Homological Percolation in a Torus

Dissertation

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Abstract

In this thesis we study extensions to higher dimensions of several versions of percolation within a torus. We include the contents of two papers, one covering independent percolation and the other covering a well-known dependent model.

Percolation traditionally studies the appearance of infinite components in random subgraphs of lattices. The canonical subgraphs studied this way are obtained by taking a constant fraction of the vertices or edges independently at random. One can build higher dimensional random complexes in this way, but it is not clear what the equivalent of the infinite component should be. Bobrowski and Skraba offered a partial answer in finite volume complexes called homological percolation. In the torus $\mathbb{T}^d$, homological percolation in dimension i means that the subcomplex contains a representative of a nontrivial element of $H_i\left(\mathbb{T}^d\right)$. In section 2 we consider analogues of both bond and site percolation in the torus and show that such percolation has a sharp threshold

function in all dimensions. We also show that percolation in half the dimension of the torus occurs at $p = 1/2$, analogous to the classical Harris-Kesten theorem.

Another percolation model of interest is the random-cluster model, which weights configurations of independent percolation according to the number of connected components. This is a particularly interesting model because it can be coupled with the Ising and Potts models of magnetism. In section 3 we study a higher dimensional version of this model introduced by Hiraoka and Shirai, which also admits a coupling to a higher dimensional Potts model. It is worth noting that the two dimensional random-cluster model is associated to a Potts lattice gauge theory, which is related to interesting questions from physics. We prove similar results about sharp thresholds for homological percolation in the random-cluster model, which sheds some light on the Wilson loops in the associated lattice gauge theory.

Dedicated to my parents

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I would first like to deeply thank my advisor, Matt Kahle. He pushed me to become the best mathematician I could be, but also reminded me not to lose sight of the big picture. All of the work collected in this thesis was heavily influenced by his mentorship, support, and eye for a good problem.

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Chapter 1: Introduction

1.1 Organization of this Thesis

This thesis is organized as follows.

In Chapter 1, we will review relevant background on classical percolation theory and algebraic topology. In Chapter 2, we prove the existence of sharp thresholds for homological percolation in higher dimensional analogues of bond and site percolation in a torus. Chapter 3 contains analogues of the results of Chapter 2 in a higher dimensional random cluster model, which we then use to study the associated Potts lattice gauge theory. In Chapter 4, we discuss future directions and open problems.

1.2 Motivation

The relationship between small scale and large scale properties is central to much of mathematics and physics. In this thesis, we consider this question in the context of percolation models originally introduced to model physics. Broadbent and Hammersley introduced bond percolation, which we will define in the next section, in order to model the flow of water through a porous rock [13]. One can imagine that at a very small scale, the cavities through which the water may pass appear at random. When the rock is relatively dense, the holes are isolated, but as more holes appear, they eventually join together to form long channels that the water can pass through.

From a theoretical perspective, these kinds of models are often interesting because they exhibit phase transitions. A familiar example of this is the physical transition between states of matter. The temperature of water, for example, is a measurement of the speed of the atoms at a small scale, and whether water is in the liquid or solid phase is observable at a large scale. Unlike many physical processes, changes in states of matter occur discontinuously as temperature is varied; there is a single critical temperature value at which the change occurs. It turns out that the mathematical model for the porous rock has a similar property, namely a single critical density value at which the holes suddenly combine in a way that is visible at a large scale.

Percolation of one-dimensional objects such as the paths of water through a rock mentioned before has been well studied and remains a highly active area of research. Less is known about percolation of higher dimensional objects. An informal analogy comparing the different models is randomly sewing together patches to make a huge quilt instead of randomly tying together segments to make a long rope. In order to make this precise, we will use the tools of algebraic topology, which measures shapes in a way that remains consistent when they are continuously deformed. The idea of studying higher dimensional percolation from a topological perspective also came from physics. Frisch and Hammersley wrote in 1963: "Nearly all extant percolation theory deals with regular interconnecting structures, for lack of knowledge of how to define randomly irregular structures. Adventurous readers may care to rectify this deficiency by pioneering branches of mathematics that might be called *stochastic geometry* or *statistical topology*."

We are also motivated by the study of lattice gauge theories, which offer discrete approximations to fundamental forces in the Standard Model of physics. These are often also thought of as one-dimensional objects, namely random assignments of certain kinds of values to the edges of a graph. However, this turns out to be connected to a type of two-dimensional percolation, which we explore in Chapter 3.

1.3 Independent Percolation

Let G be an infinite graph with vertex set V and edge set E. For historical reasons, vertices are often referred to as sites and edges are referred to as bonds. Bernoulli bond percolation on G with parameter $p \in [0, 1]$ is the random subgraph obtained by taking V and adding each edge of E independently with probability p. Bernoulli site percolation is similar, except that we take each vertex of V independently with probability p, and then add all edges of E

between the remaining vertices. Equivalently, we can designate each edge or vertex open with probability p and closed with probability $1-p$ independently, and then consider the open subgraph.

The focus of percolation theory is the event that there is an infinite component in the random subgraph. Let $\theta_G^{\text{bond}}(p)$ be the probability that there is an infinite component in bond percolation with probability p on G. Then we define the critical probability

$$p_c^{\text{bond}}(G) = \inf\left\{p : \theta_G^{\text{bond}}(p) > 0\right\} .$$

The critical probability $p_c^{\text{site}}(G)$ is defined analogously. The results we state in this section will apply to both settings, so we will only state them in the bond case. One reason that percolation is interesting is that $\theta_G^{\text{bond}}(p) \in \{0, 1\}$, which can be proven by Kolmogorov's 0-1 law, among other methods. In particular, this means that θ is discontinuous as a function of p at p_c. Such discontinuities are called phase transitions, and are of interest in physical models.

The percolation literature is substantial, and we will not attempt to cover all of it here, but we will mention a few of the important questions for background. Determining $p_c^{\text{bond}}(G)$ or $p_c^{\text{site}}(G)$ exactly is thought to be intractable on most graphs G, even among familiar lattice such as $\mathbb{Z}^d$. However, $\mathbb{Z}^2$ is a notable exception. Harris showed that $\theta_{\mathbb{Z}^2}^{\text{bond}}(1/2) = 0$ [34], and twenty years later Kesten [39] showed that $\theta_{\mathbb{Z}^2}^{\text{bond}}(1/2 + \epsilon) > 0$ for any $\epsilon > 0$, completing the proof that $p_c^{\text{bond}}(\mathbb{Z}^2) = 1/2$. An underlying philosophy is to use the self duality of

$\mathbb{Z}^2$, and this is also at the heart of much of the work in this thesis. A similar technique can be used to show that $p_c^{\text{site}}(\mathcal{T}) = 1/2$, where $\mathcal{T}$ is the triangular lattice.

We will also mention a couple of well known results characterizing the behavior of percolation when $p < p_c$ and $p > p_c$, called the subcritical and supercritical regimes respectively. In the subcritical regime, the connected components of the graph are in some sense probabilistically as small as possible. This is made precise by the following theorem due to Menshikov[45], also proven independently by Aizenman and Barsky [2]. For a vertex x and a subset S of $\mathbb{Z}^d$, denote the event that x is connected to a vertex in S by a path of edges in the open subgraph by $x \leftrightarrow S$.

Theorem 1.3.1 (Menshikov/Aizenman–Barsky)**.** *Consider bond percolation on $\mathbb{Z}^d$. If $p < \hat{p}_c$ then there exists a $\kappa(p) > 0$ so that*

$$\mathbb{P}_p\left(0 \leftrightarrow \partial[-M, M]^d\right) \leq e^{-\kappa(p)M}$$

for all $M > 0$.

We stated the theorem for $\mathbb{Z}^d$ for simplicity, but a more general version with a functionally similar bound can be found in [12]. Note that this theorem does not rule out large components in an absolute sense. Indeed, any set of k edges is open with probability p^k, so in an infinite graph one will almost surely find arbitrarily large components. However, such components will be as rare as they can possibly be in such a random model. We will use this theorem in Chapter 2 to show that homological 1-percolation in the torus $\mathbb{T}^d$ has the same critical probability as bond percolation in $\mathbb{Z}^d$.

The supercritical regime is characterized by a fundamental theorem of Aizenman, Kesten, and Newman [5] on the uniqueness of the infinite component.

Theorem 1.3.2 (Aizenman–Kesten–Newman)**.** *Consider bond percolation on $\mathbb{Z}^d$. Let $p > p_c\left(\mathbb{Z}^d\right)$. Then*

$$\mathbb{P}_p\left(\textit{there is exactly one infinite open component}\right) = 1\,.$$

Again, we have given a more narrow statement of this theorem than is necessary. A later proof given by Burton and Keane [14] showed that amenability is a sufficient condition for the uniqueness of the infinite component in almost transitive graphs. Benjamini and Schramm [9] conjectured that it is also necessary, spurring a flurry of activity in the study of nonamenable percolation. We do not use a uniqueness result directly in this thesis, but we do cite results using this important tool.

We will now mention a couple of results that are somewhat more general, but are often useful in studying percolation models. Most constructions in classical percolation involve putting together smaller paths in order to build larger paths. In order to bound the probability of such a construction from below, it is useful to be able to describe the correlations between the events describing the existence of the building blocks. In independent percolation, Harris [34] showed that many such events are positively correlated. Let X be the probability space formed by the product of n Bernoulli(p) random variables, and let μ_p be the probability measure on the power set $\mathcal{P}(X)$ defined by taking each element of X independently with probability p. That is, if $Y \subseteq X$,

$$\mu(Y) = p^{|Y|}(1-p)^{|Y|} .$$

An event B is *increasing* if

$$Y_0 \subset Y_1 \text{ and } Y_0 \in B \text{ together imply that } Y_1 \in B .$$

Theorem 1.3.3 (Harris's Inequality). *If $B_1, \ldots, B_j$ are increasing events then*

$$\mathbb{P}\left(\bigcap_{k=1}^{j} B_k\right) \geq \prod_{k=1}^{j} \mathbb{P}(B_k) .$$

We will also later use a result of Friedgut and Kalai on sharpness of thresholds [28].

Theorem 1.3.4 (Friedgut–Kalai). *Let B be an increasing event that is invariant under a transitive group action on X. There exists a constant $\rho > 0$ so that if $\mu_p(B) > \epsilon > 0$ and*

$$q \geq p + \rho \frac{\log(1/(2\epsilon))}{\log(|X|)} \tag{1.1}$$

then $\mu_q(B) > 1 - \epsilon$.

1.4 Random-cluster Percolation

Perhaps the most important dependent percolation model is the random-cluster model, sometimes called Fortuin-Kasteleyn(FK) percolation [27]. Unlike independent percolation, the random-cluster model only comes in a bond percolation version. The probability of an edge configuration ω in the random cluster model with parameters p and q on a finite graph G is given by

$$\mu_{G,p,q}(\omega) = \frac{1}{Z} p^{\eta(\omega)} (1-p)^{|E-\eta(\omega)|} q^{k(\omega)},$$

where $\eta(\omega)$ is the number of open edges in ω, $k(\omega)$ is the number of connected components of the open subgraph, and Z is a normalizing constant.

One reason that the random-cluster model is interesting from a physics perspective is that it is closely related to the Potts model of magnetism. The states of the q-state classical Potts model can be thought of as the functions $f : V \to \mathbb{Z}_q$, where the value of a given vertex is interpreted as its spin. The energy of a state f is given by the Hamiltonian

$$H_q(f) = \sum_{\{v,w\} \in E} K(f(v), f(w)) ,$$

where $K(x, y)$ is the Kronecker delta function (written this way to avoid confusion with the notion of coboundary that will be explained in the next section). Then the probability of a state is given by

$$\nu(f) = \frac{1}{Z} e^{-\beta H_q(f)} ,$$

where Z is again a normalizing constant and β is a constant representing inverse temperature. Edwards and Sokal showed that the random-cluster model with parameters p, q and the q-state Potts model with $\beta = 1 - e^{-\beta}$ can be naturally coupled [24]. Informally, the two measures are the marginals of independent bond percolation with probability p and uniform independent assignments of states to vertices, conditioned on there only being open edges between vertices of the same state. We will give a more detailed description of the coupling in the higher dimensional version.

The random-cluster model on an infinite graph can be defined as a weak limit of the random cluster model on a sequence of subgraphs [33]. In $\mathbb{Z}^d$, for example, one can take the limit in boxes $\Lambda_n := \{-n, -n+1, \ldots, n-1, n\}^d$. However, when discussing the random-cluster model on a subgraph $H \subset G$, there is some ambiguity regarding connected components that reach the boundary, since they may or may not become connected outside of H. Because of this, it is often useful to consider boundary conditions that encode these possible connections. The free random-cluster model with measure $\mu^{\mathbf{f}}_{H,p,q}$ (sometimes written $\mu^0_{H,p,q}$) makes no identifications on the boundary and simply weights by the number of connected components in the random subgraph of H. On the other extreme is the wired random-cluster model with measure $\mu^{\mathbf{w}}_{H,p,q}$ (sometimes written $\mu^1_{H,p,q}$), which identifies the vertex boundary of H in G before counting connected components in the percolation subgraph. For fixed q, it is known that the free and wired limits coincide in $\mathbb{Z}^d$ for all but countably many values of p(see Chapter 4 of [33]), but whether or not they always coincide is an open problem. In the torus, we need not worry about boundary conditions, but when we work in $\mathbb{Z}^d$ we will use the free boundary limit unless otherwise stated.

The states of different edges are now no longer independent because changing the state of a given edge may or may not change the number of connected components, depending on its surroundings. Since many standard percolation arguments rely on dealing with different parts of the graph separately, the random cluster model is much more difficult to work with. However, there has been a significant amount of progress, some quite recent, that has established versions of several of the previously mentioned results from independent percolation in this dependent setting. One reason the random cluster model is tractable at all is that when $q \geq 1$, Fortuin, Kasteleyn, and Ginibre showed that it retains the positive association property given by Harris' Lemma in the independent model [26].

Theorem 1.4.1 (Fortuin–Kasteleyn–Ginibre)**.** *Let G be a finite graph. Let $p \in [0,1]$ and $q \geq 1$. Then if $B_1, \ldots, B_j$ are increasing events then*

$$\mu_{G,p,q}\left(\bigcap_{k=1}^{j} B_k\right) \geq \prod_{k=1}^{j} \mu_{G,p,q}\left(B_k\right) .$$

It is also useful to be able to compare different random-cluster models via coupling. This is straightforward in the independent model, but requires some work when edges are dependent. Fortunately, Holley gave a useful criterion for one model to have more open edges than another in a certain sense [37]. Let E be a finite set and let μ_1, μ_2 be probability measures on $\Omega = \{0,1\}^E$. We say that μ_1 is *stochastically dominated* by μ_2 if there is a probability measure κ on $\Omega \times \Omega$ with first and second marginals μ_1 and μ_2 such that

$$\kappa\left(\{(\omega_1, \omega_2) : \omega_1 \leq \omega_2\}\right) = 1\,.$$

In this case we write $\mu_1 \leq_{\mathrm{st}} \mu_2$. The following formulation of Holley's theorem can be found as Theorem 2.3 of [33].

Theorem 1.4.2 (Holley)**.** *Let E be a finite set and let μ_1, μ_2 be strictly positive probability measures on $\Omega = \{0,1\}^E$. Suppose that for each pair $\xi, \zeta \in \Omega$ with $\xi \leq \zeta$ and each $e_0 \in E$,*

$$\begin{aligned}&\mu_1\left(\omega\left(e_0\right) = 1 : \omega\left(e\right) = \xi\left(e\right) \textit{ for all } e \in E \setminus \{e_0\}\right)\\ &\qquad \leq \mu_2\left(\omega\left(e_0\right) = 1 : \omega\left(e\right) = \zeta\left(e\right) \textit{ for all } e \in E \setminus \{e_0\}\right)\,.\end{aligned}$$

Then $\mu_1 \leq_{\mathrm{st}} \mu_2$.

There is also a version of the Harris-Kesten theorem for the random-cluster model. It is not difficult to check that the dual percolation is also distributed a random-cluster model with parameters q and $p^* = p^*(p,q)$ where

$$p^* := \frac{(1-p)\,q}{(1-p)\,q + p}\,.$$

We will prove this in a higher dimensional setting in Chapter 3. Beffara and Duminil-Copin proved that in $\mathbb{Z}^2$, the critical probability is the value of p at which $p = p^*$ [8], namely

$$p_{\text{sd}} \coloneqq \frac{\sqrt{q}}{1+\sqrt{q}}$$

at which the system is self dual.

The subcritical and supercritical regimes of the random-cluster model were much less well understood than their independent counterparts, but a recent breakthrough paper of Duminil-Copin, Raoufi, and Tassion provided the following characterization [21].

Theorem 1.4.3 (Duminil-Copin–Raoufi–Tassion)**.** *Fix $d \geq 2$ and $q \geq 1$. For a finite graph G, let $\mu^{\mathbf{w}}_{G,p,q}$ be the random-cluster model with wired boundary conditions on G. Let $\mu^{\mathbf{w}}_{\mathbb{Z}^d,p,q}$ be the random-cluster model on $\mathbb{Z}^d$ defined by the weak limit of wired subsets of $\mathbb{Z}^d$. Let $\theta(p) = \mu^{\mathbf{w}}_{\mathbb{Z}^d,p,q}(0 \leftrightarrow \infty)$. Then*

- *there exists a $c_0 > 0$ such that $\theta(p) \geq c_0 (p - \hat{p}_c)$ for any $p \geq p_c$ sufficiently close to p_c;*
- *for any $p < \hat{p}_c$, there exists a c_p such that for every $n \geq 0$,*

$$\mu^{\mathbf{w}}_{\Lambda_n,p,q}(0 \leftrightarrow \partial\Lambda_n) \leq \exp(-c_p n) .$$

Graham and Grimmett proved a dependent analogue of the Friedgut-Kalai sharp threshold result that will be useful to us in a random-cluster setting [30].

Theorem 1.4.4 (Graham–Grimmett)**.** *There exists $0 < c_1 < \infty$ such that the following holds. Let $N \geq 1$, $I = \{1, \ldots, N\}$, $\Omega = \{0,1\}^N$, and let $\mathcal{F}$*

be the set of subsets of Ω. *Let* $A \in \mathcal{F}$ *be an increasing event. Let* μ *be a positive probability measure on* $(\Omega, \mathscr{F})$ *which is monotonic and let* $X_i = \omega(i)$. *If there exists a subgroup* $\mathcal{A}$ *of the symmetric group on* N *elements* Π_N *acting transitively on* I *such that* μ *and* A *are* $\mathcal{A}$*-invariant, then*

$$\frac{d}{dp}\mu_p(A) \geq \frac{c_1\mu_p(X_1)(1-\mu_p(X_1))}{p(1-p)} \min\{\mu_p(A), 1-\mu_p(A)\} \log N\,.$$

1.5 Homology and Cohomology

In this section we will give a brief review of homology and cohomology, two fundamental invariants studied in algebraic topology. These theories exhibit a number of dualities, which are closely related to classical dualities between lattice spin models and allow them to be generalized to higher dimensions. Though this thesis is not self-contained with respect to topological background, we hope to provide enough context for a reader with minimal previous knowledge of the area to make sense of our results. More detail can be found in [35] or [20] which provides an exposition specific to the context of lattice spin models but does not cover all the material we need here. Those familiar with differential forms but not algebraic topology may find some of these concepts familiar, as they are important in a continuous version of the discrete theories developed here.

For concreteness, this section will describe homology and cohomology in spaces called cubical complexes. This is the setting for most, though not all of our results. These are composed of i-dimensional plaquettes for various i, which are isometric to the unit i-cubes $[0,1]^i$. Two references with details specific to this setting are [38, 47]. The first example is the integer lattice cubical complex $\mathbb{Z}^d$, which is a union of all i-plaquettes for $0 \leq i \leq d$ which have corners in the integer lattice. In order to convert vague geometric questions about "the number of holes" of a topological space into concrete algebraic quantities, homology theory defines a space $C_i(X)$ of linear combinations of i-plaquettes (called chains) and boundary operators ∂_i, which maps an i-plaquette to a sum of its $(i-1)$-faces. The continuous analogues of these concepts are, roughly speaking "i-dimensional spaces on which one can integrate an i-form" (called an i-current) and the geometric boundary of the space.

