mathbf{q}_r) \in \mathbb{Z}^m \times (\mathbb{Z}^n - \{0\})$, there exists $1 \leq i \leq r$ satisfying

$$\|A\mathbf{q}_i - \mathbf{p}_i\| \geq \frac{c}{\|\mathbf{q}_i\|^{\frac{n}{m}}}.$$

We want to find $\delta > 0$ such that

$$\lambda_r(g_t u_A \mathbb{Z}^d) \geq \delta,$$

for all $t \geq 0$. From the computation of product 1.1.5, this is the same as there exists $\delta > 0$, such that for all $t \geq 0$,

$$\lambda_r\left(\begin{bmatrix} e^{\frac{t}{m}}(\mathbf{p} + A\mathbf{q}) \\ e^{-\frac{t}{n}}\mathbf{q} \end{bmatrix} : \mathbf{p} \in \mathbb{Z}^m, \mathbf{q} \in \mathbb{Z}^n\right) \geq \delta$$

Suppose on the contrary that this is not possible, then for any $\delta > 0$, there exists $t \geq 0$ and linearly independent r vectors $(\mathbf{p}_1, \mathbf{q}_1), \ldots, (\mathbf{p}_r, \mathbf{q}_r) \in Z^m \times \mathbb{Z}^n$ satisfying

$$\|e^{\frac{t}{m}}(A\mathbf{q}_i + \mathbf{p}_i)\| < \delta \text{ and}$$
$$\|e^{-\frac{t}{n}}\mathbf{q}_i\| < \delta$$

for all $1 \leq i \leq r$. But the first equation times the $\frac{n}{m}$-th power of the second equation yields

$$\|A\mathbf{q}_i + \mathbf{p}_i\| \cdot \|\mathbf{q}_i\|^{\frac{n}{m}} < \delta^{\frac{m+n}{m}}$$

for $i = 1, 2, ..., r$, contradicting to (1.1.3) whenever $c < \delta^{\frac{m+n}{m}}$. □

Theorem 1.1.16. *A matrix $A \in \mathbb{R}^{m\times n}$ is singular of order r if and only if for any $\epsilon > 0$ there exists $Q_\epsilon > 0$ such that for all $Q > Q_\epsilon$, there exist r linearly independent vectors*

$(\boldsymbol{p}_1, \boldsymbol{q}_1), \ldots, (\boldsymbol{p}_r, \boldsymbol{q}_r) \in \mathbb{Z}^m \times (\mathbb{Z}^n - \{0\})$ *satisfying*

$$\|A\boldsymbol{q}_i - \boldsymbol{p}_i\| \leq \frac{\delta}{\|\boldsymbol{q}_i\|^{\frac{n}{m}}},$$
$$\textit{and } 0 < \|\boldsymbol{q}_i\| < Q.$$

for all $1 \leq i \leq d$.

Proof. We first prove the forward direction. The aim is to find Q_ϵ. Fix $\epsilon > 0$, the definition of singularity of order r is the same as saying that for any $\delta > 0$, there exists $T_\delta > 0$ such that for all $t \geq T_\delta$,

$$\lambda_r(g_t u_A \mathbb{Z}^d) < \delta.$$

In view of the equation 1.1.5, this is equivalent to that for any $\delta > 0$, there exists $T_\delta > 0$ such that for all $t \geq T_\delta$, there exist linearly independent r vectors $(\mathbf{p}_1, \mathbf{q}_1), \ldots, (\mathbf{p}_r, \mathbf{q}_r) \in Z^m \times \mathbb{Z}^n = \mathbb{Z}^d$ with

$$\begin{cases} \|e^{\frac{t}{m}}(A\mathbf{q}_i + \mathbf{p}_i)\| < \delta \\ \|e^{-\frac{t}{n}}\mathbf{q}_i\| < \delta \end{cases} \tag{1.1.6}$$

for all $1 \leq i \leq d$.

In order to find the Q_ϵ we need, we consider the system of inequality

$$\begin{cases} e^{-\frac{t}{m}}\delta \leq \epsilon Q^{-\frac{n}{m}} \\ e^{\frac{t}{n}}\delta \leq Q \end{cases} \tag{1.1.7}$$

and solve it for Q. Note that this is equivalent to

$$\delta e^{\frac{t}{n}} \leq Q \leq \left(\frac{\epsilon}{\delta}\right)^{\frac{m}{n}} e^{\frac{t}{n}} \tag{1.1.8}$$

We first fix $\delta := \frac{1}{2}\epsilon^{\frac{m}{m+n}} < \epsilon^{\frac{m}{m+n}}$ so that 1.1.8 is solvable for Q and it follows that as long as

$$Q \in I = \cup_{t \geq T_\delta} I(t),$$

where

$$I(t) := \left[\delta e^{\frac{t}{n}}, \left(\frac{\epsilon}{\delta}\right)^{\frac{m}{n}} e^{\frac{t}{n}}\right],$$

then 1.1.8 holds. Therefore, our choice of Q_ϵ in the statement of the theorem can be

$$Q_\epsilon := \frac{1}{2}\epsilon^{\frac{m}{m+n}} e^{\frac{T_\delta}{n}}.$$

For the backward direction, suppose now for any $\epsilon > 0$ there exists $Q_\epsilon > 0$ such that for all $Q > Q_\epsilon$, there exist r linearly independent vectors $(\mathbf{p}_1, \mathbf{q}_1), \ldots, (\mathbf{p}_r, \mathbf{q}_r) \in \mathbb{Z}^m \times (\mathbb{Z}^n - \{0\})$ satisfying

$$\begin{aligned} &\|A\mathbf{q}_i - \mathbf{p}_i\| \leq \frac{\delta}{\|\mathbf{q}_i\|^{\frac{n}{m}}} \text{ and} \\ &0 < \|\mathbf{q}_i\| < Q, \end{aligned} \tag{1.1.9}$$

for all $1 \leq i \leq d$.

For any $\delta > 0$, we want to find $T_\delta > 0$ such that for any $t \geq T_\delta$,

$$\lambda_r(g_t u_A \mathbb{Z}^d) < \delta.$$

Again from 1.1.5, what we need is

$$\begin{cases} \|e^{\frac{t}{m}}(A\mathbf{q}_i + \mathbf{p}_i)\| < \delta \\ \|e^{-\frac{t}{n}}\mathbf{q}_i\| < \delta \end{cases}$$

From 1.1.9, we solve

$$\begin{cases} e^{\frac{t}{m}}\epsilon Q^{-\frac{n}{m}} < \delta \\ e^{-\frac{t}{n}}Q < \delta \end{cases} \tag{1.1.10}$$

for t as

$$n \log\left(\frac{Q}{\delta}\right) < t < m \log\left(\frac{\delta}{\epsilon} Q^{\frac{n}{m}}\right)$$

This is solvable as long as we choose $\epsilon := \frac{1}{2}\delta^{\frac{m+n}{m}} < \delta^{\frac{m+n}{m}}$. Let

$$J = \cup_{Q \geq Q_\epsilon} J_Q,$$

where $J_Q := \left[n \log\left(\frac{Q}{\delta}\right), m \log\left(\frac{\delta}{\epsilon} Q^{\frac{n}{m}}\right)\right]$. It follows that the T_δ we need can be taken as

$$T_Q := n \log\left(\frac{Q_\epsilon}{\delta}\right).$$

$\square$

1.2 Fractal Dimensions and Variational Principles

1.2.1 Hausdorff and packing dimensions and a theorem of Rogers-Taylor-Tricot

The *s-dimensional Hausdorff measure* of a set $S \subset \mathbb{R}^D$ is defined to be

$$\mathscr{H}^s(S) := \lim_{\delta\to 0} \inf \left\{ \sum_{i=1}^{\infty} (\operatorname{diam} U_i)^s : \bigcup_{i=1}^{\infty} U_i \supseteq S, \operatorname{diam} U_i < \delta \right\} \tag{1.2.1}$$

The *s-dimensional packing measure* of a set S is defined as

$$\mathscr{P}^s(S) = \inf \left\{ \sum_{j\in J} \mathscr{P}_0^s(S_j) \middle| S \subseteq \bigcup_{j\in J} S_j, J \text{ countable} \right\}, \tag{1.2.2}$$

where $\mathscr{P}_0^S$, called the *s-dimensional packing pre-measure*, is defined as

$$\mathscr{P}_0^s(S) = \limsup_{\delta\to 0} \left\{ \sum_{i\in I} \operatorname{diam}(B_i)^s \middle| \begin{array}{c} \{B_i\}_{i\in I} \text{ is a countable collection} \\ \text{of pairwise disjoint closed balls with} \\ \text{diameters } \le \delta \text{ and centres in } S \end{array} \right\}. \tag{1.2.3}$$

Given the measures defined above, we define the Hausdorff dimension and packing dimension of a set $S \subset \mathbb{R}^D$ as follows:

$$\dim_{\mathrm{H}}(S) := \inf\{d \ge 0 : \mathscr{H}^d(S) = 0\} = \sup\left(\{d \ge 0 : \mathscr{H}^d(S) = \infty\} \cup \{0\}\right), \tag{1.2.4}$$

$$\dim_{\mathrm{P}}(S) := \sup\{s \ge 0 | \mathscr{P}^s(S) = +\infty\} = \inf\{s \ge 0 | \mathscr{P}^s(S) = 0\}. \tag{1.2.5}$$

Let μ be a Borel probability measure on $\mathbb{R}^d$ and $x \in \mathbb{R}^d$, we define the lower and upper pointwise dimensions of the measure μ at x by

$$\underline{\dim}_x(\mu) := \liminf_{r\to 0} \frac{\log(\mu(B(x,r)))}{\log(r)},$$

and

$$\overline{\dim}_x(\mu) := \limsup_{r\to 0} \frac{\log(\mu(B(x,r)))}{\log(r)}.$$

Remark 1.2.1. By a squeezing argument, we can replace the liminf and limsup on the right hand sides by the lininf and limsup over any sequence $r_n \to 0$ with r_n/r_{n+1} bounded. Namely, $\liminf_{r\to 0} \frac{\log(\mu(B(x,r)))}{\log(r)} = \liminf_{n\to\infty} \frac{\log(\mu(B(x,r_n)))}{\log(r_n)}$,and similarly for limsup.

The following powerful theorem introduces a method of computing the Hausdorff and packing dimensions of a set in terms of local geometric-measure-theoretic information, which plays an important role in establishing the relation between Hausdorff dimension and game:

Theorem 1.2.2 (Rogers–Taylor–Tricot, [Fal97] Proposition 2.3)**.** *Fix $d \in \mathbb{N}$ and let μ be a locally finite Borel measure on R^d. Then for every Borel set $A \subset \mathbb{R}^d$,*

- *If $\underline{\dim}_x(\mu) \geq s$ for all $x \in A$ and $\mu(A) > 0$, then $\dim_H(A) \geq s$.*
- *If $\underline{\dim}_x(\mu) \leq s$ for all $x \in A$, then $\dim_H(A) \leq s$.*
- *If $\overline{\dim}_x(\mu) \geq s$ for all $x \in A$ and $\mu(A) > 0$, then $\dim_P(A) \geq s$.*
- *If $\overline{\dim}_x(\mu) \leq s$ for all $x \in A$, then $\dim_P(A) \leq s$.*

1.2.2 Templates

Recall the definition of successive minina (Definition 1.1.13). The key idea of variational principles is to approximate successive minina function $\mathbf{h}$ by piecewise linear functions called templates, define appropriate averaging quantities for templates and study the relation between such quantities and the fractional dimension.

Definition 1.2.3 ([DFSU20])**.** An $m \times n$ *template* is a piecewise linear map $\mathbf{f} : [0, \infty) \to \mathbb{R}^d$ with the following properties:

(I) $f_1 \leq \cdots \leq f_d$.

(II) $-\frac{1}{n} \leq f_i' \leq \frac{1}{m}$ for all i.

(III) For all $j = 0, \ldots, d$ and for every interval I such that $f_j < f_{j+1}$ on I, the function $F_j := \sum_{0<i\leq j} f_i$ is convex and piecewise linear on I with slopes in

$$Z(j) := \{\frac{L_+}{m} - \frac{L_-}{n} : L_\pm \in [0, d_\pm]_{\mathbb{Z}}, L_+ + L_- = j\} \tag{1.2.6}$$

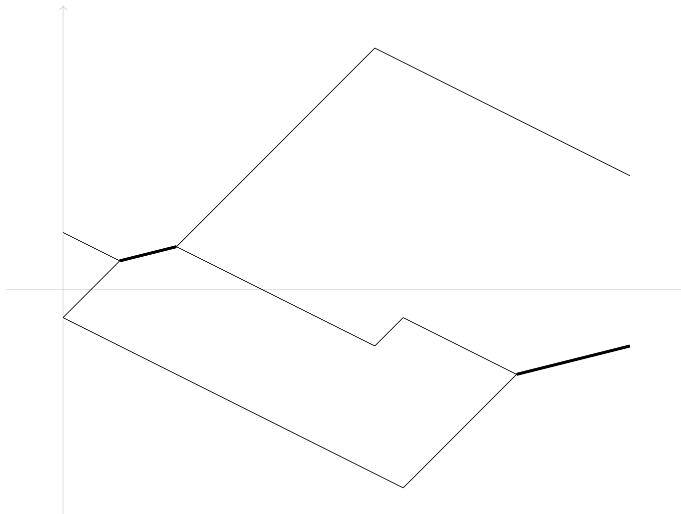

Figure 1.1: A graph of a 1×2 partial template $f = (f_1, f_2, f_3)$.

Here $d_+ := m, d_- := n, [a, b]_{\mathbb{Z}} := [a, b] \cap \mathbb{Z}$.

We use the convention that $f_0 = -\infty$ and $f_{d+1} = +\infty$. We will call the assertion that F_j is convex the *convexity condition*, and the assertion that its slopes are in $Z(j)$ the *quantized slope condition*. We denote the space of $m \times n$ templates by $\mathcal{T}_{m,n}$

Observe that $F_d = f_1 + ... + f_d$ is always a constant due to the property (III) above. A template $\mathbf{f}$ is called *balanced* if $F_d = 0$. A *partial template* is a piecewise linear map f satisfying (I)-(III) whose domain is a closed, possibly infinite, subinterval of $[0, \infty)$.

The fundamental relation between templates and successive minima functions is given as follows:

Theorem 1.2.4 ([DFSU20])**.**

(i) For every $m \times n$ matrix A, there exists an $m \times n$ template $\mathbf{f}$ such that $\mathbf{h}_A \asymp_+ \mathbf{f}$.

(ii) For every $m \times n$ template $\mathbf{f}$, there exists an $m \times n$ matrix A such that $\mathbf{h}_A \asymp_+ \mathbf{f}$.

Theorem 1.2.4(ii) asserts that for every template f, the set $\mathcal{D}(f) := \{A : \mathbf{h}_A \asymp_+ f\}$ is nonempty. It is natural to ask how big this set is in terms of Hausdorff and packing dimensions. Moreover, given a collection of templates $\mathcal{F}$, we can ask the same question about the set

$$\mathcal{D}(\mathcal{F}) := \cup_{\mathbf{f}\in\mathcal{F}} D(f).$$

1.2.3 The lower and upper contraction rates

The next important notion we need to introduce for the statement of the variational principal is the lower and upper average contraction rate of a template.

We define the lower and upper average contraction rate of a template $\mathbf{f}$ as follows. Let I be an open interval on which $\mathbf{f}$ is linear. For each $q = 1, \ldots, d$ such that $f_q < f_{q+1}$ on I, let $L_\pm = L_\pm(\mathbf{f}, I, q) \in [0, d_\pm]_\mathbb{Z}$ be chosen to satisfy $L_+ + L_- = q$ and

$$F'_q = \sum_{i=1^q} f'_i = \frac{L_+}{m} - \frac{L_-}{m}, \tag{1.2.7}$$

as guaranteed by (III) of the definition of templates. An interval of equality for f on I is an interval $(p, q]_\mathbb{Z}$, where $0 \le p < q \le d$ satisfy

$$f_p < f_{p+1} = \cdots = f_q < f_{q+1} \text{ on } I. \tag{1.2.8}$$

As before, we use the convention that $f_0 = -\infty$ and $f_{d+1} = \infty$. Note that the collection of intervals of equality forms a partition of $[1, d]_\mathbb{Z}$. If $(p, q]_\mathbb{Z}$ is an interval of equality for $\mathbf{f}$ on I, then we let $M_\pm(p, q) = M_\pm(\mathbf{f}, I, p, q)$, where

$$M_\pm(\mathbf{f}, I, p, q) = L_\pm(\mathbf{f}, I, q) - L_\pm(\mathbf{f}, I, p). \tag{1.2.9}$$

or equivalently, $M_\pm(p, q)$ are the unique integers such that

$$M_+ + M_- = q - p \text{ and } \sum_{i=p+1}^{q} f'_i = \frac{M_+}{m} - \frac{M_-}{n} \text{ on } I. \tag{1.2.10}$$

It can be shown from the definition of template that $M_{\pm} \geq 0$ by (II) of the definition of templates. Next, let

$$S_+ = S_+(\mathbf{f}, I) = \cup_{(p,q]_{\mathbb{Z}}}(p, p + M_+(p,q)]_{\mathbb{Z}} \tag{1.2.11}$$

$$S_- = S_-(\mathbf{f}, I) = \cup_{(p,q]_{\mathbb{Z}}}(p + M_+(p,q), q]_{\mathbb{Z}} \tag{1.2.12}$$

where the unions are taken over all intervals of equality for $\mathbf{f}$ on I. Note that S_+ and S_- are disjoint and satisfy $S_+ \cup S_- = [1, d]_{\mathbb{Z}}$, and that $\#(S+) = m$ and $\#(S_-) = n$.

Next, let

$$\delta(\mathbf{f}, I) = \#\{(i_+, i_-) \in S_+ \times S_- : i_+ < i_-\} \in [0, mn]_{\mathbb{Z}}, \tag{1.2.13}$$

and note that

$$mn - \delta(\mathbf{f}, I) = \#\{(i_+, i_-) \in S_+ \times S_- : i_+ > i_-\} \tag{1.2.14}$$

Definition 1.2.5. The *lower and upper average contraction rates* of $\mathbf{f}$ are the numbers

$$\underline{\delta}(\mathbf{f}) := \liminf_{T\to\infty} \Delta(\mathbf{f}, T), \tag{1.2.15}$$

and

$$\overline{\delta}(\mathbf{f}) := \limsup_{T\to\infty} \Delta(\mathbf{f}, T), \tag{1.2.16}$$

where $\Delta(\mathbf{f}, T) := \frac{1}{T}\int_0^T \delta(\mathbf{f}, t)dt$. Here we abuse notation by writing $\delta(\mathbf{f}, t) = \delta(\mathbf{f}, I)$ for all $t \in I$.

To help illustrate the definitions above, let us introduce the following example:

Example 1.2.6 (The contraction rate of zero template is mn)**.** Let $\mathbf{f}$ be the template where all of its components are zero, then the interval of linearity is $[0, \mathbb{R})$ and the only interval of equality is $(0, d]_{\mathbb{Z}}$.

(1) For $q = 0$

$$\begin{cases} L_+ + L_- = 0 \\ \frac{L_+}{m} - \frac{L_-}{n} = 0 \end{cases} \iff \begin{cases} L_+ = 0 \\ L_- = 0 \end{cases} .$$

(2) For $q = d$

$$\begin{cases} L_+ + L_- = F'_d = 0 \\ \frac{L_+}{m} - \frac{L_-}{n} = 0 \end{cases} \iff \begin{cases} L_+ = m \\ L_- = n \end{cases}.$$

Hence,

$$M_+(0,d) = M_+(\mathbf{f}, [0,\mathbb{R}), 0, d) := L_+(d) - L_+(0) = m$$

and

$$S_+ = \cup_{(p,q]_{\mathbb{Z}}}(p, p + M(p,q)] = (0, m]_{\mathbb{Z}}$$

and

$$S_- = [1, d]_{\mathbb{Z}} - S_+ = (m+1, d].$$

It follows that

$$\delta(\mathbf{f}, I) = \#\{(i_+, i_-) \in S_+ \times S_- : i_+ < i_-\} = mn.$$

Hence

$$\delta(\mathbf{f}) = \lim_{T\to\infty} \frac{1}{T} \int_0^T \delta(\mathbf{f}, t)dt \equiv mn.$$

Example 1.2.7 (The contraction rate of standard quadrilateral partial template of order r)**.** For $r \leq \min(m, n)$, let $\mathbf{f}$ be the template with

$$f_1 = \cdots = f_r < f_{r+1} = \cdots = f_d,$$

where each of the components is a piecewise linear function with two pieces defined on $I_1 := [0, \frac{m}{m+n}]$ and $I_2 := [\frac{m}{m+n}, 1]$ (intervals of linearity). For $f_1 = \cdots = f_r$ the derivative (slope) on the first interval is $-\frac{1}{n}$ and the derivative on the second interval is $\frac{1}{m}$; for $f_{r+1} = \cdots = f_d$ the derivative (slope) on the first interval is $-\frac{r}{(d-r)n}$ and the derivative on the second interval is $-\frac{r}{(d-r)m}$.

There are two intervals of equality: $(0, r]_{\mathbb{Z}}$ and $(r+1, d]_{\mathbb{Z}}$.

(1) Over the interval I_1:

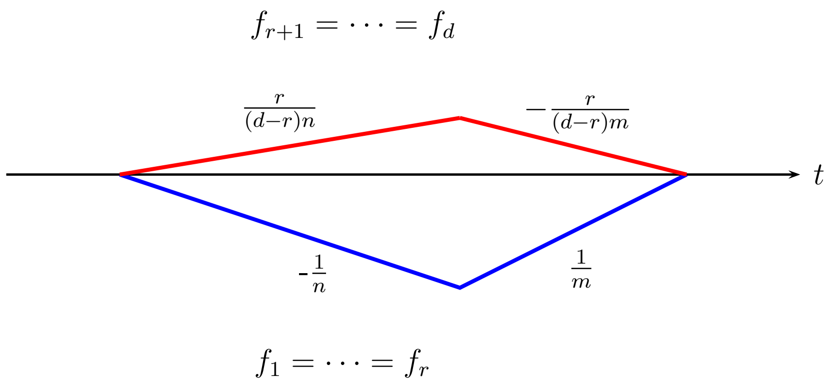

Figure 1.2: Construction of the standard partial template of order r.

(a) For $q = 0$

$$\begin{cases} L_+ + L_- = 0 \\ \frac{L_+}{m} - \frac{L_-}{n} = 0 \end{cases} \iff \begin{cases} L_+ = 0 \\ L_- = 0 \end{cases}.$$

(b) For $q = r$

$$\begin{cases} L_+ + L_- = r \\ \frac{L_+}{m} - \frac{L_-}{n} = F'_r = \sum_{i=1}^r f'_i = -\frac{r}{n} \end{cases} \iff \begin{cases} L_+ = 0 \\ L_- = r \end{cases}.$$

(c) For $q = d$

$$\begin{cases} L_+ + L_- = d \\ \frac{L_+}{m} - \frac{L_-}{n} = 0 \end{cases} \iff \begin{cases} L_+ = m \\ L_- = n \end{cases}.$$

Hence,

$$M_+(0, r) = M_+(\mathbf{f}, [0, \mathbb{R}), 0, r) := L_+(r) - L_+(0) = 0;$$

$$M_+(r, d) = M_+(\mathbf{f}, [0, \mathbb{R}), r, d) := L_+(d) - L_+(r) = m.$$

and

$$S_+ = \cup_{(p,q]_\mathbb{Z}}(p, p + M(p, q)] = (r, m + r]_\mathbb{Z}$$

and

$$S_- = [1, d]_\mathbb{Z} - S_+ = (0, r]_\mathbb{Z} \cup (m + r + 1, d]_\mathbb{Z}.$$

It follows that

$$\delta(\mathbf{f}, I_1) = \#\{(i_+, i_-) \in S_+ \times S_- : i_+ < i_-\} = m(n-r).$$

(2) Over the interval I_2:

(a) For $q = 0$

$$\begin{cases} L_+ + L_- = 0 \\ \frac{L_+}{m} - \frac{L_-}{n} = 0 \end{cases} \iff \begin{cases} L_+ = 0 \\ L_- = 0 \end{cases} .$$

(b) For $q = r$

$$\begin{cases} L_+ + L_- = r \\ \frac{L_+}{m} - \frac{L_-}{n} = F'_r = \sum_{i=1}^r f'_i = +\frac{r}{m} \end{cases} \iff \begin{cases} L_+ = r \\ L_- = 0 \end{cases} .$$

(c) For $q = d$

$$\begin{cases} L_+ + L_- = F'_d = d \\ \frac{L_+}{m} - \frac{L_-}{n} = 0 \end{cases} \iff \begin{cases} L_+ = m \\ L_- = n \end{cases} .$$

Hence,

$$M_+(0, r) = M_+(\mathbf{f}, 0, r) := L_+(r) - L_+(0) = r;$$

$$M_+(r, d) = M_+(\mathbf{f}, r, d) := L_+(d) - L_+(r) = m - r.$$

and

$$S_+ = \cup_{(p,q]_{\mathbb{Z}}} (p, p + M(p, q)] = (0, r]_{\mathbb{Z}} \cap (r, m]_{\mathbb{Z}} = (0, m]_{\mathbb{Z}}$$

$$S_- = [1, d]_{\mathbb{Z}} - S_+ = (m+1, d]_{\mathbb{Z}}.$$

It follows that

$$\delta(\mathbf{f}, I_2) = \#\{(i_+, i_-) \in S_+ \times S_- : i_+ < i_-\} = mn.$$

Therefore,

$$\int_0^1 \delta(\mathbf{f}, t)dt = \frac{n}{m+n} m(n-r) + \frac{m}{m+n} mn = mn - \frac{mnr}{m+n}.$$

Note that when $r = 1$, this is equal to $mn - \frac{mn}{m+n}$. This example plays a central role in the computation of the Hausdorff dimension of singular matrices.

Note that if $r > \min(m, n)$, then the derivatives (slopes) will violate the axiom $-\frac{1}{n} \leq f_i' \leq \frac{1}{m}, i = 1, 2, \cdots, d$ for templates.

1.2.4 The variational principles for templates

Definition 1.2.8. A collection of templates $\mathcal{F}$ is said to be *closed under finite perturbations* if whenever $g \asymp_+ \mathbf{f} \in \mathcal{F}$, we have $g \in \mathcal{F}$.

Theorem 1.2.9 (Variational principle: version 1 [DFSU20])**.** *Let $\mathcal{F}$ be a collection of templates closed under finite perturbations. Then*

$$\dim_H(\mathcal{D}(\mathcal{F})) = \sup_{\mathbf{f} \in \mathcal{F}} \underline{\delta}(\mathbf{f}) \tag{1.2.17}$$

and

$$\dim_P(\mathcal{D}(\mathcal{F})) = \sup_{\mathbf{f} \in \mathcal{F}} \overline{\delta}(\mathbf{f}). \tag{1.2.18}$$

Theorem 1.2.10 (Variational principle, version 2 [DFSU20])**.** . *Let S be a collection of Borel functions from $[0, \infty)$ to $\mathbb{R}^d$ which is closed under finite perturbations, and let $\mathcal{D}(S) := \{A : \mathbf{h}_A \in S\}$. Then*

$$\dim_H(\mathcal{D}(S)) = \sup_{\boldsymbol{f} \in S \cap \mathcal{T}_{m,n}} \underline{\delta}(\mathbf{f}) \tag{1.2.19}$$

and

$$\dim_P(\mathcal{D}(S)) = \sup_{\boldsymbol{f} \in S \cap \mathcal{T}_{m,n}} \overline{\delta}(\mathbf{f}). \tag{1.2.20}$$

Theorem 1.2.11 (Variational principle, version 3 [DFSU20])**.**

(i) Let S be a (Borel) set of $m \times n$ matrices of Hausdorff (resp. packing) dimension $> \delta$. Then there exist a matrix $A \in S$ and a template $\mathbf{f} \asymp_+ \mathbf{h}_A$ whose lower (resp. upper) average contraction rate is $> \delta$.

(ii) Let $\mathbf{f}$ *be a template whose lower (resp. upper) average contraction rate is* $> \delta$*. Then there exists a (Borel) set* S *of* $m\times n$ *matrices of Hausdorff (resp. packing) dimension* $> \delta$*, such that* $\mathbf{h}_A \asymp_+ \mathbf{f}$ *for all* $A \in S$*.*

1.2.5 Applications of the variational principles

With variational principle, one can immediate prove Schmidt's result on the Hausdorff dimention of the set of badly approximable matrices.

Indeed, by the variational principle (Theorem 1.2.10), the Hausdorff dimension of the set of badly approximable matrices is equal to the supremum of $\underline{\delta}$ over all bounded templates. Since the zero template satisfies $\underline{\delta}(0) = mn$ and any template $\mathbf{f}$ satisfies $\delta(\mathbf{f}) \leq mn$ by definition, this supremum has to be mn.

As mentioned in the introduction, the following theorem is one of the most important result of Das, Fishman, Simmons and Urbański in [DFSU20]

Theorem 1.2.12 ([DFSU20])**.** *For all* $(m, n) \neq (1, 1)$*, we have*

$$\dim_H(\mathbf{Sing}(m,n)) = \dim_P(\mathbf{Sing}(m,n)) = \delta_{m,n},$$

where $\dim_H(S)$ *and* $\dim_P(S)$ *denote the Hausdorff and packing dimensions of a set* S*, respectively, as defined in the section 1.2.1.*

Remark 1.2.13. When $m = n = 1, \dim_H(\mathbf{Sing}(m,n)) = \dim_P(\mathbf{Sing}(m,n)) = 0$, since in this case $\mathbf{Sing}(m,n)$ is simply the set of rational numbers.

Let us summarize the proof of the lower bound for this theorem in [DFSU20]. The key idea is the construction of the so-called standard templates:

Definition 1.2.14. Fix $0 \leq t_k < t_{k+1}$ and ϵ_k, $\epsilon_{k+1} \geq 0$ and let $\Delta t = \Delta t_k = t_{k+1} - t_k$ and $\Delta\epsilon = \Delta\epsilon_k = \epsilon_{k+1} - \epsilon_k$. Assume that the following formulas hold:

$$-\frac{1}{m}\Delta t \leq \Delta\epsilon \leq \frac{1}{n}\Delta t \tag{1.2.21}$$

$$\Delta\epsilon \geq -\frac{n-1}{n}\Delta t \text{ if } m = 1 \tag{1.2.22}$$

$$\Delta\epsilon \leq \frac{m-1}{2m}\Delta t \text{ if } n = 1 \tag{1.2.23}$$

$$\text{either } (n-1)(\frac{1}{n}\Delta t - \Delta\epsilon) \geq d\epsilon_k \text{ or } (m-1)(\frac{1}{m}\Delta t + \Delta\epsilon) \geq d\epsilon_k \tag{1.2.24}$$

Then the *standard (partial) template defined by the two points* (t_k, ϵ_k) and $(t_{k+1}, \epsilon_{k+1})$ is the partial template (i.e. the restriction of a template to an interval) $\mathbf{f} : [t_k, t_{k+1}] \to \mathbb{R}^d$ defined as follows:

Let $g_1, g_2 : [t_k, t_{k+1}] \to \mathbb{R}$ be two piecewise linear functions such that for $i = 1, 2$,

$$g_i(t_1) = -\epsilon_1 \text{ and } g_i(t_2) = -\epsilon_2.$$

We assign g_1 with two intervals of linearity: first with slope $-\frac{1}{n}$ and second with slope $\frac{1}{m}$. g_2 has two intervals of linearity flipped: first with slope $\frac{1}{m}$ and second with slope $-\frac{1}{n}$. So the graph of g_1 and g_2 form a parallelogram, with g_2 on the top. The existence of g_1 and g_2 is guaranteed by (1.2.21). Define $g_3 = g_4 = \cdots = g_d$ so that $g_1 + g_2 + \cdots + g_d = 0$.

For each $t \in [t_k, t_{k+1}]$ let $\mathbf{f}(t) = \mathbf{g}(t)$ if $g_2(t) \leq g_3(t)$; otherwise let $f_1(t) = g_1(t)$ and let $f_2(t) = \cdots = f_d(t)$ be chosen so that $f_1 + \cdots + f_d = 0$.

The key idea to construct a template (over $[0, \infty)$) using the standard (partial) templates we defined above, give an estimate of the contraction rates and use the variational principles. Specifically, we choose two parameters $\tau \geq 0$ and $\lambda > 1$ and let

$$t_k = \lambda^k, \epsilon_k = \tau t_k, \text{ for all } k.$$

In this case, equations 1.2.21 through 1.2.24 become

$$\tau \leq \frac{1}{n}, \tag{1.2.25}$$

$$\tau \leq \frac{m-1}{2m}, \text{ if } n = 1, \tag{1.2.26}$$

$$\text{either } (n-1)(\frac{1}{n} - \Delta\epsilon) \geq \frac{1}{\lambda - 1} d\tau \text{ or } (m-1)(\frac{1}{m} + \tau) \geq \frac{\lambda}{\lambda - 1} d\tau. \tag{1.2.27}$$

Now let $\mathbf{f}[\tau, \lambda]$ denote the template defined by gluing together the standard (partial) templates defined by the pairs of points $(t_k, -\epsilon_k)$ and $(t_{k+1}, -\epsilon_{k+1})$ for all k. From the example 1.2.7 with $r = 1$, the lower and upper contraction rates of $\mathbf{s}[(t_k, 0), (t_{k+1}, 0)]$ are both $\delta_{m,n} := mn - \frac{mn}{m+n}$. And with $\tau \to 0$, using the integral estimates, one can indeed show that this is indeed the average contraction rate over $[0, \infty)$ and the variational principle gives the lower bound $\delta_{m,n}$.

Conjecture 1.2.15 (Schmidt's conjecture on successive minima functions, [Sch82])**.** For all $2 \in k \in m$, there exists an $m \times 1$ matrix A such that

$$\lambda_{k-1}(g_t u_A \mathbb{Z}^d) \to 0 \text{ and } \lambda_{k+1}(g_t u_A \mathbb{Z}^d) \to \infty \text{ as } t \to \infty. \tag{1.2.28}$$

This conjecture was proven by Moshchevitin [Mos12]. Tushar Das, Lior Fishman, David Simmons and Mariusz Urbański [DFSU20] improved Moshchevitin's result by finding a lower bound on the the Hausdorff dimension of the set of matrices witnessing Schmidt's conjecture in the matrix framework:

Definition 1.2.16 ([DFSU20])**.** An $m \times n$ matrix A is $k-$*singular* for $2 \leq k \leq m+n-1$ if

$$\lambda_{k-1}(g_t u_A \mathbb{Z}^d) \to 0 \text{ and } \lambda_{k+1}(g_t u_A \mathbb{Z}^d) \to \infty \text{ as } t \to \infty. \tag{1.2.29}$$

Theorem 1.2.17. *For all $(m, n) \neq (1, 1)$ and for all $2 \leq k \leq m+n-1$, the Hausdorff dimension of the set of matrices A that satisfy (1.2.29) is at least*

$$\max(f_{m,n}(k), f_{m,n}(k-1))$$

where

$$f_{m,n}(k) := mn - \frac{k(m+n-k)mn}{(m+n)^2} - \{\frac{km}{m+n}\}\{\frac{kn}{m+n}\} \tag{1.2.30}$$

Here $\{x\}$ denotes the fractional part of a real number x. The same formula is valid for the set of matrices A that satisfy (1.2.29) and trivially singular (meaning entries satisfy a linear equation with rational coefficients).

1.2.6 Fractional dimensions of badly approximable matrices of order r

In this section, we shall prove the following results on the measure and fractal dimensions of badly approximable matrices of higher orders:

Theorem 1.2.18. $\mathbf{BA}_d(m,n) = \mathbb{R}^{m\times n}$. *For all $r = 1, 2, \dots, d-1$, $\mathbf{BA}_r(m,n)$ is Lebesgue null.*

Since $\mathbf{BA}_1(m,n) \subset \mathbf{BA}_2(m,n) \cdots \subset \mathbf{BA}_d(m,n) = \mathbb{R}^{m\times n}$, a natural question to ask next is how big the gaps $\mathbf{BA}_{r+1}(m,n) - \mathbf{BA}_r(m,n)$ are in terms of fractional dimensions. We have

Theorem 1.2.19. *For $1 \le r \le d-1$, we have the Hausdorff dimension for the gaps between the badly approximable matrices of order r and $r+1$ is full:*

$$\dim_H \big(\mathbf{BA}_{r+1}(m,n) - \mathbf{BA}_r(m,n)\big) = \dim_P \big(\mathbf{BA}_{r+1}(m,n) - \mathbf{BA}_r(m,n)\big) = mn.$$

To prove Theorem 1.2.18, the following lemmata are needed:

Lemma 1.2.20. *For $d = m+n$, $G = \mathrm{SL}(d,\mathbb{R})$, let P_- denote the subgroup of lower diagonal block matrices*

$$P_- := \left\{ \begin{bmatrix} A & O \\ C & D \end{bmatrix} \in \mathrm{SL}(d,\mathbb{Z}) \;\middle|\; \begin{array}{l} A, C, D \textit{ are any real matrices of size } m\times m, n\times m, n\times n \textit{ respectively,} \\ \textit{and } O \textit{ is a zero } m\times n \textit{ matrix} \end{array} \right\}$$

and U^+ denote the subgroup of unipotent upper diagonal matrices

$$U^+ := \left\{ \begin{bmatrix} I_m & B \\ O & I_m \end{bmatrix} \in \mathrm{SL}(d,\mathbb{Z}) \,\middle|\, B \textit{ is a any } m \times n \textit{ real matrix.}, \right\}$$

then the complement $G - P\_U^+$ has Haar measure zero in G.

Proof. The classical proof of this involves the theory of algebraic geometry and algebraic groups. By looking at the orbit $P\_\backslash P\_U^+$ in the irreducible projective variety $P\_\backslash G$. By the theory of algebraic groups, $P\_\backslash P\_U^+$ is Zariski open in its closure and further Zariski open, and therefore dense in $P\_\backslash G$. Therefore its complement in $P\_\backslash G$ has strictly non-full dimension. So P_U^+ has non-full dimension in G and thus of zero Haar measure.

For the self-containedness purpose,we shall give an elementary proof here that only involves linear algebra and manifold theory. We will study the set of matrices $\begin{bmatrix} X & Y \\ Z & W \end{bmatrix}$ in $\mathrm{SL}(n,\mathbb{Z})$ that can be (and cannot be) represented by matrices in P_- and U^+.

Suppose $\begin{bmatrix} A & O \\ C & D \end{bmatrix} \in P_-$ and $\begin{bmatrix} I_m & B \\ O & I_n \end{bmatrix} \in U^-$ and we have the following equation of block matrices:

$$\begin{bmatrix} X & Y \\ Z & W \end{bmatrix} = \begin{bmatrix} A & O \\ C & D \end{bmatrix} \begin{bmatrix} I_m & B \\ O & I_n \end{bmatrix} = \begin{bmatrix} A & AB \\ C & BC + D \end{bmatrix} \tag{1.2.31}$$

We have the system of equations:

$$\begin{cases} X = A \\ Z = C \\ Y = AB \\ W = BC + D \end{cases} \tag{1.2.32}$$

which can be simplified to $Y = XB$ and $W = BZ + D$. Therefore, the matrix equation 1.2.31 is solvable for A, B, C, D and if and only if $Y = XB$ is solvable for B (noticing that $W = BZ + D$ always gives the solution $D = W - BZ$). However, if $Y = XB$ is not solvable for B, then we must have $\det(X) = 0$ for the $m \times m$ matrix X from the beginning. It follows that

$$
\begin{aligned}
&G - P_{\_}U^+ \\
=& \left\{ \begin{bmatrix} X & Y \\ Z & W \end{bmatrix} \in \mathrm{SL}(d, \mathbb{R}) : \begin{bmatrix} X & Y \\ Z & W \end{bmatrix} \text{ cannot be written as } \begin{bmatrix} A & O \\ C & D \end{bmatrix} \begin{bmatrix} I_m & B \\ O & I_n \end{bmatrix} = \begin{bmatrix} A & AB \\ C & BC + D \end{bmatrix} \right\} \\
\subset& \left\{ \begin{bmatrix} X & Y \\ Z & W \end{bmatrix} \in \mathrm{SL}(d, \mathbb{R}) : \det(X) = 0 \right\}
\end{aligned}
$$

By the last term is a subvariety of $\mathrm{SL}(d, \mathbb{R})$ whose dimension is strictly less than the full dimension, and therefore of Haar measure zero since the Haar measure on Lie groups are given by the full-dimensional volume form.

□

Lemma 1.2.21 (Decomposition of Haar measure in Lie groups, Theorem 8.32 in [Kna02])**.** *Let G be a Lie group, and let S and T be closed subgroups such that $S \cap T$ is compact, multiplication $S \times T \to G$ is an open map, and the set of products ST exhausts G except possibly for a set of Haar measure 0. Let Δ_T and Δ_G denote the modular functions of T and G. Then the left Haar measures on G, S, and T can be normalized so that*

$$\int_G f(x)dx = \int_{S \times T} f(st) \frac{\Delta_T(t)}{\Delta_G(t)} ds dt,$$

for all Borel function $f \geq 0$ on G.

Remark 1.2.22. We can apply this lemma to the scenario where $S = P_-$ and $T = U^+$ since they are both closed subgroups with trivial (compact) intersection. That $P_- \times U^+ \to G$ is open follows from the fact that this map is injective (since $S = P_-$ and $T = U^+$ have trivial intersection) and thus an immersion for Lie groups. Also the product maps in Lie groups are submersions. Therefore it gives an local diffeomorphsm and thus open. Finally, the previous lemma gives $G - P_{\_}U^+$ is of measure zero.

Theorem 1.2.23 (Birkhoff's Pointwise Ergodic Theorem, the discrete version,[EW11],Theorem 2.30)**.** *Let $(X, \mathscr{B}, \mu)$ ba a probability space and $T : X \to X$ be an ergodic measure-preserving* *transformation. If $f \in L^1(X)$, then*

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} f(T^i x) = \int_X f d\mu,$$

for μ-almost every $x \in X$.

Corollary 1.2.24 (Birkhoff's Pointwise Ergodic Theorem, the continuous version)**.** *Let G be a topological group with continuous action on X,namely $G \times X \to X, (g,x) \to g.x$ is continuous and let $(X, \mathscr{B}, \mu)$ a probability space with μ a G-invariant probability measure . Suppose that $(g_t)_{t\in\mathbb{R}}$ is a one-parameter subgroup of G and that there the discrete subgroup $(g_n)_{n\in Z}$ acting ergodically on X.*

If $f \in L^1(X)$, then

$$\lim_{t\to+\infty} \frac{1}{T}\int_0^T f(g_t.x) = \int_X f d\mu,$$

for μ-almost every $x \in X$.

Proof. First, observe that from the definition of ergodicity we have immediately that for fixed $s > 0$, $(g_{sn})_{n\in Z}$ acting ergodically on X implies that g_s acts on X ergodically. Now taking $T := g_s$ and $f_s(x) := \int_0^s f(g_t.x)dt$ in the discrete version of Birkorff's ergodic theorem, noticing that f_s is again a L^1-function since s is fixed and

$$\int_X f_s(x)d\mu = \int_X \int_0^s f(g_t.x)dtd\mu = \int_0^s \int_X f(g_t.x)d\mu dt < \infty,$$

by Fubini's theorem.

The discrete version of Birkorff's ergodic theorem gives us

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} f_s(g_{is}.x) = \int_X f_s d\mu$$

Therefore for this fixed s, we have

$$\begin{aligned}
\lim_{T\to\infty}\frac{1}{T}\int_0^T f(g_t.x)dt &= \lim_{n\to\infty}\frac{1}{s\lfloor\frac{T}{s}\rfloor}\left(\sum_{i=0}^{\lfloor\frac{T}{s}\rfloor-1}\int_{is}^{(i+1)s} f(g_t.x)dt + O_x(1)\right)\\
&= \lim_{n\to\infty}\frac{1}{s\lfloor\frac{T}{s}\rfloor}\left(\sum_{i=0}^{\lfloor\frac{T}{s}\rfloor-1}\int_0^s f(g_{t+is}.x)dt + O_x(1)\right)\\
&= \lim_{n\to\infty}\frac{1}{s\lfloor\frac{T}{s}\rfloor}\left(\sum_{i=0}^{\lfloor\frac{T}{s}\rfloor-1} f_s(g_{is}.x)dt + O_x(1)\right)\\
&=\frac{1}{s}\int_X f_s d\mu \qquad\qquad \text{(by the discrete Birkorff)}\\
&=\frac{1}{s}\int_X\int_0^s f(g_t.x)dtd\mu\\
&=\frac{1}{s}\int_0^s\int_X f(g_t.x)d\mu dt = \frac{1}{s}\int_0^s\int_X f(g_t.x)d\mu dt \qquad \text{(by Fubini)}\\
&=\frac{1}{s}\int_0^s\int_X f(x)d\mu dt \qquad\qquad (\mu \text{ is } g_t\text{-invariant})\\
&=\int_X f(x)d\mu.
\end{aligned}$$

□

Lemma 1.2.25. *For any $r = 1, 2, \dots d-1$ and $\delta > 0$, the set $B_\delta^r := \{\Lambda \in \mathcal{L} : \lambda_r(\Lambda) < \delta\}$ has positive measure, where the measure is the unique $\mathrm{SL}(d,\mathbb{R})$ invariant measure in the homogeneous space $G/\Gamma := \mathrm{SL}(d,\mathbb{R})/\mathrm{SL}(d,\mathbb{Z})$.*

Proof. First we observe that for $r = 1, 2, \dots d-1$, B_δ^r is nonempty since it contains the elements

$$\frac{\delta}{2}e_1, \cdots, \frac{\delta}{2}e_{d-1},$$

which already form a linearly independent set in B_δ^r of lengths all less than δ. Note that for δ small enough, this cannot be generalized to $r = 1, 2, \dots, d$ due the Minkowski's second convex body theorem A.8.

By the continuity of λ_r on G/Γ (by the Theorem A.10), B_δ^r is open.

Any open subset on G/Γ. Indeed, since any open subset of $\mathrm{SL}(d,\mathbb{R})/\mathrm{SL}(d,\mathbb{Z})$ has countably many translations under $\mathrm{SL}(n,\mathbb{Q})$ whose union will cover the whole space, whose measure is 1, and this follows from the G-invariance of the measure. $\square$

Proof of the Theorem 1.2.18. We first notice that the condition

$$\limsup_{t\to\infty} -\mathbf{h}_{A,r}(t) < \infty.$$

is equivalent to

$$\inf_{t\geq 0} \lambda_r(g_t u_A \mathbb{Z}^d) \geq \delta,$$

for some $\delta > 0$.

When $r = d(= m+n)$, we notice that by the Theorem A.8 (Minkowski's second convex body theorem),

$$[\lambda_d(g_t u_A \mathbb{Z}^d)]^d \geq \prod_{r=1}^{d} \lambda_r(g_t u_A \mathbb{Z}^d) \asymp_d \mathrm{covol}(g_t u_A \mathbb{Z}^d) = 1.$$

Hence there exist a $\delta_d > 0$ such that for any $m \times n$ matrix A and $t \geq 0$,

$$\inf_{t\geq 0} \lambda_d(g_t u_A \mathbb{Z}^d) \geq \delta_d.$$

Hence, $\mathbf{BA}_d(m,n) = \mathbb{R}^{m\times n}$.

For $r \leq d-1$, we first observe that the set $\{\Lambda \in \mathcal{L} : \inf_{t\geq 0} \lambda_r(g_t \Lambda) \geq \delta\}$ is g_t-invariant for any $t \geq 0$. So by the ergodicity of $(g_n)_{n\in\mathbb{Z}}$-action (Theorem 1.1.7), this set has μ-measure zero or 1. So it suffices to show its complement $\{\Lambda \in \mathcal{L} : \inf_{t\geq 0} \lambda_r(g_t \Lambda) < \delta\}$ has positive measure for any $\delta > 0$. However, by the continuous version of Birkoff's ergodic theorem, Corollary 1.2.24,

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T f(g_t.\Lambda) dt = \int_X f(\Lambda) d\Lambda,$$

for μ-almost every $\Lambda \in \mathcal{L}$ (identified with $X := \mathrm{SL}(d,\mathbb{R})/\mathrm{SL}(d,\mathbb{Z})$ and $f \in L^1(\mathcal{L},\mu)$.

Now take f as the characteristic function on the set $B_\delta^r := \{\Lambda \in \mathcal{L} : \lambda_r(\Lambda) < \delta\}$, which is of positive measure by Lemma 1.2.25 and it follows that for μ-a.e. $\Lambda \in \mathcal{L}$,

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \mathbf{1}_{B_\delta^r}(g_t.\Lambda)dt > 0. \tag{1.2.33}$$

This means for almost every lattice $\Lambda \in \mathcal{L}$ and over a positive proportion of time $t \in [0,T]$ when T is large, $\lambda_r(g_t\Lambda) < \delta$. More precisely, thanks to the boundedness of $\mathbf{1}$, for μ-a.e. Λ and any $T_0 > 0$,

$$\lim_{T\to\infty} \frac{1}{T}\int_{T_0}^T \mathbf{1}_{B_\delta^r}(g_t.\Lambda)dt > 0.$$

In particular, for μ-a.e. Λ and any $T_0 > 0$, there exists $t > t_0$ such that

$$\limsup_{t\to\infty} f(g_t.\Lambda) < \delta. \tag{1.2.34}$$

This together with the fact that the set $B_\delta^r := \{\Lambda \in \mathcal{L} : \lambda_r(\Lambda) < \delta\}$, which is of positive measure gives

$$\mu\left(\{\Lambda \in \mathcal{L} : \inf_{t\geq 0} \lambda_r(g_t\Lambda) < \delta\}\right) \geq \mu\left(\{\Lambda \in \mathcal{L} : \limsup_{t\geq 0} \lambda_r(g_t\Lambda) < \delta\}\right) > 0$$

Therefore by the ergodicity of g_t-action, $\{\Lambda \in \mathcal{L} : \inf_{t\geq 0} \lambda_r(g_t\Lambda) \geq \delta\}$ has zero μ-measure in the space of unimodular lattices.

It remains for us to show that the set of matrices corresponding to the lattices $\{u_A\mathbb{Z}^d\}$ has measure zero.

To this end, we first recall the root space decomposition for $\mathfrak{sl}(d,\mathbb{R})$, cf. [Kna02] Chapter II section 1:

$$\begin{aligned}\mathfrak{sl}(d,\mathbb{R}) &= \mathfrak{h} \oplus_{i\neq j} \mathfrak{g}_{ij} \\ &= \left(\oplus_{i\neq j:1\leq i\leq m,1\leq j\leq n;\ m+1\leq i\leq m+n,m+1\leq j\leq m+n}\mathfrak{g}_{ij}\right) \oplus \mathfrak{h} \oplus \left(\oplus_{1\leq i\leq m,m+1\leq j\leq m+n}\mathfrak{g}_{ij}\right) \\ &= \mathfrak{p}_- \oplus \mathfrak{u}_+.\end{aligned}$$

Under the exponential map $\exp : \mathfrak{sl}(d,\mathbb{R}) \to \mathrm{SL}(d,\mathbb{R})$, $\mathfrak{p}_-$ corresponds to the subgroup P_- in $\mathrm{SL}(d,\mathbb{R})$, and $\mathfrak{u}_+$ corresponds to the subgroup U^+ in $\mathrm{SL}(d,\mathbb{R})$, cf. lemma 1.2.20. In particular,

$$g_t \in \exp \mathfrak{p}_-, u_A \in \exp \mathfrak{u}_+$$

Since the canonical quotient map $\pi : \mathrm{SL}(d,\mathbb{R}) \to \mathrm{SL}(d,\mathbb{R})/\mathrm{SL}(d,\mathbb{Z})$ is a local diffeomorphism of manifolds, for any $A \in M(m \times n, \mathbb{R})$, we have that the map

$$\begin{aligned} \pi \circ \exp : \mathfrak{sl}(d,\mathbb{R}) = \mathfrak{p}_- \oplus \mathfrak{u}_+ &\longrightarrow \mathrm{SL}(d,\mathbb{R})/\mathrm{SL}(d,\mathbb{Z}) \\ (X_-, X_+) &\longmapsto \exp X_- \exp X_+ \cdot u_A \mathrm{SL}(d,\mathbb{Z}) \end{aligned}$$

gives a local coordinate at the point $u_A\mathrm{SL}(d,\mathbb{Z})$.

Observing that

$$g_t \exp X_- \exp X_+ \cdot u_A \mathrm{SL}(d,\mathbb{Z}) = g_t \exp X_- g_{-t} \cdot g_t \exp X_+ \cdot u_A \mathrm{SL}(d,\mathbb{Z})$$

and that

$$\{g_t \exp X_- g_{-t}\}_{t \geq 0}$$

is bounded, since $\exp X_-$ is a lower triangular block matrix contracted to the identity matrix as $t \to \infty$ under the $\{g_t\}_{t\geq 0}$ conjugation, and in view of the inequality in the Lemma A.9, we have for fixed X_-, A

$$\lambda_r \left(g_t u_A \mathrm{SL}(d,\mathbb{Z})\right) \text{ is bounded below away from zero}$$

if and only if

$$\lambda_r \left(g_t \exp X_- \cdot u_A \mathrm{SL}(d,\mathbb{Z})\right) \text{ is bounded below away from zero.}$$

By what we proved above using the ergodicity of (g_t)-action

$$\mu\{x \in G/\Gamma : \inf_{t\geq 0} \lambda_r \left(g_t x\right) > 0\} = 0$$

Now notice that the subgroup $U^+ := \left\{ \begin{bmatrix} I_m & A \\ 0 & I_n \end{bmatrix} : A \in M(m \times n, \mathbb{R}) \right\}$ is naturally isomorphic to $M(m \times n, \mathbb{R})$ with multiplication corresponding to the addition. The Haar measure on U^+, by the uniqueness, can be identified with the Lebesgue measure on $M(m \times n, \mathbb{R})$. Notice that with the local diffeomorphism $\pi : \mathrm{SL}(d, \mathbb{R}) \to \mathrm{SL}(d, \mathbb{R})/\mathrm{SL}(d, \mathbb{Z})$, the identification also can also go from the Haar measure on $U^+\Gamma/\Gamma$ to the Lebesgue measure on $M(m \times n, \mathbb{R})$.

Now by Lemma 1.2.20, almost every element $g\Gamma$ in $G/\Gamma := \mathrm{SL}(d, \mathbb{R})/\mathrm{SL}(d, \mathbb{Z})$ has the decomposition $g\Gamma = pu\Gamma$ with $p \in P_-$ and $u \in U^+$. By the decomposition of Haar measure $dx = \frac{\Delta_{U^+}(u)}{\Delta_G(u)} dpdu$ in G and the local identification of Haar measures on G and G/Γ, we have that

$$\begin{aligned}
0 &= \int_G \mathbb{1}_{\{x \in G/\Gamma : \inf_{t \geq 0} \lambda_r(g_t x) > 0\}} dx && \text{(by the ergodicity of } g_t\text{-action)} \\
&= \int_{U^+} \int_{P_-} \mathbb{1}_{\{pu\Gamma \in G/\Gamma : \inf_{t \geq 0} \lambda_r(g_t pu\Gamma) > 0\}} \frac{\Delta_{U^+}(u)}{\Delta_G(u)} dpdu \\
& && \text{(by the decomposition of Haar measure)} \\
&= \int_{P_-} \int_{U^+} \mathbb{1}_{\{pu\Gamma \in G/\Gamma : \inf_{t \geq 0} \lambda_r(g_t pu\Gamma) > 0\}} \frac{\Delta_{U^+}(u)}{\Delta_G(u)} dudp && \text{(by the Fubini's theorem)} \\
&= \int_{P_-} \int_{U^+} \mathbb{1}_{\{pu\Gamma \in G/\Gamma : \inf_{t \geq 0} \lambda_r(g_t u\Gamma) > 0\}} \frac{\Delta_{U^+}(u)}{\Delta_G(u)} dudp \\
& && \text{(by the contraction above on } g_t\text{-contraction)} \\
&= \int_{P_-} \int_{U^+} \mathbb{1}_{\{u\Gamma \in G/\Gamma : \inf_{t \geq 0} \lambda_r(g_t u\Gamma) > 0\}} \frac{\Delta_{U^+}(u)}{\Delta_G(u)} dudp \\
& && \text{(since } pu\Gamma \in G/\Gamma \text{ if and only if } u\Gamma \in G/\Gamma\text{)}
\end{aligned}$$

Therefore,

$$\int_{U^+} \mathbb{1}_{\{u\Gamma \in G/\Gamma : \inf_{t \geq 0} \lambda_r(g_t u\Gamma) > 0\}} \frac{\Delta_{U^+}(u)}{\Delta_G(u)} du = 0,$$

and thus $\mathbb{1}_{\{u\Gamma \in G/\Gamma : \inf_{t \geq 0} \lambda_r(g_t u\Gamma) > 0\}} = 0$ for μ-almost all $u \in U^+$ or equivalently

$$\mu_{U^+}(\{pu\Gamma \in G/\Gamma : \inf_{t \geq 0} \lambda_r (g_t u\Gamma) > 0\}) = 0.$$

Finally, by the identification between the Haar measure on U^+ and the Lebesgue measure on $\mathbb{R}^{m\times n}$, we have

$$m(\{A \in \text{some neighborhood of O in } \ M(m\times n,\mathbb{R}) : \inf_{t\geq 0} \lambda_r\left(g_t u_A \Gamma\right) > 0\}) = 0$$

Note that the above argument also works if we replace Γ with $u_{A_0}\Gamma$ for some $A_0 \in M(m\times n,\mathbb{R})$ and a countable union of zero-measure sets is again of zero measure. This completes the proof of the Theorem 1.2.18.

