Retractive operadic algebras in spectra and completions

Dissertation

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By

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Abstract

Working in the context of operadic algebras in modules over the sphere spectrum, we study completions with respect to invariants centered away from the base point—that is, centered at a fixed operadic algebra Y. We show that for retractive objects admitting 0-connected structural maps $Y \to X$, the Bousfield-Kan completion map $X \to X^{\wedge}_{\Omega^k_Y \Sigma^k_Y}$ is an equivalence for $1 \leq k \leq \infty$. This generalizes completion results of Blomquist and Ching-Harper when $Y = *$. The manner of our attack will require us to pick up and develop Hovey's stabilization machinery and carefully study the homotopy theory and stabilization of categories of retractive objects.

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Table of Contents

Chapter 1

Introduction

To fix concepts, we will work with operads in spectra as developed by Harper [26], Harper-Hess [27] and Ching-Harper [15]. The word "space" will always mean simplicial set and we model spectra of spaces by symmetric spectra [37].

The main theorem of this paper is the following.

Theorem 1.0.1. *If $\mathcal{O}$ is a (-1)-connected operad in spectra, Y is a (-1)-connected $\mathcal{O}$-algebra and X is a retractive $\mathcal{O}$-algebra over Y which is 0-connected relative to Y, then the Bousfield-Kan completion map*

$$X \to X^{\wedge}_{\Omega^k_Y \Sigma^k_Y}$$

is an equivalence for $1 \leq k \leq \infty$.

The case of $k < \infty$ and $k = \infty$, which corresponds to stabilization in the sense of Goodwillie [24], must be treated separately. This theorem generalizes work of Ching-Harper in [14] and Blomquist in [7]. In [14], Ching and Harper study the special cases of our theorem for $Y = *$ and $k = \infty$, whereas in [7], Blomquist studies the special case of $Y = *$ and $1 \leq k < \infty$.

These theorems are not without antecedents. The general idea is as follows. In homotopy theory, many comparison maps come to us in the form of an adjunction. For instance, the Hurewicz map comparing homotopy groups with homology groups $\pi_*(X) \to H_*(X)$ of a pointed space X is implemented on the level of spaces by an adjunction

$$\mathsf{S}_* \underset{}{\overset{\mathbf{Z}}{\rightleftarrows}} \mathsf{sAb}$$

The unit of this adjunction $X \to \mathbf{Z}X$ implements, on the level of homotopy, the Hurewicz map $\pi_*(X) \to H_*(X)$. The appropriate thing to do with such comparison maps is to iterate them and thereby build a resolution of the original object X by a coaugmented cosimplicial object (with only coface maps shown)

$$X \longrightarrow (\mathbf{Z}X \rightrightarrows \mathbf{Z}^2 X \Rrightarrow \mathbf{Z}^3 X \cdots)$$

and glue the data of this resolution together with a homotopy limit. With this procedure, Bousfield and Kan show in [12, III.5.4] that the $\mathbf{Z}$-completion map

$$X \to X^{\wedge}_{\mathbf{Z}}$$

is an equivalence for simply-connected spaces.

The other classical comparison map of algebraic topology compares homotopy groups with homotopy groups—this is the Freudenthal suspension homomorphism

$$\pi_*(X) \to \pi_{*+1}(\Sigma X).$$

As is well-known, this map is implemented on the level of pointed spaces from the loops-suspension adjunction as the derived unit

$$X \to \Omega\Sigma X,$$

where Ω indicates the derived loops functor. By iterating this map, we may form a resolution of X

$$X \longrightarrow (\Omega\Sigma X \rightrightarrows (\Omega\Sigma)^2 X \Rrightarrow (\Omega\Sigma)^3 X \cdots)$$

which, upon taking homotopy limits, gives us the Bousfield-Kan completion map

$$X \to X^\wedge_{\Omega\Sigma}.$$

This map has been studied by Bousfield in [11] as well as Hopkins in unpublished work, but see [11]. Bousfield has shown that for finite k, the $\Omega^k\Sigma^k$-completion map

$$X \to X^\wedge_{\Omega^k\Sigma^k}$$

is an equivalence for simply-connected spaces.

There is still another classical comparison map in algebraic topology which compares the homotopy groups of a space with its stable homotopy groups

$$\pi_*(X) \to \pi_*^s(X),$$

and, as before, this map is implemented on the level of spaces by the derived unit of the stabilization adjunction

$$X \to \Omega^\infty \Sigma^\infty X.$$

In this case, Carlsson has shown in [13] that, in the case of spaces, the Bousfield-Kan completion map

$$X \to X^\wedge_{\Omega^\infty \Sigma^\infty}$$

is an equivalence for simply-connected X. This was also studied by Arone-Kankaanrinta in [2].

For $1 \leq k \leq \infty$, it is known by work of Ching-Harper [14] and Blomquist [7] that the completion map

$$X \to X^\wedge_{\Omega^k \Sigma^k}$$

is an equivalence for 0-connected $\mathcal{O}$-algebras X with $\mathcal{O}$ (-1)-connected. This forms the starting point for the work of this paper. Our main theorem generalizes the work of Blomquist and Ching-Harper. We now allow our category of $\mathcal{O}$-algebras to be centered away from the basepoint and allow for non-trivial homotopy information in degree 0. It is worth pointing out here that our method in the case of stabilization $k = \infty$ is entirely different from that of Ching-Harper in [14] and conceptually much simpler. Using model categorical methods, the crux of our stabilization approach is establishing an S_*-enriched stable fibrant replacement monad on the corresponding category of spectra of retractive $\mathcal{O}$-algebras.

Chapter 2

Model Structures and Stabilization

In this chapter, we introduce the basic framework in which our work will take place and establish necessary technical results. As these may be of independent interest, we will elaborate upon them with more generality than is strictly necessary for our results.

2.1 Categories of Retractive Objects

While almost everything here goes through for categories consisting of factorizations of a general map $f\colon c \to c'$, we restrict our attention to the case of $f = \mathrm{id}_c$, which is our primary focus in this paper.

Definition 2.1.1. Given a category C and object $c \in \mathsf{C}$, the ***category of retractive objects*** over c, denoted by $\mathsf{C}_{c//c}$ or simply $\mathsf{C}(\mathrm{id}_c)$, has as its objects all pairs of maps (s, c_0, r) where

$$s\colon c \to c_0, \quad r\colon c_0 \to c, \quad rs = \mathrm{id}_c,$$

and a morphism $f\colon (s_0, c_0, r_0) \to (s_1, c_1, r_1)$ of such objects is simply an element $f \in \hom_{\mathsf{C}}(c_0, c_1)$ compatible with the structure maps. Respectively, this may be displayed diagrammatically as

$$
\begin{tikzcd}
c \arrow[d, "s"] \arrow[dd, bend right, equal] \\
c_0 \arrow[d, "r"] \\
c
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
c \arrow[r, equal] \arrow[d, "s_0"'] & c \arrow[d, "s_1"] \\
c_0 \arrow[r, "f" description] \arrow[d, "r_0"'] & c_1 \arrow[d, "r_1"] \\
c \arrow[r, equal] & c
\end{tikzcd}
$$

where each diagram is required to commute.

Remark 2.1.2. The category $\mathsf{C}(\mathrm{id}_c)$ has a distinguished (but not necessarily unique) zero object and hence is pointed. Namely

$$
*_c = \left(
\begin{tikzcd}
c \arrow[d, "\mathrm{id}_c"] \\
c \arrow[d, "\mathrm{id}_c"] \\
c
\end{tikzcd}
\right),
$$

which, as indicated, we may choose to denote by $*_c$.

The following simple observation, while not essential, simplifies arguments in what is to come by recognizing categories of retractive objects as a special instance of another operation on categories.

Lemma 2.1.3. *For any category C and object $c \in \mathsf{C}$, the category of retractive objects over c, denoted by $\mathsf{C}(\mathrm{id}_c)$ is isomorphic to the iterated slice category $(\mathsf{C}_{/c})_{\mathrm{id}_c/}$.*

Proof. This amounts to unraveling the definitions. An object of $(\mathsf{C}_{/c})_{\mathrm{id}_c/}$ is a morphism

$$
\begin{tikzcd}
c \arrow[r, "g"] \arrow[d, "\mathrm{id}_c"'] & c_0 \arrow[d, "f"] \\
c \arrow[r, equal] & c
\end{tikzcd}
$$

which therefore forces $fg = \mathrm{id}_c$. This datum is precisely that specified by an object in $\mathsf{C}(\mathrm{id}_c)$. Similarly, since id_c is terminal in $\mathsf{C}_{/c}$, morphisms of such diagrams amount precisely to the data of a morphism in $\mathsf{C}(\mathrm{id}_c)$. Namely, a morphism f for which the following diagram commutes

$$\begin{tikzcd} c \arrow[r, "s_0"'] \arrow[d, "\mathrm{id}_c"'] \arrow[rr, bend left, "s_1"] & c_0 \arrow[r, "f"'] \arrow[d, "r_0"] & c_1 \arrow[d, "r_1"] \\ c \arrow[r, equal] & c \arrow[r, equal] & c \end{tikzcd}$$

This is precisely the data determining the morphisms of $\mathsf{C}(\mathrm{id}_c)$. The procedure is identical in the reverse direction, mutatis-mutandis, and each is functorial. These procedures are manifestly inverse. $\square$

Specifying functors into this category is also particularly easy, in a sense.

Lemma 2.1.4. *There is an isomorphism of categories*

$$\mathsf{Fun}(\mathsf{D}, \mathsf{C}(\mathrm{id}_c)) \cong \mathsf{Fun}(\mathsf{D}, \mathsf{C})(\underline{c})$$

natural in D. *More generally, there are isomorphisms*

$$\mathsf{Fun}(\mathsf{D}, \mathsf{C}_{c/}) \cong \mathsf{Fun}(\mathsf{D}, \mathsf{C})_{\underline{c}/}, \qquad \mathsf{Fun}(\mathsf{D}, \mathsf{C}_{/c}) \cong \mathsf{Fun}(\mathsf{D}, \mathsf{C})_{/\underline{c}}$$

natural in D.

Proof. The pattern of the argument is the same, mutatis-mutandis, in either the retractive or slice cases, so let us consider the retractive case.

Let $U\colon \mathsf{C}(\mathrm{id}_c) \to \mathsf{C}$ be the forgetful functor and let us denote the image of the object d under a functor $F\colon \mathsf{D} \to \mathsf{C}(\mathrm{id}_c)$ by

$$Fd = (s_d, UFd, r_d).$$

The association in one direction sends a functor $F\colon \mathsf{D} \to \mathsf{C}(\mathrm{id}_c)$ to the functor UF with the evident structure maps $S\colon \underline{c} \to UF$ and $R\colon UF \to \underline{c}$ given on an object d as $S_d = s_d$ and $R_d = r_d$, using the notation established just above. To see this is well-defined, we must check S and R are natural. For this, we must only check for $f\colon d \to d'$ that $S_{d'} = UFf \circ S_d$ and similarly $R_d = R_{d'} \circ UFf$. This follows from functorality of F since $S_d = s_d$ and $R_d = r_d$ for all objects $d \in \mathsf{D}$. The association on morphisms is defined in the evident way.

The reverse association takes a functor G with natural transformations $s\colon \underline{c} \to G$ and $r\colon G \to \underline{c}$ and assembles a functor $G\colon \mathsf{D} \to \mathsf{C}(\mathrm{id}_c)$ by mapping $G(d) = (s_d, Gd, r_d)$ and is defined on morphisms by letting $G(f)$ be the underlying morphism $G(f)$.

These two procedures are functorial and manifestly inverse and natural. $\square$

Remark 2.1.5. The above may also be seen from the properties of the join and slice constructions.

An important property of retractive categories is that many limits and colimits are computed as in C for bicomplete C. In fact, in a certain sense, all limits and colimits in $\mathsf{C}(\mathrm{id}_c)$ may be computed in C.

To fix ideas, let us consider the category of pointed spaces $\mathsf{S}_* \cong \mathsf{S}(\mathrm{id}_*)$. This following a simple example that needs no explanation.

Example 2.1.6. The coproduct (wedge product) $X \vee Y$ in S_* is computed as the pushout in S

$$\begin{tikzcd}
* \arrow[r] \arrow[d] & X \arrow[d] \\
Y \arrow[r] & X \vee_* Y
\end{tikzcd}$$

The structure map $* \to X \vee_* Y$ is simply the composite map from the pushout square.

Coequalizers may be computed in the same manner.

Example 2.1.7. The coequalizer of two maps $f, g \colon X \to Y$ in S_* may be computed in S as the colimit in C of the diagram

$$\begin{tikzcd}
* \arrow[d] \arrow[dr] & \\
X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y
\end{tikzcd}$$

where the maps $* \to X$ and $* \to Y$ are the basepoint maps. The diagram commutes since the maps f and g are pointed. The colimit of this diagram in S is simply the coequalizer of the two maps f and g—call the object Z and the coequalizing map h.

$$\begin{tikzcd}
* \arrow[d] \arrow[dr] & & \\
X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \arrow[r, "h"] & Z
\end{tikzcd}$$

If we give Z the basepoint coming from coequalizing diagram above, then we claim that

$$\begin{tikzcd}
X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \arrow[r, "h"] & Z
\end{tikzcd}$$

is coequalizing in S_*. This is, of course, a simple computation.

These two examples suggest that the retractive categories of a bicomplete category C are themselves bicomplete, as long as we guarantee that the structure maps "take care of themselves." In fact, these examples suggest a little more—namely, that limits and colimits in $\mathsf{C}(\mathrm{id}_c)$ are closely related to limits and colimits in C.

The remainder of this section is essentially an elementary categorical exercise, but we give details anyways. Let us begin by briefly collecting some relevant definitions.

Definition 2.1.8. A functor $U\colon \mathsf{E} \to \mathsf{C}$ is said to ***create limits of shape*** D if for every diagram $F\colon \mathsf{D} \to \mathsf{E}$ for which UF has a limit in C, a cone $\eta\colon \underline{e} \to F$ is a limiting cone in E if and only if $U\eta\colon \underline{Ue} \to UF$ is a limiting cone in D. In other words, $U_*\colon \mathsf{Fun}(\mathsf{D},\mathsf{E}) \to \mathsf{Fun}(\mathsf{D},\mathsf{C})$ preserves and reflects limit cones.

Definition 2.1.9. For any category D, let $\mathsf{D}^{\triangleleft} = \Delta[0] * \mathsf{D}$ and $\mathsf{D}^{\triangleright} = \mathsf{D} * \Delta[0]$, the join of categories. Here, $\mathsf{D}^{\triangleleft}$ is the category formed from D by freely adjoining an initial object and $\mathsf{D}^{\triangleright}$ the category freely formed by adjoining a terminal object. We denote these new objects $\bullet$ and call them the cone points of the categories $\mathsf{D}^{\triangleleft}$ and $\mathsf{D}^{\triangleright}$.

Remark 2.1.10. The notation $\triangleleft$ and $\triangleright$ is supposed to be evocative of the shape of the new category.

Definition 2.1.11. If C is a category having an initial object $\emptyset$ and terminal object $*$, then for any category D, we define functors

$$(-)^{\triangleleft}\colon \mathsf{Fun}(\mathsf{D},\mathsf{C}) \to \mathsf{Fun}(\mathsf{D}^{\triangleleft},\mathsf{C}), \qquad (-)^{\triangleright}\colon \mathsf{Fun}(\mathsf{D},\mathsf{C}) \to \mathsf{Fun}(\mathsf{D}^{\triangleright},\mathsf{C})$$

by declaring $F^{\triangleleft}\colon \mathsf{D}^{\triangleleft} \to \mathsf{C}$ to be the functor with the property that $F^{\triangleright}|\,\mathsf{D} = F$ and is defined on the cone point by $F^{\triangleleft}(\bullet) = \emptyset$. The functor $F^{\triangleright}$ is defined dually.

Lemma 2.1.12. *If* C *is complete category, then for any small category* D*, following diagram commutes up to natural isomorphism*

$$\begin{tikzcd}
\mathsf{Fun}(\mathsf{D},\mathsf{C}) \arrow[r, "\lim"] \arrow[d, "(-)^{\triangleright}"'] & \mathsf{C} \arrow[d, equal] \\
\mathsf{Fun}(\mathsf{D}^{\triangleright},\mathsf{C}) \arrow[r, "\lim"'] & \mathsf{C}
\end{tikzcd}$$

Moreover, $(-)^{\triangleright}$ *is an isomorphism of categories onto its image. The dual assertion holds for* C *cocomplete.*

Proof. This follows from the fact that the limit cone

$$\eta\colon \underline{\lim F} \to F$$

may be uniquely extended to a cone

$$\eta'\colon \underline{\lim F} \to F^{\triangleright}$$

which one may easily check is a limit cone. The first assertion now follows by uniqueness of adjoints. The second assertions follows easily by noting that there a unique morphism between terminal objects. □

Lemma 2.1.13. *Let* C *be a bicomplete category and* $c \in \mathsf{C}$ *any object.*

(a) $\mathsf{C}(\mathrm{id}_c)$ *is bicomplete.*

(b) The forgetful functor $U\colon \mathsf{C}(\mathrm{id}_c) \to \mathsf{C}$ *preserves all colimits (resp. limits) of shape* D *for which the inclusion* $\mathsf{D} \to \mathsf{D}^{\triangleleft}$ *(resp.* $\mathsf{D} \to \mathsf{D}^{\triangleright}$*) is final (resp. cofinal).*

(c) *The forgetful functor* $U\colon \mathsf{C}(\mathrm{id}_c) \to \mathsf{C}$ *creates colimits (resp. limits) for all functors whose domain category* D *has the property that the inclusion* $\mathsf{D} \to \mathsf{D}^{\triangleleft}$ *(resp.* $\mathsf{D} \to \mathsf{D}^{\triangleright}$*) is final (resp. cofinal).*

Remark 2.1.14. For such a category C, essentially all that goes wrong with the computation of colimits in the underlying category of C is that we have not yet provided a natural map from c into the colimiting object. To get a feel for what this is saying, one may consider that this statement is true for coproducts in pointed spaces $\mathsf{S}_* \cong \mathsf{S}^*$. This is a good example to keep in mind to avoid getting bogged down in notation while reading the following proof.

Proof. The two cases are dual, so we consider colimits.

(a) Given $F\colon \mathsf{D} \to \mathsf{C}(\mathrm{id}_c)$, let $F^{\triangleleft}\colon \mathsf{D}^{\triangleleft} \to \mathsf{C}(\mathrm{id}_c)$ be the (unique) cone corresponding to the (unique) natural transformation $\underline{*_c} \to F$ and denote the cone point of $\mathsf{D}^{\triangleleft}$ by $\bullet$.

Let c_0 be any colimit of $UF^{\triangleleft}$ in C with cone $\eta\colon UF^{\triangleleft} \to \underline{c_0}$. Let $s = \eta_\bullet\colon c \to c_0$ and let $r\colon c_0 \to c$ be the map induced from the cone $UF^{\triangleleft} \to \underline{c}$ of Lemma 2.1.4. Then $rs = \mathrm{id}_c$ since for any $d \in \mathsf{D}^{\triangleleft}$, the map s factors as $s = \eta_d \circ s_d$ where $s_d\colon c \to UF^{\triangleleft}d$ is the structure map and $r\colon c_0 \to c$ is induced by the maps $r_d\colon UF^{\triangleleft}d \to c$ of the cone $UF^{\triangleleft} \to \underline{c}$ where $r_d s_d = \mathrm{id}_c$ for all d.

We claim that (s, c_0, r) along with the maps $\eta_d\colon UFd \to c_0$ exhibiting c_0 as the colimit of $UF^{\triangleleft}$ exhibits (s, c_0, r) as the colimit of F in $\mathsf{C}(\mathrm{id}_c)$. From the analysis in the preceding paragraph, we at least know that morphisms comprising η assemble into a cone $\eta\colon F \to \underline{(s, c_0, r)}$. Note that by the construction above, the evident extension $\eta'\colon F^{\triangleleft} \to \underline{(s, c_0, r)}$ satisfies that

$$U\eta' = \eta.$$

We must now show η' is the initial such cone. Given any other cone $\tau\colon F \to \underline{(s', c', r')}$, let $\tau'\colon F^{\triangleleft} \to \underline{(s', c', r')}$ be the evident extension as before. Since $c_0 \cong \operatorname{colim} UF^{\triangleleft}$, there is a unique morphism $f\colon c_0 \to c'$ for which $\underline{f} \circ U\eta' = U\tau'$. It suffices to show that f respects the structure maps. Since the natural transformation $UF^{\triangleleft} \to \underline{c}$ factors through the map $r'\colon \underline{c'} \to \underline{c}$ on account of naturality of τ' and since $\underline{c_0} \to \underline{c}$ satisfies the same, induced by the colimit property of c_0, it follows that $r' \circ f = r$ by universal properties of the colimit. That $s' = f \circ s$ follows from the fact that $s' = U\tau'_{\bullet}$ and $s = U\eta'_{\bullet}$.

(b) This statement, for when $\mathsf{D} \to \mathsf{D}^{\triangleleft}$ is final, follows immediately from the above analysis and formula for colimits.

(c) Fix a functor $F\colon \mathsf{D} \to \mathsf{C}(\mathrm{id}_c)$ and let $\eta\colon F \to \underline{(s, c_0, r)}$ be a cone. Then η admits a (unique) extension to a cone $\eta'\colon F^{\triangleleft} \to \underline{(s, c_0, r)}$ where $F^{\triangleleft}$ is as above.

Suppose first $U\eta\colon UF \to \underline{c_0} = U\underline{(s, c_0, r)}$ is a colimit cone in C. We must show that η is a colimit cone in $\mathsf{C}(\mathrm{id}_c)$. By finality, $U(\eta')$ is a colimit cone if and only if $U\eta = U(\eta')|\,\mathsf{D}$ is a colimit cone, so we may just as well suppose that $U(\eta')\colon UF^{\triangleleft} \to \underline{c_0}$ is a colimit cone and show that η' is a colimit cone.

Note that $U\eta'_{\bullet} = s$. This is because $*_c$ is the terminal object of $\mathsf{C}(\mathrm{id}_c)$. By finality, $c_0 \cong \operatorname{colim} UF$. Hence, to provide a map $c_0 \to c$ is the same as specifying a natural transformation $UF \to \underline{c}$. Since $r\colon c_0 \to c$ commutes with the structure maps maps $U\eta_d\colon UFd \to c_0$, this map is determined by the natural transformation $UF \to \underline{c}$ of Lemma 2.1.4. Note that $rs = \mathrm{id}_c$. This is because for any $d \in \mathsf{D}^{\triangleleft}$,

To see that η' must be a colimit cone in $\mathsf{C}(\mathrm{id}_c)$, consider any other cone $\tau\colon F^{\triangleleft} \to \underline{(s', c', r')}$. As a colimit in C, there is a unique map $f\colon c \to c'$ for which $f \circ U\eta' = U\tau$. We must show it respects the structure maps. For this, note that, as before, $\tau_\bullet = s'$ and hence, $f \circ U\eta'_\bullet = U\tau_\bullet$ or, in other words, $f \circ s = s'$. By assumption, $\underline{r'} \circ \tau$ is the natural transformation $UF^{\triangleleft} \to \underline{c}$ of Lemma 2.1.4. Hence, since $U\eta'\colon UF^{\triangleleft} \to \underline{c_0}$ is the initial cone out of $UF^{\triangleleft}$, the map $f\colon c_0 \to c'$ satisfies $r' \circ f = r$.

Conversely, suppose $\eta\colon F \to \underline{(s, c_0, r)}$ is a colimit cone in $\mathsf{C}(\mathrm{id}_c)$. It follows from (a) that $U\eta\colon UF \to \underline{Uc_0}$ is a colimit cone. □

Inspecting the proof of (a) carefully reveals the following.

Lemma 2.1.15. *Suppose* C *is bicomplete.*

(a) $\mathsf{C}_{c/}$ *and* $\mathsf{C}_{/c}$ *are bicomplete.*

(b) The colimit of a diagram $F\colon \mathsf{D} \to \mathsf{C}_{c/}$ *is computed in* C *as the colimit of the augmented diagram* $F^{\triangleleft}\colon D^{\triangleleft} \to \mathsf{C}$. *The limit of a diagram* $F\colon \mathsf{D} \to \mathsf{C}_{/c}$ *is computed in* C *as the limit of the augmented diagram* $F^{\triangleright}\colon \mathsf{D}^{\triangleright} \to \mathsf{C}$.

(c) The forgetful functor $U\colon \mathsf{C}_{c/} \to \mathsf{C}$ *preserves and creates small colimits for all functors whose domain category* D *has the property that the inclusion* $\mathsf{D} \to \mathsf{D}^{\triangleleft}$ *is final. The forgetful functor* $U\colon \mathsf{C}_{/c} \to \mathsf{C}$ *preserves and creates small limits for all functors whose domain category* D *has the property that the inclusion* $\mathsf{D} \to \mathsf{D}^{\triangleright}$ *is cofinal.*

The following observation will allow us to characterize limits and colimits in categories of retractive objects.

Lemma 2.1.16. *If* D *has an initial (resp. terminal) object, the inclusion* $\mathsf{D} \to \mathsf{D}^{\triangleleft}$ *(resp.* $\mathsf{D} \to \mathsf{D}^{\triangleright}$*) is final (resp. cofinal).*

Proof. Let $\emptyset$ denote an initial object of D. For any $d \in \mathsf{D}$, the comma category $(d \downarrow \mathsf{D})$ is connected because it has an initial object $\mathrm{id}_d \colon d \to d$. If $\bullet$ denotes the cone point of $\mathsf{D}^{\triangleleft}$, then the unique map $\bullet \to \emptyset$ is an initial object in this category. $\square$

Corollary 2.1.17. *If C is complete, then the limit of a diagram $F \colon \mathsf{D} \to \mathsf{C}(\mathrm{id}_c)$ may be computed, up to isomorphism, as the limit of the unique extension $F^{\triangleright} \colon \mathsf{D}^{\triangleright} \to \mathsf{C}$ and this limit is created in C.*

The following are special cases of Lemma 2.1.16 that we will use.

Corollary 2.1.18. *Limits and colimits of punctured cubes in $\mathsf{C}(\mathrm{id}_c)$ are computed as in C. For any infinite regular cardinal κ, κ-cofiltered limits and κ-filtered colimits in $\mathsf{C}(\mathrm{id}_c)$ are computed as in C.*

2.2 Model Structures on Retractive Objects

As alluded to above, the benefit of our description of $\mathsf{C}(\mathrm{id}_c)$ as an iterated slice category is that it allows us to equip $\mathsf{C}(\mathrm{id}_c)$ with a model structure whenever C has a model structure.

Definition 2.2.1. An object c of a cocomplete category C is said to be ***κ-compact***, where κ is an infinite regular cardinal, if for any κ-filtered diagram, $F \colon J \to \mathsf{C}$,

$$\operatorname{colim} \hom_{\mathsf{C}}(c, F) \to \hom_{\mathsf{C}}(c, \operatorname{colim} F)$$

is an isomorphism. When $\kappa = \omega$, the first infinite cardinal, we say such an object is ***compact***.

When C is enriched over S or S_*, if the natural map

$$\operatorname{colim} \mathbf{Hom}(c, F) \to \mathbf{Hom}(c, \operatorname{colim} F)$$

is an isomorphism, then we say c is ***simplicially compact***—note that in all cases we will consider, it is irrelevant whether we use the pointed mapping space or the unpointed mapping space. We also say an object c is ***sequentially compact*** if for every sequence $c_0 \to c_1 \to \cdots$, the natural map

$$\operatorname{colim}_i \hom_{\mathsf{C}}(C, c_i) \to \hom_{\mathsf{C}}(c, \operatorname{colim}_i c_i)$$

is an isomorphism.

Definition 2.2.2. Following Hovey in [35], we call a cofibrantly generated model structure on a category C ***finitely generated*** if the domains and codomains of the generating cofibration and generating acyclic cofibrations are sequentially compact.

Definition 2.2.3. A category C is said to be ***locally presentable*** if it is cocomplete and there is an infinite regular cardinal κ and a set of κ-compact objects S such that every object of the category is a κ-filtered colimit of objects in S. In this case, C is also said to be ***locally κ-presentable***. When $\kappa = \omega$, the first infinite cardinal, C is said to be ***locally finitely presentable***. A cofibrantly generated model category that is locally presentable is said to be a ***combinatorial model category***.

Remark 2.2.4. The property of being locally presentable may be thought of as a point-set tameness condition, whereas being cofibrantly generated is a homotopical tameness condition. Combinatorial model categories are therefore particularly well-behaved and, in practice, most well-behaved model categories (such as simplicial sets) are combinatorial.

The following is [1, Prop. 1.57].

Proposition 2.2.5. *If* C *is a locally* κ*-presentable category, then for every* $c \in \mathsf{C}$*,* $\mathsf{C}_{/c}$ *and* $\mathsf{C}_{c/}$ *are both once again locally* κ*-presentable.*

This fact, along with the main theorems of [32] have the following interesting implication.

Proposition 2.2.6. *Suppose* C *is a locally* κ*-presentable, finitely generated model category with set of generating cofibrations* I *and acyclic cofibrations* J*.*

(a) Then $\mathsf{C}(\mathrm{id}_c)$ *is again a locally* κ*-presentable, finitely generated model category in which the classes of weak equivalences, cofibrations and fibrations in* $\mathsf{C}(\mathrm{id}_c)$ *are underlying in* C*. In particular, the set of generating cofibrations* I^c *for* $\mathsf{C}(\mathrm{id}_c)$ *consists of all maps*

$$\begin{tikzcd} c \arrow[r, equal] \arrow[d, "\mathrm{in}"'] & c \arrow[d, "\mathrm{in}"] \\ c_0 \quad c \arrow[r, "i \quad \mathrm{id}_c"] \arrow[d, "r_0+\mathrm{id}_c"'] & c_1 \quad c \arrow[d, "r_1+\mathrm{id}_c"] \\ c \arrow[r, equal] & c \end{tikzcd}$$

for which $i \in I$ *and the set of generating acyclic cofibrations* J^c *consists of all maps*

$$\begin{tikzcd} c \arrow[r, equal] \arrow[d, "\mathrm{in}"'] & c \arrow[d, "\mathrm{in}"] \\ c_0 \quad c \arrow[r, "j \quad \mathrm{id}_c"] \arrow[d, "r_0+\mathrm{id}_c"'] & c_1 \quad c \arrow[d, "r_1+\mathrm{id}_c"] \\ c \arrow[r, equal] & c \end{tikzcd}$$

for which $j \in J$.

(b) More generally, if c is any compact object of C, then any object of the form

$$\begin{tikzcd}
c \arrow[r, equal] \arrow[d, "\mathrm{in}"'] & c \arrow[d, "\mathrm{in}"] \\
c_0 \quad c \arrow[r, "i \quad \mathrm{id}_c"] \arrow[d, "r_0+\mathrm{id}_c"'] & c_1 \quad c \arrow[d, "r_1+\mathrm{id}_c"] \\
c \arrow[r, equal] & c
\end{tikzcd}$$

is compact in $\mathsf{C}(\mathrm{id}_c)$.

(c) If the domains of the generating cofibrations or generating acyclic cofibrations are cofibrant, then the same is true in $\mathsf{C}(\mathrm{id}_c)$.

(d) If C is left (resp. right) proper, then so too is $\mathsf{C}(\mathrm{id}_c)$.

Proof. (a) The description of the generating cofibrations and acyclic cofibrations follow directly from the iterated slice description of $\mathsf{C}(\mathrm{id}_c)$ of Lemma 2.1.3 and an application of the descriptions of the generating cofibrations and acyclic cofibrations for slice categories provided in [32]. Similarly, the fact that $\mathsf{C}(\mathrm{id}_c)$ is once again locally κ-presentable follows directly from directly from the iterated slice description of $\mathsf{C}(\mathrm{id}_c)$ of Lemma 2.1.3 and Proposition 2.2.5.

The only thing that remains to be shown is that $\mathsf{C}(\mathrm{id}_c)$ is almost finitely generated. To take care of this, we will show that a larger class of objects in $\mathsf{C}(\mathrm{id}_c)$ are sequentially compact. Suppose

c_0 is a compact object in C and let $r\colon c_0 \to c$. It suffices for us to show that

$$\begin{tikzcd} c \arrow[d, "\mathrm{in}"] \\ c_0 \quad c \arrow[d, "r+\mathrm{id}_c"] \\ c \end{tikzcd}$$

is compact in $\mathsf{C}(\mathrm{id}_c)$. This object is obtained from $r\colon c_0 \to c$ by applying the free functor $F\colon \mathsf{C}_{/c} \to (\mathsf{C}_{/c})_{\mathrm{id}_c\,/} \cong \mathsf{C}(\mathrm{id}_c)$ taking an object such as $r\colon c_0 \to c$ to

$$\begin{tikzcd} c \arrow[d, "\mathrm{in}"] \arrow[dd, bend right=60, equal] \\ c_0 \quad c \arrow[d, "r+\mathrm{id}_c"] \\ c \end{tikzcd}$$

and this functor is left adjoint to the forgetful functor $U\colon \mathsf{C}(\mathrm{id}_c) \cong (\mathsf{C}_{/c})_{\mathrm{id}_c\,/} \to \mathsf{C}_{/c}$. Hence, for any sequence

$$c'_0 \to c'_1 \to c'_2 \to \cdots$$

in $\mathsf{C}(\mathrm{id}_c)$, on account of how colimits of such sequences are computed by Corollary 2.1.18, we have

$$\hom_{\mathsf{C}(\mathrm{id}_c)}(c_0 \coprod c, \operatorname{colim}^{\mathsf{C}(\mathrm{id}_c)} c'_i) \cong \hom_{\mathsf{C}_{/c}}(c_0, \operatorname{colim}^{\mathsf{C}_{/c}} c'_i),$$

where we have decorated where these colimits are computed (at least on underlying objects, this does not make a difference).

But the mapping set $\hom_{\mathsf{C}_{/c}}(c_0, \operatorname{colim} c_i')$ is the pullback

$$\begin{tikzcd}
\hom_{\mathsf{C}_{/c}}(c_0, \operatorname{colim} c_i') \arrow[r] \arrow[d] & \hom_{\mathsf{C}}(c_0, \operatorname{colim}^{\mathsf{C}} c_i') \arrow[d] \\
* \arrow[r, "\{r\colon c_0 \to c\}"] & \hom_{\mathsf{C}}(c_0, c)
\end{tikzcd}$$

and, by assumption, c_0 is compact in C. Hence, this diagram is a pullback

$$\begin{tikzcd}
\hom_{\mathsf{C}_{/c}}(c_0, \operatorname{colim} c_i') \arrow[r] \arrow[d] & \operatorname{colim} \hom_{\mathsf{C}}(c_0, c_i') \arrow[d] \\
* \arrow[r, "\{r\colon c_0 \to c\}"] & \hom_{\mathsf{C}}(c_0, c)
\end{tikzcd}$$

and since filtered colimits and finite limits commute in the category of sets, this shows that the natural map $\hom_{\mathsf{C}_{/c}}(c_0, \operatorname{colim} c_i') \to \operatorname{colim} \hom_{\mathsf{C}_{/c}}(c_0, c_i')$ is an isomorphism. Hence, the natural map $\operatorname{colim} \hom_{\mathsf{C}(\mathrm{id}_c)}(c_0 \quad c, c_i') \to \hom_{\mathsf{C}(\mathrm{id}_c)}(c_0 \quad c, \operatorname{colim} c_i')$ is an isomorphism, as desired.

(b) This follows in precisely the same way as the proof just given above, mutatis-mutandis—simply replace the graphical depiction of the sequential colimit by a filtered diagram.

(c) The underlying object of the domains of the generating cofibrations and generating acyclic cofibrations are given by the following pushout in C

$$\begin{tikzcd}
\emptyset \arrow[r] \arrow[d] & c \arrow[d, "\mathrm{in}"] \\
c_0 \arrow[r] & c_0 \quad c
\end{tikzcd}$$

where $c_0 \in \mathrm{dom}(I) \cup \mathrm{dom}(J)$. Thus, since c_0 is cofibrant, the left-hand vertical arrow is a cofibration and therefore the left-most arrow is a cofibration. It follows immediately that the domains of the generating cofibrations and generating acyclic cofibrations are cofibrant in $\mathsf{C}(\mathrm{id}_c)$.

(d) The forgetful functor creates pushouts and pullbacks by Corollary 2.1.18. Moreover, the forgetful functor preserves and reflects all classes of distinguished morphisms. Suppose C is right proper. Hence, given a pullback in $\mathsf{C}(\mathrm{id}_c)$, displaying underlying objects only,

$$\begin{tikzcd} c \arrow[r, "f"] \arrow[d] & c_0 \arrow[d, two heads, "p"] \\ c_1 \arrow[r, "\sim", "f"'] & c_2 \end{tikzcd}$$

the forgetful functor sends it to a pullback in C, where since C is right proper, f is a weak equivalence. The assertion is dual for left properness. □

In homotopy theory, it is often preferable for our model categories to be suitably enriched over a monoidal model category—in particular, we might ask how a simplicial model structure on C passes to one on $\mathsf{C}(\mathrm{id}_c)$ and, in particular, a *pointed* simplicial model structure on $\mathsf{C}(\mathrm{id}_c)$. Such questions have been addressed in Schwede's thesis work [48] and later in [34].

The following definition is adapted from Hovey in [34], but see also [22] for an equivalent definition. We will only ever consider closed modules, so we have dropped the word 'closed' from the definition. Similarly, we shall only every be interested in the case when the symmetric monoidal model category in question is spaces S or pointed spaces S_*. Before giving the definition, let us make a remark.

Remark 2.2.7. There are at least two equivalent ways to define a (pointed) simplicial model category having a notion of a tensoring. As noted, we will state the one given by Hovey in [34]. As for the other possible definition such as that found in the appendices of [41] or in [47], which is predicated upon a enrichment with a tensoring instead of a module structure in the definition below, the equivalence between these definitions will follow by the equivalence specified in [25, 38]. We will say more about this equivalence in a remark following the definition.