For reasons that become apparent below, it is important to give each term in the boundary of a plaquette a sign, which corresponds to the orientation of a plaquette. The formula is relatively simple in low dimensions. A zero-plaquette is a vertex and its boundary is zero. A 1-plaquette is an edge (v_1, v_2) and its boundary is the difference $v_2 - v_1$. A two-plaquette is an oriented unit square with vertices (v_1, v_2, v_3, v_4) and its boundary is $(v_1, v_2) + (v_2, v_3) + (v_3, v_4) - (v_1, v_4)$. More generally, let $1 \leq k_1 < k_2 < \ldots < k_i \leq d$ and let $I_j = [0,1]$ for $j \in \{k_1, k_2, \ldots, k_i\}$ and $I_j = \{0\}$ for $j \in [d] \setminus k_1, k_2, \ldots, k_i$. Then $\sigma = \prod_{1 \leq j \leq d} I_j$ is an i-plaquette in $\mathbb{R}^d$, and its boundary is given by

$$\partial_i \sigma = \sum_{l=0}^{i} (-1)^{l-1} \left(\prod_{1 \leq j < k_l} I_j \times \{1\} \times \prod_{k_l < m \leq d} I_m - \prod_{1 \leq j < k_l} I_j \times \{0\} \times \prod_{k_l < m \leq d} I_m \right).$$

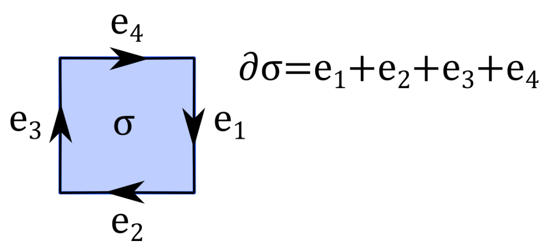

Figure 1.1: The boundary map for a two dimensional plaquette.

The reason that the sum is alternating in sign is so that the boundary operator satisfies the equation

$$\partial_i \circ \partial_{i-1} = 0\,. \tag{1.2}$$

Of particular interest are the chains $\alpha \in C_i(X)$ satisfying $\partial\alpha = 0$. Such chains are called *cycles*, the space of which is denoted $Z_i(X)$. Equation 1.2 provides one source of i-cycles, namely the *boundaries* of $(i+1)$-plaquettes. We denote the space of boundaries $B_i(X)$. However, it turns out that the most interesting cycles are the ones that are not boundaries.

To illustrate why this is the case, consider the unit square $Q = [0,1]^2 \subset \mathbb{R}^2$. In the cubical complex structure we defined earlier, we have that $Z_1(Q)$ consists of multiples of a single 1-cycle, namely the boundary of Q. But now imagine that we make our lattice spacing $1/2$ instead of 1. Now $Z_1(Q)$ contains linear combinations of the boundaries of the four 2-plaquettes contained in Q. Though Q has not changed, our choice of lattice has made the cycle space significantly more complicated.

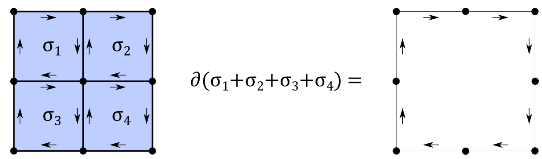

Figure 1.2: The boundary of a union of 4 plaquettes computed with the linearity of the boundary operator.

Therefore, in order to measure the shape of spaces in a way that does not depend on our choice of lattice, we define the homology group

$$H_i(X) = Z_i(X) / B_i(X) \ .$$

Now whatever cubical structure we put on Q, every 1-cycle is a boundary, so $H_1(Q) = 0$. It turns out that homology groups are invariant up to both different cell complex structures on the same space and up to continuous deformations of the space. In fact, one can show that $H_1(Q) = 0$ by continuously deforming Q to a point, which has no nonzero 1-chains, let alone 1-cycles.

For an example with nontrivial homology, consider $H_1(\partial Q)$, where ∂Q is the topological boundary of Q, i.e. the empty square formed by the union of the four sides of Q. There is still the 1-cycle from before but there are no 2-plaquettes, so $H_1(\partial Q) \neq 0$. This example illustrates the often repeating informal description of $H_i(X)$ as a measurement of the number of i-dimensional holes of X. In the case of H_1, the cycles that remain are the loops that cannot be filled in by 2-plaquettes.

Much of this thesis takes place in a cubical structure on the torus that is locally the same as the previously described complex on $\mathbb{Z}^d$. Write $\mathbb{T}_N^d = \mathbb{Z}^d / (N\mathbb{Z})^d$ as the cubical complex as the d-dimensional cubical complex made up of N^d unit d-cubes. We can now define giant cycles in a subcomplex of the torus. If $X \subset \mathbb{T}_N^d$ consists of a subset of the cubical cells of $\mathbb{T}_N^d$, then the chains groups of X are subgroups of the chain groups of $\mathbb{T}_N^d$. The giant cycles of X are the cycles that are not boundaries when considered as chains in $\mathbb{T}_N^d$. These cycles differ by boundaries from the nonzero cycles of the torus, which will be described later. In the case $i = 1$, a giant cycle is a loop that spans the torus $\mathbb{T}_N^d$ in some sense, and must therefore have length at least N. We interpret this as a finite volume analogue of an infinite path in the lattice.

So far, we have been vague about the kind of linear combinations used in the chains and the groups derived from them. Integer linear combinations are a natural choice, but one can also use coefficients from other groups. For any abelian group $\mathcal{G}$, we write the i-th homology group of X with coefficients in $\mathcal{G}$ as $H_i(X; \mathcal{G})$. Although different choices of coefficients can detect different topological features, the universal coefficient theorem tells us that knowledge of $H_i(X; \mathbb{Z})$ for each value of i determines the corresponding homology groups for any group $\mathcal{G}$. In this thesis, we will almost exclusively work with homology over the finite field of integers mod q for prime q, denoted F_q. This is the correct choice to study the higher dimensional q-state Potts models for reasons we will discuss later, and has the advantage of simplifying the algebra required, since a homology group with field coefficients is a vector space. The information gained from field coefficients is also not far from the whole picture, since again by the universal coefficient theorem, knowing $H_i(X; F_q)$ for every q and $H_i(X; \mathbb{Q})$ is enough to reconstruct $H_i(X; \mathbb{Z})$.

For an example where different coefficients produce different results, consider the identifications of the sides of the 2-plaquette in Figure 1.5. In the first case we roll up the plaquette into a cylinder and then put the opposite ends of the cylinder together to form a torus $\mathbb{T}$. In the second, we also form a cylinder but put the opposite ends of the cylinder together in the reverse orientation to form a Klein bottle $\mathbb{K}$. Consider the second homology of both spaces. In both cases there are no 3-plaquettes and only one 2-plaquette σ, so it suffices to check if the 2-chains generated by σ contain nontrivial cycles. In the torus,

$$\partial\sigma = e_1 + e_2 - e_1 - e_2 = 0\,,$$

so $H_2(\mathbb{T}; F) \simeq F$ for any field F. In the Klein bottle,

$$\partial\sigma = e_1 + e_2 + e_1 - e_2 = 2e_1\,,$$

which is nonzero over most fields, but is zero over $\mathbb{Z}_2$. Thus, $H_2(\mathbb{K}, \mathbb{Z}_2) \simeq \mathbb{Z}_2$, and $H_2(\mathbb{K}, \mathbb{Z}_q) = 0$ for any prime $q \neq 2$. Of course, there are no single plaquettes with sides identified this way in the integer lattice, but non-orientable surfaces such as the Klein bottle appear as unions of plaquettes in dimensions 4 and higher.

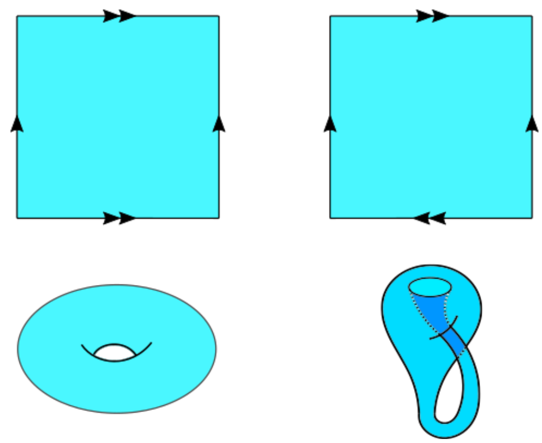

Figure 1.3: A torus $\mathbb{T}$ and a Klein bottle $\mathbb{K}$, the resulting shapes of two possible identifications of the opposite sides of a plaquette. The left plaquette is a cycle with any field coefficients and the right is only a cycle with $\mathbb{Z}_2$ coefficents.

A small algebraic change in the definitions that give the homology groups leads to cohomology. The i-th cochain group $C^i(X;F)$ is defined as the group of F-linear functions from $C_i(X;F)$ to F. Since we are using field coefficients, this is the dual vector space to $C_i(X;F)$. Then the coboundary operator $\delta^i : C^i(X;F) \to C^{i+1}(X;F)$ is defined by $\delta^i f(\alpha) = f(\partial\alpha)$ for $f \in C^i(X;F), \alpha \in C_{i+1}(X;F)$. The smooth analogues of an i-cochain and and its coboundary are a differential form and its exterior derivative. Note that if f is an i-cochain and α is an i-chain, the evaluation $f(\alpha)$ corresponds to integration and the definition of the coboundary operator corresponds to Stokes' Theorem.

As before, we define the i-th cocycle group $Z^i(X;F)$ to be the kernel of δ^i and the i-th coboundary group $B^i(X;F)$ to be the image of δ^{i+1}. Then i-th cohomology group is the quotient

$$H^i(X;F) = Z^i(X;F)/B^i(X;F) \ .$$

In the continuous setting, the analogue of the i-dimensional cohomology group is the i-dimensional de Rham cohomology of closed forms modulo exact forms. This is more than just as analogy: de Rham's theorem states that the de Rham cohomology of a smooth manifold is isomorphic to $H^i(M, \mathbb{R})$ computed using a cell complex structure on M.

In some ways cohomology does not add more information than homology already provided. The universal coefficient theorem tells us that $H_i(X;F) \simeq H^i(X, F)$ when F is a field. However, this algebraic equivalence does not preclude problems lending themselves more naturally to one perspective or the other. For example, the notion of giant surfaces that motivates the definition of giant cycles that we use is homological in nature, while we argue later that the Potts model and lattice gauge theories should be thought of as random co-chains and Wilson loop variables as cohomological quantities. It is also worth noting that although the homology and cohomology groups themselves may be isomorphic, there is a relationship between the different cohomology groups in the form of a multiplication of cocycles called the cup product that does not always have a homological analogue.

Homology and cohomology come together in global duality theorems, of which there are many versions. We primarily use variations of *Alexander duality*, the original form of which says that for a sufficiently nice subspace $X \subset S^d$,

$$H_i(X;F) \simeq H^{d-i-1}\left(S^d \setminus X;F\right)$$

for $1 \leq i \leq d-1$. Since the complement of a set of plaquettes is a thickening of the dual system by Lemma 2.1.2, we are able to use techniques related to the proof of Alexander duality to relate both the giant cycles and local cycles of the two sets. The relationship is quantified in terms of the dimension of the relevant subspaces of the homology groups, and is important in showing that the dual complex to the higher dimensional random-cluster model (defined in the next section) is itself close to a higher dimensional random-cluster model..

We will now briefly discuss the relevant information about the topology of the torus specifically. Although we most often think of the d-torus in a percolation context as a d-cube with the opposite sides identified, it is useful topologically to think of it as the product of d copies of the circle S^1. In such a product space, the Künneth formula tells us that there is an isomorphism of the form

$$\bigoplus_{i+j=k} H_i(X;F) \times H_j(Y;F) \simeq H_k(X \times Y;F) \,.$$

Since $H_1(S^1;F) \simeq H_0(S^1;F) \simeq F$ and $H_i(S^1;F) = 0$ for all $i \geq 2$, we see that

$$H_i(\mathbb{T};F) \simeq F^{\binom{d}{i}} \,.$$

Furthermore, there is a set of generating i-cycles consisting of the products of i of the d possible S^1 factors.

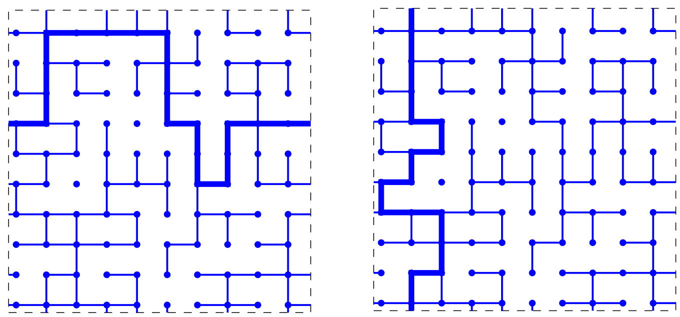

Figure 1.4: Two giant cycles for a random system of 1-dimensional plaquettes (bonds) on a 2-dimensional torus, shown in a square with opposite sides identified.

1.6 Higher Dimensional Percolation Models

Plaquette percolation was studied by Aizenman, Chayes, Chayes, Frölich, and Russo in [1] as a higher dimensional version of bond percolation in $\mathbb{Z}^d$. They proved the following generalization of the quantitative phase transition of Theorem 1.3.1.

Theorem 1.6.1 (Aizenman–Chayes–Chayes–Frölich–Russo)**.** *Let γ be a rectangular loop in $\mathbb{Z}^3$, and let V_γ be the event that γ is null-homologous. Then, for 2-dimensional plaquette percolation,*

$$\mathbb{P}_p(V_\gamma) \sim \begin{cases} \exp(-c_3(p)\mathrm{Area}(\gamma)) & p < 1 - p_c(\mathbb{Z}^3) \\ \exp(-c_4(p)\mathrm{Per}(\gamma)) & p > 1 - p_c(\mathbb{Z}^3) \end{cases}$$

for some $0 < c_3, c_4 < \infty$.

We work in an analogous random subcomplex of $\mathbb{T}_N^d$. Define the independent plaquette system $P = P\,(i, d, N, p)$ to be the random set obtained by taking

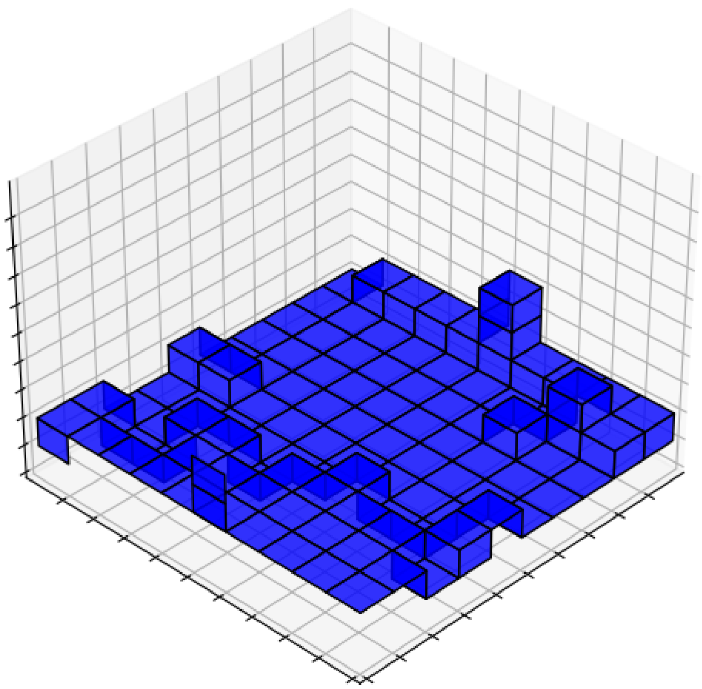

Figure 1.5: A giant cycle for 2-dimensional plaquette percolation on a 3-dimensional torus, shown in a cube with opposite sides identified.

the $(i-1)$-skeleton of $\mathbb{T}_N^d$ and adding each i-face independently with probability p. Let $\phi : P \hookrightarrow \mathbb{T}_N^d$ be the inclusion, and let $\phi_* : H_i(P) \to H_i\left(\mathbb{T}_N^d\right)$ be the induced map on homology. Also, denote by $A^\square = A^\square(i,d,N,p)$ the event that ϕ_* is nontrivial, and denote by $S^\square = S^\square(i,d,N.p)$ the event that ϕ_* is surjective (the superscript is to differentiate between our versions of bond and site percolation). For example, in Figure 1.4 the two giant cycles shown are homologous with standard generators for $H_1(\mathbb{T}^2)$, so we have the event $S^\square$.

Our higher dimensional site percolation is performed on the tiling of the torus by d-dimensional permutohedra, which was previously studied in [10]. The precise definitions are as follows. Let

$$\hat{\mathbb{R}}^d \coloneqq \left\{(x_0, x_1, ..., x_d) : \sum_{k=0}^{d} x_k = 0\right\}.$$

Recall that the root lattice $\mathcal{A}_d$ is defined by

$$\mathcal{A}_d \coloneqq \hat{\mathbb{R}}^d \cap \mathbb{Z}^{d+1}.$$

The dual lattice is then defined by

$$\mathcal{A}_d^* \coloneqq \left\{x \in \hat{\mathbb{R}}^d : \forall y \in \mathcal{A}_d, x \cdot y \in \mathbb{Z}\right\}$$

which is generated by the basis

$$B \coloneqq \{\mathbf{1} - d\boldsymbol{e}_k : 1 \leq k \leq d\}\ .$$

Let $\pi : \hat{\mathbb{R}}^d \to \mathbb{R}^d$ be the natural isometry. Then the closed Voronoi cells of $\pi(\mathcal{A}_d)$ are d-dimensional permutohedra. When $d = 2$, $\mathcal{A}_2^*$ is the triangular lattice and the permutohedra are hexagons. For the case $d = 3$, $\mathcal{A}_3^*$ is the body-centered cubic lattice and the permutohedra are truncated octahedra (see [7] for a detailed exposition).

Consider the torus $\mathbf{T}_\mathbf{N}^\mathbf{d}$ as the parallelepiped generated by $\{Nv : v \in B\}$ with opposite faces identified. Define $Q = Q(d, N, p)$ to be the random set obtained by adding each permutohedron independently with probability p. The topological justification for calling this site percolation is that the adjacency graph on the permutohedra of Q is exactly site percolation on the lattice $\mathcal{A}_d^*$. In other words, site percolation is the one-skeleton of the nerve of the cover of Q by the closed permutohedra. By the nerve theorem, Q is homotopy equivalent to this nerve and as such has the same connected components as site percolation on $\mathcal{A}_d^*$.

The giant cycle events are defined as before, except that i-dimensional giant cycles exhibit interesting behavior for all $1 \leq i \leq d-1$ (for the plaquette model $P\left(i, d, N, p\right)$, all giant cycles in homological dimensions less than i are automatically present, and there can be no giant cycles in homological dimensions exceeding i.) More precisely, let $\varphi : Q \hookrightarrow \mathbf{T}_\mathbf{N}^\mathbf{d}$ be the inclusion, and let $\varphi_{i*} : H_i\left(Q\right) \to H_i\left(\mathbf{T}_\mathbf{N}^\mathbf{d}\right)$ be the induced maps for homology in each dimension. For each i, Let $A_i^{\hexagon}$ be the event that φ_{i*} is nonzero and let $S_i^{\hexagon}$ be the event that φ_{i*} is surjective.

In both models, we will need a notion of dual percolation. In the permutohedral case, we simply define

$$Q^{\hexagon} = Q^{\hexagon}(d, N, p) \coloneqq \overline{Q^c},$$

i.e. the union of the permutohedra that are not included in Q. Defining the dual system to plaquette percolation requires more work.

To define the *dual system of plaquettes* $P^{\blacksquare} = P^{\blacksquare}(i, d, N, p)$, let $(\mathbb{T}_N^d)^{\blacksquare}$ be the regular cubical complex obtained by shifting $\mathbb{T}_N^d$ by $\frac{1}{2}$ in each coordinate direction. Each i-face of $\mathbb{T}_N^d$ intersects a unique $(d-i)$-face of $(\mathbb{T}_N^d)^{\blacksquare}$ and they meet in a single point at their centers. For example, the faces $[0,1]^i \times \{0\}^{d-i}$ and $\{1/2\}^i \times [-1/2, 1/2]^{d-i}$ intersect in the point $\left\{\frac{1}{2}\right\}^i \times \{0\}^{d-i}$.

Define the dual system $P^{\blacksquare}$ to be the subcomplex of $(\mathbb{T}_N^d)^{\blacksquare}$ consisting of all faces for which the corresponding face in $\mathbb{T}_N^d$ is not contained in P. See Figure 1.6. Observe that the distribution of $P^{\blacksquare}(i, d, N, p)$ is the same as that of $P(d-i, d, N, 1-p)$. If $B^{\blacksquare}$ is an event defined in terms of $P^{\blacksquare}$ we will write $\mathbb{P}_p\left(B^{\blacksquare}\right)$ to mean the probability of $B^{\blacksquare}$ with respect to the parameter p of P.