□

Proof of Theorem 1.2.19

To illustrate the idea of the proof, we first observe that the Examples 1.2.6 and 1.2.7 allows us to construct an “electrocardiography” template with deeper and deeper “pulse” (corresponding to quadrilateral partial templates) that spends larger and larger proportion of time with zero templates. However, the limitation for this construction, as discussed in the Example 1.2.7, is that we will violate the bounds for the derivatives of components for such templates when $r > \min(m,n)$. So the main task left for us is to lower the slopes for the templates so that they are confined in $[-\frac{1}{n}, \frac{1}{m}]$ and at the same time make sure we still have the “quantized” accumulated slope as required in the definition of templates, cf. Definition 1.2.3.

To this end, we need to following lemma from the elementary number theory, which will later allow us to generalize the bound on r to $r \leq \max(m,n)$:

Lemma 1.2.26. *For positive integers m,n,r with $r \leq \max(m,n)$, we have*

$$r - n \leq \left\lceil \frac{mr}{m+n} - 1 \right\rceil \tag{1.2.35}$$

and dually

$$r - m \leq \left\lceil \frac{nr}{m+n} - 1 \right\rceil \tag{1.2.36}$$

Proof. For the first inequality, if $r \le n$, then it trivially holds (notice that the right hand side is always nonnegative). Hence it suffices to assume $n < r \le m$.

$$\begin{aligned}
& r - n \le \left\lceil \frac{mr}{m+n} - 1 \right\rceil \\
\iff & r - n \le \left\lceil r - \frac{nr}{m+n} - 1 \right\rceil \\
\iff & r - n \le \left\lceil -\frac{nr}{m+n} - 1 \right\rceil + r && \text{(since } \lceil x + r \rceil = \lceil x \rceil + r\text{)} \\
\iff & -n \le \left\lceil -\frac{nr}{m+n} - 1 \right\rceil \\
\iff & 0 \le \left\lceil n - \frac{nr}{m+n} - 1 \right\rceil \\
\Longleftarrow & 0 \le \left\lceil n - \frac{nm}{m+n} - 1 \right\rceil && \text{(by the monotonicity of ceiling function)}
\end{aligned}$$

The nonnegativeness of the right hand side on the last line follows from $\frac{nm}{m+m} < n$. This proves the first inequality and the second follows from switching m and n.

$\square$

Proof. Recall the variational principle gives us

$$\dim_H(\mathcal{D}(S)) = \sup_{\mathbf{f} \in S \cap \mathcal{T}_{m,n}} \underline{\delta}(\mathbf{f})$$

and

$$\dim_P(\mathcal{D}(S)) = \sup_{\mathbf{f} \in S \cap \mathcal{T}_{m,n}} \overline{\delta}(\mathbf{f})$$

where S is a (Borel) collection of functions closed under finite perturbation and $\mathcal{D}(S) := \{A \in \mathbb{R}^{m \times n} : \mathbf{h}_A \in S\}$, where $\mathbf{h}_A$ is the successive minima function of A. In this proof we shall take

$$S := \{g \in C([0, \infty), \mathbb{R}^d) : \liminf_{x \to \infty} g_i(x) = -\infty, \forall i \le r; \liminf_{x \to \infty} g_i(x) > -\infty, \forall i \ge r+1\}.$$

Then it follows that

$$\mathcal{D}(S) = \mathbf{BA}_{r+1}(m, n) - \mathbf{BA}_r(m, n)$$

We first observe that for any $m \times n$ template in $\mathbf{f}$, we have by definition of lower and upper contraction rates:

$$\delta(\mathbf{f}) \leq mn.$$

In view of the variational principles,it suffices to find a template (or a sequence of templates) whose contraction rates is equal to (or approximates) mn.

Let τ_1, τ_2 be two positive number, which will represent slopes for templates, to be determined later. We now construct a piecewise linear map $\mathbf{f}$ as follows:

For convenience let us denote

$$a_k = k^k - k, b_k := k^k,$$

and $b_0 = 0$. Observe that $b_{k-1} \leq a_k$ for all $k \geq 1$.

For $n = 1, 2, 3, 4 \cdots$, we first define the restriction of $\mathbf{f}$ on $[b_k - k, b_k]$, denoted $\mathbf{q}_k := \mathbf{q}_k[\tau_1, \tau_2]$ as follows:

For the first r components of $\mathbf{q}_k$,

$$q_{k1} = q_{k2} \cdots = q_{kr} = \begin{cases} -\frac{\tau_1}{r}(x - b_k + n) & \text{if } x \in [b_k - k, \eta_k(\tau_1, \tau_2)] \\ \frac{\tau_2}{d-r}(x - b_k) & \text{if } x \in [\eta_k(\tau_1, \tau_2), b_k] \end{cases}, \tag{1.2.37}$$

where $\eta_k[\tau_1, \tau_2]$ is the point satisfies

$$\frac{-b_k + k + \eta_k(\tau_1, \tau_2)}{-\eta_k(\tau_1, \tau_2) + b_k} = \frac{\tau_2}{\tau_1}.$$

For the last $d - r$ components of $\mathbf{q}_k$, we set

$$q_{n,r+1} = \cdots = q_{nd}$$

so that

$$q_{k1} + q_{k2} + \cdots + q_{kd} \equiv 0.$$

and then we define

$$\mathbf{f} = \begin{cases} \mathbf{0} & \text{if } x \in [b_{k-1}, b_k - k] \\ \mathbf{q}_k[\tau_1, \tau_2](x) & \text{if } x \in [b_k - k, b_k] \end{cases}. \tag{1.2.38}$$

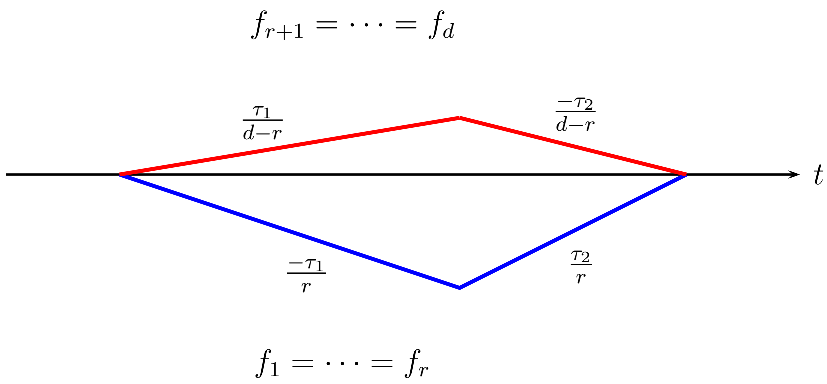

Figure 1.3: Construction of $\mathbf{q}[\tau_1, \tau_2]$.

It is easy to see that

$$\lim_{n\to\infty} f_r(\eta_k(\tau_1, \tau_2)) = -\infty,$$

and therefore

$$\liminf_{t\to+\infty} f_r(t) = -\infty.$$

Now let us see under what circumstances will $\mathbf{f}$ indeed becomes a template on $[0, \infty)$. Recall that we have three conditions for a template:

(I) $f_1 \leq \cdots \leq f_d$.

(II) $-\frac{1}{n} \leq f_i' \leq \frac{1}{m}$ for all i.

(III) For all $j = 0, \ldots, d$ and for every interval I such that $f_j < f_{j+1}$ on I, the function $F_j := \sum_{0<i\leq j} f_i$ is convex and piecewise linear on I with slopes in

$$Z(j) := \{\frac{L_+}{m} - \frac{L_-}{n} : L_\pm \in [0, d_\pm]_{\mathbb{Z}}, L_+ + L_- = j\} \tag{1.2.39}$$

Here $d_+ = m$ and $d_- = n$.

(I) is obvious from our construction. So our next goal is to find an appropriate choice τ_1 and τ_2 so that (II) and (III) hold.

The template $\mathbf{f}$ we defined above, restricted on the interval $[b_k - k, b_k]$, has two intervals of linearity, namely

$$I_1 := [b_k - k, \eta_k(\tau_1, \tau_2)], \text{and } I_2 := [\eta_k(\tau_1, \tau_2), b_k].$$

On the both of them, we have the separation of components of templates as follows:

$$-\infty =: f_0 < f_1 = f_2 = \cdots = f_r < f_{r+1} = \cdots = f_d = f_{d+1} := +\infty.$$

By the definition of templates, there are two (integer) *intervals of equality*:

$$(0, r]_{\mathbb{Z}} \text{ and } (r, d]_{\mathbb{Z}},$$

where the subscripts means intersections with the set of integers, forming a partition of the set $\{1, 2, \cdots, d\}$.

On the interval of linearity I_1, we have slopes:

$$\begin{cases} f_1'(a_k+) = \cdots = f_r'(a_k+) = -\frac{\tau_1}{r} \\ f_{r+1}'(a_k+) = \cdots = f_r'(a_k+) = +\frac{\tau_1}{d-r} \end{cases}. \tag{1.2.40}$$

Now we will follow the procedures of computing the lower and upper contractions rates:

(1) On the intervals of linearity I_1, we shall determine the possible values for $L_\pm := L_\pm(I_1, q)$ as follows:

   (a) For $q = 0$,

   $$\begin{cases} L_+ + L_- = 0 \\ \frac{L_+}{m} - \frac{L_-}{n} = F_0' = 0 \end{cases} \iff L_+ = L_- = 0.$$

   (b) For $q = d$,

   $$\begin{cases} L_+ + L_- = d = m + n \\ \frac{L_+}{m} - \frac{L_-}{n} = F_d' = 0 \end{cases} \iff L_+ = m, L_- = n.$$

   (c) For $q = r$,

   $$\begin{cases} L_+ + L_- = r \\ \frac{L_+}{m} - \frac{L_-}{n} = F_r' = -\tau_1 \end{cases}$$

Note that in this case, in order to make our solution

$$\begin{aligned} L_+ &= \frac{rm - mn\tau_1}{m+n} \\ L_- &= \frac{rn + mn\tau_1}{m+n} \end{aligned}$$

satisfy the conditions (II) and (III) in the definition of templates. We need:

$$\tau_1 > 0 \tag{1}$$

$$L_+ = \frac{rm - mn\tau_1}{m+n} \in [0, m]_{\mathbb{Z}} \tag{2}$$

$$L_- = r - L_+ = \frac{rn + mn\tau_1}{m+n} \in [0, n]_{\mathbb{Z}} \tag{3}$$

$$-\frac{1}{n} \leq -\frac{\tau_1}{r} \leq \frac{1}{m} \tag{4}$$

$$-\frac{1}{n} \leq \frac{\tau_1}{d-r} \leq \frac{1}{m} \tag{5}$$

Solving (2) for τ_1 and using (1), we get

$$\tau_1 = \frac{-(m+n)L_+ + rm}{mn} = \frac{r}{n} - L_+(\frac{m+n}{mn}) > 0 \tag{1.2.41}$$

Now we can list the possible choices for L_+, L_-, τ_1 in the following table:

L_+	0	1	$\cdots$	$\lceil \frac{rm}{m+n} - 1 \rceil$
L_-	r	$r-1$	$\cdots$	$r - \lceil \frac{rm}{m+n} - 1 \rceil$
τ_1	$\frac{r}{n}$	$\frac{r}{n} - \frac{m+n}{mn}$	$\cdots$	$(\tau_1)_{\min}$

Table 1.3: Correspondence between $L_\pm$ and τ_1 on I_1.

Under the condition $\tau_1 > 0$, the maximal possible value for L_+ is $\lceil \frac{rm}{m+n} - 1 \rceil$, which gives us the minimal positive value for τ_1. Note that here we cannot choose $L_+ = \lfloor \frac{rm}{m+n} \rfloor$ since this may result in $\tau_1 = 0$ if $\frac{rm}{m+n}$ is an integer.

Now we turn to look at the conditions (4) and (5). Given $\tau_1 > 0$, it is equivalent to saying

$$0 < \tau_1 \leq \min\{\frac{r}{n}, \frac{d-r}{m}\}$$

Note that

$$(\tau_1)_{\min} = \tau_1 = \frac{-(m+n)\lceil \frac{rm}{m+n} - 1 \rceil + rm}{mn} = \frac{r}{n} - \left\lceil \frac{rm}{m+n} - 1 \right\rceil (\frac{m+n}{mn}) \leq \frac{r}{n}$$

So it suffices to check if

$$\frac{r}{n} - \left\lceil \frac{rm}{m+n} - 1 \right\rceil (\frac{m+n}{mn}) \leq \frac{d-r}{m}$$

But this is equivalent to

$$\begin{aligned}
&\frac{r}{n} - \frac{d-r}{m} \leq \left\lceil \frac{rm}{m+n} - 1 \right\rceil \frac{m+n}{mn} \\
\iff &(\frac{1}{m} + \frac{1}{n})r - \frac{d}{m} \leq \left\lceil \frac{rm}{m+n} - 1 \right\rceil \\
\iff &r - n = r - \frac{\frac{d}{m}}{\frac{1}{m} + \frac{1}{n}} \leq \left\lceil \frac{rm}{m+n} - 1 \right\rceil
\end{aligned}$$

By our lemma above, the last line holds for $r \leq \max(m, n)$.

(2) On the intervals of linearity I_2, we shall determine the possible values for $L_\pm := L_\pm(I_2, q)$ as follows:

(a) For $q = 0$,

$$\begin{cases} L_+ + L_- = 0 \\ \frac{L_+}{m} - \frac{L_-}{n} = F'_0 = 0 \end{cases} \iff L_+ = L_- = 0.$$

(b) For $q = d$,

$$\begin{cases} L_+ + L_- = d = m + n \\ \frac{L_+}{m} - \frac{L_-}{n} = F'_d = 0 \end{cases} \iff L_+ = m, L_- = n.$$

(c) For $q = r$,

$$\begin{cases} L_+ + L_- = r \\ \frac{L_+}{m} - \frac{L_-}{n} = F_r' = +\tau_2 \end{cases} \iff \begin{cases} L_+ = \frac{rm+mn\tau_2}{m+n} \\ L_- = \frac{rn-mn\tau_2}{m+n} \end{cases}$$

A similar discussion yields the possible values for L_+, L_- and τ_2 as listed in the following table:

L_-	0	1	$\cdots$	$\lceil \frac{rn}{m+n} - 1 \rceil$
L_+	r	$r-1$	$\cdots$	$r - \lceil \frac{rn}{m+n} - 1 \rceil$
τ_2	$\frac{r}{m}$	$\frac{r}{m} - \frac{m+n}{mn}$	$\cdots$	$(\tau_2)_{\min}$

Table 1.4: Correspondence between $L_\pm$ and τ_2 on I_2.

From the derivative restrictions:

$$-\frac{1}{n} \leq \frac{\tau_2}{r} \leq \frac{1}{m}$$
$$-\frac{1}{n} \leq \frac{\tau_2}{d-r} \leq \frac{1}{m}$$

we have $\tau \leq \min\{\frac{r}{m}, \frac{d-r}{n}\}$, but $\tau_2 \leq \frac{r}{m}$ is automatic and the condition

$$(\tau_2)_{\min} = -\frac{m+n}{mn}\left\lceil \frac{rn}{m+n} - 1 \right\rceil + \frac{r}{m} \leq \frac{d-r}{n}$$

is equivalent to

$$r - m \leq \left\lceil \frac{rn}{m+n} - 1 \right\rceil.$$

Again, this is guaranteed by the proceeding lemma as long as $r \leq \max(m, n)$.

Summarizing what we have done so far, we have demonstrated that it is possible to find slopes τ_1, τ_2 such that the d-component piecewise linear function $\mathbf{f} = \mathbf{q}[\tau_1, \tau_2]$ on $[a_k, b_k] = [b_k - k, b_k]$ as constructed is indeed a partial template as long as $r \leq \max(m, n)$.

Now we will show that

$$\delta(\mathbf{f}) = mn.$$

Indeed,

$$\delta(\mathbf{f}) := \lim_{T\to\infty} \Delta(\mathbf{f}, T) := \lim_{T\to\infty} \frac{1}{T}\int_0^T \delta(\mathbf{f}, [0, t])dt,$$

where $0 < \delta(\mathbf{f}, [a_k, b_k]) \leq mn$ and $\delta(\mathbf{f}, [b_{k-1}, a_k]) = mn$.

Since

$$0 =: b_0 < b_1 < \cdots < b_k = n^n \to \infty,$$

any $T > 0$ lies in the dilating period $(b_{n-1}, b_k]$ of $\mathbf{f}$ for some $n \geq 0$. In each of such period, the average of integral $\frac{1}{T}\int_0^T \delta(\mathbf{f}, [0, T])dt$ reaches its maximum at $T = a_k$ and its minimum at $T = b_k$:

$$\frac{1}{a_k}\int_0^{a_k} \delta(\mathbf{f}, [0, t])dt \leq mn.$$

But

$$\begin{aligned}
\frac{1}{a_k}\int_0^{a_k} \delta(\mathbf{f}, [0, t])dt \geq& \frac{1}{a_k}\sum_{i=1}^{n}\int_{b_{i-1}}^{a_i} \delta(\mathbf{f}, [0, t])dt \\
=& \frac{1}{a_k}\sum_{i=1}^{n}\int_{b_{i-1}}^{a_i} mn\ dt \\
=& mn\frac{1}{k^k - k}\sum_{i=1}^{k}(i^i - i - (i-1)^{i-1}) \\
=& mn\frac{k^k}{k^k - k} - mn\frac{k(k+1)}{2(k^k - k)}
\end{aligned}$$

which converges to mn as $k \to \infty$.

Therefore, by the variational principle,

$$\dim_P(\mathbf{BA}(r+1) - \mathbf{BA}(r)) = \dim_H(\mathbf{BA}(r+1) - \mathbf{BA}(r)) = mn,$$

for all $r \leq \max(m, n)$. Since $\frac{d}{2} \leq \max(m, n)$, we have proved the theorem for at least half of the orders.

For the computation of cases when $r > \max(m, n)$, we will study the dual of the orbit of lattices $(g_t u_A \mathbb{Z}^d)_{t \geq 0}$.

Recall that for a lattice $\Lambda \subset \mathbb{R}^d$ with basis $\{\mathbf{b}_1, \cdots, \mathbf{b}_d\}$, since $\mathbf{b}_1, \cdots, \mathbf{b}_d$ are linearly independent, from linear algebra we know there exist vectors $\mathbf{b}_1^*, \cdots \mathbf{b}_d^*$, call *dual vectors* to $\mathbf{b}_1, \cdots, \mathbf{b}_d$, such that

$$\langle \mathbf{b}_i, \mathbf{b}_i^* \rangle = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}.$$

The $\mathbb{Z}$-span of dual basis vectors, namely $\Lambda^* := \mathrm{Span}\{\mathbf{b}_1^*, \cdots \mathbf{b}_d^*\}$, is called the *dual (or polar or reciprocal) lattice* to the lattice Λ.

Although defined through basis, it turns out that the dual lattices are independent of the choice of basis of the original lattice.

Proposition 1.2.27. *The dual lattice Λ^* consists of all vectors $\mathbf{b}^* \in \mathbb{R}^d$ such that $\langle \mathbf{b}^*, \mathbf{b} \rangle$ is an integer for all $\mathbf{b}$ in Λ. As a consequence, Λ^* is also the dual of Λ.*

Proof. Let $\mathbf{b}_1, \cdots, \mathbf{b}_d$ be a basis of the lattice Λ and their duals be $\mathbf{b}_1^*, \cdots, \mathbf{b}_d^*$. For any $\mathbf{b} \in \Lambda$ and any $\mathbf{c} \in \Lambda^*$, suppose

$$\mathbf{b} = s_1\mathbf{b}_1 + \cdots + s_d\mathbf{b}_d, \text{ and } \mathbf{c} = t_1\mathbf{b}_1^* + \cdots + t_d\mathbf{b}_d^*$$

with integer coefficients $s_i, t_i \in \mathbb{Z}$ for $i = 1, 2, \cdots, d$. We have immediately that

$$\langle \mathbf{b}, \mathbf{c} \rangle = s_1t_1 \cdots + s_dt_d \in \mathbb{Z}.$$

On the other hand, if $\mathbf{b}^* = u_1\mathbf{b}_1^* + \cdots + u_d\mathbf{b}_d^* \in \mathbb{R}^d$, where $u_i \in \mathbb{R}$ satisfies $\langle \mathbf{b}^*, \mathbf{b} \rangle \in \mathbb{Z}$, for any $\mathbf{b} \in \mathbb{Z}$, then in particular this holds for $\mathbf{b} = \mathbf{b}_i$, for any $i = 1, 2, \cdots, d$ and thus

$$u_i = \langle \mathbf{b}^*, \mathbf{b}_i \rangle \in \mathbb{Z}.$$

Therefore $\mathbf{b}^* \in \Lambda^*$.

$\square$

The dual lattice operator commutes nicely with an invertible linear transformation on $\mathbb{R}^d$:

Proposition 1.2.28. *Let Λ be a lattice on $\mathbb{R}^d$ and $T : \mathbb{R}^d \to \mathbb{R}^d$ be an invertible linear transformation, then we have*

$$(T\Lambda)^* = T^*\Lambda^*,$$

where $T^* = {}^tT^{-1}$ is the inverse of the transpose of T and Λ^* is the dual lattice to Λ.

Proof. If $\mathbf{b}_1, \cdots, \mathbf{b}_d$ is a basis of Λ, then $T\mathbf{b}_1, \cdots, T\mathbf{b}_d$ is a basis of $T\Lambda$. The corresponding dual basis

$$(T\mathbf{b}_1)^*, \cdots, (T\mathbf{b}_d)^*$$

satisfy

$$\begin{bmatrix} {}^t(T\mathbf{b}_1)^* \\ \vdots \\ {}^t(T\mathbf{b}_1)^* \end{bmatrix} \begin{bmatrix} T\mathbf{b}_1 & \cdots & T\mathbf{b}_d \end{bmatrix} = I_d$$

But on the other hand,

$$\begin{bmatrix} {}^t({}^tT^{-1}\mathbf{b}_1) \\ \vdots \\ {}^t({}^tT^{-1}\mathbf{b}_d) \end{bmatrix} \begin{bmatrix} T\mathbf{b}_1 & \cdots & T\mathbf{b}_d \end{bmatrix} = I_d$$

So by the uniqueness of inverse matrix, $T^*\mathbf{b}_i = {}^tT^{-1}\mathbf{b}_i = (Tb_i)^*$, for any $i = 1, 2 \cdots, d$. Therefore $(T\Lambda)^* = T^*\Lambda^*$. $\square$

We are able to use the idea of dual lattice to address the issue of higher r's, thanks to the following theorem:

Theorem 1.2.29 ([Cas97] Chapter VIII, Theorem VI)**.** *Let $\lambda_1, \cdots, \lambda_d$ be the successive minima of lattices in $\mathbb{R}^d$. Then for a lattice Λ and its dual Λ^*, we have*

$$1 \leq \lambda_r(\Lambda)\lambda_{d+1-r}(\Lambda^*) \leq d!$$

for any $r = 1, 2, \cdots d$.

Now let us return to our proof, for the flow of lattices $(g_t u_A \Lambda)$, the proposition above gives its dual as:

$$
\begin{aligned}
(g_t u_A \mathbb{Z}^d)^* =& g_t^* u_A^* (\mathbb{Z}^d)^* \\
=& {}^t g_t^{-1} \cdot {}^t u_A^{-1} \mathbb{Z}^d \\
=& {}^t \begin{bmatrix} e^{t/m} I_m & 0 \\ 0 & e^{-t/n} I_n \end{bmatrix}^{-1} \cdot {}^t \begin{bmatrix} I_m & A \\ 0 & I_n \end{bmatrix}^{-1} \mathbb{Z}^d \\
=& \begin{bmatrix} e^{-t/m} I_m & 0 \\ 0 & e^{t/n} I_n \end{bmatrix} \cdot \begin{bmatrix} I_m & 0 \\ -{}^t A & I_n \end{bmatrix} \mathbb{Z}^d
\end{aligned}
$$

Now observe that for $r > \frac{d}{2}$, $d+1-r < d+1-\frac{d}{2} = \frac{d}{2}+1 \leq \max(m,n)+1$ (which is the same as $d+1-r \leq \max(m,n)$), and we have that

$$
\begin{aligned}
& \liminf_{t\to\infty} \mathbf{h}_{A,r} = -\infty \\
\iff & \liminf_{t\to\infty} \lambda_r(g_t u_A \mathbb{Z}^d) = 0 \\
\iff & \liminf_{t\to\infty} \lambda_r(g_t u_A \mathbb{Z}^d) = 0 \\
\iff & \limsup_{t\to\infty} \lambda_{d+1-r}((g_t u_A \mathbb{Z}^d)^*) = \infty \\
\iff & \limsup_{t\to\infty} \lambda_{d+1-r} \left(\begin{bmatrix} e^{-t/m} I_m & 0 \\ 0 & e^{t/n} I_n \end{bmatrix} \cdot \begin{bmatrix} I_m & 0 \\ -{}^t A & I_n \end{bmatrix} \mathbb{Z}^d \right) = \infty \\
\iff & \limsup_{t\to\infty} \mathbf{h}_{-{}^t A, d+1-r}(t) = \infty \\
\iff & \liminf_{t\to\infty} -\mathbf{h}_{-{}^t A, d+1-r}(t) = -\infty
\end{aligned}
$$

and that

$$\begin{aligned}
&\liminf_{t\to\infty} \mathbf{h}_{A,r+1}(t) > -\infty \\
\Longleftrightarrow\ &\liminf_{t\to\infty} \lambda_{r+1}(g_t u_A \mathbb{Z}^d) > 0 \\
\Longleftrightarrow\ &\liminf_{t\to\infty} \lambda_{r+1}(g_t u_A \mathbb{Z}^d) > 0 \\
\Longleftrightarrow\ &\limsup_{t\to\infty} \lambda_{d-r}((g_t u_A \mathbb{Z}^d)^*) < \infty \\
\Longleftrightarrow\ &\limsup_{t\to\infty} \lambda_{d-r}\left(\begin{bmatrix} e^{-t/m} I_m & 0 \\ 0 & e^{t/n} I_n \end{bmatrix} \cdot \begin{bmatrix} I_m & 0 \\ -{}^t A & I_n \end{bmatrix} \mathbb{Z}^d\right) < \infty \\
\Longleftrightarrow\ &\limsup_{t\to\infty} \mathbf{h}_{-{}^t A,d-r}(t) < \infty \\
\Longleftrightarrow\ &\liminf_{t\to\infty} -\mathbf{h}_{-{}^t A,d-r}(t) > -\infty
\end{aligned}$$

Therefore, by replacing templates $\mathbf{f}$ with $-\mathbf{f}$ in our $\mathcal{S}$ and observe that if we change any $m \times n$ matrix A $D(\mathcal{S})$ to its negative transpose $-t^A$, then the Hausdorff and Packing dimensions of $D(\mathcal{S})$ will not change and the result we obtained above for $d \leq \max(m,n)$ applies. This proves

$$\dim_P(\mathbf{BA}(r+1) - \mathbf{BA}(r)) = \dim_H(\mathbf{BA}(r+1) - \mathbf{BA}(r)) = mn,$$

for $d > \max(m,n)$ and completes the proof of the theorem.

$\square$

1.3 Haar measure distribution of successive minima on the space of unimodular lattices and logarithms laws.

1.3.1 Distribution function associated to higher sucesssive minima and estimates

Proposition 1.3.1. *For a rank d unimodular lattice $\Lambda \in \mathcal{L}$, let $\lambda_i(\Lambda)$ denote its i-th successive minima ($1 \leq i \leq d$). For any $\delta > 0$, we have*

$$\mu(\{\Lambda \in \mathcal{L} : \lambda_i(\Lambda) = \delta\}) = 0,$$

where μ is the Haar measure defined on the space of unimodular lattices.

Proof. Indeed, the set $\{\Lambda \in \mathcal{L} : \lambda_i(\Lambda) = \delta\}$ is contained in

$$S_\delta := \{\Lambda \in \mathcal{L} : \text{there exists a vector } v \in \Lambda \text{ with } \|v\| = \delta\}.$$

Noticing that any unimodular lattice can be written as $g\mathbb{Z}^d$ for some $g \in \mathrm{SL}(d, \mathbb{R})$ and the local identification between Haar measure on $G = \mathrm{SL}(d, \mathbb{R})$ and $G/\Gamma = \mathrm{SL}(d, \mathbb{R})/\mathrm{SL}(d, \mathbb{Z})$, we shall look at the set

$$\begin{aligned} T_\delta :=& \{g \in G : \text{there exists a vector } v \in \mathbb{Z}^d \text{ with } \|gv\| = \delta\} \\ =& \cup_{v \in \mathbb{Z}^d} \{g \in G : \|gv\| = \delta\}. \end{aligned}$$

This is a countable union and each member in the union is a submanifold of G with lower dimension and hence of zero Haar measure. $\square$

For $x \geq 0$, it would be interesting to give an estimate for the distribution function

$$\Phi_i(\delta) := \mu(\{\Lambda \in \mathcal{L} : \lambda_i(\Lambda) < \delta\}) = \mu(\{\Lambda \in \mathcal{L} : \lambda_i(\Lambda) \leq \delta\})$$

For $i = 1$, Kleinbock and Margulis gave both lower and upper bounds for $\Phi_1(x)$ [KM99] using a generalized Siegel's formula:

Theorem 1.3.2 ([KM99], Proposition 7.1)**.** *There exists C_d, C'_d such that*

$$C_d \delta^d - C'_d \delta^{2d} \leq \Phi_1(\delta) \leq C_d \delta^d,$$

for $\delta \ll 1$.

The main result in this section is a generalization of the above result to λ_i:

Theorem 1.3.3. *For $1 \leq i < d$, there exists C_d and C'_d such that*

$$C_d \delta^{di} - o(\delta^{di}) \leq \Phi_i(\delta) := \mu(\{\Lambda \in \mathcal{L} : \lambda_i(\Lambda) \leq \delta\}) \leq C'_d \delta^{di},$$

for all $\delta \ll 1$.

Corollary 1.3.4. *For $1 \leq i \leq d-1$, we have*

$$\lim_{t\to\infty} \frac{-\log \mu\left(\{\Lambda \in \mathcal{L} : \lambda_i(\Lambda) \leq e^{-t}\}\right)}{t} = di$$

Proof. Take $\delta = e^{-t}$. $\square$

For the proof we will use a generalized version of Siegel's mean value formula in geometry of numbers: For a lattice Λ in $\mathbb{R}^d$, let $P(\Lambda)$ denote the set of primitive vectors in Λ, i.e. those vectors that are not a proper integer multiple of any other element in Λ. Given a real-valued function f on $\mathbb{R}^d$, we define a function $\hat{f}$ on the homogeneous space $X = G/\Gamma := \mathrm{SL}(d,\mathbb{R})/\mathrm{SL}(d,\mathbb{Z})$ by

$$\hat{f} := \sum_{v\in P(\Lambda)} f(v)$$

Theorem 1.3.5 (Classical Siegel's Formula [Sie45])**.** *For any $f \in L^1(\mathbb{R}^d)$, one has*

$$\int_X \hat{f} d\mu = c_d \int_{\mathbb{R}^d} f dv,$$

where $c_d = \frac{1}{\zeta(d)} := \frac{1}{\sum_{n=1}^{\infty} \frac{1}{n^d}}$.

Below is a generalization of classical Siegel's formula. First let us recall the notion of primitive tuple from geometry of numbers:

Definition 1.3.6. For $1 \leq k \leq d$, we say that an ordered k-tuple of vectors $(v_1, \ldots, v_k) \in \underbrace{\Lambda \times \cdots \times \Lambda}_{d\text{-times}}$ for a lattice $\Lambda \subset \mathbb{R}^d$ is primitive if it is extendable to a basis of Λ, and denote by $P^k(\Lambda)$ the set of all such k-tuples. Note that $P^1(\Lambda) = P(\Lambda)$ above [3].

Now for a function $f \in \mathbb{R}^{dk} = (\mathbb{R}^d)^k$, we define correspondingly

$$\hat{f}^k(\Lambda) := \sum_{(v_1,\ldots,v_k)\in P^k(\Lambda)} f(v_1, \ldots, v_k).$$

[3] Any primitive vector in a lattice can be extended to a basis of the lattice. This follows from the general fact that any element of a free abelian group which is not divisible by any integer bigger than 1 can be extended to a basis of the abelian group

Here the superscript on $\hat{f}^k$ should not be confused with the composition (power) of a function. Then we have a generalized Siegel's Formula for primitive tuples which will be helpful for us to estimate the distribution for $\lambda_i(\Lambda)$.

Theorem 1.3.7 (Generalized Siegel's Formula for primitive tuples)**.** *For* $1 \leq k < d$ *and* $\phi \in L^1(\mathbb{R}^{dk})$*, we have*

$$\int_X \hat{f}^k d\mu = c_{k,d} \int_{\mathbb{R}^{dk}} f dv_1 \cdots dv_k, \tag{1.3.1}$$

where $c_{d,k} = \frac{1}{\zeta(d)\cdots\zeta(d-k+1)}$.

Proof. Let $\{e_1, \ldots, e_d\}$ be the canonimcal basis of $\mathbb{R}^d$. For $G = \mathrm{SL}(d, \mathbb{R})$ and $\Gamma = \mathrm{SL}(d, \mathbb{Z})$ and the k-tuple $(e_1, \ldots, e_k)$, be , let

$$G_k := \{g \in G : g.e_i = e_i, \forall 1 \leq i \leq k\},$$

$$\Gamma_k := \{g \in \Gamma : g.e_i = e_i, \forall 1 \leq i \leq k\}.$$

be the stabilizer subgroup of $(e_1, \ldots, e_k)$ in G and Γ, respectively.

Now consider the subset

$$L := \{(v_1, \ldots, v_k) \in \mathbb{R}^{dk} : v_1, \ldots, v_k \text{ are linearly independent vectors in } \mathbb{R}^d\}$$

Claim 1. *L is open dense in $\mathbb{R}^{dk}$ and in particular $\mathbb{R}^{dk} - L$ is of Lebesgue measure zero.*

Proof of claim . That L is open follows from the condition that linear independence implies that $[v_1, \ldots, v_k]$ is a full-rank matrix (there exists at least one $k \times k$ submatrix with determinant zero).

To see it is dense, we observe that this is equivalent to proving that the set of full-rank $d \times k$ matrices, denoted F, is dense in $M(d \times k, \mathbb{R})$. Noticing that its complement F^c is contained in some subvariety (of stricly lower dimension)

$$\{A \in M(d \times k, \mathbb{R}) : \det(A_{k\times k}) = 0\},$$

for some $k \times k$ submatrix of A. Therefore F must be dense in $\mathbb{R}^{dk}$. #

Claim 2. *L is equal to the G-orbit of the k-tuple $(e_1, \ldots, e_k)$ in $\mathbb{R}^{dk}$.*

Proof of claim . Indeed, for any $g \in G = \mathrm{SL}(d, \mathbb{R})$, the tuple $(g.e_1, \ldots, g.e_k)$ corresponds to the first k columns of the matrix g. However, any k linearly independent vectors $v_1, \cdots, v_k$ in $\mathbb{R}^d$ ($k < d$) can be completed to a $d \times d$ matrix of determinant 1 (by adding diagonal entries to the last $d - k$ columns). #

Now consider the map

$$\phi_G : G \to L \subset \mathbb{R}^{dk}, g \mapsto (g.e_1, \ldots, g.e_k)$$

By the Orbit-Stabilizer theorem, we have the identification of homogeneous spaces $\phi'_G : G/G_k \xrightarrow{\sim} L$. Since L is open dense in $\mathbb{R}^{dk}$ and the Lebesgue measure on $\mathbb{R}^{dk}$ (viewed as a product (Lebesgue) measure on $\underbrace{\mathbb{R}^d \times \cdots \times \mathbb{R}^d}_{k\text{-times}}$) is invariant under $G = \mathrm{SL}(d, \mathbb{R})$. The pull-back of the Lebesgue measure on $\mathbb{R}^{dk}$ gives a (unique up to scalar multiple) G-invariant Haar measure μ_{G/G_k} on G/G_k (uniqueness of Haar measure on G/G_k follows from the unimodularity of G).

Claim 3. *$P^k(\mathbb{Z}^d)$ is equal to the Γ-orbit of the k-tuple $(e_1, \ldots, e_k)$ in $\mathbb{R}^{dk}$.*

Proof of claim . Let $(q_1, \ldots, q_k)$ be any k-tuple of integer vectors in $\mathbb{Z}^d$ that are extendable to a basis $\{q_1, \ldots, q_d\}$ of $\mathbb{Z}^d$, and as a basis we have $\det[q_1 \ldots q_d] = 1$ (up to adjusting the sign of the last column q_d) and hence $(q_1, \ldots, q_k)$ lies in the Γ-orbit of the k-tuple $(e_1, \ldots, e_k)$.

On the other hand, for any $(g.e_1, \ldots, g.e_k)$, where $g \in \Gamma = \mathrm{SL}(d, \mathbb{Z})$, clearly $\{g.e_1, \ldots, g.e_k\}$ can be completed to a basis $\{g.e_1 \ldots, g.e_d\}$ of $\mathbb{Z}^d$. #

It follows that the map

$$\phi_\Gamma : \Gamma \to P^k(\mathbb{Z}^d) \subset \mathbb{R}^{dk}, \gamma \mapsto (\gamma.e_1, \ldots, \gamma.e_k)$$

gives the identification of Γ-homogeneous spaces $\phi_\Gamma : \Gamma/\Gamma_k \xrightarrow{\sim} P^k(\mathbb{Z}^d)$, under which the counting measure on $P^k(\mathbb{Z}^d)$ (clearly Γ-invariant) can be pulled back to a (unique up to scalar multiples) Γ-invariant Haar measure on Γ/Γ_k, denoted μ_{Γ/Γ_k}. Note that the summation over $P^k(\mathbb{Z}^d)$ is equal to the integration with respect to the counting measure μ_{Γ/Γ_k} over Γ/Γ_k.

By the Lemma 1.6, Chapter I in [Rag72], the invariant measures on G/Γ and Γ/Γ_k give an invariant measure on Γ/Γ_k and we have the quotient integral formula

$$\begin{aligned}&\int_{G/\Gamma}\int_{\Gamma/\Gamma_k}\varphi(gl\Gamma_k)\ d\mu_{\Gamma/\Gamma_k}(l\Gamma)d\mu(g)\\ =&\int_{G/\Gamma_k}\varphi(g\Gamma_k)\ d\mu_{G/\Gamma_k}(g\Gamma)\\ =&\int_{G/G_k}\int_{G_k/\Gamma_k}\varphi(gg_k\Gamma_k)\ d\mu_{G_k/\Gamma_k}(g_k\Gamma_k)d\mu_{G/G_k}(gG_k)\end{aligned}$$

for any $\varphi \in L^1(G/\Gamma_k)$.

Now for any $f \in L^1(\mathbb{R}^{dk})$, we first identify f with a function in $L^1(G/G_k)$ via ϕ_G. For this new f, we define φ_f on G/Γ_k by setting it as of constant value on each G_k-coset:

$$\varphi_f(g\Gamma_k) := f(gG_k), \forall g \in G.$$

In other words, $\varphi_f(g\Gamma_k) = \varphi_f(h\Gamma_k)$, whenever $h^{-1}g \in G_k$.

It follows that

$$\int_{G/\Gamma_k}\varphi_f(g\Gamma_k)d\mu_{G/\Gamma_k} = \mu_{G_k/\Gamma_k}(G_k/\Gamma_k)\cdot\int_{G/G_k}f(gG_k)d\mu_{G/G_k}(gG_k) < \infty.$$

So $\varphi_f \in L^1(G/\Gamma_k)$.

Claim 4. *The inner integral on the left hand side is (recall that the integration with respect to counting measure on Γ/Γ_d is the same as the sum over primitive tuples)*

$$\int_{\Gamma/\Gamma_k}\varphi_f(gl\Gamma_k)d\mu_{\Gamma/\Gamma_k}(l) = \sum_{(v_1,\ldots,v_k)\in P^k(g\mathbb{Z}^d)} f(v_1,\ldots,v_k) =: \hat{f}^k(g\mathbb{Z}^d).$$

Proof of claim . First recall by our identification

$$
\begin{aligned}
&\sum_{(v_1,\ldots,v_k)\in P^k(g\mathbb{Z}^d)} f(v_1,\ldots,v_k) \\
&= \sum_{(he_1,\ldots,he_k)\in P^k(g\mathbb{Z}^d),h\in G} f(he_1,\ldots,he_k) \\
&= \sum_{(he_1,\ldots,he_k)\in P^k(g\mathbb{Z}^d),h\in G} f(hG_k) && \text{(viewing } f \text{ as a function on } G/G_k) \\
&= \sum_{(g^{-1}he_1,\ldots,g^{-1}he_k)\in P^k(\mathbb{Z}^d),h\in G} f(hG_k) \\
&= \sum_{(le_1,\ldots,le_k)\in P^k(\mathbb{Z}^d),l\in G} f(glG_k) && \text{(change of variable } l := g^{-1}h) \\
&= \int_{\Gamma/\Gamma_k} f(glG_k)d\mu_{\Gamma/\Gamma_k}(l\Gamma_k) && \text{(summation to integration w.r.t. counting measure)} \\
&= \int_{\Gamma/\Gamma_k} \varphi_f(gl\Gamma_k)d\mu_{\Gamma/\Gamma_k}(l\Gamma_k) && \text{(definition of } \varphi_f)
\end{aligned}
$$

#

Hence this proves

$$\int_X \hat{f}^k d\mu = c_{k,d}\int_{\mathbb{R}^{dk}} f dv_1\cdots dv_k,$$

with $c_{k,d} = \mu_{G_k/\Gamma_k}(G_k/\Gamma_k)$. □

Now we can give the distribution for $\Phi_i(\delta)$:

Proof of Theorem 1.3.3. Let B be the ball centered at 0 with radius δ.

Note that the condition $\lambda_i(\Lambda) < \delta$ implies that there are at least i linearly independent vectors in Λ lying in the open ball B. However, this does not necessarily mean they can be extended to a basis. But thanks to the Theorem A.6, there exists a basis $v_1,\ldots,v_d$ of Λ such that

$$\|v_1\| = \lambda_1(\Lambda), \|v_2\|_d \asymp_d \lambda_2(\Lambda),\ldots,\|v_d\| \asymp_d \lambda_d(\Lambda).$$

It follows that there exists a constant factor $\eta_d > 1$ such that if we dilate the ball B by η_d to a new ball B' (centered at the origin with radius $\eta_d\delta$), we have $\Lambda \cap B'$ contains i linearly independent vectors that can be extended to a basis of Λ. It follows that (since by symmetry v and $-v$ must be contained in $\Lambda \cap B'$ simultaneously and any permutation of this i-tuple also gives a new primitive i-tuple):

$$|P^i(\Lambda) \cap B'^i| \geq 2^i i!,$$

where $B'^i := \underbrace{B' \times \cdots \times B'}_{i\text{-times}} \subset \mathbb{R}^{di}$.

Now take $f := \mathbf{1}_{B'^i}$ and $\hat{f} = \hat{f}^i(\Lambda) := \sum_{(v_1,\ldots,v_i)\in P^i(\Lambda)} f(v_1,\ldots,v_i)$ counts the number of points falling into B'^i. The left hand side of the generalized Siegel's formula (Theorem 1.3.7) yields

$$\int_X \hat{f}^i d\mu \geq \int_{\{\Lambda:\lambda_i(\Lambda)\leq\delta\}} \hat{f}^i d\mu \geq 2^i i! \mu(\{\lambda : \lambda_i(\Lambda) \leq \delta\}).$$

On the other hand

$$\int_{\mathbb{R}^{di}} f dv_1 \cdots dv_i = \mathrm{Vol}(B')^i = \eta_d^i \delta^{di} c_d^i,$$

where c_d is the volume of unit ball in $\mathbb{R}^d$.

Hence we have the upper bound

$$\mu(\{\Lambda : \lambda_i(\Lambda) \leq \delta\}) \leq \frac{1}{i!}\left(\frac{\eta_d c_d}{2}\right)^i \delta^{di}$$

For the lower bound, for $1 \leq i < d-1$ and $x > 0$, let $N(i,x)$ denote the quantity

$$\min\{|P^i(\Lambda)\cap B(0,x)^i| : \Lambda \in \mathcal{L}, \Lambda\cap B(0,x) \text{ contains at least } i+1 \text{ linearly independent vectors}\},$$

namely the miminum of the number of all primitive i-tuples $(v_1,\cdots,v_i)$ with each component taken from the lattice $\Lambda \cap B(0,x)$ for all unimodular lattice Λ, given $\Lambda \cap B(0,x)$ contains at least $i+1$ linearly independent vectors. Note that by our assumption and the discussion above, $N(i,\eta_d\delta) \geq 2^i i!$.

Since one can always choose a unimodular lattice with the first $i+1$ sucessive minima small enough to be contained in $B(0,x)$ for any $x > 0$ whenever $i < d-1$, we have

Claim 5. $N(i,x) \leq 2^i(i+1)!$ *(independent of* $x \in (0,1)$*) whenever* $i < d-1$.

Proof of claim . Consider the unimodular lattice

$$\Lambda_x := \left\{ \frac{3}{4}xe_1, ... \frac{3}{4}xe_{i+1}, \frac{1}{(\frac{3}{4}x)^{\frac{i+1}{d-i-1}}}e_{i+2}, ... \frac{1}{(\frac{3}{4}x)^{\frac{i+1}{d-i-1}}}e_d \right\}.$$

Note that when $x < 1$, we have

$$\Lambda_x \cap B(0,x) = \left\{ \pm\frac{3}{4}xe_1, ..., \pm\frac{3}{4}xe_{i+1} \right\}$$

since any integer linear combination

$$n_1\frac{3}{4}xe_1 + ... + n_{i+1}\frac{3}{4}xe_{i+1}$$

with some $|n_j| \geq 2$ or at least two of $n_j \neq 0$ must be outside of $B(0,x)$. So

$$|P^i(\Lambda_x) \cap B(0,x)^i| \leq 2^i i! \binom{i+1}{i} = 2^i(i+1)!.$$

Therefore $N(i,x) \leq 2^i(i+1)!$. #

Now set $x = \delta$. The idea is to separated the integration domain into two parts: $\{\Lambda : \hat{f}^i(\Lambda) < N(i,\delta)\}$ and $\{\Lambda : \hat{f}^i(\Lambda) \geq N(i,\delta)\}$. We will see that the integration over the second domain contribute insignificantly as $\delta \to 0$. Hence,

$$\int_X \hat{f}^i d\mu = \int_{\{\Lambda:\hat{f}^i(\Lambda)<N(i,\delta)\}} \hat{f}^i d\mu + \int_{\{\Lambda:\hat{f}^i(\Lambda)\geq N(i,\delta)\}} \hat{f}^i d\mu$$

Notice that by our choice of B', f and the definition of $\hat{f}^i$, $f^i(\Lambda) = |P^i(\Lambda) \cap B^i|$, and the first term [4]

$$\begin{aligned}
\int_{\{\Lambda:\hat{f}^i(\Lambda)<N(i,\delta)\}} \hat{f}^i d\mu = & \int_{\{\Lambda:\hat{f}^i(\Lambda)<N(i,\delta)\}} \hat{f}^i d\mu \\
= & \int_{\{\Lambda:\Lambda\cap B \text{ contains no } i+1 \text{ linearly independent vectors, } \hat{f}^i(\Lambda)<N(i,\delta)\}} \hat{f}^i d\mu \\
\leq & \int_{L_i} 2^i(i+1)! d\mu \\
= & 2^i(i+1)!\mu(S_i) \qquad \text{(by the estimate from the claim above)}
\end{aligned}$$

where L_i denote the set of unimodular lattices that contain no $i+1$ linearly independent vectors but contain at least one family of primitive i-set of vectors (so that the integral will not vanish). But clearly $S_i \subset \{\Lambda : \lambda_i(\Lambda) \leq \delta\}$.

Now we look at the second term $\int_{\{\Lambda:\hat{f}^i(\Lambda)\geq N(i,\delta)\}} \hat{f}^i d\mu$. If $\hat{f}^i(\Lambda) \geq N(i,\delta)$, by definition it means the ball $B = B(0,\delta)$ contains at least $i+1$ linearly independent vectors in Λ. But again by symmetry that extra vector has to come in pairs namely $B \cap \Lambda$ has to contain both v_{i+1} and $-v_{i+1}$.

Therefore for such Λ,

$$|P^i(\Lambda) \cap B^i)| \leq \frac{1}{2}|P^{i+1}(\Lambda) \cap B^{i+1})|.$$

[4] Note that if $i = d-1$ this argument won't make sense since $N(d,\delta)$ will become zero if $\delta \to 0$ by the Minkowski's second convex body theorem A.8.

Notice that the left hand side is precisely $\hat{f}^i(\Lambda)$. Let $f_i = \mathbf{1}_{B^{i+1}}$, a function in $\mathbb{R}^{d(i+1)}$, we have

$$\begin{aligned}
\int_{\{\Lambda:\hat{f}^i(\Lambda)\geq N(i,\delta)\}} \hat{f}^i d\mu =& \frac{1}{2}\int_{\{\Lambda:\hat{f}^i(\Lambda)>N(i,\delta)\}} \hat{f_1}^{i+1} d\mu \\
\leq& \frac{1}{2}\int_X \hat{f_i}^{i+1} d\mu \\
=& \frac{1}{2}\int_{\mathbb{R}^{d(i+1)}} f_i dv_1 \cdots dv_{i+1} \qquad \text{(by the generalized Siegel's formula)} \\
=& \frac{1}{2}(\int_{\mathbb{R}^d} \mathbf{1}_B dv)^{i+1} \\
=& \frac{1}{2}(c_d\delta^d)^{i+1}
\end{aligned}$$

Therefore we obtain the lower bound

$$\begin{aligned}
\mu(\{\Lambda : \lambda_i(\Lambda) \leq \delta\}) \geq& \frac{1}{N(i,\delta)}\left(\int_X \hat{f}^i d\mu - \int_{\{\Lambda:\hat{f}^i(\Lambda)\geq N(i,\delta)\}} \hat{f}^i d\mu\right) \\
=& \frac{1}{2^i(i+1)!}\left(\int_{\mathbb{R}^{di}} f dv_1 \cdots dv_i - \int_{\{\Lambda:\hat{f}^i(\Lambda)\geq N(i,\delta)\}} \hat{f}^i d\mu\right) \\
=& \frac{1}{2^i(i+1)!}(c_d^i\delta^{di} - \frac{1}{2}c_d^{i+1}\delta^{d(i+1)}).
\end{aligned}$$

This finishes the case when $i < d-1$.

To cover the remaining case when $i = d-1$, we will study the following example:

Example 1.3.8 (Measure of a subset of $\{\Lambda : \lambda_i(\Lambda) \leq \delta\}$)**.**

In view of Iwasawa decomposition of $G = \mathrm{SL}(d,\mathbb{R})$, we shall first construct a subset of G that will shrink the first i canonical basis vectors in $\mathbb{Z}^d$: Let S_i denote the collection of all elemnents of the form kan, where $k \in SO(d,\mathbb{R})$,

$$a = \mathrm{diag}(a_1, ...a_i, a_{i+1}, ..., a_d) \in \mathrm{SL}(d,\mathbb{R}), \tag{1.3.2}$$

with $\frac{\sqrt{3}}{2}a_{j-1} \leq a_j < \frac{\delta}{\sqrt{i}}, \forall 1 \leq j \leq i$ (assuming $a_0 = 0$ as convention) and $\frac{\sqrt{3}}{2}a_{j-1} \leq a_j \leq 1$ whenever $j > i$, and

$$n = \begin{bmatrix} 1 & n_{12} & n_{13} & ... & n_{1d} \\ 0 & 1 & n_{23} & ... & n_{2d} \\ 0 & 0 & 1 & ... & n_{3d} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & ... & 1 \end{bmatrix} \tag{1.3.3}$$

with $\frac{1}{2} \leq n_{ij} \leq 1$ for all $i < j$.

Clearly, with restrictions $\frac{\sqrt{3}}{2}a_{j-1} \leq a_j$ and $\frac{1}{2} \leq n_{ij}$, S_i is contained in the Siegel domain $\Sigma := \Sigma_{\frac{2}{\sqrt{3}},\frac{1}{2}}$. The reason that we choose to restrict our set to Siegel domain is that over Siegel domain, we have very good control on the overlaps modulo the $\Gamma = \mathrm{SL}(n, \mathbb{Z})$ action thanks to Theorem B.4.

Claim 6. *$\lambda_j(S_i\mathbb{Z}^d) \leq \delta$ for all $j \leq i$.*

Proof of Claim . Indeed, for $kan \in S_i$ and $j \leq i$,

$$\begin{aligned} \|kane_j\|_2 =& \|[a_1n_{1j}, ..., a_{i-1}n_{i-1,j}, a_i]^T\|_2 && (k \text{ preserves the distance}) \\ =& \sqrt{a_1^2n_{1j}^2 + \cdots + a_{i-1}^2n_{i-1,j}^2 + a_i^i} \\ \leq& \sqrt{i \cdot \frac{\delta^2}{i}} = \delta \end{aligned}$$

#

Therefore $\pi(S_i) \subset \{\Lambda : \lambda_i(\Lambda) \leq \delta\}$. Now we shall give a lower bound estimate for the measure of $\pi(S_i)$ in G/Γ.

Let $f = \mathbf{1}_{S_i}$ denote the indicator function of $\pi(S_i)$ on G/Γ and let N_d denote the (finite) number of γ for which $\Sigma \cap F\gamma$ is nonempty, then since S_i is a subset of the Siegel set Σ, $S_i = \cup_{\gamma\in\Gamma}(S_i\gamma^{-1} \cap F)$ is a finite union of no more than N_d nonempty sets. Let m_d denotes the largest measure of these N_d sets, it follows that

$$\begin{aligned}\int_G f(g)dg =& \int_{G/\Gamma} \sum_{\gamma\in\Gamma} f(g\gamma)d(g\Gamma)\\ \leq& N_d m_d\\ \leq& N_d \mu_{G/\Gamma}(S_i).\end{aligned}$$

Now we compute $\int_G f(g)dg$ via Iwasawa decomposition in view of Theorem B.8:

$$\begin{aligned}&\int_G f(g)dg\\ =&\int_K\int_A\int_N f(kan)\rho(a)dkdadn\\ =&\int_K dk\int_0^{\delta/\sqrt{i}}\int_{\frac{\sqrt{3}}{2}a_1}^{\delta/\sqrt{i}}\cdots\int_{\frac{\sqrt{3}}{2}a_{i-1}}^{\delta/\sqrt{i}}\int_{\frac{\sqrt{3}}{2}a_i}^{1}\cdots\int_{\frac{\sqrt{3}}{2}a_{d-1}}^{1}\frac{\rho(a)}{a_1\dots a_{d-1}}da_{d-1}\dots da_1\int_{[\frac{1}{2},1]^{d(d-1)}}\prod_{i<j}dn_{ij}\\ \geq&\mathrm{Vol}(K)\frac{1}{2^{d(d-1)}}\int_0^{\delta/\sqrt{i}}\int_{\frac{\sqrt{3}}{2}a_1}^{\delta/\sqrt{i}}\cdots\int_{\frac{\sqrt{3}}{2}a_{i-1}}^{\delta/\sqrt{i}}\int_{[\frac{\sqrt{3}}{2},1]^{d-i-1}}\left(a_{d-1}^1a_{d-2}^3\cdots a_1^{2d-3}\right)da_{d-1}\dots da_1\\ =&\mathrm{Vol}(K)\frac{c_{d,i}}{2^{d(d-1)}}\int_0^{\delta/\sqrt{i}}\int_{\frac{\sqrt{3}}{2}a_1}^{\delta/\sqrt{i}}\cdots\int_{\frac{\sqrt{3}}{2}a_{i-1}}^{\delta/\sqrt{i}}\left(a_i^{2d-1-2i}\cdots a_1^{2d-3}\right)da_i\dots da_1\\ \geq&D_d\delta^{(2d-i-1)i}+o(\delta^{(2d-i-1)i})\end{aligned}$$

where the constant c_d, i comes from the integration w.r.t. the variables $a_{i+1}, ...a_d$ and the exponential $\delta^{(2d-i-1)i}$ comes from $(2d-i-1)i = 2d-1-2i+\cdots+2d-3+i$ (the last i is from the accumulation of the total order of anti-derivatives of polynomials) and when $i = d-1$, $\delta^{(2d-i-1)i} = d(d-1) = di$.

Therefore, for $i = d-1$, we have proved

$$\mu\{\Lambda : \lambda_i(\Lambda) \leq \delta\} \geq C'_d\delta^{d(d-1)},\ \text{as } \delta \ll 1.$$

This completes the proof for all $1 \leq u \leq d-1$. □

By looking at the dual lattice, we can also obtain the tail bound for this distribution.

Corollary 1.3.9. *For $1 < i \leq d$ there exist C_d and C'_d such that*

$$C_d \delta^{di} \leq \mu\left(\left\{\Lambda \in \mathcal{L} : \lambda_i(\Lambda) \geq \frac{1}{\delta}\right\}\right) \leq C'_d \delta^{di},$$

for all $\delta \ll 1$.