Definition 2.2.8. Given a closed monoidal category $(\mathsf{C}, *, \alpha, \rho, \lambda, e)$, a (closed) C-***module*** is a category D along with a cotensor ($\mathbf{hom}\colon \mathsf{C} \times \mathsf{D} \to \mathsf{D}$), tensoring ($\otimes\colon \mathsf{D} \times \mathsf{C} \to \mathsf{D}$) and hom-object $\mathbf{Hom}\colon \mathsf{D}^{\mathrm{op}} \times \mathsf{D} \to \mathsf{C}$ having natural (unenriched) isomorphisms φ and ψ

$$\mathrm{hom}_{\mathsf{D}}(X, \mathbf{hom}_{\mathsf{D}}(K, Y)) \xleftarrow[\cong]{\psi} \mathrm{hom}_{\mathsf{D}}(X \otimes K, Y) \xrightarrow[\cong]{\varphi} \mathrm{hom}_{\mathsf{C}}(K, \mathbf{Hom}_{\mathsf{D}}(X, Y))$$

along with natural isomorphisms

$$a\colon (X \otimes K) \otimes L \to X \otimes (K * L)$$

and

$$r\colon X \otimes e \to X$$

such that the following three diagrams commute

$$\begin{tikzcd}
((X \otimes K) \otimes L) \otimes M \arrow[r, "a"] \arrow[d, "a\otimes\mathrm{id}"'] & (X \otimes K) \otimes (L * M) \arrow[r, "a"] & X \otimes (K * (L * M)) \\
(X \otimes (K * L)) \otimes M \arrow[rr, "a"'] & & X \otimes ((K * L) * M) \arrow[u, "1\otimes\alpha"']
\end{tikzcd}$$

$$\begin{tikzcd}
(X \otimes e) \otimes K \arrow[rr, "a"] \arrow[dr, "r\otimes\mathrm{id}"'] & & X \otimes (e * K) \arrow[dl, "\mathrm{id}\otimes\lambda"] \\
& X * K &
\end{tikzcd}$$

$$\begin{tikzcd}
(X \otimes K) \otimes e \arrow[rr, "a"] \arrow[dr, "r"'] & & X \otimes (K * e) \arrow[dl, "\mathrm{id}\otimes\rho"] \\
& X * K &
\end{tikzcd}$$

A functor of C-modules $F\colon \mathsf{D} \to \mathsf{D}'$ is a functor F of the underlying categories along with a natural assembly map

$$\mathrm{assemb}^F_{X,K}\colon F(X) \otimes K \to F(X \otimes K),$$

which is associative and unital. A natural transformation of C-module functors $\eta\colon F \to F'$ is a natural transformation of the underlying functors which, additionally, respects the assembly maps.

If, furthermore, C is a closed monoidal model category and D a model category, then if $f\colon A \to B$ is a cofibration in D and $g\colon K \to L$ is a cofibration in C, we say that D is a C***-model category*** if the induced pushout product map

$$(A \otimes L) \coprod_{A\otimes K} (B \otimes K) \to B \otimes K$$

is a cofibration in D which is acyclic if either of f and g are acyclic cofibrations.

A few remarks are in order. The first two are easy technical points on the definition above.

Remark 2.2.9. When C is a symmetric monoidal category, it is known that the braiding $\tau\colon C*C' \xrightarrow{\cong} C'*C$ satisfies that $\rho = \lambda\circ\tau$. This implies that the first unit compatibility diagram (second diagram above) is subsumed by the second unit compatibility diagram (third diagram above) and conversely, as noted in [38].

Remark 2.2.10. If $*_{\mathsf{D}}$ is the terminal object of D, then $\mathbf{hom}_{\mathsf{D}}(K, *_{\mathsf{D}}) \cong *_{\mathsf{D}}$ naturally in K by Yoneda, since there is a natural isomorphism

$$\hom(X, \mathbf{hom}(K, *_{\mathsf{D}})) \cong \hom(X \otimes K, *_{\mathsf{D}}) = * = \hom(X, *_{\mathsf{D}}),$$

and since, therefore, all objects $\mathbf{hom}(K, *_{\mathsf{D}})$ is terminal in D for each K, there is a unique such natural isomorphism and naturality in K is immediate.

Similarly, for any object Y, $\mathbf{hom}(e, Y) \cong Y$ naturally. This likewise follows by Yoneda since there is an isomorphism

$$\hom(X, \mathbf{hom}(e, Y)) \xrightarrow{\psi^{-1}} \hom(X \otimes e, Y) \xrightarrow{(r, \mathrm{id}_Y)} \hom(X, Y)$$

natural in X. As it happens, this map is simply the one adjoint to the natural unit isomorphism $Y \otimes e \cong Y$. We defer a discussion of this to later, where we actually need it.

The last remark clarifies the precise nature of the equivalence mentioned above and our use of it.

Remark 2.2.11. Up to a suitable notion of equivalence as in [25, 38] and, the above definition may instead be taken to be a cotensored and tensored C-enriched category satisfying the pushout-product axiom above. Indeed, the natural adjunction isomorphisms φ and ψ of this definition may be made to be natural for the enriched hom-functors. See the discussion in [38, §2] and [25]. Indeed, the main theorem of [25] may be understood as saying that, for a closed symmetric monoidal category C—understood as a bicategory by way of its delooping—there is an equivalence of 2-categories via a 2-functor

$$K \colon \mathsf{C}\text{-Cat}_{\otimes} \to \mathsf{Mod}_{\mathsf{C}}$$

between tensored C-enriched categories and C-modules where, in particular, the functor K amounts only to making certain canonical choices for unit, assembly and associativity natural transformations. Effectively, this means that with some additional canonical choices, C-enriched categories, functors and natural transformations all already satisfy the module conditions. See [25, §3] or the appendix of [38] for details. More recent treatments may be found in [17, 45, 40], where it should be noted that [17] corrects an error in [45] which found its way into [40]. It should also be noted that there is an explicit construction of an inverse equivalence that is spelled out, for instance, in [25, 17, 45], among other places and, under this inverse equivalence, the resulting endofunctor on $\mathsf{Mod}_{\mathsf{C}}$ is the identity—the reverse construction is quite simple and is *exactly* the first thing the reader will think of, essentially amounting to a forgetful functor, so we leave these details to the interested reader to discover or read about themselves, from the references already mentioned.

The upshot of this is the following observations. All facts about enriched functors of tensored C-enriched categories and enriched natural transformations between them hold for functors and natural transformations of C-modules (because the functors K amounts only to certain canonical choices). There are certain canonical choices such that C-modules are tensored C-categories and for these choices module functors and natural transformations are enriched and preserved by the 2-functor $K\colon \mathsf{C}\text{-Cat}_{\otimes} \to \mathsf{Mod}_{\mathsf{C}}$.

Because this is a result that is not often spelled out in its entirety, the module characterization of functors and natural transformations will be more natural in our work in certain places and C-modules and their functors and natural transformations being somewhat more frequently occurring in nature, we will prefer the definition given above. To briefly elaborate, there may not be an obvious choice for composition if one wants to construct a simplicial model category from an ordinary model category—morally, this is because there is a rigidification problem that requires one to specify all coherences in a way compatible with the given model structure. If one, instead, looks for a module structure—so a tensoring of the sort above—then there is an essentially unique (because of the equivalence above) tensored simplicial model structure occurring on the given model category. We will defer some longer discussions about the content of this remark and what happens when there is, additionally, a cotensoring—as in the definition above—to Remarks 3.3.11 and 3.3.4.

In light of these remarks, we make the following (somewhat abusive) definition.

Definition 2.2.12. Fix a closed monoidal category $(\mathsf{C}, *, \alpha, \rho, \lambda, e)$. We will call a functor of C-modules as given in Definition 2.2.8 a C***-enriched functor***. We will call a natural transformation between functors of C-modules as given in Definition 2.2.8 a C***-enriched natural transformation***.

As mentioned before, there are two important cases of the above definition that deserve special attention.

Definition 2.2.13. If $\mathsf{C} = \mathsf{S}$ with its Cartesian monoidal model structure, then a S-model category is also called a ***simplicial model category***. If $\mathsf{C} = \mathsf{S}_*$ with its monoidal structure inherited from the smash product, then a S_*-model category is also called a ***pointed simplicial model category***.

We collect two simple observations.

Lemma 2.2.14. *Every pointed simplicial model category is naturally a simplicial model category by way of the disjoint basepoint functor* $(-)_+ \colon \mathsf{S} \to \mathsf{S}_*$.

Proof. This is because $(-)_+$ is a strong symmetric monoidal Quillen functor. ☐

Lemma 2.2.15. *Every pointed simplicial model category* C *is pointed as a category (i.e., has a zero object).*

Proof. There is an isomorphism $\hom(X \wedge *, Y) \cong \hom_{\mathsf{S}_*}(*, \mathbf{Hom}(X, Y)) \cong *$ natural in Y; hence, by Yoneda, $X \wedge * \cong \emptyset$, the initial object of C. On the other hand, the unique map $S^0 \to *$ induces for every $X \in \mathsf{C}$ a map

$$X \cong X \wedge S^0 \to X \wedge *.$$

Hence, when $X = *_{\mathsf{C}}$, the terminal object of C, this is a map

$$*_{\mathsf{C}} \to \emptyset.$$

The only way such a map can exist is if it is an isomorphism. ☐

The following idea is relatively straightforward, if one keeps in mind the special case of spaces S. We are going to generalize the process of passing a simplicial model structure on S to one on S_*. In particular, we should also like to generalize how the pointed simplicial model structure on

S_* may be obtained from its simplicial model structure. Accordingly, the key observation is that the smash product

$$X \wedge K = \frac{X \times K}{\{(*, k) \sim (x, *)\}}$$

of pointed spaces, along with its distinguished basepoint, is obtained from the product $X \times Y$ as the following pushout in S

$$\begin{tikzcd} X \quad K \arrow[r, "(\mathrm{id}_X, *)+(*, \mathrm{id}_K)"] \arrow[d] & X \times K \arrow[d] \\ * \arrow[r] & X \wedge K \end{tikzcd}$$

This sort of procedure has been observed by Hovey.

Proposition 2.2.16 ([34, Prop. 4.2.9])**.** *If* $(\mathsf{C}, \otimes, e)$ *is a symmetric monoidal model category whose terminal object* $*$ *is cofibrant, then* $(\mathsf{C}_*, \wedge, (*)_+)$ *is a symmetric monoidal model category, where* $\mathsf{C}_* = \mathsf{C}_{*/}$ *with the following constructions.*

(a) For objects $X, Y \in \mathsf{C}_*$, $X \wedge Y$ *is defined by the following pushout in* C

$$\begin{tikzcd} X \quad Y \arrow[r, "\mathrm{id}_X \otimes (* \to Y)", "+(* \to X) \otimes \mathrm{id}_Y"'] \arrow[d] & X \otimes Y \arrow[d] \\ * \arrow[r] & X \wedge Y \end{tikzcd}$$

(b) $\mathbf{Hom}_{\mathsf{C}_*}(X, Y)$ *is defined as the pullback in* C *given by*

$$\begin{tikzcd} \mathbf{Hom}_{\mathsf{C}_*}(X, Y) \arrow[r] \arrow[d] & \mathbf{Hom}_{\mathsf{C}}(X, Y) \arrow[d, "(* \to X, \mathrm{id}_Y)"] \\ * \cong \mathbf{Hom}_{\mathsf{C}}(*, *) \arrow[r, "(\mathrm{id}_*, * \to Y)"] & \mathbf{Hom}_{\mathsf{C}}(*, Y) \end{tikzcd}$$

with the structure map $* \to \mathbf{Hom}_{\mathsf{C}_*}(X, Y)$ *obtained from universal properties by way of the map* $* \to \mathbf{Hom}_{\mathsf{C}}(X, Y)$ *adjoint to* $X \otimes * \to * \to Y$ *or, equivalently, the map* $* \cong \mathbf{Hom}(X, *) \to \mathbf{Hom}(X, Y)$ *induced by* $* \to Y$*.*

(c) $\mathbf{hom}_{\mathsf{C}_*}(X, Y)$ *is defined as the pullback in* C *given by*

$$\begin{tikzcd}
\mathbf{hom}_{\mathsf{C}_*}(X, Y) \arrow[r] \arrow[d] & \mathbf{hom}_{\mathsf{C}}(X, Y) \arrow[d, "(*\to X, \mathrm{id}_Y)"] \\
* \cong \mathbf{hom}_{\mathsf{C}}(*, *) \arrow[r, "(\mathrm{id}_*, *\to Y)"] & \mathbf{hom}_{\mathsf{C}}(*, Y)
\end{tikzcd}$$

with the structure map $* \to \mathbf{hom}_{\mathsf{C}_*}(X, Y)$ *obtained from universal properties.*

Remark 2.2.17. For instance, the category of spaces satisfies the hypotheses of this proposition. This produces the usual symmetric monoidal model structure on S_* with tensor the smash product.

The same recipe, guided by the case of spaces, provides a manner of inducing a pointed simplicial model structure on C_*.

Proposition 2.2.18 ([34, Prop. 4.2.19])**.** *If* C *is a simplicial model category, then* C_* *is naturally a pointed simplicial model category with the following constructions.*

(a) For objects $X \in \mathsf{C}_*$ *and* $K \in \mathsf{S}_*$*,* $X \wedge K$ *is defined by the following pushout in* C

$$\begin{tikzcd}
X \otimes * \quad * \otimes K \arrow[r, "(\mathrm{id}_X \otimes (*\to K))", "+((*\to X)\otimes \mathrm{id}_K)"'] \arrow[d] & X \otimes K \arrow[d] \\
* \arrow[r] & X \wedge K
\end{tikzcd}$$

(b) $\mathbf{Hom}_{\mathsf{C}_*}(X,Y)$ *is defined as the pullback in* S *given by*

$$\begin{tikzcd}
\mathbf{Hom}_{\mathsf{C}_*}(X,Y) \arrow[r] \arrow[d] & \mathbf{Hom}_{\mathsf{C}}(X,Y) \arrow[d, "(*\to X,\mathrm{id}_Y)"] \\
* \arrow[r, "\{*\to Y\}"] & \mathbf{Hom}_{\mathsf{C}}(*,Y)
\end{tikzcd}$$

with the structure map $* \to \mathbf{Hom}_{\mathsf{C}_*}(X,Y)$ *picking out the evident map* $X \to * \to Y$*.*

(c) $\mathbf{hom}_{\mathsf{C}_*}(K,X)$ *is defined as the pullback in* C *given by*

$$\begin{tikzcd}
\mathbf{hom}_{\mathsf{C}_*}(K,X) \arrow[r] \arrow[d] & \mathbf{hom}_{\mathsf{C}}(K,X) \arrow[d, "(*\to K,\mathrm{id}_X)"] \\
* \cong \mathbf{hom}_{\mathsf{C}}(*,*) \arrow[r, "(\mathrm{id}_*,*\to X)"] & \mathbf{hom}_{\mathsf{C}}(*,X)
\end{tikzcd}$$

with the structure map $* \to \mathbf{hom}_{\mathsf{C}_*}(K,X)$ *obtained from universal properties.*

Remark 2.2.19. Any simplicial model category C which is pointed as a category is naturally a pointed simplicial model category by way of this construction since for such a category, $\mathsf{C} \cong \mathsf{C}_*$. Conversely, a pointed simplicial model category C acquires the structure of a simplicial model category under the strong symmetric monoidal Quillen functor $(-)_+ \colon \mathsf{S} \to \mathsf{S}_*$.

Thus, in light of Proposition 2.2.18, to provide $\mathsf{C}(\mathrm{id}_c) \cong (\mathsf{C}_{/c})_{\mathrm{id}_c/}$ with a pointed simplicial model structure, we only need to show that $\mathsf{C}_{/c}$ admits a simplicial model structure.

Lemma 2.2.20. *If* C *is a simplicial model category, then for any object* $c \in \mathsf{C}$*, so too is* $\mathsf{C}_{/c}$ *with the following constructions.*

(a) Given objects $X \in \mathsf{C}_{/c}$ *and* $K \in \mathsf{S}$*,* $X \otimes K \in \mathsf{C}_{/c}$ *has underlying object in* C $X \otimes^{\mathsf{C}} K$ *with*

structure map the evident composite

$$X \otimes K \to X \otimes * \cong X \to c,$$

where the isomorphism $X \otimes * \to X$ is the natural one and the map $X \to c$ the structure map.

(b) $\mathbf{Hom}_{\mathsf{C}_{/c}}(X,Y)$ *is defined as the pullback in* S *given by*

$$\begin{tikzcd}
\mathbf{Hom}_{\mathsf{C}_{/c}}(X,Y) \arrow[r] \arrow[d] & \mathbf{Hom}_{\mathsf{C}}(X,Y) \arrow[d, "(\mathrm{id}_X, Y\to c)"] \\
* \arrow[r, "\{X\to c\}"'] & \mathbf{Hom}_{\mathsf{C}}(X,c)
\end{tikzcd}$$

(c) $\mathbf{hom}_{\mathsf{C}_{/c}}(K,X)$ *is defined as the pullback in* C *given by*

$$\begin{tikzcd}
\mathbf{hom}_{\mathsf{C}_{/c}}(K,X) \arrow[r] \arrow[d] & \mathbf{hom}_{\mathsf{C}}(K,X) \arrow[d, "(\mathrm{id}_K, X\to c)"] \\
c \cong \mathbf{hom}_{\mathsf{C}}(*,c) \arrow[r, "(K\to *, \mathrm{id}_c)"'] & \mathbf{hom}_{\mathsf{C}}(K,c)
\end{tikzcd}$$

with the structure map to c as displayed.

Proof. First note that we may define the associativity and unit isomorphism for $\mathsf{C}_{/c}$ to be on the underlying objects the same as the one in C. To define a natural isomorphism $\hom_{/c}(X \otimes K, Y) \to$

$\hom_{/c}(X, \mathbf{hom}_{/c}(K, Y))$, observe that $\hom_{/c}$ is naturally the pullback

$$\begin{tikzcd}
\hom_{/c}(X,Y) \arrow[r] \arrow[d] & \hom_{\mathsf{C}}(X,Y) \arrow[d, "Y\to c"] \\
* \arrow[r, "\{X\to c\}"'] & \hom_{\mathsf{C}}(X,c)
\end{tikzcd}$$

and, similarly, $\mathbf{hom}_{/c}$ is naturally the pullback

$$\begin{tikzcd}
\mathbf{hom}_{\mathsf{C}_{/c}}(K,X) \arrow[r] \arrow[d] & \mathbf{hom}_{\mathsf{C}}(K,X) \arrow[d, "(\mathrm{id}_K, X\to c)"] \\
c \cong \mathbf{hom}_{\mathsf{C}}(*,c) \arrow[r, "(K\to *, \mathrm{id}_c)"'] & \mathbf{hom}_{\mathsf{C}}(K,c)
\end{tikzcd}$$

This means that the pullback diagram for $\hom_{/c}(X, \mathbf{hom}_{/c}(K, Y))$

$$\begin{tikzcd}
\hom_{/c}(X, \mathbf{hom}_{/c}(K,Y)) \arrow[r] \arrow[d] & \hom(X, \mathbf{hom}_{/c}(K,Y)) \arrow[d] \\
* \arrow[r] & \hom(X,c)
\end{tikzcd}$$

expands by continuity of the hom-functor to a pasting of pullback diagrams, using the natural isomorphism $\hom(X \otimes K, Y) \cong \hom(X, \mathbf{hom}(K, Y))$, as

$$\begin{tikzcd}
\hom_{/c}(X, \mathbf{hom}_{/c}(K,Y)) \arrow[r] \arrow[d] & \hom(X, \mathbf{hom}_{/c}(K,Y)) \arrow[r] \arrow[d] & \hom(X \otimes K, Y) \arrow[d] \\
* \arrow[r] & \hom(X,c) \arrow[r] & \hom(X \otimes K, c)
\end{tikzcd}$$

where the composite map $* \to \hom(X \otimes K, c)$ picks out the structure map $X \otimes K \to c$. Hence, since all constructions in sight are natural, we obtain a natural isomorphism $\hom_{/c}(X \otimes K, Y) \cong$

$\hom_{/c}(X, \mathbf{hom}_{/c}(K, Y))$, as desired. The argument is identical, mutatis-mutandis, for the natural isomorphism $\hom_{/c}(X \otimes K, Y) \cong \hom_{/c}(K, \mathbf{Hom}_{/c}(X, Y))$.

The unit isomorphism $r\colon X \otimes * \to X$ and the associativity isomorphism $a\colon (X \otimes K) \otimes L \to X \otimes (K \times L)$ are inherited from the ones in C. It follows easily that these provide C_* with the structure of an S_*-module.

All that remains to be checked is that the enrichment is suitably compatible with the model structure on C. Recall that pushouts in $\mathsf{C}_{/c}$ are computed as in C. Since cofibrations and weak equivalences in the slice model structure are underlying, this follows from C being a simplicial model category and the fact that the tensoring $X \otimes K$ in $\mathsf{C}_{/c}$ is the same as the one in C. □

The important, immediate consequence of the preceding discussion is the following.

Proposition 2.2.21. *If C is a simplicial model category, then for any object $c \in \mathsf{C}$, $\mathsf{C}(\mathrm{id}_c)$ is a simplicial model category and, in fact, a pointed simplicial model category.*

Proof. By Lemma 2.2.20, $\mathsf{C}_{/c}$ is a simplicial model category. In this category, there is a distinguished terminal object $\mathrm{id}_c\colon c \to c$. Hence, by Proposition 2.2.18, $(\mathsf{C}_{/c})_{\mathrm{id}_c/} \cong \mathsf{C}(\mathrm{id}_c)$ is a pointed simplicial model category. Since $\mathsf{C}(\mathrm{id}_c)$ is pointed, it inherits a simplicial model structure under the strong symmetric monoidal Quillen functor $(-)_+\colon \mathsf{S} \to \mathsf{S}_*$. □

Remark 2.2.22. In particular, this means that there are simplicial models for the loops and suspension functors, which we denote by $\Omega_c = \mathbf{hom}_{\mathsf{C}(\mathrm{id}_c)}(S^1, -)$ and $\Sigma_c = - \wedge S^1$, respectively.

It is prudent to compare the models given by this construction for loops and suspension to the ones provided by Schwede in [48]. For a simplicial model category C, Schwede gives $\mathsf{C}(\mathrm{id}_c)$ a simplicial model structure with its *simplicial* tensoring $X \otimes^c K$ defined as the pushout

$$\begin{tikzcd}
c \otimes K \arrow[r] \arrow[d] & X \otimes K \arrow[dd] \\
c \otimes * \arrow[d, "\cong"'] & \\
c \arrow[r] & X \otimes^c K
\end{tikzcd}$$

with the evident structure maps and, similarly, defines the *simplicial* mapping object $\mathbf{hom}^{\mathsf{S}}_{\mathsf{C}(\mathrm{id}_c)}(K, X)$ as the pullback

$$\begin{tikzcd}
\mathbf{hom}^{\mathsf{S}}_{\mathsf{C}(\mathrm{id}_c)}(K, X) \arrow[rr] \arrow[d] & & \mathbf{hom}_{\mathsf{C}}(K, X) \arrow[d] \\
c \arrow[r, "\cong"'] & \mathbf{hom}_{\mathsf{C}}(*, c) \arrow[r] & \mathbf{hom}_{\mathsf{C}}(K, c)
\end{tikzcd}$$

with the evident structure maps.

Lemma 2.2.23. *For a simplicial model category* C*,* $X \otimes^c K$ *is naturally isomorphic to* $X \wedge K_+$ *and* $\mathbf{hom}^{\mathsf{S}}_{\mathsf{C}(\mathrm{id}_c)}(K, X)$ *is naturally isomorphic to* $\mathbf{hom}_{\mathsf{C}(\mathrm{id}_c)}(K_+, X)$ *in* $\mathsf{C}(\mathrm{id}_c)$

Proof. Since pushouts in $\mathsf{C}(\mathrm{id}_c)$ are computed as in C on underlying objects, $X \wedge K_+$ is the pushout

$$\begin{tikzcd}
(X \otimes *) \quad (c \otimes * \quad c \otimes K) \arrow[r] \arrow[d] & X \otimes K \\
c &
\end{tikzcd}$$

where the unlabeled maps are the evident ones. We have used the natural isomorphism $c \otimes K_+ \cong c \otimes * \quad c \otimes K$. We then have a map of diagrams

$$\begin{tikzcd}
c \arrow[d, equal] & (X \otimes *) \quad (c \otimes * \quad c \otimes K) \arrow[l] \arrow[r] \arrow[d] & X \otimes * \quad X \otimes K \arrow[d, "\mathrm{id}_X \otimes (* \to K) + \mathrm{id}_{X \otimes K}"] \\
c & c \otimes K \arrow[l] \arrow[r] & X \otimes K
\end{tikzcd}$$

where the middle arrow is the map given on components by

$$(X \to c) \otimes (* \to K) \colon X \otimes * \to c \otimes K \quad \text{and} \quad \mathrm{id}_c \otimes (* \to K) + \mathrm{id}_{c \otimes K}.$$

This map is furthermore natural in $X \in \mathsf{C}(\mathrm{id}_c)$ and $K \in \mathsf{S}$ and therefore induces a natural map on pushouts. A simple check of universal properties verifies that this map $X \wedge K_+ \to X \otimes^c K$ is an isomorphism in C and commutativity of the map of diagrams enforces that the isomorphism commutes with the structure maps coming from c. That it also commutes with the structure maps to c follows in each case since the maps are induced by $\mathrm{id}_c \colon c \to c$ and

$$X \otimes K \to c \otimes * \cong c,$$

for $X \otimes^c K$ and $X \otimes * \quad X \otimes K \to c$ given on components by (displayed graphically for conciseness)

$$(X \otimes * \to c \otimes * \cong c) + (X \otimes K \to c \otimes * \cong c).$$

These maps make the following cube commute

$$\begin{tikzcd}
X \otimes * \quad (c \otimes * \quad c \otimes K) \arrow[rr] \arrow[dr] \arrow[dd] & & c \otimes K \arrow[dr] \arrow[dd] & \\
& X \otimes * \quad X \otimes K \arrow[rr] \arrow[dd] & & X \otimes K \arrow[dd] \\
c \arrow[rr, equal] \arrow[dr, equal] & & c \arrow[dr, equal] & \\
& c \arrow[rr, equal] & & c
\end{tikzcd}$$

and thus the induced map commutes with the structure maps. This means that, as objects in $\mathsf{C}(\mathrm{id}_c)$, there is a natural isomorphism $X \wedge K_+ \cong X \otimes^c K$. The argument for the mapping object is argued similarly. $\square$

Remark 2.2.24. Schwede further defines a loops functor on $\mathsf{C}(\mathrm{id}_c)$ by the pullback of

$$\begin{tikzcd} & \mathbf{hom}^{\mathsf{S}}_{\mathsf{C}(\mathrm{id}_c)}(\Delta[1], X) \arrow[d] \\ c \arrow[r] & \mathbf{hom}^{\mathsf{S}}_{\mathsf{C}(\mathrm{id}_c)}(\partial\Delta[1], X) \end{tikzcd}$$

and a suspension functor on $\mathsf{C}(\mathrm{id}_c)$ by the pushout of

$$\begin{tikzcd} X \otimes^c \partial\Delta[1] \arrow[r] \arrow[d] & X \otimes^c \Delta[1] \\ c & \end{tikzcd}$$

An immediate corollary of the preceding lemma is that these constructions are themselves naturally isomorphic to the simplicial models provided by the pointed simplicial model structure on $\mathsf{C}(\mathrm{id}_c)$. Indeed, $X \otimes^c K \cong X \wedge K_+$ naturally and $*_{\mathsf{C}} \cong X \wedge (\emptyset)_+$. Hence, the suspension functor is the pushout of

$$\begin{tikzcd} X \wedge \partial\Delta[1]_+ \arrow[r] \arrow[d] & X \wedge \Delta[1]_+ \\ X \wedge (\emptyset)_+ & \end{tikzcd}$$

and since $X \wedge -$ is a left adjoint, it follows that Schwede's suspension functor is naturally isomorphic to $X \wedge (\Delta[1]_+/\partial\Delta[1]_+) \cong X \wedge S^1$, as desired. The same analysis, mutatis-mutandis shows that the loops functor is defined appropriately—alternatively, this follows by uniqueness of adjoints.

2.3 Semi-model Categories and Left Bousfield Localization

A key part of our arguments will involve constructing the stabilization of retractive objects via Hovey's stabilization machine [36]. While many of the results of this chapter may be phrased for semi-model categories—perhaps satisfying some additional properties—we elect to avoid them whenever possible.

We take the following definition from [4].

Definition 2.3.1. A ***semi-model category*** is a bicomplete category C along with three classes of maps $\mathscr{C}$, $\mathscr{F}$ and $\mathscr{W}$ of cofibrations, fibrations and weak equivalences, respectively, which are required to satisfy the following properties.

(SM1) Fibrations are closed under pullback.

(SM2) $\mathscr{W}$ is closed under the two-out-of-three property.

(SM3) $\mathscr{C}, \mathscr{F}$ and $\mathscr{W}$ contains all isomorphisms and are closed under composition and retracts.

(SM4) Cofibrations have the left lifting property with respect to acyclic fibrations; acyclic cofibrations with cofibrant domain have the left lifting property with respect to fibrations.

(SM5) Morphisms in C admit a functorial factorization into a cofibration followed by an acyclic fibration; morphisms with cofibrant domain admit functorial factorizations into an acyclic cofibration followed a fibration.

Such a semi-model structure is said to be ***cofibrantly generated*** if there are sets of morphisms I and J such that the class of acyclic fibrations are the maps that have the right lifting property with respect to I and the class of fibrations are the maps that have the right lifting property with respect to J, the domains of I are small relative to transfinite compositions of pushouts of elements of I and the domains of J are small relative to those transfinite compositions with cofibrant domains of pushouts of elements of J.

Remark 2.3.2. Every semi-model category admits functorial fibrant *and* cofibrant replacement functors. In particular, it is only the acyclic cofibrations and fibrations that are not necessarily pinned down by lifting properties in a semi-model category—however, among maps with cofibrant domain, the strong acyclic cofibrations are pinned down as the maps having the left lifting property with respect to fibrations. In a cofibrantly generated semi-model category, this means the weakly saturated class generated by the set J does not necessarily contain all acyclic cofibrations, despite the class of fibrations being the right complement of J.

Definition 2.3.3. Following Goerss and Hopkins in [44, Def. 1.1.8], we will say a semi-model structure C is ***simplicial*** if it is simplicially enriched, tensored and cotensored such that for any strong cofibration (i.e., a cofibration with cofibrant source) $i\colon A \to B$ and fibration $p\colon X \to Y$ the pullback corner map

$$\mathbf{Hom}(B, X) \to \mathbf{Hom}(B, Y) \times_{\mathbf{Hom}(A,Y)} \mathbf{Hom}(A, X)$$

is a fibration which is acyclic if either of i or p are acyclic. We will say a pointed semi-model category is a ***pointed simplicial*** semi-model category if the analogous axiom holds.

The following is due to Batanin and White as [4, Thm. 4.2], which will allow us to localize our category of spectra in $\mathsf{Sp}^{\mathbf{N}}(\mathsf{Alg}_{\mathcal{O}}^{Y})$ at a set of maps defined further below. It is also possible to construct this localization of $\mathsf{Sp}^{\mathbf{N}}(\mathsf{Alg}_{\mathcal{O}}^{Y})$ using the cellular arguments of Harper-Zhang [28] once it has been established that the local equivalences of $\mathsf{Sp}^{\mathbf{N}}(\mathsf{Alg}_{\mathcal{O}}^{Y})$ are π_*^s-isomorphisms.

Proposition 2.3.4. *Suppose C is a locally finitely presentable, cofibrantly generated model category such that the domains of the generating cofibrations are cofibrant. Then for any set of morphisms S, the left Bousfield localization $L_S\mathsf{C}$ exists exists as a cofibrantly generated semi-model structure. The weak equivalences are the S-local equivalences, the cofibrations in $L_S\mathsf{C}$ are the same as in C and the fibrant objects of $L_S\mathsf{C}$ are the S-local objects. The set of generating cofibrations are the same as in C and the set of generating acyclic cofibrations all have cofibrant domains.*

Remark 2.3.5. Every S-local object is fibrant in C. This is because there are more weak equivalences in $L_S\mathsf{C}$, so there are more acyclic cofibrations in $L_S\mathsf{C}$ and the class of all such acyclic cofibrations contains the class of acyclic cofibrations in C.

We introduce some terminology, which we adapt from [28, 39].

Definition 2.3.6. Fix a set S of maps in a simplicial model category C and a map $f\colon X \to Y$ in C.

(a) We say f is a ***strong cofibration*** if it is a cofibration in C between cofibrant objects.

(b) We say f is a ***S-local fibration*** if it has the right lifting property with respect to every cofibration that is an S-local equivalence.

(c) We say f is ***weak S-local fibration*** if it has the right lifting property with respect to every strong cofibration that is an S-local equivalence.

As in [28, Prop. 3.6], the following implications holds.

Proposition 2.3.7. *Fix a set S of maps in a cofibrantly generated simplicial model category C and suppose that the domains of the generating acyclic cofibrations of C are cofibrant.*

(a) Every strong cofibration is a cofibration.

(b) Every weak equivalence is an S-local equivalence.

(c) Every S-local fibration is a weak S-local fibration and every weak S-local fibration is a fibration.

Proof. The proof follows as in Harper-Zhang with only minor modifications. □

Remark 2.3.8. Only the implication that every weak S-local fibration is a fibration requires the hypothesis that C be cofibrantly generated having generating acyclic cofibrations with cofibrant domain. The implication that every S-local fibration is a fibration follows since the class of S-local equivalences contain the class of weak equivalences in C. In fact, all that is required of C it have a set of J of acyclic cofibrations between cofibrant objects such that the class of fibrations is the class of maps having the right lifting property with respect to J.

The following is due to Harper and Zhang in [28, Prop. 3.8]. An analogue of this was also noticed by Barwick in [3, Lem. 1.7.1].

Proposition 2.3.9. *Fix a set S of maps in a cofibrantly generated simplicial model category C and suppose that the domains of the generating cofibrations of C are cofibrant. For a map $f\colon X \to Y$, the following are equivalent.*

(a) f is a weak S-local fibration and S-local equivalence.

(b) f *is an* S*-local fibration and* S*-local equivalence.*

(c) f *is an acyclic fibration in* C*.*

Remark 2.3.10. Similar to Remark 2.3.8 following Proposition 2.3.7, only the implication (a)⇒(c) requires the hypothesis that C is cofibrantly generated with each generating cofibrations having a cofibrant domain. Unlike Proposition 2.3.7, it is the generating cofibrations this is imposed upon. A similar weakening of this assumption as in Remark 2.3.8 is possible.

Proof. The proof of this proposition likewise follows just as in Harper-Zhang with only minor modifications. □

As it turns out, we have a better handle on this localization when the category we are localizing has a compatible simplicial structure.

Proposition 2.3.11. *Suppose* C *is a locally finitely presentable, cofibrantly generated, (pointed) simplicial model category whose generating cofibrations have cofibrant domain and let* S *be a set of maps in* C*. Then the acyclic fibrations of* $L_S\mathsf{C}$ *are precisely the acyclic fibrations in* C *and* $L_S\mathsf{C}$ *is a cofibrantly generated, (pointed) simplicial model category with the same mapping space functor as* C*.*

Proof. Say by Proposition 2.3.9, we know that acyclic fibrations of $L_S\mathsf{C}$ are precisely the acyclic fibrations in C. Since any two functorial models for a derived mapping space are connected to each other by a zig-zag of natural equivalences as a consequence of [31, Thm. 17.5.7], we may use the mapping space functor **Hom** of the original category to detect S-local objects and S-local equivalences, and we do so. Note that in a pointed simplicial model category, the pointed simplicial mapping space $\mathbf{Hom}(X, Y)$ is simply the unpointed mapping space equipped with basepoint the unique map $X \to * \to Y$.

Let $i \colon A \to B$ be a strong cofibration and $p \colon X \to Y$ an S-local fibration. In this case, all that needs to be checked is that pullback powering is a fibration of simplicial sets and is a weak equivalence when either of i or p are a weak equivalence. We consider the former first.

Let us show the pullback power map is a fibration. To do this, consider a lifting problem

$$\begin{tikzcd}
\Lambda_k[n] \arrow[r] \arrow[d, "j"'] & \mathbf{Hom}(B, X) \arrow[d, "p^{\square i}"] \\
\Delta[n] \arrow[r] \arrow[ur, dotted] & \mathbf{Hom}(B, Y) \times_{\mathbf{Hom}(A,Y)} \mathbf{Hom}(A, X)
\end{tikzcd}$$

which, by adjunction, is equivalent to

$$\begin{tikzcd}
B \otimes \Lambda_k[n] \ {}_{A\otimes\Lambda_k[n]} A \otimes \Delta[n] \arrow[r] \arrow[d, "i\square j"'] & X \arrow[d, "p"] \\
B \otimes \Delta[n] \arrow[r] \arrow[ur, dotted] & Y
\end{tikzcd}$$

but since $A \to B$ is a cofibration—thus, a cofibration in C—and j is an acyclic cofibration of simplicial sets, $i\square j$ is an acyclic cofibration in C, since C is a simplicial semi-model category. But then, in particular, $i\square j$ is a cofibration and an S-local equivalence. Since p is an S-local fibration, the dotted arrow exists.

If p is additionally a weak equivalence, then it is an acyclic fibration in C and we are done since C is a simplicial semi-model category, so we suppose that i is an acyclic cofibration between cofibrant objects. As before, to test that the resulting map is an acyclic fibration, it is enough to check it lifts against the generating cofibrations in simplicial sets. For this, we adjoint to the following lifting diagram

$$\begin{tikzcd} B\otimes\partial\Delta[n] \;{}_{A\otimes\partial\Delta[n]}\; A\otimes\Delta[n] \arrow[r] \arrow[d, "i\Box j"'] & X \arrow[d, "p"] \\ B\otimes\Delta[n] \arrow[r] \arrow[ur, dotted] & Y \end{tikzcd}$$

Since p is an S-local fibration, we must check that $i\Box j$—which is a cofibration as before—is an S-local equivalence. To do this, note that $i\Box j$ has cofibrant domain since C is simplicial semi-model category. Hence, it suffices to show that for every S-local object W, the map

$$(i\Box j)^*\colon \mathbf{Hom}(B\otimes\Delta[n], W) \to \mathbf{Hom}\left(B\otimes\partial\Delta[n] \coprod_{A\otimes\partial\Delta[n]} A\otimes\Delta[n], W\right)$$

is a weak equivalence of Kan complexes. The target of this map is the pullback

$$\begin{tikzcd} \mathbf{Hom}(B\otimes\partial\Delta[n] \;{}_{A\otimes\partial\Delta[n]}\; A\otimes\Delta[n], W) \arrow[r] \arrow[d, "(**)"'] & \mathbf{Hom}(B\otimes\partial\Delta[n], W) \arrow[d, "(*)"] \\ \mathbf{Hom}(A\otimes\Delta[n], W) \arrow[r] & \mathbf{Hom}(A\otimes\partial\Delta[n], W) \end{tikzcd}$$

which is, by adjunction, a pullback

$$\begin{tikzcd} \mathbf{Hom}(B\otimes\partial\Delta[n] \;{}_{A\otimes\partial\Delta[n]}\; A\otimes\Delta[n], W) \arrow[r] \arrow[d] & \mathbf{Hom}(\partial\Delta[n], \mathbf{Hom}(B, W)) \arrow[d] \\ \mathbf{Hom}(\Delta[n], \mathbf{Hom}(A, W)) \arrow[r] & \mathbf{Hom}(\partial\Delta[n], \mathbf{Hom}(A, W)) \end{tikzcd}$$

and this shows that the left-most vertical map $(*)$ is an acyclic fibration—this follows since $A \to B$ is an S-local cofibration between cofibrant objects and W is S-local, so that the map $\mathbf{Hom}(B, W) \to \mathbf{Hom}(A, W)$ is a weak equivalence by definition of the S-local equivalences and that it is a fibration follows from our preceding analysis since $W \to *$ is an S-local fibration. Hence, by stability under pullback, $(**)$ is an acyclic fibration.