We always use the notation $\phi : P \hookrightarrow \mathbb{T}^d$ and $\psi : P^{\blacksquare} \hookrightarrow \mathbb{T}^d$ for the respective inclusion maps, and $\phi_* : H_i(P) \to H_i\left(\mathbb{T}_N^d\right)$ and $\psi_* : H_{d-i}\left(P^{\blacksquare}\right) \to H_{d-i}\left(\mathbb{T}^d\right)$ for the induced maps on homology. Also, we consistently use notation $A^{\square} = A^{\square}(i, d, N, p)$ for the event that $\operatorname{im}\phi_* \neq 0$, $S^{\square} = S^{\square}(i, d, N, p)$ for the event

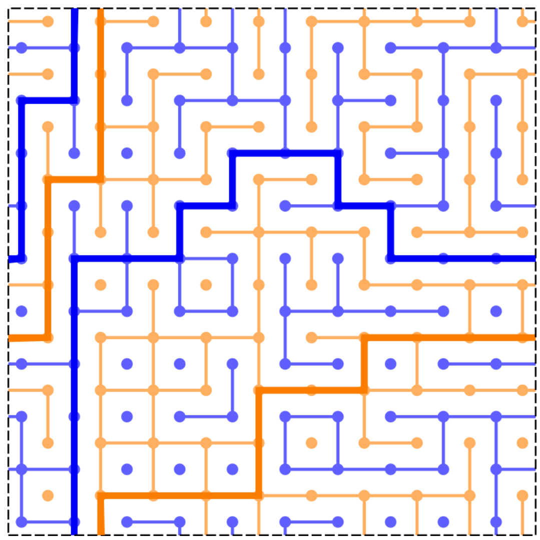

Figure 1.6: Bond percolation at criticality (i.e. $p = 1/2$) on the torus T^2_{10} in blue, with the corresponding dual system of bonds in orange. Giant cycles are shown in bold. Observe that while $\operatorname{rank} \phi_* + \operatorname{rank} \psi_* = 2$ (as required by duality), neither the bond system nor its dual has a giant cycle homologous to one of the standard basis elements of $H_1(\mathbb{T}^2)$.

that ϕ_* is surjective, and $Z^\square = Z^\square(i, d, N, p)$ for the event that ϕ_* is zero. Denote by $A^\blacksquare, S^\blacksquare$, and $Z^\blacksquare$ the corresponding events for ψ_*.

We also have a higher dimensional version of the random-cluster model. A similar model in a random simplicial complex analogous to the Linial-Meshulam complex [41] was introduced by Hiraoka and Shirai [36]. Our notation will be somewhat similar to the independent case, but since the results in the two models are confined to separate chapters, we do not expect this to be confusing. Let F_N^i be the set of i-plaquettes of $\mathbb{T}_N^d$. We say that $\omega : F_N^i \to \{0,1\}$ is a configuration, and we call the elements of $\omega^{-1}\{1\}$ and the elements of $\omega^{-1}\{0\}$ the open and closed plaquettes of ω respectively. We then define P_ω to be the union of the $(i-1)$-skeleton of $\mathbb{T}_N^d$ and the open plaquettes of ω. Let $\eta(\omega) \coloneqq |\omega^{-1}\{1\}|$ be the number of open plaquettes of ω. Let $\phi : P_\omega \hookrightarrow \mathbb{T}_N^d$ be the inclusion map and let $\phi_{i*} : H_i(P_\omega) \to H_i\left(\mathbb{T}_N^d\right)$ be the induced map on ith homology. Then we can define $\rho(\omega) \coloneqq \operatorname{rank} \phi_{i*}$, informally the number of giant cycles in P_ω. Since there is no site percolation counterpart, we will simplify notation denote the events $A = A(\omega)$ that ϕ_{i*} is nontrivial and denote $S = S(\omega)$ that ϕ_{i*} is surjective.

Now can now define the i-random-cluster model on the torus as the random complex P_ω, where ω is distributed in the following way:

$$\mu_{p,q,i,N}\left(\omega\right) \coloneqq \frac{1}{Z_{p,q,i,N}} p^{\eta(\omega)}\left(1-p\right)^{\left|F_N^i\right|-\eta(\omega)} q^{\mathbf{b}_{i-1}(P_\omega)}.$$

where $Z_{p,q,i,N}$ is a normalizing constant, chosen so that the probabilities of the possible open subgraphs sum to 1.. We also define a "balanced" version [8] which we will later show satisfies exact duality:

$$\tilde{\mu}_{p,q,i,N}\left(\omega\right) \coloneqq \frac{\left(\sqrt{q}\right)^{-\rho(\omega)}}{\tilde{Z}_{p,q,i,N}} p^{\eta(\omega)}\left(1-p\right)^{\left|F_N^i\right|-\eta(\omega)} q^{\mathbf{b}_{i-1}(P_\omega)}.$$

The two densities are absolutely continuous with respect to each other with a Radon-Nikodym derivative bounded above and below by functions of q and the same is true for their dual models. Since our arguments involve sharp thresholds, this will be a sufficient duality relationship.

Hiraoka and Shirai showed that the i-random-cluster model also satisfies the FKG inequality [36].

Theorem 1.6.2 (Hiraoka–Shirai)**.** *Let $p \in (0,1)$ and $q \geq 1$. Then $\mu_{p,q,i,N}$ satisfies the FKG lattice condition and is thus positively associated, meaning that for any events E, F that are increasing with respect to ω,*

$$\mu_{p,q,i,N}\left(E \cap F\right) \geq \mu_{p,q,i,N}\left(E\right)\mu_{p,q,i,N}\left(F\right).$$

As before, there is a coupled $(i-1)$-dimensional Potts model. Now we consider states in $C^{i-1}\left(\mathbb{T}_N^d;\mathbb{Z}_q\right)$ with Hamiltonian

$$\hat{H}(f) = \sum_{\sigma\in F_N^i} K\left(f\left(\partial\sigma\right),0\right) .$$

Then the probability of a state is

$$\nu(f) = e^{-\beta\hat{H}(f)} .$$

In this setting, Hiraoka and Shirai [36] showed an analogue of the Edwards-Sokal coupling.

Proposition 1.6.3 (Hiraoka–Shirai)**.** *Let $q \geq 1$, $p \in [0,1)$, and suppose $p = 1-e^{-\beta}$. Consider the coupling on $C^{i-1}\left(\mathbb{T}_N^d\right)\times\{0,1\}^{F_N^i}$ defined by*

$$\nu(f,\omega) \propto \prod_{\sigma\in F_N^i}\left[(1-p)\,K\left(\omega(\sigma),0\right)+pK\left(\omega(\sigma),1\right)K\left(\delta^{i-1}f(\sigma),0\right)\right] .$$

Then ν has the following marginals:

- *The first marginal is the q-state Potts model with inverse temperature β given by*

  $$\sum_{\omega\in\{0,1\}^{F_N^i}} \nu(f,\omega) \propto e^{-\beta H(f)} ,$$

  where $H(f) = -\sum_{\sigma\in F_N^i}\boldsymbol{\delta}_{\delta^{i-1}f(\sigma),0}$.

- *The second marginal is the i-random cluster model with parameters p,q given by*

  $$\sum_{f\in C^{i-1}\left(\mathbb{T}_N^d\right)} \nu(f,\omega) \propto p^{\eta(\omega)}\left(1-p\right)^{\left|F_N^i\right|-\eta(\omega)} q^{\mathbf{b}_{i-1}(P_\omega;\mathbb{Z}_q)} .$$

the first marginal of ν is the $(i-1)$-dimensional q-state Potts model with $p = 1 - e^{-\beta}$ and the second marginal is the i-random cluster model.

The case $i = 2$ is of particular interest because the coupled 1-dimensional Potts model is an example of a lattice gauge theory. Lattice gauge theories are discretized models of Euclidean Yang–Mills theory [16]. We will only work with a specific type of lattice gauge theory here, but we give more general background for context.

Let G be a complex multiplicative matrix group. For a 1-cochain $f \in C^1(T, G)$ and 2-plaquette σ with oriented edges (e_1, e_2, e_3, e_4) in cyclic order, define

$$W_\sigma(f) = \operatorname{Re}\operatorname{Tr} f(e_1) f(e_2) f(e_3) f(e_4)\ ,$$

where this is the only place in this thesis that we will use multiplicative notation, to emphasize that G may be non-abelian. Note that this is well-defined because the trace of a product of matrices is invariant to cyclic permutations of the matrices. 1-dimensional lattice gauge theory on T with gauge group G is the Gibbs measure on the co-chain group $C^1(T, G)$ induced by the Hamiltonian

$$H(f) = \sum_{\sigma \in F_N^2} W_\sigma(f) \tag{1.3}$$

That is, it is the probability measure

$$d\eta_{\beta,i,d}(f) = \frac{1}{Z} e^{-\beta H(f)} d\,\tau$$

where Z is a normalizing constant and $d\tau$ is the Haar measure on G. In this thesis, G will be a finite group and $d\tau$ will be the uniform measure. However, the most physically relevant cases are given by $G = U(1)$, $G = SU(2)$, and $G = SU(3)$ (when $i = 1$ and $d = 4$), which correspond to the the electromagnetic, weak nuclear, and strong nuclear forces, respectively. Note that the above definition can be readily generalized to define a measure on $C^i(T, G)$ for $i > 1$ when G is abelian, but that difficulties arise for non-abelian groups due to the dependence of the order of the terms in the boundary of σ in the definition of Λ_f.

1-dimensional Potts lattice gauge theory is a the probability measure on $C^1(T, G)$ given by replacing the Hamiltonian above with the simpler formula

$$\hat{H}(f) = \sum_{\sigma} K(W_\sigma(f), 1) \, .$$

In this thesis, we will focus exclusively on the cases where $G = \mathbb{Z}(q)$ for prime q, where $\mathbb{Z}(q)$ is the multiplicative group of q-th complex roots of unity. While they are not themselves physical, they have been studied extensively in the physics literature as they are more tractable and thought to present some of the same behavior as more physically relevant cases [4, 40, 29, 43]. The special cases $G = \mathbb{Z}(2)$ and $G = \mathbb{Z}(3)$ coincide with the $\mathbb{Z}(2)$ (Ising) and $\mathbb{Z}(3)$ "clock" lattice gauge theories in the sense defined in the previous paragraph after an appropriate rescaling of the coupling constant β. These models have themselves been studied in the physical and the mathematical literatures as relatively approachable examples of lattice gauge theories [42, 48, 44, 50, 51, 6, 17].

Some of the most important observables in lattice gauge theory are the Wilson loop variables. For an 1-cochain $\omega \in C^1(T, G)$ and an oriented loop $\gamma = (e_1, \ldots, e_k)$ define

$$W_\gamma(\omega) = f(e_1) \ldots f(e_k) \; .$$

An important conjecture — called the Wilson area law — is that if $\gamma = \partial\sigma$ is a 1-boundary and σ is the minimal bounding chain then the expectation of $W_{\partial\sigma}(\omega)$ should decay as $e^{-c|\sigma|}$. For $G = SU(2)$ or $SU(3)$, this conjecture is thought to be related to the phenomenon of quark confinement, that charged particles for the weak or strong nuclear forces are not seen in isolation, unlike for the electromagnetic forces [49, 18]. It was recently shown, first for Ising ($\mathbb{Z}(2)$) lattice gauge theory by Chaterjee [17] and then for lattice gauge theory with any finite gauge group by Cao [15], that the expected Wilson loop variable decays exponentially in the perimeter of the loop for sufficiently large values of β.

Chapter 2: Homological Percolation in Bernoulli Percolation on a Torus

This chapter is based on joint work with Matthew Kahle and Benjamin Schweinhart [22].

2.0.1 Main results

Our main result for plaquette percolation is that if $d = 2i$, then i-dimensional percolation is self-dual and undergoes a sharp transition at $p = 1/2$.

Theorem 2.0.1. *Suppose* $\operatorname{char}(F) \neq 2$. *If* $d = 2i$ *then*

$$\begin{cases} \mathbb{P}_p\left(A^{\square}\right) \to 0 & p < \frac{1}{2} \\ \mathbb{P}_p\left(S^{\square}\right) \to 1 & p > \frac{1}{2} \end{cases}$$

as $N \to \infty$.

Using results on bond percolation on $\mathbb{Z}^d$, we also prove dual sharp thresholds for 1-dimensional and $(d-1)$-dimensional percolation on the torus.

Theorem 2.0.2. *Suppose* $\operatorname{char}(F) \neq 2$. *Let* $\hat{p}_c = \hat{p}_c(d)$ *be the critical threshold for bond percolation on* $\mathbb{Z}^d$. *If* $i = 1$ *then*

$$\begin{cases} \mathbb{P}_p\left(A^{\square}\right) \to 0 & p < \hat{p}_c \\ \mathbb{P}_p\left(S^{\square}\right) \to 1 & p > \hat{p}_c \end{cases}$$

as $N \to \infty$.

Furthermore, if $i = d-1$ *then*

$$\begin{cases} \mathbb{P}_p\left(A^{\square}\right) \to 0 & p < 1 - \hat{p}_c \\ \mathbb{P}_p\left(S^{\square}\right) \to 1 & p > 1 - \hat{p}_c \end{cases}$$

as $N \to \infty$.

In the above, we also show that the decay of $\mathbb{P}_p\left(A^{\square}\right)$ below the threshold and $\mathbb{P}_p\left(S^{\square}\right)$ above the threshold is exponentially fast for both $i = 1$ and $i = d-1$.

For other values of i and d we show the existence of a sharp threshold function as follows. For each $N \in \mathbb{N}$, let $\lambda^{\square}(N, i, d)$ satisfy

$$\mathbb{P}_{\lambda^{\square}(N,i,d)}\left(A^{\square}\right) = \frac{1}{2}. \tag{2.1}$$

Note that $\mathbb{P}_p\left(A^{\square}\right)$ is continuous as a function of p, so such a $\lambda^{\square}(N, i, d)$ exists by the intermediate value theorem. Since the tori of different sizes do not embed nicely into each other, it is not obvious that $\lambda^{\square}(N, i, d)$ should be convergent a priori.

We should mention that this choice of $\lambda^{\square}(N, i, d)$ is somewhat arbitrary. We could replace $\frac{1}{2}$ in Equation 2.1 with any constant strictly between 0 and 1, for example, and the sharp threshold results we use would apply just as well. In several cases we show that $\lambda^{\square}(N,i,d)$ converges, and in those cases the limiting value could also be taken as a constant threshold function.

Define

$$p_c^{\square}(i,d) = \inf\left\{p : \liminf_{N\to\infty} \mathbb{P}_p\left(A^{\square}\right) > 0\right\}$$

and

$$q_c^{\square}(i,d) = \sup\left\{p : \limsup_{N\to\infty} \mathbb{P}_p\left(S^{\square}\right) < 1\right\}.$$

As we show below, we could alternatively define $p_c^{\square}(i,d)$ and $q_c^{\square}(i,d)$ as the limit supremum and limit infimum of the threshold function $\lambda^{\square}(N,i,d)$, respectively.

With the understanding that these depend on choice of i and d, which are always understood in context, we sometimes abbreviate to simply $p_c^{\square}$, $q_c^{\square}$, and $\lambda^{\square}(N)$.

Theorem 2.0.3. *Suppose* $\operatorname{char}(F) \neq 2$. *For every* $d \geq 2$, $1 \leq i \leq d-1$, *and* $\epsilon > 0$

$$\begin{cases} \mathbb{P}_{\lambda^{\square}(N)-\epsilon}\left(A^{\square}\right) \to 0 \\ \mathbb{P}_{\lambda^{\square}(N)+\epsilon}\left(S^{\square}\right) \to 1 \end{cases}$$

as $N \to \infty$.

Moreover, for every $d \geq 2$ *and* $1 \leq i \leq d-1$ *we have*

$$0 < q_c^{\square} = \liminf_{N\to\infty} \lambda^{\square}(N) \leq \limsup_{N\to\infty} \lambda^{\square}(N) = p_c^{\square} < 1\,,$$

and $p_c^{\square}(i,d)$ *has the following properties.*

(a) *(Duality)* $p_c^{\square}(i,d) + q_c^{\square}(d-i,d) = 1$.

(b) *(Monotonicity in* i *and* d*)* $p_c^{\square}(i,d) < p_c^{\square}(i,d-1) < p_c^{\square}(i+1,d)$ *if* $0 < i < d-1$.

It follows that $p_c^{\square} = q_c^{\square}$ for $i = d/2$, $i = 1$, and $i = d-1$, and we conjecture that this equality (and hence sharp threshold from a trivial map to a surjective one at a constant value of p) holds for all i and d. Bobrowksi and Skraba make analogous conjectures for the continuum percolation model in [11].

For each $N \in \mathbb{N}$, let $\lambda_i^{\hexagon}(N,d)$ satisfy

$$\mathbb{P}_{\lambda_i^{\hexagon}(N,d)}\left(A_i^{\hexagon}\right) = \frac{1}{2}\,.$$

Define

$$p_i^{\hexagon} = p_i^{\hexagon}(d) = \inf\left\{p : \liminf_{N\to\infty} \mathbb{P}_p\left(A_i^{\hexagon}\right) > 0\right\}$$

and

$$q_i^{\hexagon} = q_i^{\hexagon}(d) = \sup\left\{p : \limsup_{N\to\infty} \mathbb{P}_p\left(S_i^{\hexagon}\right) < 1\right\}.$$

Theorem 2.0.4. *Suppose* $\operatorname{char}(F) \nmid d+1$. *For every* $d \geq 2$, $1 \leq i \leq d-1$, *and* $\epsilon > 0$

$$\begin{cases} \mathbb{P}_{\lambda_i^{\hexagon}(N)-\epsilon}\left(A_i^{\hexagon}\right) \to 0 \\ \mathbb{P}_{\lambda_i^{\hexagon}(N)+\epsilon}\left(S_i^{\hexagon}\right) \to 1 \end{cases}$$

as $N \to \infty$. *For every* $d \geq 2$ *and* $1 \leq i \leq d-1$ *we have*

$$0 < q_i^{\hexagon} = \liminf_{N\to\infty} \lambda_i^{\hexagon}(N) \leq \limsup_{N\to\infty} \lambda_i^{\hexagon}(N) = p_i^{\hexagon} < 1\,.$$

In some cases the threshold converges, namely

$$\lim_{N\to\infty} \lambda_1^{\hexagon}(N) = p_1^{\hexagon} = q_1^{\hexagon} = p_c(A_d^*)\,,$$

$$\lim_{N\to\infty} \lambda_{d-1}^{\hexagon}(N) = p_{d-1}^{\hexagon} = q_{d-1}^{\hexagon} = 1 - p_c(A_d^*)\,,$$

and if d is even, then

$$\lim_{N\to\infty} \lambda_{d/2}^{\hexagon}(N) = p_{d/2}^{\hexagon} = q_{d/2}^{\hexagon} = \frac{1}{2}\,.$$

Moreover, $p_i^{\hexagon}(d)$ has the following properties.

(a) (Duality) $p_i^{\hexagon}(d) + q_{d-i}^{\hexagon}(d) = 1$.

(b) (Monotonicity in i and d) $p_i^{\hexagon}(d) < p_i^{\hexagon}(d-1) < p_{i+1}^{\hexagon}(d)$ *if* $0 < i < d-1$.

In particular, when $d = 4$, the random set Q exhibits three qualitatively distinct phase transitions at $p_c(A_4^*)$, $\frac{1}{2}$, and $1 - p_c(A_4^*)$, where $p_c(A_4^*)$ is the site percolation threshold for the lattice A_4^*.

2.0.2 Proof sketch

We provide an overview of our main argument. Much of it is the same, mutatis mutandis, whether we are working with plaquettes or permutohedra. Throughout the chapter, we shall make a note at points of substantial difference, but otherwise we only include proofs with plaquettes for brevity. In the section on topological results (Section 2.1), we show that duality holds in the sense that $\operatorname{rank}\phi_* + \operatorname{rank}\psi_* = \operatorname{rank} H_i\left(\mathbb{T}^d\right)$ (Lemma 2.1.4). This is similar in spirit to results of [10] and [11] for other models of percolation on the torus including permutohedral site percolation. In particular, at least one of the events $A^{\square}$ and $A^{\blacksquare}$ occurs, $S^{\square}$ occurs if and only if $Z^{\blacksquare}$ occurs, and $S^{\blacksquare}$ occurs if and only if $Z^{\square}$ occurs.

Our strategy is to exploit the duality between the events $S^{\square}$ and $Z^{\blacksquare} = \left(A^{\blacksquare}\right)^c$. Toward that end, we show that a threshold for $A^{\square}$ is also a threshold for $S^{\square}$ in Section 2.2. First, we use the action of the point symmetry group of the torus on the homology to show that there are constants b_0 and b_1 so that $\mathbb{P}_p\left(S^{\square}\right) \geq b_0 \mathbb{P}_p\left(A^{\square}\right)^{b_1}$. This follows from a more general result for events defined in terms of an irreducible representation of the point symmetry group (Lemma 2.2.1) and the fact that $H_i\left(\mathbb{T}^d; F\right)$ is an irreducible representation of the point symmetry group of $\mathbb{Z}^d$ assuming the characteristic of F does not equal 2 (Proposition 2.2.3). This is one point at which the argument differs for site percolation on the permutohedral lattice; to account for the symmetries of that lattice we include a different argument that assumes that the characteristic of F is not divisible by $d+1$ (Proposition 2.2.2).