Proof. Recall the notion of dual lattice (cf. 1.2.6) and notice that the dual map

$$* : X \to X, \Lambda \mapsto \Lambda^*$$

is measure-preserving. By the Theorem 1.2.29. We have

$$1 \leq \lambda_r(\Lambda)\lambda_{d+1-r}(\Lambda^*) \leq d!$$

for any $r = 1, 2, \cdots d$. Hence the corollary follows. $\square$

1.3.2 The measure of the set of unimodular lattices with k-tuple vectors avoiding a measurable set

For a more general setting, using Rogers' formulas [Rog56], Athreya and Margulis [AM09] proved the following:

Theorem 1.3.10 ([AM09], Theorem 2.2)**.** *For $d \geq 2$, there is a constant C_d such that if A is a measurable set in $\mathbb{R}^d$, with $m(A) > 0$,*

$$\mu(\{\Lambda \in \mathcal{L} : \Lambda \cap A = \varnothing\}) \leq \frac{C_d}{m(A)}.$$

For $1 \leq k \leq d$ and a unimodular lattice Λ in $\mathbb{R}^d$, let Λ^k denote the direct sum of k copies of Λ, which is again a unimodular lattice in $\mathbb{R}^{dk}$.

The main result in this section is the following:

Theorem 1.3.11. *For $d \geq 2$ and $1 \leq k \leq \frac{d}{2}$, there is a constant C_d such that if A is a measurable set in $\mathbb{R}^{dk}$, with $m(A) > 0$,*

$$\mu(\{\Lambda \in \mathcal{L} : \Lambda^k \cap A = \varnothing\}) \leq \frac{C_d}{m(A)^{\frac{1}{k}}}.$$

Before we give the proof, let us first introduce some background on Rogers' theory on the mean value formulas in the geometry of numbers.

In order to tackle with the arbitrary measurable set A, we shall use the following Rogers' symmetrization technique:

Definition 1.3.12 (Spherical Symmetrization of a measurable function)**.**
Let f be any measurable function in $\mathbb{R}^d$, we define a function f^*, called the *spherical symmetrization* of f, as follows:

$$f^*(0) := \sup_{x \in \mathbb{R}^d} f(x)$$

and if $x \neq 0$,

$$f^*(x) := \inf\{\rho \geq 0 : m(\{y \in \mathbb{R}^d : f(y) > \rho\}) \leq m(\{y \in \mathbb{R}^d : \|y\| \leq \|x\|\})\}.$$

The following example justifies the terminology spherical symmetrization:

Example 1.3.13 (Spherical symmetrization of indicator functions)**.**
For $A \subset \mathbb{R}^d$ measurable, if we take $f = \mathbf{1}_A$, then $1_A^*(0) = 1$ and

$$\begin{aligned} 1_A^*(x) &= \inf\{\rho \geq 0 : m(\{y \in \mathbb{R}^d : \mathbf{1}_A(y) > \rho\}) \leq m(\{y \in \mathbb{R}^d : \|y\| \leq \|x\|\})\} \\ &= \mathbf{1}_{B(A)}(x). \end{aligned}$$

where $B(A)$ denotes the ball in $\mathbb{R}^d$ centered at the origin that has volume equal to the measure of A, namely $m(B(A)) = m(A)$.

Theorem 1.3.14 (Special case of Theorem 1, [Rog56]). *For $d \geq 2$, let $f : \mathbb{R}^d \to \mathbb{R}$ be a nonnegative Borel measurable function, and let f^* be the function obtained from f by spherical symmetrization. Then*

$$\int \left(\sum_{v \in \Lambda - 0} f(v) \right)^2 d\mu(\Lambda) \leq \int \left(\sum_{v \in \Lambda - 0} f^*(v) \right)^2 d\mu(\Lambda).$$

Now we state a multiple-sum formula due to Rogers:

Theorem 1.3.15 (Theorem 4, [Rog55] and for the convergence part, Theorem 2, [Sch58]). *Let $1 \leq k \leq d-1$ and $f : \mathbb{R}^{dk} \to \mathbb{R}$ be a non-negative Borel measurable function in $(x_1, \dots, x_k)$. Then*

$$\begin{aligned}
&\int_X \sum_{v_1, \dots, v_k \in \Lambda} f(v_1, \dots, v_k) d\mu(\Lambda) \\
=& f(0, ..., 0) + \int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} f(x_1, \dots, x_k) dx_1 \dots dx_k \\
&+ \sum_{(\nu,\mu)=1} \sum_{q=1}^{\infty} \sum_D \left(\frac{e_1}{q} \cdots \frac{e_m}{q} \right)^d \int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} f\left(\sum_{i=1}^m \frac{d_{i1}}{q} x_i, \dots, \sum_{i=1}^m \frac{d_{ik}}{q} x_i \right) dx_1 \dots dx_m. \quad (1.3.4)
\end{aligned}$$

Here the first sum is over all partitions $(\nu, \mu) = (\nu_1, ..., \nu_m; \mu_1, ..., \mu_{k-m})$ of the numbers $\{1, ..., k\}$ into two sequences $1 \leq \nu_1 < ... < \nu_m \leq k$ and $1 \leq \mu_1 < ... < \mu_{k-m} \leq k$ with $1 \leq m \leq k-1$ (of course $\nu_i \neq \mu_j$ for any i, j).

The third sum is taken over all integer-valued $m \times k$ matrices D, such that

(1) the greatest common divisor of all entries is 1

(2) for all i, j, D satisfies

$$\begin{cases} d_{i\nu_j} = q\delta_{ij} & \text{for } i = 1, ..., m \text{ and } j = 1, ..., m \\ d_{i\mu_j} = 0 & \text{for } \mu_j < \nu_i, \ i = 1, ..., m \text{ and } j = 1, ..., k-m. \end{cases} \quad (1.3.5)$$

Finally, $e_i = (\epsilon_i, q)$, where $\epsilon_1, ..., \epsilon_m$ are the elementary divisors of D.

Moreover, if f is bounded and compactly supported, then the both sides of the equation 1.3.4 are finite.

Here for the convenience of readers, we shall give an example for the matrix D:

Example 1.3.16. Let $k = 6$ and $m = 4$ and assume the partition is given by

$$(\nu_1, \nu_2, \nu_3, \nu_4) = (2, 3, 5, 6), \text{ and } (\mu_1, \mu_2) = (1, 4).$$

The the 4×6 matrix D with respect to this partition has the form

$$\begin{bmatrix} 0 & q & 0 & * & 0 & 0 \\ 0 & 0 & q & * & 0 & 0 \\ 0 & 0 & 0 & 0 & q & 0 \\ 0 & 0 & 0 & 0 & 0 & q \end{bmatrix}$$

Note that in the 4-th column, since $\mu_2 = 4 > \nu_1 = 2$, $d_{i\mu_j}$ are not necessarily zero, however they must be chosen so that the greatest common divisor of all coefficients of D is coprime to q.

Proof of Theorem 1.3.11.

For a measurable subset $A \subset \mathbb{R}^{dk}$, let $\Sigma_{A,k} := \{\Lambda \in \mathcal{L} : \Lambda^k \cap A = \varnothing\}$ and $g_A(\Lambda) := \mathbf{1}_{\Sigma^c_{A,k}}(\Lambda)$. Let[5]

$$f_A(\Lambda) := \sum_{(v_1,\ldots,v_k)\in\Lambda^k-(0,\ldots,0)} \mathbf{1}_A(v_1, \ldots, v_k).$$

Since

$$g_A(\Lambda) = 0 \iff \Lambda \in \Sigma_{A,k} \iff \Lambda^k \cap A = \varnothing \implies f_A(\Lambda) = 0,$$

it follows that $f_A = g_A f_A$ and by Cauchy-Schwarz' theorem

$$\left(\int_X f_A d\mu\right)^2 \leq \left(\int_X f_A^2 d\mu\right)\left(\int_X g_A^2 d\mu\right)$$

Notice that since $g_A^2 = g_A$, we have

$$\mu(\Sigma_{A,k}) = 1 - \|g_A\|_1 = 1 - \|g_A\|_2^2,$$

[5]Note here we are excluding the all-zero term in the sum.

and therefore

$$\mu(\Sigma_{A,k}) \leq 1 - \frac{\|f_A\|_1^2}{\|f_A\|_2^2} \leq 1 - \frac{m(A)^2}{\|f_A\|_2^2},$$

where the last inequality follows from Theorem 1.3.15 (the remainder term is non-negative).

On the other hand, by the spherical symmetrization (Theorem 1.3.14 and Example 1.3.13), we have

$$\|f_A\|_2^2 \leq \| \sum_{\Lambda^k-(0,\ldots,0)} \mathbf{1}_{B(A)}\|_2^2,$$

where $B(A)$ is the spherical symmetrization of A in $\mathbb{R}^{dk}$. Note that the radius of this ball has magnitude $r \asymp \sqrt[dk]{m(A)}$.

It follows that

$$\mu(\Sigma_{A,k}) \leq 1 - \frac{m(A)^2}{\| \sum_{\Lambda^k-(0,\ldots,0)} \mathbf{1}_{B(A)}\|_2^2}, \tag{1.3.6}$$

It remains for us to give an estimate for the second moment of $\sum_{\Lambda^k-(0,\ldots,0)} \mathbf{1}_{B(A)}$.

Assume $1 \leq 2k \leq d-1$. By replacing k in Theorem 1.3.15 with $2k$, f with $f \times f$ and keeping all other assumptions are the same as the previous theorem (with $2k$ playing the role of k, of course), and distribute the products in the square of sums, we have

$$\begin{aligned}
&\int_X \left(\sum_{v_1,\ldots,v_k\in\Lambda} f(v_1,\ldots,v_k) \right)^2 d\mu(\Lambda) \\
=&f(0,...,0)^2 + \int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} f(x_1,\ldots,x_k) f(x_{k+1},...,x_{2k}) dx_1 \ldots dx_{2k} \\
+& \sum_{(\nu,\mu)=1} \sum_{q=1}^{\infty} \sum_{D} \left(\frac{e_1}{q} \cdots \frac{e_m}{q} \right)^d \int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} \\
& f\left(\sum_{i=1}^m \frac{d_{i1}}{q} x_i, \ldots, \sum_{i=1}^m \frac{d_{ik}}{q} x_i \right) f\left(\sum_{i=1}^m \frac{d_{i,k+1}}{q} x_i, \ldots, \sum_{i=1}^m \frac{d_{i,2k}}{q} x_i \right) dx_1 \ldots dx_m. &(1.3.7) \\
=&f(0,...,0)^2 + \left[\int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} f(x_1,\ldots,x_k) dx_1 \ldots dx_k \right]^2 \\
+& \sum_{(\nu,\mu)=1} \sum_{q=1}^{\infty} \sum_{D} \left(\frac{e_1}{q} \cdots \frac{e_m}{q} \right)^d \int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} \\
& f\left(\sum_{i=1}^m \frac{d_{i1}}{q} x_i, \ldots, \sum_{i=1}^m \frac{d_{ik}}{q} x_i \right) f\left(\sum_{i=1}^m \frac{d_{i,k+1}}{q} x_i, \ldots, \sum_{i=1}^m \frac{d_{i,2k}}{q} x_i \right) dx_1 \ldots dx_m. &(1.3.8)
\end{aligned}$$

Note that the condition $1 \leq 2k \leq d-1$ guarantees the convergence on the right hand side of equation. Now take $f = \mathbf{1}_{B_r^{dk}} = \mathbf{1}_{B(A)}$, where B_r^{dk} is a ball in $\mathbb{R}^{dk}$ with radius r whose volume is the same as that of $B(A)$ and thus A. Specifically, $r^{dk} \asymp_{d,k} m(A)$ or

$$r \asymp_{d,k} m(A)^{\frac{1}{dk}} \tag{1.3.9}$$

Then the first term is 1 and the second term can be easily computed as $m(A)^2$.

For the third term, we have

$$\begin{aligned} & f\left(\sum_{i=1}^m \frac{d_{i1}}{q}x_i, \ldots, \sum_{i=1}^m \frac{d_{ik}}{q}x_i\right) f\left(\sum_{i=1}^m \frac{d_{i,k+1}}{q}x_i, \ldots, \sum_{i=1}^m \frac{d_{i,2k}}{q}x_i\right) = 1 \\ \Longleftrightarrow & \sum_{i=1}^m \frac{d_{i1}^2}{q^2}\|x_i\|^2 + \cdots + \sum_{i=1}^m \frac{d_{ik}^2}{q^2}\|x_i\|^2 \leq r^2 \text{ and} \\ & \sum_{i=1}^m \frac{d_{i,k+1}^2}{q^2}\|x_i\|^2 + \cdots + \sum_{i=1}^m \frac{d_{i,2k}^2}{q^2}\|x_i\|^2 \leq r^2 \end{aligned}$$

If the partition is given by $1 \leq \nu_1 < \ldots < \nu_m \leq k$ and $1 \leq \mu_1 < \ldots < \mu_{k-m} \leq k$, then in view of the definition of entries in D (1.3.5), each $\|x_i\|$ for $1 \leq i \leq m$ must be in the range $[0, r]$ in order to make the integrand nonvanishing. So we can magnify each the integral as

$$\begin{aligned} & \int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} f\left(\sum_{i=1}^m \frac{d_{i1}}{q}x_i, \ldots, \sum_{i=1}^m \frac{d_{ik}}{q}x_i\right) f\left(\sum_{i=1}^m \frac{d_{i,k+1}}{q}x_i, \cdots, \sum_{i=1}^m \frac{d_{i,2k}}{q}x_i\right) dx_1 \cdots dx_m \\ \leq & \int_{B_r^d} \cdots \int_{B_r^d} 1 dx_1 \cdots dx_m \\ = & \mathrm{Vol}(B_r^d)^m = O_{d,m}(r^{dm}). \end{aligned}$$

Noticing that $m < 2k < d-1$, it follows from the convergence that the right hand side of (1.3.8) is $m(A)^2 + O_d(r^{d(2k-1)})$. So the estimate (1.3.6) becomes

$$\begin{aligned} \mu(\Sigma_{A,k}) \leq & 1 - \frac{m(A)^2}{m(A)^2 + O_d(r^{d(2k-2)})} \\ = & \frac{O_d(r^{d(2k-1)})}{m(A)^2 + O_d(r^{d(2k-1)})} \\ = & \frac{O_d(r^{d(2k-1)})}{m(A)^2} = O_d(1)\frac{m(A)^{\frac{2k-1}{k}}}{m(A)^2} && \text{(By (1.3.9))} \\ = & O_d(1)\frac{1}{m(A)^{\frac{1}{k}}} \end{aligned}$$

$\square$

1.3.3 Logarithm law associated to the higher successive minima

Now we define $\Delta_i(\Lambda) := -\log(\lambda_i(\Lambda))$. It follows from taking the negative logarithm of all sides of the equation (A.0.4) that $\Delta_i(\Lambda)$ is uniformly continuous. Recall from [KM99]:

Definition 1.3.17. For a function Δ on a G-homogeneous space X, define the tail distribution $\Phi_\Delta(z) := \mu\{x \in X : \delta(x) \geq z\}$.

For $k > 0$, we will also say that δ is k distance-like if it is uniformly continuous and in addition there exist constants C_d and C'_d such that

$$C_d e^{-kz} \leq \Phi_\Delta(z) \leq C'_d e^{-kz}, \forall z \in \mathbb{R}.$$

It follows from our Theorem 1.3.3 that Δ_i is di distance-like in the space of unimodular lattices. And as an immediate consequence of Theorem 1.7 in [KM99], we have the non-unipotent version of logarithm law:

Theorem 1.3.18. *For any nonzero $(z_1, \ldots z_d)$ with $z_1 + \cdots + z_d = 0$, and for almost all unimodular lattice Λ in $X = \mathrm{SL}(d, \mathbb{R})/\mathrm{SL}(d, \mathbb{Z})$ we have*

$$\limsup_{t\to\infty} \frac{\Delta_i(\exp(tz)\Lambda)}{\log t} = \frac{1}{di}. \tag{1.3.10}$$

For the unipotent flow and first successive minimum, Athreya and Margulis proved the following logarithm law:

Theorem 1.3.19 ([AM09], Theorem 2.1)**.** *Let $(u_t)_{t\in\mathbb{R}}$ be a unipotent one-parameter subgroup of $\mathrm{SL}(d, \mathbb{R})$ and $X := \mathrm{SL}(d, \mathbb{R})/\mathrm{SL}(d, \mathbb{Z})$. For μ-a.e. Λ in X, we have*

$$\limsup_{t\to\infty} \frac{-\log \lambda_1(h_t\Lambda)}{\log t} = \frac{1}{d}.$$

We shall generalize this theorem to higher λ_i's:

Theorem 1.3.20. *Let* $(g_t)_{t\in\mathbb{R}}$ *be an unbounded one-parameter subgroup of* $\mathrm{SL}(d,\mathbb{R})$ *and* $X := \mathrm{SL}(d,\mathbb{R})/\mathrm{SL}(d,\mathbb{Z})$. *For* μ*-a.e.* Λ *in* X, *we have*

$$\limsup_{t\to\infty}\frac{-\log\lambda_i(h_t\Lambda)}{\log t}=\frac{1}{di}.$$

This is also discovered independently by Kim and Skenderi [Kim22]. To prove this theorem, we first observe that by Borel-Cantelli Lemma, the upper bound holds for all flows:

Lemma 1.3.21 (The upper bound)**.** *For* μ*-almost every* $\Lambda \in X$, $1 \le i \le d-1$, *and any one parameter subgroup* $(h_t)_{t\in\mathbb{R}}$ *of* $G = \mathrm{SL}(d,\mathbb{R})$,

$$\limsup_{t\to\infty}\frac{-\log\lambda_i(h_t\Lambda)}{\log t}\le\frac{1}{di}.$$

Proof. For any $\epsilon > 0$, and for $k \ge 1$, let $r_k = (\frac{1}{di}+\epsilon)\log k$ and let t_k be any sequence going to ∞ as $k \to \infty$, we have by Theorem 1.3.3 and the fact that u_{t_k} is measure-preserving that

$$\mu(\{\Lambda\in X:\lambda_i(u_{t_k}\Lambda)\le e^{-r_k}\})\le C_d'(e^{r_k})^{di},$$

which is equivalent to

$$\mu(\{\Lambda\in X:-\log\lambda_i(u_{t_k}\Lambda)\ge r_k\})\le C_d'\frac{1}{k^{1+di\epsilon}}.$$

Since the summatin on the right hand side over k is finite, by Borel-Cantelli Lemma, we have

$$\mu(\limsup_{k\to\infty}\{\Lambda\in X:-\log\lambda_i(u_{t_k}\Lambda)\ge r_k\})=0.$$

Taking the complement, this means

$$\mu(\cup_N\cap_{k\ge N}\{\Lambda\in X:-\log\lambda_i(u_{t_k}\Lambda)<r_k\})=\mu(\liminf_{k\to\infty}\{\Lambda\in X:-\log\lambda_i(u_{t_k}\Lambda)<r_k\})=1.$$

In other words, for μ-almost every $\Lambda \in X$, there exists N such that $k \ge N$ implies

$$-\log\lambda_i(u_{t_k}\Lambda)<r_k:=(\frac{1}{di}+\epsilon)\log(k)$$

Since $t_k \to \infty$ is arbitrary, we have

$$\limsup_{t\to\infty} \frac{-\log \lambda_i(h_t\Lambda)}{\log t} \le \frac{1}{di}.$$

for μ-almost all Λ. $\square$

To show the lower bound, we shall use a logarithm law for hitting time of unbounded flow against the spherical shrinking target due to Kelmer and Yu [KY17].

Theorem 1.3.22 (Special Case of Theorem 1.1, [KY17])**.** *Let $\{B_t\}_{t>0}$ denote a monotone family of spherical (meaning each set B_t is invariant under the left action of $K = SO(d,\mathbb{R})$) shrinking (meaning $B_t \supset B_s$ for $t \ge s$ and $\lim_{t\to\infty} \mu(B_t)$) targets in $X := G/\Gamma := \mathrm{SL}(d,\mathbb{R})/\mathrm{SL}(d,\mathbb{Z})$. Let $\{g_m\}_{m\in\mathbb{Z}}$ denote an unbounded discrete time flow on X . Then for a.e. $\Lambda \in X$*

$$\lim_{t\to\infty} \frac{\log(\min\{m\in\mathbb{N} : g_m\Lambda \in B_t\})}{-\log(\mu(B_t))} = 1 \tag{1.3.11}$$

The quantity $\min\{m \in \mathbb{N} : g_m.x \in B_t\}$ is often called the first *hitting time* with respect to the flow $\{g_m\}$ and the target set B_t.

Proof of 1.3.20 (the lower bound). We will take the shrinking targets as

$$B_t := \{\Lambda : \lambda_i(g_m\Lambda) \le e^{-t}\}, t \ge 0.$$

These sets are clearly spherical, namely $SO(d,\mathbb{R})$-invariant since λ_i. So by Theorem 1.3.22,

$$\lim_{t\to\infty} \frac{\log \min\{m\in\mathbb{N} : \lambda_i(g_m\Lambda) \le e^{-t}\}}{-\log \mu(\{\Lambda : \lambda_i(g_m\Lambda) \le e^{-t}\})} = 1.\text{[6]} \tag{1.3.12}$$

By Corollary 1.3.4,

$$\lim_{t\to\infty} \frac{-\log \mu\left(\{\Lambda \in \mathcal{L} : \lambda_i(\Lambda) \le e^{-t}\}\right)}{t} = di \tag{1.3.13}$$

[6] The set $\{m \in \mathbb{N} : \lambda_i(g_m\Lambda) \le e^{-t}\}$ is non-empty because (g_m)-action is ergodic by the Howe-Moore theorem, and thus almost every (g_m)-orbit is dense.

Therefore by taking the product and reciprocal,

$$\lim_{t\to\infty} \frac{t}{\log \min\{m \in \mathbb{N} : \lambda_i(g_m\Lambda) \le e^{-t}\}} = \frac{1}{di}. \tag{1.3.14}$$

But observe that

$$\lambda_i \left(g_{\min\{m\in\mathbb{N}:\lambda_i(g_m\Lambda)\le e^{-t}\}}\Lambda\right) \le e^{-t}, \tag{1.3.15}$$

and therefore

$$-\log \lambda_i \left(g_{\min\{m\in\mathbb{N}:\lambda_i(g_m\Lambda)\le e^{-t}\}}\Lambda\right) \ge t, \tag{1.3.16}$$

It follows that

$$\begin{aligned}
&\limsup_{t\to\infty} \frac{-\log \lambda_i(h_t\Lambda)}{\log t} \\
\ge &\limsup_{t\to\infty} \frac{-\log \lambda_i \left(g_{\min\{m\in\mathbb{N}:\lambda_i(g_m\Lambda)\le e^{-t}\}}\Lambda\right)}{\log \min\{m \in \mathbb{N} : \lambda_i(g_m\Lambda) \le e^{-t}\}} \\
&\text{(By the definition of limsup. Here the subscript is considered as a subsequence.)} \\
\ge &\limsup_{t\to\infty} \frac{t}{\log \min\{m \in \mathbb{N} : \lambda_i(g_m\Lambda) \le e^{-t}\}} && \text{(By the inequality (1.3.16))} \\
= &\frac{1}{di}. && \text{(By the limit (1.3.14))}
\end{aligned}$$

This finishes the proof of lower bound and thus the whole logarithm law theorem. □

Chapter 2: Equidistribution of sub-lattices in $\mathbb{R}^d$

This chapter is based on the joint work with Michael Bersudsky [BX23].

2.1 Introduction

In this paper we study the limiting distribution of dense orbits of a lattice $\Gamma \leq \mathrm{SL}(m+1, \mathbb{R})$ in the space $X_{m,m+1}$ of normalized m-dimensional discrete subgroups of $\mathbb{R}^{m+1}$ with respect to a filtration given by growing norm balls (see precise definitions below). Such a research direction was originally suggested by U. Shapira as natural continuation of the work [SS17] which considers random walks on $X_{2,3}$ (see also the more recent work [GLS22] which generalizes [SS17]). Another motivation for our work is to extend the scope of applications of the duality principle in homogeneous dynamics to the ergodic theory of lattice subgroups, see Section 2.1.1 below for more details.

We start with our results in $X_{m,m+1}$. In what follows, m is a natural number strictly larger than 1. We say that $\Lambda \subset \mathbb{R}^{m+1}$ is a m-lattice if Λ is the $\mathbb{Z}$-Span of a tuple of linearly independent vectors $v_1, v_2, ..., v_m \in \mathbb{R}^m$, that is,

$$\Lambda := \mathrm{Span}_{\mathbb{Z}}\{v_1, v_2, ..., v_m\}.$$

For Λ we let

$$\mathrm{Cov}(\Lambda) := \sqrt{\det(\langle v_i, v_j \rangle)},$$

which is the area of a fundamental parallelogram of Λ. An m-lattices Λ is called *unimodular* if $\mathrm{Cov}(\Lambda) = 1$. Next, we recall the definition of the shape of lattices, a notion that was extensively studied in e.g [EMSS16; AES16], which refined the classical work of Schmidt [Sch98]. We view $\mathbb{R}^{m+1}$ as row vectors, and for a unimodular m-lattice $\Lambda \subset \mathbb{R}^{m+1}$, let $w \in \mathbb{S}^m$ be such that $w \perp \Lambda$. We choose a $\rho \in \mathrm{SO}(m+1, \mathbb{R})$ such that $w\rho = e_{m+1} := (0, \dots, 0, 1)$, and we define the shape of the pair (Λ, w) as

$$\mathbf{s}(\Lambda, w) := \Lambda\rho \begin{bmatrix} \mathrm{SO}(m, \mathbb{R}) & 0 \\ 0 & 1 \end{bmatrix}, \tag{2.1.1}$$

which is independent of the choice of ρ. By identifying the unimodular m-lattices $\Lambda \subset e_{m+1}^{\perp}$ with $X_m := \mathrm{SL}(m, \mathbb{Z})\backslash\mathrm{SL}(m, \mathbb{R})$, we view $\mathbf{s}(\Lambda, w)$ as a point in $X_m/\mathrm{SO}(m, \mathbb{R})$. Note that in general

$$\mathbf{s}(\Lambda, w) \neq \mathbf{s}(\Lambda, -w).$$

Remark 2.1.1. A more intrinsic definition of a shape of a discrete subgroup, see e.g. [SS17; Sch98], is defined by the equivalence class under the equivalence relation of scaling and rotations. When defining shape in this way, one gets a point in $X_m/\mathrm{O}(m, \mathbb{R})$, and it captures slightly less information. Our definition (2.1.1) is mainly motivated by the definition in [EMSS16; AES16], which produces a point in the more familiar space $X_m/\mathrm{SO}(m, \mathbb{R})$.

We consider

$$X_{m,m+1} := \{(\Lambda, w) : \mathrm{Cov}(\Lambda) = 1, w \in \mathbb{S}^m, w \perp \Lambda\},$$

and note that $\mathbf{s}$ defined in (2.1.1) yields a map

$$\mathbf{s} : X_{m,m+1} \to X_m/\mathrm{SO}(m, \mathbb{R}).$$

We define a right $\mathrm{SL}(m+1, \mathbb{R})$ action on $X_{m,m+1}$ using the usual right matrix multiplication by

$$(\Lambda, w).g := \left(\frac{\Lambda g}{\sqrt{\mathrm{Cov}(\Lambda g)}}, \frac{w({}^t g^{-1})}{\|w({}^t g^{-1})\|} \right), \ g \in \mathrm{SL}(m+1, \mathbb{R}),\ w \in \mathbb{S}^m, \tag{2.1.2}$$

where $\|\cdot\|$ is the usual Euclidean norm. We note that this action is transitive.

It turns out that for each lattice subgroup $\Gamma \leq \mathrm{SL}(m+1,\mathbb{R})$ and each $x_0 \in X_{m,m+1}$, the orbit $x_0\Gamma$ is dense in $X_{m,m+1}$. This follows for example by [SS17], and a more direct proof of this fact is obtained by applying a duality argument as follows — $X_{m,m+1}$ is a homogeneous space identified with $H\backslash\mathrm{SL}(m+1,\mathbb{R})$, with H as in (2.1.7). Now a Γ-orbit $Hg\Gamma$ is dense in $H\backslash\mathrm{SL}(m+1,\mathbb{R})$ if and only if the "dual" H-orbit $Hg\Gamma$ is dense in $\mathrm{SL}(m+1,\mathbb{R})/\Gamma$. By Proposition 1.5 of [SR80], it follows that all H-orbits in G/Γ are dense.

Our goal in this paper will be to compute the limiting distribution of Γ orbits in $X_{m,m+1}$ with respect to growing Hilbert-Schmidt norm balls. Namely, let $\|g\| = \sqrt{\mathrm{Trace}({}^t g g)} = \sqrt{\sum_{ij} g_{ij}^2}$ be the Hilbert-Schmidt norm of $g \in \mathrm{SL}(m+1,\mathbb{R})$, and let

$$\Gamma_T := \{\gamma \in \Gamma : \|\gamma\| \leq T\}. \tag{2.1.3}$$

For

$$x_0 := (\Lambda_0, w_0) \in X_{m,m+1},$$

consider the probability measures

$$\mu_{T,x_0} := \frac{1}{\#\Gamma_T} \sum_{\gamma \in \Gamma_T} \delta_{x_0.\gamma},\ T > 0.$$

Remark 2.1.2. When $m = 1$ the space $X_{m,m+1}$ is naturally identifies with $\mathbb{S}^1$ the unit circle in $\mathbb{R}^2$. The limiting distribution of μ_{T,x_0} in the case of $m = 1$ was obtained in [Gor03].

Our result below states that the probability measures μ_{T,x_0} converge as $T \to \infty$ to a probability measure $\tilde{\nu}_{x_0}$ depending on $x_0 \in X_{m,m+1}$ which we describe now. We observe that $X_{m,m+1}$ has a natural projection to $\mathbb{S}^m$ defined by

$$\pi_\perp(\Lambda, w) := w,$$

which endows $X_{m,m+1}$ with a fiber-bundle structure, where the fibers are isomorphic to

$$X_m := \mathrm{SL}(m,\mathbb{Z})\backslash\mathrm{SL}(m,\mathbb{R}).$$

To define the measure $\tilde{\nu}_{x_0}$, we define a measure $\mu_{x_0,w}$ on each fiber $\pi_\perp^{-1}(w)$, and we integrate those measures by the unique rotation invariant probability measure $\mu_{\mathbb{S}^m}$ on $\mathbb{S}^m$. We note that the measures $\mu_{x_0,w}$ have a slightly surprising form; they are a combination of a $\mathrm{SL}(m,\mathbb{R})$-invariant measure with a density involving the Hilbert-Schmidt norm of operators which we define now.

For an operator T from a hyperplane $U \subset \mathbb{R}^m$ to another hyperplane $V \subset \mathbb{R}^m$, we define

$$\|T\|_{\mathrm{HS}}^2 := \sum_{i=1}^{m} \|Tu_i\|^2, \tag{2.1.4}$$

where $\{u_1, u_2, ..., u_m\}$ is an orthonormal basis of U, and where the norm on the right hand side is the usual Euclidean norm on $\mathbb{R}^m$. We note that this norm is independent of the choice of an orthonormal basis $\{u_1, u_2, ..., u_m\}$. Moreover, this norm is bi-$\mathrm{SO}(m+1,\mathbb{R})$ invariant in the following sense. If $\rho_1, \rho_2 \in \mathrm{SO}(m+1,\mathbb{R})$, then $\rho_2 \circ T \circ \rho_1 : \rho_1^{-1}U \to \rho_2 V$ satisfies

$$\|\rho_2 \circ T \circ \rho_1\|_{\mathrm{HS}} = \|T\|_{\mathrm{HS}}.$$

For an ordered tuple of linearly independent vectors $B = (u_1, u_2, ..., u_m) \in \mathbb{R}^{m\times m}$ we define the linear map $T_B : \mathrm{Span}_{\mathbb{R}}\{e_1, e_2, ..., e_m\} \to \mathrm{Span}_{\mathbb{R}}\{u_1, u_2, ..., u_m\}$, by sending $e_1 \mapsto u_1, ..., e_m \mapsto u_m$. Now fix unimodular m-lattice $\Lambda_0 \subset \mathbb{R}^m$, and let $\mathscr{B}_0$ be an ordered tuple of linearly independent vectors forming a $\mathbb{Z}$-basis for Λ_0. We define for an arbitrary unimodular m-lattice $\Lambda \subset \mathbb{R}^m$,

$$\Psi_{\Lambda_0}(\Lambda) := \sum_{\mathrm{Span}_{\mathbb{Z}}\mathscr{B}=\Lambda} \frac{1}{\|T_{\mathscr{B}} \circ T_{\mathscr{B}_0}^{-1}\|_{\mathrm{HS}}^{m^2}}. \tag{2.1.5}$$

We note that Ψ_{Λ_0} is independent of the choice of basis $\mathscr{B}_0$, and we observe that by bi-$\mathrm{SO}(m+1,\mathbb{R})$ invariance of the Hilbert-Schmidt norm that the values of the function $\Psi_{\Lambda_0}(\Lambda)$ only depends on the shapes of Λ and Λ_0.

By identifying $\pi_\perp^{-1}(e_{m+1})$ with $\mathrm{SL}(m,\mathbb{Z})\backslash\mathrm{SL}(m,\mathbb{R})$, we obtain the $\mathrm{SL}(m,\mathbb{Z})\backslash\mathrm{SL}(m,\mathbb{R})$ invariant measure $\mu_{e_{m+1}}$ supported on $\pi_\perp^{-1}(e_{m+1})$ scaled such that the measure $\nu_{x_0,e_{m+1}}$ defined by

$$\nu_{x_0,e_{m+1}}(f) := \int_{\pi_\perp^{-1}(e_{m+1})} f(\Lambda, e_{m+1})\Psi_{\Lambda_0}(\Lambda)d\mu_{e_{m+1}}(\Lambda),\ f \in C_c(X_{m,m+1}),$$

is a probability measure. Then, the measure supported on $\pi_\perp^{-1}(w)$, for $w \in \mathbb{S}^m$, is defined by choosing $\rho_w \in \mathrm{SO}(m+1,\mathbb{R})$ such that $w = e_{m+1}\rho_w$, and by letting,

$$\nu_{x_0,w} := (\rho_w)_*\nu_{x_0,e_{m+1}},$$

which is the push-forward of the right translation by ρ_w via the right action of $\mathrm{SO}(m+1,\mathbb{R})$ on $X_{m,m+1}$ defined in (2.1.2). Note that $\nu_{x_0,w}$ is independent of the choice of ρ_w. Finally, we define $\tilde{\nu}_{x_0}$ by

$$\tilde{\nu}_{x_0}(f) = \int_{\mathbb{S}^m} \nu_{x_0,w}(f)d\mu_{\mathbb{S}^m}(w).$$

Theorem 2.1.3. *Let $\Gamma \leq \mathrm{SL}(m+1,\mathbb{R})$ be a lattice and fix $x_0 \in X_{m,m+1}$. Then, μ_{T,x_0} converges in the weak-* topology to $\tilde{\nu}_{x_0}$ as $T \to \infty$. In other words, for all $f \in C_c(X_{m,m+1})$, we have*

$$\lim_{T\to\infty} \frac{1}{\#\Gamma_T} \sum_{\gamma\in\Gamma_T} f(x_0 \cdot \gamma) = \int_{X_{m,m+1}} f(x)d\tilde{\nu}_{x_0}(x). \tag{2.1.6}$$

We observe that that the push-forward of $\tilde{\nu}_{x_0}$ by $\mathbf{s}$ is given by

$$\mathbf{s}_*\tilde{\nu}_{x_0} = \mathbf{s}_*\nu_{x_0,e_{m+1}}.$$

Corollary 2.1.4. *Let $\Gamma \leq \mathrm{SL}(m+1,\mathbb{R})$ be a lattice and fix $x_0 \in X_{m,m+1}$. Then, the probability measures on $X_m/\mathrm{SO}(m,\mathbb{R})$ given by*

$$\mathbf{s}_*\mu_{T,x_0} = \frac{1}{\#\Gamma_T} \sum_{\gamma\in\Gamma_T} \delta_{\mathbf{s}(x_0.\gamma)},\ T > 0,$$

converge in the weak- topology to the probability measure $\mathbf{s}_*\nu_{x_0,e_{m+1}}$, as $T \to \infty$.*

2.1.1 Connection to homogeneous dynamics - the duality principle

To simplify notation, we denote $G := \mathrm{SL}(m+1,\mathbb{R})$. We note that G-action on $X_{m,m+1}$ given in (2.1.2) is transitive, and we observe that the stabilizer subgroup of the base point

$(\mathrm{Span}_{\mathbb{Z}}\{e_1, \ldots, e_m\}, e_{m+1})$ is

$$H = \left\{ \begin{bmatrix} t^{-\frac{1}{m}} q & 0 \\ v & t \end{bmatrix} : t > 0, q \in \mathrm{SL}(m, \mathbb{Z}), v \in \mathbb{R}^m \right\}. \tag{2.1.7}$$

Then we obtain the identification $H \backslash G \cong X_{m,m+1}$.

This connects our problem to the study of distribution of orbits of closed subgroups in homogeneous spaces. The duality principle allows us to connect the equidistributional properties of the Γ-orbits on $H \backslash G$ to the equidistributional properties of the H-orbits in the dual action of H on G/Γ, see Section 1.7 of [GN14] for an extensive exposition of the existing literature on this principle. A general recipe for applying the duality principle was developed in [GN14] and [GW04]. Our approach in this paper uses a theorem of Gorodnik and Weiss ([GW04]), since it allows to prove equidistribution for every starting point.

Prior to this work, the duality principle wasn't applied to the setting in which H has infinitely many non-trivial connected components. We note that the case when H is connected algebraic was studied in great generality in [GN12; GW04], and the case where H is a lattice was studied in [Oh05] and in greater generality in [GW04].

In the section below we formulate our main result which we view as a first step towards a more general theorem in the setting which G is a semi-simple group and H is a subgroup of a parabolic group P such that in at least one of the levi-components of P appears a lattice.

2.1.2 Our general results

From now on, $G := \mathrm{SL}(m+1, \mathbb{R})$, $\Gamma \leq G$ is a lattice,

$$H := \left\{ \begin{bmatrix} t^{-\frac{1}{m}} q & 0 \\ v & t \end{bmatrix} : t > 0, q \in \Delta, v \in \mathbb{R}^m \right\}, \tag{2.1.8}$$

where $\Delta \leq \mathrm{SL}(m, \mathbb{R})$ is a lattice, and

$$P := \left\{ \begin{bmatrix} t^{-\frac{1}{m}} \eta & 0 \\ v & t \end{bmatrix} : t > 0, \eta \in \mathrm{SL}(m, \mathbb{R}), v \in \mathbb{R}^m \right\}. \tag{2.1.9}$$

Let $x_0 := Hg_0 \in H\backslash G$, and consider the one-parameter set of probability measures on $H\backslash G$

$$\mu_{T,x_0} := \frac{1}{\#\Gamma_T} \sum_{\gamma\in\Gamma_T} \delta_{x_0.\gamma},$$

for $T > 0$. Our main result below will show that μ_{T,x_0} equidistributes as $T \to \infty$ with respect to the $\tilde{\nu}_{x_0}$ the probability measure we define below.

The space $H\backslash G$ is naturally a fiber-bundle over $P\backslash G$ with respect to the natural map sending

$$\pi(Hg) := Pg,$$

(note that $P\backslash G$ is identified with $\mathbb{S}^m$ the unit sphere in $\mathbb{R}^{m+1}$ via the right action $w.g := \frac{w({}^tg^{-1})}{\|w({}^tg^{-1})\|},\ g \in G,\ w \in \mathbb{S}^m$). The fibers are isomorphic to $\Delta\backslash \mathrm{SL}(m, \mathbb{R})$. Notice that $\mathrm{SO}(m+1, \mathbb{R})$ acts transitively on $P\backslash G$ (with respect to the natural action), and we let $\mu_{P.\mathrm{SO}(m+1,\mathbb{R})}$ be the right $\mathrm{SO}(m+1, \mathbb{R})$-invariant probability on $P\backslash G = P.\mathrm{SO}(m+1, \mathbb{R})$. Now we define measures on each fiber

$$\pi^{-1}(P\rho) = HP\rho,\ \rho \in \mathrm{SO}(m+1, \mathbb{R}).$$

We write (as we may, using Iwasawa decomposition)

$$x_0 = H \begin{bmatrix} G_0 & 0 \\ v_0 & {}^1\!/\det(G_0) \end{bmatrix} \rho_0,$$

where $G_0 \in \mathrm{SL}(m, \mathbb{R})$, and $\rho_0 \in \mathrm{SO}(m+1, \mathbb{R})$, and consider the function

$$\Phi_{x_0}\left(H \begin{bmatrix} \eta & 0 \\ v & {}^1\!/\det(\eta) \end{bmatrix} \rho \right) = \sum_{q\in\Delta} \frac{1}{\|G_0^{-1} q\eta\|^{m^2}}.$$

Here $\rho \in \mathrm{SO}(m+1, \mathbb{R})$ and $\eta \in \mathrm{SL}(m, \mathbb{R})$. The experession on the right is well-defined as the Hilbert-Schmidt norm is bi-$\mathrm{SO}(m+1, \mathbb{R})$ invariant. The infinite sum is convergent due to Lemma 2.2.1 below. The "standard" fiber $\pi^{-1}(P) = H\backslash P$ is naturally identified with $\Delta\backslash \mathrm{SL}(m, \mathbb{R})$, and we let $\mu_{H.\mathrm{SL}(m,\mathbb{R}))}$ be the $\mathrm{SL}(m, \mathbb{R})$ invariant measure on $H\backslash P$ such that the measure

$$\nu_{x_0,\pi^{-1}(P)}(f) := \int_{\pi^{-1}(P)} f(y)\Phi_{x_0}(y) d\mu_{H.\mathrm{SL}(m,\mathbb{R})}(y),\ f \in C_c(H\backslash G)$$

is a probability measure. On any other fiber $\pi^{-1}(P\rho)$, $\rho \in \mathrm{SO}(m+1,R)$, we define the pushed measure

$$\nu_{x_0,\pi^{-1}(P\rho)} := \rho_* \nu_{x_0,\pi^{-1}(P)},$$

where the pushforward is via the right translation by ρ. Having defined the above measures on the base space and the fibers of the fiber bundle $H\backslash G$, we can now define our probability measure on $H\backslash G$ to be

$$\tilde{\nu}_{x_0}(f) = \int_{P.\mathrm{SO}(m+1,\mathbb{R})} \nu_{x_0,\pi^{-1}(b)}(f) d\mu_{P.\mathrm{SO}(m+1,\mathbb{R})}(b), \ \ f \in C_c(H\backslash G).$$

Theorem 2.1.5. *Let $\Gamma \leq \mathrm{SL}(m+1,\mathbb{R})$ be a lattice and fix $x_0 \in H\backslash G$. Then, μ_{T,x_0} converges in the weak-* topology to $\tilde{\nu}_{x_0}$ as $T \to \infty$.*

Theorem 2.1.3 is a particular case of Theorem 2.1.5. We leave the details to the reader.

As mentioned above, to prove our main result we will follow the method developed in [GW04]. The key ingredients are certain volume estimates and certain ergodic theorems which we present in the following section.

2.1.3 Volume estimates of expanding skew balls in H and an equidistribution theorem on G/Γ

Let us first describe the left invariant measure on H. Put

$$\Delta_{m,1} := \begin{bmatrix} \Delta & 0 \\ 0 & 1 \end{bmatrix}, A = \left\{ \begin{bmatrix} t^{-\frac{1}{m}} I_2 & 0 \\ 0 & t \end{bmatrix} : t > 0 \right\}, U := \begin{bmatrix} I_2 & 0 \\ \mathbb{R}^m & 1 \end{bmatrix} \tag{2.1.10}$$

Then the group H has the decomposition:

$$H = U \rtimes (\Delta_{m,1} \times A) = U \rtimes (A \times \Delta_{m,1}).$$

Notice that any element in H can be uniquely represented as uqa, where $u \in U$, $q \in \Delta_{m,1}(\mathbb{Z})$, $a \in A$ (note that a commutes with q), where the semidirect product is given by:

$$(u_1, q_1, a_1) \cdot (u_2, q_2, a_2) := (u_1 q_1 a_1 u_2 (q_1 a_1)^{-1}, q_1 q_2, a_1 a_2) \tag{2.1.11}$$

Now we give a formula for the left Haar measure μ on H. For $x \in H$, write $x = u_v a_t q$, where

$$q := \begin{bmatrix} q & 0 \\ 0 & 1 \end{bmatrix}^{7}, a_t := \begin{bmatrix} t^{-\frac{1}{m}} I_m & 0 \\ 0 & t \end{bmatrix}, u_v := \begin{bmatrix} I_m & 0 \\ v & 1 \end{bmatrix}, \tag{2.1.12}$$

where $q \in \Delta, t > 0, v \in \mathbb{R}^m$. Then a left Haar measure on H is given by

$$\int_H f(x) d\mu(x) = \sum_{q \in \Delta} \int_0^\infty \int_{\mathbb{R}^m} f(u_v a_t q) dv \frac{1}{t^{m+2}} dt, \tag{2.1.13}$$

where the measure dv and dt denote the Lebesgue measures on $\mathbb{R}^m$ and $\mathbb{R}$ correspondingly.

Let $\|\cdot\|$ denote the Hilbert-Schmidt norm on matrices. Namely $\|A\| := \sqrt{\operatorname{Trace}(A^T A)}$, or the square root of the sum of squares of all entries of the matrix A.

For any subgroup L of $G = \mathrm{SL}(m+1, \mathbb{R})$, let

$$L_T := \{g \in L : \|g\| < T\}. \tag{2.1.14}$$

Following [GW04] (cf. [GN12]), for $g_1, g_2 \in G$ and $T > 0$, we define the so-called "skewed balls" as follows:

$$H_T[g_1, g_2] := \{h \in H : \|g_1^{-1} h g_2\| < T\}. \tag{2.1.15}$$

Let

$$V_{q,T}[g_1, g_2] := \{h \in UAq : \|g_1^{-1} h g_2\| < T\}, \tag{2.1.16}$$

then it follows that

$$H_T[g_1, g_2] = \bigsqcup_{q \in \Delta} V_{q,T}[g_1, g_2]. \tag{2.1.17}$$

[7] Here and henceforth we shall abuse the notation q, allowing it to represent both the $m \times m$ and $(m+1) \times (m+1)$ matrices whenever its meaning is evident from the context.

For $g_1, g_2 \in G$, by using Iwasawa decomposition (for block-lower-triangular matrix), we may assume

$$g_1 := \begin{bmatrix} \mathrm{g}_1 & 0 \\ \mathbf{v}_1 & \det(\mathrm{g}_1)^{-1} \end{bmatrix} k_1, \;\; g_2 := \begin{bmatrix} \mathrm{g}_2 & 0 \\ \mathbf{v}_2 & \det(\mathrm{g}_2)^{-1} \end{bmatrix} k_2 \tag{2.1.18}$$

where $k_1, k_2 \in \mathrm{SO}(m+1, \mathbb{R})$, $\mathrm{g}_1, \mathrm{g}_2 \in \mathbb{R}^{m \times m}$, and $\mathbf{v}_1, \mathbf{v}_2 \in \mathbb{R}^{1 \times m}$.

The key volume estimates we will prove are given in the following proposition.

Proposition 2.1.6. *Let $\Gamma(z)$ be the classical Gamma function, and let*

$$C(m) := \frac{\pi^{\frac{m}{2}} m \Gamma(\frac{m^2}{2})}{2\Gamma(\frac{m^2}{2} + \frac{m}{2} + 1)}.$$

There exists $\kappa > 0$ such that for any bounded subset $B \subseteq \mathrm{SL}(m+1, \mathbb{R})$ it holds for all $g_1, g_2 \in B$ that for $q \in \Delta$,

$$\mu(V_{q,T}[g_1, g_2]) = C(m) \frac{\det(\mathrm{g}_1)^m}{\det(\mathrm{g}_2)} \frac{T^{m(m+1)}}{\|\mathrm{g}_1^{-1} q \mathrm{g}_2\|^{m^2}} + C_q O(T^{m(m+1)-\kappa}), \tag{2.1.19}$$

where $C_q > 0$, and

$$\mu(H_T[g_1, g_2]) = C(m) \frac{\det(\mathrm{g}_1)^m}{\det(\mathrm{g}_2)} T^{m(m+1)} \sum_{q \in \Delta} \frac{1}{\|\mathrm{g}_1^{-1} q \mathrm{g}_2\|^{m^2}} + O(T^{m(m+1)-\kappa}). \tag{2.1.20}$$

As an immediate corollary, we obtain the following statements which are key requirements for the method of [GW04].

Corollary 2.1.7 (Uniform volume growth for skewed balls in H, property D1 in [GW04])**.** *For any bounded subset $B \subset G$ and any $\epsilon > 0$, there are T_0 and $\delta > 0$ such that for all $T > T_0$ and all $g_1, g_2 \in B$ we have:*

$$\mu\left(H_{(1+\delta)T}[g_1, g_2]\right) \leq (1+\epsilon) \mu\left(H_T[g_1, g_2]\right). \tag{2.1.21}$$

Corollary 2.1.8 (Limit volume ratios, property D2 in [GW04])**.** *For any $g_1, g_2 \in G$. the limit*

$$\alpha(g_1, g_2) := \lim_{T \to \infty} \frac{\mu\left(H_T[g_1, g_2]\right)}{\mu\left(H_T\right)} = \frac{\det(\mathrm{g}_1)^m}{\det(\mathrm{g}_2)} \frac{\sum_{q \in \Delta} \frac{1}{\|\mathrm{g}_1^{-1} q \mathrm{g}_2\|^{m^2}}}{\sum_{q \in \Delta} \frac{1}{\|q\|^{m^2}}}, \tag{2.1.22}$$

exists and is positive and finite.

We prove Proposition 2.1.6 in Section 2.2.

Next, we give our ergodic theorem. For $F \in C_c(G/\Gamma)$ and $g_1, g_2 \in G$, we consider the measure defined by the integral

$$\mu_{T,g_1,g_2}(F) := \frac{1}{\mu\left(H_T[g_1,g_2]\right)} \int_{H_T[g_1,g_2]} F(h^{-1}g_1\Gamma)d\mu(h), \; F \in C_c(G/\Gamma). \tag{2.1.23}$$

Theorem 2.1.9. *Let μ_X be the normalized G-invariant probability measure on $X = G/\Gamma$. For all $g_1, g_2 \in G$ and all $F \in C_c(X)$*

$$\lim_{T\to\infty} \mu_{T,g_1,g_2}(F) = \int_X F(x)d\mu_X(x). \tag{2.1.24}$$

The above ergodic theorem is proven in Section 2.3. In an overview, we will establish unipotent invariance of the limiting measure, which opens the way to apply the celebrated results of Ratner (see e.g. [Rat91]), combined with the results of Shah (see e.g. [Sha94]) on the behaviour of polynomial orbits. The main technical result we prove is a certain divergence of polynomial maps in representation space, where the domain of the maps "shrinks" (see Lemma 2.3.6).

Notations and conventions

Throughout this paper, for function f and g on $\mathbb{R}$, by $f(x) = O(g(x))$ or $f(x) \ll g(x)$ we mean there is some $C > 0$ such that $|f(x)| \leq C|g(x)|$ for sufficiently large x; by $f(x) \asymp g(x)$ we mean $f(x) \ll g(x)$ and $g(x) \ll f(x)$; by $f(x) \sim g(x)$ we mean $\lim_{x\to\infty} \left|\frac{f(x)}{g(x)}\right| = 1$ or $\lim_{x\to 0} \left|\frac{f(x)}{g(x)}\right| = 1$, depending on the context.

2.2 Volume estimates of skewed balls in H

The main goal of this section is to prove Proposition 2.1.6. In our arguments below we will make use of the following Lemma.

Lemma 2.2.1. *For an integer $n \geq 1$ and $\sigma \in \mathbb{R}$, when $T \to \infty$,*

$$\int_{\mathrm{SL}(n,\mathbb{R})_T} \|g\|^\sigma dg \sim \mathrm{Vol}(\mathrm{SL}(n,\mathbb{R})/\Delta) \int_{\Delta_T} \|q\|^\sigma dq \tag{2.2.1}$$

$$\sim \begin{cases} C_n \frac{n(n-1)}{n(n-1)+\sigma} T^{n(n-1)+\sigma} & \text{if } \sigma > -n(n-1) \\ I_{n,\sigma} + O(T^{n(n-1)+\sigma}) & \text{if } \sigma < -n(n-1) \end{cases}, \tag{2.2.2}$$

where $I_{n,\sigma}$ and C_n are constants, and $d\gamma$ is the counting measure.

Remark 2.2.2. We note that our method of proof doesn't give explicitly the constants $I_{n,\sigma}$. The constants C_n come from formula A.1.15 in [DRS93]. We also note that our method of proof below also gives the asymptotics for $\sigma = -n(n-1)$, but they will not be used in our paper.

Proof. The case $\sigma = 0$ is the estimate that as $T \to \infty$, $\mathrm{Vol}(\mathrm{SL}(n,\mathbb{R})/\Delta) \cdot \#\Delta_T \sim \mathrm{Vol}(\mathrm{SL}(n,\mathbb{R})_T)$ and the estimate $\mathrm{Vol}(\mathrm{SL}(n,\mathbb{R})_T) \sim C_n T^{n(n-1)}$, which were obtained in [GN09] (see formula A.1.15 in [DRS93] for the expression of the constant C_n). For $\sigma > 0$, by Fubini argument and using A1.15 in [DRS93], we have

$$\begin{aligned}
\int_{\mathrm{SL}(n,\mathbb{R})_T} \|g\|^\sigma dg &= \int_{\mathrm{SL}(n,\mathbb{R})_T} \int_0^{\|g\|^\sigma} 1 dt dg \\
&= \int_{\mathrm{SL}(n,\mathbb{R})_T} \int_0^{\infty} \mathbf{1}_{[t<\|g\|^\sigma]} dt dg \\
&= \int_0^{\infty} \int_{\mathrm{SL}(n,\mathbb{R})_T} \mathbf{1}_{[t<\|g\|^\sigma]} dg dt \\
&= \int_0^{T^\sigma} \mathrm{Vol}([t^{\frac{1}{\sigma}} < \|g\| < T]) dt \\
&= \int_0^{T^\sigma} \mathrm{Vol}([\|g\| < T]) dt - \int_0^{T^\sigma} \mathrm{Vol}([\|g\| \leq t^{\frac{1}{\sigma}}]) dt \\
&= C_n T^{n(n-1)+\sigma} - \int_0^{T^\sigma} C_n t^{n(n-1)/\sigma} dt + o(T^{n(n-1)+\sigma}) \\
&= C_n \frac{n(n-1)}{n(n-1)+\sigma} T^{n(n-1)+\sigma} + o(T^{n(n-1)+\sigma}).
\end{aligned}$$

For $-n(n-1) \neq \sigma < 0$,

$$\begin{aligned}
\int_{\mathrm{SL}(n,\mathbb{R})_T} \|g\|^\sigma dg &= \int_{\mathrm{SL}(n,\mathbb{R})_T} \int_0^{\|g\|^\sigma} 1 dt dg \\
&= \int_{\mathrm{SL}(n,\mathbb{R})_T} \int_0^\infty \mathbf{1}_{[t<\|g\|^\sigma]} dt dg \\
&= \int_0^\infty \int_{\mathrm{SL}(n,\mathbb{R})_T} \mathbf{1}_{[t<\|g\|^\sigma]} dg dt \\
&= \int_0^\infty \mathrm{Vol}([\|g\| < \min(t^{\frac{1}{\sigma}}, T)]) dt \\
&= \int_{T^\sigma}^\infty \mathrm{Vol}([\|g\| < t^{\frac{1}{\sigma}}]) dt + \int_0^{T^\sigma} \mathrm{Vol}([\|g\| < T]) dt \\
&= \int_{T^\sigma}^1 \mathrm{Vol}([\|g\| < t^{\frac{1}{\sigma}}]) dt + \int_0^{T^\sigma} \mathrm{Vol}([\|g\| < T]) dt \\
&\qquad\qquad\qquad\qquad\qquad\qquad (\text{For } g \in \mathrm{SL}(n,\mathbb{R}),\ \|g\| \geq 1) \\
&= \int_{T^\sigma}^1 \mathrm{Vol}([\|g\| < t^{\frac{1}{\sigma}}]) dt + C_n T^{n(n-1)+\sigma} + o(T^{n(n-1)+\sigma})
\end{aligned}$$

Since $\mathrm{Vol}([\|g\| < t^{\frac{1}{\sigma}}]) \asymp t^{\frac{n(n-1)}{\sigma}}$ as $t \to 0$ (recall that here $\sigma < 0$), and since $\int_0^1 t^{\frac{n(n-1)}{\sigma}} dt$ converges, we have by dominated convergence that

$$\int_0^1 \mathrm{Vol}([\|g\| < t^{\frac{1}{\sigma}}]) dt = I_{n,\sigma}$$

for some (implicit) constant $= I_{n,\sigma}$. Now

$$\begin{aligned}
\int_{T^\sigma}^1 \mathrm{Vol}([\|g\| < t^{\frac{1}{\sigma}}]) dt =& I_{n,\sigma} - \int_0^{T^\sigma} \mathrm{Vol}([\|g\| < t^{\frac{1}{\sigma}}]) dt \\
=& I_{n,\sigma} + O\left(\int_0^{T^\sigma} t^{\frac{n(n-1)}{\sigma}} dt \right) = I_{n,\sigma} + O(T^{n(n+1)+\sigma}),
\end{aligned}$$

which concludes the proof of the estimate for $\sigma < -n(n-1)$.