We claim now that the map $\mathbf{Hom}(B \otimes \Delta[n], W) \to \mathbf{Hom}(A \otimes \Delta[n], W)$ is an acyclic fibration. This follows by exactly the same analysis we used when showing $(*)$ is an acyclic fibration.

Putting this all together, we have a diagram

$$\begin{tikzcd}
\mathbf{Hom}(B \otimes \Delta[n], W) \arrow[dr, dotted] \arrow[drr, bend left] \arrow[ddr, two heads, bend right, "\sim"'] & & \\
& \mathbf{Hom}(B \otimes \partial\Delta[n] \;{}_{A\otimes\partial\Delta[n]}\; A \otimes \Delta[n], W) \arrow[r] \arrow[d, two heads, "\sim"'] & \mathbf{Hom}(B \otimes \partial\Delta[n], W) \arrow[d, two heads, "\sim"] \\
& \mathbf{Hom}(A \otimes \Delta[n], W) \arrow[r] & \mathbf{Hom}(A \otimes \partial\Delta[n], W)
\end{tikzcd}$$

and thus by two-out-of-three, the dotted arrow is a weak equivalence and acyclic fibration, as desired.

Since any S_*-enriched model category is pointed (see Lemma 2.2.15) and has a natural S-enriched model structure for which the pointed simplicial mapping spaces are simply the simplicial mapping spaces with the zero map as a basepoint, it is easy to see nothing changes when we ask that C be a pointed simplicial model category. $\square$

A careful inspection of the argument above reveals that when C is a simplicial model category, requiring $i\colon A \to B$ to be a strong cofibration is only required to show the pullback power map is an acyclic fibration when i is an acyclic cofibration. We may therefore deduce the following.

Corollary 2.3.12. *Suppose* C *is a locally finitely presentable, cofibrantly generated, (pointed) simplicial model category whose generating cofibrations have cofibrant domain and let* S *be a set of* *maps in* C*. Then* $L_S\mathsf{C}$ *is a (pointed) simplicial semi-model category with the same set generating cofibrations and a set of generating acyclic cofibrations having cofibrant domain. Moreover, if* $i\colon A \to B$ *is any cofibration and* $p\colon X \to Y$ *an* S*-local fibration, then the pullback corner map* $p^{\square i}$ *is a fibration which is acyclic if* p *is an acyclic fibration.*

Corollary 2.3.13. *Suppose* C *is a locally finitely presentable, (pointed) simplicial semi-model category whose generating cofibrations have cofibrant domain and let* S *be a set of maps in* C*. Then* $L_S\mathsf{C}$ *admits a (pointed) simplicial monad* R *which is a fibrant replacement functor on cofibrant objects.*

Proof. The proof of the simplicial case follows exactly as in [9, Thm. 6.1] because we may assume the generating acyclic cofibrations in $L_S\mathsf{C}$ have cofibrant domains. Hence, for any simplicial set K and generating acyclic cofibration j, $j \otimes K$ is once again an acyclic cofibration. More generally, in light of [47, Thm. 13.2.1, Cor. 13.2.4, Rem. 13.4.3], it follows that the same holds in the pointed simplicial case. The point is that strong acyclic cofibrations are closed under pushout and transfinite composition when restricted to mapping to cofibrant objects. $\square$

Remark 2.3.14. The enriched cofibrant replacement comonad on $L_S\mathsf{C}$ above may simply be taken to be the on on C.

Remark 2.3.15. It follows easily that if X is cofibrant, so too is RX since cofibrations in a semi-model category are preserved under transfinite composition and pushout in a semi-model category.

Chapter 3

Stabilization of Retractive Model Structures

With this said, Hovey's stabilization machine [36] in the finitely generated case carries over without change. The purpose of this chapter is to explain these consequences and develop some more machinery that will be necessary for our completion results.

3.1 Categories of Spectra

Definition 3.1.1. Fix a pointed simplicial category C. The category of spectra $\mathsf{Sp}^{\mathbf{N}}(\mathsf{C})$ in C is the category whose objects consist of sequences $(X_0, X_1, \ldots)$ of objects in C along with structure maps $\sigma_i \colon X_i \wedge S^1 = \Sigma X_i \to X_{i+1}$—equivalently, adjoint structure maps maps $\sigma_i \colon X_i \to \Omega X_{i+1}$. The morphisms are the morphisms of sequences respecting the structure maps—equivalently, respecting the adjoint structure maps.

Definition 3.1.2. For a pointed simplicial category C, the category $\mathsf{Sp}^{\mathbf{N}}(\mathsf{C})$ admits several interesting functors. This category has (underived) shifted suspension spectrum functors

$$F_n = \Sigma^{\infty - n} \colon \mathsf{C} \to \mathsf{Sp}^{\mathbf{N}}(\mathsf{C})$$

given by

$$(F_n X)_m = \begin{cases} X \wedge (S^1)^{\wedge m-n} & m \geq n \\ * & \text{else} \end{cases}$$

where $*$ is the zero object of C which are left adjoint to the (underived) shifted infinite loop space functors

$$\mathrm{Ev}_n = \Omega^{\infty + n}$$

defined by $\mathrm{Ev}_n X = X_n$ and, furthermore, Ev_n is left adjoint to the functor $M_n \colon \mathsf{C} \to \mathsf{Sp}^{\mathbf{N}}(\mathsf{C})$ defined by

$$(M_n X)_m = \begin{cases} \Omega^{n-m} X & m \leq n \\ * & \text{else.} \end{cases}$$

Pictorially, with left adjoints on top,

$$\begin{tikzcd} \mathsf{C} \arrow[r, shift left=2, "F_n"] \arrow[r, shift right=2, "M_n"'] & \mathsf{Sp}^{\mathbf{N}}(\mathsf{C}) \arrow[l, "Ev_n" description] \end{tikzcd}$$

Lemma 3.1.3. *If C has all limits or colimits of shape D, then so too does $\mathsf{Sp}^{\mathbf{N}}(\mathsf{C})$. In particular, these limits and colimits are created in the category of sequences $\mathsf{C}^{\mathbf{N}}$ and, hence, levelwise, under* *the forgetful functor.*

Proof. In this case, the limits and colimits may be computed pointwise in $\mathsf{Sp}^{\mathbf{N}}(\mathsf{C})$. That the forgetful functor creates the relevant colimits follows from the fact that $\Sigma\colon \mathsf{C} \to \mathsf{C}$ is a left adjoint. That the forgetful functor creates the relevant colimits follows from the fact that $\Omega\colon \mathsf{C} \to \mathsf{C}$ is a right adjoint, using the adjoint structure maps. $\square$

Lemma 3.1.4. *If* C *is a pointed simplicially enriched category which is locally finitely presentable and has all pullbacks, then so too is* $\mathsf{Sp}^{\mathbf{N}}(\mathsf{C})$ *locally finitely presentable.*

Rather than making an ad-hoc argument by hand, we will refer to the following basic fact about locally finitely presentable categories to make quick work of this. First, we collect a definition.

Definition 3.1.5. A generating set $\mathcal{G}$ of a category C is said to be ***strong*** if for each object $X \in \mathsf{C}$ and each proper subobject $i\colon X_0 \to X$, there is a $G \in \mathcal{G}$ and a morphism $g\colon G \to X$ which does not factor through i.

Proposition 3.1.6 ([1, Thm. 1.11])**.** *A category* C *is locally finitely presentable if and only if it is cocomplete and has a strong generating set of compact objects* $\mathcal{G}$*. In particular, if* $\overline{\mathcal{G}}$ *denotes the closure of the full subcategory spanned by* $\mathcal{G}$ *under finite colimits, then* $\overline{\mathcal{G}}$ *is a set of compact objects and* C *is generated under filtered colimits by* $\overline{\mathcal{G}}$*.*

This is a mild condition that most well-behaved categories satisfy. This proposition may be thought of roughly as a necessary and sufficient condition for a category to admit "categories of simplices" for each object X. In simplicial sets, a simplicial set X is the filtered colimit $X \cong$

$\operatorname{colim}_{\Delta[n]\to X}\Delta[n]$, and this proposition guarantees that a similar situation occurs in the category C.

Proof of Lemma 3.1.4. It is easy to see $\mathsf{Sp}^{\mathbf{N}}(\mathsf{C})$ has all pullbacks since we can compute them levelwise by the preceding lemma.

If an object $c \in \mathsf{C}$ is compact, then so too is $F_n c$ for each c by an adjunction argument. Let $\mathcal{G}$ be a generating set of objects for C. Then $\mathsf{Sp}(\mathcal{G}) = \quad {}_{n\geq 0} F_n\mathcal{G}$ is a generating set for $\mathsf{Sp}^{\mathbf{N}}(\mathsf{C})$, we claim.

Note that in any category with pullbacks, to call a map $i\colon X_0 \to X$ a monomorphism is precisely the same as saying that the following square is a pullback

$$\begin{tikzcd}
X_0 \arrow[r, "\mathrm{id}_X"] \arrow[d, "\mathrm{id}_X"'] & X_0 \arrow[d, "i"] \\
X_0 \arrow[r, "i"'] & X
\end{tikzcd}$$

In particular, a monomorphism in $\mathsf{Sp}^{\mathbf{N}}(\mathsf{C})$ is precisely a level monomorphism. Since isomorphisms are levelwise, a proper subobject of a spectrum X is a monomorphism $i\colon X_0 \to X$ which is not an isomorphism and, hence, for some level n, is a proper subobject. Given such a proper subobject, a factorization

$$\begin{tikzcd}
F_n c \arrow[r] \arrow[d, "\mathrm{id}"'] & X_0 \arrow[d, "i"] \\
F_n c \arrow[r] & X
\end{tikzcd}$$

is equivalent, by adjunction, to a factorization

$$
\begin{tikzcd}
c \arrow[r] \arrow[d, equal] & (X_0)_n \arrow[d, "i_n"] \\
c \arrow[r] & X_n
\end{tikzcd}
$$

Since i is a proper subobject, there is an n such that i_n is a proper subobject and therefore there is an object $c \in \mathcal{G}$ and a map $c \to X_n$ for which the factorization above does not exist. It follows that the adjoint of this map $F_n c \to X$ does not factor through i. $\square$

It turns out that for a cofibrantly generated, pointed simplicial model category, its category of spectra inherits a cofibrantly generated, pointed simplicial model structure from C. This is the content of the following result from [36, Thm. 1.13, Thm. 6.3].

Proposition 3.1.7. *Let C is a cofibrantly generated, pointed simplicial model category with sets of generating cofibrations I and generating acyclic cofibrations J.*

*(a) $\mathsf{Sp}^{\mathbf{N}}(\mathsf{C})$ is a cofibrantly generated model category in the **projective model structure**, in which the weak equivalences and fibrations are levelwise.*

(b) The set of generating cofibrations for the projective model structure is given by $_{n\geq 0} F_n(I)$ and the set of generating acyclic cofibrations for the projective model structure is given by $_{n\geq 0} F_n(J)$.

(c) The S_*-tensoring is given levelwise by $(X \wedge K)_n = X_n \wedge K$ and with structure maps

$$(X_n \wedge K) \wedge S^1 \cong X_n \otimes (K \wedge S^1) \cong X_n \otimes (S^1 \wedge K) \cong (X_n \wedge S^1) \wedge K \to X_{n+1} \wedge K,$$

using the associativity isomorphism.

(d) *The S_*-cotensoring is given levelwise by $\mathbf{hom}(K, X)_n = \mathbf{hom}(K, X_n)$ and with adjoint structure maps*

$$\mathbf{hom}(K, X_n) \to \mathbf{hom}(K, \Omega X_{n+1}) \cong \mathbf{hom}(K \wedge S^1, X_{n+1}) \cong \mathbf{hom}(S^1 \wedge K, X_{n+1}) \cong \Omega\mathbf{hom}(K, X_{n+1}),$$

using the adjoint associativity isomorphism (see Remark 3.3.4).

(e) *If C is left or right proper, then so too is $\mathsf{Sp}^{\mathbf{N}}(\mathsf{C})$.*

(f) *The unit and associativity maps for the S_*-module structure on $\mathsf{Sp}^{\mathbf{N}}(\mathsf{C})$ are given levelwise.*

Remark 3.1.8. Hovey states in [36, Thm. 6.3] that the stable model structure on $\mathsf{Sp}^{\mathbf{N}}(\mathsf{C})$—under suitable hypotheses on C to guarantee its left Bousfield localization exists as a model category—is a pointed simplicial model structure. However, Hovey's argument shows that the projective model structure above is simplicial since the acyclic cofibrations in the projective model structure are acyclic cofibrations in the stable model structure, since every level equivalence is a stable equivalence. This is independent of the assumptions Hovey places upon C to guarantee its left Bousfield localization exists as a model category.

In order to begin our discussion on the stabilization of this model structure, we first introduce the class of maps we wish to invert on the level of homotopy. The class of maps is picked out by Hovey in [36, Def. 3.3.].

Definition 3.1.9. Suppose C is a cofibrantly generated, pointed simplicial model category in which the domains of the generating cofibrations are cofibrant. Let S be the set of maps in $\mathsf{Sp}^{\mathbf{N}}(\mathsf{C})$ given by

$$S = \quad \zeta_n^C : F_{n+1}(\Sigma C) \to F_n C : C \in \mathrm{dom}(I) \cup \mathrm{cod}(I), n \geq 0$$

where ζ_n^C is the map adjoint to the identity map $\mathrm{id}_{\Sigma C} \colon \Sigma C \to \Sigma C$. Then we say a map $f \colon X \to Y$ in $\mathsf{Sp}^{\mathbf{N}}(\mathsf{C})$ is a ***stable equivalence*** if it is an S-local equivalence. If the left Bousfield localization at the set S of maps of the projective model structure on $\mathsf{Sp}^{\mathbf{N}}(\mathsf{C})$ exists as a semi-model category, we will call it the ***stable semi-model structure***.

One way of thinking about this set of maps is the following, which is due to Hovey in [36].

Definition 3.1.10. Let C be a pointed simplicial model category. In $\mathsf{Sp}^{\mathbf{N}}(\mathsf{C})$, an Ω***-spectrum*** is a level-fibrant spectrum X such that the adjoint structure maps $X_n \to \Omega X_{n+1}$ are weak equivalences in C.

Proposition 3.1.11. *Suppose C is a cofibrantly generated, pointed simplicial model category in which the domains of the generating cofibrations are cofibrant. Let S be the set of maps of Definition 3.1.9. Then S-local objects are precisely the Ω-spectra.*

Proof. The functors and adjunctions of Definition 3.1.2 are all simplicial. Hence, if W is S-local, then the map

$$(\zeta_n^C)^* \colon \mathbf{Hom}_{\mathsf{Sp}}(F_n C, W) \to \mathbf{Hom}_{\mathsf{Sp}}(F_{n+1}\Sigma C, W)$$

is an equivalence for all $n \geq 0$ and $C \in \mathrm{dom}(I) \cup \mathrm{cod}(I)$. By adjunction, this means, equivalently, that the map

$$\sigma_{n*} \colon \mathbf{Hom}_{\mathsf{C}}(C, W_n) \to \mathbf{Hom}_{\mathsf{C}}(C, \Omega W_{n+1})$$

is an equivalence for all $n \geq 0$ and $C \in \mathrm{dom}(I) \cup \mathrm{cod}(I)$. Thus, it is clear the S-local objects at least contain the Ω-spectra. On the other hand, since the domains of the generating cofibrations are assumed to be cofibrant, [36, Prop. 3.2] shows that this is equivalent to requiring the adjoint structure $\sigma_n \colon W_n \to \Omega W_{n+1}$ be equivalences for $n \geq 0$. Hence, the S-local objects are precisely the Ω-spectra. $\square$

The following proposition is now an immediate consequence of Corollary 2.3.12.

Proposition 3.1.12. *Let C be a locally finitely presentable, finitely generated, pointed simplicial model category in which the domains of the generating cofibratons are cofibrant. Then the* ***stable semi-model structure*** *on $\mathsf{Sp}^{\mathbf{N}}(\mathsf{C})$ exists as a pointed simplicial semi-model category.*

We now end this section with the following stability theorem for $\mathsf{Sp}^{\mathbf{N}}(\mathsf{C})$, justifying our use of the word 'stable.'

Theorem 3.1.13 ([36, Thm. 10.3])**.** *Let C be a pointed simplicial semi-model category which is locally presentable and cofibrantly generated such that the generating cofibrations have cofibrant domains. Then the stable semi-model structure on $\mathsf{Sp}^{\mathbf{N}}(\mathsf{C})$ exists and is stable—the simplicial suspension functor $- \wedge S^1 = \Sigma$ on $\mathsf{Sp}^{\mathbf{N}}(\mathsf{C})$ is a Quillen equivalence.*

Remark 3.1.14. As before, nothing in Hovey's arguments really requires that the category C in question be left proper and cellular, in light of Proposition 2.3.11. The only part of Hovey's argument where the particulars of the model structure really comes into play is when an acyclic cofibration is pushed out along another map where all objects are cofibrant—in this case, the pushout of the acyclic cofibration remains an acyclic cofibration in a semi-model structure. See also [53, pg. 7] for a brief discussion.

3.2 The Functor Θ^∞

Under mild hypotheses, Hovey provides a characterization of the stable equivalences in $\mathsf{Sp}^{\mathbf{N}}(\mathsf{C})$ in terms of level equivalences. To do this, we first introduce a sort of spectrification functor. We first introduce some auxiliary functors.

Definition 3.2.1. Let C be a pointed simplicial model category. Define shift functors $s_+, s_- \colon \mathsf{Sp}^{\mathbf{N}}(\mathsf{C}) \to \mathsf{Sp}^{\mathbf{N}}(\mathsf{C})$ on objects as follows

$$s_-(X)_n = X_{n+1}$$

$$s_+(X)_n = \begin{cases} * & n = 0 \\ X_{n-1} & n \geq 1. \end{cases}$$

The functors s_+ and s_- commute with all limits and colimits. We denote the k-fold iterates of the shift functors by s_+^k and s_-^k.

Lemma 3.2.2 ([36, Lem. 3.8])**.** *Let C be a pointed simplicial model category.*

(a) The shift functor s_+^k is left adjoint to s_-^k.

(b) s_+^k commutes with the S_*-tensoring and cotensoring on $\mathsf{Sp}^{\mathbf{N}}(\mathsf{C})$. s_-^k commutes with the S_*-tensoring and cotensoring on $\mathsf{Sp}^{\mathbf{N}}(\mathsf{C})$—in fact, there are equalities $s_-^k(X \wedge L) = s_-^k(X) \wedge L$ and $s_-^k \mathbf{hom}(L, X) = \mathbf{hom}(L, s_-^k X)$

(c) $s_+ F_n = F_{n+1}$ and $\mathrm{Ev}_n s_- = \mathrm{Ev}_{n+1}$.

Proof. The only part of this that requires words is (b). Note that $\mathbf{hom}(L, -)$ and $- \wedge L$ are defined levelwise on the spectrum and that s_- shifts the objects and structure maps. Hence, $s_-\mathbf{hom}(L, X) = \mathbf{hom}(L, s_-X)$. On the other hand, s_+ shifts the structure maps but also introduces new objects in the spectrum in degree 0. Since $* \wedge L$ is only naturally isomorphic to $*$ (the distinguished zero object), but by a unique natural isomorphism, this shows that $s_+^k(X) \wedge L \cong s_+^k(X \wedge L)$ uniquely. □

Definition 3.2.3. Let

$$\Theta = \Omega s_- = s_-\Omega\colon \mathsf{Sp}^{\mathbf{N}}(\mathsf{C}) \to \mathsf{Sp}^{\mathbf{N}}(\mathsf{C}).$$

This is the functor given by mapping a spectrum $X = (X_0, X_1, \ldots)$ to the spectrum $(\Omega X_1, \Omega X_2, \ldots)$ with the evident structure maps. There are natural maps $i_X\colon X \to \Theta X$. Let Θ^∞ be the endofunctor of $\mathsf{Sp}^{\mathbf{N}}(\mathsf{C})$ defined by

$$\Theta^\infty X = \operatorname{colim}(X \xrightarrow{i_X} \Theta X \xrightarrow{i_{\Theta X}} \Theta^2 X \to \cdots).$$

Let $j_X\colon X \to \Theta^\infty X$ be the evident natural transformation induced from the colimiting cone.

Remark 3.2.4. Levelwise, $(\Theta^\infty X)_n \cong \operatorname{colim}(X_n \to \Omega X_{n+1} \to \Omega^2 X_{n+2} \to \cdots)$, which is the classical spectrification procedure for spectra of topological (say, CW-complexes) spaces.

Remark 3.2.5. Since $\Theta = s_- \circ \Omega = \Omega \circ s_-$ is a composite of right adjoints, it is a right adjoint to the functor $\Phi = s_+ \circ \Sigma \cong \Sigma \circ s_+$. More generally, Θ^k is right adjoint to the functor $\Phi^k = s_+^k \circ \Sigma^k$.

Lemma 3.2.6 ([36, Lem. 4.5])**.** *For any X, the maps $i_{\Theta X}$ and Θi_X coincide.*

Hovey has shown this functor exhibits excellent properties.

Proposition 3.2.7 ([36, Prop. 4.6])**.** *Suppose* C *is a finitely generated, pointed simplicial model category and suppose* Ω *preserves sequential colimits in* C*. Then the map* $\Theta^\infty X \to \Theta(\Theta^\infty X)$ *is an isomorphism. If, in addition,* C *is finitely generated and* X *is level fibrant, then* $\Theta^\infty X$ *is an* Ω*-spectrum.*

As an immediate corollary, it follows that Θ^∞ may be used to give stable fibrant replacements.

Corollary 3.2.8. *Suppose* C *is a locally finitely presentable, finitely generated, pointed simplicial model category and suppose* Ω *preserves sequential colimits in* C*. For any level fibrant spectrum* X*,* $\Theta^\infty X$ *is a stable fibrant replacement of* X*. In particular, if* R' *is any functorial level fibrant replacement functor,* $\Theta^\infty \circ R'$ *is a functorial stable fibrant replacement functor.*

This functor allows us to detect stable equivalences as level equivalences.

Proposition 3.2.9 ([36, Thm. 4.9, Cor. 4.11])**.** *Let* C *be a finitely generated, pointed simplicial model category in which the domains of the generating cofibrations are cofibrant, sequential colimits preserve finite products and* Ω *preserves sequential colimits.*

(a) If $f\colon A \to B$ *is a map in* $\mathsf{Sp}^{\mathbf{N}}(\mathsf{C})$ *such that* $\Theta^\infty f$ *is a level equivalence, then* f *is a stable equivalence.*

(b) The natural map $j_A\colon A \to \Theta^\infty A$ *is a stable equivalence.*

Remark 3.2.10. While Hovey does not specify in [36, Thm. 4.9, Cor. 4.11] that the model structure should satisfy the hypotheses of [36, Prop. 3.2], all argument still go through go through by interpreting a stable equivalence to mean a map $f\colon A \to B$ in $\mathsf{Sp}^{\mathbf{N}}(\mathsf{C})$ such that for all Ω-spectra W,

$Qf^*\colon \mathbf{Hom}(QB, W) \to \mathbf{Hom}(QA, W)$ is a weak equivalence, where Q is some choice of projective cofibrant replacement functor. Note that (a)⇒(b) by applying Θ^∞ and the preceding proposition.

The following corollary is essentially [36, Thm. 4.12].

Corollary 3.2.11. *Let C be a finitely generated, pointed simplicial model category in which the domains of the generating cofibrations are cofibrant, sequential colimits preserve finite products and Ω preserves sequential colimits. Let L be any level fibrant replacement functor.*

(a) A map $f\colon A \to B$ is a stable equivalence if and only if $\Theta^\infty f$ is a stable equivalence. In particular, Θ^∞ preserves weak equivalences.

(b) A map $f\colon A \to B$ is a stable equivalence if and only if $\Theta^\infty Lf$ is a level equivalence.

Proof. (a) We have a commutative diagram

$$\begin{array}{ccc} A & \xrightarrow[\sim]{j_A} & \Theta^\infty A \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \Theta^\infty f} \\ B & \xrightarrow[j_B]{\sim} & \Theta^\infty B \end{array}$$

The maps j_A and j_B are stable equivalences from the preceding proposition. Suppose f is a stable equivalence. Then by two-out-of-three, $\Theta^\infty f$ is a stable equivalence. Conversely, suppose $\Theta^\infty f$ is a stable equivalence, then by two-out-of-three, f is a stable equivalence.

(b) We have a commutative diagram

$$\begin{array}{ccccc} A & \xrightarrow[\sim]{} & LA & \xrightarrow[\sim]{j_{LA}} & \Theta^\infty LA \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle Lf} & & \downarrow{\scriptstyle \Theta^\infty Lf} \\ B & \xrightarrow{\sim} & LB & \xrightarrow[j_B]{\sim} & \Theta^\infty LB \end{array}$$

The maps $A \to LA$ and $B \to LB$ are level equivalences and, hence, stable equivalences. The maps j_{LA} and j_{LB} are stable equivalences from the preceding proposition. Suppose f is a stable equivalence. Then by two-out-of-three, Lf is a stable equivalence. By (a), $\Theta^\infty Lf$ is a stable equivalence. But $\Theta^\infty LB$ and $\Theta^\infty LB$ are Ω-spectra and thus local objects. Hence, $\Theta^\infty Lf$ is a level equivalence. Conversely, suppose $\Theta^\infty Lf$ is a level equivalence. Then it is a stable equivalence and by (a), Lf is a stable equivalence; by two-out-of-three, it follows that f is a stable equivalence. ☐

Remark 3.2.12. The utility of this corollary is in (b). For instance, level equivalences between Ω-spectra of pointed spaces are precisely π_*^s-isomorphisms. (b) of the preceding corollary allows us to characterize the stable equivalences in terms of algebraic data.

As an indication of a more general argument, we collect a corollary. This corollary will follow from a proposition due to Dan Dugger in [18], who attributes it to Jeff Smith.

Proposition 3.2.13 ([18, Prop. 7.3])**.** *Let C be a cofibrantly generated model category in which the domains and codomains of the generating cofibrations are compact. Then any colimit functor for filtered diagrams is a model for the point-set homotopy colimit functor.*

Proof. Given filtered diagrams $F_1, F_2 \colon \mathsf{D} \to \mathsf{C}$ and a level equivalence $\eta \colon F_1 \to F_2$, we may form, in the projective model structure on $\mathsf{Fun}(\mathsf{D}, \mathsf{C})$ the following commutative square

$$\begin{tikzcd} QF_1 \arrow[r, two heads, "\sim"] \arrow[d, "\sim"'] & F_1 \arrow[d, "\sim"] \\ QF_2 \arrow[r, two heads, "\sim"'] & F_2 \end{tikzcd}$$

where the left-most vertical map is a level equivalence by the two-out-of-three property. Taking colimits, since $\operatorname{colim} QF_1 \to \operatorname{colim} QF_2$ is an equivalence, we have a commutative diagram

$$\begin{tikzcd}
\operatorname{colim} QF_1 \arrow[r] \arrow[d, "\sim"'] & \operatorname{colim} F_1 \arrow[d, "(*)"] \\
\operatorname{colim} QF_2 \arrow[r] & \operatorname{colim} F_2
\end{tikzcd}$$

and to show that $(*)$ is a weak equivalence, it suffices to show that $\operatorname{colim} QF_i \to \operatorname{colim} F_i$ is an acyclic fibration for each $i = 1, 2$, since then, by two-out-of-three, $(*)$ is a weak equivalence.

For this, set $i = 1$, without loss of generality, and set $QF_1 = F_0$. Then for each generating cofibration $i\colon C_0 \to C_1$ and solid commutative diagram

$$\begin{tikzcd}
C_0 \arrow[r, "g_0"] \arrow[d, "i"] & \operatorname{colim} F_0 \arrow[d] \\
C_1 \arrow[r, "g_1"'] \arrow[ur, dotted] & \operatorname{colim} F_1
\end{tikzcd}$$

we must check the dotted lift exists. By compactness of the C_i, there is a $d_i \in \mathsf{D}$ and a map $f_i\colon C_i \to F_i(d_i)$ for which we have a factorization

$$\begin{tikzcd}
C_i \arrow[r, "f_i"] \arrow[rr, bend left, "g_i"] & F_i(d_i) \arrow[r, "\text{in}"] & \operatorname{colim} F_i
\end{tikzcd}$$

Similarly, the composite map $C_0 \to \operatorname{colim} F_1$ factors through some $F_1 d_2$ as

$$\begin{tikzcd}
C_0 \arrow[r, "f_2"] \arrow[rr, bend left] & F_1(d_2) \arrow[r, "\text{in}"] & \operatorname{colim} F_1
\end{tikzcd}$$

Since D is filtered, there there exists a cospan

$$\begin{tikzcd} d_0 \arrow[r, "\alpha"] & d_2 & d_1 \arrow[l, "\beta"'] \end{tikzcd}$$

and we claim that we my pick it such that the following diagram commutes

$$\begin{tikzcd}
C_0 \arrow[r, "f_0"] \arrow[d, "i"'] \arrow[rrr, bend left, "g_0"] & F_0 d_0 \arrow[r, "F_0\alpha"] & F_0 d_2 \arrow[r, "\mathrm{in}"] \arrow[d, two heads, "\eta_{d_2}"', "\sim"] & \operatorname{colim} F_0 \arrow[d] \\
C_1 \arrow[r, "f_1"'] \arrow[rrr, bend right, "g_1"'] & F_1 d_1 \arrow[r, "F_1\beta"'] & F_1 d_2 \arrow[r, "\mathrm{in}"'] & \operatorname{colim} F_1
\end{tikzcd}$$

It is at least true that the right-most square and outer rectangle commute. If the left-hand rectangle does not commute, we may at least make the following observation. The composites represented by $\eta_{d_2} \circ F_0\alpha \circ f_0$ and $F_1\beta \circ f_1 \circ i$ represent the same element of $\operatorname{colim} \hom(C_0, F_1)$ because, as maps into $\operatorname{colim} F_1$ they are determined by the two composites

$$\begin{tikzcd}
C_0 \arrow[r, "f_0"] & F_0 d_0 \arrow[r, "F_0\alpha"] & F_0 d_2 \arrow[r, "\mathrm{in}"] \arrow[d, two heads, "\sim"] & \operatorname{colim} F_0 \arrow[d] \\
 & & F_1 d_2 \arrow[r, "\mathrm{in}"'] & \operatorname{colim} F_1
\end{tikzcd}$$

and

$$\begin{tikzcd}
C_0 \arrow[d, "i"'] & & & \\
C_1 \arrow[r, "f_1"'] & F_1 d_1 \arrow[r, "F_1\beta"'] & F_1 d_2 \arrow[r, "\mathrm{in}"'] & \operatorname{colim} F_1
\end{tikzcd}$$

and we know these are equal. Hence, there is a map $\tau \colon d_2 \to d_3$ such that the following diagram commutes

$$\begin{tikzcd}
C_0 \arrow[r, "f_0"] \arrow[d, "i"'] & F_0 d_0 \arrow[r, "F_0\alpha"] & F_0 d_2 \arrow[r, two heads, "\eta_{d_2}", "\sim"'] & F_1 d_2 \arrow[d, "F_1\tau"] \\
C_1 \arrow[r, "f_1"'] & F_1 d_1 \arrow[r, "F_1\beta"'] & F_1 d_2 \arrow[r, "F_1\tau"] & F_1 d_3
\end{tikzcd}$$

Hence, by naturality of η, we have the following solid commutative diagram

$$\begin{tikzcd}
C_0 \arrow[r, "F_0(\tau\alpha)\circ f_0"] \arrow[d, "i"'] & F_0 d_3 \arrow[r, "\text{in}"] \arrow[d, two heads, "\eta_{d_3}", "\sim"'] & \operatorname{colim} F_0 \arrow[d] \\
C_1 \arrow[r, "F_1(\tau\beta)\circ f_1"'] \arrow[ur, dotted] & F_1 d_3 \arrow[r, "\text{in}"'] & \operatorname{colim} F_1
\end{tikzcd}$$

so that the dotted lift exists. This provides a lift in the original diagram and thereby shows that $\operatorname{colim} QF_i \to F_i$ is an acyclic fibration, as desired. $\square$

As promised, the functor Θ^∞ allows us to characterize stable equivalences in terms of algebraic data in good cases. Note that the following corollary does not use any of the model categorical properties of the stable model category of spectra and goes through if we only knew it was a stable model category.

Corollary 3.2.14. *The left Bousfield localization of $\mathsf{Sp} = \mathsf{Sp}^{\mathbf{N}}(\mathsf{S}_*)$ at the set of maps S described above has weak equivalences precisely the π_*^s-isomorphisms. In particular, the map $j_A \colon A \to \Theta^\infty A$ is a π_*^s-isomorphism.*

Remark 3.2.15. We must be precise about what we mean by the homotopy groups of a spectrum in this context. For a simplicial set X, its homotopy groups are the homotopy groups of its geometric realization $|X|$, since the adjunction between geometric realization and the singular simplicial set functors respects basepoints and are pointed simplicial functors. For instance, for pointed simplicial sets, $|X|$ may be defined as the coend in S_* given by the coend

$$|X| = X_n \wedge_\Delta |\Delta[n]_+| \cong |X_n \wedge_\Delta \Delta[n]_+|$$

where, for a pointed discrete set X_n, we consider it as a pointed discrete space generated by X_n in degree 0 in each of the displayed isomorphisms. The simplicial structure follows by Fubini for coends and the fact that the smash product $X \wedge -$ is a left adjoint, giving a natural isomorphism $|K| \wedge |X| \cong |K \wedge X|$. The pointed simplicial structure on the corresponding $\mathrm{Sing} = \mathrm{hom}_{\mathsf{S}_*}(|\Delta[n]_+|, -)$ arises using this isomorphism and an adjunction argument.

Thus, we could define the homotopy groups of a spectrum as $\pi_*^s(X) = \pi_{*+k}|X_{*+k}|$. However, not every model category C has such a functor with the properties these two have. We will work a little more generally.

Proof. Let L be a level fibrant replacement functor. Define the stable homotopy groups of a spectrum X to be the colimit

$$\pi_*^s(X) = \operatorname{colim} \pi_{*+k}(LX)_k = \operatorname{colim} \pi_* \Omega^k (LX)_k.$$

This is independent of the choice of functorial level fibrant replacement functor as a consequence of [31, Thm. 14.6.9]. It is, moreover, naturally isomorphic to the description given by $\operatorname{colim} \pi_{*+k}|X_k|$ because $\mathrm{Sing}\,|-|$ is a level fibrant replacement functor. Let us define, additionally,

$$\pi_*^{s,u}(X) = \operatorname{colim} \pi_{*+k} X_k = \operatorname{colim} \pi_* \Omega^k (X_k)$$

for "underived" stable homotopy groups of a spectrum X. We will say a map of f of spectra is a π_*^s-isomorphism if Lf is a $\pi_*^{s,u}$-isomorphism. This satisfies the two-out-of-three property since isomorphisms satisfy this property.

We claim the natural map $i_A\colon A \to LA$ is a π_*^s-isomorphism. To see this, it is enough to show that $Li_A\colon LA \to LLA$ is a $\pi_*^{s,u}$-isomorphism. Of course, this follows since

$$\pi_*^{s,u} LA = \operatorname{colim} \pi_* \Omega^k (LA)_k \quad \text{and} \quad \pi_*^{s,u} LLA = \operatorname{colim} \pi_* \Omega^k (L^2 A)_k$$

and since $LA \to L^2 A$ is a level weak equivalence, the map Li_A induces level π_*^s-isomorphism between these two colimits. When X is a level fibrant spectrum, a similar argument implies that the natural map $i_X\colon X \to LX$ induces an isomorphism

$$\pi_*^{s,u} X \cong \pi_*^s X = \pi_*^{s,u} LX.$$

by applying $\pi_*^{s,u}$. For level fibrant spectra X, we are therefore free to understand their stable homotopy groups prior to level fibrant replacement. For level fibrant spectra, note that the natural map $j_X\colon X \to \Theta^\infty X$ is a π_*^s-isomorphism. To see this, note that X and $\Theta^\infty X$ are both level fibrant, so we may check this using $\pi_*^{s,u}$. For this,

$$\pi_*^{s,u} X = \operatorname{colim}_k \pi_* \Omega^k X_k$$

and

$$\pi_*^{s,u} \Theta^\infty X = \operatorname{colim}_k \pi_* (\Omega^k (\operatorname{colim}_n \Omega^n X_{n+k})).$$

By Proposition 3.2.7, we know the colimit in k of the right-hand side above consists only of isomorphisms. Hence, there is a natural isomorphism

$$\pi_*^{s,u}\Theta^\infty X \cong \pi_* \operatorname{colim}_k \Omega^k X_k.$$

As a consequence of the preceding proposition, we know that the natural map $\operatorname{colim}_k \pi_*\Omega^k X_k \to \pi_* \operatorname{colim}_k \Omega^k X_k$ is an isomorphism, since homotopy groups commute with filtered homotopy colimits. This is the map implemented levelwise on the level of spaces by $j_X \colon X \to \Theta^\infty X$ and so $j_X \colon X \to \Theta^\infty X$ is a π_*^s-isomorphism.

Now fix $f \colon A \to B$ a map of spectra and consider the commutative diagram

$$\begin{tikzcd}
A \arrow[r, "\sim"] \arrow[d, "f"'] & L \arrow[r, "\sim"] \arrow[d, "Lf"'] & \Theta^\infty LA \arrow[d, "\Theta^\infty Lf"'] \\
B \arrow[r, "\sim"'] & LB \arrow[r, "\sim"] & \Theta^\infty LB
\end{tikzcd}$$

From we have seen, the maps $C \to LC$ and $LC \to \Theta^\infty LC$ are π_*^s-isomorphisms and, additionally, stable equivalences for $C = A, B$.

Suppose f is a stable equivalence. Then by two-out-of-three, Lf is a stable equivalence and it follows that $\Theta^\infty Lf$ is a level equivalence between Ω-spectra. But the level equivalences between Ω-spectra are precisely the $\pi_*^{s,u}$-isomorphisms. Since the Ω-spectra are level fibrant, $\Theta^\infty Lf$ is in fact a π_*^s-isomorphism. By two-out-of-three, Lf is a π_*^s-isomorphism and hence by two-out-of-three, f is a $\pi_a st^s$-isomorphism.

Conversely, suppose f is a π_*^s-isomorphism. Then by two-out-of-three, Lf is a π_*^s-isomorphism and so by two-out-of-three, $\Theta^\infty Lf$ is a π_*^s-isomorphism. But for level fibrant spectra, a π_*^s-

isomorphism is, in particular, a $\pi_*^{s,u}$-isomorphism and for Ω-spectra, $\pi_*^{s,u}$-isomorphisms are level equivalences. Hence, Lf is a stable equivalence by two-out-of-three and hence f is a stable equivalence by two-out-of-three. $\square$

Precisely the same pattern of argument will show the stabilization of the categories of retractive $\mathcal{O}$-algebras have as their stable equivalences the π_*^s-isomorphisms.