Recall that $\lambda^{\square}$ was chosen such that

$$\mathbb{P}_{\lambda^{\square}(N,i,d)}\left(A^{\square}\right) = \frac{1}{2}\,.$$

By the above, it follows that $\mathbb{P}_{\lambda^{\square}(N)}\left(S^{\square}\right) > b_0 \left(\frac{1}{2}\right)^{b_1}$. $S^{\square}$ is increasing and invariant under the symmetry group of $\mathbb{T}^d$ so Friedgut and Kalai's theorem on sharpness of thresholds (Theorem 1.3.4) implies that for any $\epsilon > 0$, $\mathbb{P}_{\lambda^{\square}(N)+\epsilon}\left(S^{\square}\right) \to 1$ as $N \to \infty$.

The proof of Theorem 2.0.1 is then straightforward (Section 2.3). By duality

$$\mathbb{P}_{1/2}\left(A^{\square}\right) = \mathbb{P}_{1/2}\left(A^{\blacksquare}\right) \qquad \text{and} \qquad \mathbb{P}_{1/2}\left(A^{\square}\right) + \mathbb{P}_{1/2}\left(A^{\blacksquare}\right) \geq 1\,,$$

so $\mathbb{P}_{1/2}\left(A^{\square}\right) \geq \frac{1}{2}$ for all N. It follows from the previous argument that $\mathbb{P}_p\left(S^{\square}\right) \to 1$ for $p > 1/2$. On the other hand, if $p < 1/2$ duality implies that

$$\mathbb{P}_p\left(A^{\square}\right) = 1 - \mathbb{P}_p\left(S^{\blacksquare}\right) = 1 - \mathbb{P}_{1-p}\left(S^{\square}\right) \to 0\,.$$

Next, in Section 2.4 we study the relationship between duality and convergence. Recall that

$$q_c^{\square}(i,d) = \sup\left\{p : \limsup_{N\to\infty} \mathbb{P}_p\left(S^{\square}\right) < 1\right\}\,.$$

We show that $p_c^{\square}(i,d) + q_c^{\square}(d-i,d) = 1$ by using Lemma 2.1.4 and applying Theorem 1.3.4 to $A^{\square}$ above $p_c^{\square}(d-i,d)$ and to $A^{\blacksquare}$ below $q_c^{\square}(i,d)$ (Proposition 2.4.2). It follows that the threshold for $A^{\square}$ converges if and only if $p_c^{\square}(d-i,d) + p_c^{\square}(i,d) = 1$ (Corollary 2.4.3).

In Section 2.5 we show that $p_c^{\square}(1,d)$ and $q_c^{\square}(1,d)$ coincide and equal the critical threshold for bond percolation on $\mathbb{Z}^d$ by applying classical results on connection probabilities in the subcritical and supercritical regimes (in the proofs of Propositions 2.5.3 and 2.5.1). This together with Corollary 2.4.3 demonstrates Theorem 2.0.2.

Finally, in Section 2.6, we complete the proof of Theorem 2.0.3 by showing the monotonicity property $p_c^{\square}(i,d) < p_c^{\square}(i,d-1) < p_c^{\square}(i+1,d)$ if $0 < i < d-1$, and corresponding result for the thresholds $q_c^{\square}$ (Proposition 2.6.1). This is done by comparing percolation on $\mathbb{T}_N^d$ with percolation on a subset homotopy equivalent to $\mathbb{T}^{d-1}$. The proof of the corresponding result for permutohedral site percolation is different, but the overall idea is similar (Proposition 2.6.3).

2.1 Topological Results

In this section, we discuss duality lemmas which will be useful in many of our arguments.

In [10], Bobrowski and Skraba prove a duality lemma for the permutohedral lattice. We will use their notation which, for a subcomplex $X \subset \mathbf{T_N^d}$, defines

$$\mathcal{B}_k(X) \coloneqq \operatorname{rank} \varphi_*,$$

where $\varphi_* : H_k(X) \to H_k\left(\mathbf{T_N^d}\right)$ is the map on homology induced by inclusion.

Lemma 2.1.1 (Bobrowski and Skraba)**.** *For* $0 \le k \le d$,

$$\mathcal{B}_k(Q) + \mathcal{B}_{d-k}(Q^c) = \operatorname{rank} H_k\left(\mathbb{T}^d\right).$$

This is a point at which one must consider permutohedra and plaquettes separately. We use the previous lemma in the permutohedral case, but we must prove an analogue in order to work with plaquettes. First, we show a preliminary result demonstrating a relationship between the complement of P and $P^{\blacksquare}$.

Lemma 2.1.2. $\mathbb{T}^d \setminus P$ *deformation retracts to* $P^{\blacksquare}$.

Proof. Let $T^{(j)}$ and $T_{\blacksquare}^{(j)}$ denote the j-skeletons of $\mathbb{T}_N^d$ and $\left(\mathbb{T}_N^d\right)^{\blacksquare}$, respectively, and let

$$S_j = T_{\blacksquare}^{(d-j)} \setminus T^{(j)} .$$

Observe that S_j is obtained from $T_{\blacksquare}^{(d-j)}$ by removing the central point of each $(d-j)$-cell. Also, let

$$\hat{S}_j = T_{\blacksquare}^{(d-j)} \setminus P .$$

We construct a deformation retraction from $T^d \setminus P = \hat{S}_0$ to $P^{\blacksquare}$ by iteratively collapsing $\hat{S}_j$ to $\hat{S}_{j+1}$ for $j < i$, then collapsing $\hat{S}_i$ to $P^{\blacksquare}$.

For an j-cell σ of $T_{\blacksquare}^{(j)}$ with center point q let

$$f_\sigma : \sigma \setminus \{q\} \times [0,1] \to \sigma \setminus \{q\}$$

be the deformation retraction from the punctured j-dimensional cube to its boundary along straight lines radiating from the center. Observe that the restriction of f_σ to $(\sigma \setminus P) \times [0,1]$ defines a deformation retraction from $\sigma \setminus P$ to $\partial\sigma \setminus P$ (for $j > d-i$); this is because σ intersects P in hyperplanes spanned

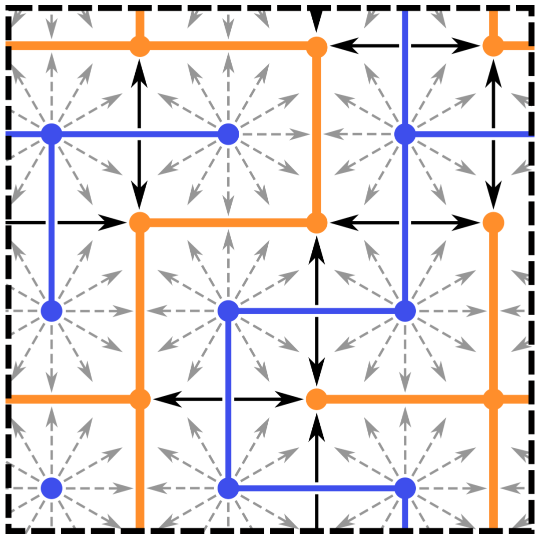

Figure 2.1: of the deformation retraction for the case $N = 3$, $d = 2, i = 1$. P is shown in blue and $P^{\blacksquare}$ in orange. $\mathbb{T}^d \setminus P$ is first retracted to $T_{\blacksquare}^{(1)} \setminus P$ via the dashed gray arrows radiating from each vertex of P, then to $P^{\blacksquare}$ by the solid black arrows radiating from the midpoints of the edges of P.

by the coordinate vectors based at q. When projecting radially from q, points inside $\sigma \cap P$ remain inside $\sigma \cap P$ and points outside of $\sigma \cap P$ remain outside of $\sigma \cap P$.

For $x \in \mathbb{T}^d$, let $\sigma(x)$ be the unique $(d-j)$-cell of $(\mathbb{T}^d_N)^{\blacksquare}$ that contains x in its interior. Define $G_j : S_j \times [0,1] \to S_j$ by

$$G_j(x,t) = \begin{cases} f_{\sigma(x)}(x,t) & x \in S_j \setminus T_{\blacksquare}^{d-j-1} \\ x & \text{otherwise}. \end{cases}$$

G_j collapses S_j to $T^{(d-j-1)}$ by retracting the punctured $(d-j)$-cells to their boundaries. It follows from the discussion in the previous paragraph that the restriction of G_j to $\hat{S}_j \times [0,1]$ defines a deformation retraction from $\hat{S}_j$ to $\hat{S}_{j+1}$.

Similarly, define $H : \hat{S}_i \times [0,1] \to \hat{S}_i$ by

$$H\left(x,t\right) = \begin{cases} f_{\sigma(x)}\left(x,t\right) & x \in \hat{S}_i \setminus P^{\blacksquare} \\ x & \text{otherwise}\,. \end{cases}$$

That is, H collapses the $(d-i)$-faces of $T_{\blacksquare}^{(d-i)}$ that are punctured by i-faces of P to deformation retract $\hat{S}_i$ to $P^{\blacksquare}$.

In summary, we can deformation retract $\mathbb{T}^d \setminus P$ to $P^{\blacksquare}$ via the function $F : T^d \setminus P \times [0,i] \to T^d \setminus P$ defined by

$$F\left(x,t\right) = \begin{cases} G_0\left(x,t\right) & t \in [0,1] \\ G_j\left(F\left(x,j\right), t-j\right) & t \in \left(j, j+1\right], 0 < j < i \\ H\left(F\left(x,i\right), t-i\right) & t \in \left(i, i+1\right]\,. \end{cases}$$

$\square$

In fact, the same deformation retraction works when P is slightly thickened, which will be useful for the next Lemma. Let P_ϵ denote the ϵ-neighborhood

$$P_\epsilon = \left\{x \in \mathbb{T}_N^d : d(x,P) \leq \epsilon\right\}.$$

Corollary 2.1.3. *For any $0 < \epsilon < 1/2$, the closure $\overline{(\mathbb{T}^d \setminus P_\epsilon)}$ deformation retracts to $P^{\blacksquare}$.*

Proof. Consider the deformation retraction as in Lemma 2.1.2 restricted to $\overline{(\mathbb{T}^d \setminus P_\epsilon)}$. When a punctured j-cell σ is retracted via f_σ, the property that points outside of $\sigma \cap P_\epsilon$ remain outside of $\sigma \cap P_\epsilon$ is preserved even though $\sigma \cap P_\epsilon$ now is a union of thickened hyperplanes. The deformation retractions

G_j and H are defined by collapsing different cells via the functions f_σ, so the restricted retraction does not pass through P_ϵ. $\square$

The next result is a key topological tool we use in many of our arguments. It is very similar to results of [10] and [11] for other models of percolation on the torus including Lemma 2.1.1 above. For convenience, let

$$D = \operatorname{rank} H_i\left(\mathbb{T}^d\right) = \binom{d}{i}.$$

Lemma 2.1.4 (Duality Lemma)**.** $\operatorname{rank}\phi_* + \operatorname{rank}\psi_* = D$. *In particular, at least one of the events* $A^\square$ *and* $A^\blacksquare$ *occurs,* $S^\blacksquare \iff Z^\square$, *and* $Z^\blacksquare \iff S^\square$.

Proof. We proceed similarly to Bobrowski and Skraba's proof of Lemma 2.1.1. Let $\epsilon = 1/4$ and define $P_\epsilon^c := \overline{\left(\mathbb{T}_N^d \setminus P_\epsilon\right)}$. Consider the diagram

$$\begin{array}{ccccccc}
H_i\left(P_\epsilon\right) & \xrightarrow{\quad i_* \quad} & H_i\left(\mathbb{T}_N^d\right) & \longrightarrow & H_i\left(\mathbb{T}_N^d, P_\epsilon\right) & \xrightarrow{\quad \delta_i \quad} & H_{i-1}\left(P_\epsilon\right) \\
\uparrow\cong & & \uparrow\cong & & \uparrow\cong & & \uparrow\cong \\
H^{d-i}\left(\mathbb{T}_N^d, P_\epsilon^c\right) & \longrightarrow & H^{d-i}\left(\mathbb{T}_N^d\right) & \xrightarrow{\ j^*\ } & H^{d-i}\left(P_\epsilon^c\right) & \xrightarrow{\delta^{d-i}} & H^{d-i+1}\left(\mathbb{T}_N^d, P_\epsilon^c\right)
\end{array}$$

Here i and j are the inclusions of P_ϵ and P_ϵ^c respectively into $\mathbb{T}_N^d$. The first isomorphism from the left is from Lefschetz Duality, the second is from Poincaré Duality, and the third is from the five lemma. (A similar diagram is used in the proof of Alexander duality). In particular note that by exactness and a diagram chase,

$$H_i(\mathbb{T}_N^d) \cong \operatorname{im} i_* \oplus \operatorname{im} j^*.$$

Furthermore, since we are considering homology with field coefficients, $\operatorname{rank} j^* = \operatorname{rank} j_*$. Now by Corollary 2.1.3, $\overline{\left(\mathbb{T}_N^d \setminus P_\epsilon\right)}$ retracts to $P^\blacksquare$, and P_ϵ clearly retracts to P, so $\operatorname{rank} \phi_* = \operatorname{rank} i_*$ and $\operatorname{rank} \psi_* = \operatorname{rank} j^*$. Putting these together gives $\operatorname{rank} \phi_* + \operatorname{rank} \psi_* = D$. $\square$

2.2 Surjectivity

The goal of this section is to show that if $p > p_c^\square$ then $\mathbb{P}_p\left(S^\square\right) \to 1$ as $N \to \infty$, where

$$p_c^\square = p_c^\square(i,d) = \inf\left\{p : \liminf_{N\to\infty} \mathbb{P}_p\left(A^\square\right) > 0\right\}.$$

First, we will prove that $\mathbb{P}_p\left(S^\square\right) \geq b_0 \mathbb{P}_p\left(A^\square\right)^{b_1}$ for some $b_0, b_1 > 0$ that do not depend on N. This argument is another point of distinction between our argument in the permutohedral lattice and the cubical lattice because the symmetries of the lattices become relevant. However, we start with a general lemma that we use in both cases.

Recall that a vector space V that is acted on by a group G is called an **irreducible representation** of G if it has no proper, non-trivial G-invariant subspaces. That is, the only subspaces W of V so that $\{gw : w \in W\} = W$ are $\{0\}$ and V.

Lemma 2.2.1. *Let V be a finite dimensional vector space and Y be a set. Let $\mathcal{A}$ be the lattice of subspaces of V. Suppose $f : \mathcal{P}(Y) \to \mathcal{A}$ is an increasing function, i.e. if $A \subset B$ then $f(A) \subset f(B)$. Let G be a finite group which* *acts on both Y and V whose action is compatible with f. That is, for each $g \in G$ and $v \in V$ $g(f(v)) = f(gv)$. Let X be a $\mathcal{P}(Y)$-valued random variable with a G-invariant distribution that satisfies the conclusion of Harris' Lemma, meaning that increasing events with respect to X are non-negatively correlated. Then if V is an irreducible representation of G, there are positive constants C_0, C_1 so that*

$$\mathbb{P}_p\left(f(X) = V\right) \geq C_0 \mathbb{P}_p\left(f(X) \neq 0\right)^{C_1} ,$$

where C_0 only depends on G and C_1 only depends on $\dim V$.

Proof. Let $A_k = \{X \in \mathcal{P}(Y) : \operatorname{rank} f(X) \geq k\}$ and $\mathcal{W}_k = f(A_k)$. For a subspace W of V let $\mathrm{Stab}(W)$ denote the stabilizer of W, $\{g \in G : gW = W\}$, and for $H \leq G$ let

$$S_k(H) = \{X : \mathrm{Stab}(f(X)) = H\} \cap A_k .$$

Then in particular, since $A_k = \sqcup_{H \leq G} S_k(H)$, there is a subgroup H' of G so that

$$\mathbb{P}_p\left(S_k(H')\right) \geq \frac{1}{c_G} \mathbb{P}_p\left(A_k\right) , \tag{2.2}$$

where c_G is the number of subgroups of G.

If $H' = G$ then

$$S_k(H') = S_k(G) = \{X : f(X) = V\}$$

because V is an irreducible representation of G, and it follows that

$$\mathbb{P}_p\left(f(X) = V\right) = \mathbb{P}_p\left(S_k(H')\right) \geq \frac{1}{c_G} \mathbb{P}_p\left(A_k\right) . \tag{2.3}$$

Otherwise, if $\mathrm{Stab}(W) = H'$ then the orbit $\{gW : g \in G\}$ contains $|G| / |H'|$ elements, where the elements of each coset of H' in G have the same action on W. Let $\mathcal{B}$ be a collection of subspaces of V that contains one element from each orbit of $\{W \in \mathcal{W}_k : \mathrm{Stab}(W) = H'\}$ so

$$f(S_k(H')) = \bigsqcup_{g \in G/H'} g\mathcal{B}.$$

Taking $B \coloneqq \{X : f(X) \in \mathcal{B}\}$, we have that

$$S_k(H') = \bigsqcup_{g \in G/H'} gB.$$

Let $g \in G \setminus H'$. The events B and gB are symmetric so

$$\mathbb{P}_p(B) = \mathbb{P}_p(gB) = \frac{|H'|}{|G|}\mathbb{P}_p(S_k(H'))) \geq \frac{1}{c_G |G|}\mathbb{P}_p(A_k)$$

using Equation 2.2. By construction, $gB \cap B \subseteq A_{k+1}$ and the Harris' Lemma-like property of X yields

$$\mathbb{P}_p(A_{k+1}) \geq \mathbb{P}_p(B \cap gB) \geq \mathbb{P}_p(B)^2 \geq \left(\frac{1}{c_G |G|}\mathbb{P}_p(A_k)\right)^2.$$

Since either the preceding equation or Equation 2.3 holds for all k, we can conclude that there are positive constants $C_0(G, V)$ and $C_1(V)$ so that

$$\begin{aligned}\mathbb{P}_p(f(X) = V) = \mathbb{P}_p(A_{\dim V}) &\geq C_0 \mathbb{P}_p(A_1)^{2\dim V - 2} \\ &= C_0 \mathbb{P}_p(f(X) \neq 0)^{C_1}.\end{aligned}$$

□

Now it suffices to check the irreducibility of the homology of the torus as a representation of the point symmetry group of each lattice, which we do separately. We begin with the case of the permutohedral lattice A_d^* whose point symmetry group is the symmetric group S_{d+1}.

Proposition 2.2.2. *Let F be a field, $d > 0$, and $1 \leq k \leq d-1$. $H_k(\mathbf{T}^{\mathbf{d}}_{\mathbf{N}}; \mathbf{F})$ is an irreducible representation of S_{d+1} if and only if* $\operatorname{char}(F) \nmid d+1$.

Proof. First, we review the action of S_{d+1} on A_d^*. The lattice $A_d^* \subset F^{d+1}$ has a basis

$$B := \{\mathbf{1} - d\boldsymbol{e}_k : 1 \leq k \leq d\}$$

where $\mathbf{1}$ is the vector whose entries all equal 1. S_{d+1} acts on F^{d+1} by permuting the coordinates, and this restricts to an action on A_d^* which permutes the elements of $B \cup \{\mathbf{1} - d\boldsymbol{e}_{d+1}\}$. The F-vector space generated by A_d^* is called the standard representation of S_{d+1}. Denote it by $\hat{F}^d$. $\hat{F}^d$ is an irreducible representation of S_{d+1} if and only if $\operatorname{char}(F) \nmid d+1$. This can be shown directly or deduced from [25], for example.

The exterior powers of the standard representation $\bigwedge^k \hat{F}^d$ give other representations of S_{d+1}. S_{d+1} acts on $v_1 \wedge \ldots \wedge v_k \in \bigwedge^k \hat{F}^d$ by $g(v_1 \wedge \ldots \wedge v_k) = gv_1 \wedge \ldots \wedge gv_k$. These representations are irreducible if and only if $\operatorname{char}(F) \nmid d+1$ by the criterion given in [25].

Consider $\mathbf{T^d_N}$ as the parallelepiped generated by $\{Nv : v \in B\}$ with opposite faces identified. The homology group $H_1\left(\mathbb{T}^d; F\right)$ is generated by the circles in the coordinate directions corresponding to the elements of B. This correspondence induces an isomorphism of S_{d+1}-representations $H_1\left(\mathbf{T^d_N}; \mathbf{F}\right) \simeq \hat{F}^d$. By the Künneth formula for homology, $H_k\left(\mathbf{T^d_N}; \mathbf{F}\right) \simeq \bigwedge^k \hat{F}^d$. This is easily seen to be an isomorphism of S_{d+1}-representations by comparing the action of S_{d+1} on the homology generators with the definition of the k^{th} exterior power of a representation. As such, the proposition follows from the previous paragraph. $\square$

Next, we consider the case of the square lattice $\mathbb{Z}^d$. The point symmetry group of $\mathbb{Z}^d$ is the hyperoctahedral group $W_d = S_2 \wr S_d$, where $\wr$ denotes the wreath product. It is generated by permutations of the coordinate directions and reflections which reverse a coordinate direction.