The proof for the statement with Δ_T is similar — use the Fubini argument with the counting measure dq instead of dg, and then apply the estimate that $\Delta_\tau \sim C_n \tau^{n(n-1)}$ as $\tau \to \infty$. $\square$

For $g_1 \in \mathrm{SL}(m+1,\mathbb{R})$ we write

$$g_1^{-1} := k_1 \begin{bmatrix} G_1 & 0 \\ G_3 & G_4 \end{bmatrix},\ g_2 := \begin{bmatrix} H_1 & 0 \\ H_3 & H_4 \end{bmatrix} k_2, \tag{2.2.3}$$

where $k_1, k_2 \in \mathrm{SO}(m+1, \mathbb{R})$, $G_1, H_1 \in \mathbb{R}^{m\times m}$, $G_3, H_3 \in \mathbb{R}^{1\times m}$ and $G_4, H_4 \in \mathbb{R}$. We proceed now to study $\mu\left(H_T[g_1, g_2]\right)$, where $H_T[g_1, g_2] := \{h \in H : \|g_1^{-1}hg_2\| < T\}$. Recall that our $H_T[g_1, g_2]$ is the disjoint union (2.1.17), so that

$$\mu\left(H_T[g_1, g_2]\right) = \sum_{q\in\Delta} \mu\left(V_{q,T}[g_1, g_2]\right), \tag{2.2.4}$$

where we recall $V_{q,T}[g_1, g_2] := \{h \in UAq : \|g_1^{-1}hg_2\| < T\}$. We will now inspect closely $V_{q,T}[g_1, g_2]$. Since the Hilbert-Schmidt norm is bi-$\mathrm{SO}(m+1, \mathbb{R})$ invariant, we may assume that k_1 and k_2 in (2.2.3) are equal to the identity matrix.

A generic term $h \in UA\gamma$ is of the form

$$h := \begin{bmatrix} I_2 & 0 \\ v & 1 \end{bmatrix} \begin{bmatrix} t^{-\frac{1}{m}} I_2 & 0 \\ 0 & t \end{bmatrix} \begin{bmatrix} q & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} t^{-\frac{1}{m}} q & 0 \\ t^{-\frac{1}{m}} vq & t \end{bmatrix},$$

where $t \neq 0, q \in \Delta, v \in \mathbb{R}^m$. Note

$$\begin{aligned} g_1 h g_2 &= \begin{bmatrix} G_1 & 0 \\ G_3 & G_4 \end{bmatrix} \begin{bmatrix} t^{-\frac{1}{m}} q & 0 \\ t^{-\frac{1}{m}} vq & t \end{bmatrix} \begin{bmatrix} H_1 & 0 \\ H_3 & H_4 \end{bmatrix} \\ &= \begin{bmatrix} t^{-\frac{1}{m}} G_1 q & 0 \\ t^{-\frac{1}{m}} G_3 q + t^{-\frac{1}{m}} G_4 vq & G_4 t \end{bmatrix} \begin{bmatrix} H_1 & 0 \\ H_3 & H_4 \end{bmatrix} \\ &= \begin{bmatrix} t^{-\frac{1}{m}} G_1 q H_1 & 0 \\ t^{-\frac{1}{m}} G_3 q H_1 + t^{-\frac{1}{m}} G_4 vq H_1 + G_4 H_3 t & G_4 H_4 t \end{bmatrix}. \end{aligned}$$

Upon taking the sum of squares and rearranging terms, we conclude that $\|g_1 h g_2\| \leq T$ is equivalent to

$$\|G_3 q H_1 + G_4 vq H_1 + G_4 H_3 t^{\frac{1}{m}+1}\|^2 \leq -|G_4 H_4|^2 t^{\frac{2}{m}+2} + t^{\frac{2}{m}} T^2 - \|G_1 q H_1\|^2 \tag{2.2.5}$$

In view of (2.2.5), let

$$D_{q,T,t} := \{v \in \mathbb{R}^m : \|G_3 q H_1 + G_4 vq H_1 + G_4 H_3 t^{\frac{1}{m}+1}\|^2 \leq -B_1^2 t^{\frac{2}{m}+2} + t^{\frac{2}{m}} T^2 - A_m(q)^2\}, \tag{2.2.6}$$

where

$$A_m(q) = \|G_1 q H_1\| \tag{2.2.7}$$

and

$$B_1 = |G_4 H_4|. \tag{2.2.8}$$

Recall the expression (2.1.13) for the left Haar measure on H, which gives

$$\mu\left(V_{q,T}[g_1, g_2]\right) := \int_0^\infty \int_{\mathbb{R}^m} \mathbf{1}_{V_{q,T}[g_1,g_2]}(u_v a_t q) dv \frac{1}{t^{m+2}} dt = \int_0^\infty \mathrm{Vol}(D_{q,T,t}) \frac{1}{t^{m+2}} dt, \tag{2.2.9}$$

where $\mathrm{Vol}(D_{q,T,t})$ is the Lebesgue measure of $D_{q,T,t}$. We observe that $D_{q,T,t}$ is the interior of an ellipse in $\mathbb{R}^m$, whose area is

$$\mathrm{Vol}(D_{q,T,t}) = \begin{cases} v_m \frac{\left(-B_1^2 t^{\frac{2}{m}+2} + t^{\frac{2}{m}} T^2 - A_m(q)^2\right)^{\frac{m}{2}}}{|G_4|^m |\det(H_1)|} & \text{if } -B_1^2 t^{\frac{2}{m}+2} + t^{\frac{2}{m}} T^2 - A_m(q)^2 > 0 \\ 0 & \text{otherwise.} \end{cases} \tag{2.2.10}$$

Here $v_m = \frac{\pi^{\frac{m}{2}}}{\Gamma(\frac{m}{2}+1)}$ denotes the volume of Euclidean ball of radius one in $\mathbb{R}^m$. We now inspect more closely (2.2.10) with the goal to shed light on how it changes as we vary γ and T. Roughly, we will show that when $A_m(q)$ is "large" with respect to T, then the volume of the ellipse is "small".

First, observe that the maximal value of $-B_1^2 t^{\frac{2}{m}+2} + t^{\frac{2}{m}} T^2$ for $t > 0$ is

$$M_T := \frac{m}{m+1} \left(\frac{1}{m+1}\right)^{\frac{1}{m}} \frac{T^{2+\frac{2}{m}}}{B_1^{\frac{2}{m}}}, \tag{2.2.11}$$

which is attained at

$$\theta(T) := \frac{1}{\sqrt{1+m}} \frac{T}{B_1}. \tag{2.2.12}$$

Then, according to (2.2.10), we get that $m(D_{\gamma,T,t}) \neq 0$ only if

$$A_m(q) < \sqrt{M_T} \asymp \frac{T^{1+\frac{1}{m}}}{B_1}. \tag{2.2.13}$$

Next, observe that $f(t) = -B_1^2 t^{\frac{2}{m}+2} + t^{\frac{2}{m}} T^2 - A_m(q)^2$ is monotonic increasing in $(0, \theta(T))$ and monotonic decreasing in $(\theta(T), \infty)$. Also, note that $f(0) < 0$ and that $f(\sqrt{m+1}\theta(T)) < 0$. Then, whenever $A_m(q)^2 < M_T$, it is easy to see that $-B_1^2 t^{\frac{2}{m}+2} + t^{\frac{2}{m}} T^2 - A_m(q)^2$ has two positive roots, which we denote by $\alpha_{q,T}$ and $\beta_{q,T}$, and the following bounds hold

$$0 < \alpha_{q,T} < \theta(T) < \beta_{q,T} < \sqrt{m+1}\theta(T), \tag{2.2.14}$$

whenever $A_m(q) < \sqrt{M_T}$. In particular,

$$\mathrm{Vol}(D_{q,T,t}) = \begin{cases} v_m \frac{\left(-B_1^2 t^{\frac{2}{m}+2} + t^{\frac{2}{m}} T^2 - A_m(q)^2\right)^{\frac{m}{2}}}{|G_4|^m |\det(H_1)|} & \text{if } t \in (\alpha_{q,t}, \beta_{q,t}) \\ 0 & \text{otherwise,} \end{cases} \tag{2.2.15}$$

and by recalling (2.2.4), we conclude that

$$\mu\left(V_{q,T}[g_1, g_2]\right) = \frac{v_m}{|G_4|^m |\det(H_1)|} \int_{\alpha_{\gamma,T}}^{\beta_{\gamma,T}} \frac{(-B_1^2 t^{\frac{2}{m}+2} + t^{\frac{2}{m}} T^2 - A_m(q)^2)^{\frac{m}{2}}}{t^{m+2}} dt. \tag{2.2.16}$$

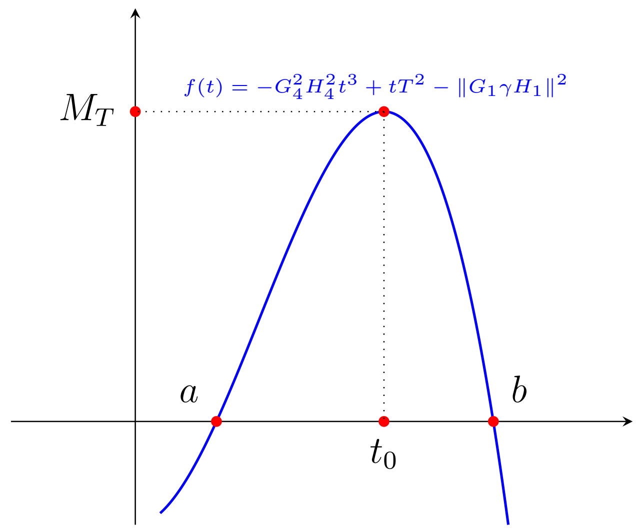

Figure 2.1: Illustration of the estimate of the roots of the polynomial when $m = 2$.

The following gives more precise bounds for the root $\alpha_{\gamma,T}$ as q and T vary. By rearranging terms in

$$-B_1^2 \alpha_{q,T}^{\frac{2}{m}+2} + \alpha_{q,T}^{\frac{2}{m}} T^2 - A_m(q)^2 = 0,$$

we get that

$$\alpha_{q,T}^{\frac{2}{m}} = \frac{A_m(q)^2}{T^2 - B_1^2 \alpha_{q,T}^2}, \tag{2.2.17}$$

and by using that $\alpha_{q,T} < \theta(T)$, see (2.2.14), we conclude that

$$\alpha_{q,T}^{\frac{2}{m}} = \frac{A_m(q)^2}{T^2 - B_1^2 \alpha_{q,T}^2} \in \left(\frac{A_m(q)^2}{T^2}, \frac{(m+1)A_m(q)^2}{mT^2} \right). \tag{2.2.18}$$

In particular, it follows that

$$\alpha_{q,T} \asymp \frac{A_m(q)^m}{T^m}, \tag{2.2.19}$$

and moreover, by (2.2.17) and (2.2.18),

$$\alpha_{q,T}^{\frac{2}{m}} = \frac{A_m(q)^2}{T^2} + O\left(\frac{1}{T^{m+1.5}}\right), \text{ when } A_m(q) \leq \sqrt{T} \tag{2.2.20}$$

Lemma 2.2.3. *It holds that*

$$\begin{aligned} \mu(V_{q,T}[g_1, g_2]) \leq & \frac{v_m}{|G_4|^m| \det(H_1)|} \frac{T^{m(m+1)}}{\|A_m(q)\|^{m^2}} && (2.2.21) \\ \leq & \frac{v_m \|G_1^{-1}\|^{m^2} \|H_1^{-1}\|^{m^2}}{|G_4|^m| \det(H_1)|} \frac{T^{m(m+1)}}{\|q\|^{m^2}}. && (2.2.22) \end{aligned}$$

In particular, when g_1, g_2 vary in a compact subset of $\mathrm{SL}(m+1, \mathbb{R})$, *it holds that*

$$\mu(V_{q,T}[g_1, g_2]) \leq T^{m(m+1)} O\left(\frac{1}{\|q\|^{m^2}}\right).$$

Proof. We have

$$\begin{aligned} & \int_{\alpha_{q,T}}^{\beta_{q,T}} \frac{(-B_1^2 t^{\frac{2}{m}+2} + t^{\frac{2}{m}} T^2 - A_m(q)^2)^{\frac{m}{2}}}{t^{m+2}} dt && (2.2.23) \\ \leq & \int_{\alpha_{q,T}}^{\beta_{q,T}} \frac{(t^{\frac{2}{m}} T^2)^{\frac{m}{2}}}{t^{m+2}} dt && (2.2.24) \\ = & T^m \int_{\alpha_{q,T}}^{\beta_{q,T}} \frac{1}{t^{m+1}} && (2.2.25) \\ \leq & T^m \frac{1}{m \alpha_{q,T}^m}. && (2.2.26) \end{aligned}$$

Using (2.2.18), we get

$$\frac{1}{\alpha_{q,T}^m} \leq \frac{T^{m^2}}{A_m(q)^{m^2}} \leq T^{m^2} \frac{\|G_1^{-1}\|^{m^2} \|H_1^{-1}\|^{m^2}}{\|\gamma\|^{m^2}},$$

which proves the claim upon recalling (2.2.16). □

We now return to $\mu\left(H_T[g_1, g_2]\right)$. We conclude by (2.2.27) that

$$\mu\left(H_T[g_1, g_2]\right) = \sum_{\substack{q \in \Delta, \\ A_m(q) < \sqrt{M_T}}} \frac{v_m}{|G_4|^m| \det(H_1)|} \int_{\alpha_{q,T}}^{\beta_{q,T}} \frac{(-B_1^2 t^{\frac{2}{m}+2} + t^{\frac{2}{m}} T^2 - A_m(q)^2)^{\frac{m}{2}}}{t^{m+2}} dt, \tag{2.2.27}$$

where M_T is given by (2.2.11). We split the sum into two parts as

$$\left(\sum_{\substack{q\in\Delta,\\ A_m(q)<\sqrt{T}}} + \sum_{\substack{q\in\Delta,\\ \sqrt{T}<A_m(q)<\sqrt{M_T}}}\right)\frac{v_m}{|G_4|^m|\det(H_1)|}\int_{\alpha_{q,T}}^{\beta_{q,T}}\frac{(-B_1^2t^{\frac{2}{m}+2}+t^{\frac{2}{m}}T^2-A_m(q)^2)^{\frac{m}{2}}}{t^{m+2}}dt. \tag{2.2.28}$$

We will now estimate the second sum. The second sum is relatively easier to estimate, and it will eventually follow that it is of lower order in T compared to the first sum.

Lemma 2.2.4. *If g_1, g_2 vary in a bounded set, then*

$$\sum_{\substack{q\in\Delta,\\ \sqrt{T}<A_m(q)<\sqrt{M_T}}}\frac{v_m}{|G_4|^m|\det(H_1)|}\int_{\alpha_{q,T}}^{\beta_{q,T}}\frac{(-B_1^2t^{\frac{2}{m}+2}+t^{\frac{2}{m}}T^2-A_m(q)^2)^{\frac{m}{2}}}{t^{m+2}}dt = O(T^{m(m+1)-m/2}) \tag{2.2.29}$$

Proof. In the following g_1, g_2 vary in a bounded set. By Lemma 2.2.3,

$$\sum_{\substack{q\in\Delta,\\ \sqrt{T}<A_m(q)<\sqrt{M_T}}}\int_{\alpha_{q,T}}^{\beta_{q,T}}\frac{(-B_1^2t^{\frac{2}{m}+2}+t^{\frac{2}{m}}T^2-A_m(q)^2)^{\frac{m}{2}}}{t^{m+2}}dt \tag{2.2.30}$$

$$\leq T^{m(m+1)}O\left(\sum_{\substack{q\in\Delta,\\ \sqrt{T}<A_m(q)}}\frac{1}{\|q\|^{m^2}}\right). \tag{2.2.31}$$

We have

$$A_m(q) = \|G_1\gamma H_1\| \leq \|G_1\|\|\gamma\|\|H_1\|,$$

where $\|G_1\|, \|H_1\|$ are bounded above since they depend continuously on g_1, g_2. Then, for some $C > 0$, we conclude that

$$O\left(\sum_{\substack{q\in\Delta,\\ \sqrt{T}<A_m(q)}}\frac{1}{\|q\|^{m^2}}\right) = O\left(\sum_{\substack{q\in\Delta,\\ C\sqrt{T}<\|q\|}}\frac{1}{\|q\|^{m^2}}\right).$$

By Lemma 2.2.1 with $n = m$ and $\sigma = -m^2$, it follows that $\sum_{q\in\Delta}\frac{1}{\|q\|^{m^2}}$ converges with the tail estimate

$$\sum_{\substack{q\in\Delta,\\ \sqrt{T}<\|q\|}}\frac{1}{\|q\|^{m^2}} = O(\sqrt{T}^{-m}),$$

which proves our claim. $\square$

We now proceed to treat the first part of the sum (2.2.28). Namely, in the following, assume that $A_m(q) \leq \sqrt{T}$.

For ϵ_1 we define $\delta = \delta(T, q, \epsilon_1)$ by

$$\delta := \frac{A_m(q)}{T^{1+\epsilon_1}}, \tag{2.2.32}$$

we define $\alpha_\delta = \alpha_\delta(q, T, \epsilon_1)$ by

$$\alpha_\delta^{\frac{1}{m}} := \alpha_{q,T}^{\frac{1}{m}} + \delta, \tag{2.2.33}$$

and for $\epsilon_2 \in (0,1)$ we let $\lambda = \lambda(q, T, \epsilon_2)$

$$\lambda := \left(\frac{A_m(q)}{T^{\epsilon_2}}\right)^{\frac{m}{m+1}}. \tag{2.2.34}$$

We consider the following partition of the integral appearing in the terms of (2.2.28),

$$\int_{\alpha_{q,T}}^{\beta_{q,T}} \frac{(-B_1^2 t^{\frac{2}{m}+2} + t^{\frac{2}{m}} T^2 - A_m(q)^2)^{\frac{m}{2}}}{t^{m+2}} dt \tag{2.2.35}$$

$$= \left(\int_{\alpha_{q,T}}^{\alpha_\delta(q,T,\epsilon_1)} + \int_{\alpha_\delta(q,T,\epsilon_1)}^{\lambda(q,T,\epsilon_2)} + \int_{\lambda(q,T,\epsilon_2)}^{\beta_{q,T}}\right) \frac{(-B_1^2 t^{\frac{2}{m}+2} + t^{\frac{2}{m}} T^2 - A_m(q)^2)^{\frac{m}{2}}}{t^{m+2}} dt \tag{2.2.36}$$

Our key point in the computation below will be that the main term among the three integrals above is the integral in the range $\int_{\alpha_\delta(q,T,\epsilon_1)}^{\lambda(q,T,\epsilon_2)}$. For the following, we note that

$$\|G_1^{-1}\|^{-1} \|H_1^{-1}\|^{-1} \|q\| \leq A_m(q), \tag{2.2.37}$$

and we note that $\|G_1^{-1}\|^{-1}$, $\|H_1^{-1}\|^{-1}$ are bounded from below when g_1, g_2 vary in a bounded set.

Lemma 2.2.5. *Suppose that g_1, g_2 vary in a bounded subset of $\mathrm{SL}(m+1, \mathbb{R})$ and fix $\epsilon_1 \in (0,1)$. Then*

$$\int_{\alpha_{q,T}}^{\alpha_\delta(q,T,\epsilon_1)} \frac{(-B_1^2 t^{\frac{2}{m}+2} + t^{\frac{2}{m}} T^2 - A_m(q)^2)^{\frac{m}{2}}}{t^{m+2}} dt = O\left(\frac{T^{m(m+1)-\epsilon_1 \frac{m+2}{2}}}{\|q\|^{m^2}}\right) \tag{2.2.38}$$

Proof. By substituting the variable $s = t^{\frac{1}{m}}$ in $-B_1^2 t^{\frac{2}{m}+2} + t^{\frac{2}{m}} T^2 - A_m(q)^2$, the expression becomes

$$f(s) := -B_1^2 s^{2m+2} + T^2 s^2 - A_m(q)^2. \tag{2.2.39}$$

Recall

$$\alpha_\delta(q,T,\epsilon_1)^{\frac{1}{m}} := \alpha_{q,T}^{\frac{1}{m}} + \delta(q,T,\epsilon_1) = \alpha_{q,T}^{\frac{1}{m}} + \frac{A_m(q)}{T^{1+\epsilon_1}},$$

and note that $f(\alpha_{q,T}^{\frac{1}{m}}) = 0$. In the following, to ease the reading, we will omit from the notations the dependencies on q, T and ϵ_1. We apply Taylor expansion to $f(s)$ in the range $s \in [\alpha^{\frac{1}{m}}, \alpha_\delta^{\frac{1}{m}}]$ at $s = \alpha^{\frac{1}{m}}$ which gives

$$\begin{aligned} f(\alpha_\delta^{\frac{1}{m}}) =& [-(2m+2)B_1^2 \alpha^{\frac{2m+1}{m}} + 2T^2 \alpha^{\frac{1}{m}}]\delta + \frac{f''(\xi)}{2}\delta^2 \\ \ll& T^2 \alpha^{\frac{1}{m}} \delta + f''(\xi)\delta^2 \end{aligned} \tag{2.2.40}$$

where $\xi \in [\alpha^{\frac{1}{m}}, \alpha_\delta^{\frac{1}{m}}]$. When g_1, g_2 are bounded, we get that B_1 is bounded above and away from zero. Then the second derivative $f''(\xi)$ over $[\alpha^{\frac{1}{m}}, \alpha_\delta^{\frac{1}{m}}]$ will be bounded as

$$\begin{aligned} |f''(\xi)| =& |-(2m+2)(2m+1)B_1^2 \xi^{2m} + 2T^2| \\ \leq& (2m+2)(2m+1)B_1^2 \alpha_\delta^{\frac{2m}{m}} + 2T^2 \\ \ll& T^2. \end{aligned} \tag{2.2.41}$$

We note that for all large T (independently of q and ϵ_1) f is monotonically increasing on the interval $(\alpha^{\frac{1}{m}}, \alpha_\delta^{\frac{1}{m}})$. To see this, recall that $\theta(T)^{\frac{1}{m}} \sim T^{\frac{1}{m}}$ is a critical point for $f(s)$, $f(s)$ is monotonically increasing in $(0, \theta(T)^{\frac{1}{m}})$, and $\alpha_\delta = o(1)$. Then

$$
\begin{aligned}
&\int_{\alpha}^{\alpha_\delta} \frac{(-B_1^2 t^{\frac{2}{m}+2} + t^{\frac{2}{m}} T^2 - A_m(q)^2)^{\frac{m}{2}}}{t^{m+2}} dt \\
\leq &\frac{1}{\alpha^{m+2}} \int_{\alpha}^{\alpha_\delta} (-B_1^2 t^{\frac{2}{m}+2} + t^{\frac{2}{m}} T^2 - A_m(q)^2)^{\frac{m}{2}} dt \\
= &\frac{1}{\alpha^{m+2}} \int_{\alpha^{\frac{1}{m}}}^{\alpha_\delta^{\frac{1}{m}}} f(s)^{\frac{m}{2}} m s^{m-1} ds \\
\ll &\frac{1}{\alpha^{m+2}} f(\alpha_\delta^{\frac{1}{m}})^{\frac{m}{2}} \alpha_\delta^{\frac{m-1}{m}} \delta && \text{(monotonicity of } f(s)) \\
\ll &\frac{1}{\alpha^{m+1+\frac{1}{m}}} f(\alpha_\delta^{\frac{1}{m}})^{\frac{m}{2}} \delta && (\alpha_\delta \ll \alpha) \\
\ll &\frac{\left\{T^2 \alpha^{\frac{1}{m}} \delta + T^2 \delta^2\right\}^{\frac{m}{2}}}{\alpha^{m+1+\frac{1}{m}}} \cdot \delta && \text{(by (2.2.40) and (2.2.41))} \\
= &\frac{\left\{T^2 \alpha_\delta^{\frac{1}{m}}\right\}^{\frac{m}{2}}}{\alpha^{m+1+\frac{1}{m}}} \cdot \delta^{\frac{m+2}{2}} \\
\ll &\frac{\left\{T^2 \alpha^{\frac{1}{m}}\right\}^{\frac{m}{2}}}{\alpha^{m+1+\frac{1}{m}}} \cdot \delta^{\frac{m+2}{2}}
\end{aligned}
$$

Using the bounds of α in (2.2.18) and by applying the definition of δ in (2.2.32), we get

$$
\frac{\left\{T^2 \alpha^{\frac{1}{m}}\right\}^{\frac{m}{2}}}{\alpha^{m+1+\frac{1}{m}}} \cdot \delta^{\frac{m+2}{2}} \ll \frac{T^{m(m+1)-\epsilon_1 \frac{m+2}{2}}}{A_m(q)^{m^2}}, \tag{2.2.42}
$$

which concludes our proof. □

We proceed to treat the middle integral, which will be the main term in (2.2.35).

Lemma 2.2.6. *Suppose that $A_m(q) \leq \sqrt{T}$, and g_1, g_2 vary in a bounded set of $\mathrm{SL}(m+1, \mathbb{R})$. Suppose that $0 < \epsilon_1 < 2\epsilon_2 < 1$, and let*

$$
\epsilon := \min\{\epsilon_1, 2\epsilon_2 - \epsilon_1, 1 + \frac{1 - 2\epsilon_2}{m+1}\}. \tag{2.2.43}
$$

Then

$$
\begin{aligned}
&\int_{\alpha_\delta(q,T,\epsilon_1)}^{\lambda(q,T,\epsilon_2)} \frac{(-B_1^2 t^{\frac{2}{m}+2} + t^{\frac{2}{m}} T^2 - A_m(q)^2)^{\frac{m}{2}}}{t^{m+2}} dt \\
= &\frac{m\Gamma(\frac{m}{2}+1)\Gamma(\frac{m^2}{2})}{2\Gamma(\frac{m^2}{2}+\frac{m}{2}+1)} \cdot \frac{T^{m(m+1)}}{A_m(q)^{m^2}} + O\left(\frac{T^{m(m+1)-\epsilon}}{\|q\|^{m^2}}\right).
\end{aligned} \tag{2.2.44}
$$

Proof. Suppose that $A_m(q) \leq \sqrt{T}$, and assume that g_1, g_2 vary in a bounded set. In particular, recall that B_1 will be bounded above. To ease the reading, we will omit from some notations the dependencies on q, T and ϵ_1, ϵ_2. The dependencies should be clear from the context. We first rewrite our integral as

$$\int_{\alpha_\delta}^{\lambda} \frac{(-B_1^2 t^{\frac{2}{m}+2} + t^{\frac{2}{m}} T^2 - A_m(q)^2)^{\frac{m}{2}}}{t^{m+2}} dt = \int_{\alpha_\delta}^{\lambda} \frac{(t^{\frac{2}{m}} T^2 - A_m(q)^2)^{\frac{m}{2}}}{t^{m+2}} \left(1 - \frac{B_1^2 t^{\frac{2}{m}+2}}{t^{\frac{2}{m}} T^2 - A_m(q)^2}\right)^{\frac{m}{2}} dt.$$

We now show that in the range $t \in (\alpha_\delta, \lambda)$ it holds that

$$\frac{B_1^2 t^{\frac{2}{m}+2}}{t^{\frac{2}{m}} T^2 - A_m(q)^2} = O\left(\frac{1}{T^{2\epsilon_2 - \epsilon_1}}\right), \tag{2.2.45}$$

Indeed,

$$\begin{aligned}
\frac{B_1^2 t^{\frac{2}{m}+2}}{t^{\frac{2}{m}} T^2 - A_m(q)^2} \ll & \frac{\lambda^{2+\frac{2}{m}}}{\alpha_\delta^{\frac{2}{m}} T^2} \\
= & \frac{\lambda^{2\frac{m+1}{m}}}{(\alpha^{\frac{1}{m}} + \delta)^2 T^2} && \text{(definition of } \alpha_\delta \text{ (2.2.33))} \\
\ll & \frac{\lambda^{2\frac{m+1}{m}}}{\alpha^{\frac{1}{m}} \delta T^2} \\
\leq & \frac{\left(A_m(q)/T^{\epsilon_2}\right)^2}{\frac{A_m(q)}{T} \frac{A_m(q)}{T^{1+\epsilon_1}} T^2} = O\left(\frac{1}{T^{2\epsilon_2 - \epsilon_1}}\right). && \text{(see (2.2.34), (2.2.18) and (2.2.32))}
\end{aligned}$$

Now in view of the estimate $(1+x)^\alpha \sim 1 + \alpha x$ as $x \to 0$, we conclude that

$$\begin{aligned}
&\int_{\alpha_\delta}^{\lambda} \frac{(t^{\frac{2}{m}} T^2 - A_m(q)^2)^{\frac{m}{2}}}{t^{m+2}} \left(1 - \frac{B_1^2 t^{\frac{2}{m}+2}}{t^{\frac{2}{m}} T^2 - A_m(q)^2}\right)^{\frac{m}{2}} dt \\
&\qquad = \left(\int_{\alpha_\delta}^{\lambda} \frac{(t^{\frac{2}{m}} T^2 - A_m(q)^2)^{\frac{m}{2}}}{t^{m+2}} dt\right) \left(1 + O\left(\frac{1}{T^{2\epsilon_2 - \epsilon_1}}\right)\right). \qquad (2.2.46)
\end{aligned}$$

We shall now estimate $\int_{\alpha_\delta}^{\lambda} \frac{(t^{\frac{2}{m}} T^2 - A_m(q)^2)^{\frac{m}{2}}}{t^{m+2}} dt$. To this end, we use the substitute $u = \frac{A_m(q)^2}{T^2 t^{\frac{2}{m}}}$ (or conversely $t = \frac{A_m(q)^m}{T^m u^{\frac{m}{2}}}$), so that

$$\int_{\alpha_\delta}^{\lambda} \frac{(t^{\frac{2}{m}} T^2 - A_m(q)^2)^{\frac{m}{2}}}{t^{m+2}} dt \tag{2.2.47}$$

$$= \int_{\alpha_\delta}^{\lambda} \frac{T^m}{t^{m+1}} \left(1 - \frac{A_m(q)^2}{T^2 t^{\frac{2}{m}}}\right)^{\frac{m}{2}} dt \tag{2.2.48}$$

$$= \frac{m}{2} \frac{T^{m^2+m}}{A_m(q)^{m^2}} \int_{A_m(q)^2/T^2 \lambda^{\frac{2}{m}}}^{A_m(q)^2/T^2 \alpha_\delta^{\frac{2}{m}}} u^{\frac{m^2}{2}-1} (1-u)^{\frac{m}{2}} du. \tag{2.2.49}$$

We will now prove that

$$\int_{A_m(q)^2/T^2\lambda^{\frac{2}{m}}}^{A_m(q)^2/T^2\alpha_\delta^{\frac{2}{m}}} u^{\frac{m^2}{2}-1}(1-u)^{\frac{m}{2}}\,du = \int_0^1 u^{\frac{m^2}{2}-1}(1-u)^{\frac{m}{2}}\,du + O\left(\frac{1}{T^{\epsilon_1}}\right) + O\left(\frac{1}{T^{1+\frac{1-2\epsilon_2}{m+1}}}\right)$$
$$= \frac{m\Gamma(\frac{m}{2}+1)\Gamma(\frac{m^2}{2})}{2\Gamma(\frac{m^2}{2}+\frac{m}{2}+1)} + O\left(\frac{1}{T^{\epsilon}}\right). \tag{2.2.50}$$

where in the last equality we used the classical formula for the beta function. Notice that by (2.2.46), by (2.2.49) and by (2.2.50) the proof is complete.

To prove (2.2.50) it suffices to estimate the differences between the endpoints of the corresponding integral, namely it suffices to estimate $\left|1-\frac{A_m(q)^2}{T^2\alpha_\delta^{\frac{2}{m}}}\right|$ and $\left|\frac{A_m(q)^2}{T^2\lambda^{\frac{2}{m}}}-0\right| = \frac{A_m(q)^2}{T^2\lambda^{\frac{2}{m}}}$. We have

$$\begin{aligned}
\left|1-\frac{A_m(q)^2}{T^2\alpha_\delta^{\frac{2}{m}}}\right| &= \left|\frac{T^2\alpha_\delta^{\frac{2}{m}}-A_m(q)^2}{T^2\alpha_\delta^{\frac{2}{m}}}\right| \\
&= \left|\frac{(T^2\alpha^{\frac{2}{m}}-A_m(q)^2)+(2\alpha^{\frac{1}{m}}\delta+\delta^2)T^2}{T^2\alpha_\delta^{\frac{2}{m}}}\right| && \text{(using definition of } \alpha_\delta\text{, see (2.2.33))} \\
&= \frac{\left|{}^1/T^{m-0.5}+(2\alpha^{\frac{1}{m}}\delta+\delta^2)T^2\right|}{T^2\alpha_\delta^{\frac{2}{m}}} && \text{(see (2.2.20))} \\
&\le \frac{\left|{}^1/T^{m-0.5}+(2\alpha^{\frac{1}{m}}\delta+\delta^2)T^2\right|}{\left|T^2\alpha^{\frac{2}{m}}\right|} \\
&= O\left(\frac{1}{T^{\epsilon_1}}\right), && \text{(we have } T^2\alpha^{\frac{2}{m}} = O(1) \text{ and } (2\alpha^{\frac{1}{m}}\delta+\delta^2)T^2 = O(\tfrac{1}{T^{\epsilon_1}})\text{)}
\end{aligned}$$

and finally

$$\frac{A_m(q)^2}{T^2\lambda^{\frac{2}{m}}} = \frac{A_m(q)^2}{T^2\left({}^{A_m(q)}/T^{\epsilon_2}\right)^{\frac{2}{m+1}}} = \frac{A_m(q)^{2-\frac{2}{m+1}}}{T^{2-\frac{2\epsilon_2}{m+1}}} \underbrace{=}_{A_m(q)\le\sqrt{T}} O\left(\frac{1}{T^{1+\frac{1-2\epsilon_2}{m+1}}}\right). \tag{2.2.51}$$

□

For the last integral, we have the following.

Lemma 2.2.7. *Suppose that g_1, g_2 vary in a bounded set and fix $\epsilon_2 \in (0,1)$. Then*

$$\int_{\lambda(q,T,\epsilon_2)}^{\beta_{q,T}} \frac{(-B_1^2 t^{\frac{2}{m}+2}+t^{\frac{2}{m}}T^2-A_m(q)^2)^{\frac{m}{2}}}{t^{m+2}}dt = T^{m(m+1)-m^2(1-\frac{\epsilon_2}{m+1})}O\left(\frac{1}{\|q\|^{\frac{m^2}{m+1}}}\right). \tag{2.2.52}$$

Proof. Again, to ease the reading, we will omit from the notations the dependencies on q,T and ϵ_2. We have

$$\int_{\lambda}^{\beta} \frac{(-B_1^2 t^{\frac{2}{m}+2} + t^{\frac{2}{m}} T^2 - A_m(q)^2)^{\frac{m}{2}}}{t^{m+2}} dt \tag{2.2.53}$$

$$\leq \int_{\lambda}^{\beta} \frac{(t^{\frac{2}{m}} T^2)^{\frac{m}{2}}}{t^{m+2}} dt = \int_{\lambda}^{\beta} \frac{T^m}{t^{m+1}} dt \tag{2.2.54}$$

$$\leq \frac{T^m}{m\lambda^m} = \frac{T^m}{m(A_m(q)/T^{\epsilon_2})^{\frac{m^2}{m+1}}} = \frac{T^{m+\frac{\epsilon_2 m^2}{m+1}}}{mA_m(q)^{\frac{m^2}{m+1}}} \tag{2.2.55}$$

□

Proof of Proposition 2.1.6. We first conclude the estimate for the volume of $V_{q,T}[g_1, g_2]$. By combining Lemmata 2.2.5 - 2.2.7, we conclude that there exists $\kappa > 0$ and $C_q > 0$ (which can be explicitly determined by optimizing ϵ_1 and ϵ_2) such that whenever g_1, g_2 vary in a bounded set, we have

$$\mu(V_{q,T}[g_1, g_2]) \underbrace{=}_{(2.2.16)} \frac{v_m}{|G_4|^m|\det(H_1)|} \int_{\alpha_{q,T}}^{\beta_{q,T}} \frac{(-B_1^2 t^{\frac{2}{m}+2} + t^{\frac{2}{m}} T^2 - A_m(q)^2)^{\frac{m}{2}}}{t^{m+2}} dt$$

$$= \left(\int_{\alpha_{q,T}}^{\alpha_\delta(q,T,\epsilon_1)} + \int_{\alpha_\delta(q,T,\epsilon_1)}^{\lambda(q,T,\epsilon_2)} + \int_{\lambda(q,T,\epsilon_2)}^{\beta_{q,T}} \right) \frac{v_m}{|G_4|^m|\det(H_1)|} \int_{\alpha_{q,T}}^{\beta_{q,T}} \frac{(-B_1^2 t^{\frac{2}{m}+2} + t^{\frac{2}{m}} T^2 - A_m(q)^2)^{\frac{m}{2}}}{t^{m+2}} dt$$

$$\underbrace{=}_{\text{Lemmata } 2.2.5-2.2.7} \frac{v_m}{|G_4|^m|\det(H_1)|} \frac{m\Gamma(\frac{m}{2}+1)\Gamma(\frac{m^2}{2})}{2\Gamma(\frac{m^2}{2}+\frac{m}{2}+1)} \frac{T^{m(m+1)}}{A_m(\gamma)^{m^2}} + C_\gamma O(T^{m(m+1)-\kappa}).$$

Now we compute the total volume $\mu\left(H_T[g_1, g_2]\right)$.

$$\mu\left(H_T[g_1, g_2]\right) = \sum_{\substack{q\in\Delta,\\ A_m(q)^2<M_T}} \mu\left(V_{q,T}[g_1, g_2]\right) \tag{2.2.56}$$

$$\underbrace{=}_{\text{Lemma } 2.2.4 \text{ and } (2.2.37)} \sum_{\substack{q\in\Delta,\\ \|q\|<C\sqrt{T}}} \mu\left(V_{q,T}[g_1, g_2]\right) + O(T^{m(m+1)-m/2}). \tag{2.2.57}$$

As above, we split the computation of the volume of $V_{q,T}[g_1, g_2]$ into the sum of the three integrals

$$\left(\int_{\alpha_{q,T}}^{\alpha_\delta(q,T,\epsilon_1)} + \int_{\alpha_\delta(q,T,\epsilon_1)}^{\lambda(q,T,\epsilon_2)} + \int_{\lambda(q,T,\epsilon_2)}^{\beta_{q,T}} \right) \frac{v_m}{|G_4|^m|\det(H_1)|} \int_a^b \frac{(-B_1^2 t^{\frac{2}{m}+2} + t^{\frac{2}{m}} T^2 - A_m(q)^2)^{\frac{m}{2}}}{t^{m+2}} dt.$$

But now, in order to conclude the volume estimate (2.1.20), we need to use Lemma 2.2.1 when summing over $\gamma \in \Delta$ the estimates depending on $\|\gamma\|$ that appear in lemmata 2.2.5 - 2.2.7. For the summation over γ of the estimates in lemmata 2.2.5 - 2.2.6, note that by Lemma 2.2.1, the infinite sum $\sum_{q\in\Delta}\frac{1}{\|q\|^{m^2}}$ converges. For the estimate appearing in Lemma 2.2.7, observe that

$$\sum_{\substack{q\in\Delta,\\ \|q\|<C\sqrt{T}}}\frac{T^{m+\frac{\epsilon_2 m^2}{m+1}}}{A_m(q)^{\frac{m^2}{m+1}}} \tag{2.2.58}$$

$$=T^{m+\frac{\epsilon_2 m^2}{m+1}}O(\sqrt{T}^{m(m-1)-\frac{m^2}{m+1}}) \tag{2.2.59}$$

$$=O(T^{0.5(m^2+m)-\frac{(0.5-\epsilon_2)m^2}{m+1}}). \tag{2.2.60}$$

As we may choose any ϵ_1,ϵ_2 with $0<\epsilon_1<2\epsilon_2<2$ (this restriction appears only in Lemma 2.2.6), it follows that there exists a $\kappa>0$ such that

$$H_T[g_1,g_2]=\frac{v_m}{G_4^m|\det(H_1)|}\frac{m\Gamma(\frac{m}{2}+1)\Gamma(\frac{m^2}{2})}{2\Gamma(\frac{m^2}{2}+\frac{m}{2}+1)}T^{m(m+1)}\sum_{q\in\Delta}\frac{1}{A_m(q)^{m^2}}+O(T^{m(m+1)-\kappa}).$$

To obtain the exact expressions appearing in Proposition 2.1.6, recall that $A_m(q)=\|G_1qH_1\|$ and note that ν_m the volume of the unit ball in m-space is $\frac{\pi^{\frac{m}{2}}}{\Gamma(\frac{m}{2}+1)}$ □

2.3 Proof of equidistribution along skewed H-balls

We start by reducing Theorem 2.1.9 to the following equidistribution statement along skewed balls of the connected component of H. Consider for $g_1,g_2\in\mathrm{SL}(m+1,\mathbb{R})$ and $T>0$ the measures

$$\mu^{\circ}_{T,g_1,g_2}(F):=\frac{1}{\mu(V_T[g_1,g_2])}\int_{V_T[g_1,g_2]}F(a_t^{-1}u_v^{-1}g_1\Gamma)dv\frac{1}{t^{m+2}}dt. \tag{2.3.1}$$

Theorem 2.3.1. *For all $g_1,g_2\in\mathrm{SL}(m+1,\mathbb{R})$ and all $F\in C_c(X)$ it holds that*

$$\lim_{T\to\infty}\mu^{\circ}_{T,g_1,g_2}(F)=\mu_X(F).$$

Proof of Theorem 2.1.9 assuming Theorem 2.3.1. In view of (2.1.17), we can decompose $\mu_{T,g_1,g_2}(F)$ as the following convex linear combination:

$$\mu_{T,g_1,g_2}(F) = \sum_{q\in\Delta} \frac{\mu\left(V_{q,T}[g_1,g_2]\right)}{\mu\left(H_T[g_1,g_2]\right)} \frac{1}{\mu\left(V_{q,T}[g_1,g_2]\right)} \int_{V_{q,T}[g_1,g_2]} F(q^{-1}a_t^{-1}u_v^{-1}g_1\Gamma)\frac{1}{t^{m+2}}dvdt \quad (2.3.2)$$

Notice that

$$\mu(V_{q,T}[g_1,g_2]) = \mu(V_T[g_1,qg_2]), \quad (2.3.3)$$

and observe that

$$\frac{1}{\mu\left(V_{q,T}[g_1,g_2]\right)} \int_{V_{q,T}[g_1,g_2]} F(q^{-1}a_t^{-1}u_v^{-1}g_1\Gamma)\frac{1}{t^{m+2}}dvdt = \mu^{\circ}_{T,g_1,qg_2}(L_{q^{-1}}(F)). \quad (2.3.4)$$

By assuming Theorem 2.3.1, we have for all $q \in \Delta$,

$$\lim_{T\to\infty} \mu^{\circ}_{T,g_1,qg_2}(L_{q^{-1}}(F)) = \mu_X(L_{q^{-1}}(F)) \underbrace{=}_{\text{invariance}} \mu_X(F).$$

We denote

$$c_{q,T} := \frac{\mu\left(V_{q,T}[g_1,g_2]\right)}{\mu\left(H_T[g_1,g_2]\right)}. \quad (2.3.5)$$

Then clearly,

$$\sum_{q\in\Delta} c_{q,T} = 1,\ \forall T,\ \text{ and } c_{q,T} \leq 1,\ \forall q, T.$$

Importantly, by Lemma 2.2.4 and by (2.1.20) there is $C > 0$ such that for all $T > 0$

$$c_{q,T} \leq \frac{C}{\|q\|^{m^2}}.$$

By Lemma 2.2.1, $\sum_{q\in\Delta} \frac{1}{\|q\|^{m^2}}$ converges. Then for arbitrary small $\epsilon > 0$, we may find N_ϵ such that

$$\sum_{\substack{q\in\Delta,\\ \|q\|>N_\epsilon}} c_{q,T} \leq \epsilon,\ \forall T > 0.$$

Thus,

$$\begin{aligned}
|\mu_{T,g_1,g_2}(F) - \mu_X(F)| &= \left| \sum_{q\in\Delta} c_{q,T}(\mu^{\circ}_{T,g_1,qg_2}(F) - \mu_X(F)) \right| &(2.3.6)\\
&\leq \sum_{q\in\Delta} c_{q,T} \left| \mu^{\circ}_{T,g_1,qg_2}(F) - \mu_X(F) \right| &(2.3.7)\\
&\leq \sum_{\substack{q\in\Delta,\\ \|q\|\leq N_\epsilon}} \left| \mu^{\circ}_{T,g_1,qg_2}(F) - \mu_X(F) \right| + 2\epsilon\|F\|_\infty. &(2.3.8)
\end{aligned}$$

Then, by assuming Theorem 2.3.1, we get that

$$\limsup_{T\to\infty} |\mu_{T,g_1,g_2}(F) - \mu_X(F)| \leq 2\epsilon\|F\|_\infty,$$

which concludes our proof. □

In the rest of the section we will be proving Theorem 2.3.1. Let $\overline{X} = \overline{G/\Gamma}$ denote the one-point compactification of $X = G/\Gamma$. Our plan is to show that if a finite measure η on $\overline{X}$ is a weak-* limit of the measures μ°_{T,g_1,g_2}, $T > 0$ (recall that by the Banach-Alaoglu theorem, there's always a weak-* limit)[8], then

(1) There's no escape of mass, namely $\eta(\infty) = 0$, and

(2) η is G-invariant.

As a result we must have $\eta|_X = \mu_X$ the unique G-invariant probability on X.

There are essentially two key facts that stand behind our proof of the above statements. The first is that the measures μ°_{T,g_1,g_2} involve integration along families of polynomial trajectories, which allows to utilize deep results due to Shah on the behavior of polynomial trajectories in G/Γ. We note that those results are generalization of the celebrated results of Dani-Margulis [DM93] building on the linearisation technique. The second key fact is that η is invariant by a unipotent group, see Theorem 2.3.2. This opens the way for the application

[8] recall if any convergent subsequence of a bounded sequence converges to the same limit, then so does the original bounded sequence.

of Ratner's theorems on measure rigidity [Rat91] in combination with the results of Shah. We note that our apporach is similar to the one taken by Gorodnik in [Gor03]. The essential difference between our approach compared to [Gor03] is in our treatment of the divergence of the polynomial trajectories in the representation space, see Section 2.3.6.

2.3.1 The U-invariance of the measure along skewed balls of V

The main goal here is to prove that any weak-* limit of μ°_{T,g_1,g_2} is $U := \begin{bmatrix} I_m & 0 \\ \mathbb{R}^m & 1 \end{bmatrix}$-invariant. For any $u_0 \in U$, consider $L_{u_0}(F)(x) := F(u_0^{-1}x)$.

Theorem 2.3.2. *For all $g_1, g_2 \in \mathrm{SL}(m+1,\mathbb{R})$ and all $F \in C_c(X)$ it holds that*

$$\lim_{T\to\infty} \left(\mu^{\circ}_{T,g_1,g_2}(F) - \mu^{\circ}_{T,g_1,g_2}(L_{u_0}(F)\right) = 0. \tag{2.3.9}$$

The following lemma will be needed:

Lemma 2.3.3. *Let $f : \mathbb{R}^d \to \mathbb{R}$ be a bounded continuous function. Let $E \subset \mathbb{R}^d$ be an ellipsoid with surface area S, then for $y \in \mathbb{R}^d$ we have the estimate:*

$$\left| \int_E [f(v) - f(y+v)]dv \right| \leq \|f\|_\infty \|y\| S. \tag{2.3.10}$$

Proof. Indeed,

$$\begin{aligned} \left| \int_E [f(v) - f(y+v)]dv \right| &= \left| \int_E f(v)dv - \int_E f(v)dv \right| \\ &\leq \int_{E\triangle E-y} |f(v)|dv \\ &\leq \|f\|_\infty \|y\| S, \qquad (f \text{ is bounded}) \end{aligned}$$

where the last line follows from the Theorem 1 of [Sch10]. □

Proof of Theorem 2.3.2. Recall that

$$V_T[g_1, g_2] = \{u_v a_t : t > 0,\ v \in D_{T,t}\},$$

where $D_{T,t} := D_{I_m,T,t}$ which is given by (2.2.6). We denote $\alpha_T := \alpha_{I_m,T}$ and $\beta_T := \beta_{I_m,T}$ the roots of $-B_1^2 t^{\frac{2}{m}+2} + t^{\frac{2}{m}} T^2 - A_m(q)^2$ which determine $D_{T,t}$, see Section 2.2. Let $F \in C_c(X)$ and fix $u_{v_0} = \begin{bmatrix} I_m & 0 \\ v_0 & 1 \end{bmatrix} \in U$. We have

$$
\begin{aligned}
&\left| \int_{V_T[g_1,g_2]} F(h^{-1}g_1\Gamma)d\mu(h) - \int_{V_T[g_1,g_2]} F(u_{v_0}^{-1}h^{-1}g_1\Gamma)d\mu(h) \right| \\
=&\left| \int_{V_T[g_1,g_2]} \left(F(h^{-1}g_1\Gamma) - F(u_{v_0}^{-1}h^{-1}g_1\Gamma) \right) d\mu(h) \right| \\
=&\left| \int_{\alpha_T}^{\beta_T} \int_{D_{T,t}} \left(F(a_t^{-1}u_v^{-1}g_1\Gamma) - F(u_{v_0}^{-1}a_t^{-1}u_v^{-1}g_1\Gamma) \right) dv \frac{1}{t^{m+2}} dt \right| \\
=&\left| \int_{\alpha_T}^{\beta_T} \int_{D_{T,t}} \left(F(a_t^{-1}u_v^{-1}g_1\Gamma) - F(a_t^{-1}[a_t u_{v_0}^{-1}a_t^{-1}]u_v^{-1}g_1\Gamma) \right) dv \frac{1}{t^{m+2}} dt \right| \\
=&\left| \int_{\alpha_T}^{\beta_T} \int_{D_{T,t}} \left(F(a_t^{-1}u_{-v}g_1\Gamma) - F(a_t^{-1}u_{-t^{\frac{1}{m}+1}v_0-v}g_1\Gamma) \right) dv \frac{1}{t^{m+2}} dt \right| \\
=&\left| \left(\int_{\alpha_T}^{c_T} + \int_{c_T}^{\beta_T} \right) \int_{D_{T,t}} \left(F(a_t^{-1}u_v^{-1}g_1\Gamma) - F(a_t^{-1}u_{-t^{\frac{1}{m}+1}v_0-v}g_1\Gamma) \right) dv \frac{1}{t^{m+2}} dt \right|,
\end{aligned}
$$

where the auxiliary parameter c_T is chosen by

$$c_T := \frac{1}{T^{\frac{m}{2}}}. \tag{2.3.11}$$

To estimate the integral $\int_{c_T}^{\beta_T}$ we use the trivial bound

$$
\begin{aligned}
&\left| \int_{D_{T,t}} \left(F(a_t^{-1}u_{-v}g_1\Gamma) - F(a_t^{-1}u_{-t^{\frac{1}{m}+1}v_0 q^{-1}-v}g_1\Gamma) \right) dv \right| \\
\leq& \int_{D_{T,t}} 2\|F\|_\infty dv = 2\|F\|_\infty v_m \frac{(-B_1^2 t^{\frac{2}{m}+2} + t^{\frac{2}{m}} T^2 - A_m^2)^{\frac{m}{2}}}{G_4^m |\det(H_1)|}, \qquad \text{(by (2.2.10))}
\end{aligned}
$$

so that the integral $\int_{c_T}^{\beta_T}$ will be bounded by

$$
\begin{aligned}
&\int_{c_T}^{\beta_T} 2\|F\|_\infty v_m \frac{(-B_1^2 t^{\frac{2}{m}+2} + t^{\frac{2}{m}} T^2 - A_m^2)^{\frac{m}{2}}}{G_4^m |\det(H_1)|} \frac{1}{t^{m+2}} dt \\
\ll& \int_{c_T}^{\beta_T} \frac{T^m}{t^{m+1}} dt \\
\ll& T^{\frac{m^2}{2}+m}. \qquad \text{(using (2.3.11))}
\end{aligned}
$$

After normalization by $V_T[g_1,g_2] \asymp T^{m(m+1)}$, this integral vanishes as $T \to \infty$.

Now it remains to estimate the integral $\int_{\alpha_T}^{c_T}$. For that we use Lemma 2.3.3.

$$\begin{aligned}
&\left|\int_{\alpha_T}^{c_T}\int_{D_{T,t}}\left(F(a_t^{-1}u_{-v}g_1\Gamma)-F(a_t^{-1}u_{-t^{\frac{1}{m}+1}v_0-v}g_1\Gamma)\right)dv\frac{1}{t^{m+2}}dt\right| \\
\leq&\|F\|_\infty\int_{\alpha_T}^{c_T}\left(\|-t^{\frac{1}{m}+1}v_0\|\sqrt{\frac{-B_1^2t^{\frac{2}{m}+2}+t^{\frac{2}{m}}T^2-A_m^2}{G_4^m|\det(H_1)|}}^{m-1}\right)\frac{1}{t^{m+2}}dt \qquad \text{(by Lemma 2.3.3)} \\
=&\|F\|_\infty\int_{\alpha_T}^{c_T}\|t^{\frac{1}{m}-m-1}v_0\|\sqrt{\frac{-B_1^2t^{\frac{2}{m}+2}+t^{\frac{2}{m}}T^2-A_m^2}{G_4^m|\det(H_1)|}}^{m-1}dt \\
\leq&\|F\|_\infty\int_{\alpha_T}^{c_T}\|\alpha_T^{\frac{1}{m}-m-1}v_0\|\sqrt{\frac{c_T^{\frac{2}{m}}T^2}{G_4^m|\det(H_1)|}}^{m-1}dt \\
\ll&\ c_T\alpha_T^{\frac{1}{m}-m-1}(c_T^{\frac{1}{m}}T)^{m-1} \\
\ll&T^{m(m+1)-\frac{5}{2}}. \qquad \text{(using (2.2.18) the range for the root } a)
\end{aligned}$$

Again, this term goes to zero as $T\to\infty$, after normalization by $V_T[g_1,g_2]\asymp T^{m(m+1)}$. $\square$

2.3.2 The non-escape of mass

Recall $G=\mathrm{SL}(m+1,\mathbb{R})$ and $\Gamma\leq G$ is a lattice. Let $\mathfrak{g}$ be the Lie algebra of G. For positive integers d and n, denote by $\mathcal{P}_{d,n}(G)$ the set of functions $\varphi:\mathbb{R}^n\to G$ such that for any $a,b\in\mathbb{R}^n$, the map

$$\tau\in\mathbb{R}\mapsto\mathrm{Ad}(\varphi(\tau a+b))\in\mathfrak{g} \tag{2.3.12}$$

is a polynomial of degree at most d with respect to some basis of $\mathfrak{g}$.

Let $\mathcal{V}_G=\sum_{i=1}^{\dim\mathfrak{g}}\bigwedge^i\mathfrak{g}$. There is a natural action of G on $\mathcal{V}_G$ induced from the adjoint representation (in other words, we are considering a representation $\pi:G\to\mathrm{GL}(\mathcal{V}_G)$ but sometimes omit the symbol π). Fix a norm $\|\cdot\|$ on $\mathcal{V}_G$. For a Lie subgroup H of G with Lie algebra $\mathfrak{h}$, take a unit vector $p_{\mathrm{H}}\in\bigwedge^{\dim\mathfrak{h}}\mathfrak{h}$.

Theorem 2.3.4 (Special case of the theorems 2.1 and 2.2 in [Sha96], combined)**.** *With notations above, there exist closed subgroups $U_i (i=1,2,...,l)$ such that each U_i is the unipotent* *radical of a parabolic subgroup, $U_i\Gamma$ is compact in $X = G/\Gamma$ and for any $d, n \in \mathbb{N}$, $\epsilon, \delta > 0$, there exists a compact set $C \subset G/\Gamma$ such that for any $\varphi \in \mathcal{P}_{d,n}(G)$ and a bounded open convex set $D \subset \mathbb{R}^n$, one of the following holds:*

(1) there exist $\gamma \in \Gamma$ and $i = 1, ..., l$ such that $\sup_{v\in D} \|\varphi(v)\gamma.p_{U_i}\| \leq \delta$;

(2) $\mathrm{Vol}(v \in D : \varphi(v)\Gamma \notin C) < \epsilon \mathrm{Vol}(D)$, where Vol is the Lebesgue measure on $\mathbb{R}^n$.

Fix $g_1, g_2 \in \mathrm{SL}(m+1, \mathbb{R})$. We shall consider the family of polynomial maps which appear in the integration along the measures μ°_{T,g_1,g_2} which are given by

$$\varphi_t(v) := a_t^{-1} u_v^{-1} g_1, \tag{2.3.13}$$

where $t > 0$ and $v \in \mathbb{R}^m$. For each fixed t, the map $\varphi_t(\cdot)$ is in $\mathcal{P}_{2,m}(G)$. Our strategy is to investigate how $\varphi_t(v)$ fails the condition 1 of the Theorem 2.3.4 by studying the expanding phenomenon of the map $\varphi_t(v)$. This will be the key fact which will allow to prove the non-escape of mass.