3.3 The QX Construction and Enriched Stable Fibrant Replacement Monads

In this section, we show that, under suitable hypotheses on the category C, the functor Θ^∞ may be used to compare the classical (derived) stabilization construction $QX = \operatorname{hocolim} \Omega^k\Sigma^k X$ with (derived) stabilization modeled as $\Omega^\infty\Sigma^\infty X$. Moreover, we show that Θ^∞ may be used to construct an S_*-enriched stable fibrant replacement monad for the stable model structure on $\mathsf{Sp}^{\mathbf{N}}(\mathsf{C})$, under suitable hypotheses on C.

Theorem 3.3.1. *Let C be a finitely generated, pointed simplicial model category in which the domains of the generating cofibrations are cofibrant, sequential colimits preserve finite products and Ω preserves sequential colimits.*

(a) Let R be an S_*-enriched fibrant replacement monad on C. Then R prolongs to a level fibrant, S_*-enriched fibrant replacement monad on $\mathsf{Sp}^{\mathbf{N}}(\mathsf{C})$, which we also denote R.

(b) Set $\Omega = \Omega R$ with R as above. Let

$$QX = \operatorname{colim}(X \to \Omega_Y \Sigma_Y X \to \Omega_Y^2 \Sigma_Y^2 X \to \cdots)$$

where the bonding maps of the colimit are obtained from the derived unit maps. Then there is a natural comparison map

$$c = c_X \colon QX \to \Omega^\infty \Theta^\infty R \Sigma^\infty X$$

which is a weak equivalence when X is cofibrant.

(c) The comparison map c respects the structure maps $X \to QX$ and $X \to \Omega^\infty \Theta^\infty R \Sigma^\infty X$—in other words, it makes the evident diagram commute. In particular, for cofibrant X, this means comparison equivalence respects the structure map $X \to QX$ and the derived unit map $X \to \Omega^\infty \Theta^\infty R \Sigma^\infty X$.

Remark 3.3.2. We will assume X is cofibrant in the proof—the resulting construction will necessarily be natural in X. By Corollary 3.2.8, the evident map $Z \to \Theta^\infty RZ$ is a stable fibrant replacement of a spectrum Z (hence, a stable equivalence), so the map $X \to \Omega^\infty \Theta^\infty R \Sigma^\infty X$ in part (c) really is the derived unit map.

Proof. In light of [47, Thm. 13.2.1, Cor. 13.2.4, Rem. 13.4.3], we may pick a level fibrant replacement functor on C which is a pointed simplicial monad. This means that, in particular, the monad's natural transformations are all pointed simplicial natural transformations—or, more simply, S_*-natural transformations. As in [37, Def. 1.2.8], this means that the natural objectwise weak equivalences

$$u\colon \mathrm{id} \xrightarrow{\sim} R \quad \text{and} \quad \mu = \mu_R\colon R^2 \xrightarrow{\sim} R$$

respecting the assembly natural transformations

$$\mathrm{assemb}_{X,K}\colon F(X) \wedge K \to F(X \wedge K)$$

equipped on each functor appearing—for id, the assembly maps are the identity. Equivalently, they respect the adjoint assembly natural transformations (defined further below)

$$\widetilde{\mathrm{assemb}}_{X,K}\colon F(\mathbf{hom}(K,X)) \to \mathbf{hom}(K,FX).$$

Note that u and μ_R are objectwise weak equivalences. That μ_R is an objectwise weak equivalence follows from the unit diagram for the monad R, since the unit map u is an objectwise weak equivalence as R is a fibrant replacement functor. In fact, for cofibrant X, the assembly map $R(X) \wedge K \to R(X \wedge K)$ is a weak equivalence; equivalently, the adjoint assembly map $R(\mathbf{hom}(K,X)) \to \mathbf{hom}(K,RX)$ is weak equivalence for fibrant X. These follow using the unit pointed simplicial natural transformation.

$$\begin{tikzcd} \mathbf{hom}(K,X) \arrow[r, equal] \arrow[d, "\mathrm{unit}"'] & \mathbf{hom}(K,X) \arrow[d, "\mathbf{hom}(K,\mathrm{unit})"] \\ R\mathbf{hom}(K,X) \arrow[r, "\widetilde{\mathrm{assemb}}"'] & \mathbf{hom}(K,RX) \end{tikzcd} \qquad (*)$$

The diagram commutes because u is a pointed simplicial natural transformation, the vertical arrows are weak equivalences because C is a pointed simplicial model category (all pointed simplicial sets are cofibrant) and X is fibrant.

Note that to say $\mu_R\colon R^2 \xrightarrow{\sim} R$ is a pointed simplicial natural transformation means that the following diagram commutes for $X \in \mathsf{Alg}_{\mathcal{O}}^{Y}$ and $K \in \mathsf{S}_*$

$$\begin{tikzcd}
R^2\mathbf{hom}(K,X) \arrow[r, "R\widetilde{\mathrm{assemb}}_{X,K}"] \arrow[d, "\mu_R"'] & R\mathbf{hom}(K,RX) \arrow[r, "\widetilde{\mathrm{assemb}}_{RX,K}", "\sim"'] & \mathbf{hom}(K,R^2X) \arrow[d, "\mathbf{hom}(K{,}\mu_R)"] \\
R\mathbf{hom}(K,X) \arrow[rr, "\widetilde{\mathrm{assemb}}_{X,K}"'] & & \mathbf{hom}(K,RX)
\end{tikzcd}$$

Now, as promised, we define the adjoint assembly map

$$R\mathbf{hom}(K,X) \xrightarrow{\widetilde{\mathrm{assemb}}} \mathbf{hom}(K,RX)$$

to be the map adjoint to

$$R(\mathbf{hom}(K,X)) \wedge K \xrightarrow{\mathrm{assemb}} R(\mathbf{hom}(K,X) \wedge K) \xrightarrow{R\varepsilon} RX$$

using the counit of the adjunction. Explicitly, it is the map

$$R(X^K) \xrightarrow{\eta} (R(X^K) \wedge K)^K \xrightarrow{\mathrm{assemb}^K} (R(X^K \wedge K))^K \xrightarrow{(R\varepsilon)^K} (RX)^K.$$

It follows using naturality of the unit map and the triangle identities that the given assembly map

$RX \wedge K \to R(X \wedge K)$ is the map adjoint to

$$RX \xrightarrow{R\eta} R(\mathbf{hom}(K, X \wedge K)) \xrightarrow{\widetilde{\text{assemb}}} \mathbf{hom}(K, R(X \wedge K))$$

using the unit of the adjunction. This follows by meditating upon the following commutative diagram

$$\begin{tikzcd}
RX \wedge K \arrow[r, "R\eta\wedge K"] \arrow[d, "\text{assemb}"'] & R\mathbf{hom}(K, X \wedge K) \wedge K \arrow[r, "\widetilde{\text{assemb}}\wedge K"] \arrow[d, "\text{assemb}"] & \mathbf{hom}(K, R(X \wedge K)) \wedge K \arrow[d, "\varepsilon"] \\
R(X \wedge K) \arrow[r, "R(\eta\wedge K)"'] & R(\mathbf{hom}(K, X \wedge K) \wedge K) \arrow[r, "R\varepsilon"] & R(X \wedge K)
\end{tikzcd}$$

The left-hand square commutes by naturality of the assembly map. As for the right-hand square, it commutes by taking adjoints and using the triangle identities. By definition, the adjoint of the counterclockwise composite is $\widetilde{\text{assemb}}$. As for the clockwise composite, its adjoint fits into a commutative diagram using naturality of η as

$$\begin{tikzcd}
R((X \wedge K)^K) \arrow[r, "\widetilde{\text{assemb}}"] \arrow[d, "\eta"'] & (R(X \wedge K))^K \arrow[d, "\eta"] \arrow[dr, equal] & \\
(R((X \wedge K)^K) \wedge K)^K \arrow[r, "(\widetilde{\text{assemb}}\wedge K)^K"'] & ((R(X \wedge K))^K \wedge K)^K \arrow[r, "\varepsilon^K"'] & (R(X \wedge K))^K
\end{tikzcd}$$

The triangle commutes by the triangle identities. The square commutes by naturality of η. Hence, the adjoint composite is simply $\widetilde{\text{assemb}}$, as desired.

This choice of R and the particular properties it enjoys is really the key point. The rest of the proof of this theorem amounts to checking that everything that should commute really does commute.

As in [37, Def. 1.2.8], R prolongs to a pointed simplicial endofunctor of $\mathsf{Sp}^{\mathbf{N}}(\mathsf{C})$ and the S_*-natural transformations $u\colon \mathrm{id} \to R$ and $\mu\colon R^2 \to R$ prolong to ones on $\mathsf{Sp}^{\mathbf{N}}(\mathsf{C})$—the prolongation of u and μ are defined to be given levelwise by u and μ as defined in C. By abuse of notation, call this prolongment R. That the prolongation of R remains an S_*-enriched functor may be seen using the underlying assembly map for $R\colon \mathsf{C} \to \mathsf{C}$—it is a simple matter of applying the associativity axiom for this functor several times to see that the assembly map prolongs to natural transformation of functors $\mathsf{Sp}^{\mathbf{N}}(\mathsf{C}) \times \mathsf{S}_* \to \mathsf{Sp}^{\mathbf{N}}(\mathsf{C})$. Explicitly, this may be seen by meditating upon the following (sparsely) labeled diagram

$$\begin{tikzcd}
R(X_{n+1} \wedge K) & R((X_n \wedge S^1) \wedge K) \arrow[l] & R(X_n \wedge (S^1 \wedge K)) \arrow[l] & R(X_n \wedge (K \wedge S^1)) \arrow[l] & R((X_n \wedge K) \wedge S^1) \arrow[l] \\
 & & R(X_n) \wedge (S^1 \wedge K) \arrow[u] \arrow[d] \arrow[ull, dotted] & R(X_n) \wedge (K \wedge S^1) \arrow[l, "\mathrm{id} \wedge \tau"'] \arrow[u] \arrow[dr, dotted, no head] & \\
R(X_{n+1}) \wedge K \arrow[uu] & R(X_n \wedge S^1) \wedge K \arrow[l, "R(\sigma_n) \wedge \mathrm{id}"] \arrow[uu, "\mathrm{assemb}"'] & (R(X_n) \wedge S^1) \wedge K \arrow[l] & (R(X_n) \wedge K) \wedge S^1 \arrow[r] \arrow[lll, dotted, bend left] & R(X_n \wedge K) \wedge S^1 \arrow[uu]
\end{tikzcd}$$

where the dotted arrows are the composites which are the relevant structure maps. The diagram commutes by naturality of the assembly map and the associativity condition the assembly map satisfies. It is manifestly natural in the spectrum X and pointed simplicial set K. Hence, the associativity condition for the prolonged assembly map is satisfied as all relevant natural transformations are given levelwise. The unit condition for the prolongation of R is likewise satisfied. It follows easily that the prolongation of the maps $u\colon \mathrm{id} \to R$ and $\mu\colon R^2 \to R$ are level equivalences and themselves S_*-natural transformations on $\mathsf{Sp}^{\mathbf{N}}(\mathsf{C})$ as all relevant natural transformations are defined levelwise.

Then, in particular, R is an S_*-enriched level fibrant replacement functor and a monad. Since the colimits in question for QX and $\Omega^\infty\Theta^\infty R\Sigma^\infty X$ are, in particular, homotopy colimits, it suffices to exhibit a level equivalence between them by Proposition 3.2.13.

To begin, we claim the following diagram commutes.

$$\begin{tikzcd}
RX \arrow[r, "\eta"] \arrow[d, "R\eta"'] & \Omega\Sigma RX \arrow[d, "\Omega\mathrm{assemb}"] \\
R\Omega\Sigma X \arrow[r, "\widetilde{\mathrm{assemb}}"'] & \Omega R\Sigma X
\end{tikzcd} \tag{$**$}$$

This is the dual version of what we just showed above and, consequently, it has what is essentially a dual proof. Indeed, to see this, note that the clockwise composite is adjoint to

$$\begin{tikzcd}
\Sigma RX \arrow[r, "\Sigma\eta"] \arrow[dr, equal] & \Sigma\Omega\Sigma RX \arrow[r, "\Sigma\Omega\mathrm{assemb}"] \arrow[d, "\varepsilon"] & \Sigma\Omega R\Sigma X \arrow[d, "\varepsilon"] \\
& \Sigma RX \arrow[r, "\mathrm{assemb}"'] & R\Sigma X
\end{tikzcd}$$

where the square commutes since ε is natural. For the counterclockwise composite, we have seen that the assembly map is adjoint to this composite, whence the square commutes. Note that $\Omega^\infty\Theta^\infty R\Sigma^\infty X$ is the colimit of the sequence of maps $\Omega^k(\widetilde{\mathrm{assemb}}\circ R\eta)\colon \Omega^k R\Sigma^k X \to \Omega^{k+1}R\Sigma^{k+1}X$. Hence, in all degrees $k \geq 0$, $(**)$ shows that this map is, equivalently $\Omega^k(\Omega\mathrm{assemb}\circ\eta)$.

Define an auxiliary sequence

$$RX \to \Omega R\Sigma X \to \Omega^2 R^2\Sigma^2 X \to \cdots \tag{$\star$}$$

as follows. The first map is the composite

$$RX \to R\Omega\Sigma X \to \Omega R\Sigma X$$

of $(**)$. In higher degrees,

$$\Omega^k R^k \Sigma^k X \to \Omega^{k+1} R^{k+1} \Sigma^{k+1} X$$

is the composite

$$\begin{tikzcd}
\Omega^k R^k \Sigma^k X \arrow[r, "\Omega^k R^k \eta"] \arrow[rrr, bend left] & \Omega^k R^k \Omega\Sigma^{k+1} X \arrow[r, "\Omega^k R^k \Omega u"] & \Omega^k R^k \Omega R \Sigma^{k+1} X \arrow[r, "\Omega^k \widetilde{\mathrm{assemb}}^{\times k}"] & \Omega^{k+1} R^{k+1} \Sigma^{k+1} X.
\end{tikzcd}$$

where $\widetilde{\mathrm{assemb}}^{\times k}$ is the evident k-fold application of the assembly map

$$(R \circ \cdots \circ R)(\Omega Z) \xrightarrow{R^{k-1}\widetilde{\mathrm{assemb}}} R^{k-1}\Omega R Z \to \cdots \to \Omega R^k Z.$$

This is natural in Z and has the property that $\widetilde{\mathrm{assemb}}^{\times k-1} \circ R^{k-1}\widetilde{\mathrm{assemb}} = \widetilde{\mathrm{assemb}}^{\times k}$.

There is a naturally occurring level equivalence between the colimit defining QX and $(\star)$. In degree 0 the map is the unit $u\colon X \to RX$ and in degree 1 the map is the identity. Commutativity of the relevant square is then enforced by naturality of the unit map and $(*)$.

In higher degrees $k \geq 2$, it is formed using the natural $(k-1)$-fold assembly assembly maps $(\Omega R)^k \Sigma^k Z \xrightarrow{\sim} \Omega^k R^k \Sigma^k Z$, which are weak equivalences. The commutativity of the square involving the first and second terms of the sequences is immediate. For $k \geq 3$, compatibility of the maps is likewise essentially immediate by naturality of the n-fold assembly map and naturality of u.

We claim that there is a level equivalence from the sequence $(\star)$ to the sequence defining $\Omega^\infty\Theta^\infty R\Sigma^\infty X$. The first two terms (the 0th and 1st objects in the sequences) may be taken to be the identity. We let the next map $\Omega^2 R^2\Sigma^2 X \to \Omega^2 R\Sigma^2 X$ be $\Omega^2\mu_R$. To see this makes the evident diagram commute, note that we have the following solid commutative diagram

$$
\begin{tikzcd}
\Omega R\Sigma X \arrow[r, "\Omega R\eta"] \arrow[dd, equal] \arrow[drr, dotted] & \Omega R\Omega\Sigma^2 X \arrow[r, "\Omega R\Omega u"] \arrow[d, "\Omega\widetilde{\text{assemb}}"] & \Omega R\Omega R\Sigma^2 X \arrow[d, "\Omega\widetilde{\text{assemb}}"] \\
 & \Omega^2 R\Sigma^2 X \arrow[r, "\sim", "\Omega^2 Ru"'] \arrow[dr, equal] & \Omega^2 R^2\Sigma^2 X \arrow[d, "\Omega^2\mu_R"', "\sim"] \\
\Omega R\Sigma X \arrow[r, "\Omega\eta"] \arrow[ur, dotted, bend left] & \Omega^2\Sigma R\Sigma X \arrow[r, "\Omega^2\text{assemb}"'] \arrow[u, "\Omega^2\text{assemb}"] & \Omega^2 R\Sigma^2 X
\end{tikzcd}
$$

The dotted maps composite maps are the bonding maps for each sequence, where we have used $(**)$ to see this for the bottom map. The big rectangle on the left commutes by applying Ω to $(**)$. The top small square commutes on account of naturality of $\widetilde{\text{assemb}}$. The bottom small square commutes because $\mu_R \circ Ru = \mathrm{id}_R$, because R is a monad.

In general, we define the maps

$$\Omega^n R^n\Sigma^n X \xrightarrow{\sim} \Omega^n R\Sigma^n X$$

to be induced by the $(n-1)$-fold R multiplication $R^n \xrightarrow{\sim} R$ going from right to left. These will produce weak equivalences between the objects since Ω is a right Quillen functor.

However, we do not yet know this is a well-defined map of spectra. To get a feeling for the general pattern of the argument, we consider the very next case of $n = 3$. This contains all the ingredients of the general pattern. We must use both $(**)$ and $(* * *)$. Consider the following commutative diagram

$$\begin{tikzcd}
\Omega^2 R^2 \Sigma^2 X \arrow[r] \arrow[dddd, squiggly, "\Omega^2 \mu_R"', "\sim"] \arrow[ddr, squiggly, "\Omega^2 R(\widetilde{\text{assemb}} \circ R\eta)"] \arrow[rrrr, bend left, dotted] & \Omega^2 R^2 \Omega \Sigma^3 X \arrow[r, "\Omega^2 R^2 \Omega u"] \arrow[dd] & \Omega^2 R^2 \Omega R \Sigma^3 X \arrow[r, "\Omega^2 R\, \widetilde{\text{assemb}}"] & \Omega^2 R \Omega R^2 \Sigma^3 X \arrow[r, "\Omega^2\, \widetilde{\text{assemb}}"] \arrow[dd, "\Omega^2 R \Omega \mu_R"'] & \Omega^3 R R^2 \Sigma^3 X \arrow[dd, "\sim"', "\Omega^3 R \mu_R"] \\
 & & & & \\
 & \Omega^2 R \Omega R \Sigma^3 X \arrow[rr, equal] & & \Omega^2 R \Omega R \Sigma^3 X \arrow[r, squiggly, "\Omega^2\, \widetilde{\text{assemb}}"'] & \Omega^3 R^2 \Sigma^3 X \arrow[dd, squiggly, "\sim"', "\Omega^3 \mu_R"] \\
 & & \Omega^2 R \Omega \Sigma^3 X \arrow[ddrr, squiggly, "\Omega^2\, \widetilde{\text{assemb}}"] & & \\
\Omega^2 R \Sigma^2 X \arrow[urr, squiggly, "\Omega^2 R\eta"] \arrow[rr, "\Omega^2 \eta"'] \arrow[rrrr, bend right, dotted] & & \Omega^3 \Sigma R \Sigma^2 X \arrow[rr, "\Omega^3 \text{ assemb}"'] & & \Omega^3 R \Sigma^3 X
\end{tikzcd}$$

The dotted composite maps are the bonding maps for each sequence. The upper rectangle commutes because of $(* * *)$. The upper-right square commutes by naturality of the assembly map. The lower triangle commutes by $(**)$. Thus, the whole diagram commutes because the squiggled composites are equal. This is a consequence of naturality of η and the fact that μ_R is an S_*-natural transformation.

Explicitly, we may expand the squiggly part to the following commutative diagram.

$$\begin{tikzcd}
\Omega^2 R^2 \Sigma^2 X \arrow[r, squiggly, "\Omega^2 R^2 \eta"] \arrow[d, squiggly, "\Omega^2 \mu_R"', "\sim"] & \Omega^2 R^2 \Omega \Sigma^3 X \arrow[r, squiggly, "\Omega^2 R \widetilde{\mathrm{assemb}}"] \arrow[d, "\sim"', "\Omega^2 \mu_R"] & \Omega^2 R \Omega R \Sigma^3 X \arrow[r, squiggly, "\Omega^2 \widetilde{\mathrm{assemb}}"] & \Omega^3 R^2 \Sigma^3 X \arrow[d, squiggly, "\sim"', "\Omega^3 \mu_R"] \\
\Omega^2 R \Sigma^2 X \arrow[r, squiggly, "\Omega^2 R \eta"'] & \Omega^2 R \Omega \Sigma^3 X \arrow[rr, squiggly, "\Omega^2 \widetilde{\mathrm{assemb}}"'] & & \Omega^3 R \Sigma^3 X
\end{tikzcd}$$

The left-most square commutes by naturality of η and the right-most since μ_R is an S_*-natural transformation.

For the next step of $n = 4$, one first recognizes the overall application of Ω^2 to the diagrams above. Thus, by removing Ω^2 and then applying $\Omega^3 R$ to the diagrams above, one fits the map $\Omega^3 R \mu_R$ of the right-most column in the big diagram above into a bigger commutative diagram. By then using naturality of μ_R, this shows the next step commutes as well. This pattern continues and the argument is finished by an induction.

This establishes an explicit sequence of weak equivalences for cofibrant X

$$QX \xrightarrow{\sim} \operatorname{colim}(\star) \xrightarrow{\sim} \Omega^\infty \Theta^\infty R \Sigma^\infty X$$

between the colimits as a consequence of Proposition 3.2.13; therefore it composes to an equivalence $QX \xrightarrow{\sim} \Omega^\infty \Theta^\infty R \Sigma^\infty X$, as desired. Moreover, the map $QX \to \Omega^\infty \Theta^\infty R \Sigma^\infty X$ is clearly natural in X because all maps used to construct the level equivalence between the underlying sequences for the colimits are natural in X.

We now wish to show that, for cofibrant X, the derived unit map $X \to \Omega^\infty \Theta^\infty R \Sigma^\infty X$ factors as $X \to QX \to \Omega^\infty \Theta^\infty R\Sigma^\infty X$; in other words, that the following diagram commutes

$$\begin{tikzcd}
X \arrow[r] \arrow[d, equal] & QX \arrow[d] \\
X \arrow[r] & \Omega^\infty \Theta^\infty R\Sigma^\infty X
\end{tikzcd}$$

Note that the map $X \to \Omega^\infty \Sigma^\infty X = \mathrm{Ev}_0 \Sigma^\infty X$ is the identity map. The map $\Omega^\infty \Sigma^\infty X \to \Omega^\infty \Theta^\infty R\Sigma^\infty X$ is simply the composite map $X \xrightarrow{\sim} RX \to \operatorname{colim} \Omega^k R\Sigma^k X$ where the latter map is the structure map of the colimit and the former the unit map for R. Since $X \to QX$ is the structure map for the colimit and, on the level of underlying colimit sequences, the map in degree 0 for $QX \to \Omega^\infty \Theta^\infty R\Sigma^\infty X$ is induced by the unit map for R, this follows immediately. $\square$

Remark 3.3.3. This is effectively a point-set version of [29, Lem. 2.10(c)] for compactly generated ∞-categories qua quasicategories. Namely that stabilization as $P_1 \mathrm{Id}$ agrees with $\Omega^\infty \Sigma^\infty$. In fact, the underlying ∞-category of $\mathsf{Alg}_{\mathcal{O}}^{Y}$ and really, let us say, any pointed simplicial, cofibrantly generated model category having domains and codomains of the generating cofibrations simplicially small and cofibrant and the domains and codomains of the generating acyclic cofibrations simplicially compact—is compactly generated in the sense of [41, Def. 5.5.7.1]. The underlying ∞-category of such a model category is indeed bicomplete by standard facts and the domains and codomains of the generating cofibrations generate the underlying ∞-category under filtered colimits as a consequence of [43, Cor. 5.1]. A variant of Proposition 3.2.13 shows that filtered homotopy limits of fibrant objects are once again fibrant, which additionally shows that domains and codomains of the generating cofibrations are indeed compact in the underlying ∞-category. These conclusions follows from say [42, Tag 01LE] and some model categorical reasoning we are suppressing for brevity.

Hence, filtered homotopy colimits commute with finite homotopy limits in this ∞-category and it therefore supports a good theory of functor calculus.

For us, the utility of this theorem is that we may now understand the connectivity of the (derived) maps $X \to \Omega^\infty\Sigma^\infty X$ in a precise way. In particular, for L a stable fibrant replacement monad, $\Omega^\infty_Y L\Sigma^\infty_Y$ is an *iterable* model, which will be amenable to our completion methods.

Remark 3.3.4. In [37, Def. 1.2.9] and in our proof above, we have used a notion of a pointed simplicially enriched natural transformation $F \to G$ and pointed simplicially enriched functors that is slightly different from the standard definition —namely a pointed simplicially enriched functor is one equipped with assembly maps and unit maps for the S_* action which are required to make the two evident diagram commute and a natural transformation which respects the assembly maps $F(X) \wedge K \to F(X \wedge K)$. Since our categories are tensored and cotensored over pointed simplicial sets, this is essentially equivalent to the usual formulation. As before, a good place for a much more detailed and thorough discussion on this may be found in [25, 38].

However, there is still a simple way to see this in the case of pointed simplicial sets. Fix such a functor $F\colon \mathsf{C} \to \mathsf{D}$ between *tensored* S_*-enriched categories. For fixed X, the maps the maps $F_{X,Y}\colon \mathbf{Hom}(X,Y) \to \mathbf{Hom}(FX,FY)$ are natural in Y and so give a natural transformation $F_{X,-}\colon \mathbf{Hom}(X,-) \to \mathbf{Hom}(FX,F-)$. Since $\mathbf{Hom}(X,-) = \hom(X \wedge \Delta[-]_+, Y)$, such a natural transformation is equivalent, by Yoneda, to specifying assembly maps $FX \wedge \Delta[n]_+ \to F(X \wedge \Delta[n]_+)$ which are natural in Δ. This is likewise natural in X since the maps $F_{X,Y}$ are natural in X as well—in particular, this follows by making judicious choices for Y. In the general case, a pointed simplicial set K is the coend

$$K \cong \Delta[n]_+ \wedge_\Delta K_n \cong \int^{[n]\in\Delta} \left(\bigvee_{*\neq\sigma\in K_n} \Delta[n]_+ \right),$$

where K_n is considered as a discrete pointed simplicial set say with basepoint $* \in K_n$. Note that $\Delta[n]_+ \wedge K_n$ is simply a coproduct (wedge) in S_* of copies of $\Delta[n]_+$ with the copy indexed by the basepoint collapsed—$\Delta[n]_+ \wedge K_n \cong \quad {}_{*\neq\sigma\in K_n} \Delta[n]_+$. This decomposition of the simplicial set as a coend is, furthermore, natural in K. Hence, since $X \wedge Y$ is a left adjoint in both variables,

$$FX \wedge K \cong (FX \wedge \Delta[n]_+) \wedge_\Delta K_n \cong \int^{[n]\in\Delta} \left(\bigvee_{*\neq\sigma\in K_n} FX \wedge \Delta[n]_+ \right)$$

and there are maps $FX \wedge \Delta[n]_+ \wedge K_n \to F(X \wedge K)$ given by

$$FX \wedge \Delta[n]_+ \wedge K_n \to F(X \wedge \Delta[n]_+) \wedge K_n \cong \bigvee_{*\neq\sigma\in K_n} F(X \wedge \Delta[n]_+) \xrightarrow{F(\mathrm{id}_X \wedge \sigma)} F(X \wedge K).$$

Since the assembly map is natural in X and Δ, these give a cone to $F(X \wedge K)$ and thus a map $FX \wedge K \to F(X \wedge K)$ which is natural in X and K. Using properties of colimits and the fact that the tensoring $- \wedge -$ is a left adjoint in each variable, it is not hard to see that this assembly map respects the associativity and unit isomorphisms, appropriately defined. See, for instance, [47, Props. 10.1.4, 10.1.5], the appendix of [38], [25] or [17, 45] for treatments.

In fact, the naturality condition on any such assembly map *forces* us to define it this way—at least when the categories in question are tensored as shall always be the case for us. This follows from the evident universal properties. Since $F(X) \wedge K$ is the coend above, a map $F(X) \wedge K \to$

$F(X \wedge K)$ is completely determined by its values on the terms $F(X) \wedge \Delta[n]_+$. By naturality, then, the following diagram would have to commute

$$\begin{tikzcd}
F(X) \wedge \Delta[n]_+ \arrow[r, "\text{assemb}"] \arrow[d, "F(X)\wedge\sigma"'] & F(X \wedge \Delta[n]_+) \arrow[d, "F(X\wedge\sigma)"] \\
F(X) \wedge K \arrow[r, dotted, "\text{assemb}"'] & F(X \wedge K)
\end{tikzcd}$$

which shows that the assembly map $F(X) \wedge K \to F(X \wedge K)$ is completely determined by the assembly maps $F(X) \wedge \Delta[n]_+ \to F(X \wedge \Delta[n]_+)$.

For pointed simplicially enriched categories, the usual notion of a pointed simplicial natural transformation is simply a choice of map $\eta_c \colon Fc \to Gc$ which make the naturality diagram of pointed mapping spaces commute. The equivalence with respecting assembly maps. This is equivalently to the usual notion—this is argued similarly to the above with the very same judicious choice indicated.

We now investigate investigate the properties of the functor Θ^∞ a little more carefully.

Theorem 3.3.5. *Let C be a finitely generated, pointed simplicial model category in which the domains of the generating cofibrations are cofibrant, sequential colimits preserve finite products and Ω preserves sequential colimits. The functor Θ^∞ naturally has the structure of an S_*-enriched endofunctor. It, moreover, has the structure of a pointed simplicial monad whose unit and assembly maps are isomorphisms.*

Remark 3.3.6. As Θ^∞ is a colimit of S_*-enriched functors, this is no surprise. The real work is verifying Θ^∞ is, additionally an S_*-enriched monad and for this we may as well be explicit about the S_*-enriched structure Θ^∞ acquires.

Proof. Since $\mathsf{Sp}^{\mathbf{N}}(\mathsf{C})$ is not only enriched but tensored and cotensored over pointed simplicial sets, it suffices to provide a natural assembly map $a\colon \Theta^\infty(X)\wedge K \to \Theta^\infty(X\wedge K)$ such that the following associativity and unit diagrams commute

$$\begin{tikzcd}
(\Theta^\infty(X)\wedge K)\wedge L \arrow[r, "\alpha"] \arrow[d, "a\wedge \mathrm{id}"'] & \Theta^\infty(X)\wedge(K\wedge L) \arrow[r, "a"] & \Theta^\infty(X\wedge(K\wedge L)) \\
\Theta^\infty(X\wedge K)\wedge L \arrow[rr, "a"'] & & \Theta^\infty((X\wedge K)\wedge L \arrow[u, "\Theta^\infty\circ\alpha"']
\end{tikzcd}$$

$$\begin{tikzcd}
\Theta^\infty X\wedge S^0 \arrow[rr, "a"] \arrow[dr, "u"'] & & \Theta^\infty(X\wedge S^0) \arrow[dl, "\Theta^\infty\circ u"] \\
& \Theta^\infty X &
\end{tikzcd}$$

where u is the unit map and α the associativity isomorphism for the simplicial action. A nice discussion on this equivalence may be found in [38, 25] and additionally [47].

To do this then, let us define a map $\Theta^\infty(X)\wedge K\to\Theta^\infty(X\wedge K)$ natural in X and K.

Since $\mathsf{Sp}^{\mathbf{N}}(\mathsf{C})$ is simplicial, $(\operatorname{colim}_k \Theta^k X)\wedge K \cong \operatorname{colim}_k(\Theta^k(X)\wedge K)$ and it suffices to produce maps $\Theta^k(X)\wedge K\to\Theta^k(X\wedge K)$. Recall from Remark 3.2.5 that $\Theta^k=\Omega^k\circ s_-^k$; it is easy to see that s_-^k is naturally a simplicial functor since $s_-^k(X\wedge K)=s_-^k(X)\wedge K$ (the simplicial action is levelwise in the spectrum). Now, by Remark 3.2.5, this is the same as providing a map $\Sigma^k(\Theta^k(X)\wedge K)\to s_-^k(X\wedge K)$. The left-hand side is, by using the associativity natural isomorphism and the twist natural isomorphism,

$$\Sigma^k(\Theta^k(X)\wedge K)=(\Omega^k s_-^k(X)\wedge K)\wedge S^k\cong(\Sigma^k\Omega^k s_-(X))\wedge K$$

and the counit map $\Sigma^k \Omega^k s_-^k(X) \to s_-^k(X)$ provides a map

$$\Sigma^k(\Theta^k(X) \wedge K) \to s_-^k X \wedge K = s_-^k(X \wedge K).$$

For each k, let $\Theta^k(X) \wedge K \to \Theta^k(X \wedge K)$ be the map adjoint to this one. This is natural in X and K because the associativity and twist isomorphisms are natural in X and K and the map $\varepsilon_X \wedge \mathrm{id}_K$ is natural in X and K. We must check that the following diagram commutes for $k \geq 1$

$$\begin{tikzcd}
\Theta^{k+1}(X) \wedge K \arrow[r] & \Theta^{k+1}(X \wedge K) \\
\Theta^k(X) \wedge K \arrow[r] \arrow[u, "\Theta^k(i_X)\wedge K", "=i_{\Theta^k X} \wedge K"'] & \Theta^k(X \wedge K) \arrow[u, "i_{\Theta^k(X\wedge K)}", "=\Theta^k i_{X \wedge K}"']
\end{tikzcd}$$

where the upwards maps are the labeled natural maps arising from applying Θ^k to the adjoint structure maps $X_{n+k} \wedge K \to \Omega(X_{n+k+1} \wedge K)$ or, in other words, applying Θ^k to the map $X \wedge K$ to $\Omega(s_- X \wedge K)$. Writing $\Theta^k = \Omega^k \circ s_-^k$, this diagram commutes if and only if its adjoint

$$\begin{tikzcd}
\Sigma^k(\Theta^k(\Theta(X)) \wedge K) \arrow[r] & s_-^k \Theta(X \wedge K) \\
\Sigma^k(\Theta^k(X) \wedge K) \arrow[r] \arrow[u] & s_-^k(X \wedge K) \arrow[u]
\end{tikzcd}$$

commutes, and this commutes by naturality of the counit, and the associativity and twist isomorphisms—the general flavor of the argument is immediate from the case of $k = 1$ and for cases of $k \geq 2$, the only additional step uses the fact that the counit of the (Σ^k, Ω^k)-adjunction is the k-fold iterate of the counit for the (Σ, Ω)-adjunction.

We must also check the case of $k = 0$, which corresponds to requiring that the the following diagram commutes

$$\begin{array}{ccc}
\Theta(X) \wedge K & \longrightarrow & \Theta(X \wedge K) \\
{\scriptstyle i_X \wedge K}\uparrow & & \uparrow{\scriptstyle i_{X \wedge K}} \\
X \wedge K & = \!= \!= & X \wedge K
\end{array}$$

This is essentially by definition, since the adjoint structure maps for $X \wedge K$ factor through the suspension of the assembly maps just given. Explicitly, the adjoint of the clockwise composite is given levelwise by

$$\Sigma(X_n \wedge K) \xrightarrow{\Sigma(\sigma_n^X \wedge K)} \Sigma(\Omega X_{n+1} \wedge K) \cong \Sigma\Omega(X_{n+1}) \wedge K \xrightarrow{\varepsilon \wedge K} X_{n+1} \wedge K$$

and since σ_n^X is the map adjoint to the structure map $\sigma_n^X\colon \Sigma X_n \to X_{n+1}$, this composite is easily seen to be $\sigma_n^{X \wedge K}$, as desired.

Now we must check that the action map is compatible with the simplicial action. For this, it is also easy to check since $- \wedge K$ is a left adjoint and the action map is built levelwise between the colimits defining Θ^∞. The unit map is similarly defined levelwise between the colimits and the evident unit diagram commutes using the fact that $- \wedge S^0$ is a left adjoint.

To see that Θ^∞ is, additionally, a pointed simplicial monad, recall that the natural map $j_{\Theta^\infty X}\colon \Theta^\infty X \to (\Theta^\infty)^2 X$ is an isomorphism—this is the structure map $\Theta^\infty X \to \Theta^\infty(\Theta^\infty X)$. Define a unit map

$$u\colon \mathrm{id} \to \Theta^\infty \qquad \text{by} \qquad u_X = j_X$$

and define

$$\mu\colon (\Theta^\infty)^2 X \to \Theta^\infty X \qquad \text{by} \qquad \mu_X = j_{\Theta^\infty X}^{-1}.$$

We claim this gives Θ^∞ the structure of a monad.

Recall that the natural maps $i_X \colon X \to \Theta X$ satisfy that $i_{\Theta^n X} = \Theta^n i_X$—this is [36, Lem. 4.5]. Now, note that since Ω and thus Θ commute with sequential colimits, a quick computation shows that the natural map $i_{\Theta^\infty X} \colon \Theta^\infty X \to \Theta\Theta^\infty X$ is the map induced by way of the following commutative diagram

$$\begin{tikzcd} \Theta^k X \arrow[r, "i_{\Theta^k X}", "= \Theta^k i_X"'] \arrow[d, "\mathrm{in}_{\Theta^k X}"'] & \Theta^{k+1} X \arrow[d, "\Theta\, \mathrm{in}_{\Theta^k X}"] \\ \Theta^\infty X \arrow[r, dotted, "\cong", "i_{\Theta^\infty X}"'] & \Theta\Theta^\infty X \end{tikzcd}$$

Since $\mathrm{in}_X = j_X$, this means that

$$\Theta j_X = i_{\Theta^\infty X} j_X$$

we claim. To see this, note that Θ commutes with directed colimits. Hence, $\Theta\Theta^\infty X$ has colimit cone given by the maps $\Theta\,\mathrm{in}_{\Theta^k X} \colon \Theta^{k+1} X \to \Theta\Theta^\infty X$, from which this follows in the case of $k = 0$ (i.e., $\Theta^0 = \mathrm{id}$).

The map $i_{\Theta^\infty X}$ has inverse given on the components of the colimits $(\Theta\Theta^\infty X)_n \to (\Theta^\infty X)_n$ simply by the identity maps $\mathrm{id} \colon \Theta^{k+1} X \to \Theta^{k+1} X$ or, in other words, $\mathrm{id} \colon \Omega^k X_{n+k} \to \Omega^k X_{n+k}$ —this follows from a simple check of universal properties. All together, this means that $i^{-1}_{\Theta^\infty X} \circ \Theta j_X = \mathrm{in}_{\Theta X}$ and this pattern holds in the sense that

$$\mathrm{in}_{\Theta^k X} = \Theta^k j_X.$$

This shows that $\Theta^\infty u_X = u_{\Theta^\infty X}$ or, in other words, $\Theta^\infty j_X = j_{\Theta^\infty X}$ by writing this as

$$\Theta^k i_{\Theta^\infty X} \circ \mathrm{in}_{\Theta^k} = \Theta^k j_X$$

and taking colimits in k—the left-hand side is $j_{\Theta^\infty X}$ and the right-hand side is $\Theta^\infty j_X$.