Proposition 2.2.3. *Let F be a field, $d > 0$, and $1 \leq i \leq d-1$. $H_i(\mathbb{T}^d_N; F)$ is an irreducible representation of W_d if and only if* $\operatorname{char}(F) \neq 2$.

Proof. Consider $\mathbb{T}^d_N$ as the cube generated by $\{N\boldsymbol{e}_k\}_{1\leq k\leq d}$ with opposite sides identified. For similar reasons as in the proof of Proposition 2.2.2, $H_i\left(T^d_N; F\right)$ is isomorphic to the W_d-representation $\bigwedge^i F^d$. We will give a direct proof that this is an irreducible representation of W_d by showing that if $w \in \bigwedge^i F^d \setminus \{\mathbf{0}\}$ then $\langle W_d w\rangle = \langle \bigwedge^i F^d\rangle$.

Let w be an arbitrary non-zero element of $\bigwedge^i F^d \setminus \{\mathbf{0}\}$ by dividing by the leading coefficient if necessary we may write

$$w = \boldsymbol{e}_{l_{1,1}} \wedge \ldots \wedge \boldsymbol{e}_{l_{1,i}} + \ldots + c_m \boldsymbol{e}_{l_{m,1}} \wedge \ldots \wedge \boldsymbol{e}_{l_{m,i}} .$$

Let $\sigma_v \in S_d \leq W_d$ be a permutation so that

$$\sigma_w \left(\boldsymbol{e}_{l_{1,1}} \wedge \ldots \wedge \boldsymbol{e}_{l_{1,i}} \right) = \boldsymbol{e}_1 \wedge \boldsymbol{e}_2 \wedge \ldots \wedge \boldsymbol{e}_i.$$

For each $1 \leq k \leq d$, let $\rho_k \in W_d$ be the reflection about the hyperplane orthogonal to $\boldsymbol{e}_k$, and let $f_k(v) \coloneqq v + \rho_k(v)$ for $v \in \bigwedge^i F^d$. Then

$$f_{i+1} \left(f_{i+2} \left(\ldots f_d \left(\sigma_w \left(w \right) \right) \right) \right) = 2^{d-i} \boldsymbol{e}_1 \wedge \boldsymbol{e}_2 \wedge \ldots \wedge \boldsymbol{e}_i.$$

$\operatorname{char}(F) \neq 2$ so $2^{d-i} \neq 0$ and thus $\boldsymbol{e}_1 \wedge \boldsymbol{e}_2 \wedge \ldots \wedge \boldsymbol{e}_i \in \langle W_d w \rangle$. But then using the action of $S_d \leq W_d$, we can obtain a basis for $\bigwedge^i F^d$, so $\langle W_d w \rangle = \bigwedge^i F^d$ for any non-zero w and the action of W_d is irreducible. $\square$

We can combine Lemma 2.2.1 with the preceding propositions to obtain the following corollary.

Corollary 2.2.4. *Then there are constants $C_0, C_1 > 0$ not depending on N, i such that*

$$\mathbb{P}_p \left(S_i^{\pentagon} \right) \geq C_0 \mathbb{P}_p \left(A_i^{\pentagon} \right)^{C_1}$$

and

$$\mathbb{P}_p \left(S^{\square} \right) \geq C_0 \mathbb{P}_p \left(A^{\square} \right)^{C_1} .$$

It is worth noting that Lemma 2.2.1 is more general than some of our other tools. For example, in the case of continuum percolation studied in [11], this Lemma can be used to show the analogue of Corollary 2.2.4, even in the absence of stronger duality results.

Proposition 2.2.5. *Let $\{Y_N\}_{N\in\mathbb{N}}$ be a sequence of finite sets with $|Y_N| \to \infty$, each of which has a transitive action by a symmetry group H_N. Also, let $R(N,p)$ be the random set obtained by including each element of Y_N independent with probability p, and suppose there are functions $f_N : \mathcal{P}(Y_N) \to V$ which satisfy the hypotheses of Lemma 2.2.1 for some fixed symmetry group G. Assume that G is a subgroup of H_N for all N and that the action of H_N/G on V is trivial. If $f_N(\varnothing) = 0$ and $f_N(Y_N) \neq 0$ for all sufficiently large N then there exists a threshold function $\lambda(N)$ so that for any $\epsilon > 0$*

$$\mathbb{P}(f_N(R(N,\lambda(N)-\epsilon)) = 0) \to 1$$

$$\mathbb{P}(f_N(R(N,\lambda(N)+\epsilon)) = V) \to 1$$

as $N \to \infty$.

Proof. For a fixed value of N, $\mathbb{P}(R(N,p))$ is an increasing, continuous function of p with $\mathbb{P}(R(N,0)) = 0$ and $\mathbb{P}(R(N,1)) = 1$ for all sufficiently large N. By the intermediate value theorem we can choose $\lambda(N)$ so that for all sufficiently large N,

$$\mathbb{P}\left(f_N\left(R\left(N,\lambda\left(N\right)\right)\right)\neq 0\right)=1/2\,.$$

Then by Lemma 2.2.1, there exist $C_0, C_1 > 0$ such that

$$\mathbb{P}\left(f_N\left(R\left(N,\lambda\left(N\right)\right)\right)=V\right)\geq C_0\mathbb{P}\left(f_N\left(R\left(N,\lambda\left(N\right)\right)\right)\neq 0\right)^{C_1}=\frac{C_0}{2^{C_1}}>0\,.$$

Choose an ϵ_0 between 0 and $\frac{C_0}{2^{C_1}}$. Note that the event $\{f_N\left(R\left(N,p\right)\right)=V\}$ is increasing in p and invariant under the action of H_N. By assumption, H_N acts transitively on X, so the hypotheses of Theorem 1.3.4 are met. Let $\epsilon>0$. Re-arranging Equation 1.1 gives that $\mathbb{P}\left(f_N\left(R\left(N,\lambda\left(N\right)+\epsilon\right)\right)=V\right)>1-\delta$ when

$$\log\left(|Y_N|\right)>\frac{\rho\log\left(1/\left(2\delta\right)\right)}{\epsilon}\,.$$

On the other hand, the event $\{f\left(R\left(N,p\right)^c\right)=0\}$ is also increasing, so by a similar argument, $\mathbb{P}\left(f_N\left(R\left(N,\lambda\left(N\right)-\epsilon\right)\right)=0\right)\to 1\,.$ $\square$

In our models of interest, this tells us that $\lambda^{\square}\left(N\right)$ and $\lambda^{\hexagon}\left(N\right)$ are sharp threshold functions of N for the appearance of all giant cycles. From the definitions of p_c and q_c, we also obtain the inequalities

$$q_c^{\square}=\liminf_{N\to\infty}\lambda^{\square}\left(N\right)\leq\limsup_{N\to\infty}\lambda^{\square}\left(N\right)=p_c^{\square}$$

and

$$q_c^{\hexagon}=\liminf_{N\to\infty}\lambda^{\hexagon}\left(N\right)\leq\limsup_{N\to\infty}\lambda^{\hexagon}\left(N\right)=p_c^{\hexagon}\,.$$

We can then describe the behavior of both models below q_c and above p_c.

Corollary 2.2.6. *If* $p_1 > p_c^{\square}(i,d)$ *and* $p_2 > p_c^{\hexagon}(i,d)$ *then*

$$\mathbb{P}_{p_1}\left(S^{\square}\right) \to 1$$

and

$$\mathbb{P}_{p_2}\left(S^{\hexagon}\right) \to 1$$

as $N \to \infty$. *Conversely, if* $p_1 < q_c^{\square}(i,d)$ *and* $p_2 < q_c^{\hexagon}(i,d)$ *then*

$$\mathbb{P}_{p_1}\left(A^{\square}\right) \to 0$$

and

$$\mathbb{P}_{p_2}\left(A^{\hexagon}\right) \to 0$$

as $N \to \infty$.

2.3 The Case $d = 2i$

We now prove Theorem 2.0.1, that $p_c^{\square}(i,2i) = 1/2$ is a sharp threshold for $A^{\square}$ when $d = 2i$. The proof of the corresponding result for the site percolation model is nearly identical.

Proof of Theorem 2.0.1. Half-dimensional plaquette percolation is self-dual so $\mathbb{P}_{1/2}\left(A^{\square}\right) = \mathbb{P}_{1/2}\left(A^{\blacksquare}\right)$. By Lemma 2.1.4 at least one of the events $A^{\square}$ and $A^{\blacksquare}$ must occur. Therefore,

$$2\mathbb{P}_{1/2}\left(A^{\square}\right) = \mathbb{P}_{1/2}\left(A^{\square}\right) + \mathbb{P}_{1/2}\left(A^{\blacksquare}\right) \geq 1$$

and

$$\mathbb{P}_{1/2}\left(A^{\square}\right) \geq 1/2$$

for all N. It follows that $p_c^{\square} \leq 1/2$. Thus, if $p > 1/2$ then

$$\mathbb{P}_p\left(S^{\square}\right) \to 1$$

as $N \to \infty$, and if $p < 1/2$ then

$$\mathbb{P}_p\left(A^{\square}\right) \to 0$$

as $N \to \infty$ by Corollary 2.2.6. $\square$

2.4 Sharpness and Duality

In this section, we combine the Duality Lemma (Lemma 2.1.4) with Corollary 2.2.6 to examine the behavior of $\mathbb{P}_p\left(A^{\square}\right)$ below $q_c^{\square}(i,d)$ and above $p_c^{\square}(i,d)$. We also relate these thresholds to $p_c^{\square}(d-i,d)$ and $q_c^{\square}(d-i,d)$. Direct analogues of these statements hold for site percolation model hold by very similar arguments, and we do not state them separately here.

We remind the reader that

$$q_c^{\square}(i,d) = \sup\left\{p : \limsup_{N\to\infty} \mathbb{P}_p\left(S^{\square}\right) < 1\right\}.$$

First, Corollary 2.2.6 above has the following corollary.

Corollary 2.4.1. $q_c^{\square}(i,d) \leq p_c^{\square}(i,d)$.

Now we show a partial duality result for any i and d.

Proposition 2.4.2.

$$p_c^{\square}(i,d) + q_c^{\square}(d-i,d) = 1\,.$$

Proof. Let $p > p_c^{\square}(i,d)$. Then

$$\begin{aligned}
\mathbb{P}_p\left(A^{\blacksquare}\right) &= 1 - \mathbb{P}_p\left(Z^{\blacksquare}\right) && \text{by definition} \\
&= 1 - \mathbb{P}_p\left(S^{\square}\right) && \text{by Lemma 2.1.4} \\
&\to 0 && \text{by Corollary 2.2.6}
\end{aligned}$$

as $N \to \infty$. Therefore, $1 - p \le q_c^{\square}(d-i,d)$ for all $p > p_c^{\square}(i,d)$ and

$$p_c^{\square}(i,d) + q_c^{\square}(d-i,d) \ge 1\,. \tag{2.4}$$

Until now, we have suppressed the dependence of probabilities of events on N. To work with subsequences in this argument, denote the probability of an event B for $P(i,d,N,p)$ by $\mathbb{P}_{p,N}(B)$.

Let $p < p_c^{\square}(i,d)$. Then there is a subsequence $\{n_1, n_2, \ldots\}$ of $\mathbb{N}$ for which

$$\mathbb{P}_{p,n_k}\left(A^{\square}\right) \to 0\ .$$

By Lemma 2.1.4,

$$\mathbb{P}_{p,n_k}\left(S^{\blacksquare}\right) \to 1$$

so

$$\limsup_{N\to\infty} \mathbb{P}_p\left(S^{\blacksquare}\right) = 1$$

and $1 - p \geq q_c^{\square}(i, d)$ for all $p < p_c^{\square}(i, d)$. Therefore,

$$p_c^{\square}(i, d) + q_c^{\square}(d - i, d) \leq 1$$

which holds with equality by Equation 2.4. $\square$

Propositions 2.2.5 and 2.4.2 show that duality between $p_c^{\square}(i, d)$ and $p_c^{\square}(d - i, d)$ is equivalent to the convergence of the threshold function $\lambda^{\square}(N)$.

Corollary 2.4.3. *The following are equivalent.*

(a) $\lim_{N \to \infty} \lambda^{\square}(N)$ *exists.*

(b) $p_c^{\square}(i, d) = q_c^{\square}(i, d)$.

(c) $p_c^{\square}(i, d) + p_c^{\square}(d - i, d) = 1$.

In the next section, we demonstrate that the statements of Corollary 2.4.3 hold in the cases $i = 1$ and $i = d - 1$.

2.5 The Cases $i = 1$ and $i = d - 1$

We show that $p_c^{\square}(1, d)$ and $q_c^{\square}(1, d)$ coincide and equal the critical threshold for bond percolation on $\mathbb{Z}^d$, denoted here by $\hat{p}_c = \hat{p}_c(d)$. As in the previous sections, the proofs for the corresponding results for the site percolation model are nearly identical and we do not state them here.

In the subcritical regime, we apply Theorem 1.3.1 to show that the probability of a giant one-cycle limits to zero as $N \to \infty$ when $p < \hat{p}_c$.

Proposition 2.5.1. $q_c^{\square}(1, d) \geq \hat{p}_c$

Proof. Let $p < \hat{p}_c$ and let $M = \lfloor N/2 \rfloor$. For a vertex x of $\mathbb{T}_N^d$, denote by A_x the event that there is a connected, giant 1-cycle containing an edge adjacent to x. If A_0 occurs then $0 \leftrightarrow \partial[-M, M]^d$ because a 1-cycle contained in $[-M, M]^d$ is null-homologous in $\mathbb{T}_N^d$. Therefore,

$$\mathbb{P}_p(A_x) \leq e^{-\kappa(p)M} \tag{2.5}$$

for all vertices x of $\mathbb{T}_N^d$, using translation invariance and Theorem 1.3.1.

Let X be the number of vertices in $\mathbb{T}_N^d$ that are contained in a connected, giant 1-cycle. $A^{\square} = \{X \geq 1\}$ so

$$\begin{aligned}
\mathbb{P}_p\left(A^{\square}\right) &= \mathbb{P}_p(X \geq 1) & \\
&\leq \mathbb{E}_p(X) & \text{by Markov's Inequality} \\
&= \sum_{x \in \mathbb{T}_N^d} \mathbb{P}_p(A_x) & \\
&\leq N^D e^{-\kappa(p)M} & \text{using Equation 2.5} \\
&= N^d e^{-\kappa(p)\lfloor N/2 \rfloor} &
\end{aligned}$$

which goes 0 as $N \to \infty$.

$\square$

In the supercritical regime, we use the following lemma on crossing probabilities inside a rectangle (which is Lemma 7.78 in [31]).

Lemma 2.5.2. *Let $p > \hat{p}_c$. Then there is an $L > 0$ and a $\delta > 0$ so that if $N > 0$ and $x \in [0, N-1]^{d-1} \times [0, L]$, the probability that 0 is connected to x inside $P \cap \left([0, N-1]^{d-1} \times [0, L]\right)$ is at least δ.*

Proposition 2.5.3. $p_c^{\square}(1, d) \leq \hat{p}_c$

Proof. Let $p > \hat{p}_c$, and let B be the event that there is a path of edges of P connecting 0 to $(N-1)\,\boldsymbol{e}_1 = (N-1, 0, \ldots, 0)$ inside of $[0, N-1]^d$. By the previous lemma, there is a $\delta > 0$ so that $\mathbb{P}_p(B) \geq \delta$.

If B occurs, then the path obtained by adding the edge between $(N-1)\,\boldsymbol{e}_1$ and $N\boldsymbol{e}_1 = 0$ is a giant 1-cycle. It follows that

$$\mathbb{P}_p\left(A^{\square}\right) \geq p\mathbb{P}_p(B) \geq p\delta$$

for any choice of N. Therefore, $p_c^{\square}(1, d) \leq p$ for all $p \geq \hat{p}_c$ and $p_c^{\square}(1, d) \leq \hat{p}_c$. $\square$

The proof of Theorem 2.0.2 is completed by combining Propositions 2.5.1 and 2.5.3 with Corollary 2.4.3.

Proof of Theorem 2.0.2. Propositions 2.5.1 and 2.5.3 show that

$$p_c^{\square}(1,d) \leq \hat{p}_c \leq q_c^{\square}(1,d) \ .$$

However, $q_c^{\square}(1,d) \leq p_c^{\square}(1,d)$ by Corollary 2.4.1 so

$$p_c^{\square}(1,d) = q_c^{\square}(1,d) = \hat{p}_c \, .$$

Therefore, it follows from Corollary 2.4.3 that $\hat{p}_c$ and $1-\hat{p}_c$ are sharp thresholds for 1-dimensional and $(d-1)$-dimensional percolation on the $\mathbb{T}^d$, respectively.

□

2.6 Monotonicity

Next, we prove that the critical probabilities $p_c^{\square}(i,d)$ are strictly increasing in i and strictly decreasing in d. This will complete the proof of Theorem 2.0.3. Here again we will need to differentiate between the cubical and permutohedral lattices.

First we consider the cubical case, in which we compare percolation on $\mathbb{T}_N^d$ with the thickened $d-1$-dimensional slice $\mathbb{T}_N^d \cap \{0 \leq x_1 \leq 1\}$. Compare the first part of the proof to that of Lemma 4.9 of [11].

Proposition 2.6.1. *For* $0 < i < d-1$,

$$p_c^{\square}(i,d) < p_c^{\square}(i,d-1) < p_c^{\square}(i+1,d) \ .$$

Proof. First, we will show that

$$p_c^{\square}(i,d) \leq p_c^{\square}(i,d-1) \leq p_c^{\square}(i+1,d) \ . \tag{2.6}$$

Let $T = \mathbb{T}_N^d \cap \{x_1 = 0\}$. T is a torus of dimension $d-1$ and, by a standard argument, the map on homology $\alpha_* : H_j(T) \to H_j\left(\mathbb{T}^d\right)$ induced by the inclusion $T \hookrightarrow \mathbb{T}^d$ is injective for all j. $P \cap T$ is distributed as $P(i, d-1, N, p)$.

Define $A_{d-1}^{\square}$ to be the event that $\gamma_* : H_i(P \cap T) \to H_i(T)$ is non-zero, where γ_* is induced by the inclusion $P \cap T \hookrightarrow T$. If $A_{d-1}^{\square}$ holds then $\alpha_* \circ \phi_*$ is also non-zero, as α_* is injective. But $\alpha_* \circ \gamma_* = \phi_* \circ \beta_*$, where β_* is the map on homology $\beta_* : H_i(P \cap T) \to H_i(P)$ induced by the inclusion $P \cap T \hookrightarrow P$, so ϕ_* is also non-zero. It follows that $A_{d-1}^{\square} \implies A^{\square}$. Therefore $p_c^{\square}(i, d-1) \geq p_c^{\square}(i, d)$ by the definition of that threshold.

Observe that $H_i\left(\mathbb{T}^d\right)$ is generated by the images of the maps on homology $H_i\left(\mathbb{T}^d \cap \{x_j = 0\}\right) \to H_i\left(\mathbb{T}^d\right)$ induced by the inclusions $\mathbb{T}^d \cap \{x_j = 0\} \hookrightarrow \mathbb{T}^d$ as j ranges from 1 to d. Denote by S_j the event that the map $H_i(P \cap \{x_j = 0\}) \to H_i\left(\mathbb{T}^d \cap \{x_j = 0\}\right)$ induced by inclusion is surjective and let $q > q_c^{\square}(i, d-1)$. Then there is a subsequence $(n_1, n_2, \ldots)$ of $\mathbb{N}$ so that

$$\mathbb{P}_{p,n_k}(S_j) \to 1$$

as $k \to \infty$ for $j = 1, \ldots, d$. As $S \subset \cap_j S_j$, Harris's Inequality implies that $\mathbb{P}_{p,n_k}(S) \to 1$ also. Therefore, $p > q_c^{\square}(i, d)$ and $q_c^{\square}(i, d-1) \geq q_c^{\square}(i, d)$ for all i and d. Combining this inequality (for a different choice of i and d) with Proposition 2.4.2 we obtain

$$p_c^{\square}(i, d-1) = 1 - q_c^{\square}(d-i-1, d-1) \leq 1 - q_c^{\square}(d-i-1, d) = p_c^{\square}(i+1, d)\,, \quad (2.7)$$

which shows Equation 2.6.