Denote by $\mathcal{H}_\Gamma$ the family of all proper closed connected subgroups H of G such that $\Gamma \cap \mathrm{H}$ is a lattice in H, and $\mathrm{Ad}(\mathrm{H} \cap \Gamma)$ is Zariski-dense in $\mathrm{Ad}(\mathrm{H})$. We have the following important theorem:

Theorem 2.3.5 (Theorem 1.1 [Rat91], Theorems 2.1 and 3.4 [DM93]. see also Section 3 and Proposition 4.1 in [Sha96])**.** *The set $\mathcal{H}_\Gamma$ is countable. For any $\mathrm{H} \in \mathcal{H}_\Gamma$, $\Gamma.p_{\mathrm{H}}$ is discrete.*

Consider the finite set of unipotent radicals of parabolic subgroups, denoted $U_1, ..., U_l$ appearing in Theorem 2.3.4. By Theorem 2.3.5, we have that $\Gamma.p_{U_i}$ is discrete in $\mathcal{V}_G$ for all i.

Write $\mathcal{V}_G = \mathcal{V}_0 \bigoplus \mathcal{V}_1$, where $\mathcal{V}_0$ is the space of vectors fixed by G and $\mathcal{V}_1$ is its G-invariant complement (exists because every finite-dimensional representation of a semisimple Lie group is completely reducible). Denote by Π the projection of $\mathcal{V}_G$ onto $\mathcal{V}_1$ with kernel $\mathcal{V}_0$, and note that by Lemma 17 in [Gor03] it holds that $\Pi(\Gamma.p_{U_i})$ is discrete for all i. Since p_{U_i} is not fixed by G (this is because the action is through conjugation and U_i's are not normal subgroups in the simple group G), $\Pi(\Gamma.p_{U_i})$ does not contain 0. So it follows that

$$\inf_{x\in\cup_{i=1}^{l}\Gamma.p_{U_i}} \|\Pi(x)\| := r > 0. \tag{2.3.14}$$

The following lemma will play a crucial role in proving $\eta(\infty) = 0$:

Lemma 2.3.6. *Let $B \subseteq \mathbb{R}^m$ be a bounded set, and let $\chi \in (0,1)$. Then for any $\nu > 0$, there exist $t_0 > 0$ such that for any $0 < t < t_0$, any $\xi \in B$ and any $x \in \mathcal{V}_G$ such that $\|\Pi(x)\| \geq r$, it holds*

$$\sup_{v\in D\left(t^{\frac{1-\chi}{m}}\right)+\xi} \|a_t^{-1}u_v^{-1}g_1.x\| > \nu, \tag{2.3.15}$$

where $D(\beta)$ is the ball of radius β in $\mathbb{R}^m$ centered at the origin.

The idea of the proof of the lemma to decompose the elements $a_t^{-1}u_v^{-1}$ into m elements, each belongs to a subgroup isomorphic to $\mathrm{SL}(2,\mathbb{R})$ and then use the representation theory of $\mathrm{SL}(2,\mathbb{R})$ for each component. We note that a similar approach was used in [KW06].

Let $E_{i,j}$ be the $(m+1)\times(m+1)$ matrix where the (i,j)-th entry is 1 while all other entries are zero. Consider the following copy of $\mathrm{SL}(2,\mathbb{R})$ in $\mathrm{SL}(m+1,\mathbb{R})$

$$\mathrm{SL}^{(j)}(2,\mathbb{R}) := \{I_{m+1}+(a-1)E_{jj}+bE_{m+1,j}+cE_{j,m+1}+(d-1)E_{m+1,m+1} : ad-bc=1\}, \tag{2.3.16}$$

for $j = 1, 2, ..., m$. We will denote

$$\pi^{(j)}\begin{bmatrix} a & b \\ c & d \end{bmatrix} := I_{m+1} + (a-1)E_{jj} + bE_{m+1,j} + cE_{j,m+1} + (d-1)E_{m+1,m+1}. \tag{2.3.17}$$

We have the following observation which we leave the reader to verify.

Lemma 2.3.7. *Let $\sigma : \{1, 2, \dots m\} \to \{1, 2, \dots, m\}$ be a permutation. Then it holds that*

$$a_t^{-1} u_v^{-1} = \prod_{j=1}^{m} \pi^{(\sigma(j))} \left(\begin{bmatrix} t^{\frac{1}{m}} & 0 \\ 0 & t^{-\frac{1}{m}} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -t^{-\frac{m-j}{m}} v_{\sigma(j)} & 1 \end{bmatrix} \right), \tag{2.3.18}$$

where $v = (v_1, ..., v_m)$ and $t > 0$.

The decomposition (2.3.18) reduces the proof of Lemma 2.3.6 to the study of the expansion of elements of the form

$$\begin{bmatrix} t^{\frac{1}{m}} & 0 \\ 0 & t^{-\frac{1}{m}} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ y & 1 \end{bmatrix}$$

in representations of $\mathrm{SL}(2, \mathbb{R})$. In the following we review some of the basic facts on the $\mathrm{SL}(2, \mathbb{R})$-irreducible representations, and then we will proceed with the proof of Lemma 2.3.6.

Recall that if π is an $(n+1)$-dimensional irreducible representation of $\mathrm{SL}(2, \mathbb{R})$ and π' is the induced Lie algebra representation, then there exists a basis $v_0, v_1, ..., v_n$ such that

$$\pi'(H)(v_i) = (n - 2i)v_i, i = 0, 1, ..., n; \tag{2.3.19}$$

$$\pi'(X)(v_i) = i(n - i + 1)v_{i-1}, i = 0, 1, ..., n; \tag{2.3.20}$$

$$\pi'(Y)(v_i) = v_{i+1}, i = 0, 1, ..., n \ (v_{n+1} = 0). \tag{2.3.21}$$

where $H = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, X = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, and $Y = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ form a generating set of $\mathfrak{sl}(2, \mathbb{R})$.

Under the matrix Lie group-Lie algebra correspondence,

$$\pi \left(\begin{bmatrix} t^{\frac{1}{m}} & 0 \\ 0 & t^{-\frac{1}{m}} \end{bmatrix} \right) = \exp(\pi'(\log(t^{\frac{1}{m}})H))$$

has the matrix representation

$$\begin{bmatrix} t^{\frac{n}{m}} & & & \\ & t^{\frac{n-2}{m}} & & \\ & & \ddots & \\ & & & t^{-\frac{n}{m}} \end{bmatrix} \tag{2.3.22}$$

and

$$\pi \left(\begin{bmatrix} 1 & 0 \\ -y & 1 \end{bmatrix} \right) = \exp(\pi'(-yY))$$

has the matrix representation

$$\begin{bmatrix} 1 & & & \\ p_{21}(y) & 1 & & \\ \vdots & \ddots & \ddots & \\ p_{nn}(y) & \cdots & p_{n1}(y) & 1 \end{bmatrix} \tag{2.3.23}$$

where $p_{kl}(y)$ is a monomial in y of degree l. Both matrices are under the basis $v_0, v_1, ..., v_n$. Observe that the line

$$W := \mathbb{R}v_n$$

is the fixed subspace of $\pi\left(\begin{bmatrix} 1 & 0 \\ -y & 1 \end{bmatrix}\right)$. This space is important for us since it's the eigenspace of the matrices $\pi\left(\begin{bmatrix} t^{\frac{1}{m}} & 0 \\ 0 & t^{-\frac{1}{m}} \end{bmatrix}\right)$ which corresponds to the largest eigenvalue as $t \to 0$. We denote by pr_W the orthogonal projection on W with respect to some scalar product on $\mathcal{V}_n$.

The following lemma is a special case of [Gor03, Lemma 13], which is essentially [Sha96, Lemma 5.1].

Lemma 2.3.8. *Let $\pi : \mathrm{SL}(2, \mathbb{R}) \to \mathcal{V}_n$ be the $n+1$-dimensional irreducible representation of $\mathrm{SL}(2, \mathbb{R})$, and let*

$$\Theta(y) := \pi\left(\begin{bmatrix} 1 & 0 \\ -y & 1 \end{bmatrix}\right).$$

Fix a bounded interval I. Then there exists a constant $c_0 > 0$ such that for any $\beta \in (0, 1)$, $\tau \in \mathrm{I}$ and $\mathbf{x} \in \mathcal{V}_n$,

$$\sup_{n \in \Theta([0,\beta]+\tau)} \|\mathrm{pr}_W(n\mathbf{x})\| \geq c_0 \beta^n \|\mathbf{x}\|. \tag{2.3.24}$$

We have the following corollary.

Corollary 2.3.9. *Suppose that $\pi : \mathrm{SL}(2, \mathbb{R}) \to \mathcal{V}$ is a representation, fix an interval I and let $\chi \in (0, 1)$. Then for all $t \in (0, 1)$ the following holds.*

(1) If π is irreducible, then for all $\tau \in \mathrm{I}$ it holds that

$$\sup_{y \in [0, t^{\frac{1-\chi}{m}}]+\tau} \left\| \pi\left(\begin{bmatrix} t^{\frac{1}{m}} & 0 \\ 0 & t^{-\frac{1}{m}} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -y & 1 \end{bmatrix}\right) \mathbf{x} \right\| \gg t^{-\frac{\chi}{m}} \|\mathbf{x}\|, \tag{2.3.25}$$

for all $\mathbf{x} \in \mathcal{V}$, where the implied constant depends on π only.

(2) If π is any representation, then for all $\tau \in \mathrm{I}$

$$\sup_{y\in[0,t^{\frac{1-\chi}{m}}]+\tau} \left\| \pi\left(\begin{bmatrix} t^{\frac{1}{m}} & 0 \\ 0 & t^{-\frac{1}{m}} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -y & 1 \end{bmatrix} \right) \mathbf{x} \right\| \gg \|\mathbf{x}\|, \tag{2.3.26}$$

for all $\mathbf{x} \in \mathcal{V}$, where the implied constant depends on π only.

Proof. In the following, for convenience, we will omit the representation symbol π. Suppose that $\mathcal{V}$ is the $n+1$'th irreducible representation. Then for all $y \in \mathbb{R}$ and all $\mathbf{x} \in \mathcal{V}$

$$\begin{aligned} & \left\| \begin{bmatrix} t^{\frac{1}{m}} & 0 \\ 0 & t^{-\frac{1}{m}} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -y & 1 \end{bmatrix} .\mathbf{x} \right\| \\ \geq & C \left\| \mathrm{pr}_W \left(\begin{bmatrix} t^{\frac{1}{m}} & 0 \\ 0 & t^{-\frac{1}{m}} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -y & 1 \end{bmatrix} .\mathbf{x} \right) \right\| \quad \text{(by the boundedness of the linear operator } \mathrm{pr}_W) \\ = & C \left\| \begin{bmatrix} t^{\frac{1}{m}} & 0 \\ 0 & t^{-\frac{1}{m}} \end{bmatrix} .\mathrm{pr}_W \left(\begin{bmatrix} 1 & 0 \\ -y & 1 \end{bmatrix} .\mathbf{x} \right) \right\| \\ & \qquad \text{(since the matrix (2.3.22) commutes with } \mathrm{pr}_W = \mathrm{pr}_{\mathbb{R}e_n}) \\ = & C t^{-\frac{n}{m}} \left\| \mathrm{pr}_W \left(\Theta(y)\mathbf{x} \right) \right\| \qquad \text{(see the last component of (2.3.22))} \end{aligned}$$

Then, part 1 of our corollary follows by Lemma 2.3.8. To conclude part 2, note that we may always decompose $\mathcal{V}$ as $\mathcal{V}_0 \oplus \mathcal{V}_1$ where $\mathcal{V}_0$ is the space of fixed vectors, and $\mathcal{V}_1$ is an invariant complement. By further decomposing $\mathcal{V}_1$ into irreducible representations, and applying part 1 of the corollary in each component, we get the result. □

The proof for Lemma 2.3.6

Recall $\mathcal{V}_G = \mathcal{V}_0 \oplus \mathcal{V}_1$ where $\mathcal{V}_0$ is the space of $\mathrm{SL}(m+1,\mathbb{R})$-fixed vectors, and $\mathcal{V}_1$ is it's invariant complement. Note that in order to prove Lemma 2.3.6, it's enough to verify that for a fixed $r > 0$, a bounded set $B \subset \mathbb{R}^m$ and $\chi \in (0,1)$ it holds that

$$\min_{\mathbf{x}\in\mathcal{V}_1,\ \|\mathbf{x}\|\geq r} \ \sup_{v\in D\left(t^{\frac{1-\chi}{m}}\right)+\xi} \|a_t^{-1}u_v^{-1}\mathbf{x}\| \gg t^{-\chi/m},\ \forall \xi \in B \tag{2.3.27}$$

where $t \in (0,1)$. As $\mathcal{V}_1$ is invariant by each $\mathrm{SL}^{(j)}(2,\mathbb{R})$ action, we may further decompose $\mathcal{V}_1$ by

$$\mathcal{V}_1 = \mathcal{V}_0^{(j)} \oplus \mathcal{V}_1^{(j)} \tag{2.3.28}$$

where $\mathcal{V}_0^{(j)}$ is the subspace of fixed vectors of the action of $\mathrm{SL}^{(j)}(2,\mathbb{R})$ for $j = 1, 2, ..., m$ and $\mathcal{V}_1^{(j)}$ is its invariant complement. For any $\mathbf{x} \in \mathcal{V}_1$ we write

$$\mathbf{x} = \mathbf{x}_0^{(j)} + \mathbf{x}_1^{(j)}, j = 1, 2, ..., m. \tag{2.3.29}$$

where $\mathbf{x}_i^{(j)} \in \mathcal{V}_i^{(j)}, i = 1, 2, ..., m$ from above.

Lemma 2.3.10. *It holds that* $\inf_{\mathbf{x}\in\mathcal{V}_1, \|\mathbf{x}_1\|\geq r} \left(\|\mathbf{x}_1^{(1)}\| + \cdots + \|\mathbf{x}_1^{(m)}\|\right) = Lr$ *for some* $L > 0$

Proof. We recall that UA is an epimorphic group, cf. [SW00]. This simply means that if $\mathbf{x} \in \mathcal{V}_1$ is fixed by UA, then it's fixed by all of $\mathrm{SL}(m+1,\mathbb{R})$. But in such case, $\mathbf{x} = 0$, since $\mathcal{V}_1$ is an invariant complement of the fixed vectors. In particular, this implies that $\bigcap_{i=1}^m \mathcal{V}_0^{(i)} = \{0\}$. Since the sphere of radius 1 is compact and as $\bigcap_{i=1}^m \mathcal{V}_0^{(i)} = \{0\}$, we have

$$\inf_{\mathbf{x}_1\in\mathcal{V}_1, \|\mathbf{x}_1\|=1} \left(\|\mathbf{x}_1^{(1)}\| + \cdots + \|\mathbf{x}_1^{(m)}\|\right) = L > 0.$$

By scaling, it is easy to see for $r \geq 1$,

$$\begin{aligned}\inf_{\mathbf{x}_1\in\mathcal{V}_1, \|\mathbf{x}_1\|=r} \left(\|\mathbf{x}_1^{(1)}\| + \cdots + \|\mathbf{x}_1^{(m)}\|\right) =& r \inf_{\mathbf{x}_1\in\mathcal{V}_1, \|\mathbf{x}_1\|=1} \left(\|\mathbf{x}_1^{(1)}\| + \cdots + \|\mathbf{x}_1^{(m)}\|\right) \\ & \text{(by linearity and scaling)} \\ =& Lr > 0.\end{aligned}$$

□

Now take $\mathbf{x} \in \mathcal{V}_1$ and fix j such that $\|\mathbf{x}_1^{(j)}\| \geq Lr$. We pick a permutation $\sigma : \{1, ..., m\} \to \{1, ..., m\}$ such that $\sigma(m) = j$. Using Lemma 2.3.7 we write

$$a_t^{-1} u_v^{-1} \mathbf{x} = \prod_{l=1}^{m-1} \pi^{(\sigma(l))} \left(\begin{bmatrix} t^{\frac{1}{m}} & 0 \\ 0 & t^{-\frac{1}{m}} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -t^{-\frac{m-l}{m}} v_{\sigma(l)} & 1 \end{bmatrix} \right) \pi^{(j)} \left(\begin{bmatrix} t^{\frac{1}{m}} & 0 \\ 0 & t^{-\frac{1}{m}} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -v_j & 1 \end{bmatrix} \mathbf{x} \right). \tag{2.3.30}$$

Note that

$$\left(\left[0, \frac{1}{\sqrt{m}} t^{\frac{1-\chi}{m}}\right] + \mathrm{I}_1 \right) \times ... \times \left(\left[0, \frac{1}{\sqrt{m}} t^{\frac{1-\chi}{m}}\right] + \mathrm{I}_m \right) \subseteq D\left(t^{\frac{1-\chi}{m}}\right) + B,$$

for some intervals I_l, $1 \leq l \leq m$. Then (2.3.27) is obtained by applying Corollary 2.3.9 as follows: We decompose $\mathcal{V}_1^{(j)}$ into $\mathrm{SL}^{(j)}(2,\mathbb{R})$-irreducible representations, and we assume (without loss of generality, as all norms are equivalent) that our norm $\|\cdot\|$ on $\mathcal{V}_1$ is obtained by taking the sup norm with respect to a basis of $\mathcal{V}_1$ composed out of a basis for $\mathcal{V}_0^{(j)}$ and bases for each of the $\mathrm{SL}^{(j)}(2,\mathbb{R})$-irreducible spaces. Then, using Corollary 2.3.9 (1), we have for all $\tau_j \in \mathrm{I}_j$ that

$$\sup_{v_j \in [0, t^{\frac{1-\chi}{m}}]+\tau_j} \left\| \pi^{(j)} \left(\begin{bmatrix} t^{\frac{1}{m}} & 0 \\ 0 & t^{-\frac{1}{m}} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -y & 1 \end{bmatrix} \right) \mathbf{x} \right\| \gg t^{-\frac{\chi}{m}} r, \tag{2.3.31}$$

and by further applying Corollary 2.3.9,(2) when taking the supremum of (2.3.30) over the parameters $\left(\left[0, \frac{1}{\sqrt{m}} t^{\frac{1-\chi}{m}}\right] + \mathrm{I}_1\right) \times \ldots \times \left(\left[0, \frac{1}{\sqrt{m}} t^{\frac{1-\chi}{m}}\right] + \mathrm{I}_m\right)$, we obtain (2.3.27).

Proof of $\eta(\infty) = 0$

Recall that η is a weak-* limit of the measures

$$\mu^{\circ}_{T,g_1,g_2}(F) := \frac{1}{\mu\left(V_T[g_1,g_2]\right)} \int_{\alpha_T}^{\beta_T} \int_{D_{T,t}} F\left(a_t^{-1} u_v^{-1} g_1 \Gamma\right) dv \frac{1}{t^{m+2}} dt, \; F \in C_c(G/\Gamma), \tag{2.3.32}$$

where we recall that $D_{T,t}$ is the ellipsoid centered at $-G_4^{-1}G_3 - H_3 H_1^{-1} t^{\frac{1}{m}+1} \in \mathbb{R}^m$ given by

$$D_{T,t} := \{v \in \mathbb{R}^m : \|G_3 H_1 + G_4 v H_1 + G_4 H_3 t^{\frac{1}{m}+1}\|^2 \leq -B_1^2 t^{\frac{2}{m}+2} + t^{\frac{2}{m}} T^2 - A_m^2\}, \tag{2.3.33}$$

$A_m := A_m(I_m)$ which was defined in (2.2.7), and $0 < \alpha_T := \alpha_{I_m,T} < \beta_T := \beta_{I_m,T}$ are the two positive roots of $-B_1^2 t^{\frac{2}{m}+2} + t^{\frac{2}{m}} T^2 - A_m^2$. We note that $D_{T,t}$ contains the ball with the same center whose radius is equal to the *shortest radius* of the ellipse, which is

$$R_{T,t} := C_{g_1,g_2} \sqrt{-B_1^2 t^{\frac{2}{m}+2} + t^{\frac{2}{m}} T^2 - A_m^2},$$

for some constant $C_{g_1,g_2} > 0$. The displacement of the center from the origin has two parts: one part ($-G_4^{-1}G_3$) is constant and the other part ($-H_3 H_1^{-1} t^{\frac{1}{m}+1}$) involves t but is bounded when $t \in (0,1)$. To use Lemma 2.3.6 in our proof for $\eta(\infty) = 0$, we would like to have

$R_{T,t} > t^{\frac{1-\chi}{m}}$. But, when t approaches to either α_T or to β_T, $R_{T,t}$ becomes too small. Thus we need to truncate the range for t as following. Using Lemmata 2.2.5 and 2.2.7 together with the estimate of $\mu(V_T[g_1, g_2])$ in (2.1.19), we have that for $\epsilon_1, \epsilon_2 \in (0,1)$ there exists some $\kappa = \kappa(\epsilon_1, \epsilon_2) > 0$ such that for any fixed $F \in C_c(G/\Gamma)$ it holds

$$\mu^{\circ}_{T,g_1,g_2}(F) = \frac{1}{\mu\left(V_T[g_1,g_2]\right)} \int_{\alpha_\delta(T,\epsilon_1)}^{\lambda(T,\epsilon_2)} \int_{D_{T,t}} F\left(a_t^{-1} u_v^{-1} g_1 \Gamma\right) dv \frac{1}{t^{m+2}} dt + O(T^{-\kappa}). \tag{2.3.34}$$

Lemma 2.3.11. *Fix $\chi \in (0,1)$. Then, there exist $0 < \epsilon_1 < \epsilon_2 < 1$ such that for all T large enough, we have that for all $t \in (\alpha_\delta(T,\epsilon_1), \lambda(T,\epsilon_2))$ it holds that*

$$R_{T,t} > t^{\frac{1-\chi}{m}} \tag{2.3.35}$$

Proof. To prove the inequality, we estimate from below the left hand side, and estimate from above the right hand side. For the right hand side, we have for $t \in (\alpha_\delta(T,\epsilon_1), \lambda(T,\epsilon_2))$ that

$$\lambda(T,\epsilon_2)^{\frac{1-\chi}{m}} \geq t^{\frac{1-\chi}{m}}. \tag{2.3.36}$$

By definition of λ, see (2.2.34), we have

$$\lambda(T,\epsilon_2)^{\frac{1-\chi}{m}} \asymp \frac{1}{T^{\frac{\epsilon_2(1-\chi)}{m+1}}} \tag{2.3.37}$$

We now estimate $R_{T,t}$. As we already noted, the function $f(t) := -B_1^2 t^{\frac{2}{m}+2} + t^{\frac{2}{m}} T^2 - A_m^2$ has one critical point θ_T in the range $t > 0$, and $f(t)$ is monotonically increasing in $(0, \theta_T)$, where we recall that $\theta_T \asymp T$. Since $\lambda(T,\epsilon_2) = o(1)$ as $T \to \infty$, we conclude that for all $t \in (\alpha_\delta(T,\epsilon_1), \lambda(T,\epsilon_2))$ it holds

$$\sqrt{-B_1^2 t^{\frac{2}{m}+2} + t^{\frac{2}{m}} T^2 - A_m^2} \geq \sqrt{-B_1^2 \alpha_\delta(T,\epsilon_1)^{\frac{2}{m}+2} + \alpha_\delta(T,\epsilon_1)^{\frac{2}{m}} T^2 - A_m^2}. \tag{2.3.38}$$

Note that as $T \to \infty$ (by (2.2.33) and by (2.2.20))

$$\alpha_\delta(T,\epsilon_1)^{\frac{2}{m}} = \left(\alpha_T^{\frac{1}{m}} + \frac{A_m}{T^{1+\epsilon_1}}\right)^2 \asymp A_m^2 \left(\frac{1}{T^2} + \frac{2}{T^{2+\epsilon_1}}\right).$$

Then, as $T \to \infty$, we have

$$\sqrt{-B_1^2 \alpha_\delta(T,\epsilon_1)^{\frac{2}{m}+2} + \alpha_\delta(T,\epsilon_1)^{\frac{2}{m}} T^2 - A_m^2} \asymp \frac{1}{T^{\epsilon_1/2}}. \tag{2.3.39}$$

We are free to choose ϵ_1 and ϵ_2 in the interval $(0,1)$ as we wish. In particular, we as well may assume that

$$\epsilon_1/2 < \frac{\epsilon_2(1-\chi)}{m+1}.$$

With the latter choice, we obtain our claim by the estimates (2.3.39) and (2.3.37). □

We are now ready to prove $\eta(\infty) = 0$. By the above Lemma 2.3.11 and Lemma 2.3.6, we conclude that the first outcome in Theorem 2.3.4 for the translated ellipsoids $D = D_{T,t}$ fails in the range $t \in (\alpha_\delta(T,\epsilon_1), \lambda(T,\epsilon_2))$ for all T large enough. Here ϵ_1, ϵ_2 are fixed such that Lemma 2.3.11 holds. Thus, by the second outcome of Theorem 2.3.4, for arbitrarily small $\epsilon > 0$ there's a compact $C \subset G/\Gamma$ such that

$$\mathrm{Vol}(v \in D_{T,t} : \varphi_t(v)\Gamma \notin C) < \epsilon \mathrm{Vol}(D_{T,t}), \ \forall t \in (\alpha_\delta(T,\epsilon_1), \lambda(T,\epsilon_2)). \tag{2.3.40}$$

Now take $F \in C_c(G/\Gamma)$ with $\mathbf{1}_C \le F \le 1$. As $T \to \infty$,

$$\begin{aligned}
\mu^\circ_{T,g_1,g_2}(F) =& \frac{1}{\mu\left(V_T[g_1,g_2]\right)} \int_{V_T[g_1,g_2]} F(h^{-1}g_1\Gamma) d\mu(h) \\
=& \frac{1}{\mu\left(V_T[g_1,g_2]\right)} \int_{\alpha_\delta(T,\epsilon_1)}^{\lambda(T,\epsilon_2)} \int_{D_{T,t}} F\left(\varphi_t(v)\Gamma\right) dv \frac{1}{t^{m+2}} dt + O(T^{-\kappa}) \qquad \text{(by (2.3.34))} \\
\ge& \frac{1}{\mu\left(V_T[g_1,g_2]\right)} \int_{a_\delta(T,\epsilon_1)}^{\lambda(T,\epsilon_2)} \int_{D_{T,t}} \mathbf{1}_C\left(\varphi_t(v)\Gamma\right) dv \frac{1}{t^{m+2}} dt + O(T^{-\kappa}) \\
\ge& \frac{1}{\mu\left(V_T[g_1,g_2]\right)} \int_{a_\delta(T,\epsilon_1)}^{\lambda(T,\epsilon_2)} (1-\epsilon)\mathrm{Vol}(D_{T,t}) \frac{1}{t^{m+2}} dt + O(T^{-\kappa}) \qquad \text{(by (2.3.40))} \\
=& (1-\epsilon) \frac{1}{\mu\left(V_T[g_1,g_2]\right)} \int_{\alpha_T}^{\beta_T} \mathrm{Vol}(D_{T,t}) \frac{1}{t^{m+2}} dt + O(T^{-\kappa}) \\
& \qquad \text{(by Lemmata 2.2.5 and 2.2.7)} \\
=& 1 - \epsilon + O(T^{-\kappa}).
\end{aligned}$$

Therefore,

$$\liminf_{T\to\infty} \mu^\circ_{T,g_1,g_2}(F) \ge 1 - \epsilon \tag{2.3.41}$$

so that

$$\eta(\infty) \le \limsup_{T\to\infty} \mu^{\circ}_{T,g_1,g_2}(\mathrm{support}(F)^c) \le \epsilon \tag{2.3.42}$$

Since ϵ is arbitrary $\eta(\infty) = 0$.

2.3.3 Proof of G-invariance

Recall $U = \begin{bmatrix} I_m & 0 \\ \mathbb{R}^m & 1 \end{bmatrix}$. For a closed subgroup H of G, denote

$$N(\mathrm{H}, U) := \{g \in G : Ug \subset g\mathrm{H}\}, \tag{2.3.43}$$

$$S(\mathrm{H}, U) := \cup_{\mathrm{H}' \subsetneq \mathrm{H}, \mathrm{H}' \in \mathcal{H}_\Gamma} N(\mathrm{H}', U). \tag{2.3.44}$$

Consider,

$$Y := \bigcup_{\mathrm{H} \in \mathcal{H}_\Gamma} N(\mathrm{H}, U)\Gamma = \bigcup_{\mathrm{H} \in \mathcal{H}_\Gamma} [N(\mathrm{H}, U) - S(\mathrm{H}, U)]\Gamma \subset G/\Gamma, \tag{2.3.45}$$

where $\mathcal{H}_\Gamma$ was defined above Theorem 2.3.5. The equality holds since for any $g \in N(\mathrm{H}, U)$, if g is also in $S(\mathrm{H}, U)$, then g must belong to $N(\mathrm{H}', U)$ for some $\mathrm{H}' \subsetneq H$ (note for Lie subgroups, this condition means $\dim \mathrm{H}' < \dim \mathrm{H}$) and $\mathrm{H}' \in \mathcal{H}_\Gamma$. Since H' has strictly lower dimension, by repeating this argument we see eventually, g will fall into some $N(\tilde{\mathrm{H}}, U) - S(\tilde{\mathrm{H}}, U)$ (with $S(\tilde{\mathrm{H}}, U)$ possibly empty when $\tilde{\mathrm{H}}$ has minimal dimension).

Now we perform the ergodic decomposition of the U-invariant measure η. By Theorem 2.2 of [MS95], each ergodic component of η is either G-invariant or supported on $Y \cup \{\infty\}$. Thus, in order to show that η is G-invariant, it is sufficient to prove the following lemma.

Lemma 2.3.12. *$\eta(Y) = 0$.*

By the discreteness of $\mathcal{H}_\Gamma$ (Theorem 2.3.5), it suffices to show that for each fixed $\mathrm{H} \in \mathcal{H}_\Gamma$ it holds

$$\eta([N(\mathrm{H}, U) - S(\mathrm{H}, U)]\Gamma) = 0. \tag{2.3.46}$$

Since $[N(\mathrm{H},U)-S(\mathrm{H},U)]\Gamma$ is a countable union of compact subsets in G/Γ (See [MS95] Proposition 3.1), it suffices to show $\eta(C)=0$ for any compact subset C of $[N(\mathrm{H},U)-S(\mathrm{H},U)]\Gamma$.

The main tool in the proof of the latter statement is the following consequence of Proposition 5.4 in [Sha94]. We use in the following the same notations as in Section 2.3.2.

Theorem 2.3.13. *Let $d,n\in\mathbb{N},\epsilon>0,\mathrm{H}\in\mathcal{H}_\Gamma$. For any compact set $C\subset[N(\mathrm{H},U)-S(\mathrm{H},U)]\Gamma$, there exists a compact set $F\subset\mathcal{V}_G$ such that for any neighborhood Φ of F in $\mathcal{V}_G$, there exists a neighborhood Ψ of C in G/Γ such that for any $\varphi\in\mathcal{P}_{d,n}(G)$ and a bounded open convex set $D\subset\mathbb{R}^n$, one of the following holds:*

(1) There exist $\gamma\in\Gamma$ such that $\varphi(D)\gamma.p_{\mathrm{H}}\subset\Phi$.

(2) $\mathrm{Vol}(t\in D:\varphi(t)\Gamma\in\Psi)<\epsilon\mathrm{Vol}(D)$, where Vol is the Lebesgue measure on $\mathbb{R}^n$.

Fix $\epsilon>0$ and $\mathrm{H}\in\mathcal{H}_\Gamma$. Recall that $\varphi_t(\cdot)$ which was defined in (2.3.13) is in $\mathcal{P}_{2,m}$. Let $C\subset[N(\mathrm{H},U)-S(\mathrm{H},U)]\Gamma$ be a compact set and take a compact set $F\subset\mathcal{V}_G$ satisfying the outcome of Theorem 2.3.13. By the same argument as in the Section 4.3 with Lemma 2.3.6 applied, the first outcome of Theorem 2.3.13 fails for all T large enough. Then, for $t\in(\alpha_\delta(T,\epsilon_1),\lambda(T,\epsilon_2))$ for all T large enough,

$$\mathrm{Vol}(v\in D_{T,t}:\varphi_t(v)\Gamma\in\Psi)<\epsilon\mathrm{Vol}(D_{T,t}), \tag{2.3.47}$$

where $C \subseteq \Psi$ is the compact neighborhood from Theorem 2.3.13. Now let $f \in C_c(G/\Gamma)$ be such that $\mathbf{1}_C \leq f \leq 1$ and support$(f) \subset \Psi$, it follows that

$$
\begin{aligned}
\eta(C) \leq& \limsup_{T\to\infty} \mu^{\circ}_{T,g_1,g_2}(f) \\
=& \limsup_{T\to\infty} \frac{1}{\mu\left(V_T[g_1,g_2]\right)} \int_{\alpha_T}^{\beta_T} \int_{D_{T,t}} f\left(\varphi_t(v)\Gamma\right) dv \frac{1}{t^{m+2}} dt \\
=& \limsup_{T\to\infty} \frac{1}{\mu\left(V_T[g_1,g_2]\right)} \int_{\alpha_\delta(T,\epsilon_1)}^{\lambda(T,\epsilon_2)} \int_{D_{T,t}} f\left(\varphi_t(v)\Gamma\right) dv \frac{1}{t^{m+2}} dt \\
& \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(by Lemmata 2.2.5 and 2.2.7)} \\
\leq& \limsup_{T\to\infty} \frac{1}{\mu\left(V_T[g_1,g_2]\right)} \int_{\alpha_\delta(T,\epsilon_1)}^{\lambda(T,\epsilon_2)} \int_{D_{T,t}} \mathbf{1}_\Psi\left(\varphi_t(v)\Gamma\right) dv \frac{1}{t^{m+2}} dt \\
\leq& \limsup_{T\to\infty} \frac{1}{\mu\left(V_T[g_1,g_2]\right)} \int_{\alpha_\delta(T,\epsilon_1)}^{\lambda(T,\epsilon_2)} \epsilon \mathrm{Vol}(D_{T,t}) \frac{1}{t^{m+2}} dt \\
\leq& \epsilon.
\end{aligned}
$$

Since $\epsilon > 0$ is arbitrary, $\eta(C) = 0$ and thus $\eta(Y) = 0$.

2.4 Proof of Theorem 2.1.5: The limiting measure on $H\backslash G$.

We follow Section 2.5 of [GW04]. First, we provide an explicit measurable section

$$\sigma : H\backslash G \to Y \subset G,$$

and then we define a measure ν_Y on Y such that

$$dg = d\mu d\nu_Y, \tag{2.4.1}$$

where $d\mu$ is the left Haar measure on H given by (2.1.13) and dg is the Haar measure on G normalized such that $\mathrm{Vol}(G/\Gamma) = 1$

To define the section, let $\mathcal{F}_m \subset \begin{bmatrix} \mathrm{SL}(m,\mathbb{R}) & 0 \\ 0 & 1 \end{bmatrix}$ denote a measurable fundamental domain of

$$\begin{bmatrix} \Delta & 0 \\ 0 & 1 \end{bmatrix} \Big\backslash \begin{bmatrix} \mathrm{SL}(m,\mathbb{R}) & 0 \\ 0 & 1 \end{bmatrix} \Big/ \begin{bmatrix} \mathrm{SO}(m,\mathbb{R}) & 0 \\ 0 & 1 \end{bmatrix} \cong \Delta\backslash \mathrm{SL}(m,\mathbb{R})/\mathrm{SO}(m,\mathbb{R}), \tag{2.4.2}$$

and consider the product

$$Y := \mathcal{F}_m \cdot \mathrm{SO}(m+1,\mathbb{R}).$$

Then, we claim that the product map $H \times Y \to G$ is a Borel isomorphism, which defines a section σ identifying $H\backslash G$ with Y. The surjectivity is clear from the block-wise Iwasawa decomposition. We only verify the injectivity here:

Suppose $h_1 g_1 s_1 = h_2 g_2 s_2$ where $h_i \in H, g_i \in \mathcal{F}_m$ and $s_i \in \mathrm{SO}(m+1,\mathbb{R})$. Then

$$\begin{bmatrix} * & 0 \\ * & * \end{bmatrix} \ni g_2^{-1} h_2^{-1} h_1 g_1 = s_2 s_1^{-1} \in \mathrm{SO}(m+1,\mathbb{R}),$$

and it follows from the definition of $\mathrm{SO}(m+1,\mathbb{R})$ that both side must lie in $\begin{bmatrix} \mathrm{SO}(m,\mathbb{R}) & 0 \\ 0 & 1 \end{bmatrix}$. So $h_2^{-1} h_1 = g_2 s_2 s_1^{-1} g_1^{-1} \in \begin{bmatrix} \mathrm{SL}(m,\mathbb{R}) & 0 \\ 0 & 1 \end{bmatrix}$. But $\begin{bmatrix} \mathrm{SL}(m,\mathbb{R}) & 0 \\ 0 & 1 \end{bmatrix} \cap H = \begin{bmatrix} \Delta & 0 \\ 0 & 1 \end{bmatrix}$. Hence $q g_1 s_1 = g_2 s_2$ for some $q \in \begin{bmatrix} \Delta & 0 \\ 0 & 1 \end{bmatrix}$. It follows that $g_2^{-1} q g_1 = s_2 s_1^{-1} \in \begin{bmatrix} \mathrm{SL}(m,\mathbb{R}) & 0 \\ 0 & 1 \end{bmatrix} \cap \mathrm{SO}(m+1,\mathbb{R}) = \begin{bmatrix} \mathrm{SO}(m,\mathbb{R}) & 0 \\ 0 & 1 \end{bmatrix}$. Hence $g_1 = g_2, s_1 = s_2$ by the definition of fundamental domain, and $h_1 = h_2$.

Next, let $T := \left\{ \begin{bmatrix} t^{-\frac{1}{m}}\eta & 0 \\ v & t \end{bmatrix} : \eta \in \mathrm{SL}(m,\mathbb{R}),\ v \in \mathbb{R}^m,\ t > 0 \right\}$ and $S := \mathrm{SO}(m+1,\mathbb{R})$. On T we define the Haar measure $d\tau = dv \frac{dt}{t^{m+2}} d\eta$, where $d\eta$ is the Haar measure on $\mathrm{SL}(m,\mathbb{R})$ defined through standard Iwasawa decomposition, under which $\mathrm{SO}(m,\mathbb{R})$ has volume 1. We note the formula

$$\int_{\mathrm{SL}(m,\mathbb{R})} \varphi(\eta) d\eta = \int_{\mathcal{F}_m} \sum_{q \in \Delta} \int_{\mathrm{SO}(m,\mathbb{R})} \varphi(q\eta\rho') d\rho' d\eta. \tag{2.4.3}$$

On S we take $d\rho$ to be the Haar measure probability measure. Then, by using Theorem 8.32 of [Kna02], we unfold a Haar measure on G (with some implicit normalization) as follows

$$\begin{aligned}
&\int_G f(g)dg \\
&= \int_S \int_T f(\tau\rho)d\tau d\rho \\
&= \int_S \int_{\mathrm{SL}(m,\mathbb{R})} \int_0^\infty \int_{\mathbb{R}^m} f(u_v a_t \eta\rho)dv\frac{dt}{t^{m+2}}d\eta d\rho && \text{(by definition of } d\tau) \\
&= \int_S \sum_{q\in\Delta} \int_{\mathcal{F}_m} \int_{\mathrm{SO}(m,\mathbb{R})} \int_0^\infty \int_{\mathbb{R}^m} f(u_v a_t q\eta\rho'\rho)dv\frac{dt}{t^{m+2}}d\eta d\rho' d\rho && \text{(by formula (2.4.3))} \\
&= \int_S \sum_{q\in\Delta} \int_{\mathcal{F}_m} \int_0^\infty \int_{\mathbb{R}^m} f(u_v a_t q\eta\rho)dv\frac{dt}{t^{m+2}}d\eta d\rho && \text{(invariance of } d\rho) \\
&= \int_S \sum_{q\in\Delta} \int_0^\infty \int_{\mathbb{R}^m} \int_{\mathcal{F}_m} f(u_v a_t q\eta\rho)d\eta dv\frac{dt}{t^{m+2}}d\rho \\
&= \int_H \int_S \int_{\mathcal{F}_m} f(h\eta\rho)d\eta d\rho d\mu(h) \\
&= \int_H \int_{\mathcal{F}_m} \int_S f(h\eta\rho)d\rho d\eta d\mu(h)
\end{aligned}$$

Then it follows that the measure ν_Y on Y defined by

$$d\nu_Y := d\rho d\tilde{\eta},$$

where $d\tilde{\eta} := \frac{1}{\mathrm{Vol}(G/\Gamma)}d\eta$, satisfies (2.4.1).

Fix $Hg_0 \in H\backslash G$. By identifying $H\backslash G$ with Y, we define a measure on $H\backslash G$ via

$$d\nu_{Hg_0}(Hg) := \alpha(\sigma(Hg_0), \sigma(Hg))d\nu_Y(\sigma(Hg)). \tag{2.4.4}$$

where $\alpha(\cdot,\cdot)$ is given in (2.1.22).

In view of Theorem 2.2 and Corollary 2.4 in [GW04] (duality principle), an immediate consequence of Theorem 2.1.9 is the following

Corollary 2.4.1. *Fix $g_0 \in G$. For any compactly supported $\varphi \in C_c(H\backslash G)$,*

$$\lim_{T\to\infty} \frac{1}{\mu(H_T)} \sum_{\gamma\in\Gamma_T} \varphi(Hg_0.\gamma) = \int_{H\backslash G} \varphi(Hg)d\nu_{Hg_0}(Hg). \tag{2.4.5}$$

Now we would like to replace the normalization factor $\mu(H_T)$ by $\#\Gamma_T$. By Theorem 1.7 of [GN12],

$$\lim_{T\to\infty} \frac{\mathrm{Vol}(G_T)}{\#\Gamma_T} = 1. \tag{2.4.6}$$

We notice that by formula A.1.15 in [DRS93] and by Proposition 2.1.6, the limit

$$L := \lim_{T\to\infty} \frac{\mu(H_T)}{\mathrm{Vol}(G_T)}$$

exists. Thus we conclude

$$\lim_{T\to\infty} \frac{\mu(H_T)}{\#\Gamma_T} = \lim_{T\to\infty} \frac{\mu(H_T)}{\mathrm{Vol}(G_T)} \frac{\mathrm{Vol}(G_T)}{\#\Gamma_T} = \lim_{T\to\infty} \frac{\mu(H_T)}{\mathrm{Vol}(G_T)} = L, \tag{2.4.7}$$

which shows that

$$\lim_{T\to\infty} \frac{1}{\#\Gamma_T} \sum_{\gamma\in\Gamma_T} \varphi(Hg_0.\gamma) = L \int_{H\backslash G} \varphi(Hg) d\nu_{Hg_0}(Hg).$$

Our goal now will be to show that $L\nu_{Hg_0}(H\backslash G) = 1$. We first show that the total measure $\nu_{Hg_0}(H\backslash G)$ is independent of the choice of the base point Hg_0. Namely $\nu_{Hg_0}(H\backslash G) = \nu_H(H\backslash G)$ for all $Hg_0 \in H\backslash G$. For $g_0, y \in Y$, we consider the decompositions

$$g_0 := \begin{bmatrix} G_0 & 0 \\ 0 & 1 \end{bmatrix} \rho_0,\ y := \begin{bmatrix} \eta & 0 \\ 0 & 1 \end{bmatrix} \rho, \tag{2.4.8}$$

where $G_0, \eta \in \mathcal{F}_m$ and $\rho_0, \rho \in \mathrm{SO}(m+1, \mathbb{R})$. Hence α in (2.1.22) takes a more simpler form (note $g_0^{-1} := \rho_0^{-1} \begin{bmatrix} G_0^{-1} & 0 \\ 0 & 1 \end{bmatrix}$)

$$\alpha(g_0, y) = \frac{\sum_{q\in\Delta} \frac{1}{\|G_0^{-1} q\eta\|^{m^2}}}{\sum_{q\in\Delta} \frac{1}{\|q\|^{m^2}}}.$$

Using the invariance of the $\mathrm{SO}(m+1,\mathbb{R})$ invariance of the Hilbert-Schmidt norm, we compute

$$
\begin{aligned}
&\nu_{Hg_0}(H\backslash G)\\
=&\int_Y \alpha(g_0,y)d\nu_Y\\
=&\frac{1}{\sum_{q\in\Delta}\frac{1}{\|q\|^{m^2}}}\int_{\mathrm{SO}(m+1,\mathbb{R})}\int_{\mathcal{F}_m}\sum_{q\in\Delta}\frac{1}{\|G_0^{-1}q\eta\|^{m^2}}d\tilde{\eta}d\rho && \text{(definition of } d\nu_Y)\\
=&\frac{1}{\sum_{q\in\Delta}\frac{1}{\|q\|^{m^2}}}\int_{\mathrm{SL}(m,\mathbb{R})}\frac{1}{\|G_0^{-1}\eta\|^{m^2}}d\tilde{\eta} && \text{(by formula (2.4.1))}\\
=&\frac{1}{\sum_{q\in\Delta}\frac{1}{\|q\|^{m^2}}}\int_{\mathrm{SL}(m,\mathbb{R})}\frac{1}{\|\eta\|^{m^2}}d\tilde{\eta} && \text{(invariance of } d\tilde{\eta})\\
=&\nu_H(H\backslash G).
\end{aligned}
$$

This also confirms that $\nu_{Hg_0}(H\backslash G)$ is finite. Finally, we prove

Proposition 2.4.2. $\nu_{Hg_0}(H\backslash G)=\nu_H(H\backslash G)=\lim_{T\to\infty}\frac{\mathrm{Vol}(G_T)}{\mu(H_T)}=\frac{1}{L}$.

Proof. ν_H is a Radon measure since is finite and Borel. So for any $\epsilon>0$, we can choose $f_\epsilon\in C_c(H\backslash G)$ with support B_ϵ (note B_ϵ is bounded) such that (recall that Y is a lift of $H\backslash G$ to G such that $H\times Y\to G$ is a Borel isomorphism)

$$\int_Y |f_\epsilon(Hy)-1|\alpha(e,y)d\nu_Y(y)=\int_{H\backslash G}|f_\epsilon(Hg)-1|d\nu_H(Hg)\le\epsilon. \tag{2.4.9}$$

As in [GW04], we observe that

$$
\begin{aligned}
\frac{1}{\mu(H_T)}\int_{G_T}|f_\epsilon(Hg)-1|dg=&\frac{1}{\mu(H_T)}\int_Y\int_{\{h:\|hy\|<T\}}|f_\epsilon(Hy)-1|d\mu(h)d\nu_Y(y) && (2.4.10)\\
=&\frac{1}{\mu(H_T)}\int_Y|f_\epsilon(Hy)-1|\mu(\{h:\|hy\|<T\})d\nu_Y(y) && (2.4.11)\\
=&\int_Y|f_\epsilon(Hy)-1|\frac{\mu(H_T[e,y])}{\mu(H_T)}d\nu_Y(y) && (2.4.12)
\end{aligned}
$$

Recall that $\lim_{T\to\infty}\frac{\mu(H_T[e,y])}{\mu(H_T)}=\alpha(e,y)$. We will use below the dominated convergence theorem, and for that we will now show that the integrand $\frac{\mu(H_T[e,y])}{\mu(H_T)}$ is bounded by a function in $L^1(Y)$ for large T. Recall that $\mu(H_T) \sim T^{m(m+1)}$ and it only depends on the variable T (constant over Y).

Here since $g_1 = e$ and $g_2 = y$, $A_m(q) = \|G_0^{-1} q\eta\| = \|q\eta\|$ in view of (2.4.8). Recall $Y = \mathcal{F}_m \cdot \mathrm{SO}(m+1, \mathbb{R})$ and the factor $\mathrm{SO}(m+1, \mathbb{R})$ does not affect the finiteness of the integral over Y, therefore in this proof we shall ignore the $\mathrm{SO}(m+1, \mathbb{R})$ part and only integrate against the variable $H_1 \in \mathcal{F}_m$.

By (2.2.21) and by (2.2.4), we see

$$\frac{\mu(H_T[e,y])}{\mu(H_T)} \ll \sum_{q\in\Delta} \frac{1}{\|q\eta\|^{m^2}}. \tag{2.4.13}$$

By Lemma 2.2.1, we get that $\Psi(y) := \sum_{q\in\Delta} \frac{1}{\|q\eta\|^{m^2}} \in L^1(Y)$ (more precisely, to see that this is a L^1 function, use (2.4.3) and then Lemma 2.2.1).

By the dominant convergence theorem, the second term satisfies

$$\lim_{T\to\infty} \int_Y |f_\epsilon(Hy) - 1| \frac{\mu(H_T[e,y])}{\mu(H_T)} d\nu_Y(y) = \int_Y |f_\epsilon(Hy) - 1| d\nu_H(y) \le \epsilon \tag{2.4.14}$$

Therefore, by triangular inequality

$$\begin{aligned}
&\limsup_{T\to\infty} \left| \frac{\mathrm{Vol}(G_T)}{\mu(H_T)} - \nu_H(H\backslash G) \right| \\
\le &\limsup_{T\to\infty} \left| \frac{\mathrm{Vol}(G_T)}{\mu(H_T)} - \frac{1}{\mu(H_T)} \int_{G_T} f_\epsilon(Hg)dg \right| \\
&+ \limsup_{T\to\infty} \left| \frac{1}{\mu(H_T)} \int_{G_T} f_\epsilon(Hg)dg - \int_{H\backslash G} f_\epsilon(Hg)d\nu_H(g) \right| \\
&+ \limsup_{T\to\infty} \left| \int_{H\backslash G} f_\epsilon(Hg)dg - \nu_H(H\backslash G) \right| \\
\le &\epsilon + 0 + \epsilon \qquad \text{(the middle term vanishes because of Theorem 2.3, [GW04])}
\end{aligned}$$

Now let $\epsilon \to 0$, and this finishes the proof. □

Therefore, $\tilde{\nu}_{Hg_0} := L\nu_{Hg_0}$ is a probability measure, and we conclude

$$\lim_{T\to\infty} \frac{1}{\#\Gamma_T} \sum_{\gamma\in\Gamma_T} \varphi(Hg_0.\gamma) = \int_{H\backslash G} \varphi(Hg) d\tilde{\nu}_{Hg_0}(g). \tag{2.4.15}$$

Chapter 3: Equidistribution of o-minimal curves in homogeneous spaces

This chapter is based on part of a joint work in progress with Michael Bersudsky and Nimish Shah.

3.1 Introduction

Throughout this chapter, $G = \mathrm{SL}(n, \mathbb{R})$ and $\Gamma = \mathrm{SL}(n, \mathbb{Z})$.

Definition 3.1.1. We say that a curve $\varphi : [0, \infty) \to G$ (not necessarily a one-parameter subgroup), has the *homogeneous equidistribution property*, if for all $x \in G/\Gamma$ there exists a closed subgroup $F \leq G$ and a $y \in G/\Gamma$ such that Fy has a finite F-invariant volume, and for any compactly supported function $f \in C_c(G/\Gamma)$

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T f(\varphi(t)x)dt = \int_{Fy} f d\mu_F, \tag{3.1.1}$$

where μ_F is the F-invariant probability measure on Fy.

Along this direction of research, Shah [Sha94] established the homogeneous equidistribution property for multivariate polynomial trajectories in homogeneous spaces general real algebraic group with a product type assumption and later generalized by Zhang [Zha23] to trajectories without product type assumption (but with an additional regularity condition on the domain of averaging). For more general curves, Peterzil and Starchenko established the equidistribution results for definable curves in complex and real tori [PS18a] a polynomially bounded o-minimal structure with dense image in nilmanifolds [PS18b].

In this chapter, we establish homogeneous equidistribution property for single variable unipotent definable curves. A curve

$$\varphi(t) = [f_{i,j}(t)]_{1\leq i,j\leq n} \tag{3.1.2}$$

is definable in a polynomially bounded o-minimal structure, if each $f_{i,j} : [0,\infty) \to \mathbb{R}$ is definable in such a structure.

Theorem 3.1.2. *Supose that $\varphi(t) \in U$ for all $t \in [0,\infty)$, where $U \leq G$ is a unipotent upper-triangular subgroup. Then φ has homogeneous equidistribution property.*

Another result of independent interest which stands behind our equidistrbution theorems is the following (C,α)-good property. This property stands behind many applications in homogeneous dynamics, as first was noted in [KM98].

Theorem 3.1.3. *Let $\mathscr{F}$ be a vector space of functions definable in a polynomially bounded o-minimal structure such that for all $f \in \mathscr{F} \smallsetminus 0$ it holds that*

$$\lim_{t\to\infty} f(t) \neq 0.$$

Then, there exists $C,\alpha > 0$ and $A \geq 1$ such that for all $f \in \mathscr{F} \smallsetminus 0$ and all $I \subseteq [A,\infty)$

$$|\{t \in I : |f(t)| \leq \epsilon\}| \leq C\left(\frac{\epsilon}{\|f\|_I}\right)^{\alpha} \cdot |I|$$

This property is well-known for polynomials, and is new in the above setting.

3.2 (C,α)-good property of definable functions in a polynomially bounded o-minimal structure

The (C,α)-good property (Theorem 3.1.3) will follow from the following inequalities for suprememums of a function on nested intervals. This form of inequalities are well-known in the literature as Remez-type inequalities, see [Rem36].

Definition 3.2.1. Let $\delta \in (0,1)$. A family of real functions $\mathscr{F}$ defined on $[A,\infty)$ is called *δ-good* if there exists a constant $M(\delta)$ depending on δ only such that

$$\frac{\|f\|_I}{\|f\|_{I_\delta}} \leq M(\delta), \tag{3.2.1}$$

for all $f \in \mathscr{F}$ and all bounded sub-intervals $I_\delta \subset I \subseteq [A,\infty]$ satisfying $|I_\delta| = \delta|I|$.

The following is our key theorem which stands behind Theorem 3.1.3.

Theorem 3.2.2. *Let $\mathscr{F}$ be a finite dimensional vector space of functions definable in a polynomially bounded o-minimal structure such that for all $f \in \mathscr{F} \setminus 0$,*

$$\lim_{t\to\infty} f(t) \neq 0.$$

Namely, $\mathscr{F}$ doesn't contain functions decaying to 0. Then, there exists $A > 0$ such that for all $\delta \in (0,1)$ and all $f \in \mathscr{F} \setminus 0$ it holds that f restricted to $[A,\infty)$ is δ-good.

We first prove Theorem 3.1.3 by assuming Theorem 3.2.2, and the rest of the section will be dedicated to proving Theorem 3.2.2.

3.2.1 Proving Theorem 3.1.3

We recall that for a definable function f in a polynomially bounded o-minimal structure it holds that either $f : [0,\infty) \to \mathbb{R}$ is eventually constantly zero, or there exist $r \neq 0$ such that $\lim_{t\to\infty} f(t)/t^r = c \neq 0$, see [Mil94a]. We will denote this exponent by:

$$\deg(f) := r. \tag{3.2.2}$$

We have the following elementary observation.

Lemma 3.2.3. *Suppose that $\mathscr{F}$ is a finite dimensional vector space of functions $f : [0,\infty) \to \mathbb{R}$ definable in a polynomially bounded o-minimal structure of dimension $n+1$. Then there exist a basis $\{f_0, f_1, ..., f_n\}$ of $\mathscr{F}$, where*

$$\deg(f_0) < \deg(f_1) < ... < \deg(f_n).$$

In particular, $\mathscr{F}$ doesn't include functions decaying to zero if and only if $0 \leq \deg(f_0)$.

For the rest of the section,

$$\mathscr{F} = \mathrm{Span}\{f_0, f_1, ..., f_n\}, \tag{3.2.3}$$

where

$$0 \leq \deg(f_0) < \deg(f_1) < ... < \deg(f_n),$$

and we denote

$$r_i = \deg(f_i).$$

In addition, we assume without loss of generality that

$$\lim_{t \to \infty} \frac{f_i(t)}{t^{r_i}} = 1.$$

In what follows, for a tuple $(c_0, c_1, ..., c_n) \in \mathbb{R}^{n+1}$, we will denote

$$f_{\underline{c}} = c_0 f_0 + c_1 f_1 + ... + c_n f_n. \tag{3.2.4}$$

We have the following Corollary from Theorem 3.2.2.

Corollary 3.2.4. *Let $\mathscr{F}$ be a finite dimensional vector space of functions definable in a polynomially bounded o-minimal structure such that for all $f \in \mathscr{F} \smallsetminus 0$,*

$$\lim_{t \to \infty} f(t) \neq 0.$$

Let $A \geq 1$ such that the outcome of Theorem 3.2.2 holds. Then, there exists $\lambda, r > 0$ such that for all $x \geq 1$, $f \in \mathscr{F} \smallsetminus 0$ and $I' \subseteq I \subseteq [A, \infty)$ such that $\frac{|I|}{|I'|} \leq x$ it holds that

$$\|f\|_I \leq \lambda x^r \|f\|_{I'} \tag{3.2.5}$$

Proof. Consider the definable set:

$$S := \{(x, y) : x, y > 0, \text{ such that } , \frac{\|f_{\underline{c}}\|_I}{\|f_{\underline{c}}\|_{I'}} \leq y, \forall \underline{c} \neq 0, I' \subseteq I \subseteq [A, \infty), \frac{|I|}{|I'|} \leq x\}.$$

By Theorem 3.2.2, the projection of S to the first coordinate includes $[1, \infty)$. Then, by the definable choice theorem there exists a definable function $\phi : [1, \infty) \to S$ in the polynomially bounded o-minimal structure. Since ϕ is polynomially bounded, the result follows. $\square$

We are now ready to prove the C, α-good property.