The upshot, or slogan, here is the following.

The transfinite composites j_X exhibit the same properties as the natural maps $i_{\Theta^k X}\colon \Theta^k X \to \Theta^{k+1} X$ that they are built out of, transfinitely.

To see that the associativity diagram for the asserted monad structure on Θ^∞ commutes, note that this amounts to showing that $j^{-1}_{(\Theta^\infty)^2 X} = \Theta^\infty j^{-1}_{\Theta^\infty X}$ or, in other words, $j_{(\Theta^\infty)^2 X} = \Theta^\infty j_{\Theta^\infty X}$. Hence, this follows from the above. The above discussion likewise verifies the unit condition.

To see that u is an S_*-isomorphism is relatively straightforward. It follows since, on the level of colimits, the unit diagram already commutes—this is essentially by definition, since $i_{\Theta^n X} = \Theta^n i_X$ by Lemma 3.2.6 and, as we have seen above, the following diagram commutes for all X

$$\begin{tikzcd}
\Theta(X) \wedge K \arrow[r] & \Theta(X \wedge K) \\
X \wedge K \arrow[u] \arrow[r, equal] & X \wedge K \arrow[u]
\end{tikzcd}$$

That μ is an S_*-natural transformation (isomorphism, even) is slightly more opaque. Since

$\mu = (\Theta^\infty j_X)^{-1}$ is an isomorphism, it is equivalent for us to show the following diagram commutes.

$$\begin{tikzcd}
(\Theta^\infty\Theta^\infty(X)) \wedge K \arrow[r, "\widetilde{\mathrm{assemb}}"] & \Theta^\infty(\Theta^\infty(X) \wedge K) \arrow[r, "\Theta^\infty\widetilde{\mathrm{assemb}}"] & \Theta^\infty\Theta^\infty(X \wedge K) \\
\Theta^\infty(X) \wedge K \arrow[u, "\Theta^\infty j_{X \wedge K}", "=j_{\Theta^\infty X \wedge K}"'] \arrow[rr, "\widetilde{\mathrm{assemb}}"'] & & \Theta^\infty(X \wedge K) \arrow[u, "=j_{\Theta^\infty(X \wedge K)}", "\Theta^\infty j_{X \wedge K}"']
\end{tikzcd}$$

On the level of underlying colimit sequences, the map $\Theta^\infty j_X \wedge K$ is induced by the maps $(\Theta^k j_X) \wedge K$. Thus, the following solid diagram then commutes

$$\begin{tikzcd}
\Theta^k(\Theta^\infty(X)) \wedge K \arrow[r, "\widetilde{\mathrm{assemb}}"] & \Theta^k(\Theta^\infty(X) \wedge K) \arrow[r, "\Theta^k\widetilde{\mathrm{assemb}}"] & \Theta^k(\Theta^\infty(X \wedge K)) \\
\Theta^k X \wedge K \arrow[u, "(\Theta^k j_X)\wedge K"] \arrow[r, "\widetilde{\mathrm{assemb}}"] & \Theta^k(X \wedge K) \arrow[u, "\Theta^k(j_X \wedge K)"'] \arrow[ur, dotted, "\Theta^k j_{X \wedge K}"'] &
\end{tikzcd}$$

Upon taking colimits, the outside part of this diagram induces the associativity diagram we wish to show commutes. Since the unit condition is an S_*-enriched functor, this implies that $\widetilde{\mathrm{assemb}} \circ (j_X \wedge K) = j_{X \wedge K}$. Hence, the dotted arrow does indeed make this diagram commute, whence the conclusion. $\square$

Remark 3.3.7. In fact, the functors Θ^k are S_*-enriched functors for essentially the very same reason. The adjoint assembly maps $\Theta^k\mathbf{hom}(K, X) \to \mathbf{hom}(K, \Theta^k X)$ are given as follows. First, note that there is a natural assembly isomorphism $s^k_-(X) \wedge K \cong s^k_-(X \wedge K)$, as the smash product is defined levelwise—in fact, this map is simply an equality. Hence, there is a natural assembly isomorphism $s^k_-\mathbf{hom}(K, X) \cong \mathbf{hom}(K, s^k_- X)$ defined as the map adjoint to

$$s^k_-\mathbf{hom}(K, X) \wedge K \cong s^k_-(\mathbf{hom}(K, X) \wedge K) \xrightarrow{s^k_-\varepsilon} s^k_- X,$$

and this is easily seen to be an isomorphism—an equality, even. The adjoint assembly map is then simply the composite of natural isomorphisms

$$\Theta^k \mathbf{hom}(K, X) = \Omega^k \mathbf{hom}(K, s_-^k X) \cong \mathbf{hom}(S^k \wedge K, s_-^k X) \cong \mathbf{hom}(K, \Theta^k X),$$

using the twist isomorphism and the adjoint associativity isomorphism. We will say more about this associativity isomorphism in a remark following the theorem statement below and in Remark 3.3.11.

Theorem 3.3.8. *Let C be a locally finitely presentable, finitely generated, pointed simplicial model category in which the domains of the generating cofibrations are cofibrant, sequential colimits preserve finite products and Ω preserves sequential colimits. Suppose the domains and codomains of the generating acyclic cofibrations are simplicially compact. There exists a level fibrant, pointed simplicial fibrant replacement monad L on $\mathsf{Sp}^{\mathbf{N}}(\mathsf{C})$ such that $\Theta^\infty \circ L$ is a pointed simplicial, stable fibrant replacement monad.*

Remark 3.3.9. Throughout the proof, we will implicitly identify the k-fold application of the functor Ω—namely, Ω^k—with the functor $\mathbf{hom}(S^k, -)$ under the natural associativity isomorphism

$$\alpha\colon \mathbf{hom}(K \wedge L, X) \xrightarrow{\cong} \mathbf{hom}(K, \mathbf{hom}(L, X)).$$

This is the map adjoint to

$$X^{K \wedge L} \otimes K \otimes L \xrightarrow{\alpha} X^{K \wedge L} \otimes K \wedge L \xrightarrow{\varepsilon} X.$$

We are at liberty to do this because for an S_*-enriched functor the adjoint assembly maps are compatible with α. While this adjoint associativity should be easy to believe, we will nevertheless give an extended discussion on this point in Remark 3.3.11 following the proof.

Proof. Let R be an S_*-enriched fibrant replacement monad on C. In this case, we may assume R commutes with filtered colimits—this follows exactly as in [16, Lem. 1.3], except Garner's small object argument is run with respect to the generating acyclic cofibrations and we restrict to the map to the terminal (zero) object. Note that Garner's small object argument runs if the category is locally presentable, but the commutation with filtered colimits step only requires the domains and codomains of the generating sets of cofibrations to be simplicially compact. Let L be the prolongation of R. Then L also commutes with filtered colimits since these are computed objectwise. As we have seen in Theorem 3.3.1, L remains an S_*-enriched level fibrant replacement monad on the category of spectra. Hence, $\Theta^\infty L$ and $L\Theta^\infty$ are S_*-enriched functors.

In particular, we claim there is an S_*-natural transformation $L\Theta^\infty \to \Theta^\infty L$. To see this, note that since L is S_*-enriched, it as has natural assembly maps

$$L(\mathbf{hom}(K,X)) \xrightarrow{\widetilde{\mathrm{assemb}}_{K,X}} \mathbf{hom}(K,LX).$$

Note that $\Theta^\infty = \operatorname{colim}\Theta^k$ where $\Theta^k = s_-^k\Omega^k = \Omega^k s_-^k$. The functor s_-^k shifts a spectrum by $(s_-^k X)_n = X_{n+k}$, and this functor commutes with all limits and colimits, as these are computed objectwise. Hence, $Ls_-^k = s_-^k L$, since L is a functor built out of various colimits, all of which s_-^k naturally commutes with. We define the map $L\Theta^\infty X \to \Theta^\infty LX$ as follows. First note that

$L\Theta^\infty X \cong \operatorname{colim}_k L\Theta^k X$ since L commutes with filtered colimits. In particular,

$$\operatorname{colim}_k L\Theta^k X = \operatorname{colim}_k Ls_-^k \Omega^k X = \operatorname{colim}_k s_-^k L\Omega^k X.$$

Using the assembly map, we obtain a map $s_-^k L\Omega^k X \to s_-^k \Omega^k LX = \Theta^k LX$. These maps commute with the structure maps of the colimit by naturality of $\widetilde{\text{assemb}}$ and the manner in which the adjoint structure maps $LX_n \to \Omega LX_{n+1}$ structure maps are defined for L—namely, this map is the composite

$$LX_n \to L\Omega X_{n+1} \xrightarrow{\widetilde{\text{assemb}}} \Omega LX_{n+1},$$

and the relevant commutative diagram is

$$\begin{tikzcd}
L\Omega^k\Omega X_{n+1} \arrow[r, "\widetilde{\text{assemb}}"] & \Omega^k L\Omega X_{n+1} \arrow[r, "\Omega^k\widetilde{\text{assemb}}"] & \Omega^{k+1}LX_{n+1} \\
L\Omega^k X_n \arrow[u, "L\Omega^k\sigma_n^X"] \arrow[r, "\widetilde{\text{assemb}}"] & \Omega^k LX_n \arrow[u, "\Omega^k L\sigma_n^X"'] \arrow[ur, "\Omega^k\sigma_n^{LX}"'] &
\end{tikzcd}$$

A separate analysis is needed for $k = 0$, in which case the bottom map above is the identity. Commutativity of the square when $k = 0$ follows since L is the prolongation of an S_*-enriched functor on C. In particular, this first square is given levelwise by

$$\begin{tikzcd}
LX_n \arrow[r, "L\sigma_n^X"] \arrow[d, equal] & L\Omega X_{n+1} \arrow[d, "\widetilde{\text{assemb}}"] \\
LX_n \arrow[r, "\sigma_n^{LX}"'] & \Omega LX_{n+1}
\end{tikzcd}$$

and this diagram commutes precisely because of how the structure maps for the prolongation of L are defined.

In particular, in the colimit, this gives us a natural assembly map

$$\widetilde{\text{assemb}}^{\infty} = \text{swap}\colon L\Theta^{\infty}X \to \Theta^{\infty}LX.$$

Now we must show that this assembly map is S_*-natural in X.

Recall that Θ^k is an S_*-enriched functor as well. Since the swap map for $L\Theta^{\infty}$ is built from the swap maps $L\Theta^k$ under a colimit, the easiest way to see that swap: $L\Theta^{\infty} \to \Theta^{\infty}L$ is S_*-natural is to check that it is S_*-natural for the assembly maps $L\Theta^k \to \Theta^k L$, since it then follows for swap upon taking colimits. To check compatibility with the S_*-module adjoint assembly maps for $L\Theta^k$ and $\Theta^k L$, since s^k_- pulls out of each, this amounts to showing the following diagram commutes

$$\begin{tikzcd}
L\Omega^k\mathbf{hom}(K,X) \arrow[r, "\widetilde{\text{assemb}}"] \arrow[d, "\cong"] & \Omega^k L\mathbf{hom}(K,X) \arrow[d, "\Omega^k\widetilde{\text{assemb}}"] \\
L\mathbf{hom}(K,\Omega^k X) \arrow[d, "\widetilde{\text{assemb}}"'] & \Omega^k\mathbf{hom}(K,LX) \arrow[d, "\cong"'] \\
\mathbf{hom}(K,L\Omega^k X) \arrow[r, "\mathbf{hom}(K{,}\widetilde{\text{assemb}})"'] & \mathbf{hom}(K,\Omega^k LX)
\end{tikzcd}$$

where the top and bottom horizontal maps are simply coming from the natural assembly map $L\Theta^k \to \Theta^k L$ and the columns are S_*-module adjoint assembly maps for these functors. Note that adjoint assembly map $\widetilde{\text{assemb}}$ for L is suitably associative—this is the adjoint property of associative on the level of the S_*-tensoring. More precisely, as s^k_- pulls out of everything and unpacking the assembly map for Ω^k, we may blow this diagram up to the following one.

$$\begin{tikzcd}
L\Omega^k(X^K) \arrow[r, "\widetilde{\mathrm{assemb}}"] \arrow[d, "L\alpha^{-1}"'] & \Omega^k L(X^K) \arrow[r, "\Omega^k \widetilde{\mathrm{assemb}}"] & \Omega^k(L(X)^K) \arrow[r, "\alpha^{-1}"] & L(X)^{S^k \wedge K} \arrow[r, "\tau^*"] & L(X)^{K \wedge S^k} \arrow[ddll, bend left, "\alpha"] \\
L(X^{S^k \wedge K}) \arrow[r, "L\tau^*"'] \arrow[d, "\widetilde{\mathrm{assemb}}"'] \arrow[urrr, "\widetilde{\mathrm{assemb}}"] & L(X^{K \wedge S^k}) \arrow[r, "L\alpha"'] \arrow[d, "\widetilde{\mathrm{assemb}}"] \arrow[urrr, "\widetilde{\mathrm{assemb}}"'] & L((\Omega^k X)^K) \arrow[r, "\widetilde{\mathrm{assemb}}"] & (L\Omega^k X)^K \arrow[dl, "\widetilde{\mathrm{assemb}}^K"'] & \\
L(X)^{S^k \wedge K} \arrow[r, "\tau^*"'] & L(X)^{K \wedge S^k} \arrow[r, "\alpha"'] & (\Omega^k L(X))^K & &
\end{tikzcd}$$

Each subdiagram here commutes by naturality of the adjoint assembly map $\widetilde{\mathrm{assemb}}$ for L or the fact that L is an S_*-enriched functor, using the adjoint associativity condition for $\widetilde{\mathrm{assemb}}$.

Give $\Theta^\infty L$ the S_*-enriched monad structure derived from the composition of units maps $X \to LX \to \Theta^\infty LX$ and with multiplication $\mu\colon \Theta^\infty L \Theta^\infty L \to \Theta^\infty L$ using the swap S_*-natural transformation just constructed. If this is indeed a monad, then it is an S_*-monad as all natural transformations appearing in its unit and multiplication maps are S_*-natural. It is now a straightforward—if not tedious—exercise to show this natural swap map is a distributive law as in [6] from the properties of the various assembly maps $\widetilde{\mathrm{assemb}}$ of L and thus that $\Theta^\infty L$ is an S_*-enriched monad. For the sake of completeness, we conclude with this argument.

For $L^2\Theta^\infty$, we wish for the diagram

$$\begin{tikzcd}
L^2\Theta^\infty \arrow[r, "\mathrm{swap}"] \arrow[rr, bend left, "\mathrm{swap}"] \arrow[d, "\mu^L\Theta^\infty"'] & L\Theta^\infty L \arrow[r, "\mathrm{swap}"] & \Theta^k L^2 \arrow[d, "\Theta^\infty \mu^L"] \\
L\Theta^\infty \arrow[rr, "\mathrm{swap}"'] & & \Theta^\infty L
\end{tikzcd}$$

to commute, and this follows since for each k, the underlying diagrams

$$\begin{tikzcd}
L^2\Theta^k \arrow[r, "\text{swap}"] \arrow[rr, bend left, "\text{swap}"] \arrow[d, "\mu^L\Theta^k"'] & L\Theta^k L \arrow[r, "\text{swap}"] & \Theta^k L^2 \arrow[d, "\Theta^k\mu^L"] \\
L\Theta^k \arrow[rr, "\text{swap}"'] & & \Theta^k L
\end{tikzcd}$$

commute. To see this, note that since $\Theta^k = s_-^k \Omega_k$, we may write this as s_-^k applied to the following diagram

$$\begin{tikzcd}
L^2\Omega^k \arrow[r, "\text{swap}"] \arrow[rr, bend left, "\text{swap}"] \arrow[d, "\mu^L\Omega^k"'] & L\Omega^k L \arrow[r, "\text{swap}"] & \Omega^k L^2 \arrow[d, "\Omega^k\mu^L"] \\
L\Omega^k \arrow[rr, "\text{swap}"'] & & \Omega^k L
\end{tikzcd}$$

and this commutes since μ^L is an S_*-natural transformation. Both unit maps are transfinite composites, so similar reasoning shows compatibility of the units. For the unit for Θ^∞, the following diagram commutes

$$\begin{tikzcd}
& LX \arrow[dl, "Lj_X"'] \arrow[dr, "j_{LX}"] & \\
L\Theta^\infty X \arrow[rr, "\text{swap}"'] & & \Theta^\infty LX
\end{tikzcd}$$

for the following reason. Since L is commutes with colimits, being built out of various colimits, the following is a colimiting diagram for $L\Theta^\infty X$, displaying only the first structure map

$$\begin{tikzcd}
LX \arrow[r, "Li_X"] \arrow[rrr, bend left, "Lj_X"] & L\Theta X \arrow[r, "L\Theta i_X"] & \cdots & L\Theta^\infty X
\end{tikzcd}$$

and the following diagram commutes from what we saw before in our analysis of the swap map.

$$
\begin{tikzcd}
LX \arrow[r, "Li_X"] \arrow[d, equal] \arrow[rrr, bend left, "Lj_X"] & L\Theta X \arrow[r, "Li_{\Theta X}"] \arrow[d, "\widetilde{\text{assemb}}"] & \cdots \arrow[d, "\widetilde{\text{assemb}}"] & L\Theta^\infty X \arrow[d, "\text{swap}"] \\
LX \arrow[r, "i_{LX}"'] \arrow[rrr, bend right, "j_{LX}"'] & \Theta LX \arrow[r, "i_{\Theta LX}"] & \cdots & \Theta^\infty LX
\end{tikzcd}
$$

To see the following diagram commutes

$$
\begin{tikzcd}
 & \Theta^\infty X \arrow[dl, "u^L_{\Theta^\infty X}"'] \arrow[dr, "\Theta^\infty u^L_X"] & \\
L\Theta^\infty X \arrow[rr, "\text{swap}"'] & & \Theta^\infty LX
\end{tikzcd}
$$

note that since L commutes with colimits, the left-hand map is in fact the colimit of the following commuting ladder of maps, we claim.

$$
\begin{tikzcd}
X \arrow[r, "i_X"] \arrow[d, "u^L_X"] \arrow[rrrr, bend left, "\text{in}_X"] & \Theta X \arrow[r, "i_{\Theta X}"] \arrow[d, "u^L_{\Theta X}"] \arrow[rrr, bend left, "\text{in}_{\Theta X}"] & \Theta^2 X \arrow[r] \arrow[d, "u^L_{\Theta^2 X}"] & \cdots \arrow[d] & \Theta^\infty X \arrow[d, dotted, "u^L_{\Theta^\infty X}"] \\
LX \arrow[r, "Li_X"'] \arrow[rrrr, bend right] & L\Theta X \arrow[r, "Li_{\Theta X}"'] \arrow[rrr, bend right] & \Theta^2 X \arrow[r] & \cdots & L\Theta^\infty X
\end{tikzcd}
$$

The squares all commutes by naturality of u^L. To show that $u^L_{\Theta^\infty X}$ is the colimiting map, we note that by universal properties there is only one such map making the evident diagram commute, so it suffices to show that the following diagram commutes for all $k \geq 0$

$$\begin{tikzcd}
\Theta^k X \arrow[r, "\mathrm{in}_{\Theta^k X}"] \arrow[d, "u^L_{\Theta^k X}"] & \Theta^\infty X \arrow[d, "u^L_{\Theta^\infty X}"] \\
L\Theta^k X \arrow[r, "L\,\mathrm{in}_{\Theta^k X}"'] & L\Theta^\infty X
\end{tikzcd}$$

and this follows precisely from naturality of u^L. Hence, $u^L_{\Theta^\infty X}$ is the colimit of the unit maps $u^L_{\Theta^k X}$. As before, this reduces us to checking commutativity at the level of the underlying diagrams defining various maps. But then, since $u^L\colon \mathrm{id} \to L$ is, in particular, an S_*-natural transformation, the following diagram commutes by pulling out copies of s^k_-

$$\begin{tikzcd}
\Theta^k X \arrow[r, equal] \arrow[d, "u^L_{\Theta^k X}"'] & \Theta^k X \arrow[d, "\Theta^k u^L_X"] \\
L\Theta^k X \arrow[r, "\widetilde{\mathrm{assemb}}"'] & \Theta^k L X
\end{tikzcd}$$

which shows that the composite

$$\Theta^k X \xrightarrow{u^L_{\Theta^k X}} L\Theta^k X \xrightarrow{\widetilde{\mathrm{assemb}}} \Theta^k L X \tag{$*$}$$

is simply $\Theta^k u^L_X$. the composite map $\Theta^\infty X \to L\Theta^\infty X \to \Theta^\infty L X$ is induced by $(*)$, and since the composite $(*)$ is simply $\Theta^k u^L_X$, it follows by taking the colimit in k the resulting map is precisely $\Theta^\infty u^L_X$, which is what was to be shown.

Recall that

$$\mu^{\Theta^\infty}_X = j^{-1}_{\Theta^\infty X} = (\Theta^\infty j_X)^{-1}.$$

Thus, for the last diagram, it is equivalent to show the following diagram commutes

$$\begin{tikzcd}
L(\Theta^\infty)^2 \arrow[r, "\text{swap}"] & \Theta^\infty L\Theta^\infty \arrow[r, "\text{swap}"] & (\Theta^\infty)^2 L \\
L\Theta^\infty \arrow[u, "Lj_{\Theta^\infty}"] \arrow[rr, "\text{swap}"] & & \Theta^\infty L \arrow[u, "j_{\Theta^\infty L}"', "=\Theta^\infty j_L"]
\end{tikzcd}$$

The map $Lj_{\Theta^\infty X} = L\Theta^\infty j_X$ is induced by the maps $L\Theta^k j_X \colon L\Theta^k X \to L\Theta^k\Theta^\infty X$ and the map $\Theta^\infty j_L$ is induced by the maps $\Theta^k j_{LX} \colon \Theta^k LX \to \Theta^k\Theta^\infty LX$. However, we saw that $j_{LX} = \operatorname{swap} \circ Lj_X$ in the course of this proof. Hence,

$$\Theta^k j_{LX} = \Theta^k \operatorname{swap} \circ \Theta^k Lj_X.$$

We therefore have the following commutative diagram

$$\begin{tikzcd}
L\Theta^k\Theta^\infty X \arrow[r, "\text{swap}"] & \Theta^k L\Theta^\infty X \arrow[r, "\Theta^k \text{swap}"] & \Theta^k\Theta^\infty LX \\
L\Theta^k X \arrow[u, "L\Theta^k j_X"] \arrow[r, "\text{swap}"'] & \Theta^k LX \arrow[u, "\Theta^k Lj_X"] \arrow[ur, "\Theta^k j_{LX}"'] &
\end{tikzcd}$$

and this shows that, on the level of the underlying colimiting sequences, the two composites are equal—hence, the diagram commutes, as desired. □

Remark 3.3.10. What is remarkable is that this composite is therefore a pointed simplicial, stable fibrant replacement monad for the stable semi-model structure on $\mathsf{Sp}^{\mathbf{N}}(\mathsf{C})$; that is, an honest, enriched, spectrification monad for the stable semi-model structure—we are not guaranteed that this exists in general when we do left Bousfield localization of a semi-model structure. This heavily exploits the fact that we may suppose L is an S_*-enriched, level fibrant replacement monad that is *prolonged* from one on C as well as the particularly simple and amenable description of Θ^∞. The only downside is that we cannot possibly expect the unit map $X \to \Theta^\infty LX$ to be a cofibration, in general.

It was convenient for us to use a slightly different notion of an S_*-enriched functor above. It is worthwhile, albeit quite tedious, to say a few words about this.

Remark 3.3.11. Consider $F\colon \mathsf{C} \to \mathsf{C}$ some S_*-enriched endofunctor such as L or Θ^∞ above—while the restriction that the domain and codomain of F be the same is not necessary, it simplifies our discussion. Let $\alpha\colon (X \otimes K) \otimes L \to X \otimes (K \wedge L)$ be the associativity isomorphism for the simplicial tensor on C.

We can define the adjoint associativity natural transformation $\alpha\colon X^{K\wedge L} \to (X^L)^K$ to be the adjoint to

$$(X^{K\wedge L} \otimes K) \otimes L \xrightarrow{\alpha} X^{K\wedge L} \otimes (K \wedge L) \xrightarrow{\varepsilon} X.$$

Namely, the map

$$\begin{tikzcd}
((X^{K\wedge L} \otimes K \otimes L)^L)^K \arrow[r, "(\alpha^L)^K"] & ((X^{K\wedge L} \otimes K \wedge L)^L)^K \arrow[r, "(\varepsilon^L)^K"] & (X^L)^K \\
(X^{K\wedge L} \otimes K)^K \arrow[u, "\eta^K"] & & \\
X^{K\wedge L} \arrow[u, "\eta"] \arrow[uurr] & &
\end{tikzcd}$$

This map is inverse to the map $\beta\colon (X^L)^K \to X^{K\wedge L}$ adjoint to

$$(X^L)^K \otimes (K \wedge L) \xrightarrow{\alpha^{-1}} (X^L)^K \otimes K \otimes L \xrightarrow{\varepsilon \otimes L} X^L \otimes L \xrightarrow{\varepsilon} X.$$

In particular, this means it is the composite

$$\begin{tikzcd}
(X^L)^K \arrow[r, "\eta"] \arrow[ddr] & ((X^L)^K \otimes (K \wedge L))^{K\wedge L} \arrow[r, "(\alpha^{-1})^{K\wedge L}"] & ((X^L)^K \otimes K \otimes L)^{K\wedge L} \arrow[d, "(\varepsilon\otimes L)^{K\wedge L}"] \\
& & (X^L \otimes L)^{K\wedge L} \arrow[d, "\varepsilon^{K\wedge L}"] \\
& & X^{K\wedge L}
\end{tikzcd}$$

Consider the composite $\alpha\beta\colon (X^L)^K \to (X^L)^K$. This is the identity by naturality of α, η, ε and the triangle identities. Concisely and with somewhat imprecise labeling, the crucial point is that the following diagram commutes.

$$\begin{tikzcd}
& (X^L)^K \arrow[r, "\eta"] \arrow[d] & ((X^L)^K \otimes K \wedge L)^{K\wedge L} \arrow[d] \\
& (((X^L)^K \otimes K \otimes L)^L)^K \arrow[r, "\eta"] \arrow[d, "\alpha^{-1}"] \arrow[ddl, equal] & ((((X^L)^K \otimes K \wedge L)^{K\wedge L} \otimes K \otimes L)^L)^K \arrow[d, "\alpha^{-1}"] \\
& (((X^L)^K \otimes K \wedge L)^L)^K \arrow[r, "\eta"] \arrow[dr, equal] & ((((X^L)^K \otimes K \wedge L)^{K\wedge L} \otimes K \wedge L)^L)^K \arrow[d, "\varepsilon"] \\
(((X^L)^K \otimes K \otimes L)^L)^K & & ((X^L)^K \otimes K \wedge L)^L)^K \arrow[ll, "\alpha"]
\end{tikzcd}$$

The first vertical arrows are the evident composites involving the unit of the adjunction. That the composite $\beta\alpha$ is the identity follows from similar considerations.

Now consider the endofunctor F above. The unit condition is simpler than the associativity condition. If the unit natural isomorphism is $\rho\colon X \otimes S^0 \to X$, let $\rho\colon X \to X^{S^0}$ be the adjoint natural isomorphism—it remains a natural isomorphism because it is inverse to the map $\varepsilon \circ \rho^{-1}\colon X^{S^0} \to X$.

Write the adjoint unit diagram

$$\begin{tikzcd}
FX \arrow[r, "F(\rho)"] \arrow[d, equal] & F(X^{S^0}) \arrow[d, "\widetilde{\text{assemb}}"] \\
FX \arrow[r, "\rho"'] & F(X)^{S^0}
\end{tikzcd}$$

and note that on account of how the assembly map is itself suitably adjoint to the adjoint assembly map as in the proof of Theorem 3.3.1, the adjoint of this diagram is, slightly blown up,

$$\begin{tikzcd}
F(X \wedge S^0) \arrow[rr, bend left, "F(\rho \wedge S^0)"] & & \\
F(X) \wedge S^0 \arrow[u, "\text{assemb}"] \arrow[r, "F(\rho) \wedge S^0"] \arrow[d, equal] & F(X^{S^0}) \wedge S^0 \arrow[r, "\text{assemb}"] & F(X^{S^0} \wedge S^0) \arrow[d, "F(\varepsilon)"] \\
F(X) \wedge S^0 \arrow[rr, "\rho"'] & & F(X)
\end{tikzcd}$$

and $\varepsilon \circ \rho \wedge S^0 = \rho$, essentially by definition. Hence, this diagram commutes since F is an S_*-enriched functor.

The adjoint associativity condition for the functor F is to require the following diagram commutes

$$\begin{tikzcd}
F(X^{K \wedge L}) \arrow[r, "F\alpha"] \arrow[d, "\widetilde{\text{assemb}}"'] & F((X^L)^K) \arrow[r, "\widetilde{\text{assemb}}"] & F(X^L)^K \arrow[d, "\widetilde{\text{assemb}}^K"] \\
F(X)^{K \wedge L} \arrow[rr, "\alpha"'] & & (F(X)^L)^K
\end{tikzcd}$$

Adjointing the counterclockwise composite and using the definition of α and $\widetilde{\text{assemb}}$ as adjoints, the resulting composite augments to a commutative diagram

$$\begin{tikzcd}
F(X^{K\wedge L})\otimes K\otimes L \arrow[r, "\alpha"] \arrow[d, "\widetilde{\mathrm{assemb}}\otimes K\otimes L"] & F(X^{K\wedge L})\otimes K\wedge L \arrow[r, "\mathrm{assemb}"] \arrow[d, "\widetilde{\mathrm{assemb}}\otimes K\wedge L"] & F(X^{K\wedge L}\otimes K\wedge L) \arrow[d, "F(\varepsilon)"] \\
F(X)^{K\wedge L}\otimes K\otimes L \arrow[r, "\alpha"'] & F(X)^{K\wedge L}\otimes K\wedge L \arrow[r, "\varepsilon"'] & F(X)
\end{tikzcd}$$

To see that the right-hand square commutes, note that the clockwise composite is the adjoint of $\widetilde{\mathrm{assemb}}\colon F(X^{K\wedge L})\to F(X)^{K\wedge L}$—the counterclockwise composite has the same adjoint by virtue of the triangle identities—this is completely analogous to the verification of $(**)$ in the proof of Theorem 3.3.1. Similarly, we may augment the adjoint of the clockwise composite as the following solid commutative diagram.

$$\begin{tikzcd}
F(X^{K\wedge L})\otimes K\otimes L \arrow[r, "F(\alpha)\otimes K\otimes L"] \arrow[d, "\mathrm{assemb}"'] & F((X^L)^K)\otimes K\otimes L \arrow[r, "\widetilde{\mathrm{assemb}}\otimes K\otimes L"] \arrow[d, "\mathrm{assemb}"] & F(X^L)^K\otimes K\otimes L \arrow[d, "\varepsilon\otimes L"] & \\
F(X^{K\wedge L}\otimes K)\otimes L \arrow[r, "F(\alpha\otimes K)\otimes L"] \arrow[d, "\mathrm{assemb}"'] & F((X^L)^K\otimes K)\otimes L \arrow[r, "F(\varepsilon)\otimes L"] \arrow[d, "\mathrm{assemb}"] & F(X^L)\otimes L \arrow[r, "\mathrm{assemb}"] \arrow[d, "\widetilde{\mathrm{assemb}}\otimes L"] & F(X^L\otimes L) \arrow[d, "F(\varepsilon)"] \\
F(X^{K\wedge L}\otimes K\otimes L) \arrow[r] & F((X^L)^K\otimes K\otimes L) \arrow[urr] & F(X)^L\otimes L \arrow[r, "\varepsilon"] & F(X)
\end{tikzcd}$$

and the right-most square on top commutes because the counterclockwise composite is the adjoint of $\widetilde{\mathrm{assemb}}$ tensored with L and the clockwise composite is the very same since $\varepsilon\circ\widetilde{\mathrm{assemb}}$ is the adjoint of assembly map composed with F of the counit from what we saw in Theorem 3.3.1. The same reasoning shows the last big square on the bottom commutes. The quadrilateral commutes by naturality of the assembly maps.

Consider the counterclockwise, outside composite above and call it $(\star)$. Since F is an S_*-enriched functor, the left-hand column of $(\star)$ fits into the associativity diagram

$$\begin{tikzcd}
F(X^{K\wedge L}) \otimes K \otimes L \arrow[r, "\alpha"] \arrow[d, "\mathrm{assemb}"] & F(X^{K\wedge L}) \otimes K \wedge L \arrow[dd, "\mathrm{assemb}"] \\
F(X^{K\wedge L} \otimes K) \otimes L \arrow[d, "\mathrm{assemb}"] & \\
F(X^{K\wedge L} \otimes K \otimes L) \arrow[r, "F(\alpha)"] & F(X^{K\wedge L} \otimes K \wedge L)
\end{tikzcd}$$

We wish to show the adjoint map now under consideration is equal to $F(\varepsilon) \circ \mathrm{assemb} \circ \alpha$. From the counterclockwise direction in the associativity diagram above, it now suffices to show that $\varepsilon \circ \alpha = \varepsilon \circ \varepsilon \otimes L \circ \alpha \otimes K \otimes L$, we claim.

To see this, suppose it is so. Then by applying F, it follows that $(\star)$ is equal to the counterclockwise composite of the associativity diagram above post-composed with $F(\varepsilon)$. Hence, by commutativity of this diagram, $(\star)$ is equal to $F(\varepsilon) \circ \mathrm{assemb} \circ \alpha$, as desired.

The adjoint of the left-hand side is the adjoint associativity isomorphism, by definition. The adjoint of the right-hand side is the very same, it happens. This follows because, upon taking the adjoint, we get a commutative diagram

$$\begin{tikzcd}
X^{K\wedge L} \arrow[r, "\eta"] \arrow[d, "\alpha"] & (X^{K\wedge L} \otimes K)^K \arrow[r, "\eta^K"] \arrow[d, "\alpha"] & (X^{K\wedge L} \otimes K \otimes L)^L)^K \arrow[d, "\alpha"] \\
(X^L)^K \arrow[r, "\eta"] \arrow[dr, equal] & ((X^L)^K \otimes K)^K \arrow[r, "\eta^K"] \arrow[d, "\varepsilon^K"] & (((X^L)^K \otimes K \otimes L)^L)^K \arrow[d, "(\varepsilon\otimes L)^L)^K"] \\
 & (X^L)^K \arrow[r, "\eta^K"'] \arrow[dr, equal] & ((X^L \otimes L)^L)^K \arrow[d, "(\varepsilon^L)^K"] \\
 & & (X^L)^K
\end{tikzcd}$$

which shows the adjoint is likewise α, as claimed.

Arguments similar in spirit to the above show that the adjoint condition to be an S_*-enriched natural transformations between S_*-enriched functors—namely, respect for the the adjoint assembly map—holds and is equivalent to the usual tensor condition, at least when the module category in question is both tensored and cotensored over S_*. At this point, we hope the reader has seen enough to believe the adjoint conditions are equivalent and omit a further discussion on this, leaving it as an easy exercise for the interested reader.

Chapter 4

Spectra of Retractive Operadic Algebras

We now specialize many of the preceding results to $\mathsf{Alg}_{\mathcal{O}}$. Along the way, we will pick up some nice facts about these categories of $\mathcal{O}$-algebras. This chapter, along with the preceding one, comprise one part of the technical heart of this paper. We work in the context of operads in symmetric spectra, using the framework of of [26].

4.1 Properties of the Category $\mathsf{Alg}_{\mathcal{O}}$

Rather than making certain constructions by hand, we will again refer to Definition 3.1.5 and Proposition 3.1.6, with which we can easily show that categories of $\mathcal{O}$-algebras in spectra are locally finitely presentable.

Proposition 4.1.1. *The categories Sp^{Σ} and $\mathsf{Alg}_{\mathcal{O}}$ are locally finitely presentable with strong sets* *of compact generators*

$$\mathcal{G}_\Sigma = \{F_n(\Delta[m]_+) : n, m \in \mathbf{N}\} \quad \text{and} \quad \mathcal{G}_\mathcal{O} = \{\mathcal{O}(F_n(\Delta[m]_+)) : n, m \in \mathbf{N}\},$$

respectively.

Proof. Since both categories are bicomplete, it is enough to show, in light of Proposition 3.1.6, that $\mathcal{G}_\Sigma = \{F_n(\Delta[m]_+) : n, m \in \mathbf{N}\}$ and $\mathcal{G}_\mathcal{O} = \{\mathcal{O}(F_n(\Delta[m]_+)) : n, m \in \mathbf{N}\}$ are strong generators of compact objects. It is clear that both sets consist of compact objects by an adjunction argument, so we need only show that the given sets are strong generators for the categories.

Note that in any category with pullbacks, to call a map $i \colon X_0 \to X$ a monomorphism is precisely the same as saying that the following square is a pullback

$$\begin{tikzcd} X_0 \arrow[r, "\mathrm{id}_X"] \arrow[d, "\mathrm{id}_X"'] & X_0 \arrow[d, "i"] \\ X_0 \arrow[r, "i"'] & X \end{tikzcd}$$

Since limits in $\mathsf{Alg}_\mathcal{O}$ are created in Sp^Σ under forgetful functors U, the forgetful functor preserves and reflects monomorphisms. Hence, if $i \colon X_0 \to X$ is a proper subobject in $\mathsf{Alg}_\mathcal{O}$, then so too is $Ui \colon UX_0 \to UX$ in Sp^Σ. If we know that $\mathcal{G}_\Sigma$ is a strong generating set, then we can find some map $f \colon F_n(\Delta[m]_+) \to UX$ for which there is no factorization

$$\begin{tikzcd} F_n(\Delta[m]_+) \arrow[r] \arrow[d, "\mathrm{id}"'] & UX_0 \arrow[d, tail, "Ui"] \\ F_n(\Delta[m]_+) \arrow[r, "f"'] & UX \end{tikzcd}$$

But then, by adjunction, there can be no factorization

$$\begin{tikzcd} \mathcal{O}(F_n(\Delta[m]_+)) \arrow[r] \arrow[d, "\mathrm{id}"'] & X_0 \arrow[d, tail, "i"] \\ \mathcal{O}(F_n(\Delta[m]_+)) \arrow[r, "f^\sharp"'] & X \end{tikzcd}$$

so it suffices to prove $\mathcal{G}_\Sigma$ is a strong generating set. For this, simply note that if $i\colon X_0 \to X$ is a proper subobject of a symmetric spectrum, then there is a simplex $x \in (X_n)_m$ not in the image of i for some $n, m \in \mathbf{N}$. It follows that the map $h\colon F_n(\Delta[m]_+) \to X$ specified by this simplex does not factor through i. Similarly, if $f, g\colon X_0 \to X$ are such that $f \neq g$, then there is a simplex $x \in (X_n)_m$ for some $n, m \in \mathbf{N}$ for which $f(x) \neq g(x)$ and, once again, letting $F_n(\Delta[m]_+) \to X$ be the map specified by this simplex shows that $fh \neq gh$. □

Remark 4.1.2. Alternatively, noting that $\mathsf{Alg}_{\mathcal{O}}$ is the category of algebras for the monad $U\mathcal{O} \circ (-)$ associated to the adjunction

$$\mathsf{Sp}^\Sigma \underset{U}{\overset{\mathcal{O}\circ(-)}{\rightleftarrows}} \mathsf{Alg}_{\mathcal{O}},$$

we have that $U\mathcal{O}\circ(-)$ preserves filtered colimits since $\mathcal{O}\circ(-)$ is a left adjoint and by [46, Prop. 2.3.5] and [15, Prop. 2.16], the forgetful functor creates filtered colimits, the monad $U\mathcal{O} \circ (-)$ preserves filtered colimits and thus, since Sp^Σ is locally finitely presentable, it follows by [10, Thm. 5.5.9] that $\mathsf{Alg}_{\mathcal{O}}$ is locally finitely presentable as well.