It will be useful later in the argument to observe that these inequalities, together with Theorem 2.0.2 and known lower bounds on $\hat{p}_c$ (see [12], for example), imply that

$$0 < p_c^{\square}(1,d) \leq p_c^{\square}(i,d) \leq p_c^{\square}(i,i+1) < 1\,. \tag{2.8}$$

Furthermore, we can show $p_c^{\square}(i,d) < p_c^{\square}(i,d-1)$ using the thicker cross-section $T' = \mathbb{T}_N^d \cap \{0 \leq x_1 \leq 1\}$. Note that an i-face v of T is in the boundary of a unique $(i+1)$-face $w(v)$ of T' that is not contained in T (for example, if $v = \{0\} \times [0,1]^i \times \{0\}^{d-i-1}$, then $w(v) = [0,1]^{i+1} \times \{0\}^{d-i-1}$). The idea is to sometimes add v to T when the other i-faces of $w(v)$ are present, effectively increasing the percolation probability in T by a small amount. However, we must be careful to do so in a way so that the i-faces remain independent.

The i-faces of T' are divided into three subsets: those included in T, those which are perpendicular to T (that is, i-faces not included in T which intersect T in their boundary), and those parallel to T (that is, i-faces of the form $v + \boldsymbol{e}_1$, where v is an i-face of T). For an i-face v of T, let $J(v)$ be the set of all perpendicular i-faces that meet v at an $i-1$ face. v, $v+\boldsymbol{e}_1$, and $J(v)$ are the i-faces of the $(i+1)$-face $w(v)$. Also, for a perpendicular i-face u of T', let $K(u) = \{v : u \in J(v)\}$. Note that for any u and v

$$|J(v)| = 2i \qquad \text{and} \qquad |K(u)| = 2(d-i)\,. \tag{2.9}$$

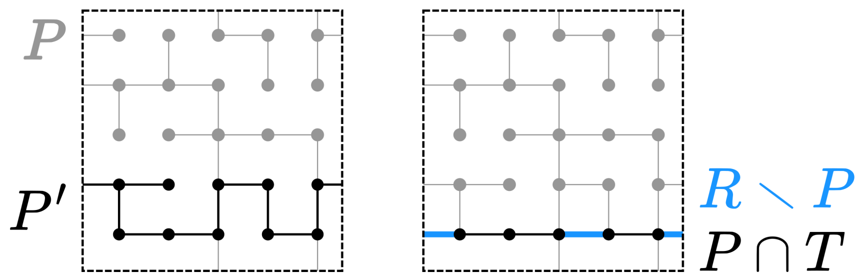

Figure 2.2: The setup in the proof of Proposition 2.6.1 for the case $d = 2, i = 1$. On the left, P' is shown in black and the remaining faces of P are depicted in gray. On the right, $P \cap T$ is in black and the additional faces of R are shown in blue. Note that giant cycles exist in P, P', and R, but not in $P \cap T$.

We define a coupling between i-dimensional plaquette percolation P' on T' with probability p and i-dimensional percolation R with probability $p + p(1-p)q^{2i}$ on T, where $q = q(p)$ is chosen to satisfy $p = 1-(1-q)^{2(d-i)}$. For all pairs (v, u) where v is an i-face of T and $u \in J(v)$, define independent Bernoulli random variables $\kappa(u, v)$ to be 1 with probability q and 0 with probability $1-q$. Let $P' \subset T'$ be the subcomplex containing the $i-1$-skeleton of T' where each i-face in T or parallel to T is included independently with probability p, and the other i-faces u of T' are included if $\kappa(u, v) = 1$ for at least one $v \in K(u)$. Observe that

$$\mathbb{P}(u \in P') = 1 - \mathbb{P}\left(\cap_{v \in K(u)} \{\kappa(u, v) = 0\}\right) = 1 - (1-q)^{2(d-i)} = p$$

(using Equation 2.9), and that the faces u are included independently. That is, P' is percolation with probability p on T'. On the other hand, define $R \subset T$ by starting with all faces of $P' \cap T$ and adding an i-face $v \notin P'$ if $v + \boldsymbol{e}_1 \in P'$

and $\kappa(v, u) = 1$ for all $u \in J(v)$. Then R is percolation on T with probability $p + p(1-p)q^{2i} > p$. See Figure 2.2.

As $p + p(1-p)q^{2i}$ is a continuous function of p and $0 < p_c^{\square}(i, d-1) < 1$ (Equation 2.8), we can choose p to satisfy

$$0 < p < p_c^{\square}(i, d-1) < p + p(1-p)q^{2i} < 1\,.$$

Then

$$\mathbb{P}_{p+p(1-p)q^{2i}}\left(\xi_* \text{ is non-trivial}\right) \to 1 \tag{2.10}$$

as $N \to \infty$ by the definition of $p_c^{\square}(i, d-1)$, where $\xi_* : H_i(R) \to H_i(T)$ is the map on homology induced by the inclusion $R \hookrightarrow T$.

Extend P' to plaquette percolation P on all of $\mathbb{T}_N^d$ by including the i-faces in $\mathbb{T}_N^d \setminus T'$ independently with probability p. If σ is an i-cycle of R we can write

$$\sigma = \sum_j a_j u_j + \sum_k b_k v_k$$

where $u_j \notin P$ and $v_k \in P$ for all j and k. Then, by construction, we can form a corresponding i-cycle σ' of P by setting

$$\sigma' = \sigma + \sum_j a_j \partial w\left(u_j\right)\,.$$

σ and σ' are homologous in $\mathbb{T}^d$, so $\alpha_* \circ \xi_*\left([\sigma]\right) = \phi_*\left([\sigma']\right)$ In particular, if ξ_* is non-trivial then ϕ_* is non-trivial as well. Using Equation 2.10, it follows that

$$\mathbb{P}_p\left(A^{\square}\right) \geq \mathbb{P}_{p+p(1-p)q^{2i}}\left(\xi_* \text{ is non-trivial}\right) \to 1\,,$$

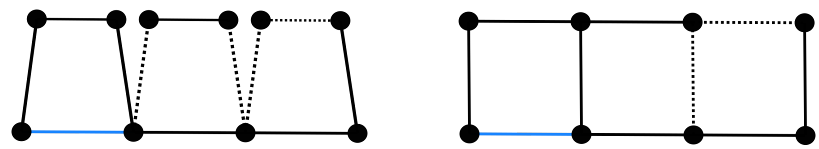

Figure 2.3: Percolation on T'' (left) mapping to percolation on T' (right) for $i = 1, d = 2$. The blue edges are in $R \setminus P$.

as $N \to \infty$. Therefore,

$$p_c^{\square}(i,d) \leq p < p_c^{\square}(i,d-1) \ .$$

We can define similar couplings between percolations on $\mathbb{T}_N^d \cap \{x_j = 0\}$ and on $\mathbb{T}_N^d \cap \{0 \leq x_j \leq 1\}$ for $j = 1 \ldots d$. Combining these couplings with the argument leading to Equation 2.7 yields $q_c^{\square}(i,d) < q_c^{\square}(i,d-1)$. Then from Proposition 2.4.2 we obtain

$$p_c^{\square}(i,d-1) = 1 - q_c^{\square}(d-i-1,d-1) < 1 - q_c^{\square}(d-i-1,d) = p_c^{\square}(i+1,d) \,.$$

$\square$

An alternative approach to the proof of Proposition 2.6.1 is to construct a third space T'' by attaching a new $i+1$-cube to each i-face of T along one of the cube's i-faces. We can define inhomogeneous percolation P'' on T'' by starting with the $i-1$-skeleton of T'', adding each i-face of T and and each i-face parallel to T independently with probability p, and adding the perpendicular i-faces independently with probability q (these faces play the same role as the random variables $\kappa(u,v)$ above). Giant cycles in P'' are ones that are mapped non-trivially to $H_i(T'')$ by the map on homology induced by the inclusion $P'' \hookrightarrow T''$, and they appear at a lower value of p than $p_c^{\square}(i,d-1)$ (precisely when they appear in R as defined above). The proof is finished by observing that the quotient map $\pi : T'' \to T'$ identifying the corresponding perpendicular faces of neighboring cubes induces an injective map on homology, and therefore the existence of giant cycles in P'' implies the existence of giant cycles in P. This idea is illustrated in Figure 2.3. Note that our definition of giant cycles in T'' can be adapted to give a more general notion of homological percolation in the i-skeleton of a cubical or simplicial complex whose i-dimensional homology is nontrivial.

Now we consider the permutohedral lattice. The idea of the proof is again to find a copy of $\mathbb{T}^{d-1}$ within $\mathbb{T}^d$, but here the correct embedding is slightly less obvious.

Before beginning the proof, it will be useful to discuss the combinatorial structure of the permutohedron. Recall that $\hat{\mathbb{R}}^d \subset \mathbb{R}^{d+1}$ is the subspace

$$\hat{\mathbb{R}}^d := \left\{(x_0, x_1, ..., x_d) : \sum_{k=0}^{d} x_k = 0\right\}.$$

We review the expositions of [19] and [52]. The d-permutohedron centered at the origin in $\hat{\mathbb{R}}^d$ has vertices obtained from permuting the coordinates of

$$\frac{1}{2d+1}\left(d, d-2, d-4, \ldots, -d+2, -d\right).$$

Let σ be this permutohedron centered at 0. It is enough to understand the geometry of σ because the other permutohedra of the lattice are translates of σ. The k-faces of σ correspond to ordered partitions of the coordinate indices into $d-k+1$ subsets $M_1, \ldots, M_{d-k+1}$, where every coordinate in places M_j is smaller than every coordinate in M_l for $j < l$. We will abuse notation and identify these partitions with the faces associated to them. For example, the 4-permutohedron has a 2-face $\{\{1,3\},\{4\},\{2,5\}\}$ containing the vertices, and by extension all points, satisfying the property that the two smallest coordinates are in positions 1 and 3, and the next smallest coordinate is in position 4. This face has 1-subfaces corresponding to refinements of its ordered partition, namely $\{\{1\},\{3\},\{4\},\{2,5\}\}$, $\{\{3\},\{1\},\{4\},\{2,5\}\}$, $\{\{1,3\},\{4\},\{2\},\{5\}\}$, and $\{\{1,3\},\{4\},\{5\},\{2\}\}$. Since antipodal vertices have reversed coordinate order, it is not hard to check that opposite k-faces correspond to the same partition with reversed blocks.

We will be also interested in the combinatorial structure of the permutohedral lattice as it relates to translates of a fixed $(d-1)$-face. Let f be the $(d-1)$-face corresponding to the partition $\{\{1,2,\ldots,d\},\{d+1\}\}$. Then f is itself a $(d-1)$-permutohedron with subfaces corresponding to ordered partitions of $\{1,2,\ldots,d\}$. Let f' be the opposite $(d-1)$-face, i.e. the one corresponding to $\{\{d+1\},\{1,2,\ldots,d\}\}$. Consider $(d-1)$-face paths between f and f', meaning sequences $(f_1, f_2, \ldots, f_m)$ of $(d-1)$-faces such that $f_1 = f$, $f_m = f'$, and $f_k \cap f_{k+1}$ is a $(d-2)$-face for each $1 \leq k \leq m-1$. Then there is no path of length 3, but for a $(d-2)$-face $h \subset f$ corresponding to an ordered partition $\{A, B, \{d+1\}\}$, there is a path $(f, \{A, B \cup \{d+1\}\}, \{A \cup \{d+1\}, B\}, f')$ of length 4 passing through h. Thus, the other $(d-1)$-faces can be decomposed into one pair for each $(d-2)$-face of f, each pair consisting of a neighbor of f and a neighbor of f'.

We start with an easy lemma.

Lemma 2.6.2. *Let σ_1 and σ_2 be the two other permutohedra adjacent to σ along $\{A, B \cup \{d+1\}\}$ and $\{A \cup \{d+1\}, B\}$ respectively. Then $\sigma_1 \cap \sigma_2$ is a translate of f. In fact, letting v be such that $\sigma + v = \sigma_1$, we have $\sigma_1 \cap \sigma_2 = f' + v$.*

Proof. By Lemma 3.4 of [19], the centers of the permutohedra are in general position, so a $(d-k)$-cell of the Voronoi complex is an intersection of exactly $(k+1)$ top dimensional cells. In particular, there are exactly 3 $(d-1)$-faces among all permutohedra of the lattice that contain $\{A, \{d+1\}, B\}$. We will show that the face not contained in σ is $\sigma_1 \cap \sigma_2 = f' + v$.

Let $G = \sigma \cap \sigma_1 = \{A, B \cup \{d+1\}\} \subset \sigma$ and $G' = \{B \cup \{d+1\}, A\} \subset \sigma$, and let $T_v : \hat{\mathbb{R}}^d \to \hat{\mathbb{R}}^d$ be translation by v. Also, denote the reflections about the line spanned by v and the hyperplane orthogonal to v by ρ_1 and ρ_2, respectively. Then $G = T_v(G')$ since opposite $(d-1)$-faces of σ have reversed ordered partitions. Moreover, G and G' are orthogonal to v so $T_v |_{G'} = \rho_2 |_{G'}$.

Now, consider the action of ρ_1 on the $(d-2)$-faces of G'. ρ_1 sends a $(d-2)$ face to the face opposite to it in G', which can be obtained by reversing the subpartitions within each of $B \cup \{d+1\}$ and A. This can be seen from the fact that opposite vertices of G' are maximally far apart via edge paths within G', so they must have reversed coordinates within $B \cup \{d+1\}$ and A.

Then $\rho_2 \circ \rho_1$ is reflection about the origin (the antipodal map), so since antipodal $(d-2)$-faces have reversed ordered partitions,

$$\begin{aligned}\{A, \{d+1\}, B\} = \rho_2 \circ \rho_1 (\{B, \{d+1\}, A\}) &= \rho_2 (\{\{d+1\}, B, A\}) \\ &= T_v (\{\{d+1\}, B, A\}) \subset f' + v.\end{aligned}$$

But $f' + v$ is not contained in σ, so it must be the third $(d-1)$-face adjacent to $\{A, \{d+1\}, B\}$ (in addition to $\sigma \cap \sigma_1$ and $\sigma \cap \sigma_2$, which are both contained in σ) and therefore $\sigma_1 \cap \sigma_2 = f' + v$. $\square$

Now we are ready to prove the monotonicity of $p_i^{\hexagon}(d)$ in i and d.

Proposition 2.6.3. $p_i^{\hexagon}(d) < p_i^{\hexagon}(d-1) < p_{i+1}^{\hexagon}(d)$

Proof. First we prove $p_i^{\hexagon}(d) \leq p_i^{\hexagon}(d-1) \leq p_{i+1}^{\hexagon}(d)$. Unlike the case of plaquettes in $\mathbb{Z}^d$, there is no obvious isometric embedding of the $(d-1)$-dimensional permutohedral lattice into the d-dimensional permutohedral lattice. We will instead find a set of d-permutohedra that is combinatorially and homotopy equivalent to the $(d-1)$-lattice. The idea is to associate each

d-permutohedron with a fixed $(d-1)$-face and then project a thickened $(d-1)$-surface of permutohedra orthogonally to that face.

Now we can begin constructing the sublattice. Let w be the vector pointing to the center of f. Let $\mathbb{L}_{d-1} \simeq A_{d-1}^*$ be the sublattice of A_d^* generated by the vectors of A_d^* orthogonal to w and let $\mathbb{L}_d$ be the sublattice generated by $\mathbb{L}_{d-1} \cup \{2w\}$. Call an equivalence class of the d-permutohedra under the action of translation by $2\mathbb{Z}w$ a pile of permutohedra. For any permutohedron θ, define

$$B_\theta = \{\theta' : \theta' \cap \theta \neq \emptyset\},$$

the union of the permutohedra that intersect θ. Identifying permutohedra with their centers, let

$$S := \bigcup_{\theta \in \mathbb{L}_{d-1}} B_\theta.$$

Then take S' be the set of permutohedra intersecting the upper envelope of S with respect to w. In other words, S' is the union of one permutohedron of S from each pile such that for each pile Π we have

$$(S' \cap \Pi) \cdot w = \sup_{\theta \in S \cap \Pi} \theta \cdot w.$$

Alternatively, one can construct S' explicitly. For a permutohedron θ, let v_θ be such that $\sigma + v_\theta = \theta$. Define

$$U_\theta := \{\theta' \in A_d^* : \theta' \cap (f + v_\theta) \neq \emptyset\} \setminus \theta.$$

Then we can write

$$S' = \bigcup_{\theta \in \mathbb{L}_{d-1}} U_\theta.$$

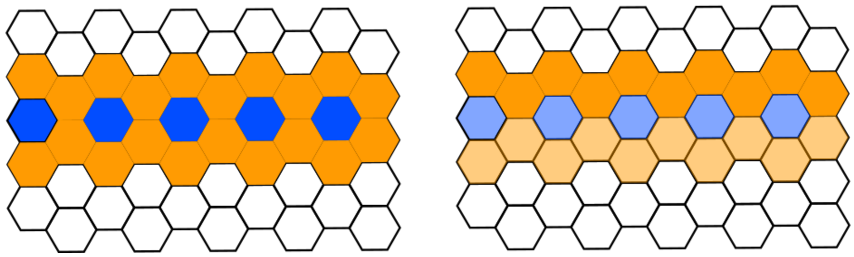

Figure 2.4: The set S (left) in orange with the centers of each ball in blue. Taking the highest element of S in each column gives S' (right).

Now we will show the homotopy equivalence respecting the cell structure via the nerve theorem, seen in Corollary 4G.3 of [35]. Let $\mathcal{U}$ be the open cover of S' induced by the permutohedra it contains. Since the permutohedra are convex, it is a good cover and therefore S' is homotopy equivalent to $\mathcal{N}\mathcal{U}$. Then we compare this to the the cover $\mathcal{V}$ of the $(d-1)$-dimensional permutohedral lattice induced by its $(d-1)$-permutohedra and the corresponding nerve $\mathcal{N}\mathcal{V}$.

Let θ be a permutohedron of S'. Let $f_\theta = f + v_\theta$ and let $f'_\theta = f' + v_\theta$. Lastly, let h be an arbitrary $(d-2)$-dimensional face of f_θ and let h' be the corresponding $(d-2)$-dimensional of f'_θ obtained via translation by $-2w$. By the definition of S', there is at most one other permutohedron of S' containing h, since the one adjacent to θ along f_θ would be in the same pile. The same is true for h', and by Lemma 2.6.2 we cannot have both because this would again be two elements of the same pile. However, one or the other must be present because adjacent piles of S are connected by $(d-1)$-faces and taking the upper

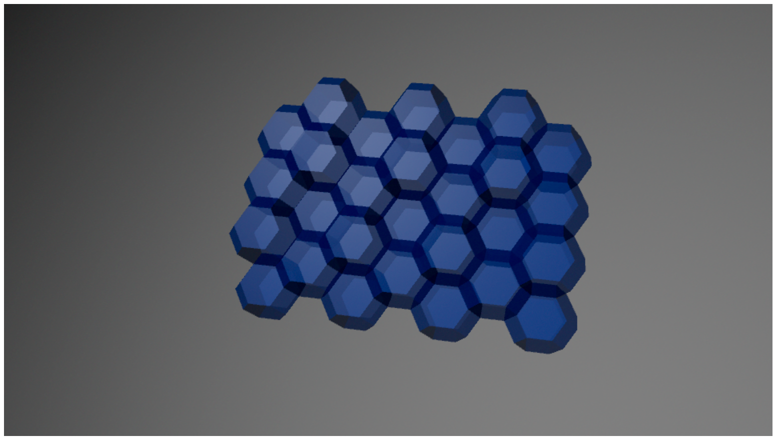

Figure 2.5: A portion of S' in 3 dimensions.

envelope preserves this property. Thus, the permutohedra of S' adjacent to θ are in bijection with adjacent $(d-1)$-permutohedra to a fixed permutohedron $\theta^{d-1} \in A^*_{d-1}$. Then it only remains to check that the intersections are the same in each case.

Let $\alpha_1, \alpha_2, \ldots, \alpha_k \in A^*_{d-1}$ be $d-1$-permutohedra adjacent to θ^{d-1}. For each j, take $\gamma_j \in S'$ to be the d-permutohedron in the pile corresponding to α_j. Again using Lemma 3.4 of [19], any k pairwise adjacent permutohedra intersect at a $(d-k+1)$-face and any set that is not pairwise adjacent has empty intersection. From the construction of S', $\gamma_j \cap \gamma_l = \emptyset$ if and only if $\alpha_j \cap \alpha_l = \emptyset$. Thus, $\{\gamma_1, \ldots, \gamma_k, \theta\}$ are pairwise adjacent if and only if $\left\{\alpha_1, \ldots, \alpha_k, \theta^{d-1}\right\}$ are, and so we have

$$\left(\bigcap_{j \leq k} \gamma_j\right) \cap \theta \neq \emptyset \iff \left(\bigcap_{j \leq k} \alpha_j\right) \cap \theta^{d-1} \neq \emptyset.$$

We have then shown that the $\mathcal{NU} \simeq \mathcal{NV}$. Furthermore, we have shown that there is a bijection between the permutohedra of S' and those of A^*_{d-1} such that for any $\mathcal{U}' \subset \mathcal{U}$, the corresponding $\mathcal{V}' \subset \mathcal{V}$ satisfies $\mathcal{NU}' \simeq \mathcal{NV}'$.