Proof of Theorem 3.1.3. By o-minimality, the number of connected components of the sub level sets

$$\{t \in I : |f(t)| \leq \epsilon\},$$

where $\epsilon > 0$ and $f \in \mathscr{F} \setminus 0$ is bounded uniformly, say by K. Let

$$I' \subseteq \{t \in I : |f(t)| \leq \epsilon\}$$

be an interval of maximum length. Then

$$L \leq K|I'|,$$

where

$$L := |\{t \in I : |f(t)| \leq \epsilon\}|.$$

The latter inequality implies,

$$\frac{|I|}{|I'|} \leq \frac{|I|}{L/K}.$$

By Corollary 3.2.5, we get

$$\|f\|_I \leq \lambda \left(\frac{|I|}{L/K}\right)^r \|f\|_{I'} \leq \lambda \left(\frac{|I|}{L/K}\right)^r \epsilon.$$

Reordering the latter inequality, we get that

$$|\{t \in I : |f(t)| \leq \epsilon\}| = L \leq \lambda^{\frac{1}{r}} K \left(\frac{\epsilon}{\|f\|_I}\right)^{\frac{1}{r}} |I|.$$

$\square$

3.2.2 Proving Theorem 3.2.2

The following is the fundamental well known example of δ-good functions. We present it's proof for completeness.

Proposition 3.2.5. *Let $\mathscr{F} := \mathrm{Span}_{\mathbb{R}}\{1, x, ..., x^n\} \setminus 0$, then $\mathscr{F}$ is δ-good on $\mathbb{R}$. More precisely:*

$$\frac{\|c_0 + c_1 x + \cdots + c_n x^n\|_I}{\|c_0 + c_1 x + \cdots + c_n x^n\|_{I_\delta}} \leq (n+1)\frac{n^n}{\delta^n}, \tag{3.2.6}$$

for all $c_0, c_1, ..., c_n \in \mathbb{R}$ not all zero, interval $I \subset \mathbb{R}$ and sub-interval $I_\delta \subset I$ satisfying $|I_\delta| = \delta|I|$

Proof. Let $f := c_0 + c_1 x + \cdots + c_n x^n$ and $I = [a, a+T]$. Put $t_i = t_0 + \frac{i\delta T}{n}$ for $i = 0, 1, ..., n$, $a \leq t_0 < t_1 < \cdots < t_n \leq a + T$, and $I_\delta = [t_0, t_n]$. Then by polynomial interpolation of f at points $t_0, t_1, ..., t_n$, we have

$$f(t) = \sum_{j=0}^{n} \frac{\prod_{i\neq j}(t - t_i)}{\prod_{i\neq j}(t_j - t_i)} f(t_j). \tag{3.2.7}$$

Now, by triangle inequality

$$\left|\frac{\prod_{i\neq j}(t - t_i)}{\prod_{i\neq j}(t_j - t_i)}\right| \leq \frac{n^n}{\delta^n},\ \forall t \in I.$$

Thus

$$|f(t)| \leq (n+1)\frac{n^n}{\delta^n} \max|f(t_i)| \leq (n+1)\frac{n^n}{\delta^n}\|f\|_{I_\delta},\ \forall t \in I,$$

and the statement follows. □

We will be using the following facts:

- The derivative f' of any definable function $f : [0, \infty) \to \mathbb{R}$ exists and continuous for all large enough t, and moreover f' is definable.

- For a definable function $f : [0, \infty) \to \mathbb{R}$ in a polynomially bounded o-minimal structure with $f(t) \sim t^r$ where $r \neq 0$, we have $f'(t) \sim rt^{r-1}$, as $t \to \infty$, see [Mil94b, Proposition 3.1].

Lemma 3.2.6. *There exists $A > 0$ such that for any interval $I \subset [A, \infty)$ it holds that $\|f_{\underline{c}}\|_I > 0$ for all $\underline{c} \neq 0$.*

Proof. We first note that there's no loss in generality in assuming that $f_0(t) \equiv 1$ after possibly dividing $f_{\underline{c}}$ by f_0. In fact, since $f_0(t) \sim t^{r_0}$, we get that $f_0(t) > 0$ for all large t. Moreover $\deg(f_i/f_0) = r_i - r_0$.

We now prove the statement for $n = 1$, and then argue by induction. Let $A_0 > 0$ be such that $f_0'(t) \neq 0$ for all $t \geq A_0$. Assume that for an interval $I \subset [A, \infty)$ we have $\|c_0 + c_1 f\|_I = 0$. Since $c_0 + c_1 f(t) = 0,\ \forall t \in I$, we get that

$$\frac{d}{dt}\left(c_0 + c_1 f(t)\right) = c_1 f'(t) = 0,\ \forall t \in I,$$

and as $f'(t) \neq 0$ for all $t \in [A, \infty)$, it follows that $c_1 = 0$. As a consequence, $c_0 = 0$ as-well.

Now let $n \geq 2$, $0 < r_1 < r_2 < \cdots < r_n$ be a fixed sequence of real numbers, and assume that $f_1, ..., f_n$ are definable in a polynomially bounded o-minimal structure such that for $1 \leq i \leq n-1$

$$f_i(t) \sim t^{r_i},$$

as $t \to \infty$. Observe that in some ray $J = [A_1, \infty]$ we have $f_1'(t) \neq 0,\ \forall t \in J$, and for all $2 \leq i \leq n$ the functions $\frac{f_i'(t)}{f_1'(t)}$ are well defined in J. Note that

$$h_i(t) := \frac{r_1}{r_2}\frac{f_i'(t)}{f_1'(t)} \sim t^{r_i'},$$

where $r_i' := r_i - r_1$. By induction, there's $A > 0$ such that if

$$\|c_1 + c_2 h_2 + ... + c_n h_n\|_I = 0$$

for some $I \subset [A, \infty)$ then $c_1 = \cdots = c_n = 0$. Now if

$$\|c_0 + c_1 f_1 + c_2 f_2 + ... + c_n f_n\|_I = 0,$$

then

$$\|c_1 + c_2\frac{r_1}{r_2}h_2 + ... + c_n\frac{r_1}{r_n}h_n\|_I = 0.$$

It then follows that $c_i = 0$ for all i. $\square$

In what follows we will take A such that $f_0(t), f_1(t), ..., f_n(t)$ are continuously differentiable $(n+1)$ times for all $t \geq A$. We consider the Wronskian matrix

$$D(t) = D(f_0, ..., f_n)(t) \tag{3.2.8}$$

which is the $(n+1)\times(n+1)$ matrix $D(t)$ is whose 0-th row is $D_0(t) = (f_0(t), f_1(t), \ldots, f_n(t))$ and its and i-th row is $D_i(t) = D_0^{(i)}(t)$, the i-th derivative of the 0-th row at t, for $1 \leq i \leq n$.

Lemma 3.2.7. *There exists an $A > 0$ such that $D(t)$ is non-singular for all $t \geq A$.*

Proof. If $n = 0$, the statement is trivial. Assume that $D(f_1, ..., f_{n-1})(t)$ is not singular for all $t \geq T_0$. Since $D(t) = D(f_1, ..., f_n)(t)$ is a definable function, it's either non-zero for all large t or eventually is constantly zero, say $D(t) = 0, \forall t \geq T_1$. Suppose for contradiction the latter case. Then by a classical result of Bôcher [Bôc01], we get that $f_0, f_1, ..., f_n$ are linearly dependent in any sub-interval of $[T_1, \infty)$. This contradicts Lemma 3.2.6.

$\square$

In the following we will make use of the fact:

- A definable function $f : [0, \infty) \to \mathbb{R}$ either converges as $t \to \infty$ or diverges to ∞ or to $-\infty$ as $t \to \infty$.

Proof of Theorem 3.2.2. Let A as in Lemma 3.2.7. Fix $\delta \in (0, 1)$, and assume for contradiction that $\mathscr{F} \smallsetminus 0$ is not δ-good on $[A, \infty)$. Consider the following definable subset:

$$\mathcal{A} := \left\{(s, \underline{c}, a, T, \alpha) : \begin{array}{c} s \geq 1,\ \underline{c} \neq 0,\ a > A,\ T > 0, \\ a \leq \alpha \leq a + T - \delta T,\ \frac{\|f_{\underline{c}}\|_{[a,a+T]}}{\|f_{\underline{c}}\|_{[\alpha,\alpha+\delta T]}} > s \end{array}\right\}. \tag{3.2.9}$$

By the assumption, the projection of $\mathcal{A}$ to the first coordinate is $\mathbb{R}_{\geq 1}$. By the choice function theorem, there exists a polynomially bounded definable curve $\phi : [1, \infty) \to \mathcal{A}$, meaning each of the coordinate components of

$$\phi(s) = (\underline{c}(s), a(s), T(s), \alpha(s))$$

is a polynomially bounded o-minimal function in s. In particular, we have

$$\frac{a(s)}{T(s)} \asymp s^{\kappa}. \tag{3.2.10}$$

We will now discuss two cases for κ.

Case 1: if $\kappa \leq 0$. In this case, we treat two sub-cases in which $T(s)$ is bounded or not.

Case 1.1. *$T(s)$ is bounded in s.* Then $a(s)$ is also bounded, and by o-minimality that $T(s)$ and $a(s)$ converge as $s \to \infty$. We first observe that $\lim_{s\to\infty} T(s) \neq 0$. In fact, if we assume by contradiction that $\lim_{s\to\infty} T(s) = 0$, since we have that $a(s) \geq A \geq 1$, we will get that $\lim_{s\to\infty} \frac{a(s)}{T(s)} = \infty$, which is a contradiction to the assumption that $\kappa \leq 0$. Now let $\lambda(s) = \|\underline{c}(s)\|_1$ be the l^1 norm of the coefficients. Denote

$$\hat{\underline{c}} := \frac{\underline{c}}{\lambda},$$

and observe that by o-minimality, since $\hat{\underline{c}}(s)$ is bounded, $\hat{\underline{c}}(s)$ converges to some vector $\mathbf{v}$ with $\|\mathbf{v}\|_1 = 1$. Also, the end-points of the intervals $[\alpha(s), \alpha(s) + \delta T(s)] \subset [a(s), a(s) + T(s)]$ converge to the end-points of some intervals, say $I_\delta \subset I \subseteq [A, \infty)$ where $\frac{|I_\delta|}{|I|} = \delta$. Now

$$\frac{\|f_{\hat{\underline{c}}(s)}\|_{[a(s),a(s)+T(s)]}}{\|f_{\hat{\underline{c}}(s)}\|_{[\alpha(s),\alpha(s)+\delta T(s)]}} = \frac{\|f_{\underline{c}(s)}/\lambda(s)\|_{[a(s),a(s)+T(s)]}}{\|f_{\underline{c}(s)}/\lambda(s)\|_{[\alpha(s),\alpha(s)+\delta T(s)]}} = \frac{\|f_{\underline{c}(s)}\|_{[a(s),a(s)+T(s)]}}{\|f_{\underline{c}(s)}\|_{[\alpha(s),\alpha(s)+\delta T(s)]}} > s.$$

Namely:

$$\frac{\|f_{\hat{\underline{c}}(s)}\|_{[a(s),a(s)+T(s)]}}{s} \geq \|f_{\underline{c}(s)}\|_{[\alpha(s),\alpha(s)+\delta T(s)]},$$

and by taking the limit $s \to \infty$, we get

$$0 = \|f_{\mathbf{v}}\|_{I_\delta},$$

which is a contradiction.

Case 1.2. *$T(s)$ is unbounded in s.* Then, by o-minimality $T(s) \to \infty$. Since $f_i(x) \sim x^{r_i}$, there exists $0 < \nu_i$ such that for all $t \in [0,1]$, we have

$$\begin{aligned}\frac{f_i(a+tT)}{T^{r_i}} &= \frac{(a+Tt)^{r_i}}{T^{r_i}} + O\left(\frac{(a+Tt)^{r_i-\nu_i}}{T^{r_i}}\right) \\ &= \left(\frac{a}{T}+t\right)^{r_i} + O\left(T^{-\nu_i}\left(\frac{a}{T}+t\right)^{r_i-\nu_i}\right).\end{aligned}$$

Now, $\lim_{s\to\infty} T(s) = \infty$ and $\frac{a(s)}{T(s)}$ is bounded in s. Thus,

$$\frac{f_i(a(s)+tT(s))}{T(s)^{r_i}} = \left(\frac{a(s)}{T(s)}+t\right)^{r_i} + O\left(T(s)^{-\nu}\right),\ \forall 0 \le i \le n,\ \forall t \in [0,1],$$

where $\nu = \min\{\nu_i\} > 0$. In particular,

$$\begin{aligned}f_{\underline{c}(s)}(a(s)+tT(s)) =& c_0(s)f_0(a(s)+tT(s)) + \cdots + c_n(s)f_n(a(s)+tT(s)) \\ =& (c_0(s)T(s)^{r_0})\left(\frac{a(s)}{T(s)}+t\right)^{r_0} + \cdots + (c_n(s)T(s)^{r_n})\left(\frac{a(s)}{T(s)}+t\right)^{r_n} + O(C(s)T(s)^{-\nu}),\end{aligned}$$

where $C(s) = \max\{c_i(s)T(s)^{r_i}\}$. We consider:

$$\varphi_{\underline{c}(s)}(t) := f_{\underline{c}(s)}(a(s)+tT(s)),\ t \in [0,1].$$

Since $\frac{a(s)}{T(s)}$ is bounded, we have $\lim_{s\to\infty} \frac{a(s)}{T(s)} = x_0$ (here it's possible that $x_0 = 0$). We put

$$\lambda(s) := \|(c_0(s)T(s)^{r_0}, ..., c_n(s)T(s)^{r_n})\|_1,$$

and we observe that

$$\lim_{s\to\infty} \frac{\varphi_{\underline{c}(s)}(t)}{\lambda(s)} = v_0\left(x_0+t\right)^{r_0} + \cdots + v_n\left(x_0+t\right)^{r_n},$$

uniformly in $t \in [0,1]$, where $(v_0, ..., v_1)$ is the limiting vector of $\frac{1}{\lambda(s)}(c_0(s)T(s)^{r_0}, ..., c_n(s)T(s)^{r_n})$ having $\|(v_0, ..., v_n)\|_1 = 1$. Now

$$s < \frac{\|f_{\underline{c}(s)}\|_{[a(s),a(s)+T(s)]}}{\|f_{\underline{c}(s)}\|_{[\alpha(s),\alpha(s)+\delta T(s)]}} = \frac{\|\varphi_{\underline{c}(s)}\|_{[0,1]}}{\|\varphi_{\underline{c}(s)}\|_{I_\delta(s)}} = \frac{\|\varphi_{\underline{c}(s)}/\lambda(s)\|_{[0,1]}}{\|\varphi_{\underline{c}(s)}/\lambda(s)\|_{I_\delta(s)}} \tag{3.2.11}$$

where $I_\delta(s) := \left(\frac{\alpha(s)}{T(s)} - a(s), \frac{\alpha(s)+\delta T(s)}{T(s)} - a(s)\right) \subseteq [0,1]$, where $|I_\delta(s)| = \delta,\ \forall s \geq 1$. We conclude that:

$$\lim_{s\to\infty} \|\varphi_{\underline{c}(s)}/\lambda(s)\|_{I_\delta(s)} = 0$$

As $s \to \infty$, we have that $I_\delta(s)$ converges to a sub-interval J of $[0,1]$ of relative length δ, and we get that

$$v_0\,(x_0+t)^{r_0} + \cdots + v_n\,(x_0+t)^{r_n} = 0,\ \forall t \in J,$$

which is a contradiction since not all the coefficients v_i are zero.

Case 2: if $\kappa > 0$. In an overview, in this case, the idea is to approximate $f_{\hat{\underline{c}}}$ in the intervals $[a, a+T]$ using Taylor expansion and to apply Proposition 3.2.5 to get a contradiction.

As above, let $\lambda(s) = \|\underline{c}(s)\|_1$ be the l^1-norm of the coefficients. We denote: $\hat{\underline{c}} := \frac{\underline{c}}{\lambda}$, and we note that

$$\frac{\|f_{\hat{\underline{c}}(s)}\|_{[a(s),a(s)+T(s)]}}{\|f_{\hat{\underline{c}}(s)}\|_{[\alpha(s),\alpha(s)+\delta T(s)]}} = \frac{\|f_{\underline{c}(s)}\|_{[a(s),a(s)+T(s)]}}{\|f_{\underline{c}(s)}\|_{[\alpha(s),\alpha(s)+\delta T(s)]}} > s.$$

We now consider two cases: $a(s)$ is bounded or $a(s) \to \infty$.

Case 2.1. *$a(s)$ is bounded.* We observe that $\lim_{s\to\infty} T(s) = 0$. In fact, this follows since $a(s) \geq A \geq 1$, and $\frac{a(s)}{T(s)} \sim s^\kappa \to \infty$ because $\kappa > 0$. We have that $\lim_{s\to\infty} a(s) = x_0 \geq A$, and $\lim_{s\to\infty} \hat{\underline{c}}(s) = \mathbf{v}$ with $\|\mathbf{v}\|_1 = 1$.

Recall that $f_0, f_1, ..., f_n$ are $(n+1)$-times continuously differentiable for all $a \in [A, \infty)$. Then we may consider the Taylor polynomial $Q_{n,a(s)}(t)$ for $f_{\hat{\underline{c}}(s)}$ centered at $a(s) \in [A, \infty)$ of degree n. Since $a(s), \hat{\underline{c}}(s)$ and $T(s)$ are bounded, we get that $|f^{(n+1)}_{\hat{\underline{c}}(s)}(\xi)|$ is uniformly bounded in the range $\xi \in [a(s), a(s)+T(s)]$. By Taylor's theorem there exists $\xi \in [a(s), a(s)+T(s)]$ such that

$$\begin{aligned} |f_{\hat{\underline{c}}(s)}(t) - Q_{n,a(s)}(t)| =& \left| \frac{f^{(n+1)}_{\hat{\underline{c}}(s)}(\xi)}{(n+1)!} (t-a(s))^{n+1} \right| \\ =& O(T(s)^{n+1}) \end{aligned}$$

for all $t \in [a(s), a(s)+T(s)]$. Denote $I(s) := [a(s), a(s)+T(s)]$, and $I_\delta(s) := [\alpha(s), \alpha(s)+\delta T(s)]$. We have:

$$\begin{aligned}
\|f_{\hat{\underline{c}}(s)}\|_{I(s)} =& \|Q_{n,a(s)}\|_{I(s)} + O(T(s)^{n+1}) && \text{(By Taylor approximation)}\\
\leq& M_n(\delta)\|Q_{n,a(s)}\|_{I_\delta(s)} + O(T(s)^{n+1}) && \text{(By Proposition 3.2.5)}\\
\leq& M_n(\delta)(\|f_{\hat{\underline{c}}(s)}\|_{I_\delta(s)} + O(T(s)^{n+1}) && \text{(By Taylor approximation)}\\
=& M_n(\delta)\|f_{\hat{\underline{c}}(s)}\|_{I_\delta(s)}\left[1 + O\left(\frac{T(s)^{n+1}}{\|f_{\hat{\underline{c}}(s)}\|_{I_\delta(s)}}\right)\right]
\end{aligned}$$

Namely,

$$\frac{\|f_{\underline{c}(s)}\|_{I(s)}}{\|f_{\underline{c}(s)}\|_{I_\delta(s)}} \ll 1 + O\left(\frac{T(s)^{n+1}}{\|f_{\hat{\underline{c}}(s)}\|_{I_\delta(s)}}\right).$$

We now show that

$$\lim_{s\to\infty} \frac{T(s)^{n+1}}{\|f_{\hat{\underline{c}}(s)}\|_{I_\delta(s)}} = 0. \tag{3.2.12}$$

This outcome gives a contradiction to our assumption $s \leq \frac{\|f_{\underline{c}(s)}\|_{I(s)}}{\|f_{\underline{c}(s)}\|_{I_\delta(s)}}$. To prove (3.2.12), first recall the following general fact: if $q(x) = c_0 + c_1 x + ... + c_k x^k$ is a polynomial of degree k and J is an interval of length $|J| < 1$, then

$$\|q\|_J \gg \|(c_0, c_1, ..., c_k)\|_\infty |J|^k. \tag{3.2.13}$$

Now $Q_{n,a(s)}$ is a polynomial of degree n whose coefficients are given by $D(a(s))\hat{\underline{c}}(s)$, where $D = D(f_0, ..., f_n)$ is the Wronskian matrix. According to Lemma 3.2.7, using the fact that $a(s)$ is bounded, the l^1-norm of it's coefficients is uniformly bounded from below. Then

$$\|f_{\hat{\underline{c}}(s)}\|_{I_\delta(s)} + O(T(s)^{n+1}) = \|Q_{n,a(s)}\|_{I_\delta(s)} \gg T(s)^n. \tag{3.2.14}$$

Since $T(s) \to 0$, we get (3.2.12).

Case 2.1. *$a(s)$ is unbounded.* Here $a(s) = s^\theta$, for $\theta > 0$. We recall that for each $N \in \mathbb{N}$ there's x_N such that f_i will be continuously differentiable N times in a ray $[A_N, \infty)$, see

[DM96]. We have by [Mil94b] for a non-negative integer l,

$$f_i^{(l)}(x) \ll x^{r_i - l}.$$

Then we conclude that for integers N such that $r_i - (N+1) < 0$ for all $1 \leq i \leq n$, we have for all $t \in [a(s), a(s) + T(s)]$ that

$$f_{\underline{\hat{c}}(s)}^{(N+1)}(t) = O(a(s)^{r_n - (N+1)}).$$

By Taylor's theorem:

$$\begin{aligned}
|f_{\underline{\hat{c}}(s)}(t) - Q_{N,a(s)}(t)| &= O(a(s)^{r_n - (N+1)} T(s)^{N+1}) \\
&= O\left(a(s)^{r_n} \left(\frac{T(s)}{a(s)} \right)^{N+1} \right) \\
&= O\left(s^{r_n \theta} s^{-(N+1)\kappa} \right). && \left(\text{using that } \tfrac{a(s)}{T(s)} \sim s^{\kappa}\right)
\end{aligned}$$

The function $\Psi(s) := \|f_{\underline{\hat{c}}(s)}\|_{[\alpha(s), \alpha(s) + \delta T(s)]}$ is positive and definable in a polynomially bounded o-minimal structure, which implies that there's $\eta > 0$ such that

$$\|f_{\underline{\hat{c}}(s)}\|_{[\alpha(s), \alpha(s) + \delta T(s)]} \gg s^{-\eta}, \text{ as } s \to \infty \tag{3.2.15}$$

We take N large enough such that

$$\nu := r_n \theta - (N+1)\kappa > \eta,$$

where η as in (3.2.15). Then:

$$\begin{aligned}
\|f_{\underline{\hat{c}}(s)}\|_{I(s)} \leq & \|Q_{N,a(s)}\|_{I(s)} + O(s^{-\nu}) && \text{(By Taylor approximation)} \\
\leq & M_N(\delta) \|Q\|_{I_\delta(s)} + O(s^{-\nu}) && \text{(By Proposition 3.2.5)} \\
\leq & M_N(\delta) \|f_{\underline{\hat{c}}(s)}\|_{I_\delta(s)} + O(s^{-\nu}) && \text{(By Taylor approximation)} \\
= & M_N(\delta) (\|f_{\underline{\hat{c}}(s)}\|_{I_\delta(s)} \left[1 + O\left(\frac{s^{-\nu}}{\|f_{\underline{\hat{c}}(s)}\|_{I_\delta(s)}} \right) \right] \\
\ll & M_N(\delta) \|f_{\underline{\hat{c}}(s)}\|_{I_\delta} \left[1 + O\left(\frac{s^{-\nu}}{s^{-\eta}} \right) \right] && \text{(By (3.2.15))}
\end{aligned}$$

This is a contradiction to our assumption $\|f_{\underline{\hat{c}}(s)}\|_{I(s)} \geq s \|f\|_{I_\delta(s)}$. $\square$

3.2.3 (C, α)-good property of o-minimal curves in representations

Our goal in ths section is to prove the following proposition.

Proposition 3.2.8. *Let $\theta : G \to \mathrm{GL}(m, \mathbb{R})$ be an algebraic homomorphism, and fix a norm $\|\cdot\|$ on $\mathbb{R}^m$. Let $\varphi : [0, \infty) \to G$ be either contained in a unipotent group for all t. Then the following holds:*

(1) For $v \in \mathbb{R}^m$ we have $\lim_{t\to\infty} \theta(\varphi(t))v = v$ or $\lim_{t\to\infty} \theta(\varphi(t))v = \infty$.

(2) There exists $A, C, \alpha > 0$ such that for all v the function $\Theta(t) := \|\theta(\varphi(t))v\|$ is (C, α)-good in $[A, \infty)$.

In order to prove Proposition 3.2.8, we have the following statement.

Lemma 3.2.9. *Let $U \leq \mathrm{GL}(m, \mathbb{R})$ be the unipotent group of upper-triangular matrices, and let $\psi : [0, \infty) \to U$ be a continuous curve definable in a polynomially bounded o-minimal structure. Then there exist upper triangular unipotent matrices $\sigma(t)$ and $u(t)$ such that*

$$\psi(t) = \sigma(t)u(t) \tag{3.2.16}$$

where $\lim_{t\to\infty} \sigma(t) = I_m$ and the real span of entries of $u(t)$ in each row form a real linear space containing either constant functions or functions converging to ∞ in absolute value as $t \to \infty$.

Proof. We prove this by induction on m. $m = 1$ is is trivial. Now let $m > 1$, and consider

$$\psi(t) = \begin{bmatrix} 1 & f_{1,2} & \cdots & f_{1,m-1} & f_{1,m} \\ 0 & 1 & \cdots & f_{2,m-1} & f_{2,m} \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & f_{m-1,m} \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix}. \tag{3.2.17}$$

Using the induction hypothesis, we assume without loss of generality that the $(m-1)\times(m-1)$ sub-matrix on the left upper corner of ψ satisfies the outcome of the lemma. If i is such that $\mathrm{Span}_{\mathbb{R}}\{1, f_{i,i+1}, ..., f_{i,m-1}, f_{i,m}\} - \{0\}$ contains a non-trivial linear combination yielding function decaying to zero, then, there's coefficients $c_{i,i1}, ..., c_{i,m} \in \mathbb{R}$ not all zero such that

$$h_i(t) := c_{i,i} + c_{i,i+1} f_{i,i+1} + \cdots + c_{i,m} f_{i,m}$$

which decays to zero. By induction hypothesis $c_{i,m} \neq 0$ so without loss of generality we may always assume $c_{i,m} = 1$. Consider

$$\sigma_i^{-1}(t) = \begin{bmatrix} 1 & & & & \\ & 1 & & & -h_i(t) \\ & & \ddots & & \\ & & & 1 & \\ & & & & 1 \end{bmatrix}, \tag{3.2.18}$$

where the function h_i is in the i-th row. We have that the span of the functions in first row of $u_i(t) := \sigma_i^{-1}(t)\varphi(t)$ is as the span of $\{1, f_{i,i+1}, ..., f_{i,m-1}\}$, which contains functions that are either constant or converge to infinity in absolute value. Also, $\lim_{t\to\infty} \sigma_i(t) = I_m$. Let $i_1, ..., i_k$ be the indices of the rows in $\varphi(t)$ whose span contains decaying functions. We conclude that

$$u(t) := \sigma_{i_1}^{-1}(t) \cdots \sigma_{i_k}^{-1}(t)\varphi(t)$$

and

$$\sigma(t) := \sigma_{i_1}(t) \cdots \sigma_{i_k}(t),$$

satisfy the outcome of the lemma.

$\square$

Proof of Proposition 3.2.8 for curves in a unipotent group. Suppose that $U \leq G$ is a unipotent subgroup and $\varphi : [0, \infty) \to U$ is a continuous curve definable in a polynomially bounded structure. Let $\theta : G \to \mathrm{GL}(m, \mathbb{R})$ be an algebraic homomorphism. Then $\theta(U)$ is a unipotent subgroup and up-to conjugation, $\theta(U)$ is contained in the group of unipotent upper-triangular matrices in $\mathrm{GL}(m, \mathbb{R})$. Moreover, θ is a polynomial map and therefore $\psi(t) := \theta(\varphi(t))$ is definable. The rest follows from Lemma 3.2.9. $\square$

3.3 Non-escape of mass

Let $X\cup\{\infty\}$ denote the one-point compactification of $X := G/\Gamma$. For $T>0$ and $\Lambda\in X$, let

$$\mu_{\gamma,\Lambda,T}(f) := \frac{1}{T}\int_0^T f(\gamma(t).\Lambda)dt. \tag{3.3.1}$$

Since $\{\mu_{\gamma,\Lambda,T}\}_{T>0}$ is a family of probablity measure and by Banach-Alaoglu theorem, there exists a subsequence $T_i\to\infty$ such that μ_{Λ,T_i} has a weak-star limit $\mu_{\Lambda,\infty}$. With an abuse of notations we shall drop the subscripts in below.

For a rank k sublattice $\Delta\subset\mathbb{Z}^n$, let $\|\Delta\|$ denote the volume of the quotient space $R\Delta/\Delta$. This definition can be interpreted using the exterior algebra of $\mathbb{R}^n$: if Δ is generated by $\mathbf{v}_1,...,\mathbf{v}_k$, then $\Delta=\|\mathbf{v}_1\wedge\cdots\wedge\mathbf{v}_k\|$, where the norm is the standard Euclidean norm on the wedge product defined through inner product.

The following powerful theorem on quantitative non-divergence due to Kleinbock [Kle07] will be needed. Let $P(\mathbb{Z}^n)$ denote the set of primitive sublattices of $\mathbb{Z}^n$ (whose bases can be extended to a basis of $\mathbb{Z}^n$).

Theorem 3.3.1. *Suppose an interval $B\subset\mathbb{R}$, $C,\alpha>0$, $0<\rho<1$ and a continuous map $h:B\to\mathrm{SL}(n,\mathbb{R})$ are given. Assume that for any $\Delta\subset P(\mathbb{Z}^n)$, we have*

(1) the function $x\to\|h(x)\Delta\|$ is (C,α)-good on B, and

(2) $\sup_{x\in B}\|h(x)\Delta\|\geq\rho^{\mathrm{rank}(\Delta)}$

then for any $\epsilon<\rho$,

$$|\{x\in B:\lambda_1(h(x)\mathbb{Z}^n)\leq\epsilon\}|\leq Cn2^n\left(\frac{\epsilon}{\rho}\right)^\alpha|B|, \tag{3.3.2}$$

where $\lambda_1(\cdot)$ is the function that outputs the length of the shortest nonzero vector of an Euclidean lattice.

The follow classical fact for representations of algebraic groups will also be used in the proof (Cf. Theorem 9.18 [Mil17]):

Lemma 3.3.2. *Let V be an $\mathbb{R}$-vector space and $\theta : G \to \mathrm{GL}(V)$ be an algebraic homomorphism. If $u \in G$ is unipotent, then so is $\varphi(u)$ as a linear map on V.*

Lemma 3.3.3. $\mu_{\Lambda,\infty}(\infty) = 0$*, for all $\Lambda \in X$.*

Proof. For any exterior representation $\rho : G \to \mathrm{GL}(\wedge^k \mathbb{R}^n) \cong \mathrm{GL}(\mathbb{R}^N)$.

Fix a norm $\|\cdot\|$ on $\mathbb{R}^N$. We will denote below for $r > 0, \mathbf{v} \in \mathbb{R}^N$ by $B_r(\mathbf{v}) \subseteq \mathbb{R}^N$ the ball of radius r centered at $\mathbf{v}$. Since φ is contained in a unipotent group or is polynomially regular, we get by Proposition 3.2.8 that there exists $A > 0$ such that $\Theta_{\mathbf{v}}(t) := \|\varphi(t)\mathbf{v}\|$ is (C, α)-good for all $\mathbf{v} \in \mathbb{R}^N \setminus 0$ and $t \in [A, \infty)$.

We identify $\mathrm{SL}(N, \mathbb{R})/\mathrm{SL}(N, \mathbb{Z})$ with the space of unimodular lattices

$$\mathcal{L}_N := \{\Lambda := \mathrm{Span}_{\mathbb{Z}}\{v_1, ..., v_n\} \mid \det(v_{i,j}) = 1\}.$$

Consider

$$\mathcal{B}_\epsilon := \{\Lambda \in \mathcal{L}_N \mid (\Lambda \setminus 0) \cap B_\epsilon(0) \neq \emptyset\}.$$

By Mahler's Criterion ([BM00], Theorem 3,2), $\mathcal{L}_N \setminus \mathcal{B}_\epsilon$ is compact for all $\epsilon > 0$ and thus $\mathcal{B}_\epsilon$ is a neighborhood of ∞.

To use Theorem 3.3.1, we assume the base point $\Lambda = g\mathbb{Z}^N$ and $h(t) = \varphi(t)g$ and $\rho := \inf_{\Delta \in P(\mathbb{Z}^N)} \|h(A)\Delta\|^{1/\mathrm{rank}(\Delta)}$.

Now it suffices to connect condition (1) to Proposition 3.3.1. To this end, we observe that for any rank k sublattice of $\mathbb{Z}^N$ with a $\mathbb{Z}$-basis $\mathbf{v}_1, \cdots, \mathbf{v}_k$,

$$h(t)\Delta = h(t)\mathbf{v}_1 \wedge \cdots \wedge h(t)\mathbf{v}_k.$$

The action of $h(t)$ on the linear space $\wedge_{i=1}^k \mathbb{R}^N$ is unipotent by Lemma 3.3.2 and therefore there exist $C, \alpha > 0$ that only depend on $\varphi(t)$ such that $t \mapsto \|h(t)\Delta\|$ is (C, α)-good.

It follows from Theorem 3.3.1 that there exist $C, \alpha > 0$ that for an interval $I \subset \mathbb{R}$,

$$|\{t \in I : \lambda_1(\varphi(t)\Lambda) \leq \epsilon\}| \leq C \left(\frac{\epsilon}{\rho}\right)^{\alpha} |I| \tag{3.3.3}$$

Now we take $I = [A, T]$ and f be a continuous bump function on $\mathrm{SL}(N, \mathbb{R})/\mathrm{SL}(N, \mathbb{Z})$ compactly supported on $\mathcal{B}_\epsilon$ and constantly equal to 1 on $\mathcal{B}_{\epsilon/2}$. It follows that for large T,

$$\mu(\infty) \leq \frac{1}{T}\int_A^T f(\varphi(t)\Lambda)dt \leq \frac{1}{T}\int_A^T \mathbf{1}_{\mathcal{B}_\epsilon}(\varphi(t)\Lambda)dt \leq C_N \left(\frac{\epsilon}{\rho}\right)^{\alpha}.$$

This completes the proof.

□

3.4 Unipotent invariance of limiting measure

An important ingredient of the linearization techique is to show the limiting measure is invariant under a one-parameter unipotent subgroup of G.

For the following, for $r < 1$, consider

$$T_{r,s}(t) := \left(t^{1-r} + (1-r)s\right)^{\frac{1}{1-r}} = t + st^r + o(t^r), \text{ where } s \in \mathbb{R}, \tag{3.4.1}$$

and let

$$T_{1,s}(t) := st, \text{ where } s > 0. \tag{3.4.2}$$

The following is a summary of the results we require from [**Poulios_thes**]

Proposition 3.4.1 (Poulious' Thesis)**.** *Let $\varphi : [0, \infty) \to \mathrm{SL}(n, \mathbb{R})$ be an unbounded curve definable in a polynomially bounded o-minimal structure. Then there exists a unique $r \leq 1$ such that the limit*

$$M_\varphi := \lim_{t\to\infty} t^r \cdot \varphi'(t) \cdot \varphi(t)^{-1} \tag{3.4.3}$$

exists and non-zero. We have that M_φ is nilpotent $\iff$ $r < 1$. M_φ is diagonalizable $\iff$ $r = 1$. Moreover, let $\mathfrak{p} = \mathrm{Span}_{\mathbb{R}}\{M_\varphi\} \leq \mathfrak{sl}(n, \mathbb{R})$, and $P = \exp \mathfrak{p}$. Then:

(1) $r < 1 \iff$ *for each* $s \in \mathbb{R}$*, the limit*

$$\rho(s) := \lim_{t\to\infty} \varphi(T_{r,s}(t))\varphi(t)^{-1}, \tag{3.4.4}$$

exists. In this case ρ *defines an isomorphism* $\mathbb{R} \to P$*. Here* $\mathbb{R}$ *denotes the additive group of real numbers.*

(2) $r = 1 \iff$ *for each* $s > 0$*, the limit*

$$\rho(s) := \lim_{t\to\infty} \varphi(T_{1,s}(t))\varphi(t)^{-1}, \tag{3.4.5}$$

exists. In this case ρ *defines an isomorphism* $\rho : \mathbb{R}_{>0} \to P$*. Here* $\mathbb{R}_{>0}$ *denote the multiplicative group of positive real numbers.*

Notice that when $\varphi(t)$ is contained in a unipotent subgroup, it follows that the matrix M_φ is nilpotent. In particular, it follows from the proposition that $r < 1$.

Next, we will show the invariance of limiting measure under this unipotent subgroup. To this end, we need the following elementary lemma from Calculus:

Lemma 3.4.2. *Let :* $l : (0,\infty) \to \mathbb{R}$ *be a differentiable function such that* $\lim_{t\to\infty} l'(t) = 0$ *as* $t \to \infty$*. Then, for any bounded continuous function* $f : \mathbb{R} \to \mathbb{R}$*, we have*

$$\frac{1}{T}\int_0^T [f(t + l(t)) - f(t)]dt = 0 \tag{3.4.6}$$

Proof. We don't necessarily have $\lim_{t\to\infty} f(t+l(t))/f(t) = 1$. Let $y = g(t) := t + l(t)$. Then by change of variables formula,

$$\begin{aligned}
&\frac{1}{T}\int_0^T [f(t+l(t)) - f(t)]dt \\
=&\frac{1}{T}\int_0^T f(y)dg^{-1}(y) - \frac{1}{T}\int_0^T f(t)dt \\
=&\frac{1}{T}\int_0^T f(y)dg^{-1}(y) - \frac{1}{T}\int_0^T f(t)dt \\
=&\frac{1}{T}\int_0^T f(y)\frac{1}{g'(g^{-1}(y))}dy - \frac{1}{T}\int_0^T f(t)dt \\
=&\frac{1}{T}\int_0^T f(y)\frac{1}{1+l'(g^{-1}(y))}dy - \frac{1}{T}\int_0^T f(t)dt \\
=&\frac{1}{T}\int_0^T f(t)\left[\frac{1}{1+l'(g^{-1}(t))} - 1\right]dt
\end{aligned}$$

Since $\frac{g(t)}{t} = \frac{y}{g^{-1}(y)}$, $\lim_{y\to\infty} g^{-1}(y) = \infty$, then last limit is zero by the boundedness of f and integral-truncation trick. □

Lemma 3.4.3. *Let $G \le \mathrm{SL}(n,\mathbb{R})$ be a closed subgroup and let $\Gamma \le G$ be a discrete subgroup. Suppose that $\varphi : [0,\infty) \to G$ is an unbounded curve definable in a polynomially bounded o-minimal structure such that (3.4.3) holds for $r < 1$. Let ρ as in (3.4.5). Then*

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T \{f(\rho(s)\varphi(t)x) - f(\varphi(t)x)\}dt = 0, \tag{3.4.7}$$

for all $f \in C_c(G/\Gamma)$, $x \in G/\Gamma$ and $s \in \mathbb{R}$.

Proof. For any $f \in C_c(G/\Gamma)$, $\epsilon > 0$, take T_ϵ such that

$$|f(\rho(s)\varphi(t)x) - f(\varphi(T_{r,s}(t))| \le |f(\rho(s)\varphi(t)x) - f(\varphi(T_{r,s}(t))\varphi(t)^{-1}]\varphi(t)x)| \le \epsilon/2,$$

for $t \ge T_\epsilon$. Now

$$\begin{aligned}
&\frac{1}{T}\int_0^T |f(\rho(s)\varphi(t)x) - f(\varphi(t)x)|dt \\
\le&\frac{1}{T}\int_0^{T_\epsilon} |f(\rho(s)\varphi(t)x) - f(\varphi(t)x)|dt + \frac{1}{T}\int_{T_\epsilon}^T |f(\rho(s)\varphi(t)x) - f(\varphi(t)x)|dt \\
\le&\frac{1}{T}\int_0^{T_\epsilon} |f(\rho(s)\varphi(t)x) - f(\varphi(t)x)|dt + \frac{1}{T}\int_{T_\epsilon}^T |f(\varphi(T_{r,s}(t)) - f(\varphi(t)x)|dt| + \frac{\epsilon}{2}
\end{aligned}$$

Now take $T \to \infty$. The first term goes to zero by boundedness of the function f and the middle term goes to zero by Lemma 3.4.2. $\square$

We call the i, j index of an upper-triangular matrix $b \in \mathrm{SL}(n, \mathbb{R})$ of the form

$$b = \begin{bmatrix} f_{1,1} & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots & \vdots & & \vdots & \vdots & & \vdots \\ \vdots & \ddots & \ddots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & f_{i,i} & 0 & \cdots & 0 & \boldsymbol{f_{i,j}} & \cdots & f_{i,n} \\ \vdots & & \vdots & \ddots & \ddots & & & \vdots & & \vdots \\ \vdots & & \vdots & & \ddots & \ddots & & \vdots & & \vdots \\ \vdots & & \vdots & & & \ddots & \ddots & \vdots & & \vdots \\ \vdots & & \vdots & & & & 0 & f_{j,j} & \cdots & f_{j,n} \\ \vdots & & \vdots & & & & & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & f_{n,n} \end{bmatrix}, \tag{3.4.8}$$

where $f_{i,j} \neq 0$, the *first non-zero off-diagonal entry*. More precisely, (i, j) is the first non-zero entry amoung the off-diagonal entries according to the following lexicographic order on $\mathbb{N}^2$:

$$(i, j) \prec (k, l) \iff i < k \text{ or } (i = k \text{ and } j < l). \tag{3.4.9}$$

Let $B \leq \mathrm{SL}(n, \mathbb{R})$ be the group of upper-triangular matrices. Let $b : [0, \infty) \to B$ be a definable curve. Recall that each definable function $f(t)$ is either zero for all large enough t or $f(t)$ either positive or negative for all large t. Thus, for all large enough t there exists a unique first non-zero off-diagonal entry in $b(t)$, or $b(t)$ is diagonal for all large t. We will refer to this entry as the first non-zero off-diagonal entry of the curve $b(t)$.

Lemma 3.4.4. *Let $\varphi : [0, \infty) \to \mathrm{SL}(n, \mathbb{R})$ be a definable curve. Then there's a definable curve $b : [0, \infty) \to B$, such that $b(t)$ is either diagonal for all large t, or the first non-zero off-diagonal entry $f_{i,j}$ satisfies*

(1) $\deg f_{i,i} \neq \deg f_{i,j}$, *and*

(2) $\deg f_{i,j} > \deg f_{j,j}$,

and importantly,

$$\varphi(t) = \sigma(t)b(t)C, \tag{3.4.10}$$

where $C \in \mathrm{SL}(n, \mathbb{R})$ is constant, and $\sigma(t) \in \mathrm{SL}(n, \mathbb{R})$ is convergent as $t \to \infty$.

Proof. Using the KAN decomposition, we first write $\varphi(t) \in \mathrm{SL}(n, \mathbb{R})$, as $\varphi(t) = k(t)B(t)$, where $k(t) \in \mathrm{SO}(n, \mathbb{R})$ and $B(t) \in B$ an upper-triangular matrix. Since $k(t)$ is obtained by performing the Gram-Schimdt process in the columns of φ, we conclude that $k(t)$ is definable. As a consequence, $B(t)$ is definable. Since $k(t)$ is bounded and definable, $\lim_{t\to\infty} k(t)$ exists. For all t large, $B(t)$ is either diagonal, or $B(t)$ takes the form of (3.4.8). To obtain the outcome of the lemma, we consider the following algorithm:

(1) If $\deg(f_{i,j}) = \deg(f_{i,i})$, then subtract the i-th column from the j-th column. This amounts to multiplying by a constant unipotent matrix from the right. The obtained matrix is the same, besides that the i, j-th entry is replaced with $f_{i,j} - f_{i,i}$. There are two possibilities now:

   (a) $f_{i,i} - f_{i,j}$ is eventually zero. This means that in the obtained matrix, the first non-zero off-diagonal entry has a larger index. Then one repeats step 1. again.

   (b) $\deg(f_{i,i} - f_{i,j}) \neq 0$. In this case, we get that $\deg(f_{i,i} - f_{i,j}) \neq \deg(f_{i,i})$, which fulfills the first requirement of Lemma 3.4.4. One continues then with the following step.

(2) If $\deg(f_{i,j}) \leq \deg(f_{j,j})$. Then subtract $\frac{f_{i,j}}{f_{j,j}}$ times the j-th row from the i-th row. This amounts to multiplying from the left by a unipotent matrix converging to identity as $t \to \infty$. The i, j-th entry in the resulting matrix is now replaces with zero, and the eventually first non-zero off-diagonal entry has a larger index. One now repeats step 1. again.

The algorithm ends with a finitely many steps with either a diagonal matrix, or a definable curve $b(t)$ satisfying the requirements of the lemma. $\square$

Definition 3.4.5. A curve $\{\varphi(t)\} \subset G \subset \mathrm{SL}(n,\mathbb{R})$ is called *essentially diagonal* it has the decomposition of Lemma 3.4.4 with $b(t)$ eventually diagonal.

Proposition 3.4.6. *An unbounded continuous curve $\varphi : [0,\infty) \to \mathrm{SL}(n,\mathbb{R})$ definable in a polynomially bounded o-minimal structure is essentially diagonalizable if and only if the unique $r \in \mathbb{R}$ such that $\lim_{t\to\infty} t^r \varphi'(t)\varphi(t)^{-1}$ exists in $\mathfrak{gl}(n,\mathbb{R}) - \{0\}$ is equal to 1.*

Proof. Using Lemma 3.4.4 we write:

$$\varphi(t) = \sigma(t)b(t)C,$$

where $\lim_{t\to\infty} \sigma(t) = g \in \mathrm{SL}(n,\mathbb{R})$. Let $T_{s,r}(t)$ be is as in (3.4.1)–(3.4.2), where $r \leq 1$. We note that the limit

$$\lim_{t\to\infty} \varphi(T_{s,r}(t))\varphi(t)^{-1}$$

exists, if and only if the limit

$$\lim_{t\to\infty} b(T_{s,r}(t))b(t)^{-1}$$

exists, and according to Proposition 3.4.1, we have

$$\lim_{t\to\infty} \varphi(T_{s,r}(t))\varphi(t)^{-1} = gC\exp(sM_b)(gC)^{-1}.$$

Thus, there's no loss in generality in assuming that $\varphi(t) = b(t)$ is either eventually diagonal or upper-triangular satisfying the conditions of the eventually first non-zero off- diagonal entry.

Since (i,j) is the first non-zero off diagonal entry in $b(t)$ (for all large enough t), we observe that (i,j)-th entry in the matrix $b'(t)b(t)^{-1}$ is

$$\frac{-f'_{i,i}f_{i,j} + f'_{i,j}f_{i,i}}{f_{i,i}f_{j,j}} = \left(\frac{f_{i,j}}{f_{i,i}}\right)' \cdot \frac{f_{i,i}}{f_{j,j}} \tag{3.4.11}$$

Since $r_{i,i} := \deg f_{i,i} \neq r_{i,j} := \deg f_{i,j}$, we have that $\deg \left(\frac{f_{i,j}}{f_{i,i}}\right)' = r_{i,j} - r_{i,i} - 1$ (see Theorem C.14). Thus,

$$\deg\left(\left(\frac{f_{i,j}}{f_{i,i}}\right)' \cdot \frac{f_{i,i}}{f_{j,j}}\right) = r_{i,j} - r_{j,j} - 1.$$

Recall by Lemma 3.4.4 that $r_{i,j} - r_{j,j} > 0$. Thus the (i,j)-th entry in $tb'(t)b(t)^{-1}$ is unbounded. As a consequence, $r < 1$. □

3.5 Linearization

3.5.1 Thin neighborhood of tubes — the singular sets

In this section we state the general definitions and results on a special class of closed subgroups in G which plays a key role in linearization technique.

Definition 3.5.1. Let $\mathcal{H}$ be the class of all closed connected proper subgroups H of G such that the identity component $\Gamma^0 \subset H$, $H/H \cap \Gamma$ admits an H-invariant probability measure and the subgroup W_H generated by all unipotent one-parameter subgroups of H acts ergodically on $H/H \cap \Gamma$ with respect to the H-invariant probability measure.

Theorem 3.5.2. *([Rat91, Theorem 1.1]) The collection $\mathcal{H}$ is countable.*

Let $\pi : G \to G/\Gamma$ be the canonical projection and let W be a subgroup generated by one-parameter unipotent subgroups of G contained in W. For $H \in \mathcal{H}$, define

$$\begin{aligned} N(H,W) =& \{g \in G : W \subset gHg^{-1}\}; \\ S(H,W) =& \bigcup_{H' \in \mathcal{H}, H' \subsetneq H} N(H',W); \\ T_H(W) =& \pi(N(H,W) - S(H,W)). \end{aligned}$$

The following is a consequence of Ratner's theorem ([MS95, Theorem 2.2]) describing probability measures invariant under the subgroup W given as above.

Theorem 3.5.3. *Let $\mathcal{P}$ denote the space of regular Borel probability measure on $X = G/\Gamma$ and let W be a subgroup which is generated by one-parameter unipotent subgroups of G contained in W. Assume that $\mu \in \mathcal{P}(X)$ is a W-invariant measure.*

For every $H \in \mathcal{H}$, let μ_H denote the restriction of μ on $T_H(W)$. Then the following holds.

(1) For all Borel measurable subsets $A \subset X$,

$$\mu(A) = \sum_{H \in \mathcal{H}^*} \mu_H(A),$$

where $\mathcal{H}^* \subset \mathcal{H}$ is a countable set consisting of one representative from each Γ-conjugacy class of elements in H.

(2) Each μ_H is W-invariant. For any W-ergodic component $\mu \in \mathcal{P}(X)$ of μ_H, there exists a $g \in N(H,W)$ such that μ is the (unique) gHg^{-1}-invariant probability measure on the closed orbit $gH\Gamma/\Gamma$.

As a consequence, there exists $H \in \mathcal{H}$ such that

$$\mu(\pi(N(H,W))) > 0 \text{ and } \mu(\pi(S(H,W))) = 0.$$

Moreover, almost every W-ergodic component of μ on $\pi(N(H,W))$ is a measure of the form $g_*\mu_H$, where $g \in N(H,W) \setminus S(H,W)$ and μ_H is a finite H-invariant measure on $\pi(H)$. In particular, if H is a normal subgroup of G then μ is invariant under H.

Let $H \in \mathcal{H}$. Let $\mathfrak{g}, \mathfrak{h}$ be the Lie algebras of G and H, respectively. Let $d = \dim \mathfrak{h}$ and $V_H = \bigwedge^d \mathfrak{g}$. Consider the adjoint representation of G on $V_H = \bigwedge^d \mathfrak{g}$. Fix a vector $p_H \in \bigwedge^d \mathfrak{h} \setminus \{0\}$. Also define a continuous map $\eta_H : G \to V_H$ by $\eta_H(g) = g \cdot p_H = \mathrm{Ad}_g.p_H$ (with $\mathrm{Ad}_g(\bigwedge_{i=1}^n x_i) := \bigwedge_{i=1}^n \mathrm{Ad}_g(x_i)$ and extended to V_H by multi-linearity). Define

$$N^1(H) := \eta_H^{-1}(p_H) = \{g \in N(H) : \det(\mathrm{Ad}_g|_{\mathfrak{h}}) = 1\}, \tag{3.5.1}$$

where $N(H)$ is the normalizer of H in G.

Put $\Gamma_H = N(H) \cap \Gamma$. It follows that for any $\gamma \in \Gamma_H$, we have $\gamma(H\Gamma/\Gamma) = H\Gamma/\Gamma$ and hence γ preserves the volume of $H\Gamma/\Gamma$. Therefore, $\gamma.p_H = \pm p_H$.

Now we define $\overline{V}_H = V_H/\{\pm 1\}$ if $\Gamma_H.p_H = \{\pm p_H\}$ and $\overline{V}_H = V_H$ if $\Gamma_H.p_H = p_H$. The action of G factors through the quotient map of V_H onto $\overline{V}_H$. Let $\overline{p}_H$ denote the image of p_H in $\overline{V}_H$ and define $\overline{\eta}_H : G \to \overline{V}_H$ as $\overline{\eta}_H(g) = g.\overline{p}_H$ for all $g \in G$. Then $\Gamma_H = \overline{\eta}(\overline{p}_H) \cap \Gamma$.

For any subset Z of G/Γ, define

$$\mathrm{Rep}(Z) := \{g.\overline{p}_H : g \in G, \pi(g) \in Z\} \subset \overline{V}_H. \tag{3.5.2}$$

Theorem 3.5.4. *([DM93, Theorem 3.4]) Let $H \in \mathcal{H}$, then*

(1) the orbit $\Gamma.\overline{p}_H$ is discrete in $\overline{V}_H$ and hence closed;

(2) $N^1(H)\Gamma$ is closed in G/Γ;

(3) For any compact set $Z \subset G/\Gamma$, the set Rep(x) is discrete in $\overline{V}_H$.

(4) For any compact set $Z \subset G/\Gamma$, the set Rep(Z) is closed in $\overline{V}_H$.

(5) The map $\phi : G/\Gamma_H \to G/\Gamma \times \overline{V}_H$ defined by

$$\phi(g\Gamma_H) = (\pi(g), \overline{\eta}_H(g)), \forall g \in G, \tag{3.5.3}$$

is proper.

Recall that W is a subgroup of G which is generated by unipotent one-parameter subgroups of G contained in W. Let A_H denote the Zariski closure of $\overline{\eta}_H(N(H,W))$ in $\overline{V}_H$. Evidently, $N(H,W)$ is contained in the preimage $\overline{\eta}_H^{-1}(A_H)$. Indeed, we have the following:

Lemma 3.5.5. *([DM93, Proposition 3.2]) Let $H \in \mathcal{H}$, then $\overline{\eta}_H^{-1}(A_H) = N(H,W)$.*

Proposition 3.5.6. *([MS95, Proposition 3.2]) Let D be a compact subset of $A_H \subset \overline{V}_H$. Define*

$$S(D) = \{g \in \overline{\eta}_H^{-1}(D) : g\gamma \in \overline{\eta}_H^{-1}(D) \text{ for some } \gamma \in \Gamma - \Gamma_H\} \subset G. \tag{3.5.4}$$

Then the following holds:

(1) $S(D) \subset S(H,W)$;

(2) $\pi(S(D))$ is closed in X;

(3) For any compact set $K \subset X - \pi(S(D))$, there exists a neighborhood Φ of D in $\overline{V}_H$ such that every $y \in \pi(\overline{\eta}_H^{-1}(\Phi)) \cap K$ has a unique representative in Φ; that is, the set $\overline{\eta}_H(\pi^{-1}(y)) \cap \Phi$ consists of a single element.

3.5.2 A dichotomy theorem

Theorem 3.5.7. *For $H \in \mathcal{H}$, let a compact set $C \subset A_H$ and a $0 < \epsilon < 1$ be given. Then there exists a closed subset $\mathcal{S}$ of X contained in $\pi(S(H, W))$ such that the following holds: For a given compact set $K \subset X - \mathcal{S}$, there exists a neighborhood Ψ of C in $\overline{V}_H$ such that for any unbounded unipotent definable curve $\{\gamma(t)\}_{t>A}$ of G in a polynomially bounded o-minimal structure and any $x \in X$, at least one of the following is satisfied:*

(1) There exists $w \in \overline{\eta}_H(\pi^{-1}(x)) \cap \overline{\Psi}$ and bounded $\{\delta(t)\}_{t>A} \subset G$ with $\lim_{t\to\infty} \delta(t) = e$, such that

$$\{\delta(t)^{-1}\gamma(t)\} \subset G_w := \{g \in G : g.w = w\}. \tag{3.5.5}$$

In other words, there exists $g \in G$, such that $\delta(t)^{-1}\gamma(t).x \subset gN^1(H)\Gamma$.

(2) For all large $T > 0$,

$$|\{t \in [A, T] : \gamma(t).x \in K \cap \pi(\overline{\eta}_H^{-1}(\Psi))\}| \leq \epsilon T \tag{3.5.6}$$

Proof. For given compact set C and $0 < \epsilon < 1$, we will show (3.5.6) holds whenever (3.5.5) fails.

For $H \in \mathcal{H}$, let A_H and $V := \overline{V}_H$ be as in Section 3.5.1. Since A_H is a real algebraic subvariety of V. By the Hilbert Basis Theorem, A_H is a set of zeros of a finitely many polynomials $f_1, ..., f_r$, there exists a real polynomial function $p := f_1^2 + \cdots + f_r^2$ on V such that

$$A_H := \{\mathbf{v} \in V : p(\mathbf{v}) = 0\}. \tag{3.5.7}$$

In particular, A_H is contained in a co-dimension one hyperplane $V' \subset V$. Without loss of generality, we may assume $V = \mathbb{R}^{\dim V}$ and $V' = \mathbb{R}^{\dim V-1} \times 0$.

Let $R_1 > 0$ be the smallest number such that $[-R_1, R_1]^{\dim V-1}$ is the smallest closed cube containing C. We then determine R_2, ϵ_2, and ϵ_1 (see the end of the proof) consecutively and define $\Phi := [-R_1, R_1]^{\dim V-1} \times [-\epsilon_1, \epsilon_1] \supset D$ and $\Psi := [-R_2, R_2]^{\dim V-1} \times [-\epsilon_2, \epsilon_2] \supset C$ and prove a relative-time property for C and D.