As is shown in [27, Thm. 6.18], the category $\mathsf{Alg}_{\mathcal{O}}$ is a cofibrantly generated simplicial model category in both the positive stable and positive flat stable model structures, building upon the work of Harper in [26]. Since every positive stable cofibration is a positive flat stable cofibration, we will focus on the latter. To do this, we first introduce some notation, following Harper in [26].

Definition 4.1.3. Let $\Sigma = \quad {}_{n\geq 0}\Sigma_n$. For each $m \geq 0$ and subgroup $H \leq \Sigma_m$, let $G^H_m \colon \mathsf{S}_* \to \mathsf{S}^\Sigma_*$ be the functor sending a pointed space X to the symmetric sequence concentrated in degree m as $\Sigma_m \cdot_H \underline{X}$ where $\underline{X}$ has the trivial H-action. Explicitly, this is given by

$$G^H_m(X)_n = \begin{cases} (\Sigma_m/H)_+ \wedge X & n = m, \\ * & \text{else.} \end{cases}$$

This functor is left adjoint to the functor

$$\mathrm{Ev}^H_m = \lim_H \circ \mathrm{Res}^{\Sigma_m}_H \circ \mathrm{Ev}_m \colon \mathsf{S}^\Sigma_* \to \mathsf{S}_*$$

sending a symmetric sequence $(X_0, X_1, \ldots)$ to $\mathrm{Ev}^H_m(X) = \lim_H \mathrm{Res}^{\Sigma_m}_H X_m$, which we will also denote by X^H_m.

The positive flat stable model structure has the following explicit description, due to Harper in [26], following the recipe given by Schwede and Shipley in [50] for the model structures established by Shipley in [49]. First, we give the positive flat stable model structure on symmetric spectra Sp^Σ.

Definition 4.1.4. The positive flat stable model structure on Sp^Σ is the cofibrantly generated, simplicial model category having as its weak equivalences the stable equivalences and generating cofibrations

$$I = \quad S \otimes G^H_m(\partial\Delta[k]_+) \to S \otimes G^H_m(\Delta[k]_+) : m \geq 1,\ k \geq 0,\ H \leq \Sigma_m \text{ a subgroup}$$

and generating acyclic cofibrations of two types $J = J_I \cup J_{II}$, where the ***type one*** maps are

$$J_I = \quad S \otimes G_m^H(\Lambda_r[k]_+) \to S \otimes G_m^H(\Delta[k]_+) : m \geq 1,\ r \geq 0,\ k \geq 0,\ H \leq \Sigma_m \text{ a subgroup}$$

and the ***type two*** maps are

$$J_{II} = \bigcup_{n \geq 1} K_n$$

where

$$K_n = \{c_n \square (\partial\Delta[k]_+ \to \Delta[k]_+) : k \geq 0\}.$$

Here, the square indicates the pushout product map and $c_n \colon F_{n+1}S^1 \to F_nS^0$ is the map obtained in the pushout

$$\begin{tikzcd}
F_{n+1}S^1 \wedge \Delta[0]_+ \arrow[r, "\lambda_n"] \arrow[d] & F_nS^0 \arrow[d, "c_n"] \\
F_{n+1}S^1 \wedge \Delta[1]_+ \arrow[r] & M\lambda_n
\end{tikzcd}$$

where $\lambda_n = \lambda \wedge \mathrm{id}_{F_nS^0} \colon F_{n+1}S^1 \to F_nS^0$ and λ is the map $F_1S^1 \to F_0S^0$ adjoint to the identity map id_{S^1}.

The category $\mathsf{Alg}_{\mathcal{O}}$ inherits a model structure from the positive flat stable model structure on Sp^{Σ} in the following way.

Definition 4.1.5. The ***positive flat stable model structure*** on $\mathsf{Alg}_{\mathcal{O}}$ is the cofibrantly generated, simplicial model category having as its weak equivalences and fibrations the underlying stable equivalences and fibrations in the positive flat stable model structure on Sp^{Σ}. Note that this model structure is *not* stable. This has generating cofibrations

$$I^{\mathcal{O}} = \mathcal{O}(I)$$

and generating acyclic cofibrations

$$J^{\mathcal{O}} = \mathcal{O}(J_I) \cup \mathcal{O}(J_{II}).$$

where I, J_I and J_{II} are the sets of maps defined in the preceding definition.

Let us declare $S \otimes G_n^H = F_n^H$.

Lemma 4.1.6. *For each $n \geq 0$ and $H \leq \Sigma_n$, the functor $S \otimes G_n^H = F_n^H \colon \mathsf{S}_* \to \mathsf{Sp}^\Sigma$ is a left Quillen functor where Sp^Σ is equipped with the positive flat stable model structure.*

Proof. From the analysis given in [26], the functor F_n^H is left adjoint to the functor sending a spectrum X to $X_n^H = \lim_H \operatorname{Res}_H^{\Sigma_n} X_n$, thinking of X_n as a functor $\Sigma_n \to \mathsf{S}_*$. Since S_* is cofibrantly generated, it suffices to show that F_n^H preserves the generating cofibrations and generating acyclic cofibrations—since every other cofibration is a retract of one built out of transfinite compositions of these—and this occurs essentially by definition. $\square$

This has the following consequence.

Lemma 4.1.7. *The domains of the generating cofibrations and generating acyclic cofibrations are cofibrant in $\mathsf{Alg}_{\mathcal{O}}$.*

Proof. Harper constructs this model structure in [26] according to the recipe provided by Schwede and Shipley in [50]. It follows that $\mathcal{O}$ is a left Quillen functor and so it suffices to show that the corresponding generating cofibrations and acyclic cofibrations in Sp^Σ in the positive flat stable model structure are cofibrant. Since F_n^H is a left Quillen functor, this is true for the generating acyclic cofibrations J_I and generating cofibrations I, so it remains to show that it the generating acyclic cofibration J_{II} have cofibrant domain.

For this, taking $H = e$ in the pushout defining $M\lambda_n$, it follows that c_n is a cofibration in the positive flat stable model structure on Sp^Σ. Since this model structure is simplicial, the map $F_n S^0 \wedge \partial\Delta[k]_+ \to F_n S^0 \wedge \Delta[k]_+$ for $n \geq 1$ is likewise a positive flat stable cofibration and hence the basechange map

$$M\lambda_n \wedge \partial\Delta[k]_+ \to F_n S^0 \wedge \Delta[k]_+ \coprod_{F_n S^0 \wedge \partial\Delta[k]_+} M\lambda_n \wedge \partial\Delta[k]_+$$

is a positive flat stable cofibration. The source is cofibrant since $M\lambda_n$ is and thus the target is cofibrant. But the target is the domain of the map $c_n \Box (\partial\Delta[k]_+ \to \Delta[k]_+)$, which shows that the domains of the generating acyclic cofibrations of the second type have cofibrant domains. $\square$

The generating cofibrations and acyclic cofibrations are likewise all compact. To prove this, the following two lemmas are needed.

Lemma 4.1.8. *Suppose X has finitely many non-degenerate simplices. Then X is compact in S. If X is additionally pointed, then X is compact in S_*.*

Proof. This follows easily by induction on the smallest integer n for which $X = \operatorname{sk}_n X$, using the skeletal filtration. $\square$

Lemma 4.1.9. *Let $m \geq 0$ and $H \leq \Sigma_m$ be a subgroup. The functor Ev_m^H commutes with filtered colimits.*

Proof. The functors $\mathrm{Ev}_m \colon \mathsf{S}_*^{\Sigma} \to \mathsf{S}_*^{\Sigma_m}$ and $\mathrm{Res}_H^{\Sigma_m} \colon \mathsf{S}_*^{\Sigma_m} \to \mathsf{S}_*^H$ create colimits, so we must check that $\lim_H \colon \mathsf{S}_*^H \to \mathsf{S}_*$ commutes with filtered colimits. This follows since H is a finite group and filtered colimits and finite limits commute in sets. $\square$

Proposition 4.1.10. *The category $\mathsf{Alg}_{\mathcal{O}}$ is finitely generated. In fact, the domains and codomains of the generating cofibrations and acyclic cofibrations are all compact objects.*

Remark 4.1.11. Since the forgetful functor creates filtered colimits by [46, Prop. 2.3.5] and [15, Prop. 2.16], it suffices to prove something slightly weaker—that $\mathcal{O} \circ (-)$ preserves compact objects. We will not explicitly make this reduction even though it is implicit in the following argument.

Proof. Fix a filtered diagram $G \colon \mathsf{D} \to \mathsf{Alg}_{\mathcal{O}}$. Let us consider the domains and codomains of generating cofibrations or generating acyclic cofibrations of the first type to begin with. An object of this sort has the form $\mathcal{O}(F_n^H(A))$. We then have adjunction isomorphisms $\operatorname{colim} \hom(\mathcal{O}(A), G) \cong \operatorname{colim} \hom(F_n^H(A), UG) \cong \operatorname{colim} \hom(A, (UG)_n^H)$. But now $A \in \mathsf{S}_*$ is a compact object and, hence, the natural map

$$\operatorname{colim} \hom(A, (UG)_n^H) \to \hom(A, \operatorname{colim}(\lim_H \mathrm{Res}_H^{\Sigma_n}(UG)_n))$$

is an isomorphism. But now finite limits and filtered colimits commute in the category of sets and, hence, also in the category of pointed sets. It follows easily that

$$\hom(A, \operatorname{colim}((UG)_n^H)) \cong \hom(A, (\operatorname{colim} UG)_n^H) \cong \hom(F_n^H(A), \operatorname{colim} UG).$$

Note that the natural map $\operatorname{colim} UG \to U \operatorname{colim} G$ is an isomorphism, since G is filtered and by [46, Prop. 2.3.5] and [15, Prop. 2.16], the forgetful functor creates filtered colimits. Hence,

$$\hom(F_n^H(A), \operatorname{colim} UG) \cong \hom(F_n^H(A), U \operatorname{colim} G) \cong \hom(\mathcal{O}(F_n^H(A)), \operatorname{colim} G).$$

Unraveling this amounts to the following commutative diagram

$$\begin{tikzcd}
\operatorname{colim} \hom(\mathcal{O}(F_n^H(A)), G) \arrow[r] \arrow[d, "\cong"'] & \hom(\mathcal{O}(F_n^H(A)), \operatorname{colim} G) \arrow[d, "\cong"] \\
\operatorname{colim} \hom(A, (UG)_n^H) \arrow[r, "\cong"'] & \hom(A, \operatorname{colim}((UG)_n^H))
\end{tikzcd}$$

which, since all marked arrows are isomorphism, so too must the top arrow be an isomorphism, which is the natural map we sought to show is an isomorphism.

It therefore remains to show that the domains and codomains of the generating acyclic cofibrations of type two $\mathcal{O}(J_{II})$ are compact. We will argue this in the case of the target, with the argument for the source having the same shape, mutatis-mutandis.

To see that targets $\mathcal{O}(M\lambda_n \wedge \Delta[k]_+)$ are compact, note that since $M\lambda_n$ is a pushout and since

$- \wedge \Delta[k]_+ = - \wedge F_0\Delta[k]_+$, we may write this object as the pushout

$$\begin{array}{ccc} F_{n+1}(S^1 \wedge \Delta[0]_+ \wedge \Delta[k]_+) & \longrightarrow & F_n(S^0 \wedge \Delta[k]_+) \\ \downarrow & & \downarrow \\ F_{n+1}(S^1 \wedge \Delta[1]_+ \wedge \Delta[k]_+) & \longrightarrow & M\lambda_n \wedge \Delta[k]_+ \end{array}$$

Since $S^1 \wedge \Delta[0]_+ \wedge \Delta[k]_+$, $S^1 \wedge \Delta[1]_+ \wedge \Delta[k]_+$ and $S^0 \wedge \Delta[k]_+$ have finitely many non-degenerate simplices, they are compact in S_*. By universal properties, it follows that $\hom(\mathcal{O}(M\lambda_n \wedge \Delta[k]_+), G) \cong \hom(M\lambda_n \wedge \Delta[k]_+, UG)$ is the pullback

$$\begin{array}{ccc} & & \hom(F_n(S^0 \wedge \Delta[k]_+, UG) \\ & & \downarrow \\ \hom(F_{n+1}(S^1 \wedge \Delta[1]_+ \wedge \Delta[k]_+), UG) & \longrightarrow & \hom(F_{n+1}(S^1 \wedge \Delta[0]_+ \wedge \Delta[k]_+), UG) \end{array}$$

Hence, since, as above, U creates filtered colimits, since finite limits and filtered colimits commute in the category of sets and since each object in the above fork is compact, $\operatorname{colim} \hom(\mathcal{O}(M\lambda_n \wedge \Delta[k]_+), G)$ is the pullback

$$\begin{array}{ccc} & & \hom(\mathcal{O}(F_n(S^0 \wedge \Delta[k]_+), \operatorname{colim} G) \\ & & \downarrow \\ \hom(\mathcal{O}(F_{n+1}(S^1 \wedge \Delta[1]_+ \wedge \Delta[k]_+)), \operatorname{colim} G) & \longrightarrow & \hom(\mathcal{O}(F_{n+1}(S^1 \wedge \Delta[0]_+ \wedge \Delta[k]_+)), \operatorname{colim} G) \end{array}$$

and so it that the natural map

$$\operatorname{colim} \hom(\mathcal{O}(M\lambda_n \wedge \Delta[k]_+), G) \to \hom(\mathcal{O}(M\lambda_n \wedge \Delta[k]_+), \operatorname{colim} G)$$

is an isomorphism. $\square$

Proposition 4.1.12. *Filtered colimits and finite limits commute in* $\mathsf{Alg}_{\mathcal{O}}$*.*

Proof. The forgetful functor creates both filtered colimits and limits in Sp^{Σ} where they commute.

$\square$

Proposition 4.1.13. *For any finite, pointed simplicial set* K*, the mapping object* $\mathbf{hom}_{\mathsf{Alg}_{\mathcal{O}}}(K,-)$ *preserves filtered colimits.*

Proof. Let $U\colon \mathsf{Alg}_{\mathcal{O}} \to \mathsf{Sp}^{\Sigma}$ be the forgetful functor. Here, the mapping object in $\mathsf{Alg}_{\mathcal{O}}$ is defined as in [27, §6.1]. Namely, $\mathbf{hom}_{\mathsf{Alg}_{\mathcal{O}}}(S^1, X)$ is the mapping symmetric spectrum $\mathbf{Hom}_S(\Sigma^\infty S^1, X)$ where the map $\mathcal{O} \circ (\mathbf{Hom}_S(\Sigma^\infty S^1, X)) \to \mathbf{Hom}_S(\Sigma^\infty S^1, X)$ is the map adjoint to the composite

$$\mathcal{O} \circ (\mathbf{Hom}_S(\Sigma^\infty S^1, X)) \wedge \Sigma^\infty S^1 \xrightarrow{\nu} \mathcal{O} \circ (\mathbf{Hom}_S(\Sigma^\infty S^1, X) \wedge \Sigma^\infty S^1) \xrightarrow{\mathcal{O}\circ(\mathrm{ev})} \mathcal{O} \circ (X) \xrightarrow{\mu_X} X.$$

However, $\mathbf{Hom}_S(\Sigma^\infty K, X) \cong \mathbf{hom}(K, X)$ natural in K and X, where $\mathbf{hom}(K,X)_n = \mathbf{hom}_{S_*}(K, X_n)$ with the evident Σ_n action. This is because a simple computation shows that, in symmetric spectra, the smash product $X \wedge \Sigma^\infty K \cong X \wedge K$ natural in X and K, where the right-hand is the pointed simplicial action. Hence, there are natural isomorphism

$$\hom(X, \mathbf{Hom}_S(\Sigma^\infty K, Y)) \cong \hom(X \wedge K, Y) \cong \hom(X, \mathbf{hom}(K, Y))$$

natural in X, K and Y. By Yoneda, this means there is a natural isomorphism $\mathbf{Hom}_S(\Sigma^\infty -, -) \cong \mathbf{hom}(-,-)$. Hence, $\mathbf{hom}_{\mathsf{Alg}_{\mathcal{O}}}(K, X)$ is the $\mathcal{O}$-algebra whose underlying spectrum is the sequence $(\mathbf{hom}_{\mathsf{S}_*}(K, X_0), \mathbf{hom}_{\mathsf{S}_*}(K, X_1), \ldots)$.

Since filtered colimits in $\mathsf{Alg}_{\mathcal{O}}$ are created in spectra and since $U\mathbf{hom}_{\mathsf{Alg}_{\mathcal{O}}} = \mathbf{hom}$, it suffices to show that $\mathbf{hom}(K, -)$ commutes with filtered colimits in spectra. This now follows since, in spectra, colimits are computed levelwise in S_* and

$$\mathbf{hom}_{\mathsf{S}_*}(K, \operatorname{colim}_i A_i) \cong \operatorname{colim}_i \mathbf{hom}_{\mathsf{S}_*}(K, A_i)$$

since $S^1 \wedge \Delta[n]_+$ is a compact object in S_* for each $n \geq 0$ by Lemma 4.1.8. In particular, this means that the natural map

$$\begin{aligned}&\operatorname{colim}_i(\mathbf{hom}_{\mathsf{S}_*}(K, A_i))_n = \operatorname{colim} \hom(K \wedge \Delta[n]_+, A_i)\\ &\to \hom(K \wedge \Delta[n]_+, \operatorname{colim} A_i) = (\mathbf{hom}_{\mathsf{S}_*}(K, \operatorname{colim} A_i))_n\end{aligned}$$

is an isomorphism for all n. $\square$

Proposition 4.1.14. *The domains and codomains of the generating cofibrations and generating acyclic cofibrations of* $\mathsf{Alg}_{\mathcal{O}}$ *are simplicially compact.*

Proof. We have seen they are compact, we now wish to show for any such domain or codomain, which therefore has the form $\mathcal{O}(X)$, $\mathcal{O}(X) \wedge \Delta[n]_+$ is a compact object. Harper and Hess endow $\mathsf{Alg}_{\mathcal{O}}$ with the structure of a simplicial model category in [27, Thm. 6.18]. The relevant constructions are as follows. The simplicial tensoring $X \otimes K$ is given by Harper and Hess [27, Def. 6.2] as the reflexive coequalizer (which is therefore computed in the underlying category of spectra)

$$\mathcal{O}(X) \otimes K = \operatorname{colim}(\mathcal{O}(\mathcal{O}(X) \wedge K_+) \underset{d_1}{\overset{d_0}{\leftleftarrows}} \mathcal{O}(\mathcal{O}(\mathcal{O}(X))) \wedge K_+)$$

The smash products appearing in this coproduct are the ones arising from the simplicial tensoring on the category of symmetric spectra as in [37]. The map d_1 is induced from the action on X and the map d_0 is induced by the operad multiplication map $\mu\colon \mathcal{O} \circ \mathcal{O} \to \mathcal{O}$ and the assembly map $\nu\colon \mathcal{O}(X) \wedge K_+ \to \mathcal{O}(X \wedge K_+)$ induced from the diagonal maps $K \to K^{\times n}$. The mapping object $\mathbf{hom}_{\mathsf{Alg}_{\mathcal{O}}}(K, X)$ has as its underlying spectrum $\mathbf{hom}_{\mathsf{Sp}^\Sigma}(K, X) = \mathbf{Hom}_{\mathsf{Sp}^\Sigma}(\Sigma^\infty K_+, X)$ with action map adjoint in Sp^Σ to the composite

$$\mathcal{O}(\mathbf{hom}(K, X)) \wedge K_+ \xrightarrow{\nu} \mathcal{O}(\mathbf{hom}(K, X) \wedge K_+) \xrightarrow{\mathcal{O}(\varepsilon)} \mathcal{O}(X) \xrightarrow{\mu_X} X.$$

We claim that for any X, $\mathcal{O}(X) \otimes K \cong \mathcal{O}(X \wedge K_+)$, natural in X and K. This is an adjunction argument.

$$\begin{aligned} &\hom_{\mathsf{Alg}_{\mathcal{O}}}(\mathcal{O}(X) \otimes K, Y) \cong \hom_{\mathsf{Alg}_{\mathcal{O}}}(\mathcal{O}(X), \mathbf{hom}_{\mathsf{Alg}_{\mathcal{O}}}(K, Y)) \cong \hom_{\mathsf{Sp}^\Sigma}(X, U\mathbf{hom}_{\mathsf{Alg}_{\mathcal{O}}}(K, Y)) \\ &= \hom_{\mathsf{Sp}^\Sigma}(X, \mathbf{hom}_{\mathsf{Sp}^\Sigma}(K, UY)) \cong \hom_{\mathsf{Alg}_{\mathcal{O}}}(X \wedge K_+, UY) \cong \hom_{\mathsf{Alg}_{\mathcal{O}}}(\mathcal{O}(X \wedge K_+), Y). \end{aligned}$$

All isomorphisms above are natural in X, K and Y from which we obtain an isomorphism $\mathcal{O}(X) \otimes K \to \mathcal{O}(X \wedge K_+)$. In particular, if $X \in \mathsf{Sp}^\Sigma$ is a compact object, then $X \wedge \Delta[n]_+$ is compact by an adjunction argument. Since the forgetful functor creates filtered colimits, $\mathcal{O}(X \wedge \Delta[n]_+)$ is a compact $\mathcal{O}$-algebra. It follows that the domains and codomains of the generating cofibrations and generating acyclic cofibrations are simplicially compact. $\square$

4.2 Properties of the Category $\mathsf{Alg}_{\mathcal{O}}^{Y}$

We use the preceding results to deduce properties of the category retractive $\mathcal{O}$-algebras.

Corollary 4.2.1. *The category $\mathsf{Alg}_{\mathcal{O}}^{Y}$ is locally finitely presentable and the domains of the generating cofibrations and acyclic cofibrations are cofibrant.*

Proof. $\mathsf{Alg}_{\mathcal{O}}$ is locally finitely presentable by Proposition 4.1.1 and Proposition 2.2.5. The statement about the domains and the generating sets follows from Proposition 2.2.6. $\square$

Corollary 4.2.2. *Filtered colimits and finite limits commute in $\mathsf{Alg}_{\mathcal{O}}^{Y}$ for any Y.*

Proof. Recall that this is true in $\mathsf{Alg}_{\mathcal{O}}^{Y}$ by Proposition 4.1.12. The forgetful functor $U\colon \mathsf{Alg}_{\mathcal{O}}^{Y} \to \mathsf{Alg}_{\mathcal{O}}$ creates filtered colimits. Let D be filtered and J be finite and consider a diagram $F\colon \mathsf{D} \times \mathsf{J} \to \mathsf{Alg}_{\mathcal{O}}^{Y}$. Let $F^{\triangleright}\colon \mathsf{D} \times \mathsf{J}^{\triangleright} \to \mathsf{Alg}_{\mathcal{O}}^{Y}$ be the evident extension as in Definition 2.1.11. Then $\lim_j F^{\triangleright}(-,j) \cong \lim_j F(-,j)$. Since the forgetful functor $U\colon \mathsf{Alg}_{\mathcal{O}}^{Y} \to \mathsf{Alg}_{\mathcal{O}}$ creates limits of shape $\mathsf{J}^{\triangleright}$ and colimits of shape D, where they commute, it follows that filtered colimits and finite limits commutes in $\mathsf{Alg}_{\mathcal{O}}^{Y}$. $\square$

Corollary 4.2.3. *For any finite, pointed simplicial set K, the mapping object $\mathbf{hom}_{\mathsf{Alg}_{\mathcal{O}}^{Y}}(K,-)$ preserves filtered colimits.*

Proof. According to our recipe for building $\mathbf{hom}_{\mathsf{Alg}_{\mathcal{O}}^{Y}}(K,-)$, it is enough to show that $\mathbf{hom}_{\mathsf{Alg}_{\mathcal{O}/Y}}(K,-)$ preserves filtered colimits, but according to our recipe for building this object, it suffices to show $\mathbf{hom}_{\mathsf{Alg}_{\mathcal{O}}}(K,-)$ preserves filtered colimits, and so we are done by Proposition 4.1.13. $\square$

Corollary 4.2.4. *The domains and codomains of the generating cofibrations and generating acyclic cofibrations in* $\mathsf{Alg}_{\mathcal{O}}^{Y}$ *are simplicially compact.*

Proof. By Propositions 4.1.14 and 2.2.6, it suffices to prove that for any retractive $\mathcal{O}$-algebra over Y of the form

$$\begin{tikzcd} Y \arrow[d, "\mathrm{in}"] \\ \mathcal{O}(X) \quad Y \arrow[d, "\alpha+\mathrm{id}_Y"] \\ Y \end{tikzcd}$$

with X simplicially compact in $\mathsf{Alg}_{\mathcal{O}}$ is once again simplicially compact. Recall that the coproduct in the middle is taken in $\mathsf{Alg}_{\mathcal{O}}$. We claim that for $K \in \mathsf{S}_*$,

$$(\mathcal{O}(X) \coprod Y) \overset{Y}{\wedge} K \cong \mathcal{O}(X \wedge K_+) \coprod_{\mathcal{O}(X)} Y$$

natural in X and K. First note that, as $\mathcal{O}$-algebras, $(\mathcal{O}(X) \quad Y) \otimes K \cong \mathcal{O}(X \wedge K_+) \quad (Y \otimes K)$. This is because $- \otimes K$ is a left adjoint and therefore commutes with colimits. Now, to see this, note that by Lemma 2.2.20 and Proposition 2.2.18, $(\mathcal{O}(X) \quad Y) \overset{Y}{\wedge} K$ is computed as the following pushout in $(\mathsf{Alg}_{\mathcal{O}})_{/Y}$—hence, the following pushout in $\mathsf{Alg}_{\mathcal{O}}$—given by

$$\begin{tikzcd} ((\mathcal{O}(X) \quad Y) \otimes *) \quad {}^{\mathsf{Alg}_{\mathcal{O}}}\, Y \otimes K \arrow[r] \arrow[d] & (\mathcal{O}(X) \quad Y) \otimes K \arrow[d] \\ Y \arrow[r] & (\mathcal{O}(X) \quad Y) \overset{Y}{\wedge} K \end{tikzcd}$$

but we may write this as

$$\begin{tikzcd}
(\mathcal{O}(X \wedge S^0) \quad Y \otimes *) \quad Y \otimes K \arrow[rr, "\mathcal{O}(X\wedge(S^0\to K_+))+Y\otimes(*\to K)", "+\,\mathrm{in}_{Y\otimes K}"'] \arrow[d] & & \mathcal{O}(X \wedge K_+) \quad Y \otimes K \arrow[dd] \\
(\mathcal{O}(X) \quad Y) \quad Y \arrow[d] & & \\
Y \arrow[rr] & & (\mathcal{O}(X) \quad Y) \overset{Y}{\wedge} K
\end{tikzcd}$$

By universal properties, a cone out of the above pushout fork with cone point Z is the same as a map $Y \to Z$ and a map $\mathcal{O}(X \wedge K_+) \to Z$ commuting with the maps from $Y \cong Y \otimes *$ and thus out of the pushout indicated. Hence,

$$(\mathcal{O}(X) \coprod Y) \overset{Y}{\wedge} K \cong \mathcal{O}(X \wedge K_+) \coprod_{\mathcal{O}(X)} Y.$$

The structural map $Y \to (\mathcal{O}(X \wedge K_+) \quad_{\mathcal{O}(X)} Y)$ is simply the structural map of the coproduct and the map $(\mathcal{O}(X \otimes K_+) \quad_{\mathcal{O}(X)} Y) \quad Y \to Y$ is induced by universal properties by the map $\mathrm{id}\colon Y \to Y$ and the evident composite

$$\mathcal{O}(X \wedge K_+) \to \mathcal{O}(X \wedge \Delta[0]_+) = \mathcal{O}(X \wedge S^0) \cong \mathcal{O}(X) \xrightarrow{\alpha} Y,$$

where the first map is induced by the pointed surjective map $K \quad \{+\} \to \{0\} \quad \{+\}$. Since U commutes with filtered colimits, it suffices to show that for compact $\mathcal{O}(X)$, $\mathcal{O}(X \wedge (\Delta[n]_+)_+) \quad_{\mathcal{O}(X)} Y$ is compact in $\mathsf{Alg}_{\mathcal{O}}^{Y}$. Note that $(K_+)_+ \cong K_+ \quad^{\mathsf{S}_*} S^0$, where this coproduct occurs in S_* as indicated.

Hence, for a spectrum X,

$$X \wedge (K_+)_+ \cong X \wedge (K_+ \coprod S^0) \cong X \wedge K_+ \coprod X,$$

with the final coproduct occurring in spectra. Since, $\mathcal{O} \circ (-)$ is a left adjoint, it happens that

$$\mathcal{O}(X \wedge (\Delta[n]_+)_+) \coprod_{\mathcal{O}(X)} Y \cong (\mathcal{O}(X \wedge \Delta[n]_+) \coprod \mathcal{O}(X)) \coprod_{\mathcal{O}(X)} Y \cong \mathcal{O}(X \wedge \Delta[n]_+) \coprod Y.$$

We are thereby reduced to showing that $\mathcal{O}(X \wedge \Delta[n]_+) \quad Y$ is compact. As an object in $\mathsf{Alg}_{\mathcal{O}}^Y$, it is obtained from the left adjoint $(\mathsf{Alg}_{\mathcal{O}})_{/Y} \to \mathsf{Alg}_{\mathcal{O}}^Y$ described in the proof of Proposition 2.2.6. An adjunction argument reduce us to showing that $\mathcal{O}(X \wedge \Delta[n]_+)$ is compact in $(\mathsf{Alg}_{\mathcal{O}})_{/Y}$. However, the map $\mathcal{O}(X \wedge \Delta[n]_+) \to Y$ factors through the map $\mathcal{O}(X \wedge \Delta[n]_+) \to \mathcal{O}(X \wedge S^0)$ induced by the unique, surjective, pointed map $\Delta[n]_+ \to S^0$. Hence, an element of $\hom_{(\mathsf{Alg}_{\mathcal{O}})_{/Y}}(\mathcal{O}(X \wedge \Delta[n]_+), Z)$ is, by adjunction, precisely the same as a map in spectra $f\colon X \wedge \Delta[n]_+ \to UZ$ (U the forgetful functor) making the following diagram commute

$$\begin{tikzcd}
X \wedge \Delta[n]_+ \arrow[r, dotted, "f"] \arrow[d, "X \wedge c"] & UZ \arrow[d, "Ug"] \\
X \wedge S^0 \arrow[r, "\alpha \circ \rho"] & UY
\end{tikzcd}$$

where $Ug = g$ is the structure map of Z and is a map of $\mathcal{O}$-algebras and $c\colon \Delta[n]_+ \to S^0$ is the unique pointed, surjective map. In other words, the set of such fillers f is, up to isomorphism, simply the pullback

$$\begin{tikzcd}
\hom_{(\mathsf{Alg}_{\mathcal{O}})_{/Y}}(\mathcal{O}(X\wedge\Delta[n]_+),Z) \arrow[r] \arrow[d] & \hom_{\mathsf{Sp}^\Sigma}(X\wedge\Delta[n]_+,UZ) \arrow[d, "Ug_*"] \\
* \arrow[r, "\alpha\circ\rho\circ X\wedge c"'] & \hom_{\mathsf{Sp}^\Sigma}(X\wedge\Delta[n]_+,UZ)
\end{tikzcd}$$

where $\rho\colon X\wedge S^0\to X$ is the unit isomorphism. Since X is compact in Sp^Σ, $X\wedge\Delta[n]_+$ is compact in Sp^Σ. A quick argument using commutativity of filtered colimits and finite products in sets shows that $X\wedge\Delta[n]_+$ is compact.

In particular, chasing back through the adjunction, this shows that the domains and codomains of the generating cofibrations and generating acyclic cofibrations are simplicially compact, as

$$\mathbf{Hom}(\mathcal{O}(X)\coprod Y,Z)_n=\hom_{\mathsf{Alg}_{\mathcal{O}}^Y}((\mathcal{O}(X)\coprod Y)\overset{Y}{\wedge}\Delta[n]_+,Z).$$

$\square$

We are now ready to construct the stabilization of categories of retractive $\mathcal{O}$-algebras over Y. Later, we will impose the mild assumption that Y is bifibrant. For now, this is not needed.

Proposition 4.2.5. *Given an $\mathcal{O}$-algebra Y, the category of retractive $\mathcal{O}$-algebras over Y $\mathsf{Alg}_{\mathcal{O}}^Y$ is a locally finitely presentable, finitely generated, pointed simplicial, model category such that the domains and codomains of all generating cofibrations and generating acyclic cofibrations are simplicially compact and have cofibrant domains.*

Proof. The proposition follows as a simple matter of stringing together our preceding results. $\square$

Proposition 4.2.6. *The stable semi-model structure on* $\mathsf{Sp}^{\mathbf{N}}(\mathsf{Alg}_{\mathcal{O}}^{Y})$ *exists and is a pointed simplicial, stable semi-model category. The weak equivalences are characterized as the* π_*^s*-isomorphisms. Here, we may take* $\pi_n^s(X) = \operatorname{colim}_k \pi_n(\Omega_Y^k L X_{n+k})$ *for any level fibrant replacement functor* L*.*

Proof. The first assertion is Proposition 3.1.12 and Theorem 3.1.13. The second assertion follows exactly as in Corollary 3.2.14, mutatis-mutandis. The fact that we are only working with a semi-model structure, in this case, does not factor into the proof. □

Remark 4.2.7. The characterization of the S-local equivalences for the stable semi-model structure on $\mathsf{Sp}^{\mathbf{N}}(\mathsf{Alg}_{\mathcal{O}}^{Y})$ is independent of the existence of the stable model structure. All that was

Chapter 5

Higher Blakers-Massey Theorems and Homotopy Excision and Their Consequences

In this chapter, we investigate consequences of the higher Blakers-Massey and homotopy excision theorems of Ching and Harper in [15]. We make the following convention in this chapter.

Convention 5.0.1. Wherever $\mathsf{Alg}_{\mathcal{O}}^{Y}$ is mentioned, the object Y and spectral operad $\mathcal{O}$ of the retractive category satisfy the following hypotheses, unless otherwise specified. The object Y is (-1)-connected and bifibrant (cofibrant and fibrant). The operad $\mathcal{O}$ is (-1)-connected—in other words, for each $k \geq 0$, the spectrum of k-ary operations $\mathcal{O}[k]$ is (-1)-connected.

Remark 5.0.2. Any operad in spectra arising from one in spaces satisfies this property, such as the E_k-operads and the E_∞-operad.

5.1 Retractive Forms of Homotopy Excision and Higher Blakers-Massey Theorems

To begin, let us see that the variants of of the main theorems in [15] hold. This will follow from a few useful lemmas.

Lemma 5.1.1. *If* $\mathsf{Fun}(\mathsf{D}, \mathsf{C})$ *admits the injective model structure, then so too does* $\mathsf{Fun}(\mathsf{D}, \mathsf{C}(\mathrm{id}_c))$ *and the natural isomorphism* $\mathsf{Fun}(\mathsf{D}, \mathsf{C}(\mathrm{id}_c)) \cong \mathsf{Fun}(\mathsf{D}, \mathsf{C})(\underline{\mathrm{id}_c})$ *respects the model structures. The evident dual assertion likewise holds.*

Proof. The isomorphism of Lemma 2.1.4 is essentially an identity, so this follows immediately by noting that the cofibrations and weak equivalences of the injective model structure on $\mathsf{Fun}(\mathsf{D}, \mathsf{C}(\mathrm{id}_c))$ are objectwise in the sense that they are preserved and reflected by the forgetful functor $\mathsf{Fun}(\mathsf{D}, \mathsf{C}(\mathrm{id}_c)) \to \mathsf{Fun}(\mathsf{D}, \mathsf{C})$. $\square$

Lemma 5.1.2. *Suppose* $\mathsf{Fun}(\mathsf{D}, \mathsf{C})$ *admits the injective model structure. Then for any fibrant* $F \colon \mathsf{D} \to \mathsf{C}(\mathrm{id}_c)$*, the functor* $F \colon \mathsf{D}^{\triangleright} \to \mathsf{C}(\mathrm{id}_c)$ *obtained from the procedure of Definition 2.1.11 is likewise fibrant in* $\mathsf{Fun}(\mathsf{D}^{\triangleright}, \mathsf{C}(\mathrm{id}_c))$*. If* c *is additionally fibrant in* C*, then* $UF \colon \mathsf{D}^{\triangleright} \to \mathsf{C}$ *is fibrant in* $\mathsf{Fun}(\mathsf{D}^{\triangleright}, \mathsf{C})$*. The evident dual assertion with the projective model structure likewise holds.*

Proof. Denote the cone point of $\mathsf{D}^{\triangleright}$ be $*$ and let $A \to B$ be an acyclic cofibration of functors $\mathsf{D}^{\triangleright} \to \mathsf{C}(\mathrm{id}_c)$. Since $F(*) = \mathrm{id}_c$ is the zero object, there is a unique lift

$$\begin{tikzcd} A(*) \arrow[r, dashed] \arrow[d] & F(*) = \mathrm{id}_c \arrow[d, equal] \\ B(*) \arrow[r, dashed] & \mathrm{id}_c \end{tikzcd}$$

where the dashed arrows are the unique ones. Since $*$ is terminal in $\mathsf{D}^{\triangleright}$ it follows easily that any lift

$$\begin{tikzcd}
A|\,\mathsf{D} \arrow[r] \arrow[d] & F \arrow[d] \\
B|\,\mathsf{D} \arrow[r] \arrow[ur] & \underline{\mathrm{id}_c}
\end{tikzcd}$$

which exists as $A \to B$ restricts to an acyclic cofibration, extends uniquely to a lift

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & F^{\triangleright} \arrow[d] \\
B \arrow[r] \arrow[ur] & \underline{\mathrm{id}_c}
\end{tikzcd}$$

which means F is fibrant.