The strict inequality $p_i^{\hexagon}(d) < p_i^{\hexagon}(d-1)$ can be obtained by a similar proof to the plaquette case. For a permutohedron $\theta \in S'$, it is easy to check that if there is a giant cycle in $Q \cup \theta$, there is also a giant cycle in $Q \cup (B_G(\theta, 1) \setminus S')$. There is overlap between the added permutohedra, but we deal with this in the same way as the overlap in the plaquette construction. This also gives the strict inequality $p_i^{\hexagon}(d-1) < p_{i+1}^{\hexagon}(d)$ by duality as before. $\square$

Proof of Theorem 2.0.3. By Equation 2.8, $q_c^{\square}(i,d), p_c^{\square}(i,d) \in (0,1)$. The remaining statements follow from Corollary 2.2.6 and Propositions 2.4.2 and 2.6.1.

$\square$

Note that we could alternatively show that $p_c^{\square}(i,d), q_c^{\square}(i,d) \in (0,1)$ by modifying the proof of Proposition 2.5.1 to work for the lattice of i-plaquettes in $\mathbb{Z}^d$ and using a Peierls-type argument to obtain the bound

$$\frac{1}{2d-i+1} \leq q_c^{\square}(i,d) \leq p_c^{\square}(i,d) \leq 1 - \frac{1}{d+i+1}.$$

Proof of Theorem 2.0.4. Theorem 2.0.4 follows from the proofs of Theorems 2.0.1, 2.0.2, and 2.0.3 with the adjustments for the permutohedral lattice noted throughout the chapter. In particular, Lemma 2.1.4 and Propositions 2.2.3 and 2.6.1 are replaced by Lemma 2.1.1 and Propositions 2.2.2 and 2.6.3 respectively. $\square$

Chapter 3: Homological Percolation in Random-Cluster Percolation on a Torus

This chapter is based on joint work with Benjamin Schweinhart [23].

Recall that the i-random cluster model is defined by the following measure on configurations of plaquettes:

$$\mu_{p,q,i,N}(\omega) := \frac{1}{Z_{p,q,i,N}} p^{\eta(\omega)} (1-p)^{\left|F_N^i\right|-\eta(\omega)} q^{\mathbf{b}_{i-1}(P_\omega)},$$

where Z is a constant, $\eta(\omega)$ is the number of open plaquettes, F_N^i is the total number of i-plaquettes in the torus $\mathbb{T}_N^d$, and $\mathbf{b}_{i-1}(P_\omega)$ is the $(i-1)$st Betti number of the subcomplex composed of the $(i-1)$-skeleton together with the open plaquettes of ω.

3.1 Main Results

Recall that

$$p^* := \frac{(1-p)q}{(1-p)q+p}$$

and

$$p_{\mathrm{sd}} \coloneqq \frac{\sqrt{q}}{1+\sqrt{q}}\,.$$

Our first result is a higher dimensional analogue of the following result of Beffara and Duminil-Copin [8] for self-dual random cluster percolation.

Theorem 3.1.1 (Beffara–Duminil-Copin)**.** *Let $q \geq 1$. Then the critical probability for the random cluster model with parameter q in $\mathbb{Z}^2$ is p_{sd}.*

Theorem 3.1.2. *Suppose $q \geq 1$ and* $\operatorname{char}(F) \neq 2$. *If $d = 2i$, then*

$$\begin{cases} \mathbb{P}_p\left(A\right) \to 0 & p < p_{\mathrm{sd}}(q) \\ \mathbb{P}_p\left(S\right) \to 1 & p > p_{\mathrm{sd}}(q) \end{cases}$$

as $N \to \infty$.

We also show that there are dual sharp phase transitions for $i = 1$ and $i = d-1$ that are consistent with the critical probability for the random-cluster model in $\mathbb{Z}^d$, assuming a conjecture about the continuity of the critical probability in slabs.

Let

$$S_k \coloneqq \mathbb{Z}^2 \times \{-k, -k+1, \ldots, k\}^{d-2} \subset \mathbb{Z}^d\,.$$

Fix $q \geq 1$ and let $p_c\left(S_k\right)$ be the critical probability for the random-cluster model with parameter q on S_k with free boundary conditions. This can be constructed by a limit of free random-cluster models on

$$S_{k,l} \coloneqq \{-l, -l+1, \dots, l\}^2 \times \{-k, -k+1, \dots, k\}^{d-2} \ .$$

Since $S_{k,l} \subset S_{k+1,l}$, it follows that $p_c(S_k)$ is decreasing in k. Then let

$$p_c^{\text{slab}} = \lim_{k \to \infty} p_c(S_k) \ .$$

Let $\hat{p}_c = \hat{p}_c(q, d)$ be the critical threshold for the random-cluster model with parameter q on $\mathbb{Z}^d$. These two critical values are conjectured to coincide [46].

Conjecture 3.1.3. *For all $q \geq 1$,*

$$p_c^{\text{slab}} = \hat{p}_c$$

Theorem 3.1.4. *Let $q \geq 1$ and* $\operatorname{char}(F) \neq 2$. *Then the following statements hold:*

If $i = 1$ then

$$\begin{cases} \mathbb{P}_p(A) \to 0 & p < \hat{p}_c \\ \mathbb{P}_p(S) \to 1 & p > p_c^{\text{slab}} \end{cases}$$

as $N \to \infty$.

Furthermore, if $i = d-1$ then

$$\begin{cases} \mathbb{P}_p(A) \to 0 & p < (\hat{p}_c)^* \\ \mathbb{P}_p(S) \to 1 & p > \left(p_c^{\text{slab}}\right)^* \end{cases}$$

as $N \to \infty$.

If Conjecture 3.1.3 is true, then there is a sharp threshold at $\hat{p}_c = p_c^{\text{slab}}$.

In general dimensions we have a sharp threshold function defined as follows: Let $\lambda = \lambda(q, i, d, N)$ satisfy

$$\mathbb{P}_{\lambda,q}(A) = 1/2$$

and let $p_l = p_l(q, i, d) := \liminf_{N\to\infty} \lambda(q, i, d, N)$ and $p_u = p_u(q, i, d) := \limsup_{N\to\infty} \lambda(q, i, d, N)$.

Theorem 3.1.5. *Suppose* $q \geq 1$ *and* $\mathrm{char}(F) \neq 2$. *For every* $d \geq 2$, $1 \leq i \leq d-1$, *and* $\epsilon > 0$

$$\begin{cases} \mathbb{P}_{\lambda-\epsilon}(A) \to 0 \\ \mathbb{P}_{\lambda+\epsilon}(S) \to 1 \end{cases}$$

as $N \to \infty$.

Moreover, for every $d \geq 2$ *and* $1 \leq i \leq d-1$ *we have*

$$0 < p_l \leq p_u < 1,$$

and p_l *and* p_u *have the following properties.*

(a) (Duality) $p_u(q, i, d) = (p_l(q, d-i, d))^*$.

(b) (Monotonicity in i *and* d*)* $p_u(q, i, d) < p_u(q, i, d-1) < p_u(q, i+1, d)$ *for* $0 < i < d-1$.

In the infinite volume case the coupling between the 2-random-cluster model and the associated Potts lattice gauge theory (which is a 1-dimensional Potts model) leads to a short alternative proof of area law and perimeter law for Wilson loops at high and low temperatures respectively.

Theorem 3.1.6. *Suppose $q \geq 1$ is a prime integer, let γ be a rectangular loop in $\mathbb{Z}^4$, and let W_γ be the Wilson loop variable for γ. Also, let ϕ be sampled from the q-state Potts lattice gauge theory. For all $\beta > 0$ there exist constants $C_2(\beta, q), C_3(\beta, q) > 0$ so that*

$$\exp(-C_2(\beta, q)\mathrm{Area}(\gamma)) \leq \mathbb{E}_{\beta,q}(W_\gamma) \leq \exp(-C_3(\beta, q)\mathrm{Per}(\gamma)) . \tag{3.1}$$

Furthermore, there exist $0 < \beta_1 \leq \beta_2 < \infty$ so that

$$\log\left(\mathbb{E}_{\beta,q}(W_\gamma)\right) \begin{cases} \to -C_2(\beta, q)\mathrm{Area}(\gamma) & \beta > \beta_2 \\ = \Theta\left(-\mathrm{Per}(\gamma)\right)) & \beta < \beta_1 \end{cases} .$$

3.2 Homological i-random-cluster percolation

The i-random cluster model considered in this chapter and the Bernoulli plaquette model in Chapter 2 differ in their distributions, but have the same set of possible states. Thus, we can use the same topological tools for specific configurations without modification. However, it is less clear what the distribution of the dual model is in a dependent setting. We show that it is a $(d - i)$-random cluster model with related parameters.

We first consider duality in the balanced i-random cluster model, which we recall has measure

$$\tilde{\mu}_{p,q,i,N}(\omega) := \frac{\left(\sqrt{q}\right)^{-\rho(\omega)}}{\tilde{Z}_{p,q,i,N}} p^{\eta(\omega)} (1-p)^{\left|F_N^i\right| - \eta(\omega)} q^{\mathbf{b}_{i-1}(P_\omega)} .$$

Lemma 3.2.1. *The balanced i-random-cluster model satisfies*

$$\tilde{\mu}_{p,q,i,N}\left(\omega^{\blacksquare}\right) \overset{d}{=} \tilde{\mu}_{p^*,q,d-i,N}\left(\omega\right) \, .$$

Proof. The idea of the proof is the same as in the classical random-cluster model. We need only take care to keep track of the giant cycles and local cycles separately, since they behave differently under duality.

Let $a_k = \dim\ker\phi_{k*}, a_k^{\blacksquare} = \dim\ker\psi_{k*}, b_k = \operatorname{rank}\phi_{k*}$, and $b_k^{\blacksquare} = \operatorname{rank}\psi_{k*}$. We think of a_i as counting the local k-cycles and b_k as counting the global ones. Furthermore, let $\mathbf{b}_k = \dim H_k\left(P_\omega\right)$ and $\mathbf{b}_k^{\blacksquare} = H_k\left(P_\omega^{\blacksquare}\right)$. Then we immediately have

$$a_k + b_k = \mathbf{b}_k, \quad a_k^{\blacksquare} + b_k^{\blacksquare} = \mathbf{b}_k^{\blacksquare} \tag{3.2}$$

for each $1 \leq k \leq d$, and by Lemma 2.1.4, we have

$$b_i + b_{d-i}^{\blacksquare} = \dim H_i(\mathbb{T}_N^d). \tag{3.3}$$

Theorem 3.44 of [35] gives the isomorphism

$$H^{d-i-1}\left(P_\omega\right) \cong H_{i+1}\left(\mathbb{T}_N^d, \mathbb{T}_N^d \setminus P_\omega\right)$$

Combining this with the long exact sequence of relative homology, we obtain the following commutative diagram:

$$\begin{tikzcd}
H_{i+1}\left(\mathbb{T}^d_N \setminus P^{\blacksquare}_\omega\right) \arrow[r] & H_{i+1}\left(\mathbb{T}^d_N\right) \arrow[r, "\varphi"] \arrow[d, "\cong"'] & H_{i+1}\left(\mathbb{T}^d_N, \mathbb{T}^d_N \setminus P^{\blacksquare}_\omega\right) \arrow[r, "\chi"] \arrow[d, "\cong"'] & H_i\left(\mathbb{T}^d_N \setminus P^{\blacksquare}_\omega\right) \arrow[r, "\epsilon"] & H_i\left(\mathbb{T}^d_N\right) \\
& H^{d-i-1}\left(\mathbb{T}^d_N\right) \arrow[r] & H^{d-i-1}\left(P^{\blacksquare}_\omega\right) & &
\end{tikzcd}$$

By Lemma 2.1.2 and the definition of the plaquette system,

$$H_{i+1}\left(\mathbb{T}^d_N \setminus P^{\blacksquare}_\omega\right) \cong H_{i+1}\left(P_\omega\right) \cong 0\,,$$

so φ is surjective. Also by Lemma 2.1.2,

$$H_i\left(\mathbb{T}^d_N \setminus P^{\blacksquare}_\omega\right) \cong H_i\left(P_\omega\right)\,.$$

Then since ϵ is the map on homology induced by the inclusion $\left(\mathbb{T}^d_N \setminus P^{\blacksquare}_\omega\right) \hookrightarrow \mathbb{T}^d_N$, its image is isomorphic to the space of giant cycles of P_ω. Thus, χ restricts to an isomorphism between vector spaces of dimension $\mathbf{b}^{\blacksquare}_{d-i-1} - b^{\blacksquare}_{d-i-1}$ and $\mathbf{b}_i - b_i$ respectively, so

$$a_i = a^{\blacksquare}_{d-i-1}\,. \tag{3.4}$$

Now by the coupling of the plaquette and dual plaquette systems we have

$$\eta(\omega) + \eta\left(\omega^{\blacksquare}\right) = \left|F_N^i\right|, \tag{3.5}$$

and by the Euler–Poincaré formula we have

$$\sum_{j=0}^{d} (-1)^j \mathbf{b}_j = \sum_{j=0}^{i-1} (-1)^j \left|F_N^j\right| + \eta(\omega) \ . \tag{3.6}$$

In particular, since $\left|F_N^j\right|$ is constant for $1 \leq j \leq i-1$ and $\mathbf{b}_j$ is constant for $1 \leq j \leq i-2$, there is a constant $C_4 = C_4(i, d, N)$ such that

$$\mathbf{b}_i - \mathbf{b}_{i-1} = \eta(\omega) + C_4 \, . \tag{3.7}$$

It is not crucial to the argument, but we can simplify the upcoming calculation slightly to note from the bijection between plaquettes and dual plaquettes that

$$\left|F_N^i\right| = \left|F_N^{d-i}\right| \ . \tag{3.8}$$

Lastly, we recall the following property of p^*,

$$\frac{pp^*}{(1-p)(1-p^*)} = q \, . \tag{3.9}$$

Now by combining (3.2)–(3.9), we compute

$$
\begin{aligned}
\tilde{\mu}_{p,q,i,N} &= \frac{\left(\sqrt{q}\right)^{-b_i}}{Z} p^{\eta(\omega)} \left(1-p\right)^{\left|F_N^i\right| - \eta(\omega)} q^{\mathbf{b}_{i-1}} \\
&= \frac{\left(1-p\right)^{\left|F_N^i\right|}}{Z} \left(\sqrt{q}\right)^{-b_i} \left(\frac{p}{1-p}\right)^{\eta(\omega)} q^{\mathbf{b}_{i-1}} \\
&= \frac{q^c \left(1-p\right)^{\left|F_N^i\right|}}{Z} \left(\sqrt{q}\right)^{-b_i} \left(\frac{p}{q\left(1-p\right)}\right)^{\eta(\omega)} q^{\mathbf{b}_i} \\
&= \frac{q^c \left(1-p\right)^{\left|F_N^i\right|}}{Z} \left(\sqrt{q}\right)^{-b_i} \left(\frac{q(1-p)}{p}\right)^{-\eta(\omega)} q^{a_i+b_i} \\
&= \frac{q^c \left(1-p\right)^{\left|F_N^i\right|}}{Z} \left(\sqrt{q}\right)^{-b_i} \left(\frac{p^*}{1-p^*}\right)^{-\eta(\omega)} q^{a_i+b_i} \\
&= \frac{q^{c+\binom{d}{i}/2} \left(1-p\right)^{\left|F_N^i\right|}}{Z} \left(\sqrt{q}\right)^{b^{\blacksquare}_{d-i}} \left(\frac{p^*}{1-p^*}\right)^{\eta\left(\omega^{\blacksquare}\right) - \left|F_N^i\right|} q^{a^{\blacksquare}_{d-i-1} - b^{\blacksquare}_{d-i}} \\
&= \frac{q^{c-b^{\blacksquare}_{d-i-1}+\binom{d}{i}/2} \left(1-p\right)^{\left|F_N^i\right|}}{Z \left(p^*\right)^{\left|F_N^i\right|}} \left(\sqrt{q}\right)^{-b^{\blacksquare}_{d-i}} \left(p^*\right)^{\eta\left(\omega^{\blacksquare}\right)} \left(1-p^*\right)^{\left|F_N^{d-i}\right| - \eta\left(\omega^{\blacksquare}\right)} q^{\mathbf{b}^{\blacksquare}_{d-i-1}} \\
&\propto \tilde{\mu}_{p^*,q,i,N}\left(\omega^{\blacksquare}\right)
\end{aligned}
$$

as desired. □

3.2.1 The case $d = 2i$

The arguments in this section proceed similarly to their counterparts in Chapter 2, but we include them for completeness.

Proposition 3.2.2. *Let λ be defined as before so that $\mu_{\lambda,q}\left(A\right) = 1/2$ for each N. The for any $\epsilon > 0$, we have*

$$\mu_{\lambda-\epsilon,q}\left(A\right) \to 0$$

and

$$\mu_{\lambda+\epsilon,q}(S) \to 1$$

as $N \to \infty$.

Proof. We first prove the second statement. Note that increasing events with respect to μ are positively correlated by Theorem 1.6.2 and that the group of symmetries of the torus contains an irreducible representation of $H_i\left(\mathbb{T}_N^d\right)$ by Proposition 2.2.3. Then by Lemma 2.2.1,

$$\mu_{p,q}(S) \geq C_0 \mu_{p,q}(A)^{C_1},$$

where C_0, C_1 do not depend on N. In particular there is a $\delta > 0$ such that $\mu_{\lambda,q}(S) \geq \delta$ for all N.

Now we show that $\mu(S)$ has a sharp threshold. By Theorem 1.6.2, μ satisfies the FKG lattice condition and is thus monotonic [32]. Then since the symmetries of $\mathbb{T}_N^d$ act transitively on the plaquettes, we apply Theorem 1.4.4 to obtain

$$\frac{d}{dp}\mu_{p,q}(S) \geq \frac{c_1}{q}\min\left\{\mu_{p,q}(S), 1-\mu_{p,q}(S)\right\}\log\left|F_N^i\right|.$$

In particular, for p close to λ, $\frac{d}{dp}\mu_{p,q}(S) \geq \frac{c_1\delta}{q}\log\left|F_N^i\right|$. By integrating this inequality, we have $\mu_{\lambda+\epsilon,q}(S) \to 1$ as $N \to \infty$.

To obtain the first statement, we apply the same argument to the dual system. Since $\mu_{\lambda,q}(A) = 1/2$, it follows that $\mu^*_{(\lambda),q}(A) \geq 1/2$ by Lemma 2.1.4. Then since p^* is continuous as a function of p, the above argument shows that $\mu_{(\lambda-\epsilon)^*,q}(S) \to 1$ as $N \to \infty$. By another application of Lemma 2.1.4, we then have $\mu_{\lambda-\epsilon,q}(A) \to 0$ as $N \to \infty$.

□

Corollary 3.2.3. *If $p_0 > p_u(q)$, then*

$$\mu_{p_0,q}(S) \to 1$$

as $N \to \infty$. If $p_0 < p_l(q)$, then

$$\mu_{p_0,q}(A) \to 0$$

as $N \to \infty$.

Proof of Theorem 3.1.2. By self duality and Lemma 2.1.4,

$$\mu_{p_{\mathrm{sd}},q}(A) \geq 1/2\,.$$

In particular, $p_u \leq p_{\mathrm{sd}}$. Then by monotonicity and Corollary 3.2.3,

$$\mu_{p,q}(S) \to 1$$

as $N \to \infty$ for all $p > p_{\mathrm{sd}}$. Since p^* is decreasing as a function of p with fixed point p_{sd}, applying Lemma 2.1.4 again gives

$$\mu_{p,q}(A) \to 1$$

as $N \to \infty$ for all $p < p_{\mathrm{sd}}$. □

Proposition 3.2.4. *For any $d \geq 2$, $1 \leq i \leq d-1$,*

$$p_u(q,i,d) = (p_l(q,d-i,d))^* \; .$$

Proof. This follows from Lemma 3.2.1, Lemma 2.1.4, and Corollary 3.2.3 in a similar manner to Proposition 2.4.2. □

3.2.2 The Cases $i = 1$ and $i = d-1$

Since we have exponential decay of subcritical clusters by Theorem 1.4.3, one inequality is a straightforward adaptation of Proposition 2.5.1.