Put $\Omega = \pi(\overline{\eta}_H^{-1}(\Psi) \cap K$ and define

$$J = \{t \geq T_0 : \gamma(t).x \in \overline{\Omega}\} \tag{3.5.8}$$

Then for every $t \in J$, there exists a unique $w = w(t) \in \overline{\eta}_H(\pi^{-1}(x)) \subset V$ such that $\gamma(t).w := \rho(\gamma(t)).w \in \Phi$, in which case $\gamma(t).w \in \overline{\Psi}$.

Since $s \mapsto \gamma(s).w$ is a polynomially bounded definable map, it is either convergent to a constant map fixing w or unbounded.

In the first case, it is well-known from the theory of algebraic groups that $\rho(\gamma(t))$ is still unipotent as a linear map on V, we have the stable-unstable decomposition $\rho(\gamma(t)) = S(t)U(t)$ with $S(t) \to \mathrm{Id}$. Since the map $G \to G.\overline{p}_H$ is open, there exists $\delta(t) \to e$ in G such that $\rho(\delta^{-1}(t)\gamma(t)) = U(t)$, and that $\delta(t)^{-1}\gamma(t) \subset G_w := \{g \in G : g.w = w\}$. This becomes the first outcome of the theorem; or we have

$$\|\gamma(s).w\| \to \infty \tag{3.5.9}$$

as $s \to \infty$. We will show in this case (3.5.6) holds.

For every $t \in J$, we define $I(t) := [t^-, t^+]$ to be the largest closed interval in $[T_0, T]$ containing t such that

(1) $\gamma(s).w \in \overline{\Phi}$ for all $s \in I(t)$;

(2) $\gamma(t^+).w \in \overline{\Phi} - \Phi$ or $t^+ = T$;

(3) $t^- \in J$.

For the case $t^+ = T$, we denote the corresponding maximal interval by $I_{\text{last}} := I(t)$.

For any $s_1, s_2 \in I(t)$, we have by the maximality of $I(t)$ that

$$\text{either } I(s_1) = I(s_2) \text{ or } I(s_1) \cap I(s_2) = \varnothing \tag{3.5.10}$$

Now we have the decomposition let $[T_o, T] = \bigsqcup_k I(t_k) \bigsqcup I_{\text{last}}$. We will show by (C, α)-goodness that the relative-time property is satisfied on each interval. To this end, we need to discuss a few cases:

Let $\mathbf{n}$ denote the unit vector perpendicular to the hyperplane V'.

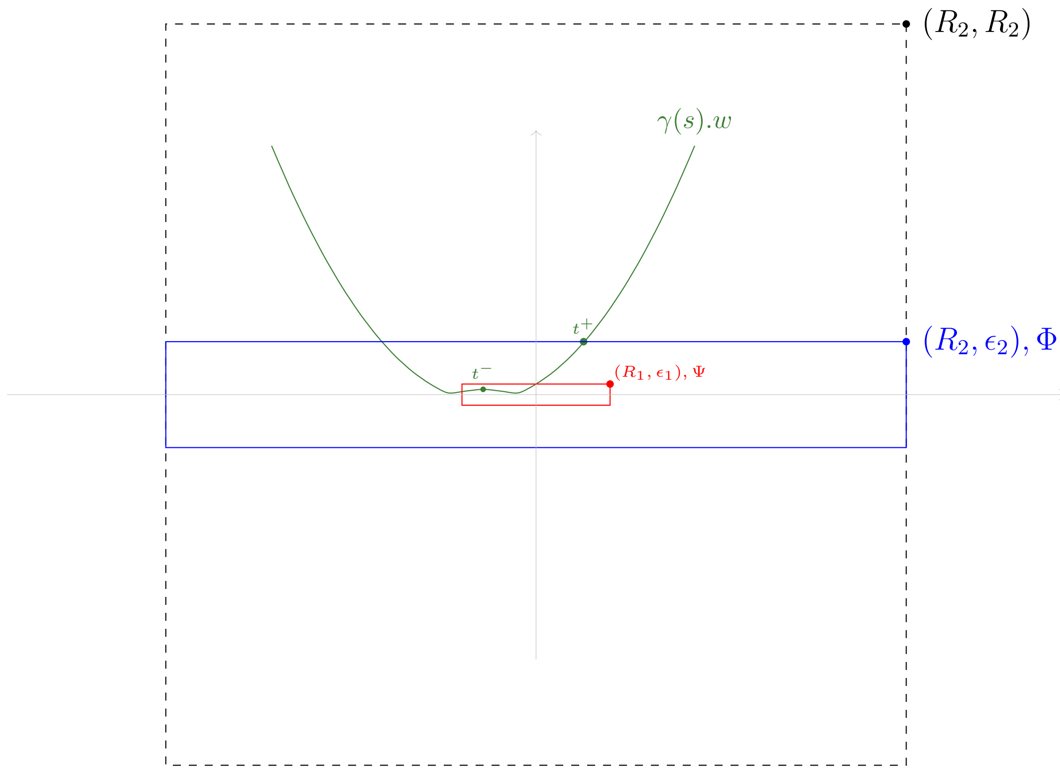

Figure 3.1: Segment of trajectory leaving R_2-box from the ceiling (with $\dim V = 2$)

Case 1: If $\gamma(s).w$ on some $I(t_k) = [t^-, t^+]$ satisfies:

$$|\mathbf{n} \cdot \gamma(s).w| \le \epsilon_2, \forall s \in I(t) \quad \text{and} \quad |\mathbf{n} \cdot \gamma(t^+).w| = \epsilon_2, \tag{3.5.11}$$

noticing that by our construction there exists $\nu > 0$ and $C_0, \alpha_0 > 0$ such that $t^\nu |\mathbf{n} \cdot \gamma(t).w|$ is (C_0, α_0)-good for some C_0 and α and that

$$\sup_{s \in [t^-, t^+]} |s^\nu \mathbf{n} \cdot \gamma(s).w| = |(t^+)^\nu \mathbf{n} \cdot \gamma(t^+).w| = (t^+)^\nu \epsilon_2,$$

we have

$$\begin{aligned}
|I(t) \cap J| \le & |\{t \in I(t) : |\mathbf{n} \cdot \gamma(s).w| \le \epsilon_1\}| \\
\le & |\{s \in I(t) : s^\nu |\mathbf{n} \cdot \gamma(s).w| \le s^\nu \epsilon_1\}| \\
\le & |\{s \in I(t) : s^\nu |\mathbf{n} \cdot \gamma(s).w| \le (t^+)^\nu \epsilon_1\}| \\
\le & C_0 \left(\frac{(t^+)^\nu \epsilon_1}{\sup_{s \in [t^-, t^+]} |s^\nu \mathbf{n} \cdot \gamma(s).w|} \right)^{\alpha_0} |I(t)| \\
\le & C_0 \left(\frac{\epsilon_1}{\epsilon_2} \right)^{\alpha_0} |I(t)|.
\end{aligned}$$

Case 2: If $\gamma(s).w$ on some $I(t_k) = [t^-, t^+]$ satisfies:

$$\|\gamma(s).w\| \le R_2, \forall s \in I(t) \quad \text{and} \quad \|\gamma(t^+).w\| = R_2, \tag{3.5.12}$$

$$\begin{aligned}
|I(t) \cap J| \le & |\{s \in I(t) : \|\gamma(s).w\| \le R_1\}| \\
\le & C_1 \left(\frac{R_1}{\sup_{s \in I(t)} \|\gamma(s).w\|} \right)^{\alpha_1} |I(t)| \\
= & C_1 \left(\frac{R_1}{R_2} \right)^{\alpha_1} |I(t)|.
\end{aligned}$$

Case 3: We now consider $\gamma(s).w$ on the interval $I_{\text{last}} = [t^-, t^+] = [t^-, T]$. By discreteness, there are only finitely many $w = w(t) \in \overline{\eta}_H(\pi^{-1}(x)) \subset V$ with $\|w\| \le R_2$, denoted $w_1, w_2, ..., w_N$ (note that at most one of them corresponds to $t \in J$). Therefore, by the unboundedness of $\gamma(s).w_i$, there exists $T_1 > T_0$ such that for any $s \ge T_1$,

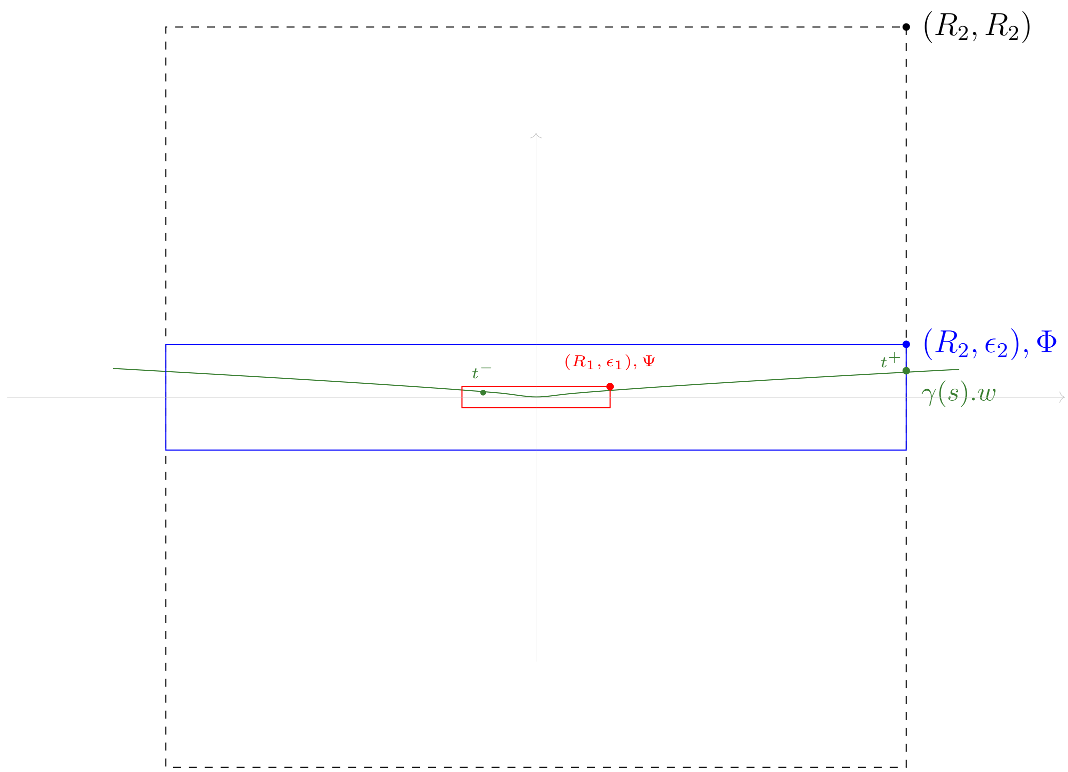

Figure 3.2: Segment of trajectory leaving R_2-box from the side (with $\dim V = 2$).

$\|\gamma(s).w_i\| \geq R_1, i = 1, 2, ..., N$ and therefore we can ignore them on $[T_1, T]$ and assume the base point $w = w(t)$ satisfies $\|w\| \geq R_2$. Now

$$\begin{aligned}
|I(t) \cap J| \leq & \{s \in [T_0, T] : \|\gamma(s).w\| \leq R_1\} \\
\leq & C_1 \left(\frac{R_1}{\sup_{s\in[T_0,T]} \|\gamma(s).w\|} \right)^{\alpha_1} |I(t)| \\
\leq & C_1 \left(\frac{R_1}{\|\gamma(T_0).w\|} \right)^{\alpha_1} |I(t)| = C_1 \left(\frac{R_1}{\|\gamma(T_0)^{-1}\|^{-1}\|w\|} \right)^{\alpha_1} |I(t) \\
\leq & C_1 \left(\frac{R_1\|\gamma(T_0)^{-1}\|}{R_2} \right)^{\alpha_1} |I(t)| \\
\leq & C_1 \left(\frac{R_1\|\gamma(T_0)^{-1}\|}{R_2} \right)^{\alpha_1} T
\end{aligned}$$

Now for the given R_1, we first choose R_2 so that $C_1 \left(\frac{R_1}{R_2}\right)^{\alpha_1} < \epsilon$ and that $C_1 \left(\frac{R_1\|\gamma(T_0)^{-1}\|}{R_2}\right)^{\alpha_1} < \epsilon$. Then we choose $\epsilon_2 = R_2/2$ and $\epsilon_1 < \epsilon_2$ so that $C_0 \left(\frac{\epsilon_1}{\epsilon_2}\right)^{\alpha_0} < \epsilon$. □

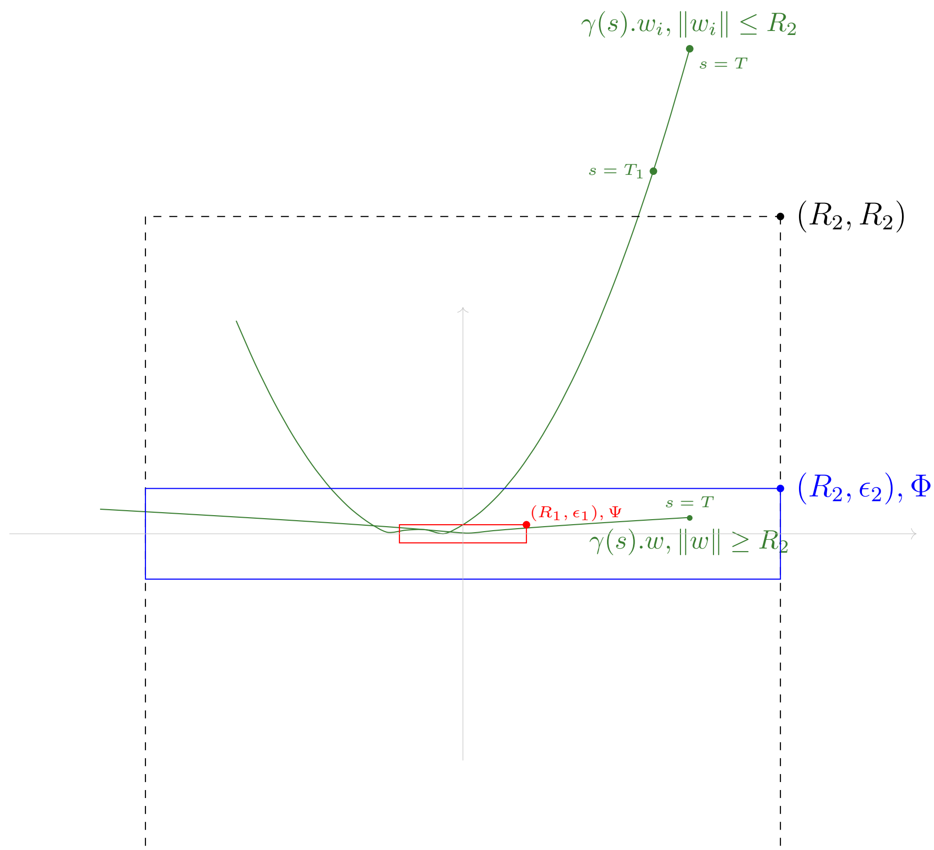

Figure 3.3: The last segment of trajectory with different base points w ($\dim V = 2$)

3.6 Proof of Theorem 3.1.2

3.6.1 Lifting properties

Lemma 3.6.1. *Let G be a locally compact Hausdorff group and H be closed normal subgroup of G and Γ be a lattice in G. Furthermore, assume $H \cap \Gamma$ is a lattice in H. Let $q : G/\Gamma \to G/H\Gamma \cong \frac{G/H}{H\Gamma/H}$ be the natural quotient map. For an H-invariant probability measure μ on G/Γ, we have*

$$\int_{G/\Gamma} f(g\Gamma) d\mu(g\Gamma) = \int_{G/H\Gamma} \int_{Hg\Gamma/\Gamma} f(gh\Gamma) d(h\Gamma) dq_*\mu(gH\Gamma)$$

where $q_*\mu$ is the push-forward of μ under q.

Proof. By ergodic decomposition, we have

$$\int_{G/\Gamma} f(g\Gamma)d\mu(g\Gamma) = \int_{G/H\Gamma}\int_{Hg\Gamma/\Gamma} f(gh\Gamma)d\mu^{\mathcal{A}}_{gH\Gamma}(h\Gamma)dq_*\mu(gH\Gamma), \tag{3.6.1}$$

where $\mu^{\mathcal{A}}_{gH\Gamma}$ is the conditional measure concentrated at the $\mathcal{A}$-atom of x relative to the sub-sigma-algebra $q^{-1}(\mathscr{B}(G/H\Gamma)) \subset \mathscr{B}(G/\Gamma)$. Given μ is H-invariant, we would like to show $\mu^{\mathcal{A}}_{gH\Gamma}$ is also H-invariant. Thus by the uniqueness of H-invariant measure on $Hg\Gamma/\Gamma$, we have $d\mu^{\mathcal{A}}_{gH\Gamma}(h\Gamma) = d(h\Gamma)$ for $q_*\mu$ almost every $gH\Gamma \in G/H\Gamma$. To show this, we recall from the construction of conditional measures that for $q_*\mu$ a.e. $gH\Gamma \in G/H\Gamma$, $\mu^{\mathcal{A}}_{gH\Gamma}$ is the unique probability measure on $gH\Gamma$ satisfying

$$\int_{G/H\Gamma} \mu^{\mathcal{A}}_{gH\Gamma}(f)dq_*\mu(gH\Gamma) = \mu(f) = h_*\mu(f) = \int_{G/H\Gamma} h_*\mu^{\mathcal{A}}_{gH\Gamma}(f)dq_*\mu(gH\Gamma)$$

and thus we have for every $h \in H$, $h_*\mu^{\mathcal{A}}_{gH\Gamma} = \mu^{\mathcal{A}}_{gH\Gamma}$ for $q_*\mu$ a.e. $gH\Gamma$. Let $S := \{(h, gH\Gamma) : h_*\mu^{\mathcal{A}}_{gH\Gamma} \neq \mu^{\mathcal{A}}_{gH\Gamma}\}$. Now by Fubini's theorem,

$$0 = \int_H \int_{G/H\Gamma} \mathbf{1}_S dq_*\mu d\mu_H = \int_{G/H\Gamma}\int_H \mathbf{1}_S d\mu_H dq_*\mu$$

Therefore for almost every $gH\Gamma \in G/H\Gamma$, we have $h_*\mu^{\mathcal{A}}_{gH\Gamma} = \mu^{\mathcal{A}}_{gH\Gamma}$ for μ_H a.e. $h \in H$. But by approximation, it follows that $h_*\mu^{\mathcal{A}}_{gH\Gamma} = \mu^{\mathcal{A}}_{gH\Gamma}$ for every $h \in H$. $\square$

The following corollary is immediate from Lemma 3.6.1

Corollary 3.6.2. *If $q_*\mu = \delta_{g_0H\Gamma}$, the Dirac measure supported at $g_0H\Gamma \in G/H\Gamma$, then μ is supported on $g_0H\Gamma/\Gamma \subset G/\Gamma$.*

Corollary 3.6.3. *If $q_*\mu$ is G/H-invariant, then μ is G-invariant.*

Proof. For any $g_0 \in G$, let $L_{g_0}\phi(g) = \phi(g_0^{-1}g)$ be the left translation operator. Then

$$\begin{aligned}
&\int_{G/\Gamma} L_{g_0}(f(g\Gamma))d\mu(g\Gamma) \\
&= \int_{G/H\Gamma} P(L_{g_0}(f))(gH\Gamma)dq_*\mu(gH\Gamma) && \text{(by Lemma 3.6.1)} \\
&= \int_{G/H\Gamma}\int_{H\Gamma/\Gamma} L_{g_0}(f)(gh\Gamma)d(h\Gamma)dq_*\mu(gH\Gamma) \\
&= \int_{G/H\Gamma}\int_{H\Gamma/\Gamma} f(g_0^{-1}gh\Gamma)d(h\Gamma)dq_*\mu(gH\Gamma) \\
&= \int_{G/H\Gamma} P(f)(g_0^{-1}gH\Gamma)dq_*\mu(gH\Gamma) \\
&= \int_{G/H\Gamma} L_{g_0^{-1}H}P(f)(gH\Gamma)dq_*\mu(gH\Gamma) \\
&= \int_{G/H\Gamma} P(f)(gH\Gamma)dq_*\mu(gH\Gamma) && \text{(by the } G/H\text{-invariance of } q_*\mu)
\end{aligned}$$

□

Proof of Theorem 3.1.2. Let W be the maximal subgroup generated by unipotent elements under which the limiting measure μ is invariant.

Let $\mu = \mu_{\gamma,\Lambda,\infty}$ denote the limiting measure. Let W be the subgroup of G preserving μ. In particular, this group contains PS_γ.

By Theorem 3.5.3, we have that there exists $H \in \mathcal{H}$ such that

$$\mu(\pi(N(H,W))) > 0 \text{ and } \mu(\pi(S(H,W))) = 0.$$

Let $C_1 \subset N(H,W) - S(H,W)$ be a compact set such that $\pi(C_1) \cap \pi(\pi(S(H,W))) = \varnothing$ and that $\mu(\pi(C_1)) = \alpha$ for some $\alpha > 0$. Now we apply Theorem 3.5.7 for $C = \overline{\eta}_H(C_1)$ and $\alpha = \epsilon/2$. Let $\mathcal{S}$ be as in Theorem 3.5.7. Then there exist a compact neighborhood K of $\pi(C_1)$ in X such that $K \cap \mathcal{S} = \varnothing$. Put $\Omega = \pi(\overline{\eta}_H^{-1}(\Psi)) \cap K$ where Ψ is as in Theorem 3.5.7. Since $\Psi \supset C$, $\Omega \supset \pi(C_1)$ and therefore $\mu(\Omega) \geq 2\epsilon > \epsilon$, contradicting to the second outcome of Theorem 3.5.7 as $T \gg 1$.

Therefore $\{\delta^{-1}(t)\gamma(t)\}_{t\geq A} \subset g_1 N^1(H) g_1^{-1}$ for some $g \in G$. By Theorem 3.5.4, the orbit $g_1 N^1(H)\Gamma$ is closed. Now put $G_1 := g_1 N^1(H) g_1^{-1}$, $H_1 = g_1 H g_1^{-1}$ and $\Gamma_1 = \Gamma \cap G_1$. Clearly H_1 is normal in G_1.

Since $H_1\Gamma_1$ is closed in G_1, the subgroup $H_1\Gamma_1/\Gamma_1 = \Gamma_1 H_1/\Gamma_1$ is closed (and hence discrete) in G_1/Γ_1 and we can view $G_1/H_1\Gamma_1$ as a G_1/H_1 homogeneous space. Let

$$q : G_1/\Gamma_1 \to \frac{G_1/H_1}{H_1\Gamma_1/H_1} \cong G_1/H_1\Gamma_1$$

be the natural quotient map. Define a map $q_* : \mathscr{P}(G_1/\Gamma_1) \to \mathscr{P}(G_1/H_1\Gamma_1)$ such that for any $\nu \in \mathscr{P}(G_1/\Gamma_1)$ and any Borel measurable subset $A \subset G_1/H_1\Gamma_1$, $q_*(\nu)(A) = \nu(q^{-1}(A))$. Then q_* is continuous.

The following lemma will be needed:

Lemma 3.6.4. *Let $q : G_1/\Gamma_1 \to G_1/H_1\Gamma_1$ be the quotient map as above where $G_1/H_1\Gamma_1$ is viewed as a G_1/H_1 homogeneous space. Let ν be a regular Borel measure on G_1/Γ_1. If $q_*(\nu)$ is invariant under $gH_1 \in G_1/H_1$, then ν is invariant under g.*

Let $\iota : G_1/H_1 \to \mathrm{SL}(n_1, \mathbb{R})$ be a regular algebraic group embedding. Then the map

$$\tilde{\gamma}(t) := \iota \circ \pi_1 \circ \gamma(t) : [A, \infty) \to G_1 \to G_1/H_1 \hookrightarrow \mathrm{SL}(n_1, \mathbb{R})$$

is again definable in a polynomially bounded o-minimal structure.

Case 1: $\tilde{\gamma}(t)$ is bounded (and thus convergent by definability). Suppose $\tilde{\gamma}(t) \to g_0 \in \mathrm{SL}(n_1, \mathbb{R})$, then the orbit $\gamma(t)H_1\Gamma_1 \to x_0 = \iota^{-1}(g_0)H_1\Gamma_1/H_1 \in G_1/H_1\Gamma_1$. Now the limiting measure associated to $\gamma(t)H_1\Gamma_1$ is $q_*(\mu) = \delta_{x_0}$. It follows that $\mu = (g_0)_*\mu_{H_1\Gamma_1}$, where $\mu_{H_1\Gamma_1}$ denotes the H_1-invariant Haar measure on $H_1\Gamma_1/\Gamma_1$.

Case 2: $\tilde{\gamma}(t)$ is unbounded. In this case, the Peterzil-Starchenko subgroup $PS_{\tilde{\gamma}} \subset \iota(G_1/H_1)$ is a non-trivial one-parameter subgroup. This is unipotent by the Poulious' condition and Proposition 3.4.6. $q_*(\mu)$ is invariant under the generator $\tilde{g} \in PS_{\tilde{\gamma}}$. Write

$\iota^{-1}(\tilde{g}) = gH_1$, then by Lemma 3.6.4, we must have μ is invariant under $g \notin H_1$. Since $W \subset H_1$, $g_1 \notin W$, which is a contradiction to the maximality of W. $\square$

Appendix A: More on Successive Minima

In this appendix we prove a few results on successive minima for a lattice that are known to experts but whose proofs are hard to find in the standard literature on geometry of numbers, for example [Cas97] and [SC89] . The goal is to relate the basis of a lattice to successive minima.

Recall that for a positive integer d and a lattice $\Lambda \subset \mathbb{R}^d$, and for each $j = 1, \ldots, d$, The j-th minimum of a lattice $\Lambda \subset \mathbb{R}^d$, denoted $\lambda_j(\Lambda)$, is the infimum of λ such that the set $\{r \in \Lambda : \|r\| \leq \lambda\}$ contains j linearly independent vectors. (with respect to the l^2 norm on $\mathbb{R}^d$).

A natural question is, can the successive minima always attained by a basis of the rank d lattice Λ? In other words, does there exist a basis $\{v_1, \ldots, v_d\}$ of Λ such that

$$\|v_j\| = \lambda_j, \text{ for } j = \{1, 2, \ldots, d\}.$$

The answer is positive if and only if $d \leq 4$, as are shown in the following theorem and example

Theorem A.1. *Let Λ be a lattice $\mathbb{R}^d$. Assume that $d \leq 4$, then there exist a basis $\{v_1, \ldots, v_d\}$ of Λ such that*

$$\|v_j\| = \lambda_j, \text{ for } j = \{1, 2, \ldots, d\}.$$

The case when $d = 1$ is trivial. To prove this theorem for the cases $d = 2, 3$, we need the following lemma from Euclidean geometry.

Lemma A.2.

(1) The minimal distance from any point in the interior of a parallelogram in $\mathbb{R}^2$ to its vertices is always strictly less than the maximal length of the edges of the parallelogram.

(2) The minimal distance from a point in the interior of a parallelepiped in $\mathbb{R}^3$ to its vertices is always strictly less than the maximal length of three linearly independent vectors form by the vertices , with at least two of them being the edges of the parallelopiped. In particular, these three vector will span the three dimensional lattice spanned by this parallelopiped.

Proof.

For the part (1), observe that a parallelogram $ABCB'$ can be divided into two triangles ABC and $AB'C$, and any point D in the interior of $ABCB'$ must fall in either the triangle ABC or the triangle $AB'C$

By drawing a line perpendicular to the line AC through the point B, we easily see

$$|BD| \leq |BE| < \max\{|AB|, |BC|\}.$$

For the part (2), first observe that a parallelepiped can be divided into six tetrahedra and any point x in the interior of the parallelepiped, say $ABCDEFGH$ must fall into one of the six.

If X falls in the tetrahedra $AFEH$. It follows from the first part of this lemma that

$$|EX| \leq |EY| \leq \max\{|EA|, |EZ|\} < \max\{|EH|, |EF|, |EA|\},$$

If X falls in the tetrahedra $DHAF$. It follows from the first part of this lemma that

$$|DX| \leq |DY| \leq \max\{|DA|, |DZ|\} < \max\{|DA|, |DH|, |DF|\},$$

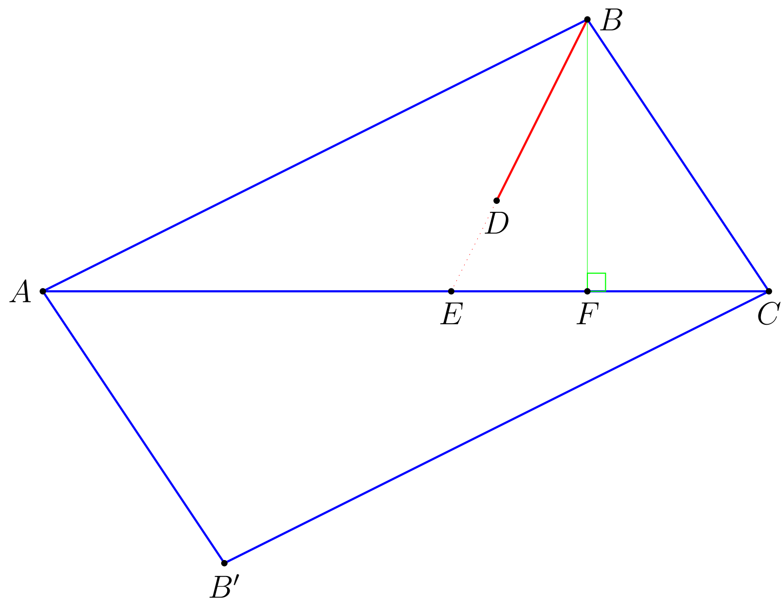

Figure A.1: The parallelogram case

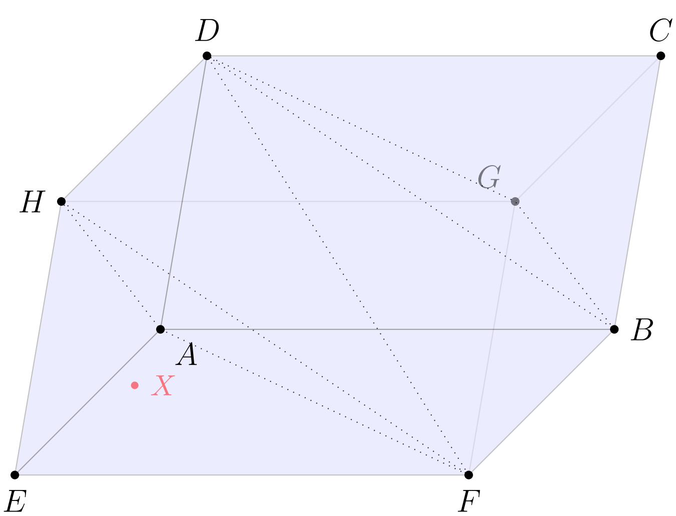

Figure A.2: X must fall into one of six tetrahedra.

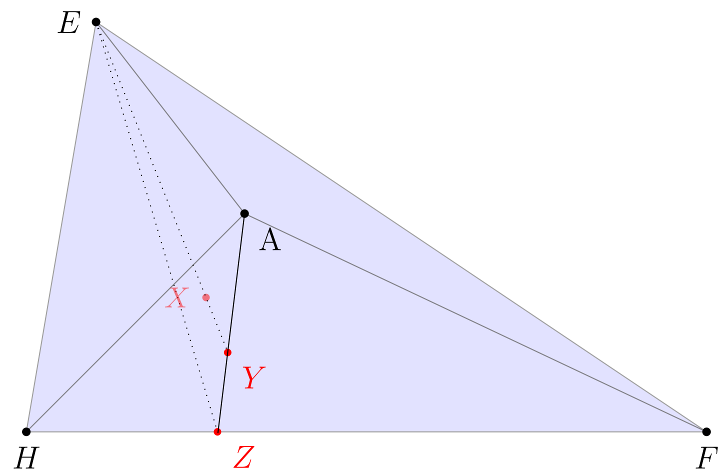

Figure A.3: Construction of points Y, Z in the proof.

If X falls in the tetrahedra $BAFD$. It follows from the first part of this lemma that

$$|BX| \leq |BY| \leq \max\{|BA|, |EZ|\} < \max\{|BA|, |BF|, |BD|\},$$

where the construction of auxiliary points and segments as illustrated in the figure A.3 above.

□

Proof of the theorem for the case d=2,3:

We do this for $d = 3$ and the case $d = 2$ is only simpler. Let v_1, v_2, v_3 be any linearly independent vectors in Λ such that

$$\|v_j\| = \lambda_j, \text{ for } j = \{1, 2, 3\}.$$

Let Λ_0 be the lattice spanned by those three vectors. Consider the fundamental domain

$$F := \{t_1 v_1 + t_2 v_2 + t_3 v_3 : t_i \in [0, 1)\}$$

of Λ_0.

The closure of F is the parallelepipiped spanned by vectors v_1, v_2, v_3 at the origin of $\mathbb{R}^3$ and it follows that $\mathbb{R}^3 = F + \Lambda_0$.

Suppose on the contrary that Λ_0 is a proper sublattice of Λ, then there exists a vector $x \in \Lambda - \Lambda_0$. By translating x with a vector in λ_0, we may assume without loss of generality that

$$x = t_1 v_1 + t_2 v_2 + t_3 v_3,$$

where $t_i \in (0, 1)$ (t_i cannot be equal to zero since $x \notin \Lambda_0$). So x is in the interior of the parallelepipiped.

By our lemma above, noticing that the length of each edge in the parallelepipiped is equal to $\|v_1\|, \|v_2\|$ or $\|v_3\|$, there exists a vertex w of the parallelepipiped spanned by vectors v_1, v_2, v_3 at the origin such that

$$\|x - w\| < \max\{\|v_j\| : j = 1, 2, 3\} = \|v_3\|.$$

Translating the vector $x - w$ to the origin. It follows that $x - w, v_1, v_2$ are still linearly independent and this lead to the contradiction to the assumption that $v_3 = \lambda_3(\lambda)$. Therefore we must have

$$\Lambda_0 = \Lambda,$$

namely v_1, v_2, v_3 form a basis of λ.

$\square$

We need the following lemma for the proof of $d = 4$ case:

Lemma A.3. *For $k \geq 1$ and $y_i \in \mathbb{R}$ for all $1 \leq i \leq k$, we have the identity*

$$S_k := \sum x_1 x_2 \cdots x_k = 1,$$

where the sum is over all 2^k possible choices of $x_i = y_i$ or $x_i = 1 - y_i$.

Proof. We perform induction on k.

When $k = 1$, this sum is simply $1 - y_1 + y_1 = 1$.

Assume $S_{k-1} = 1$. For general k, we observe that $S_k = y_i S_{k-1} + (1 - y_i) S_{k-1} = S_{k-1} = 1$. □

Proof of the theorem for the case d=4:

The idea of proof is essentially due to Noam Elkies [Elk]. Let v_1, v_2, v_3, v_4 be any linearly independent vectors in Λ such that

$$\|v_j\| = \lambda_j(\Lambda), \text{ for } j = \{1, 2, 3, 4\}.$$

As in the proof of cases $d = 2, 3$, we let Λ_0 denote the lattice spanned by v_1, v_2, v_3, v_4.

If λ_0 is a proper sublattice of λ, then there exist $v \in \Lambda - \Lambda_0$ such that

$$v = \sum_{i=1}^{4} t_i v_i, t_i \in \mathbb{R}$$

and without loss of generality, we may assume that $t_i \in [0, 1)$ for any $i = 1, 2, 3, 4$.

Namely v lives in

$$\mathscr{P} := \left\{ \sum_{i=1}^{4} t_i v_i, t_i \in [0, 1) \right\}.$$

Claim 7. *For any $v_0 \in \Lambda_0$, and any $v \in \Lambda - \Lambda_0$*

$$\|v - v_0\| \geq \|v_i\|, \ \forall i = 1, 2, 3, 4$$

Proof of Claim. If $\|v - v_0\| < \|v_{i_0}\|$ for some $i_0 \in \{1, 2, 3, 4\}$, then v_i, $i \in \{1, 2, 3, 4\} \setminus \{i_0\}$, together with $v - v_0$ would be still linearly independent (since $v \in \Lambda - \Lambda_0$) and form a new system of successive minima with strictly smaller λ_i. #

Claim 8. *For any $v \in \mathscr{P}$ we have*

$$\min_{v_0 \in \Lambda_0} \|v - v_0\|^2 \leq \frac{1}{4} \sum_{i=1}^{4} \|v_i\|^2.$$

Proof of Claim.

The vertices of $\mathscr{P}$ form the set:

$$\mathscr{V} := \left\{ \sum_{i=1}^{4} n_i v_i : n_i \in \{0,1\} \right\}.$$

In view of the preceding lemma, the idea here is to find a weighted sum of squared distances from v to each vertices in $\mathscr{V}$. For $v = t_1v_1 + t_2v_2 + t_3v_3 + t_4v_4$, we associae the weights $w(v_0)$ to each $\|v - v_0\|^2$ where $v_0 \in \mathscr{V}$:

If $v_0 = \sum_{i=1}^{4} n_i v_i, n_i \in \{0,1\}$, then $w(v_0) := \prod_{i=1}^{4}((2t_i - 1)n_i + (1 - t_i))$. For example, if $v_0 = v_2 + v_3$, then $v_0 = (1 - t_1)t_2t_3(1 - t_4)$.

It follows immediately from the preceding lemma that

$$\prod_{v_0 \in V} w(v_0) = \sum_{x_i = 1 - t_i \text{ or } t_i} x_1x_2x_3x_4 = 1.$$

The claim then follows from the following subclaim:

$$\sum_{v_0 \in \mathscr{V}} w(v_0)\|v - v_0\|^2 = \sum_{i=1}^{4} t_i(1 - t_i)\|v_i\|^2 \leq \sum_{i=1}^{4} \frac{1}{4}\|v_i\|^2.$$

Indeed, if we write $\|v - v_0\|^2 = \langle \sum_i (t_i - n_i)v_i, \sum_i (t_i - n_i)v_i \rangle$ and in view of lemma for the case when $k = 3$, the coefficient for each $\langle v_i, v_i \rangle$ is

$$(1 - t_i)t_i^2 + t_i(1 - t_i)^2 = (1 - t_i)t_i.$$

In view of lemma for the case when $k = 2$, the coefficient for each $\langle v_i, v_j \rangle$ where $i \neq j$, is

$$2(1 - t_i)(1 - t_j)t_it_j + 2(1 - t_i)t_jt_i(t_j - 1) + 2t_i(1 - t_j)(t_i - 1)t_j + 2t_it_j(t_i - 1)(t_j - 1) = 0.$$

#

a

Now we find the minimum of

$$\sum_{v_0 \in \mathscr{V}} \|v - v_0\|^2,$$

for $v = t_1 v_1 + \cdots + t_4 v_4 \in \mathscr{P}, t_1, t_2, t_3, t_4 \in [0, 1)$.

Indeed,

$$\sum_{v_0 \in \mathscr{V}} \|v - v_0\|^2 = \sum_{n_i \in \{0,1\}} \left\langle \sum_{i=1}^{4} (t_i - n_i) v_i, \sum_{i=1}^{4} (t_i - n_i) v_i \right\rangle$$

For the moment, we assume that (t_1, t_2, t_3, t_4) can take any value in $\mathbb{R}^4$ and this problem becomes a standard optimization problem without constraints.

Now taking the partial derivative with respect to t_i, for $i = 1, 2, 3, 4$, we obtain the following system of linear equations:

$$\begin{aligned}
&\frac{\partial}{\partial t_i} \sum_{v_0 \in \mathscr{V}} \|v - v_0\|^2 \\
&= \frac{\partial}{\partial t_i} \sum_{n_i \in \{0,1\}} \left\langle \sum_{i=1}^{4} (t_i - n_i) v_i, \sum_{i=1}^{4} (t_i - n_i) v_i \right\rangle \\
&= \sum_{n_i \in \{0,1\}} \left\langle v_i \sum_{i=1}^{4} (t_i - n_i) v_i \right\rangle.
\end{aligned}$$

We may write the solution to the critical points of this four variable function in its matrix form as:

$$\begin{bmatrix} \langle v_1, v_1 \rangle & \langle v_1, v_2 \rangle & \langle v_1, v_3 \rangle & \langle v_1, v_4 \rangle \\ \langle v_2, v_1 \rangle & \langle v_2, v_2 \rangle & \langle v_2, v_3 \rangle & \langle v_2, v_4 \rangle \\ \langle v_3, v_1 \rangle & \langle v_3, v_2 \rangle & \langle v_3, v_2 \rangle & \langle v_3, v_4 \rangle \\ \langle v_4, v_1 \rangle & \langle v_4, v_2 \rangle & \langle v_4, v_3 \rangle & \langle v_4, v_4 \rangle \end{bmatrix} \begin{bmatrix} 2t_1 - 1 \\ 2t_2 - 1 \\ 2t_3 - 1 \\ 2t_4 - 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Since $\{v_1, v_2, v_3, v_4\}$ are linearly independent, the coefficient matrix, as a Gram matrix, is nondegenerate and the unique solution to this equation is

$$t_1 = t_2 = t_3 = t_4 = \frac{1}{2}.$$

The second derivative test gives immediately that this is a local, and thus global minimum for the function, and its miminum value is

$$\begin{aligned}
\min_{v\in\mathbb{R}^4}\sum_{v_0\in\mathscr{V}}\|v-v_0\|^2 &= \sum_{n_i\in\{0,1\}}\left\langle\sum_{i=1}^{4}(t_i-n_i)v_i,\sum_{i=1}^{4}(t_i-n_i)v_i\right\rangle\\
&= \sum_{n_i\in\{0,1\}}\left\langle\sum_{i=1}^{4}(\frac{1}{2}-n_i)v_i,\sum_{i=1}^{4}(\frac{1}{2}-n_i)v_i\right\rangle\\
&= \sum_{n_i\in\{0,1\}}\left\langle\sum_{i=1}^{4}(\frac{1}{2}-n_i)v_i,\sum_{i=1}^{4}(\frac{1}{2}-n_i)v_i\right\rangle\\
&= 4\sum_{i=1}^{4}\|v_i\|^2,
\end{aligned}$$

where the last equality follows from the cancellations in the cross terms $\langle v_i, v_j\rangle$ whenever $i\neq j$. It follows that (noticing that $|\mathscr{V}| = 16$)

$$\begin{aligned}
\min_{v\in\Lambda-\Lambda_0}\min_{v_0\in\Lambda_0}\|v-v_0\|^2 &\leq \min_{v\in\Lambda-\Lambda_0}\frac{1}{16}\sum_{v_0\in\mathscr{V}}\|v-v_0\|^2\\
&\leq \frac{1}{4}\sum_{i=1}^{4}\|v_i\|^2 \qquad \text{(by the Claim 8)}\\
&\leq \max\{\|v_j\| : j=1,2,3,4\}
\end{aligned}$$

Now combining this with Claim 7 above yields

$$\max\{\|v_j\| : j=1,2,3,4\} \leq \frac{1}{4}\sum_{i=1}^{4}\|v_i\|^2 \leq \max\{\|v_j\| : j=1,2,3,4\}, \tag{A.0.1}$$

and thus

$$\|v_1\| = \|v_2\| = \|v_3\| = \|v_4\|.$$

Claim 9. $\langle v_i, v_j\rangle = 0$ *for all* $i\neq j$.

Proof of Claim. Let us summarize what we have obtained so far:

We proved that if $v_1, v_2.v_3, v_4$ are linearly independent and $\|v_j\| = \lambda_j, j = 1,2,3,4$, then any vector $v\in\mathscr{P}\cap\Lambda-\Lambda_0$ must be of the form:

$$\frac{1}{2}(v_1+v_2+v_3+v_4),$$

Since $\mathscr{P}$ is a fundamental domain of Λ, it follows that

$$\Lambda = \frac{1}{2}(v_1 + v_2 + v_3 + v_4) + \Lambda_0.$$

On the other hand, from the inequality

$$\|v_i\|^2 = \lambda_i(\Lambda)^2 \leq \left\| \frac{1}{2}(\pm v_1 \pm v_2 \pm v_3 \pm v_4) \right\|^2,$$

we have

$$\sum_{1 \leq i < j \leq 4} \pm \langle v_i, v_j \rangle \geq 0.$$

By symmetry,

$$\sum_{1 \leq i < j \leq 4} \pm \langle v_i, v_j \rangle = 0. \tag{A.0.2}$$

If we view this equation as a linear system with $\binom{4}{2} = 6$ variables $\langle v_i, v_j \rangle$ and the coefficient matrix

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & -1 \\ 1 & 1 & 1 & 1 & -1 & -1 \\ 1 & 1 & 1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & -1 & -1 & -1 & -1 & -1 \end{bmatrix}$$

is clearly of rank 6, which forces

$$\langle v_i, v_j \rangle = 0,$$

for all $i \neq j$. #

Hence either

$$\Lambda = \Lambda_0 = \mathrm{Span}_{\mathbb{Z}}\{v_1, v_2, v_3, v_4\}$$

or v_i's are of equal length and mutually orthogonal

$$\Lambda = \mathrm{Span}_{\mathbb{Z}}\{v_1, v_2, v_3, \frac{1}{2}(v_1 + v_2 + v_3 + v_4)\} \supsetneq \Lambda_0$$

In either case, it is possible to find a basis of Λ corresponding to the four successive minima of the lattice, as desired. This completes the proof of the case $d = 4$.

$\square$

The following example shows that the theorem above fails for $d \geq 5$.

Example A.4. Let $d \geq 5$ and consider the lattice Λ spanned by

$$e_1, e_2, \ldots, e_{d-1}, \frac{1}{2}(e_1 + \cdots + e_d),$$

where e_i is the canonical basis vector of $\mathbb{R}^d$ whose i-th component is 1 while all the other components are zero.

It is easy to see that Λ contains $\mathbb{Z}^d$ since

$$e_d = 2 \cdot \frac{1}{2}(e_1 + \cdots + e_d) - e_1 - \cdots - e_{d-1} \in \Lambda.$$

Observe that $\lambda_i(\Lambda) = 1, \forall i = 1, 2, \ldots, d$ since the closed unit ball at the origin contains exactly d linearly independent vectors $e_1, \cdots, e_d$ with equal length 1.

On the other hand, we cannot find a basis $v_1, \cdots, v_d$ of Λ satisfying

$$\|v_i\| = 1, \ \forall i = 1, 2, \cdots, d.$$

This is because each vector in Λ is either of the form e_i or of the form

$$\frac{1}{2}(\pm n_i e_i \pm n_j e_j \pm n_k e_k \pm n_p e_p \pm n_q e_q),$$

where n_i, n_j, n_k, n_p, n_q are all nonzero integers.

Since $\Lambda \supsetneq \mathbb{Z}^d$, the basis vectors of Λ cannot only be in the former case. But for the latter case, the sum of squares of coefficients is at least $\frac{5}{4}$, contradicting to $\|v_i\| = 1, \ \forall i =$

$1, 2, \cdots, d$. Therefore, for $d \geq 5$, it is not true that the successive minima of a lattice can be realized by a basis of the lattice. $\square$

However, with a compromise, we can still choose a basis whose lengths are equivalent to the successive minima of the lattice. To this end, we need the following lemma (the proof given here mimics the proof of Theorem 11.33. in [EW11]):

Lemma A.5. *Let Λ be a lattice in $\mathbb{R}^d$ and $v_1 \in \Lambda$ be a vector with $\|v_1\| = \lambda_1(\Lambda)$. In other words, this is a nonzero vector in Λ of shortest length. Let π_1 be the projection of $\mathbb{R}^d$ onto $v_1^{\perp}$, the hyperplane in $\mathbb{R}^d$ orthogonal to v_1, then we have to following statements:*

(1) $\|\pi_1(v)\| \geq \frac{\sqrt{3}}{2}\|v_1\|, \forall v \in \Lambda$;

(2) $\pi_1(\Lambda)$ *is a lattice* [9] *in* $v_1^{\perp}$ *with covolume* $\frac{covol(\lambda)}{\|v_1\|}$;

Proof. For (1) and (2), we suppose on the contrary that there is a vector $v \in \Lambda$ such that

$$\|\pi_1(v)\| < \frac{\sqrt{3}}{2}\|v_1\|.$$

The orthogonal decomposition of $\mathbb{R}^d$ gives

$$v = \pi_1(v) + tv_1,$$

for some $t \in \mathbb{R}$.

Since $\pi_1(v + nv_1) = \pi_1(v)$, for all $n \in \mathbb{Z}$, by replacing v with $v + nv_1$ for some n, we may assume $v = \pi_1(v) + tv_1$ with $t \in [-\frac{1}{2}, \frac{1}{2})$.

[9] The projections of lattices are not always lattices of the corresponding subspaces. For example, the projection of the standard lattice $\mathbb{Z}^2$ onto the irrational line $y = \sqrt{2}x$ is no longer a lattice with respect to that line, which can be deduced from Dirichlet's simultaneous Diophantine approximation theorem.

For (1), since v_1 is perpendicular to $\pi_1(v)$, by the Pythagorean's theorem

$$\begin{aligned}\|v\|^2 &= \|\pi(v)\|^2 + t^2\|v_1\|^2 \\ &< \frac{3}{4}\|v_1\|^2 + \frac{1}{4}\|v_1\|^2 \\ &= \|v_1\|^2,\end{aligned}$$

which contradicts to the choice of $\|v_1\|$ as a minimal-length nonzero vector of Λ. This proves (1).

To see (2), we first observe that from (1), all vectors in $\pi_1(\Lambda)$ are bounded $\frac{\sqrt{3}}{2}\|v_1\|$ away from zero (This gives the discreteness). On the other hand clearly $\pi_1(\Lambda)$ contains $d-1$ linearly independent vectors. So by definition, $\pi_1(\Lambda)$ is a lattice in $v_1^{\perp}$ and it makes sense from now to talk about its fundamental domain, covolume and success minima.

We shall first study the relation between the fundamental domain of Λ and $\pi_1(\Lambda)$. Let F_1 be a fundamental domain of $\pi_1(\Lambda)$.

Claim 10. *$F := F_1 + [0,1)v_1$ is a fundamental domain of Λ*

Proof of Claim. For any $x \in \mathbb{R}^d$,$\pi_1 \in v_1^{\perp}$. By the definition of fundamental domain $\pi_1(\Lambda)$, there exists a vector $v \in \Lambda$ such that

$$\pi_1(x - v) \in F.$$

It follows that $x - v - \pi_1(x - v) \in \mathbb{R}v_1$. Since $\pi_1(v_1) = 0$, there exists $n \in \mathbb{Z}$ and $t \in [0,1)$ such that

$$x - v - \pi_1(x - v) \in \mathbb{R}v_1 = nv_1 + tv_1.$$

Therefore, $x - v - nv_1 = tv_1 + \pi_1(x - v) \in [0,1)v_1 + F_1$. Namely for any vector x in $\mathbb{R}^d$, we can find a translation of x by a vector in Λ that falls into $[0,1)v_1 + F_1 =: F$.

On the other hand, if $x - v'$ and $x - v''$ are both in F with $v', v'' \in \Lambda$, we would like to see $v' = v''$.

Suppose:

$$\begin{cases} x - v' = t'v_1 + y' \\ x - v'' = t''v_1 + y'' \end{cases},$$

where $t', t'' \in [0, 1)$ and $y', y'' \in F_1$.

Applying π_1 to both sides, we get

$$\begin{cases} y' = \pi_1(x) - \pi_1(v') \\ y'' = \pi_1(x) - \pi_1(v'') \end{cases}.$$

Since F_1 is a fundamental domain for $\pi_1(\mathbb{R}^d)/\pi_1(\Lambda)$, the translation is unique and $\pi_1(v') = \pi_1(v'')$. So $v' - v'' \in \mathbb{Z}v_1$.

But $v' - v'' = (x - v'') - (x - v') \in [0, 1)v_1 + F_1 - \big([0, 1)v_1 + F_1\big) = (-1, 1)v_1 + (F_1 - F_1)$, so it forces $v' = v''$. #

Now since v_1 is orthogonal to all vectors in F_1, it follows that

$$\begin{aligned} \infty > \operatorname{covol}(\Lambda) &= m(F) \\ &= m([0, 1)v_1 + F_1) \\ &= \|v_1\| \cdot m(F_1) \\ &= \|v_1\| \cdot \operatorname{covol}(\pi_1(\Lambda)) \end{aligned}$$

This proves (2).

□

Theorem A.6. *Let Λ be a lattice in $\mathbb{R}^d$. Then there exist a basis $v_1, v_2, \ldots, v_d$ of Λ such that*

$$\|v_1\| = \lambda_1(\Lambda), \|v_2\|_d \asymp_d \lambda_2(\Lambda), \ldots, \|v_d\| \asymp_d \lambda_d(\Lambda).$$

Here $A \asymp_d B$ means there exist positive constants c_d, C_d depending only on d such that

$$c_d|A| \le |B| \le C_d|A|.$$

Proof. We shall prove this by induction on d. The case $d = 1$ is obvious.

Assume the statement holds for lattices with rank less than or equal to $d-1$. For a rank d lattice Λ in $\mathbb{R}^d$, let v_1 be any nonzero vector in Λ satisfying $\|v_1\| = \lambda_1(\Lambda)$ and let π_1 be the projection of $\mathbb{R}^d$ onto $v_1^\perp$, the hyperplane in $\mathbb{R}^d$ orthogonal to v_1 as in the previous lemma.

Now applying the induction hypothesis to the $d-1$ dimensional hyperplane $v_1^\perp$ and the rank $d-1$ lattice $\pi_1(\Lambda)$ contained in $v_1^\perp$ yields a basis $w_2 \dots w_d$ of $\pi_1(\Lambda)$ with

$$\|w_2\| = \lambda_1(\pi_1(\Lambda)), \|w_3\| \asymp_d \lambda_2(\pi_1(\Lambda)), \dots, \|w_d\| \asymp_d \lambda_{d-1}(\pi_1(\Lambda)).$$

By the monotonicity of $\lambda_j' s$, we know

$$\|w_2\| \lesssim_d \cdots \lesssim_d \|w_d\|.$$

Our next step is to choose some $v_2, \dots, v_d$ in Λ as preimages of $w_2, \dots, w_d$ under π_1 such that $v_1, v_2 \dots, v_d$ form a basis of Λ. We start by choosing $v_2, \cdots, v_d$ to be any $d-1$ vectors in $\mathbb{R}^d$ with

$$\pi_1(v_j) = w_j, 2 \le j \le d.$$

It follows that $v_1, \cdots, v_d$ are $\mathbb{R}$-linearly independent and thus form an $\mathbb{R}$-linear basis of $\mathbb{R}^d$.

For any $v \in \Lambda$,

$$\begin{aligned}\pi_1(v) &= n_2 w_2 + \cdots + n_d w_d \\ &= n_2 \pi_1(v_2) + \cdots + n_d \pi_1(v_d) \\ &= \pi_1(n_2 v_2 + \cdots + n_d v_d).\end{aligned}$$

So $\pi_1[v - (n_2 v_2 + \cdots + n_d v_d)] = 0$ and

$$v = n_2 v_2 + \cdots + n_d v_d + t v_1,$$

for some $t \in \mathbb{R}$. But since $v \in \Lambda$, $tv_1 \in \Lambda$ and thus $t = 0$ or ± 1 since v_1 by our choice is a minimal nonzero vector.

Therefore, $\Lambda = \mathrm{Span}_{\mathbb{Z}}\{v_1, \dots v_d\}$. Namely $v_1, \dots, v_d$ indeed form a basis for Λ.

Observe that replacing each v_j with $v_j + n_j v_1, n_j \in \mathbb{Z}$ does not change the nature that $v_1, \cdots, v_d$ form a basis of Λ. Since $v_j = w_j + t_j v_1$ for some $t_j \in \mathbb{R}$, by carefully choosing n_j we may assume $t_j \in [-\frac{1}{2}, \frac{1}{2})$.

It follows that for $2 \le j \le d$,

$$\begin{aligned}\|w_j\| \le \|v_j\| &\le \|w_j\| + |t_j|\|v_1\| \\ &\le \|w_j\| + \frac{1}{2}\frac{2}{\sqrt{3}}\|w_2\| \\ &\lesssim_d (1 + \frac{1}{\sqrt{3}})\|w_j\|,\end{aligned}$$

where the second inequality follows from the previous lemma with $\pi_1(v_2) = w_2$. So $\|v_j\| \asymp_d \|w_j\|, j = 2 \dots, d$.

Next, we observe that $\lambda_{j-1}(\pi_1(\Lambda)) \le \lambda_j(\Lambda)$. This is because if $v_1, v_2', \dots v_j'$ represent the first j successive minima vectors in Λ, then their projection images (excluding $\pi_1(v_1) = 0$), $\pi_1(v_1), \pi_1(v_2'), \dots \pi_1(v_j')$ are still linearly independent in $v_1^{\perp}$ and

$$\begin{cases} \|\pi_1(v_2')\| \le \lambda_j(\Lambda) \\ \quad \vdots \\ \|\pi_1(v_j')\| \le \lambda_j(\Lambda) \end{cases},$$

which implies $\lambda_{j-1}(\pi_1(\Lambda)) \le \lambda_j(\Lambda)$. Therefore

$$\|v_j\| \asymp_d \|w_j\| \asymp_d \lambda_{j-1}(\pi_1(\Lambda)) \le \lambda_j(\Lambda)$$

for $j = 2, \dots, d$.