If c is fibrant, then since by the preceding lemma, the fibrations of $\mathsf{Fun}(\mathsf{D}^{\triangleright}, \mathsf{C}(\mathrm{id}_c))$ are simply the fibrations of $\mathsf{Fun}(\mathsf{D}^{\triangleright}, \mathsf{C})(\underline{\mathrm{id}_c})$, which are created by the forgetful functor, it suffices to show that $\underline{c} \to \underline{*_{\mathsf{C}}}$ is a fibration in $\mathsf{Fun}(\mathsf{D}^{\triangleright}, \mathsf{C})$ and this follows since $\mathsf{D}^{\triangleright}$ has a terminal object $*$, so it is easy to observe that for any acyclic cofibration $A \to B$ in

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & \underline{c} \arrow[d] \\
B \arrow[r] \arrow[ur, dotted] & \underline{*_{\mathsf{C}}}
\end{tikzcd}$$

the dotted lift is completely determined by a lift

$$\begin{tikzcd}
A(*) \arrow[r] \arrow[d] & c \arrow[d, two heads] \\
B(*) \arrow[r] \arrow[ur, dotted] & *_{\mathsf{C}}
\end{tikzcd}$$

and this dotted arrow exists as c is fibrant in C. $\square$

Lemma 5.1.3. *Suppose* C *is a model admitting the injective model structure for all small categories* D*. Suppose* $c \in \mathsf{C}$ *fibrant. Then for any category* D *such that* $\mathsf{D} \to \mathsf{D}^{\triangleright}$ *is cofinal the forgetful functor* $U\colon \mathsf{C}(\mathrm{id}_c) \to \mathsf{C}$ *creates homotopy limits of shape* D*.*

Remark 5.1.4. This lemma has the evident dual with cofibrant c and homotopy colimits. $\mathsf{Alg}_{\mathcal{O}}^{Y}$ admits projective and injective model structure for all small categories as it is cofibrantly generated and combinatorial, as we have seen. Note that we do not require $\mathsf{D} \to \mathsf{D}^{\triangleright}$ to be homotopy left cofinal—we only need to assume the path-connected criterion (i.e., π_0 criterion) on the relevant slice categories.

Proof. Let $F\colon \mathsf{D} \to \mathsf{C}(\mathrm{id}_c)$ and suppose without loss of generality F is fibrant in $\mathsf{C}(\mathrm{id}_c)$. The preceding lemmas now have the following consequence. The homotopy limit of a fibrant functor $F\colon \mathsf{D} \to \mathsf{C}(\mathrm{id}_c)$ is the same as the homotopy limit of the associated $F^{\triangleright}\colon \mathsf{D}^{\triangleright} \to \mathsf{C}(\mathrm{id}_c)$. In particular, both of these homotopy limits may be computed as their limits for which we have $\lim F \cong \lim F^{\triangleright}$ and, moreover, for $U\colon \mathsf{Alg}_{\mathcal{O}}^{Y} \to \mathsf{Alg}_{\mathcal{O}}$ the forgetful functor, it follows that $UF^{\triangleright}$ and UF are still fibrant functors. Hence,

$$\operatorname{holim} UF \simeq \lim UF \cong U \lim F \cong U \lim F^{\triangleright} \cong \lim UF^{\triangleright} \simeq \operatorname{holim} UF^{\triangleright}$$

and both $U \lim F \simeq U \operatorname{holim} F$ and $U \lim F^{\triangleright} \simeq U \operatorname{holim} F^{\triangleright}$. It follows that

$$\operatorname{holim} UF \simeq U \operatorname{holim} F \simeq U \operatorname{holim} F^{\triangleright} \simeq \operatorname{holim} UF^{\triangleright}. \tag{$*$}$$

Now suppose $\underline{X} \to F$ is a homotopy limit cone in $\mathsf{Alg}_{\mathcal{O}}^{Y}$. Then $U\underline{X} \to UF$ is a homotopy limit cone as $X \to \lim F \simeq \operatorname{holim} F$ is a weak equivalence and the induced map $U\underline{X} \to \lim UF \simeq \operatorname{holim} UF$ is a weak equivalence. It follows immediately from $(*)$ that this map is $U(X \to \lim F)$ and and hence is a weak equivalence as U creates these. Conversely, if $\underline{X} \to F$ is such that $U\underline{X} \to UF$ is a homotopy limit cone, then since U creates weak equivalences, it follows immediately that $X \to \operatorname{holim} F$ is a weak equivalence by $(*)$. $\square$

Thus, this lemma, and its evident dual, allows us to avail ourselves of the higher homotopy excision and higher Blakers-Massey theorems for structured ring spectra of Ching and Harper [15] in the retractive case under mild hypotheses on Y. We recall the necessary results below, but before doing this, we collect the obvious corollary of this. As always, part of this corollary has a dual which we suppress.

Corollary 5.1.5. *The forgetful functor $U\colon \mathsf{Alg}_{\mathcal{O}}^{Y} \to \mathsf{Sp}^{\Sigma}$ creates filtered homotopy colimits and homotopy limits of cubes with the initial vertex removed. More generally, U creates homotopy limits of shape D for any D for which the inclusion $\mathsf{D} \to \mathsf{D}^{\triangleright}$ is cofinal.*

Proof. By the above, it suffices to that $U\colon \mathsf{Alg}_{\mathcal{O}} \to \mathsf{Sp}^{\Sigma}$ creates homotopy limits and filtered homotopy colimits. For this, note that filtered colimits in $\mathsf{Alg}_{\mathcal{O}}$ and Sp^{Σ} are already homotopy colimits by Proposition 3.2.13, so since U creates filtered colimits, it is enough to verify this for homotopy limits and this argument is essentially identical to the one given in the preceding lemma since the forgetful functor $U\colon \mathsf{Alg}_{\mathcal{O}} \to \mathsf{Sp}^{\Sigma}$ creates fibrations and weak equivalences. $\square$

Warning 5.1.6. While $\mathsf{Alg}_{\mathcal{O}}$ is right proper, we will still need Y to be both cofibrant and fibrant in $\mathsf{Alg}_{\mathcal{O}}$ to guarantee the compatibility of homotopy limits and colimits in $\mathsf{Alg}_{\mathcal{O}}^{Y}$ with homotopy limits and colimits computed in $\mathsf{Alg}_{\mathcal{O}}$, where the higher Blakers-Massey theorems apply.

Theorem 5.1.7 (Higher homotopy excision for structured ring spectra)**.** *Let $\mathcal{O}$ be an operad in $\mathcal{R}$-modules and W a nonempty finite set. Let $\mathcal{O}$ be a strongly ∞-cocartesian W-cube of $\mathcal{O}$-algebras (resp. left $\mathcal{O}$-modules). Assume that $\mathcal{R}, \mathcal{O}, \mathcal{X}_\emptyset$ are (-1)-connected. Let $k_i \geq -1$ for each $i \in W$. If each $\mathcal{X}_\emptyset \to \mathcal{X}_{\{i\}}$ is k_i-connected ($i \in W$), then*

(a) $\mathcal{X}$ is l-cocartesian in $\mathsf{Mod}_{\mathcal{R}}$ (resp. SymSeq) with $l = |W| - 1 + \quad {}_{i\in W} k_i$,

(b) $\mathcal{X}$ is k-cartesian with $k = \quad {}_{i\in W} k_i$.

Proof. This is [15, Thm. 1.6]. □

Theorem 5.1.8 (Higher Blakers-Massey theorem for structured ring spectra)**.** *Let $\mathcal{O}$ be an operad in $\mathcal{R}$-modules and W a nonempty finite set. Let $\mathcal{X}$ be a W-cube of $\mathcal{O}$-algebras (resp. left $\mathcal{O}$-modules). Assume that $\mathcal{R}, \mathcal{O}, \mathcal{X}_\emptyset$ are (-1)-connected, and suppose that*

(i) for each nonempty subset $V \subset W$, the V-cube $\partial_\emptyset^V \mathcal{X}$ (formed by all maps in $\mathcal{X}$ between $\mathcal{X}_\emptyset$ and $\mathcal{X}_V$) is k_V-cocartesian,

(ii) $-1 \leq k_U \leq k_V$ for each $U \subset V$.

Then $\mathcal{X}$ is k-cartesian, where k is the minimum of $-|W| + \quad {}_{V\in\lambda}(k_V + 1)$ over all partitions λ of W by nonempty sets.

Proof. This is [15, Thm. 1.7]. □

Theorem 5.1.9 (Higher dual Blakers-Massey theorem for structured ring spectra)**.** *Let $\mathcal{O}$ be an operad in $\mathcal{R}$-modules and W a nonempty finite set. Let $\mathcal{X}$ be a W-cube of $\mathcal{O}$-algebras (resp. left $\mathcal{O}$-modules). Assume that $\mathcal{R}, \mathcal{O}, \mathcal{X}_\emptyset$ are (-1)-connected, and suppose that*

(i) *for each nonempty subset* $V \subset W$*, the* V*-cube* $\partial^W_{W-V}\mathcal{X}$ *(formed by all maps in* $\mathcal{X}$ *between* $\mathcal{X}_{W-V}$ *and* $\mathcal{X}_W$*) is* k_V*-cartesian,*

(ii) $-1 \leq k_U \leq k_V$ *for each* $U \subset V$*.*

Then $\mathcal{X}$ *is* k*-cocartesian, where* k *is the minimum of* $k_W + |W| - 1$ *and* $|W| + \quad {}_{V\in\lambda} k_V$ *over all partitions* λ *of* W *by nonempty sets not equal to* W*.*

Proof. This is [15, Thm. 1.11]. □

To this list, we add the following.

Corollary 5.1.10. *The preceding theorems hold in* $\mathsf{Alg}^Y_{\mathcal{O}}$*.*

Proof. This is a consequence of Lemma 5.1.3. □

We also have the following proposition, which is a corollary of homotopy excision, that implies by elementary arguments—one of which we give below—that the cubical homotopy of retractive, (-1)-connected $\mathcal{O}$-algebras and (-1)-connected maps between them enjoys the same properties just like that of spaces.

Proposition 5.1.11. *Let* Y *be a cofibrant* $\mathcal{O}$*-algebra. If* $\mathcal{X}\colon \mathcal{P}(\underline{2}) \to \mathsf{Alg}^Y_{\mathcal{O}}$ *is a homotopy pushout cube such that* $\mathcal{X}_\emptyset$ *is* (-1)*-connected and each* $\mathcal{X}_\emptyset \to \mathcal{X}_{\{i\}}$ *is* k_i*-connected with* $k_i \geq -1$*, then the map* $\mathcal{X}_i \to \mathcal{X}_{\{1,2\}}$ *is* k_j*-connected, where* $i, j \in \{1, 2\}$*,* $i \neq j$*.*

Proof. We may just as well work in $\mathsf{Alg}_{\mathcal{O}}$. Letting $U\colon \mathsf{Alg}_{\mathcal{O}} \to \mathsf{Sp}^{\Sigma}$ be the forgetful functor, we have, by homotopy excision, the following diagram in Sp^Σ.

$$\begin{tikzcd}
U\mathcal{X}_\emptyset \arrow[r, "k_1"] \arrow[d, "k_2"'] & U\mathcal{X}_{\{1\}} \arrow[d, "k_2"] \arrow[ddr, bend left] & \\
U\mathcal{X}_{\{2\}} \arrow[r, "k_2"'] \arrow[drr, bend right] & X \arrow[dr, "k_1+k_2+1"] & \\
& & U\mathcal{X}_{\{1,2\}}
\end{tikzcd}$$

where X is the homotopy pushout in spectra of this fork. Since $k_i \geq -1$, $k_1 + k_2 + 1 \geq -1$ and it follows that the arrow $U\mathcal{X}_{\{i\}} \to U\mathcal{X}_{\{1,2\}}$ is k_j-connected where $i, j \in \{1, 2\}$, $i \neq j$. $\square$

Again, informally, this means the cubical homotopy theory of (-1)-connective objects and (-1)-connected maps between them have the same types of connectivity properties as spaces do when all objects and maps involved—except, perhaps, the retract object Y—are (-1)-connected. As an example, we show that this category of (-1)-connected objects and maps is closed under all cubical homotopy colimits (the case of cubical homotopy limits follows from the analogous case as pushed into spectra). We will only ever be interested in this for the case that Y is also (-1)-connected.

Corollary 5.1.12. *Let Y be a cofibrant $\mathcal{O}$-algebra. If $\mathcal{X}\colon \mathcal{P}_1(\underline{n}) \to \mathsf{Alg}_{\mathcal{O}}^Y$ lands in the full subcategory of (-1)-connected objects and (-1)-connected maps between them. Then* $\operatorname{hocolim} \mathcal{X}$ *is (-1)-connected, and, moreover, each map $\mathcal{X}_U \to$* $\operatorname{hocolim} \mathcal{X}$ *is (-1)-connected.*

Remark 5.1.13. In fact, if $m \geq -1$, the same assertion holds by the very same argument if, instead, we restrict to the full subcategory of m-connected objects and m-connected maps between them.

Proof. Without loss of generality, we may assume our n-cubes $\mathcal{X}$ are punctured cofibration cubes—as in [15, Def. 3.4], this means each for each proper subset $V \subsetneq \underline{n}$, the map $\operatorname{colim}_{\mathcal{P}_1(V)} \mathcal{X} \to$

$\operatorname{colim}_{\mathcal{P}(V)} \mathcal{X} \cong \mathcal{X}_V$ is a cofibration. In particular, this means every object $\mathcal{X}_U$ is cofibrant and every map $\mathcal{X}_T \to \mathcal{X}_U$ where $T \subset U \subsetneq \underline{n}$ is a cofibration. The strict colimit of such cubes compute homotopy colimits as the cofibrant objects in the projective model structure on $\mathcal{P}_1(\underline{n})$ and therefore, in particular, for *any* subset $V \subset \underline{n}$, $\operatorname{colim}_{\mathcal{P}_1(V)} \mathcal{X} \simeq \operatorname{hocolim}_{\mathcal{P}_1(V)} \mathcal{X}$. A good reference for the properties of this particular projective model structure may be found in [21, §10.13].

The proof is by induction on n. If $n = 0, 1$, there is essentially nothing to prove, noting that we reserve the word connectivity to mean connectivity relative to the map $* \to X$. The true base case of $n = 2$ is taken care of by the hard work homotopy excision. Now suppose the statement is true for punctured n-cubes and let us consider the case punctured $(n+1)$-cubes. Fix a punctured, cofibration $(n+1)$-cube $\mathcal{X}\colon \mathcal{P}_1(\underline{n+1}) \to \mathsf{Alg}_{\mathcal{O}}^Y$.

Any subset $S \subset \underline{n}$ of size $n-1$ determines a codimension 1 face of $\mathcal{X}$ as the face spanned the subsets $\emptyset \subset \underline{n}$ and $\{k\} \subset \underline{n}$ where $k \in S$. There are n-such faces. We adapt the notation of Ching-Harper [15, Def. 3.3] to the present situation. In this case, it is the cube S-cube $\partial_{\emptyset}^{S} \mathcal{X}$ defined on objects $U \subset S$ by

$$U \mapsto (\partial_{\emptyset}^{S} \mathcal{X})_U = \mathcal{X}_U, \quad U \subset S.$$

Note that while $\mathcal{X}$ is not properly a cube, since have excluded the final vertex, this still makes sense. Note that

$$\mathcal{P}(\underline{n}) \times \mathcal{P}(\underline{1}) \cong \mathcal{P}(S) \times \mathcal{P}(\{k\}) \cong \mathcal{P}(\underline{n+1}).$$

More generally, for any *proper* subset $W \subsetneq \underline{n+1}$, we define $\partial_W \mathcal{X}$ to the be the punctured $((n+$

$1) - |W|)$-cube given on objects U for any $W \subset U \subsetneq \underline{n+1}$ (note the properness of $U \subsetneq \underline{n+1}$) by

$$U \mapsto (\partial_{\emptyset}\mathcal{X})_U = \mathcal{X}_U.$$

Pick any codimension 1 face $\partial_{\emptyset}^{S}\mathcal{X}$ of $\mathcal{X}$. Say this is determined by excluding $k \in \underline{n+1}$ from S (the choice of k is immaterial). Form a cube

$$\mathcal{X} \colon \mathcal{P}(S) \times \mathcal{P}(\{k\}) \cong \mathcal{P}(\underline{n+1}) \to \mathsf{Alg}_{\mathcal{O}}^{Y},$$

which we view as an S-cube (i.e., n-cube) of 1-cubes, built on objects as follow. For any *proper* subset $V \subsetneq S$, $\mathcal{X}_V = \mathcal{X}_V$ and on S, $\mathcal{X}_S = \operatorname{colim} \partial_{\emptyset}^{S}\mathcal{X}$. For each *proper* subset $U \subsetneq \underline{n}$ containing k, $\mathcal{X}_U = \mathcal{X}_U$ and on $\underline{n+1}$, $\mathcal{X}_{\underline{n+1}} = \operatorname{colim} \mathcal{X}_{\{k\}}$. For *proper* subsets $U \subsetneq S$, the map $\mathcal{X}_U \to \mathcal{X}_{U \cup \{k\}}$ is simply the map $\mathcal{X}_U \to \mathcal{X}_{U\cup\{k\}}$. The map $\mathcal{X}_S \to \mathcal{X}_{\underline{n+1}}$ is the map induced by universal properties. By induction,

$$\mathcal{X}_S = \operatorname{colim} \partial_{\emptyset}^{S}\mathcal{X} = \operatorname{colim}_{\mathcal{P}_1(S)} \mathcal{X} \simeq \operatorname{hocolim}_{\mathcal{P}_1(S)} \mathcal{X}.$$

and is therefore (-1)-connected and the maps $\mathcal{X}_U \to \mathcal{X}_S$ are also (-1)-connected. Similarly, by induction,

$$\mathcal{X}_{\underline{n+1}} = \operatorname{colim} \partial_{\{k\}}\mathcal{X} = \operatorname{colim}_{\mathcal{P}_1(\underline{n+1}\setminus\{k\})} \mathcal{X} \simeq \operatorname{hocolim}_{\mathcal{P}_1(\underline{n+1}\setminus\{k\})} \mathcal{X}$$

and is therefore (-1)-connected and for $k \in U \subsetneq \underline{n+1}$, the maps $\mathcal{X}_U \to \mathcal{X}_{\underline{n+1}}$ are (-1)-connected.

An argument with universal properties shows that $\operatorname{colim} \mathcal{X} = \operatorname{hocolim} \mathcal{X}$ is the pushout

$$\begin{tikzcd}
\mathcal{X}_S \arrow[r] \arrow[d] & \mathcal{X}_{\underline{n+1}} \arrow[d] \\
\mathcal{X}_S \arrow[r] & \operatorname{hocolim} \mathcal{X}
\end{tikzcd} \tag{$*$}$$

where the map $\mathcal{X}_S \to \mathcal{X}_S$ is the comparison map induced by universal properties. This is because the map $\mathcal{X}_S \to \mathcal{X}_{\underline{n+1}}$ is precisely what is encoded by the cone from the punctured diagram $\mathcal{X}|\,\mathcal{P}_1(S)$ to $\mathcal{X}_{\underline{n+1}}$, and this cone factors through the map of punctured cubes $\mathcal{X}|\,\mathcal{P}_1(S) \to \partial_k \mathcal{X}$. Hence a cone out of the above pushout fork encodes a cone $\partial_k \mathcal{X} \to \underline{Z}$ as well as a cone $\partial_\emptyset^S \mathcal{X} \to \underline{Z}$ compatible with the map of punctured cubes $\mathcal{X}|\,\mathcal{P}_1(S) \to \partial_k \mathcal{X}$. This is because the map $\mathcal{X}_S = \operatorname{colim}_{\mathcal{P}_1(S)} \mathcal{X} \to \mathcal{X}_S$ is precisely the same as the commutative diagram $\partial_\emptyset^S \mathcal{X}$ represents.

We claim that $(*)$ is, in fact, a *homotopy* pushout in $\mathsf{Alg}_{\mathcal{O}}^Y$. To see this, note that every object of $(*)$ is cofibrant. From our assumption that $\mathcal{X}$ is a cofibration cube, the map $\mathcal{X}_S \to \mathcal{X}_S$ is a cofibration. That the map $\mathcal{X}_S \to \mathcal{X}_{\underline{n+1}}$ is a cofibration follows since it is induced by applying the colimit functor to a cofibration of punctured n-cubes in the projective model structure on $\mathsf{Fun}(\mathcal{P}_1(\underline{n}), \mathsf{Alg}_{\mathcal{O}}^Y)$. Alternatively, all objects in $(*)$ are cofibrant and the map $\mathcal{X}_S \to \mathcal{X}_S$ is a cofibration, it follow by standard model categorical results such as [41, Prop. A.2.4.4.(i)], the strict pushout is indeed a homotopy pushout.

In particular, since every map appearing in $\mathcal{X}$ is (-1)-connected, the map $\mathcal{X}_\emptyset \to \mathcal{X}_S$ is (-1)-connected being a composite of such maps similarly the map $\mathcal{X}_{\{k\}} \to \mathcal{X}_{\underline{n+1}}$ is (-1)-connected.

Hence, from the induction hypothesis, there is a factorization with connectivities displayed

$$\begin{tikzcd} \mathcal{X}_\emptyset \arrow[r, "-1"] \arrow[dr, "-1"'] & \mathcal{X}_S \arrow[d] \\ & \mathcal{X}_S \end{tikzcd}$$

which shows the natural map $\mathcal{X}_S \to \mathcal{X}_S$ in $(*)$ is (-1)-connected as well. Similarly, the map

$$\mathcal{X}_S \to \mathcal{X}_{\underline{n+1}}$$

is (-1)-connected since all maps in $\mathcal{X}$ are (-1)-connected, so there is a factorization with connectivities displayed

$$\begin{tikzcd} \mathcal{X}_\emptyset \arrow[r, "-1"] \arrow[d, "-1"'] \arrow[dr, "-1"] & \mathcal{X}_{\{k\}} \arrow[d, "-1"] \\ \mathcal{X}_S \arrow[r] & \mathcal{X}_{\underline{n+1}} \end{tikzcd}$$

which shows the composite map $\mathcal{X}_\emptyset \to \mathcal{X}_{\underline{n+1}}$ is (-1)-connected and therefore the natural map $\mathcal{X}_S \to \underline{\mathcal{X}}_{\underline{n+1}}$ in $(*)$ is (-1)-connected as well.

This foregoing work shows that $(*)$ satisfies the hypotheses of Proposition 5.1.11 and so we may apply it to $(*)$. It follows that the map $\mathcal{X}_{\underline{n+1}} \to \operatorname{hocolim} \mathcal{X}$ is (-1)-connected and the map $\mathcal{X}_S \to \operatorname{hocolim} \mathcal{X}$ is (-1)-connected. Since $\mathcal{X}_S$ is (-1)-connected by induction hypothesis, it follows that so too is $\operatorname{hocolim} \mathcal{X}$. Since all maps in the cube $\mathcal{X}$ are (-1)-connected, this shows that for any

$U \subsetneq \underline{n+1}$ such that $k \in U$ or U admits a map $U \subset U \cup \{k\} \neq \underline{n+1}$,

$$\begin{tikzcd} \mathcal{X}_U \arrow[r, "-1"] \arrow[dr] & \mathcal{X}_{\underline{n+1}} \arrow[d, "-1"] \\ & \operatorname{hocolim} \mathcal{X} \end{tikzcd}$$

and thus the map $\mathcal{X}_U \to \operatorname{hocolim} \mathcal{X}$ is (-1)-connected. The only remaining map excluded by this analysis is $\mathcal{X}_S \to \operatorname{hocolim} \mathcal{X}$, which is already known to be (-1)-connected by homotopy excision. $\square$

5.2 Consequences of the Higher Blakers-Massey Theorems and Homotopy Excision

Now we collect the necessary consequences of our new tools. First, we collect some notation and definitions.

Definition 5.2.1. Fix an S_*-enriched fibrant replacement monad F and an S_*-enriched cofibrant replacement comonad on C on $\mathsf{Alg}_{\mathcal{O}}^{Y}$. Define

$$\Omega_Y := \Omega_Y F \qquad \text{and} \qquad \Sigma_Y := \Sigma_Y C.$$

Definition 5.2.2. Let us say a retractive $\mathcal{O}$-algebra $X \in \mathsf{Alg}_{\mathcal{O}}^{Y}$ is ***k-connected relative to*** Y if the structure map $Y \to X$ is k-connected. We will say X is ***k-connected*** if the map $* \to X$ in spectra is k-connected

Remark 5.2.3. The point is that by taking $Y = *$, this recovers the usual notion of connectivity, so long as $\mathcal{O}$ is reduced (i.e., $\mathcal{O}[0] = *$) so that the category of $\mathcal{O}$-algebras is pointed by point in spectra.

Proposition 5.2.4. *Consider $\mathsf{Alg}_{\mathcal{O}}^{Y}$ as in Convention 5.0.1.*

(a) Suppose Y (-1)-connected and suppose $X \in \mathsf{Alg}_{\mathcal{O}}^{Y}$ is k-connected relative to Y where $k \geq -1$. For each $n \geq 1$, $\Sigma_Y^n X$ is $(k+n)$-connected relative to Y. Moreover, Σ_Y^n increases connectivity of $-1 \leq m$-connected maps between (-1)-connected objects relative to Y by n.

(b) Let $X \in \mathsf{Alg}_{\mathcal{O}}^{Y}$ and suppose X is k-connected relative to Y. Then $\Omega_Y^n X$ is $(k-n)$-connected relative to Y. Moreover, Ω_Y^n decreases connectivity of maps by n.

Proof. (a) Since $Y \to X$ is k-connected, $X \to Y$ is $(k+1)$-connected. Moreover, X is 0-connected as an $\mathcal{O}$-algebra. We may computed $\Sigma_Y X$ as the homotopy pushout of the fork

$$\begin{tikzcd} X \arrow[r, "k+1"] \arrow[d, "k+1"'] & Y \\ Y & \end{tikzcd}$$

so by Proposition 5.1.11, the map $Y \to \Sigma_Y X$ is $(k+1)$-connected. Since $\Sigma_Y X$ is a retractive object, $\Sigma_Y X \to Y$ is $(k+2)$-connected. By repeating this process, the general statement follows.

To see that Σ_Y increases connectivity of maps between such objects, let $f \colon X \to Z$ be an n-connected map of retractive $\mathcal{O}$-algebras over Y and assume that X and Z are $k_X \geq -1$ and $k_Z \geq -1$ connected relative to Y, respectively. We have the following commutative diagram with connectivities displayed.

$$\begin{tikzcd}
X \arrow[rr, "k_X+1"] \arrow[d, "k_X+1"'] \arrow[ddr, "n"] & & Y \arrow[d, "k_X+1"'] \arrow[ddr, "\infty"] & \\
Y \arrow[rr, "k_X+1"] \arrow[ddr, "\infty"'] & & \Sigma_Y X \arrow[ddr, dotted] & \\
& Z \arrow[rr, "k_Z+1"] \arrow[d, "k_Z+1"] & & Y \arrow[d, "k_Z+1"] \\
& Y \arrow[rr] & & \Sigma_Y Z
\end{tikzcd}$$

Let us consider the solid portion of the above cube as punctured 2-cube of 1-cubes $\mathcal{X}\colon \mathcal{P}_1(\underline{2}) \to \mathsf{Fun}(\mathcal{P}(\underline{1}), \mathsf{Alg}_{\mathcal{O}})$. Let us write this as $\mathcal{X}(U, V)$ where $U \subsetneq \underline{2}$. The back face of the cube above corresponds to $V = \emptyset$ and the front face corresponds to $V = \{1\}$. By [15, Prop. 3.8] and [23, Prop. 1.22], it follows that the map $\Sigma_Y f$ is n_0-connected where

$$n_0 = \min \{2 - |U| - 1 + k_U : U \subsetneq \underline{2}\}.$$

Since the only map that is not infinitely connected is f, this means

$$n_0 = 2 - 0 - 1 + n = n + 1,$$

as desired. By iterating this, Σ_Y^n will raise connectivity of maps by n.

(b) Similarly, $\Omega_Y X$ may be computed as the homotopy pullback of the fork

$$\begin{tikzcd} & Y \arrow[d, "k"] \\ Y \arrow[r, "k"'] & X \end{tikzcd}$$

Since this homotopy pullback is created in $\mathsf{Alg}_{\mathcal{O}}$ and hence in Sp^{Σ}, the map $\Omega_Y X \to Y$ is k-connected. Since $\Omega_Y X$ is a retractive object, the map $Y \to \Omega_Y X$ is $(k-1)$-connected. That Ω_Y decreases connectivity of maps by 1 follows by the argument dual to the one given above. As before we can repeat this to obtain the general result. $\square$

The following corollary has also been observed by Beardsley and Lawson in [5].

Corollary 5.2.5 (Retractive Hurewicz)**.** *Suppose Y is (-1)-connected and cofibrant in $\mathsf{Alg}_{\mathcal{O}}$. If $X \in \mathsf{Alg}_{\mathcal{O}}^{Y}$ is k-connected relative to Y where $k \geq -1$, then derived unit map $X \to \Omega_Y \Sigma_Y X$ is $(2k+2)$-connected and $\Omega_Y \Sigma_Y X$ is k-connected relative to Y.*

Proof. This map is implemented on the level of homotopy as the map into the homotopy pullback of the homotopy pushout displayed in the diagram below.

$$\begin{tikzcd} X \arrow[rr, "k+1"] \arrow[dd, "k+1"'] \arrow[dr, dotted] & & Y \arrow[dd, "k+1"] \\ & \Omega_Y \Sigma_Y X \arrow[ur, "k+1"] \arrow[dl, "k+1"] & \\ Y \arrow[rr, "k+1"'] & & \Sigma_Y X \end{tikzcd}$$

By the higher Blakers-Massey theorem, the dotted arrow is n-connected where

$$n = \min\{(-2 + (k+2) + (k+2), -2 + \infty\} = 2k + 2,$$

as claimed. Since $\Omega_Y \Sigma_Y X \to Y$ is $(k+1)$-connected, the map $Y \to \Omega_Y \Sigma_Y X$ is k-connected since the composite $Y \to \Omega_Y \Sigma_Y X \to Y$ is the identity. $\square$

Lemma 5.2.6. *Suppose Y is (-1)-connected. Suppose $X \in \mathsf{Alg}_{\mathcal{O}}^Y$ is k-connected relative to Y where $k \geq -1$. The map $\Omega_Y^n \Sigma_Y^n X \to \Omega_Y^{n+1} \Sigma_Y^{n+1} X$ obtained from the derived unit map $\Sigma_Y^n X \to \Omega_Y \Sigma_Y^{n+1} X$ is $(2k+n+2)$-connected.*

Proof. Since $\Sigma_Y^n X$ is $(k+n)$-connected relative to Y, the derived unit map $\Sigma_Y^n X \to \Omega_Y \Sigma_Y^{n+1} X$ is $(2(k+n)+2)$-connected. Looping this down n-times decreases connected by n and hence the map is $(2k+n+2)$-connected, as desired. $\square$

Corollary 5.2.7. *Suppose Y is (-1)-connected. If $X \in \mathsf{Alg}_{\mathcal{O}}^Y$ is k-connected relative to Y where $k \geq -1$, then derived unit map $X \to \Omega_Y^n \Sigma_Y^n X$ is $(2k+2)$-connected.*

Proof. This follows by observing that, on the point-set level, the unit of the underived (Σ_Y^n, Ω_Y^n)-adjunction is the composite map

$$X \to \Omega_Y \Sigma_Y X \to \Omega_Y^2 \Sigma_Y^2 X \to \cdots \to \Omega_Y^n \Sigma_Y^n X,$$

where the maps $\Omega_Y^\ell \Sigma_Y^\ell X \to \Omega_Y^{\ell+1} \Sigma_Y^{\ell+1} X$ are obtained by applying Ω_Y^ℓ to the unit map $\Sigma_Y^\ell X \to \Omega_Y \Sigma_Y^{\ell+1} X$. The analogous thing is true for the derived version, for which we know that maps

$\Omega_Y^\ell \Sigma_Y^\ell X \to \Omega_Y^{\ell+1}\Sigma_Y^{\ell+1} X$ are $(2k+\ell+2)$-connected. Hence, the composite map $X \to \Omega_Y^n \Sigma_Y^n X$ is $(2k+2)$-connected. $\square$

Corollary 5.2.8. *Suppose Y is (-1)-connected. If $X \in \mathsf{Alg}_{\mathcal{O}}^Y$ is k-connected relative to Y where $k \geq -1$, then derived unit map $X \to \Omega_Y^\infty \Sigma_Y^\infty X$ is $(2k+2)$-connected.*

Proof. This is one of the important consequences of Theorem 3.3.1 alluded to previously. Since for cofibrant X, the derived unit map $X \to \Omega_Y^\infty \Sigma_Y^\infty X$ is, equivalently, for a choice of a fibrant replacement monad, the map into the colimit $\operatorname{colim} \Omega_Y^k R^k \Sigma_Y^k X$, it is enough to observe that the maps $\Omega_Y^\ell \Sigma_Y^\ell X \to \Omega_Y^{\ell+1}\Sigma_Y^{\ell+1} X$ have increasing connectivity. In particular, the map $X \to \Omega_Y \Sigma_Y X$ is $(2k+2)$-connected. This now follows from the fact that homotopy groups commute with filtered homotopy colimits. $\square$

Remark 5.2.9. Is is important to note that when $Y \to X$ is (-1)-connected, then $X \to \Omega_Y \Sigma_Y X$ is 0-connected. This ends up throwing a wrench in the strategy (explained in the next chapter) we intend to use to analyze completions. This is because we will not be able to show that the maps to the completion tower have increasing connectivity and thus that the completion tower converges strongly.

Chapter 6

Completions With Respect to $\Omega_Y^k \Sigma_Y^k$ and Stabilization

We are now ready to begin investigating completion phenomena, and the point of this chapter is to prove the main theorems of this paper. Our strategy is motivated by Dundas [19, §2.6], subsequently written up by Goodwillie-Dundas-McCarthy in [20, §A.8.3]. This strategy has also been deployed in [8].

Fixing a S_*-enriched fibrant and cofibrant replacement monad F and comonad C n $\mathsf{Alg}_{\mathcal{O}}^Y$, we may build a cosimplicial resolution of a retractive $\mathcal{O}$-algebra X over Y using the associated functors Σ_Y^k and Ω^k. Here, we may use an S_*-enriched stable fibrant replacement monad on $\mathsf{Sp}^{\mathbf{N}}(\mathsf{Alg}_{\mathcal{O}}^Y)$ for the case of stabilization (i.e., $k = \infty$). We suppress this now in favor of ease of exposition. This resolution assembles into a coaugmented cosimplicial object using the adjunction (Σ^k_Y, Ω^k_Y)

$$X \longrightarrow (\Omega^k_Y \Sigma^k_Y X \rightrightarrows (\Omega^k_Y \Sigma^k_Y)^2 X \Rrightarrow (\Omega^k_Y \Sigma^k_Y)^3 X \cdots)$$

and the appropriate thing to do with such resolutions is to glue the datum of the resolution together in a homotopical manner—in other words, taking homotopy limits, we obtain the ***Bousfield-Kan completion*** map

$$X \to X^{\wedge}_{\Omega^k_Y \Sigma^k_Y}$$

where the target is the ***Bousfield-Kan completion*** of X with respect to $\Omega^k_Y \Sigma^k_Y$.

To make this precise, we need everything above to be sufficiently derived. Fortunately, this may done for every case $1 \leq k \leq \infty$. This follows by work of Riehl-Blumberg [9] and Blomquist [7] for $1 \leq k < \infty$. When $k = \infty$, we have shown that $\mathsf{Alg}^Y_{\mathcal{O}}$ admits an S_*-enriched stable fibrant replacement monad. While this functor does not have the property that its unit map is an acyclic cofibration in the stable semi-model structure on $\mathsf{Alg}^Y_{\mathcal{O}}$, this is irrelevant to the construction of the cosimplicial objects associated to the fundamental adjunction $(\Sigma^\infty_Y, \Omega^\infty_Y)$.

Convention 6.0.1. Whenever $\mathsf{Alg}^Y_{\mathcal{O}}$ is mentioned, we assume Y is cofibrant and fibrant in $\mathsf{Alg}^Y_{\mathcal{O}}$ as well as (-1)-connected. We additionally assume the spectral operad $\mathcal{O}$ is (-1)-connected. We furthermore restrict our attention only to the subcategory of objects in $\mathsf{Alg}^Y_{\mathcal{O}}$ that are at least (-1)-connected relative to Y. This is the same as Convention 5.0.1 but with the addition of a connectivity assumption on Y and the objects in $\mathsf{Alg}^Y_{\mathcal{O}}$.

6.1 The General Strategy

The key to analyzing this is the following observation, which is proved in [13, §6], [52] and was deployed by Hopkins in [33].

Proposition 6.1.1. *For each $n \geq 0$, let $[n]$ denote the set of elements $\{0, \ldots, n\}$. The composite*

$$\ell_n \colon \mathcal{P}_0([n]) \cong P\Delta[n] \to \Delta^{\leq n}_{\mathsf{res}} \subset \Delta^{\leq n}$$

is homotopy left cofinal. Here, $P\Delta[n]$ denotes the poset of non-degenerate simplices of $\Delta[n]$ and $\Delta^{\leq n}_{\mathsf{res}}$ is the restriction of $\Delta^{\leq n}$ to the coface maps. The composite is given on objects by $U \mapsto [|U|-1]$ and on arrows by $[|U|-1] \cong U \subset V \cong [|V|-1]$ in the sense given in the following remark.

Remark 6.1.2. In particular, this reduces our computations to punctured $(n+1)$-cubes. Given a coaugmented, truncated cosimplicial object $d^0 \colon X_{-1} \to X$, with $X \in \mathsf{C}^{\Delta}$, the corresponding truncated punctured 3-cube, along with the coaugmentation maps making is a cube, are given under the above composite as

$$\begin{tikzcd}
\emptyset \arrow[rr, dotted, "d^0"] \arrow[dr, dotted, "d^0"] \arrow[dd, dotted, "d^0"'] & & \{2\} \arrow[dr, "d^0"] \arrow[dd, "d^0"] & & & X_{-1} \arrow[rr, dotted, "d^0"] \arrow[dr, dotted, "d^0"] \arrow[dd, dotted, "d^0"'] & & X_0 \arrow[dr, "d^0"] \arrow[dd, "d^0"] & \\
& \{1\} \arrow[rr, "d^1"] \arrow[dd, "d^0"] & & \{1,2\} \arrow[dd, "d^0"] & & & X_0 \arrow[rr, "d^1"] \arrow[dd, "d^0"] & & X_1 \arrow[dd, "d^0"] \\
\{0\} \arrow[rr, "d^1"] \arrow[dr, "d^1"'] & & \{0,2\} \arrow[dr, "d^1"] & & \overset{X([|-|-1])}{\longmapsto} & X_0 \arrow[rr, "d^1"] \arrow[dr, "d^1"'] & & X_1 \arrow[dr, "d^1"] & \\
& \{0,1\} \arrow[rr, "d^2"'] & & \{0,1,2\} & & & X_1 \arrow[rr, "d^2"'] & & X_3
\end{tikzcd}$$

The right-hand side is the composite functor $X \circ \ell_2$. The left-hand side displays the coface cube corresponding the map $\mathcal{P}_0([n]) \to \Delta^{\leq n}$, where the size of the set indicates which elements $[k] \in \Delta$ the element of $\mathcal{P}_0([n])$ maps to. The coface maps d^i should be interpreted as skipping the i-th element in the (ordered) set, so that all displayed maps are simply subset inclusions.