On the other hand,

$$\begin{aligned}\lambda_j(\Lambda) &\le \max\{\|v_1\|, \dots, \|v_j\|\} \\ &\lesssim_d \max\{\|w_2\|, \dots, \|w_j\|\} \\ &\lesssim_d \|w_j\| \\ &= \lambda_{j-1}(\pi_1(\Lambda)).\end{aligned}$$

Therefore

$$\|v_j\| \asymp_d \lambda_j(\Lambda), j = 1, 2, \dots d.$$

The proof is complete by the induction hypothesis. $\square$

Remark A.7. In practice, the basis in the theorem can be achieve by the Minkowski reduced basis. A basis $\{b_1, \dots, b_d\}$ of a lattice $\Lambda \subset \mathbb{R}^d$ is called *Minkowski reduced* if for each $1 \leq i \leq d$, b_i is the shortest nonzero vector in the lattice such that i linearly independent vectors $\{b_1, ...b_i\}$ can be extended to a basis of the lattice. See [Hel85] for an algorithm to produce a Minkowski reduced basis. Interestingly, it is still not know whether the construction of shortest vectors in a lattice with respect to the l^2 norm is NP-hard or not (But the answer is affirmative for the l^∞-norm [Boa81]. Moreover, the l^2 case is proved to be NP-hard for randomized algorithms in [Ajt98]).

As another corollary to our Lemma A.5, we can prove the classical Minkowski's Second Convex Body Theorem:

Theorem A.8 (Minkowski's Second Convex Body Theorem, 1896 [Min96])**.** *Let $\Lambda \subset \mathbb{R}^d$ be a lattice and let $\lambda_k(\Lambda)$ denote the k-th successive minima of Λ. Then*

$$\lambda_1(\Lambda) \cdots \lambda_d(\Lambda) \asymp_d \mathit{covol}(\Lambda).$$

Proof. Like we did in the previous proof, we still proceed by induction. The case $d = 1$ is obvious.

Assume the statement holds for lattices with rank less than or equal to $d - 1$. Let v_1 be any nonzero vector in Λ satisfying $\|v_1\| = \lambda_1(\Lambda)$ and let π_1 be the projection of $\mathbb{R}^d$ onto $v_1^\perp$, the hyperplane in $\mathbb{R}^d$ orthogonal to v_1 as in the previous lemma.

Now applying Lemma A.5 (1) to the $d-1$ dimensional hyperplane $v_1^\perp$ and the rank $d-1$ lattice $\pi_1(\Lambda)$ contained in $v_1^\perp$ yields a basis $w_2 \dots w_d$ of $\pi_1(\Lambda)$ with

$$\|w_2\| = \lambda_1(\pi_1(\Lambda)), \|w_3\| \asymp_d \lambda_2(\pi_1(\Lambda)), \dots, \|w_d\| \asymp_d \lambda_{d-1}(\pi_1(\Lambda)).$$

By the induction hypothesis

$$\|w_2\| \cdots \|w_d\| \asymp_d \lambda_1(\pi_1(\Lambda)) \cdots \lambda_{d-1}(\pi_1(\Lambda)) \asymp_d \operatorname{covol}(\pi_1(\Lambda)).$$

On the other hand, from the proof of the Theorem A.6, we know

$$\|w_j\| \asymp_d \|v_j\| \asymp_d \lambda_j(\Lambda), j = 2, \dots d.$$

Since $\|v_1\| = \lambda_1(\Lambda)$ by construction, by Lemma A.5 (2),

$$\operatorname{covol}(\Lambda) = \operatorname{covol}(\pi_1(\Lambda)) \cdot \|v_1\| \asymp_d \lambda_1(\Lambda) \cdots \lambda_d(\Lambda).$$

□

Next, we study the continuity of successive minima on the space of lattices.

Lemma A.9. *Let $b \in \mathrm{SL}(d, \mathbb{R})$ and $\|b\|_{op}$ denotes the operator norm of b, then we have for all $i = 1, 2, \dots, d$ and unimodular lattice Λ, the inequality*

$$\lambda_i(b\Lambda) \leq \|b\|_{op} \lambda_i(\Lambda) \tag{A.0.3}$$

Proof. For $i = 1, 2, \dots, d$, let $v_1, \dots, v_i$ denote the i linearly independent vectors in $\mathbb{R}^d$ such that

$$\|v_i\| = \lambda_i(\Lambda).$$

Consider the vectors $bv_1, \dots, bv_i$. Since $b \in \mathrm{SL}(d, \mathbb{R})$, $bv_1, \dots, bv_i$ are again linearly independent. From $\|bv_i\| \leq \|b\|_{op} \|v_i\|$ it follows that $bv_1, \dots, bv_i$ are contained in a ball of radius $\|b\|_{op} \lambda_i(\Lambda)$. So it follows that $\lambda_i(b\Lambda) \leq \|b\|_{op} \lambda_i(\Lambda)$. □

Theorem A.10. $\lambda_i(\cdot)$ *are continuous functions on the space of unimodular lattices* $\mathcal{L}$ *for* $i = 1, 2, \cdots d$.

Proof. We may identify $\mathcal{L}$ with the homogeneous space $G/\Gamma := \mathrm{SL}(d,\mathbb{R})/\mathrm{SL}(d,\mathbb{Z})$ By the Lemma A.9, we have for any $b, c \in \mathrm{SL}(d,\mathbb{R})$,

$$\frac{1}{\|b\|_{op}}\lambda_i(b\Lambda) \leq \lambda_i(\Lambda) \leq \|c\|_{op}\lambda_i(c^{-1}\Lambda). \tag{A.0.4}$$

For any $\Lambda \in \mathcal{L}$, we may write $\Lambda = g\mathbb{Z}^d$ for some $g \in \mathrm{SL}(d,\mathbb{R})$, identified with $g\Gamma$. For any convergent sequence of lattices

$$g_i\Gamma \to g\Gamma, t \to \infty \tag{A.0.5}$$

which is equivalent to the convergence $g^{-1}g_j\Gamma \to \Gamma$.

Let d denote any right-invariant metric on G and define a metric d' on G/Γ by

$$d'(g\Gamma, h\Gamma) := \inf_{\gamma_1,\gamma_2\in\Gamma} d(g\gamma_1, h\gamma_2).$$

For each i, we may choose $\gamma_i \in \Gamma$ as the element closest to $g^{-1}g_i$, namely

$$d(g^{-1}g_j, \gamma_j) = \min_{\gamma\in\Gamma} d(g^{-1}g_i, \gamma) = d'(g^{-1}g_j\Gamma, \Gamma).$$

It follows that the condition $d'(g_j\Gamma, g\Gamma) \to 0$ is equivalent to $d(g^{-1}g, \gamma_j) \to 0$. Therefore, by replacing the representative g_j in $g_j\Gamma$ with $g_i\gamma$. We may assume $g_j \to g$ for the equation A.0.5.

Now taking $b = g_jg^{-1}$ and $c = b^{-1}$ in the inequality A.0.4, we have

$$\lambda_i(g_jg^{-1}\Lambda) \to \lambda(\Lambda).$$

and therefore λ_i is continuous on G/Γ.

□

Remark A.11. For $G = \mathrm{SL}(d,\mathbb{R})$ and $\Gamma = \mathrm{SL}(d,\mathbb{Z})$. There is a right G-invariant (Riemannian) metric dist on G/Γ. We speculate the following inequality holds:

$$|\lambda_i(g\mathbb{Z}^d) - \lambda_i(h\mathbb{Z}^d)| \leq C_d \mathrm{dist}(g\Gamma, h\Gamma), \forall g, h \in G, \forall i,$$

where C_d is a constant depending only on d.

Appendix B: The Siegel Sets and Invariant Measure on the Space $\mathrm{SL}(d,\mathbb{R})/\mathrm{SL}(d,\mathbb{Z})$ and the Computation of the Constant $c_{d,k}$ for the Generalized Siegel's Formula

In this appendix we shall recall a few definitions and results on Siegel sets and the probability Haar measure on the space $\mathrm{SL}(d,\mathbb{R})/\mathrm{SL}(d,\mathbb{Z})$ and use them to compute the coefficient in the generalized Siegel's formula 1.3.7 in Chapter 2.

The main reference for the following is [BM00] Chapter V and [Fol15] Section 2.6.

Let $K := SO(d,\mathbb{R})$,

$$A := \{\mathrm{diag}(a_1, ...a_d) : a_1 \cdots a_d = 1, a_i > 0, \forall i = 1, 2, \dots, d\},$$

the diagonal subgroup of $\mathrm{SL}(d,\mathbb{R})$ with positive entries and

$$N := \{(n_{ij}) \in \mathrm{SL}(d,\mathbb{R}) : n_{ii} = 1, n_{ij} = 0, \forall i < j\},$$

the subgroup of upper triangular unipotent matrices in G. We have

Theorem B.1 (Iwasawa Decomposition)**.** *The product map*

$$K \times A \times N \to G, (k, a, n) \to kan$$

is a homeomorphism.

Definition B.2 (Siegel Sets in $\mathrm{SL}(d,\mathbb{R})$).

A Siegel Set in $\mathrm{SL}(d,\mathbb{R})$ is a set $\Sigma_{t,u}$ of the form

$$\Sigma_{t,u} := KA_tN_u,$$

where $t,u>0$ and A_t and N_u are given by

$$A_t := \left\{ \mathrm{diag}(a_1, ...a_d) \in A : \frac{a_i}{a_{i+1}} \leq \frac{2}{\sqrt{3}}, i = 1,2,...d-1 \right\}$$

and

$$N_u := \{(n_ij) \in N : |n_{ij}| \leq u, \forall i < j\}$$

It turns out that $\Sigma_{t,u}$ can cover the fundamental domain of $G := \mathrm{SL}(d,\mathbb{R})$ under the action of $\Gamma := \mathrm{SL}(d,\mathbb{Z})$:

Theorem B.3. *For $t \geq \frac{2}{\sqrt{3}}$ and $u \geq \frac{1}{2}$, we have $G = \Sigma_{t,u}\Gamma$. As a result $\Sigma_{t,u}$ contains a fundamental domain of G/Γ.*

Another important fact about Siegel sets is that it only intersects finitely many of its Γ-translates

Theorem B.4. *Fix t and u, then for all but finitely many $\gamma \in \Gamma$, we have*

$$\Sigma_{t,u}\gamma \cap \Sigma_{t,u} = \varnothing.$$

In particular, all but finitely many $\gamma \in \Gamma$ satisfies

$$\Sigma_{t,u} \cap F\gamma = \varnothing.$$

Now we turn to look at the Haar measure on G. Let $B = AN \cong N \rtimes_c A$ (note that as sets $AN = NA$) be the semidirect product of A and N with conjugation as action:

$$c : A \to \mathrm{N}, a \mapsto c_a,$$

where $c_a(n) = ana^{-1}$. In other words, the product in AN is defined by

$$a_1n_1 \cdot a_2n_2 := (a_1a_2)(n_1a_1n_2a_1^{-1}).$$

Proposition B.5. *$da := \frac{da_1}{a_1} \dots \frac{da_{d-1}}{a_{d-1}}$, with the right hand side identified with the standard Lebesgue measure on $\mathbb{R}^{d-1}$, is a bi-invariant Haar measure on A.*

Proof. For $a' = \mathrm{diag}(a'_1, a'_2, \dots, a'_d) \in A$, we have $a'a = \mathrm{diag}(a'_1 a_1, a'_2 a_2, \dots, a'_d a_d)$. Hence,

$$d(a'a) = \frac{d(a'_1 a_1)}{a'_1 a_1} \cdots \frac{d(a'_{d-1} a_{d-1})}{a'_{d-1} a_{d-1}} = da.$$

□

Proposition B.6. *$dn := \prod_{i<j} dn_{ij}$, with the right hand side identified with the standard Lebesgue measure on $\mathbb{R}^{d(d-1)/2}$, is a bi-invariant Haar measure on N.*

Proof. For $n' = (n'_{ij}) \in N$, the (i,j)-th entry of $(n'_{ij})(n_{ij})$ is

$$n_{ij} + (n'_{i,i+1} n_{i+1,j} + \cdots + n'_{i,j-1} n_{j-1,j}) + n'_{ij},$$

whose partial derivative w.r.t. n_{ij} is 1. So by the $\frac{d(d-1)}{2}$-dimensional change of variable formula with Jacobian the identity matrix, we obtain the left invariance $d(n'n) = dn$. The right invariance is similar.

□

Proposition B.7. *$\rho(a)dadn$ is a right invariant Haar measure on B, where the coefficient $\rho(a) := \prod_{i<j} \frac{a_i}{a_j}$.*

Proof. For $a'n', an \in N \rtimes_c A =: B$, and for any continuous function f with compact support on AN, identified with $\mathbb{R}^{d-1} \times \mathbb{R}^{d(d-1)/2}$ via the previous propositions,

$$\begin{aligned}\int_A \int_N f(ana'n')\rho(a)dadn &= \int_A \int_N f(aa'a'^{-1}na'n')\rho(a)dadn \\ &= \int_A \int_N f(aa'(a'^{-1}na')n')\rho(a)dadn.\end{aligned}$$

Making a change of variable $n \mapsto a'na'^{-1}$, whose Jacobian can be easily computed as $\rho(a') = \prod_{i<j} \frac{a'_i}{a'_j}$, this is equal to

$$\int_N \int_A f(aa'nn')\rho(a)d(a'na'^{-1})da = \int_N \int_A f(aa'nn')\rho(a)\rho(a')dnda.$$

Making change of variables $a \mapsto aa'^{-1}$ and then $n \to nn'^{-1}$ and noticing that da, dn are bi-invariant and that ρ is a group character, the above is equal to

$$\begin{aligned}\int_N \int_A f(ann')\rho(aa'^{-1})\rho(a')dnda &= \int_N \int_A f(an)\rho(a)dnda \\ &= \int_A \int_N f(an)\rho(a)dadn.\end{aligned}$$

This proves the right invariance of the measure $\rho(a)dadn$ on B. □

Theorem B.8. *Let dk denote a (finite) Haar measure on K. If we identify $G = \mathrm{SL}(d, \mathbb{R})$ with $KB = KAN$ via the Iwasawa decomposition (Theorem B.1), then $\rho(a)dkdadn$ gives a bi-invariant Haar measure on G.*

Now we define the Haar measure on G/Γ:

Theorem and Definition B.9 (Haar meaure on G/Γ)**.** *Let F be any compactly supported continuous function on G/Γ, then there exists a compacted supported continuous function f on G such that*

$$F(g\Gamma) := \sum_{\gamma \in \Gamma} f(g\gamma).$$

Define

$$\int_X F(g\Gamma)d(g\Gamma) := \int_G f(g)dg. \tag{B.0.1}$$

The right hand side $\int_G f(g)dg$ is independent of the choice of f by unfolding the integral using the quotient integral formula (Theorem 2.51 in [Fol15]). Therefore by the theory of Radon measures on locally compact Hausdorff spaces ([Fol07] Chapter 7), the equation B.0.1 defines a left G-invariant (and thus bi-invariant by the unimodularity) Haar measure on G/Γ.

For the Haar measure on $K = \mathrm{SO}(d, \mathbb{R})$ and the scaling, since the map

$$\mathrm{SO}(d, \mathbb{R}) \to S^{d-1}, g \mapsto ge_1$$

has

$$\mathrm{Stab}_{e_1}(G) = \begin{bmatrix} 1 & \mathbb{R}^{1\times d-1} \\ 0 & \mathrm{SO}(d-1, \mathbb{R}) \end{bmatrix},$$

we have the identification $\mathrm{SO}(d, \mathbb{R})/\mathrm{SO}(d-1, \mathbb{R}) \cong S^{d-1}$. We use this identification and induction to stipulate:

$$\mathrm{Vol}(K) = \mu_K(\mathrm{SO}(d, \mathbb{R})) := \prod_{i=1}^{d-1} \mathrm{Vol}(S^i) = \prod_{i=1}^{d-1} \frac{\pi^{\frac{i}{2}}}{\Gamma(\frac{i}{2}+1)}. \tag{B.0.2}$$

Theorem B.10. *Every Siegel set $\Sigma_{t,u} \in \mathrm{SL}(d.\mathbb{R})$ has finite Haar measure in G and it follows from Theorem B.3 that the Haar measure defined above is finite. Therefore $SL(d, \mathbb{Z})$ is a lattice in* $\mathrm{SL}(d, \mathbb{R})$.

Before we compute the coefficient in the generalized Siegel's formula, let us first recall the notion of admissible functions and Poisson summation formula:

Definition B.11. A function $f : \mathbb{R}^d \to \mathbb{R}$ is called *admissible* if there exist constants $c_1, c_2 > 0$ such that both $|f(x)|$ and $|\hat{f}(x)|$ are bounded by $\frac{c_1}{(1+\|x\|)^{d+c_2}}$, where $f\hat{f}(t) := \int_{\mathbb{R}^d} f(x)e^{2\pi i\langle x,t\rangle}dx$ is the Fourier transform of f.

Theorem B.12 (Poisson Summation Formula)**.** *Given any unimodular lattice $\Lambda \in \mathbb{R}^d$, a vector v and an admissible function $f : \mathbb{R}^d \to \mathbb{R}$, we have*

$$\sum_{x\in\Lambda} f(x+v) = \sum_{w\in\Lambda^*} e^{-2\pi i\langle v,w\rangle}\hat{f}(t),$$

where Λ^* is the dual lattice of Λ, cf. 1.2.6.

Proposition B.13. *As in the proof of Theorem 1.3.7, let* $\{e_1, \ldots, e_d\}$ *be the canonimcal basis of* $\mathbb{R}^d$*. For* $G = \mathrm{SL}(d, \mathbb{R})$ *and* $\Gamma = \mathrm{SL}(d, \mathbb{Z})$ *and the* k*-tuple* $(e_1, \ldots, e_k)$*, be , let*

$$G_k := \{g \in G : g.e_i = e_i, \forall 1 \leq i \leq k\},$$

$$\Gamma_k := \{g \in \Gamma : g.e_i = e_i, \forall 1 \leq i \leq k\}.$$

be the stabilizer subgroup of $(e_1, \ldots, e_k)$ *in* G *and* Γ*, respectively. Let* dg *denote the Haar measure on* G *(scaled as above) and* $dg_k := d\mu_{G_k}(g_k)$*,*$d(g\Gamma) := d\mu_{G/\Gamma}(g\Gamma)$*,* $d\mu_{G/\Gamma_k}(g\Gamma_k)$ *denoted the induced Haar measures on* $G_k, G/\Gamma$ *and* G/Γ_k *respectively. Then,*

$$\mu_{G_k/\Gamma_k}(G_k/\Gamma_k) = \frac{1}{\zeta(d-k+1)\cdots\zeta(d)}$$

Proof.

We start from the case when $k = 1$. In this case,

$$G_1 := \mathrm{Stab}_G\{e_1\} = \{g \in \mathrm{SL}(d, \mathbb{R}) : ge_1 = e_1\} = \begin{bmatrix} 1 & \mathbb{R}^{1\times(d-1)} \\ 0 & \mathrm{SL}(d-1, \mathbb{R}) \end{bmatrix}$$
$$\Gamma_1 := \mathrm{Stab}_\Gamma\{e_1\} = \{g \in \mathrm{SL}(d, \mathbb{R}) : ge_1 = e_1\} = \begin{bmatrix} 1 & \mathbb{Z}^{1\times(d-1)} \\ 0 & \mathrm{SL}(d-1, \mathbb{Z}) \end{bmatrix}$$

For the computation, we first consider the Fourier transform of compactly supported functions.

Let $f \in C_c(\mathbb{R}^d)$, namely a continuous function with compact support. Furthermore, assume that f is K-invariant and $f(0) \neq \hat{f}(0) := \int_{\mathbb{R}^d} f(x)dx$. Such function exists. For example, there exists $\eta \in (0, 1)$ such that

$$f(x) = \begin{cases} \frac{1-\eta\|x\|}{(1+\|x\|)^{d+1}} & \text{if } x \in B[0,1] \\ 0 & \text{if } x \notin B[0,1] \end{cases}$$

satisfy $f(0) \neq \hat{f}(0)$. Other properties are immediate.

Let $\tilde{F}(g) : G \to \mathbb{R}$ be defined as

$$\tilde{F}(g) := \sum_{v\in\mathbb{Z}^d} f(gv).$$

It follows that $\tilde{F}$ is bounded and for any $\gamma \in \Gamma = \mathrm{SL}(d, \mathbb{Z})$,

$$F(g\gamma) = \sum_{\mathbb{Z}^d} f(gv\gamma) = \sum_{\mathbb{Z}^d} f(gv) = F(g).$$

The Γ-invariance of $\tilde{F}$ induces a function $F : G/\Gamma \to \mathbb{R}$ by

$$F(g\Gamma) := \sum_{v \in \mathbb{Z}^d} f(gv).$$

Consider the following decomposition of $\mathbb{Z}^d$:

$$\mathbb{Z}^d = \{0\} \bigsqcup \bigsqcup_{\gamma\Gamma_1 \in \Gamma/\Gamma_1} \bigsqcup_{j=1}^{\infty} je_1. \tag{B.0.3}$$

It follows that $F \in C_c(G/\Gamma)$ and that

$$\begin{aligned}
\int_{G/\Gamma} F(g\Gamma)d(g\Gamma) &= \int_{G/\Gamma} \sum_{v \in \mathbb{Z}^d} f(gv)d(g\Gamma) \\
&= \int_{G/\Gamma} f(0)d(g\Gamma) + \int_{G/\Gamma} \sum_{\gamma\Gamma_1 \in \Gamma/\Gamma_1} \sum_{j=1}^{\infty} f(jg\gamma e_1)d(g\Gamma_1) \\
&= \int_{G/\Gamma} f(0)d(g\Gamma) + \int_{G/\Gamma_1} \sum_{j=1}^{\infty} f(jg\gamma e_1)d(g\Gamma_1) \\
&= f(0)\mu(G/\Gamma) + \sum_{j=1}^{\infty} \int_{G/\Gamma_1} f(jge_1)d(g\Gamma_1) \qquad \text{(B.0.4)}
\end{aligned}$$

To treat the section part of the sum above, we introduce the following subgroups :

$$\begin{aligned}
W_1 &:= \begin{bmatrix} 1 & 0 \\ 0 & SL(d-1, \mathbb{R}) \end{bmatrix}, \\
U_1 &:= \begin{bmatrix} 1 & \mathbb{R}^{1 \times (d-1)} \\ 0 & I_{d-1} \end{bmatrix}, \\
A_t &:= \begin{bmatrix} t & 0 \\ 0 & t^{-\frac{1}{d-1}} \end{bmatrix}.
\end{aligned}$$

The measures on them are canonical ones: μ_{W_1} is identified with the Haar measure on $\mathrm{SL}(d-1, \mathbb{R})$, again defined through Iwasawa decomposition above; μ_{U_1} is identified with the Lebesgue measure on $\mathbb{R}^{d-1}$; and the measure A_t is identified with $\frac{dt}{t}$ on $\mathbb{R}_{>0}$.

Clearly $G_1 = W_1 U_1$.

Claim 11. *$t^d\frac{dt}{t}dwdu$ defines a right invariant measure on $A_tW_1U_1$.*

Proof of Claim. Indeed, for any continuous function f with compact support defined on $A_tW_1U_1$, identified with $\mathbb{R}_{t>0}\times \mathrm{SL}(d-1,\mathbb{R})\times\mathbb{R}^{d-1}$ and $a=\mathrm{diag}(t,t^{-\frac{1}{d-1}}I_{d-1})$, $w'\in W_1$ and $u'\in U_1$. Notice that $awa'^{-1}=w$ and the Jacobian of the map $u'\mapsto aua'^{-1}$ is $(t^{1+\frac{1}{d-1}})^{d-1}=t^{d-1}$, the same change of variable argument for the integral

$$\int_A\int_{W_1}\int_{U_1} f(awua'w'u')dadwdu$$

gives the right invariance of $t^d\frac{dt}{t}dwdu$. #

Observe that

$$K\cap AW_1U_1=\begin{bmatrix}1 & 0\\ 0 & \mathrm{SO}(d-1,\mathbb{R})\end{bmatrix}\cong \mathrm{SO}(d-1,\mathbb{R})$$

and that the map

$$K\times G_1\to KG_1,(k,g)\mapsto k^{-1}g$$

has its fiber at the identity equal to $K\cap AW_1U_1\cong\mathrm{SO}(d-1,\mathbb{R})$, we have by the quotient integral formula and the proof of Theorem 8.32 in [Kna02], for any compactly supported continuous function ϕ on G,

$$\int_G\phi(jge_1)d(g\Gamma)=\frac{1}{\mathrm{Vol}(\mathrm{SO}(d-1,\mathbb{R}))}\int_{KA_tW_1U_1}\phi(jkawue_1)dkt^ddadwdu.$$

Since any compactly supported function Φ on G/Γ_1 can be expressed as

$$\Phi(g\Gamma_1)=\sum_{\gamma\in\Gamma_1}\phi(g\gamma),$$

for some compactly supported continuous function ϕ on G, we have by quotient integral formula and the uniqueness of Haar measure on homogeneous space G/Γ_1

$$\int_{G/\Gamma_1}f(jge_1)d(g\Gamma)=\int_{KA_tW_1U_1/\Gamma_1}f(jkawue_1)dkt^ddad(wu\Gamma_1)$$

where $f \in C_c(\mathbb{R}^d)$ as above and $j \geq 1$.

Now it follows from B.0.4 that

$$
\begin{aligned}
&\int_{G/\Gamma} F(g\Gamma)d(g\Gamma) \\
=&f(0)\mu(G/\Gamma) + \sum_{j=1}^{\infty} \int_{G/\Gamma_1} f(jge_1)d(g\Gamma_1) \\
=&f(0)\mu(G/\Gamma) + \frac{1}{\mathrm{Vol}(\mathrm{SO}(d-1,\mathbb{R}))} \sum_{j=1}^{\infty} \int_K \int_{A_t W_1 U_1/\Gamma_1} f(jkawue_1)dkt^d dad(wu\Gamma_1) \\
=&f(0)\mu(G/\Gamma) + \frac{\mathrm{Vol}(\mathrm{SO}(d,\mathbb{R}))}{\mathrm{Vol}(\mathrm{SO}(d-1,\mathbb{R}))} \sum_{j=1}^{\infty} \int_{A_t W_1 U_1/\Gamma_1} f(jawue_1)t^d dad(wu\Gamma_1) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (f \text{ is } K\text{-invariant by assumption}) \\
=&f(0)\mu(G/\Gamma) + \mathrm{Vol}(S^{d-1}) \sum_{j=1}^{\infty} \int_{A_t W_1 U_1/\Gamma_1} f(jawue_1)t^d dad(wu\Gamma_1) \\
=&f(0)\mu(G/\Gamma) + \mathrm{Vol}(S^{d-1}) \sum_{j=1}^{\infty} \int_{A_t} \int_{W_1 U_1/\Gamma_1} f(jawue_1)t^d da \\
=&f(0)\mu(G/\Gamma) + \mathrm{Vol}(S^{d-1}) \sum_{j=1}^{\infty} \int_{A_t} \int_{W_1 U_1/\Gamma_1} f(jtwue_1)t^{d-1} dt \\
=&f(0)\mu(G/\Gamma) + \mathrm{Vol}(S^{d-1})\mathrm{Vol}(\mathrm{SL}(d-1,\mathbb{R})/\mathrm{SL}(d-1,\mathbb{Z})) \sum_{j=1}^{\infty} \int_0^{\infty} f(jte_1)t^{d-1}dt \\
&\qquad (\mathrm{Vol}(\mathrm{SL}(d-1,\mathbb{R})/\mathrm{SL}(d-1,\mathbb{Z})) = \mathrm{Vol}(W_1U_1/\Gamma_1) \text{ since } \mathrm{Vol}(\mathbb{R}^d/\mathbb{Z}^d)\text{=1.}) \\
=&f(0)\mu(G/\Gamma) + \mathrm{Vol}(S^{d-1})\mathrm{Vol}(\mathrm{SL}(d-1,\mathbb{R})/\mathrm{SL}(d-1,\mathbb{Z})) \sum_{j=1}^{\infty} \frac{1}{j^d} \int_0^{\infty} f(te_1)t^{d-1}dt \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (\text{change of variable } t \mapsto \tfrac{t}{j})
\end{aligned}
$$

Note that the above only works $d > 2$. For the case when $d = 2$, this $\mathrm{Vol}(\mathrm{SL}(d-1,\mathbb{R})/\mathrm{SL}(d-1,\mathbb{Z}))$ has to be replaced by 1.

Claim 12.

$$
Vol(S^{d-1}) \int_0^{\infty} f(te_1)t^{d-1}dt = \hat{f}(0),
$$

where $\hat{f}(0) = \int_{\mathbb{R}^d} f(x)dx$ is the Fourier transform of f at 0.

Proof of Claim. Indeed, notice that f is K-invariant (rotation invariant), so

$$f(te_1) = f(x), \forall x \in \mathbb{R}^d \text{ with } \|x\| = t.$$

By the spherical coordinate in $\mathbb{R}^d$, we have

$$\begin{aligned}
x_1 &= r\cos(\varphi_1) \\
x_2 &= r\sin(\varphi_1)\cos(\varphi_2) \\
x_3 &= r\sin(\varphi_1)\sin(\varphi_2)\cos(\varphi_3) \\
&\vdots \\
x_{d-1} &= r\sin(\varphi_1)\cdots\sin(\varphi_{d-2})\cos(\varphi_{d-1}) \\
x_d &= r\sin(\varphi_1)\cdots\sin(\varphi_{d-2})\sin(\varphi_{d-1}).
\end{aligned}$$

where $0 \le \phi_{d-1} \le 2\pi$ and $0 \le \phi_i \le \pi$ for all $i \le d-1$. and

$$\begin{aligned}
\int_{\mathbb{R}^d} f(x)dx =& \int_0^\infty \cdots \int_0^\infty f(x_1, \ldots, x_d)dx_1 \cdots dx_d \\
=& \int_{[0,\pi]^{d-1}\times[0,2\pi]} \int_0^\infty f(x_1, \ldots, x_d) \left| \det \frac{\partial(x_i)}{\partial(r, \varphi_j)} \right| dr\, d\varphi_1\, d\varphi_2 \cdots d\varphi_{d-1} \\
=& \int_{[0,\pi]^{d-1}\times[0,2\pi]} \int_0^\infty f(re_1)r^{d-1}\sin^{d-2}(\varphi_1)\sin^{d-3}(\varphi_2)\cdots\sin(\varphi_{d-2})\, dr\, d\varphi_1\, d\varphi_2 \cdots d\varphi_{n-1} \\
=& \mathrm{Vol}(S^{d-1}) \int_0^\infty f(re_1)r^{d-1}dr.
\end{aligned}$$

#

Therefore, we have

$$\begin{aligned}
\int_{G/\Gamma} F(g\Gamma)d(g\Gamma) =& f(0)\mu(G/\Gamma) + \mathrm{Vol}(S^{d-1})\mathrm{Vol}(\mathrm{SL}(d-1,\mathbb{R})/\mathrm{SL}(d-1,\mathbb{Z})) \sum_{j=1}^{\infty} \frac{1}{j^d} \int_0^\infty f(te_1)t^{d-1}dt \\
=& f(0)\mu(G/\Gamma) + \hat{f}(0)\mathrm{Vol}(\mathrm{SL}(d-1,\mathbb{R})/\mathrm{SL}(d-1,\mathbb{Z})) \sum_{j=1}^{\infty} \frac{1}{j^d} \\
=& f(0)\mu(G/\Gamma) + \hat{f}(0)\mathrm{Vol}(\mathrm{SL}(d-1,\mathbb{R})/\mathrm{SL}(d-1,\mathbb{Z}))\zeta(d). \qquad \text{(B.0.5)}
\end{aligned}$$

In other to find $\mathrm{Vol}(\mathrm{SL}(d-1,\mathbb{R})/\mathrm{SL}(d-1,\mathbb{Z}))$, we shall look at the dual version of the above equation.

For any $g \in G$, $g\mathbb{Z}^d$ defines a lattice and its dual lattice is $g^*\mathbb{Z}^d = {}^tg^{-1}\mathbb{Z}^d$ (Proposition 1.2.28). But the automorphism

$$* : G \to G, g \mapsto g^*$$

clearly preserves the Haar measure on G and $\gamma\mathbb{Z}^d = \gamma^*\mathbb{Z}^d$ for all $\gamma \in \Gamma$. So it also preserves the Haar measure on G/Γ.

On the other hand, by the Poisson summation formula:

$$F(g\Gamma) = \sum_{v\in\mathbb{Z}^d} f(gv) = \sum_{v\in\mathbb{Z}^d} \hat{f}(g^*v) =: \hat{F}(g^*).$$

Since $\hat{\hat{f}}(0) = f(0)$, by replacing f in the recursion equation B.0.5 by $\hat{f}$, we have

$$\begin{aligned}
&f(0)\mu(G/\Gamma) + \hat{f}(0)\mathrm{Vol}(\mathrm{SL}(d-1,\mathbb{R})/\mathrm{SL}(d-1,\mathbb{Z}))\zeta(d)\\
=&\int_{G/\Gamma} F(g\Gamma)d(g\Gamma) = \int_{G/\Gamma} \hat{F}(g\Gamma)d(g\Gamma)\\
=&\hat{f}(0)\mu(G/\Gamma) + \hat{\hat{f}}(0)\mathrm{Vol}(\mathrm{SL}(d-1,\mathbb{R})/\mathrm{SL}(d-1,\mathbb{Z}))\zeta(d)\\
=&\hat{f}(0)\mu(G/\Gamma) + f(0).
\end{aligned}$$

Since we have chosen $f(0) = \hat{f}(0)$ at the beginning, this yields:

$$\mathrm{Vol}(G/\Gamma) = \mathrm{Vol}(\mathrm{SL}(d,\mathbb{R})/\mathrm{SL}(d,\mathbb{Z})) = \zeta(d)\mathrm{Vol}(\mathrm{SL}(d-1,\mathbb{R})/\mathrm{SL}(d-1,\mathbb{Z})) = \zeta(d)\mathrm{Vol}(G_1/\Gamma_1),$$

for $d > 2$ and with our discussion above (before the claim),

$$\mathrm{Vol}(\mathrm{SL}(2,\mathbb{R})/\mathrm{SL}(2,\mathbb{Z})) = \zeta(2).$$

By induction, we have

$$\mathrm{Vol}(G/\Gamma) = \zeta(d)\cdots\zeta(d-k+1)\mathrm{Vol}(G_k/\Gamma_k)$$

This gives the constant we need for the generalized Siegel formula as well as

$$\mathrm{Vol}(G/\Gamma) = \zeta(d) \cdots \zeta(2).$$

$\square$

Appendix C: Preliminaries in model theory

In this appendix we provide necessary background in polynomially bounded *o*-minimal structures.

The notion of o-minimal structures dates back to Alexander Grothendieck in 1980's where he tried to give an axiomatic approach to tame topology [Gro97]. In this appendix we introduce basic notions for o-minimal structures which are needed to formulate our theory in the last Chapter. The main reference for this section is [Dri98].

Definition C.1. Let S be a set and $\mathscr{A}$ be a family of subsets of S. A *finite boolean combination* of $\mathscr{A}$ is obtained by a finite application of intersections, unions, and complements to a finite subset of $\mathscr{A}$. $\mathscr{A}$ is called a *boolean algebra of subsets of S* if $\mathscr{A}$ is closed under finite boolean combinations of its members.

Definition C.2. A *semi-algebraic set*, $X \subset \mathbb{R}^n$, is a finite boolean combination of sets of the form

$$\{x \in \mathbb{R}^n : f(x) = 0\} \quad \text{or} \quad \{x \in \mathbb{R}^n : g(x) > 0\}, \tag{C.0.1}$$

where $f, g \in \mathbb{R}[x_1, ..., x_n]$.

Semi-algebraic sets has a lot of nice properties showing its stability ("tame") under various operations:

Theorem C.3 (Properties of semi-algebraic sets)**.** *Let A be a semi-algebraic set in $\mathbb{R}^n$, then*

- *the closure, interior and boundary of A are also semi-algebraic;*
- *A has finitely many connected components, each of which is again semi-algebraic.*
- *(Tarski-Seidenberg Theorem) if $n > m$ and $\pi : \mathbb{R}^n \to \mathbb{R}^m$ is a projection onto its first m-coordinates, then $\pi(A)$ is again semi-algebraic.*

Definition C.4 (Structure and definability)**.** Given the field of real numbers $\mathbb{R}$, we define a structure $\mathscr{S}$ on $\mathbb{R}^n$ to be a sequence $(\mathscr{S}_n)_{n\in\mathbb{N}}$ ($\mathscr{S}_n \subset \mathscr{P}(\mathbb{R}^n)$, the power set of $\mathbb{R}^d$) satisfying the following axioms: for each $n \geq 0$,

(1) (boolean axiom) $\mathscr{S}_n$ is a boolean algebra of subsets of $\mathbb{R}^n$;

(2) (diagonal axiom) The diagonal of $\mathbb{R}^n$ is in $\mathscr{S}_n$, namely $\{(x_1, ..., x_n) : x_i = x_j\} \in \mathscr{S}_n$ for all $i \neq j$.

(3) (lifting axiom) if $A \in \mathscr{S}_n$, then the sets $A \times \mathbb{R}$ and $\mathbb{R} \times A$ are in S_{n+1};

(4) (projection axiom) if $A \in \mathscr{S}_{n+1}$ and if $\pi : \mathbb{R}^{n+1} \to \mathbb{R}^n$ is the projection map onto the first n coordinates, then $\pi(A) \in \mathscr{S}_n$.

(5) (operation axiom) For $A_1, B_1 \in \mathscr{S}_1$,the graph of addition $\{(x, y, x + y) : x \in A_1, y \in B_1\}$ and multiplication $\{(x, y, xy) : x \in A_1 . y \in B_1\}$ are in $\mathscr{S}_3$.

The elements (sets) in $\cup_{n\in\mathbb{N}}\mathscr{S}_n$ are called *definable* with respect to the structure $\mathscr{S}$. A function $f : \mathbb{R} \to \mathbb{R}$ is called definable if its graph $\text{graph}(f) := \{(x, f(x)) : x \in \mathbb{R}\}$ is definable.

Definition C.5. A structure $\mathscr{S}$ over $\mathbb{R}$ is said to be *o-minimal* if

(O1) (points and intervals axiom) $\{r\} \in \mathscr{S}_1$ for all $r \in \mathbb{R}$ and for all $x, y \in \mathbb{R}$ with $x < y$, we have the graph $\{(x, y) \in R^2 : x < y\} \in \mathscr{S}_2$; and

(O2) ($\mathscr{S}_1$ axiom) The only sets in $\mathscr{S}_1$ are finite unions of points and open intervals in $\mathbb{R}$.

Example C.6. The structure of semi-algebraic sets, $\mathbb{R}_{\text{alg}}$, are the smallest o-minimal structure on $(\mathbb{R}, \cdot, +)$.

The following theorem is crucial in our proof of goodness of definable functions in a polynomially bounded o-minimal structure:

Theorem C.7 (Definable choice function)**.** *If $S \subset R^{m+n}$ is definable and $\pi : R^{m+n} \to R^m$ the projection on the first m coordinates, then there is a definable map $f : \pi(S) \to R^n$ such that graph$(f) \subset S$.*

Definition C.8. Given a family of real valued functions, $\mathcal{F}$, the smallest structure on $(\mathbb{R}, \cdot, +)$ containing graph(f) for all $f \in \mathcal{F}$, is denoted $(\mathbb{R}, \cdot, +, \mathcal{F})$.

Definition C.9. A structure $\mathscr{S} = (\mathscr{S}_n)_{n\in\mathbb{N}}$ on $\mathbb{R}$ is *polynomially bounded* if for every definable $f : \mathbb{R} \to \mathbb{R}$ there exists $m \in \mathbb{N}$ such that $f(t) = O(t^m)$ as $t \to \infty$.

Example C.10. The structure of real numbers with restricted analytic functions, denoted

$$\mathbb{R}_{\text{an}} := (\mathbb{R}, \cdot, +, \mathcal{A}),$$

is constructed as follows:

Let $\mathbb{R}[[x_1, ..., x_m]]_{[-1,1]^m}$ denote the ring of all power series in m-variables $x_1, ..., x_m$ convergent in a neighborhood of $[-1, 1]^m \subset \mathbb{R}^m$. For each $f \in \mathbb{R}[[x_1, ..., x_m]]_{[-1,1]^m}$, put

$$\tilde{f}(x) = \begin{cases} f(x) & x \in [-1, 1]^m, \\ 0 & x \notin [-1, 1]^m. \end{cases} \tag{C.0.2}$$

Finally, let $\mathcal{A} := (\tilde{f})_{f\in\mathbb{R}[[x_1,...,x_m]]_{[-1,1]^m}}$.

$(\mathbb{R}_{\text{an}}, (x \mapsto x^r)_{r\in\mathbb{R}})$, where $x^r := 0$ whenever $x \leq 0$ is the largest known polynomially bounded o-minimal structure on $(\mathbb{R}, \cdot, +)$ to our knowledge [Mil95].

Proposition C.11 (Growth Uniform Asymptotics)**.** *Let f be a polynomially bounded o-minimal real function, then there exists $C, r \in \mathbb{R}$ such that*

$$f(x) = Cx^r + o(x^r) \tag{C.0.3}$$

for $x \gg 1$.

Proposition C.12. *Let $\mathbb{R}_+ = (0, \infty)$ and $P : \mathbb{R}_+^k \to \mathbb{R}$ be a function of the form:*

$$P(x_1, \ldots, x_k) = \sum_{r_1,\ldots,r_k} C_{r_1,\ldots,r_k} x_1^{r_1} \cdots x_2^{r_k}, \tag{C.0.4}$$

where $r_1, ..., r_k \in \mathbb{R}$ and the sum is finite. If $f_1, ..., f_k$ are polynomially bounded o-minimal, then so is $P(f_1, ..., f_k)$.

It is known that polynomially bounded *o*-minimal definable functions are piecewise smooth and monotone ([Mil94b]).

Theorem C.13. *Let $f : \mathbb{R}_+ \to \mathbb{R}$ be a polynomially bounded o-minimal definable function. Then there exist $0 < a_0 < \cdots < a_n < a_{n+1} = \infty$ such that $f|(a_k, a_{k+1})$ is differentiable and monotone (including constant).*

Theorem C.14 ([Mil94b], special case of Proposition 3.1)**.** *Let $f : \mathbb{R}_+ \to \mathbb{R}$ be a polynomial bounded o-minimal definable function and suppose $f(x) = Cx^r + o(x^r)$. Then $f(tx) \sim t^r f(x)$, $f(x + t) \sim f(x)$ for each $t > 0$. If $r \neq 0$, then $xf'(x)/f(x) \sim r$.*

Bibliography

[Ajt98] Miklós Ajtai. "The Shortest Vector Problem in L^2 is NP-Hard for Randomized Reductions (Extended Abstract)". *Proceedings of the Thirtieth Annual ACM Symposium on Theory of Computing*. STOC '98. Dallas, Texas, USA: Association for Computing Machinery, 1998, pp. 10–19. ISBN: 0897919629. URL: https://doi.org/10.1145/276698.276705.

[AES16] Menny Aka, Manfred Einsiedler, and Uri Shapira. *Integer points on spheres and their orthogonal lattices*. Inventiones mathematicae 206 (2016), pp. 379–396.

[AM09] Jayadev S. Athreya and Gregory A. Margulis. *Logarithm laws for unipotent flows, I*. Journal of Modern Dynamics 3.3 (2009), pp. 359–378.

[BM00] M. Bachir Bekka and Matthias Mayer. *Ergodic Theory and Topological Dynamics of Group Actions on Homogeneous Spaces*. Cambridge University Press (2000). DOI: https://doi.org/10.1007/978-3-319-05792-7.

[BE02] Manuel Benito and J. Javier Escribano. *An Easy Proof of Hurwitz's Theorem*. The American Mathematical Monthly 109.10 (2002), pp. 916–918. DOI: 10.1080/00029890.2002.11919929.

[BX23] Michael Bersudsky and Hao Xing. *Limiting distribution of dense orbits in a moduli space of rank m discrete subgroups in $(m+1)$-space*. 2023. arXiv: 2307.12085 [math.DS].

[Boa81] Pve Boas. *Another NP-complete partition problem and the complexity of computing short vectors in a lattice* (1981).

[Bôc01] Maxime Bôcher. *Certain cases in which the vanishing of the Wronskian is a sufficient condition for linear dependence*. Trans. Amer. Math. Soc. 2.2 (1901), pp. 139–149. ISSN: 0002-9947,1088-6850. DOI: 10.2307/1986214. URL: https://doi.org/10.2307/1986214.

[Cas97] J. W. S. Cassels. *An Introduction to the Geometry of Numbers*. Springer, 1997.

[Che11] Yitwah Cheung. *Hausdorff dimension of the set of singular pairs*. Annals of Mathematics 173 (2011), pp. 127–167.

[CC16] Yitwah Cheung and N. Chevallier. *Hausdorff dimension of singular vectors*. Duke Mathematical Journal 165 (2016), pp. 2273–2329.

[Dan85] S. G. Dani. *Divergent trajectories of flows on homogeneous spaces and Diophantine approximation*. Journal für die reine und angewandte Mathematik (Crelles Journal) 1985 (1985), pp. 55–89.

[DM93] S. G. Dani and G. A. Margulis. "Limit distributions of orbits of unipotent flows and values of quadratic forms". 1993.

[DFSU20] Tushar Das, Lior Fishman, David Simmons, and Mariusz Urbański. *A variational principle in the parametric geometry of numbers* (2020). arXiv: `1901.06602 [math.NT]`.

[DE14] Anton Deitmar and Siegfried Echterhoff. *Principles of Harmonic Analysis*. Springer (2014). DOI: `https://doi.org/10.1007/978-3-319-05792-7`.

[Dir42] P. G. Lejeune Dirichlet. *Verallgemeinerung eines Satzes aus der Lehre von den Kettenbr¨uchen nebst einige Anwendungen auf die Theorie der Zahlen*. S.-B. Preuss. Akad. Wiss (1842), pp. 93–95.

[Dri98] L. P. D. van den Dries. *Tame Topology and O-minimal Structures*. London Mathematical Society Lecture Note Series. Cambridge University Press, 1998. DOI: `10.1017/CBO9780511525919`.

[DM96] Lou van den Dries and Chris Miller. *Geometric categories and o-minimal structures*. Duke Math. J. 84.2 (1996), pp. 497–540. ISSN: 0012-7094,1547-7398. DOI: `10.1215/S0012-7094-96-08416-1`. URL: `https://doi.org/10.1215/S0012-7094-96-08416-1`.

[DRS93] William Duke, Zeév Rudnick, and Peter Sarnak. *Density of integer points on affine homogeneous varieties*. Duke Math. J. 71 (1993), pp. 143–179.

[EMSS16] Manfred Einsiedler, Shahar Mozes, Nimish Shah, and Uri Shapira. *Equidistribution of primitive rational points on expanding horospheres*. Compositio Mathematica 152 (2016), pp. 667–692.

[EW11] Manfred Einsiedler and Thomas Ward. *Ergodic theory: with a view towards number theory*. Springer, 2011.

[Elk] Noam D. Elkies. *Does every 4-dimensional lattice have a minimal system that is also a lattice basis?* Mathematics Stack Exchange. URL: `https://math.stackexchange.com/q/3448605`.

[Fal97] Kenneth Falconer. *Techniques in fractal geometry*. John Wiley & Sons, Ltd., Chichester, 1997.

[Fol15] Gerald B. Folland. *A course in Abstract Harmonic Analysis, 2nd edition*. Chapman and Hall/CRC (2015). DOI: `https://doi.org/10.1007/978-3-319-05792-7`.

[Fol07] Gerald B. Folland. *Real Analysis: Modern Techniques and Their Applications, 2nd Edition*. Wiley (2007).

[Gor03] Alexander Gorodnik. *Lattice action on the boundary of $SL(n,\mathbb{R})$*. Ergodic Theory and Dynamical Systems 23 (2003), pp. 1817–1837.

[GLS22] Alexander Gorodnik, Jialun Li, and Cagri Sert. *Stationary measures for $\mathrm{SL}_2(\mathbb{R})$-actions on homogeneous bundles over flag varieties*. 2022. arXiv: `2211.06911 [math.DS]`.

[GN09] Alexander Gorodnik and Amos Nevo. "Counting lattice points". 2009.

[GN12] Alexander Gorodnik and Amos Nevo. *Counting lattice points*. Journal für die reine und angewandte Mathematik (Crelles Journal) (2012).

[GN14] Alexander Gorodnik and Amos Nevo. *Ergodic theory and the duality principle on homogeneous spaces*. Geometric and Functional Analysis 24 (2014), pp. 159–244.

[GW04] Alexander Gorodnik and Barak Weiss. *Distribution of lattice orbits on homogeneous varieties*. GAFA Geometric And Functional Analysis 17 (2004), pp. 58–115.

[Gro97] Alexander Grothendieck. *Esquisse d'un Programme, section 5. English translation available in Geometric Galois Actions I (edited by L. Schneps and P. Lochak)*. LMS Lecture Notes (1997).

[Hel85] Bettina Helfrich. *Algorithms to construct minkowski reduced and hermite reduced lattice bases*. Theoretical Computer Science 41 (1985), pp. 125–139. ISSN: 0304-3975. DOI: `https://doi.org/10.1016/0304-3975(85)90067-2`. URL: `https://www.sciencedirect.com/science/article/pii/0304397585900672`.

[KKLM17] Shirali Kadyrov, Dmitry Kleinbock, E. Lindenstrauss, and G. Margulis. *Singular systems of linear forms and non-escape of mass in the space of lattices*. Journal d'Analyse Mathématique 133 (Dec. 2017). DOI: `10.1007/s11854-017-0033-4`.

[KY17] Dubi Kelmer and Shucheng Yu. *Shrinking target problems for flows on homogeneous spaces*. Transactions of the American Mathematical Society (2017).

[Kim22] Seungki Kim. *Mean value formulas on sublattices and flags of the random lattice*. Journal of Number Theory 241 (2022), pp. 330–351. ISSN: 0022-314X. DOI: `https://doi.org/10.1016/j.jnt.2022.03.013`. URL: `https://www.sciencedirect.com/science/article/pii/S0022314X22000920`.

[KM98] D. Y. Kleinbock and G. A. Margulis. *Flows on homogeneous spaces and Diophantine approximation on manifolds*. Ann. of Math. (2) 148.1 (1998), pp. 339–360. ISSN: 0003-486X,1939-8980. DOI: `10.2307/120997`. URL: `https://doi.org/10.2307/120997`.

[Kle07] Dmitry Kleinbock. *Quantitative nondivergence and its Diophantine applications*. Clay Mathematics Proceedings (2007).

[KM99] Dmitry Kleinbock and G. A. Margulis. *Logarithm laws for flows on homogeneous spaces*. Inventiones mathematicae 138 (1999), pp. 451–494.

[KW06] Dmitry Kleinbock and Barak Weiss. *Dirichlet's theorem on diophantine approximation and homogeneous flows*. Journal of Modern Dynamics 2 (2006), pp. 43–62.

[KW10] Dmitry Kleinbock and Barak Weiss. *Modified Schmidt games and Diophantine approximation with weights*. Advances in Mathematics 223.4 (2010), pp. 1276–1298. ISSN: 0001-8708. DOI: `https://doi.org/10.1016/j.aim.2009.09.018`. URL: `https://www.sciencedirect.com/science/article/pii/S0001870809003120`.

[Kna02] Anthony W. Knapp. *Lie Groups: Beyond an Introduction, 2nd Edition*. Springer, 2002.

[Mil94a] Chris Miller. *Expansions of the real field with power functions*. Ann. Pure Appl. Logic 68.1 (1994), pp. 79–94. ISSN: 0168-0072,1873-2461. DOI: 10.1016/0168-0072(94)90048-5. URL: https://doi.org/10.1016/0168-0072(94)90048-5.

[Mil95] Chris Miller. *Infinite Differentiability in Polynomially Bounded O-Minimal Structures*. Proceedings of the American Mathematical Society 123.8 (1995), pp. 2551–2555. URL: http://www.jstor.org/stable/2161287.

[Mil94b] Christopher Lee Miller. "Polynomially bounded o-minimal structures". PhD thesis. University of Illinois at Urbana-Champaign, 1994.

[Mil17] J. S. Milne. *Algebraic Groups: The Theory of Group Schemes of Finite Type over a Field*. Cambridge Studies in Advanced Mathematics. Cambridge University Press, 2017.

[Min96] Hermann Minkowski. *Geometrie der Zahlen*. Leipzig : Teubner (1896).

[Mos12] N.G. Moshchevitin. *Proof of W.M. Schmidt's conjecture concerning successive minima of a lattice*. Journal of the London Mathematical Society 86.1 (Mar. 2012), pp. 129–151. ISSN: 0024-6107. URL: https://doi.org/10.1112/jlms/jdr076.

[MS95] Shahar Mozes and Nimish A. Shah. *On the space of ergodic invariant measures of unipotent flows*. Ergodic Theory and Dynamical Systems 15 (1995), pp. 149–159.

[Oh05] Hee Oh. *Lattice action on finite volume homogeneous spaces*. J. Korean Math. Soc 42 (July 2005), pp. 635–653. DOI: 10.4134/JKMS.2005.42.4.635.

[PS18a] Ya'acov Peterzil and Sergei Starchenko. *Algebraic and o-minimal flows on complex and real tori*. Advances in Mathematics 333 (2018), pp. 539–569. ISSN: 0001-8708. DOI: https://doi.org/10.1016/j.aim.2018.05.040. URL: https://www.sciencedirect.com/science/article/pii/S0001870818302226.

[PS18b] Ya'acov Peterzil and Sergei Starchenko. *O-minimal flows on nilmanifolds*. Duke Mathematical Journal (2018). URL: https://api.semanticscholar.org/CorpusID:53135635.

[Rag72] Madabusi S. Raghunathan. *Discrete Subgroups of Lie Groups*. Springer Berlin, Heidelberg, 1972.

[Rat91] Marina Ratner. *On Raghunathan's Measure Conjecture*. Annals of Mathematics 134.3 (1991), pp. 545–607. (Visited on 03/05/2023).

[Rem36] E. J. Remez. *Sour une propriété de polynômes de Tchebysheff,* Communicationes le l'Inst. des Sci. Kharkow 13 (1936), pp. 93–95.

[Rog55] Claude Ambrose Rogers. *Mean values over the space of lattices*. Acta Mathematica 94 (1955), pp. 249–287.

[Rog56] Claude Ambrose Rogers. *The Number of Lattice Points in a Set*. Proceedings of The London Mathematical Society (1956), pp. 305–320.

[SR80] S.G.Dani and S. Raghavan. *Orbits of euclidean frames under discrete linear groups*. Israel J. Math. 36 (1980), pp. 300–320.

[SS17] Oliver Sargent and Uri Shapira. *Dynamics on the space of 2-lattices in 3-space*. Geometric and Functional Analysis 29 (2017), pp. 890–948.

[Sch82] W.M. Schmidt. *Open problems in Diophantine approximation. In: Diophantine Approximations and Transcendental Numbers*. Progr. Math. 31 (1982).

[Sch58] Wolfgang Schmidt. *On the Convergence of Mean Values Over Lattices*. Canadian Journal of Mathematics 10 (1958), pp. 103–110. DOI: 10.4153/CJM-1958-013-2.

[Sch69] Wolfgang M. Schmidt. *Badly approximable systems of linear forms*. Journal of Number Theory 1 (1969).

[Sch66] Wolfgang M. Schmidt. *On Badly Approximable Numbers and Certain Games*. Transactions of the American Mathematical Society 123.1 (1966), pp. 178–199.

[Sch98] Wolfgang M. Schmidt. *The distribution of sublattices of $\mathbb{Z}_m$*. Monatshefte für Mathematik 125 (1998), pp. 37–81.

[Sch10] Daria Schymura. *An upper bound on the volume of the symmetric difference of a body and a congruent copy*. Advances in Geometry 14 (Oct. 2010). DOI: 10.1515/advgeom-2013-0029.

[Sha96] Nimish A. Shah. *Limit distributions of expanding translates of certain orbits on homogeneous spaces*. Proceedings of the Indian Academy of Sciences - Mathematical Sciences 106 (1996), pp. 105–125.

[Sha94] Nimish A. Shah. *Limit distributions of polynomial trajectories on homogeneous spaces*. Duke Mathematical Journal 75 (1994), pp. 711–732.

[SW00] Nimish A. Shah and Barak Weiss. *On actions of epimorphic subgroups on homogeneous spaces*. Ergodic Theory and Dynamical Systems 20 (2000), pp. 567–592.

[Sie45] Carl Ludwig Siegel. *A Mean Value Theorem in Geometry of Numbers*. Annals of Mathematics 46.2 (1945), pp. 340–347. ISSN: 0003486X. URL: http://www.jstor.org/stable/1969027 (visited on 10/18/2022).

[SC89] Carl Ludwig Siegel and Komaravolu Chandrasekharan. *Lectures on the Geometry of Numbers*. Springer, 1989.

[Zha23] Han Zhang. *Locally unipotent invariant measures and limit distribution of a sequence of polynomial trajectories on homogeneous spaces*. Discrete Contin. Dyn. Syst. 43.2 (2023), pp. 747–770. DOI: 10.3934/dcds.2022168. URL: https://doi.org/10.3934/dcds.2022168.