In particular, this means that for $X \in \mathsf{C}^{\Delta}$, ℓ_n induces a weak equivalence

$$\operatorname{holim}_{\Delta^{\leq n}} X \to \operatorname{holim}_{\mathcal{P}_0([n])} X,$$

at least on the level of homotopy categories.

To make this precise on a point-set level, we will adopt the Bousfield-Kan model for homotopy limit. This has the following consequence.

Corollary 6.1.3. *Given a simplicial model category C, and $X \in \mathsf{C}^{\Delta}$ objectwise fibrant, the natural map induced by ℓ_n*

$$\operatorname{holim}^{\mathsf{BK}}_{\Delta^{\leq n}} X \to \operatorname{holim}^{\mathsf{BK}}_{\mathcal{P}_0([n])} X,$$

is a weak equivalence.

Remark 6.1.4. A nice discussion of the Bousfield-Kan model may be found in [12, 30, 51].

The reason why this will be useful for us is the following proposition, which really amounts to our grand strategy.

Proposition 6.1.5. *Fix a coaugmented, cosimplicial object $X_{-1} \to X$ with X objectwise fibrant in $\mathsf{Alg}^{Y}_{\mathcal{O}}$.*

(a) There is a commutative diagram

$$\begin{tikzcd}
 & \operatorname{holim}_n \operatorname{holim}_{\Delta^{\le n}} X \simeq \operatorname{holim}_{\Delta} X \arrow[d] \\
 & \vdots \arrow[d] \\
 & \operatorname{holim}_{\Delta^{\le 2}} X \simeq \operatorname{holim}_{\mathcal{P}_0([2])} X \arrow[d] \\
 & \operatorname{holim}_{\Delta^{\le 1}} X \simeq \operatorname{holim}_{\mathcal{P}_0([1])} X \arrow[d] \\
X_{-1} \arrow[r] \arrow[ur, bend left] \arrow[uur, bend left] \arrow[uuuur, bend left] & \operatorname{holim}_{\Delta^{\le 0}} X \simeq \operatorname{holim}_{\mathcal{P}_0([0])} X \simeq X_0
\end{tikzcd}$$

where the comparison map $X_{-1} \to \operatorname{holim}_n \operatorname{holim}_{\Delta^{\le n}} X$ is equivalent to the comparison map $X_{-1} \to \operatorname{holim}_{\Delta} X$.

(b) If the maps $(*_n)$ are strictly increasing in connectivity, then the map $X_{-1} \to \operatorname{holim}_{\Delta} X$ is a weak equivalence.

Proof. Using the Bousfield-Kan model, each homotopy limit of fibrant objects is again fibrant. Hence, the first statement follows commutativity of limits and the second follows since Y is fibrant, so homotopy limits may be computed in spectra as a consequence of Lemma 5.1.3 and the remark immediately following it. The sequence satisfies the Mittag-Leffler condition on homotopy groups and therefore the $\lim^1$-exact sequence is simply an isomorphism. □

Remark 6.1.6. Thus, to study the connectivity of the Bousfield-Kan completion map, we will study the the connectivity of the maps $(*_n)$ when the cosimplicial object is built from derived composites of $\Omega^k_Y \Sigma^k_Y$. In light of Remark 6.1.2, the connectivity of the maps $(*_n)$ are precisely the cartesianness of the corresponding coface $(n+1)$-cube. This will be our strategy.

Let us now formally define these coface cubes.

Definition 6.1.7. Given a coaugmented cosimplicial object $X_{-1} \to X$ with $X \in c\mathsf{C}$, the ***coface*** $(n+1)$***-cube*** associated to X is the $(n+1)$-cube formed from composite functor $X \circ \ell_n$ and the augmentation map $X_{-1} \to X\ell_n(\{k\}) = X_0$ for each $0 \leq k \leq n$.

6.2 Low-Dimensional Examples

As a warm-up, let us consider the case of $k = 1$ with X 0-connected relative to Y. In this case, the coaugmented cosimplicial object $\mathcal{R}(X)$ associated to X has the form

$$\mathcal{R}(X): \qquad X \xleftarrow{\sim} CX \longrightarrow (\Omega_Y^k \Sigma_Y^k X \rightrightarrows (\Omega_Y^k \Sigma_Y^k)^2 X \Rrightarrow (\Omega_Y^k \Sigma_Y^k)^3 X \cdots)$$

The zig-zag from X is essentially unavoidable but irrelevant—this zig-zag is the derived unit. If X is bifibrant, then so too is its cofibrant replacement under our cofibrant and fibrant replacement scheme and so the map from CX has a homotopy inverse. To avoid clutter, we may assume without loss of generality that X is bifibrant and thereby consider the coaugmentation to come from X itself. Let us call this cosimplicial object $\mathcal{R}(X)$ as indicated.

Suppose X is 0-connected relative to Y. The first map to analyze in the tower of Proposition 6.1.5 is the map $X \to \Omega_Y \Sigma_Y X$ and this is the Hurewicz map which is 2-connected by Corollary 5.2.5. Indeed, from what we have seen in Corollary 5.2.5, Lemma 5.2.6 and Proposition 5.2.4, every coface map $\mathcal{R}(X)_k \to \mathcal{R}(X)_{k+1}$ is $k+1$-connected

Let $\mathcal{X}$ be the 1-cube $X \to \Omega_Y \Sigma_Y X$. The next step is to analyze $X \to \operatorname{holim}_{\mathcal{P}_0([1])} \mathcal{R}(X)$. This is then the square

$$\begin{tikzcd} \mathcal{X} : \arrow[d] & \mathcal{X}_{\emptyset} \arrow[r] \arrow[d] & \mathcal{X}_{\{1\}} \arrow[d] \\ \Omega_Y \Sigma_Y \mathcal{X} : & \Omega_Y \Sigma_Y \mathcal{X}_{\emptyset} \arrow[r] & \Omega_Y \Sigma_Y \mathcal{X}_{\{1\}} \end{tikzcd}$$

The downwards maps are the derived unit maps for the adjunction. It is difficult to analyze the cartesianness of this cube directly, so we will augment this diagram. Let C be the homotopy cofiber in $\mathsf{Alg}_{\mathcal{O}}^{Y}$ of $\mathcal{X}$ and $\mathcal{C}$ be the cube $Y \to C$. Then the following square

$$\begin{tikzcd} \mathcal{X} : \arrow[d] & \mathcal{X}_{\emptyset} \arrow[r] \arrow[d] & \mathcal{X}_{\{1\}} \arrow[d] \\ \mathcal{C} : & Y \arrow[r] & C \end{tikzcd}$$

is ∞-cocartesian, being a homotopy pushout. Since $\mathcal{X}_{\emptyset} \to \mathcal{X}_{\{1\}}$ is 2-connected and $\mathcal{X}_{\emptyset} \to Y$ is 1-connected, it follows that $Y \to C$ is 2-connected and $\mathcal{X}_{\{1\}} \to C$ is 1-connected. We now consider the diagram

$$\begin{tikzcd} \mathcal{X} \arrow[r, "(a)"] \arrow[d, "(*)"'] & \mathcal{C} \arrow[d, "(c)"] \\ \Omega_Y \Sigma_Y \mathcal{X} \arrow[r, "(b)"'] & \Omega_Y \Sigma_Y \mathcal{C} \end{tikzcd}$$

This commutes by naturality of the unit map. We want to analyze the cartesianness of $(*)$. To do so, we will analyze the cartesianness of the cubes labeled by (a), (b) and (c) in the indicated order.

By the higher Blakers-Massey theorem, cube (a) is n-cartesian where

$$n = \min\{-2 + (2+1) + (1+1), \infty\} = 3$$

Cube (b) is handled similarly, Since Σ_Y increases connectivity of maps by 1, it increases cocartesianness by 1. Hence, the cube $\Sigma_Y \mathcal{X} \to \Sigma_Y \mathcal{C}$ is ∞-cocartesian and by the higher Blakers-Massey theorem, it is n-cartesian where

$$n = \min\{-2 + (3+1) + (2+1), \infty\} = 5.$$

Since Ω_Y decreases connectivity of maps by 1 and therefore cartesianness by 1, this implies (b) is 4-cartesian.

Finally, for cube (c), since $\Omega_Y Y \simeq Y \simeq \Sigma_Y$, this is equivalent to the cube

$$\begin{array}{ccc} Y & \longrightarrow & C \\ \| & & \downarrow \\ Y & \longrightarrow & \Omega_Y \Sigma_Y C \end{array}$$

The map $C \to \Omega_Y \Sigma_Y C$ is 6-connected since we have seen C is 2-connected relative to Y. Taking homotopy fibers in $\mathsf{Alg}_{\mathcal{O}}^Y$ horizontally, this becomes the map

$$\Omega_Y C \to \Omega_Y \Omega_Y \Sigma_Y C$$

which is 5-connected as Ω_Y decreases connectivity of maps. Since the total homotopy fiber is the iterated homotopy fiber, this shows that $\mathcal{C} \to \Omega_Y \Sigma_Y \mathcal{C}$ is 5-cartesian.

Amalgamating these results, we have

$$\begin{tikzcd}
\mathcal{X} \arrow[r, "3"] \arrow[d, "(*)"'] \arrow[dr] & \mathcal{C} \arrow[d, "5"] \\
\Omega_Y \Sigma_Y \mathcal{X} \arrow[r, "4"'] & \Omega_Y \Sigma_Y \mathcal{C}
\end{tikzcd}$$

It follows that the composite $\mathcal{X} \to \Omega_Y \Sigma_Y \mathcal{C}$ is 3-cartesian, and, hence, cube $(*)$ is 3-cartesian.

To get a feeling for the cases following from this one, let us continue and consider the next step. Keep $\mathcal{X}$ as above and let $\mathcal{Z} = (\mathcal{X} \to \Omega_Y \Sigma_Y \mathcal{X})$ be the cube $(*)$ of the preceding step. The relevant coface cube now has the form

$$\begin{tikzcd}
\mathcal{Z}: \arrow[d] & \mathcal{X} \arrow[r, "(*)"] \arrow[d] & \Omega_Y \Sigma_Y \mathcal{X} \arrow[d] \\
\Omega_Y \Omega_Y \mathcal{Z}: & \Omega_Y \Sigma_Y \mathcal{X} \arrow[r] & \Omega_Y \Sigma_Y \mathcal{X}
\end{tikzcd}$$

The downwards maps are the derived unit maps. The top cube $(*)$ is the one from the preceding step. As before, it is difficult to analyze the cartesianness of this cube directly so we augment it in the analogous manner.

Let C be the iterated cofiber of $\mathcal{Z}$, computed in $\mathsf{Alg}_{\mathcal{O}}^{Y}$ and let $\mathcal{C}$ be the 2-cube

$$\begin{tikzcd}
Y \arrow[r, equal] \arrow[d, equal] & Y \arrow[d] \\
Y \arrow[r] & C
\end{tikzcd}$$

As before, the cube $\mathcal{Z} \to \mathcal{C}$ is ∞-cocartesian and we augment our picture to

$$\begin{tikzcd} \mathcal{Z} \arrow[r, "(a)"] \arrow[d, "(*)"'] & \mathcal{C} \arrow[d, "(c)"] \\ \Omega_Y \Sigma_Y \mathcal{Z} \arrow[r, "(b)"'] & \Omega_Y \Sigma_Y \mathcal{C} \end{tikzcd}$$

and proceed in analyzing this cube as in the preceding case.

This time, however,it is not obvious what the connectivity of C is relative to Y. The key to analyzing this will be to determine the cocartesianness of $\mathcal{C}$ indirectly. If we estimate the cocartesianness of $\mathcal{Z}$, then since $\mathcal{Z} \to \mathcal{C}$ is ∞-cocartesian, we will be able to estimate the cocartesianness of $\mathcal{C}$ by [15, Prop. 3.8]. To do this, we use the higher dual Blakers-Massey theorem.

The 2-cube $\mathcal{Z}$ is the cube $(*)$ and therefore 3-cartesian. Writing out $\mathcal{Z}$ with all connectivities displayed,

$$\begin{tikzcd} X \arrow[r, "2"] \arrow[d, "2"'] & \Omega_Y \Sigma_Y X \arrow[d, "3"] \\ \Omega_Y \Sigma_Y X \arrow[r, "4"'] & \Omega_Y^2 \Sigma_Y^2 X \end{tikzcd}$$

Hence, by the higher dual Blakers-Massey theorem, $\mathcal{Z}$ is n-cocartesian where

$$n = \min\{5 + 2 - 1, 2 + 3 + 4\} = \min\{6, 9\} = 6.$$

It follows from [15, Prop. 3.8] that $\mathcal{C}$ is 6-cocartesian and, hence, the maps $Y \to C$ are 6-connected.

Recall that we are analyzing the following diagram of 2-cubes

$$\begin{tikzcd}
\mathcal{Z} \arrow[r, "(a)"] \arrow[d, "(*)"'] & \mathcal{C} \arrow[d, "(c)"] \\
\Omega_Y \Sigma_Y \mathcal{Z} \arrow[r, "(b)"'] & \Omega_Y \Sigma_Y \mathcal{C}
\end{tikzcd}$$

Consider cube (a). We apply the higher Blakers-Massey theorem 5.1.8. To organize our work, we will adopt the notation k_U of this theorem. For us, the cube $\mathcal{Z}$ corresponds to $\partial_{\emptyset}^{\{2,3\}}(\mathcal{Z} \to \mathcal{C})$. The 3-cube (a) is n-cartesian where n is the minimum of

$$\begin{aligned}
-3 + (k_{\{1\}} + 1) + (k_{\{2\}} + 1) + (k_{\{3\}} + 1) &= 1 + 2 + 2 = 5 \\
-3 + (k_{\{1\}} + 1) + (k_{\{2,3\}} + 1) &= -1 + 1 + 5 = 5 \\
-3 + (k_{\{2\}} + 1) + (k_{\{1,3\}} + 1) &= -1 + 2 + 3 = 4 \\
-3 + (k_{\{3\}} + 1) + (k_{\{1,2\}} + 1) &= -1 + 2 + 3 = 4 \\
-3 + (k_{\{1,2,3\}} + 1) &= \infty
\end{aligned}$$

Thus $n = 4$. Note that the computation of $k_{\{1,2\}}$ and $k_{\{1,3\}}$ uses the fact that for a retractive object, if $Y \to X$ is n-connected, then $X \to Y$ is $(n+1)$-connected or, alternatively, [15, Prop. 3.8]. In general, the cocartesianness of a cube $\mathcal{X} \to \underline{Y}$ of the sort considered above may be estimated by [15, Prop. 3.8] as precisely the cocartesianness of $\mathcal{X}$ plus 1, since the constant cube $\underline{Y}$ is already ∞-cocartesian. For cube (b), we proceed as before. Since Σ_Y raises connectivity of maps by 1 and, hence, cocartesianness by 1, the cube $\Sigma_Y \mathcal{Z} \to \Sigma_Y \mathcal{C}$ is ∞-cocartesian and by the higher dual

Blakers-Massey theorem, it is n-cartesian where n is the minimum of

$$
\begin{aligned}
-3 + (k_{\{1\}} + 1) + (k_{\{2\}} + 1) + (k_{\{3\}} + 1) &= 2 + 3 + 3 = 8 \\
-3 + (k_{\{1\}} + 1) + (k_{\{2,3\}} + 1) &= -1 + 2 + 6 = 7 \\
-3 + (k_{\{2\}} + 1) + (k_{\{1,3\}} + 1) &= -1 + 3 + 4 = 6 \\
-3 + (k_{\{3\}} + 1) + (k_{\{1,2\}} + 1) &= -1 + 3 + 4 = 6 \\
-3 + (k_{\{1,2,3\}} + 1) &= \infty
\end{aligned}
$$

Hence, $n = 6$. Since Ω_Y decreases connectivity of maps by 1 and therefore cartesianness by 1, this implies (b) is 5-cartesian.

Cube (c) may be written as

$$
\begin{tikzcd}
Y \arrow[rr, equal] \arrow[dr, equal] \arrow[dd] & & Y \arrow[dr] \arrow[dd] & \\
& Y \arrow[rr] \arrow[dd] & & C \arrow[dd] \\
Y \arrow[rr, equal] \arrow[dr, equal] & & Y \arrow[dr] & \\
& Y \arrow[rr] & & \Omega_Y \Sigma_Y C
\end{tikzcd}
$$

We computed that C is 6-connected relative to Y. Hence, the map $C \to \Omega_Y \Sigma_Y C$ is 14-connected. Taking homotopy fibers horizontally and then into the page gives the map

$$\Omega_Y^2(C) \to \Omega_Y^2(\Omega_Y \Sigma_Y C)$$

which is therefore 12-connected. Hence $\mathcal{C} \to \Omega_Y \Sigma_Y \mathcal{C}$ is 12-cartesian. Putting this together

$$\begin{tikzcd}
\mathcal{Z} \arrow[r, "4"] \arrow[d, "(*)"'] \arrow[dr, "4"] & \mathcal{C} \arrow[d, "12"] \\
\Omega_Y \Sigma_Y \mathcal{Z} \arrow[r, "6"'] & \Omega_Y \Sigma_Y \mathcal{C}
\end{tikzcd}$$

it follows that the composite $\mathcal{Z} \to \Omega_Y \Sigma_Y \mathcal{C}$ is 4-cartesian and, hence, the map $(*)$ is 4-cartesian.

This has shown that the maps $(*_0)$, $(*_1)$ and $(*_2)$ in the tower of Proposition 6.1.5 exhibit increasing connectivity for $Y \to X$ 0-connected. The first map is 2-connected, the second 3-connected and the third 4-connected.

6.3 Retractive Higher Freudenthal Suspension and Retractive Uniformity Correspondence

The sticking points of the general case are evident from our estimations above. We must estimate, in general, the connectivity of the total homotopy cofiber relative to Y and we must estimate the cartesianness of cubes $\mathcal{X} \to \mathcal{C}$. We need a uniform way to do this. In lieu of making an ad hoc argument, we will recognize the fundamental feature at play here. We follow closely the arguments of [8].

Let us recall the following definitions from [19, 20]; see also [8].

Definition 6.3.1. Let C be a category. A ***T-subcube*** of a W-cube $\mathcal{X}\colon \mathcal{P}(W) \to \mathsf{C}$ is a T-cube arising as the composite of $\mathcal{X}$ with an injection $\iota\colon \mathcal{P}(T) \to \mathcal{P}(W)$. If $|T| = d$, then we also refer to a T-subcube of $\mathcal{X}$ as a ***d-subcube***. Note that we permit $d = 0$ (i.e., $T = \emptyset$).

Definition 6.3.2. Given $f\colon \mathbf{N} \to \mathbf{N}$ any function, we say a W-cube $\mathcal{X}\colon \mathcal{P}(W) \to \mathsf{Alg}_{\mathcal{O}}^{Y}$ is ***f*-cartesian** (resp. ***f*-cocartesian**) if every d-subcube of $\mathcal{X}$ is $f(d)$-cartesian (resp. $f(d)$-cocartesian).

We need the following easy observations.

Lemma 6.3.3. *Let $f\colon \mathbf{N} \to \mathbf{N}$ be a function and suppose $\mathcal{X}\colon \mathcal{P}(W) \to \mathsf{Alg}_{\mathcal{O}}^{Y}$ is f-cartesian (resp. f-cocartesian), then map of $\mathcal{X}$ is 0-connected and every object $\mathcal{X}_U$ is (-1)-connected. In particular, each map $Y \to \mathcal{X}_U$ is (-1)-connected (resp. 0-connected).*

Proof. For the maps, this follows since $f(d) \geq 0$ for all $d \in \mathbf{N}$. For the objects, in the cartesian case, note that 0-subcubes of $\mathcal{X}$ correspond to the objects and the homotopy limit of the empty diagram in $\mathsf{Alg}_{\mathcal{O}}^{Y}$ is simply Y. Hence, in the cocartesian case, each map $Y \to \mathcal{X}_U$ is 0-connected and in the cartesian case, each map $\mathcal{X}_U \to Y$ is 0-connected which means the structure map $Y \to \mathcal{X}_U$ is (-1)-connected. Since we assumed Y is (-1)-connected, this means each object is (-1)-connected, in each case because we have a composite of (-1)-connected maps

$$* \xrightarrow{-1} Y \xrightarrow{-1} X.$$

$\square$

Lemma 6.3.4. *Let $f\colon \mathbf{N} \to \mathbf{N}$ be a function. If $\mathcal{X}\colon \mathcal{P}(W) \to \mathsf{Alg}_{\mathcal{O}}^{Y}$ is an f-cartesian (resp. f-cocartesian) W-cube, then for any subcube $\mathcal{Y}$ of $\mathcal{X}$, $\mathcal{Y}$ is f-cartesian (resp. f-cocartesian).*

Proof. This is likewise a simple matter of unpacking definitions. $\square$

The essential feature of our computation above is the following theorem. Note that this is also proved in the non-retractive setting in [7, Thm. 3.4].

Theorem 6.3.5 (Higher Retractive Freudenthal Suspension)**.** *Let $k \geq 1$ Suppose $\mathcal{X} \colon \mathcal{P}(W) \to \mathsf{Alg}_{\mathcal{O}}^{Y}$ is a $((k+1)(\mathrm{id}+1))$-cartesian W-cube. Then for each $1 \leq r < \infty$, so too is $\mathcal{X} \to \Omega_Y^r \Sigma_Y^r \mathcal{X}$ obtained by applying the derived unit.*

Before giving the proof, let us observe a corollary.

Corollary 6.3.6. *Let $k \geq 0$ and X be k-connected relative to Y. The coface $(n+1)$-cube associated to the cosimplicial resolution of X from the derived (Ω_Y^n, Σ_Y^n)-adjunction is $((k+1)(\mathrm{id}+1))$-cartesian.*

Proof. Since X is at least 0-connected relative to Y, the map $X \to Y$ is at least 1-connected. The coface $(n+1)$-cube is built from X by iterated the Freudenthal suspension map $X \to \Omega_Y^n \Omega_Y^n X$, so this follows from Theorem 6.3.5. □

Theorem 6.3.5 will be a consequence of this next essential feature of our above computation. This correspondence is proved in the non-retractive setting in [7, Prop. 3.3].

Proposition 6.3.7 (Retractive Uniformity Correspondence)**.** *Let $k \geq 0$. A W-cube $\mathcal{X} \colon \mathcal{P}(W) \to \mathsf{Alg}_{\mathcal{O}}^{Y}$ is $((k+1)(\mathrm{id}+1))$-cartesian if and only if it is $((k+2)\,\mathrm{id}+k)$-cocartesian (equivalently, $((k+2)(\mathrm{id}+1)-2)$-cocartesian).*

Proof. When $|W| = 0$, this amounts to saying that a retractive object has structure map $X \to Y$ $k+1$-connected if and only if its other structure map $Y \to X$ k-connected and this is true. When $|W| = 1$, the numbers in question are $(k+1)(1+1) = 2k+2$ and $(k+2)(2)-2 = 2k+4-2 = 2k+2$ and this therefore checks out as cartesianness and cocartesianness of 1-cubes is simply connectivity of maps. Thus, we may suppose without loss of generality that $|W| \geq 2$.

(⇒) We induct on $|W|$ where we may suppose $|W| \geq 2$, say $|W| = n$. By induction hypothesis, all we need to check is that $\mathcal{X}$ is $((k+2)n+k)$-cocartesian. By the higher dual Blakers-Massey theorem, this is m-cocartesian where

$$m = \min \left\{ \{(k+1)(n+1)+n-1\} \cup \left\{ n + \sum_{V\in\lambda} (k+1)(|V|+1) : \lambda \in \mathbf{Par}_{\neq\emptyset,\neq W}(W) \right\}) \right\}.$$

Since any partition λ of W, where $|W| = n$ into non-empty subsets has the property $\sum_{V\in\lambda} |V| = n$,

$$n + \sum_{V\in\lambda} (k+1)(|V|+1) = n + (k+1) \sum_{V\in\lambda} (|V|+1) = n + (k+1)\,|\lambda| + (k+1)n = (k+2)n + (k+1)\,|\lambda|$$

and $|\lambda| \geq 1$, so the minimum of this sum over the partitions λ is $(k+2)n+(k+1)$. Then

$$\begin{aligned} m &= \min\{(k+1)(n+1)+n-1, (k+2)n+(k+1)\} = \min\{(k+2)n+k, (k+2)n+k+1\} \\ &= (k+2)n+k, \end{aligned}$$

as desired.

(⇐) We induct on $|W|$ where we may suppose $|W| \geq 2$, say $|W| = n$. By induction hypothesis, all we need to check is that $\mathcal{X}$ is $((k+1)(n+1))$-cartesian. By the higher Blakers-Massey theorem, $\mathcal{X}$ is m-cartesian where m is the minimum

$$\begin{aligned} m &= \min \left\{ -n + \sum_{V\in\lambda} (((k+2)\,|V| + k) + 1) : \lambda \in \mathbf{Par}_{\neq\emptyset}(W) \right\} \\ &= \min \left\{ -n + (k+1)\,|\lambda| + (k+2) \sum_{V\in\lambda} |V| : \lambda \in \mathbf{Par}_{\neq\emptyset}(W) \right\} \end{aligned}$$

Once again, ${}_{V\in\lambda} |V| = n$, so

$$m = \min \quad -n + (k+1)\,|\lambda| + (k+2)n : \lambda \in \mathbf{Par}_{\neq\emptyset}(W) \quad .$$

Hence, taking λ to be the coarsest partition consisting of only W minimizes this. But then

$$m = -n + (k+1) + (k+2)n = (k+1)n + (k+1) = (k+1)(n+1)$$

as claimed. $\square$

With this we can prove Theorem 6.3.5. The manner of attack will be closely related to the cases we worked out above.

Proof of Theorem 6.3.5. We induct on $|W|$. If $|W| = 0$, this is simply the retractive Freudenthal suspension theorem, so suppose $n = |W| \geq 1$. Let C be the iterated homotopy cofiber of $\mathcal{X}$ and let $\mathcal{C}$ be the W-cube with $\mathcal{C}_U = Y$ for $U \neq W$ and $\mathcal{C}_W = C$. Then $\mathcal{X} \to \mathcal{C}$ is ∞-cocartesian and we consider the following commutative diagram

$$\begin{tikzcd}
\mathcal{X} \arrow[r] \arrow[d, "(*)"'] & \mathcal{C} \arrow[d] \\
\Omega_Y^r \Sigma_Y^r \mathcal{X} \arrow[r] & \Omega_Y^r \Sigma_Y^r \mathcal{C}
\end{tikzcd}$$

where the name of the game is to estimate the cartesianness of $(*)$. By the uniformity correspondence, Proposition 6.3.7, we know $\mathcal{X}$ is $((k+2)\,\mathrm{id} + k)$-cocartesian. In particular, $\mathcal{X}$ is $((k+2)n+k)$-cocartesian and thus $\mathcal{C}$ is $((k+2)n+k)$-cocartesian by [15, Prop. 3.8], which amounts to saying that C is $((k+2)n+k)$-connected relative to Y.

For any d-subcube $T \subset \mathcal{P}(W)$ of $\mathcal{X}$ such that $W \notin T$ (so $|T| = d \leq n-1$), it follows that $\mathcal{X}|T \to \mathcal{C}|T \simeq \underline{Y}$ is $((k+2)d+k+1)$-cocartesian by [15, Prop. 3.8]. If $W \in T$, then $\mathcal{C}|T$ is $((k+2)n+k)$-cocartesian from the above and for any $k \geq 0$ and $d \leq n-1$, $(k+2)d+k+1 \leq (k+2)n+k$, we claim. This is a simple matter of arithmetic as

$$\begin{aligned}(k+2)d+k+1 &\leq (k+2)(n-1)+k+1 = (k+2)n+k-(k+2)+1 \\ &= (k+2)n+k-k-2+1 = (k+2)n-1 \leq (k+2)n+k\end{aligned}$$

Exactly as claimed.

Hence, for each such T where $|T| = d < n$ with $W \in T$, as $\mathcal{X}|T$ is $((k+2)d+k)$-cocartesian and $\mathcal{C}|T$ is *at least* $((k+2)d+k+1)$-cocartesian, it follows by [15, Prop. 3.8] that $\mathcal{X}|T \to \mathcal{C}|T$ is *at least* $((k+2)+d+k+1)$-cocartesian. Since $(k+2)d+k \leq (k+2)d+k+1$ for all $k \geq 0$, it follows by the higher Blakers-Massey theorem, the whole cube $\mathcal{X} \to \mathcal{C}$ is *at least* m-cartesian where

$$\begin{aligned} m &= \min\left\{-n + \underset{V\in\lambda}{} (((k+2)\,|V|+k)+1) : \lambda \in \mathbf{Par}_{\neq\emptyset}(W)\right\} \\ &= \min\ \ -n+(k+2)n+(k+1)\,|\lambda| : \lambda \in \mathbf{Par}_{\neq\emptyset}(W)\end{aligned}$$

and this is minimized by taking $\lambda = \{W\}$ the coarsest partition. Hence, $\mathcal{X} \to \mathcal{C}$ is at least m-cocartesian where

$$m = -n+(k+2)n+(k+1) = (k+1)n+(k+1) = (k+1)(n+1).$$

Since Σ_Y^r raises cocartesianness by r, it follows similarly that $\Sigma_Y^r \mathcal{X} \to \Sigma_Y^r \mathcal{C}$ is at least $((k+1)(n+1)+r)$-cartesian. Hence, $\Omega_Y^r \Sigma_Y^r \mathcal{X} \to \Omega_Y^r \Sigma^r \mathcal{C}$ is *at least* $((k+1)(n+1))$-cartesian.

Since C is $((k+2)n+k)$-connected relative to Y, $C \to \Omega_Y^r \Omega_Y^r C$ is $(2(k+2)n+2k+2)$-connected by Corollary 5.2.7. By taking iterated fibers, it follows that the map

$$\Omega_Y^n(C) \to \Omega_Y^n(\Omega_Y^r \Sigma_Y^r C)$$

is $(2(k+2)n+2k+2-n)$-connected. In other words, $((2k+3)n+2k+2)$-connected. Hence, $\mathcal{C} \to \Omega_Y^r \Omega_Y^r \mathcal{C}$ is $((2k+3)n+2k+2)$-cartesian.

We claim that for all $n \geq 2$ and $k \geq 0$, $(k+1)(n+1) \leq (2k+3)n+2k+2$ and this follows since it is, equivalently, the assertion that $n \leq kn+3n+k+1$ and this evidently holds even with $k=0$.

Putting this all together then, with cartesianness labeled, we have

$$\begin{tikzcd}
\mathcal{X} \arrow[r, "(k+1)(n+1)"] \arrow[d, "(*)"'] \arrow[dr, dashed] & \mathcal{C} \arrow[d, "(2k+3)n+2k+2"] \\
\Omega_Y^r \Sigma_Y^r \mathcal{X} \arrow[r, "(k+1)(n+1)"'] & \Omega_Y^r \Sigma_Y^r \mathcal{C}
\end{tikzcd}$$

where the composite map is $((k+1)(n+1))$-cartesian by [15, Prop. 3.9]. It follows that $(*)$ is $((k+1)(n+1))$-cartesian, as claimed.

We showed this in the case that the subcube is the whole cube. The analysis is the same, almost verbatim, on all subcubes, and this gives the result. $\square$

6.4 Proof of the Main Theorem

In particular, by way of our grand strategy and Corollary 6.3.6, the preceding proves the following.

Theorem 6.4.1. *Let $1 \leq r < \infty$. If $X \in \mathsf{Alg}_{\mathcal{O}}^{Y}$ is 0-connected relative to Y, then the Bousfield-Kan completion map*

$$X \to X^{\wedge}_{\Omega^r_Y \Sigma^r_Y}$$

is an equivalence.

We now turn our attention to the case of completion with respect to stabilization $\Omega_Y^\infty \Sigma_Y^\infty$. In a certain sense, this is a special case of what we have shown for $\Omega_Y^k \Sigma_Y^k$. This is made precise by the following corollary of the retractive higher Freudenthal suspension theorem.

We will give two proofs of this corollary, with the second deferred to a remark. The first is somewhat reminiscent of maneuvers Goodwillie makes in [24]. The second is based on stability of $\mathsf{Sp}^{\mathbf{N}}(\mathsf{Alg}_{\mathcal{O}}^{Y})$. The equivalence of the two arguments is guaranteed by Theorem 3.3.1.

Corollary 6.4.2 (Higher Stabilization)**.** *If $\mathcal{X}$ is a $((k+1)(\mathrm{id}+1))$-cartesian n-cube where $k, n \geq 0$, then so too is $\mathcal{X} \to \Omega_Y^\infty \Sigma_Y^\infty \mathcal{X}$, where the map is the derived unit.*

Proof. Without loss of generality, we may suppose $\mathcal{X}$ is an n-cube of cofibrant objects. In this case, we may suppose, without loss of generality, that $\Omega_Y^\infty \Sigma_Y^\infty \mathcal{X} = \operatorname{colim} \Omega_Y^k \Sigma_Y^k \mathcal{X}$ by Theorem 3.3.1. Let us consider the cartesianness of the whole cube first.

Note that filtered homotopy colimits and finite homotopy limits of punctured cubes in $\mathsf{Alg}_{\mathcal{O}}^{Y}$ are computed in $\mathsf{Alg}_{\mathcal{O}}$, where they are computed in spectra, where they commute, and so they commute in $\mathsf{Alg}_{\mathcal{O}}^{Y}$ as a consequence of Corollary 5.1.5. Recall that the derived unit map $\mathcal{X} \to \Omega_Y^k \Sigma_Y^k \mathcal{X}$ factors through the derived unit maps $\mathcal{X} \to \Omega_Y \Sigma_Y \mathcal{X} \to \cdots \to \Omega_Y^k \Sigma_Y^k \mathcal{X}$ as in Corollary 5.2.7. Denote the $(n+1)$-cube

$$\mathcal{Z}_k = \quad \mathcal{X} \to \Omega_Y^k \Sigma_Y^k \mathcal{X} \quad .$$

Hence, we have a commutative diagram

$$\begin{tikzcd}
\mathcal{X}_\emptyset \arrow[r, equal] \arrow[d, "(k+1)(n+2)"'] & \mathcal{X}_\emptyset \arrow[d, "(k+1)(n+2)"] \\
\operatorname*{holim}_{\mathcal{P}_0(\underline{n+1}))} \mathcal{Z}_k \arrow[r, "(k+1)(n+2)"'] & \operatorname*{holim}_{\mathcal{P}_0(\underline{n+1}))} \mathcal{Z}_{k+1}
\end{tikzcd}$$

where each of the vertical maps are $((k+1)(n+2))$-connected by Theorem 6.3.5. Hence, the horizontal map is also $((k+1)(n+2))$-connected. By commutativity of filtered homotopy colimits and finite homotopy limits, the map $\mathcal{X}_\emptyset \to \operatorname{holim}_{\mathcal{P}_0(\underline{n+1})}(\mathcal{X} \to \Omega_Y^\infty \Sigma_Y^\infty \mathcal{X})$ may be written as the homotopy colimit of the map of sequences

$$\begin{tikzcd}
\mathcal{X}_\emptyset \arrow[r, equal] \arrow[d, "(k+1)(n+2)"'] & \mathcal{X}_\emptyset \arrow[r, equal] \arrow[d, "(k+1)(n+2)"] & \mathcal{X}_\emptyset \arrow[r, equal] \arrow[d, "(k+1)(n+2)"] & \cdots \\
\operatorname*{holim}_{\mathcal{P}_0(\underline{n+1}))} \mathcal{Z}_1 \arrow[r, "(k+1)(n+2)"'] & \operatorname*{holim}_{\mathcal{P}_0(\underline{n+1}))} \mathcal{Z}_2 \arrow[r, "(k+1)(n+2)"'] & \operatorname*{holim}_{\mathcal{P}_0(\underline{n+1}))} \mathcal{Z}_3 \arrow[r, "(k+1)(n+2)"'] & \cdots
\end{tikzcd}$$

Since homotopy groups commute with filtered homotopy colimits, this shows that $\mathcal{X}_\emptyset \simeq \operatorname{hocolim} \mathcal{X}_\emptyset \to \operatorname{hocolim}_k \operatorname{holim}_{\mathcal{P}_0(\underline{n+1}))}(\mathcal{X} \to \Omega_Y^k \Sigma_Y^k \mathcal{X}) \simeq \Omega_Y^\infty \Sigma_Y^\infty \mathcal{X}$ is $((k+1)(n+2))$-connected. This shows that $\mathcal{X} \to \Omega_Y^\infty \Sigma_Y^\infty \mathcal{X}$ is $((k+1)(n+2))$-cartesian. Repeating this argument on every subcube gives the result. $\square$

Remark 6.4.3. Alternatively, this can be shown using stability of $\mathsf{Sp}^{\mathbf{N}}(\mathsf{Alg}_{\mathcal{O}}^{Y})$, since Σ_Y^∞ preserves connectivity of objects and maps. Hence, by the uniformity correspondence, if $\mathcal{Y}$ is $((k+1)(\mathrm{id}+1))$-cartesian $(n+1)$-cube, then it is $((k+2)\,\mathrm{id}+k)$-cocartesian and so $\Sigma^\infty_Y\mathcal{Y}$ is $((k+2)\,\mathrm{id}+k)$-cocartesian. By applying [15, Prop. 3.10] to each subcube, it follows that $\Sigma^\infty_Y\mathcal{Y}$ is $((k+2)\,\mathrm{id}+k-\mathrm{id}+1) = ((k+1)\,\mathrm{id}+k+1)$-cartesian. Hence, $\Omega^\infty_Y\Sigma^\infty_Y\mathcal{Y}$ is $((k+2)\,\mathrm{id}+k-\mathrm{id}+1)$-cartesian or, in other words, $((k+1)\,\mathrm{id}+k+1)$-cartesian. Hence, by taking $\mathcal{Y} = (\mathcal{X} \to \mathcal{C})$ as in the proof of Theorem 6.3.5, the very same argument of Theorem 6.3.5 proves the preceding corollary.

In particular, by way of our grand strategy, this proves the following.

Theorem 6.4.4. *If $X \in \mathsf{Alg}^Y_{\mathcal{O}}$ is 0-connected relative to Y, then the Bousfield-Kan completion map*

$$X \to X^\wedge_{\Omega^\infty_Y\Sigma^\infty_Y}$$

is an equivalence.

Proof. The coface $(n+1)$-cube is built from X by iterating the derived unit, starting first with the map $X \to \Omega^\infty_Y\Sigma^\infty_Y X$, which is at least 2-connected. The preceding corollary then shows the labeled maps of Proposition 6.1.5 are increasing in connectivity. $\square$

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