Proposition 3.2.5. $p_l(1,d) \geq \hat{p}_c(d)$

Proof. Let $p < \hat{p}_c$ and let $M = \lfloor N/2 \rfloor$. For a vertex x of $\mathbb{T}_N^d$, let A_x the event that a giant 1-cycle passes through x. Since $[-M, M]$ is contractible in $\mathbb{T}_N^d$, $A_0 \subset \left\{0 \leftrightarrow_{\mathbb{T}_N^d} \partial\Lambda_M\right\}$. Moreover, $\mu^w_{\Lambda_M,p,q}$ stochastically dominates $\mu_{p,q,1,d}|_{\Lambda_M}$, so by Theorem 1.4.3 and translation invariance,

$$\mu_{p,q,1,d}(A_x) \leq \mu^w_{\Lambda_M,p,q}(0 \leftrightarrow \Lambda_M) \leq \exp(c_p M) \tag{3.10}$$

for all vertices x of $\mathbb{T}_N^d$.

Let X be the number of vertices in $\mathbb{T}_N^d$ that are contained in a giant 1-cycle. Since $A = \{X \geq 1\}$,

$$
\begin{aligned}
\mu_{p,q,1,d}(A) = & \ \mu_{p,q,1,d}(X \geq 1) & \\
\leq & \ \mathbb{E}_{\mu}(X) & \text{by Markov's Inequality} \\
= & \sum_{x \in \mathbb{T}_N^d} \mu_{p,q,1,d}(A_x) & \\
\leq & \ N^d e^{-c_p M} & \text{by Equation 3.10} \\
= & \ N^d e^{-c_p \lfloor N/2 \rfloor} \to 0 &
\end{aligned}
$$

as $N \to \infty$.

□

We will now assume Conjecture 3.1.3 in order to prove the reverse inequality. Let

$$\Lambda_{n,k} \coloneqq [-n,n]^2 \times [-k,k]^{d-2} \cap \mathbb{Z}^d \subset S_k \,.$$

Let $D_{n,k} \coloneqq \{v \in \Lambda_{n,k} : v \sim u \text{ for some } u \in S_k \setminus \Lambda_{n,k}\}$ be the boundary of $\Lambda_{n,k}$ in S_k.

Lemma 3.2.6. *Fix $q \geq 1, d \geq 2, k \geq 1$. There is a $C_5 > 0$ such that for any $p > p_c(S_k)$ sufficiently close to $p_c(S_k)$ and n sufficiently large,*

$$\mu^f_{\Lambda_{n,k},p,q}(0 \leftrightarrow D_{n,k}) \geq C_5 \left(p - p_c(S_k)\right) \,.$$

The proof follows from the proof of Theorem 1.4.3 in [21], replacing Λ_n with $\Lambda_{n,k}$.

Lemma 3.2.7. *Fix $q \geq 1, d \geq 2, k \geq 1,\ p > p_c(S_k)$. Let $\Lambda = [-3n, 3n] \times [-2n, 2n] \times [-k, k]^{d-2} \cap \mathbb{Z}^d$. There is a constant $C_6 > 0$ not depending on n such that*

$$\mu^f_{\Lambda,p,q}((-n, 0, \dots, 0) \leftrightarrow (n, 0, \dots, 0)) \geq C_6 \,.$$

Proof. Consider the random-cluster model defined by $\mu^f_{\Lambda,p,q}$. Define two crossing events

$$H_+ \coloneqq \{(\mathbf{0}) \leftrightarrow \{(n, j, x_3, \dots, x_d) : 0 \leq j \leq n, -k \leq x_3, \dots, x_d \leq k\}\} \,,$$

$$H_- \coloneqq \{(\mathbf{0}) \leftrightarrow \{(-n, j, x_3, \dots, x_d) : 0 \leq j \leq n, -k \leq x_3, \dots, x_d \leq k\}\} \,,$$

$$V_+ \coloneqq \{(\mathbf{0}) \leftrightarrow \{(j, n, x_3, \dots, x_d) : 0 \leq j \leq n, -k \leq x_3, \dots, x_d \leq k\}\} \,,$$

and

$$V_- \coloneqq \{(\mathbf{0}) \leftrightarrow \{(-j, n, x_3, \dots, x_d) : 0 \leq j \leq n, -k \leq x_3, \dots, x_d \leq k\}\} \,.$$

By Lemma 3.2.6 and symmetry, there is a $C_7 > 0$ not depending on n such that $\mu^f_{\Lambda,p,q}(H_+) = \mu^f_{\Lambda,p,q}(H_-) = \mu^f_{\Lambda,p,q}(V_+) = \mu^f_{\Lambda,p,q}(V_-) \geq C_7$.

Let $v_0 = (-n, 0, \dots, 0)$ and $w_o = (n, 0, \dots, 0)$. Let $\pi_{12} : \mathbb{Z}^d \to \mathbb{Z}^2$ be the projection onto the first two coordinates. Our aim will be to create overlapping paths (i.e. paths with intersecting images under π_{12}) coming from v_0 and w_0, which are then close enough to be connected with positive probability. We will do this by the following recursive process: Fix an arbitrary ordering of the finite paths in $\mathbb{Z}^d$. Given v_t, let v'_{t+1} be the endpoint on $\partial\Lambda_n(v_t)$ of the minimal path witnessing H_+ if t is odd and V_+ if t is even. Then let v_{t+1} be the projection of v'_{t+1} onto the hyperplane $\{x_3 = x_4 = \ldots = x_d = 0\}$. If no such path exists, we set $v_j = v_t$ for all $j > t$. We obtain w_{t+1} from w_t similarly, except that we replace H_+ and V_+ and H_- and V_- respectively. Let F_+ be the event that $v_8 \neq v_7$ and F_- the event that $w_8 \neq w_7$. Note that by the FKG inequality and symmetry, $\mu^f_{\Lambda,p,q}(F_+) = \mu^f_{\Lambda,p,q}(F_-) \geq C_7^8$.

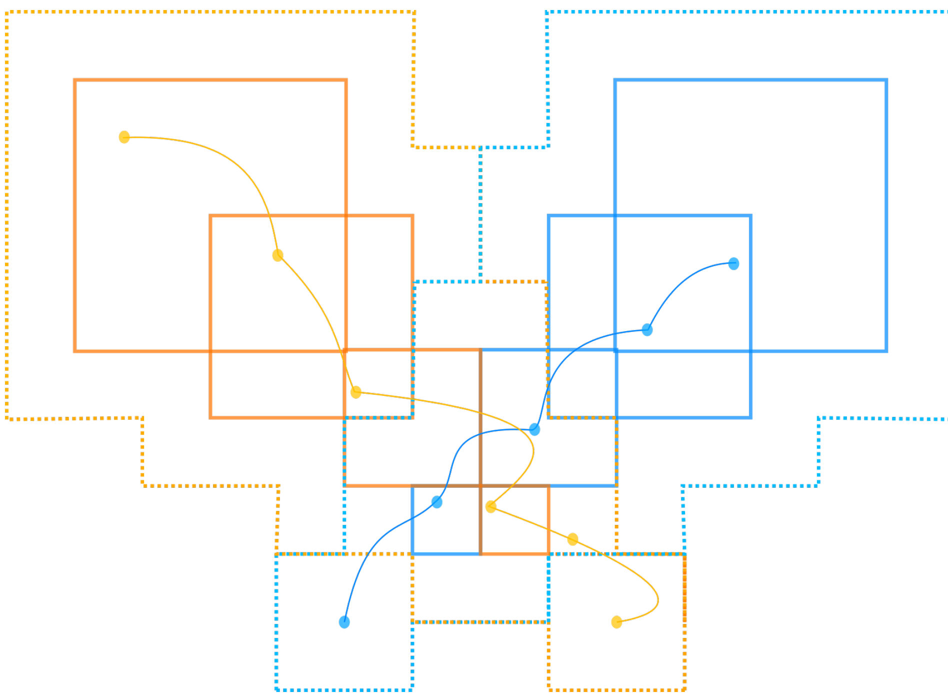

Figure 3.1: The two paths constructed in Lemma 3.2.7. The blue vertices are the v_{2k}'s, the blue boxes are the B_{2k}'s, and the region enclosed by the blue dotted loop is B. The corresponding orange objects are the w_{2k}'s, D_{2k}'s, and D respectively.

We now consider the possible paths created when F_+ holds. It is not difficult to check that $\pi_{12}(v_{2t})$ must lie in a square of side length $nt/2$ with lower left corner at $\pi_{12}(v_0) + t(1/2, 1/2)$. Explicitly,

$$\pi_{12}(v_2) \in B_2 \coloneqq [-n/2, 0] \times [n/2, n] \ ,$$

$$\pi_{12}(v_4) \in B_4 \coloneqq [0, n] \times [n, 2n] \ ,$$

$$\pi_{12}(v_6) \in B_6 \coloneqq [n/2, 2n] \times [3n/2, 3n] \ ,$$

and

$$\pi_{12}(v_8) \in B_8 \coloneqq [n, 3n] \times [2n, 4n] \ .$$

Likewise,

$$\pi_{12}(w_2) \in D_2 \coloneqq [0, n/2] \times [n/2, n] \ ,$$

$$\pi_{12}(w_4) \in D_4 \coloneqq [-n, 0] \times [n, 2n] \ ,$$

$$\pi_{12}(w_6) \in D_6 \coloneqq [-2n, -n/2] \times [3n/2, 3n] \ ,$$

and

$$\pi_{12}(w_8) \in D_8 \coloneqq [-3n, -n] \times [2n, 4n] \ .$$

We claim that on the event F_+ the image of the open bonds under π_{12} contain intersecting paths between $\pi_{12}(v_0)$ and $\pi_{12}(w_0)$. Let $B_0 = \{\pi_{12}(v_0)\}$ and $D_0 = \{\pi_{12}(v_0)\}$. By construction, we have a path P_+ between $\pi_{12}(v_0)$ and $\pi_{12}(v_6)$ that remains within L_1 distance $n/2$ of $\bigcup_{t=0}^{3} B_{2t}$ and a path P_- between $\pi_{12}(w_0)$ and $\pi_{12}(w_6)$ that remains within L_1 distance $n/2$ of $\bigcup_{t=0}^{3} D_{2t}$.

Let

$$B := \left\{ y \in \mathbb{R}^2 : d\left(y, \bigcup_{t=0}^{4} B_{2t} \right) \leq n/2 \right\}$$

and

$$D := \left\{ y \in \mathbb{R}^2 : d\left(y, \bigcup_{t=0}^{4} D_{2t} \right) \leq n/2 \right\},$$

where $d(y, z)$ is the L^1 distance. Then since

$$d(B_0, D) = d(B_8, D) = d(D_0, B) = d(D_8, B) = n/2\,,$$

the endpoints of P_+ and P_- are outside of $B \cap D$. Thus, the boundary of $B \cap D$ is partitioned into four pieces, with crossings between opposite pieces. These crossings must then intersect by a standard argument that can be found in chapter 3 of [12], for example.

We now turn these overlapping sets of open edges into a path in S_k between v_0 and w_0. Any two points in S_k with the same image under π_{12} are at graph distance at most $2k(d-2)$. In our earlier construction, we had paths between v_t and a vertex with the same image under π_{12} as v_{t+1} for each t and likewise for each w_t. Therefore, we can add at most $17(2k(d-2))$ open edges in order to connect v_0 to v_8, connect w_0 to w_8, and then connect these two paths at their point of overlap.

Let $p > r > p_c(S_k)$. Given a configuration ω, define

$$S^m(\omega) = \left\{ \omega' : \sum_{e \in E(G)} |\omega(e) - \omega'(e)| \leq m \right\}.$$

For an event F, we then define

$$E^m(F) = \{\omega : S^m(\omega) \cap F \neq \emptyset\} \ .$$

Then by Theorem 3.45 of [32] and the FKG inequality there is a constant C_8 such that

$$\begin{aligned}
&\mu^f_{\Lambda,p,q}\left((-n,0,\ldots,0) \leftrightarrow (n,0,\ldots,0)\right) \\
&\quad \geq C_8^{34k(d-2)} \mu^f_{\Lambda,p,q}\left(E^{(34k(d-2))}\left((-n,0,\ldots,0) \leftrightarrow (n,0,\ldots,0)\right)\right) \\
&\quad \geq C_8^{34k(d-2)} \mu^f_{\Lambda,p,q}\left(F_+ \cap F_-\right) \\
&\quad \geq C_7^{16} C_8^{34k(d-2)} \ .
\end{aligned}$$

Since this bound does not depend on n we are done. $\square$

Proposition 3.2.8. *Suppose Conjecture 3.1.3 holds. Then* $p_u(1,d) \leq \hat{p}_c(d)$

Proof. Let $p > \hat{p}_c(d)$. Since we are assuming Conjecture 3.1.3, there is a k such that $p > p_c(S_k)$. Let $N \geq 2k$. We will construct a giant cycle by using Lemma 3.2.7 to connect the centers of three pairwise overlapping boxes in the torus, each of diameter $\lfloor 2N/3 \rfloor$. If N is not divisible by 3, the starting and ending points of this constructed path may not exactly match. However, they will be at graph distance at most 3, and are therefore connected with probability at least $\left(\frac{p}{q}\right)^3$. We will therefore assume that N is divisible by 3 in the remainder of the proof for simplicity. We apply Lemma 3.2.7 to copies of

$$\Lambda := [-N,N] \times [-2N/3, 2N/3] \times [-k,k]^{d-2} \cap \mathbb{Z}^d$$

centered at $u_1 = (N/3, \ldots, 0)$, $u_2 = (N, \ldots, 0)$, and at $u_3\,(-N/3, 0, \ldots, 0)$ to connect $v_- = (-2N/3, 0, \ldots, 0)$, $\mathbf{0}$, and $v_+ = (2N/3, 0, \ldots, 0)$. If the events $\left\{\mathbf{0} \leftrightarrow_{\Lambda(u_1)} v_+\right\}, \left\{v_+ \leftrightarrow_{\Lambda(u_2)} v_-\right\}$, and $\left\{v_- \leftrightarrow_{\Lambda(u_3)} \mathbf{0}\right\}$ all occur, then there is an open path that is homotopic to the standard generator of $H_1\left(\mathbb{T}_N^d\right)$ contained in $\{x_2 = x_3 = \ldots = x_d = 0\}$. Thus,

$$A \supseteq \left\{\mathbf{0} \leftrightarrow_{\Lambda(u_1)} v_+\right\} \cap \left\{v_+ \leftrightarrow_{\Lambda(u_2)} v_-\right\} \cap \left\{v_- \leftrightarrow_{\Lambda(u_3)} \mathbf{0}\right\}.$$

We then apply the FKG inequality to bound

$$\begin{aligned}
\mu_{\mathbb{T}_N^d,p,q}(A) &\geq \mu_{\mathbb{T}_N^d,p,q}\left(\left\{\mathbf{0} \leftrightarrow_{\Lambda(u_1)} v_+\right\} \cap \left\{v_+ \leftrightarrow_{\Lambda(u_2)} v_-\right\} \cap \left\{v_- \leftrightarrow_{\Lambda(u_3)} \mathbf{0}\right\}\right) \\
&\geq \mu^f_{\Lambda(u_1),p,q}(\mathbf{0} \leftrightarrow v_+)\, \mu^f_{\Lambda(u_2),p,q}(v_+ \leftrightarrow v_-)\, \mu^f_{\Lambda(u_3),p,q}(v_- \leftrightarrow \mathbf{0}) \\
&\geq C_6^3.
\end{aligned}$$

Note that the final bound is uniform in N. By Proposition 3.2.2, we then have $p \geq \lambda(N)$ for all sufficiently large N, so $p \geq p_u(1, d)$ as desired. □

3.2.3 Monotonicity

We now show that the i-random cluster model also has monotone critical probabilities when we vary d or when we vary i and d in parallel. First we compare i-random-cluster percolation in a full complex to percolation in a subcomplex.

Lemma 3.2.9. *Fix $q \geq 1$. Let X, Y be finite i-dimensional cell complexes with $X \subset Y$ and let μ_X and μ_Y be i-random-cluster measures with parameter q on X and Y respectively. Then $\mu_Y|_X$ stochastically dominates μ_X.*

Proof. Let $\sigma \subset X$ be an i-cell. Given a configuration ω on Y, let Y_ω be the induced subcomplex of Y and let X_ω be subcomplex on X induced by $\omega|_X$. Let $\iota : X \hookrightarrow Y$ be the inclusion and let $\iota_* : H_i(X) \to H_i(Y)$ be the induced map on ith homology. Notice that ι_* is injective since Y has no $(i+1)$-cells. Then we have

$$\mathbf{b}_i\left(X_{\omega^\sigma}\right) - \mathbf{b}_i\left(X_{\omega_\sigma}\right) \geq \mathbf{b}_i\left(Y_{\omega^\sigma}\right) - \mathbf{b}_i\left(Y_{\omega_\sigma}\right) .$$

Then by the Euler–Poincaré formula we see that

$$\mathbf{b}_{i-1}\left(X_{\omega_\sigma}\right) - \mathbf{b}_{i-1}\left(X_{\omega^\sigma}\right) \geq \mathbf{b}_{i-1}\left(Y_{\omega_\sigma}\right) - \mathbf{b}_{i-1}\left(Y_{\omega^\sigma}\right) . \tag{3.11}$$

Recall that for any i-cell $\sigma \subset Z$,

$$\mu_Z\left(\omega(\sigma) = 1 | \omega(Z \setminus \sigma)\right) = \begin{cases} p & \mathbf{b}_{i-1}\left(Z_{\omega_\sigma}\right) - \mathbf{b}_{i-1}\left(Z_{\omega^\sigma}\right) = 0 \\ \frac{p/q}{1-p+p/q} & \mathbf{b}_{i-1}\left(Z_{\omega_\sigma}\right) - \mathbf{b}_{i-1}\left(Z_{\omega^\sigma}\right) = 1 \end{cases} \tag{3.12}$$

Now let ξ be a configuration on X and ζ be a configuration on Y with $\zeta|_X \geq \xi$. Then we have

$$\begin{aligned}
&\mu_X\left(\omega(\sigma) = 1 \mid \omega(\tau) = \xi(\tau) \text{ for all } \tau \in X \setminus \sigma\right) \\
&\qquad \leq \mu_X\left(\omega(\sigma) = 1 \mid \omega(\tau) = \zeta(\tau) \text{ for all } \tau \in X \setminus \sigma\right) \quad \text{by FKG} \\
&\qquad \leq \mu_Y\left(\omega(\sigma) = 1 \mid \omega(\tau) = \zeta(\tau) \text{ for all } \tau \in Y \setminus \sigma\right) \quad \text{by (3.11) and (3.12)}
\end{aligned}$$

By Theorem 1.4.2, μ_X stochastically dominates $\mu_Y|_X$ as desired. □

Now we can prove an analogue of Proposition 2.6.1.

Proposition 3.2.10. *For all* $1 \leq i \leq d-1$,

$$p_u(q,i,d) < p_u(q,i,d-1) < p_u(q,i+1,d) \ .$$

Proof. The topological properties of any given configuration of plaquettes are identical to those discussed in Proposition 2.6.1. We will therefore only modify the probabilistic arguments as necessary.

First we will show

$$p_u(q,i,d) < p_u(q,i,d-1) \, .$$

Our strategy will be to define a sequence of models between the random-cluster model on $\mathbb{T}_N^d$ and $\mathbb{T}_N^{d-1}$ in which the giant cycle space of each model stochastically dominates the giant cycle space of the one before. More precisely, for a configuration of plaquettes ω, let $G(\omega)$ be the associated subspace of giant cycles of P_ω in $H_i\left(\mathbb{T}_N^d\right)$. Then we say $\mu_1 \leq_G \mu_2$ if there is a coupling κ of μ_1 and μ_2 such that

$$\kappa\left(\left\{(\omega_1,\omega_2) : G(\omega_1) \subseteq G(\omega_2)\right\}\right) = 1 \, .$$

Note that $\mu_1 \leq_{\mathrm{st}} \mu_2$ implies $\mu_1 \leq_G \mu_2$.

Let $\mathcal{T}_0 = \mathbb{T}_N^d$, $\mathcal{T}_1 = \mathbb{T}_N^d \cap \{x_1 \in [0,1]\}$, and $\mathcal{T}_2 = \mathbb{T}_N^d \cap \{x_1 = 0\}$. For $j = 0,1,2$, let $\mu_{\mathcal{T}_j}$ be such that $\mu_{\mathcal{T}_j}|_{\mathcal{T}_j}$ is the random-cluster model on $\mathcal{T}_j$ and $\mu_{\mathcal{T}_j}|_{\mathbb{T}_N^d \setminus \mathcal{T}_j}$ sets all plaquettes to be closed almost surely. By Lemma 3.2.9, we have

$$\mu_{\mathcal{T}_0} \geq_{\mathrm{st}} \mu_{\mathcal{T}_1} \, .$$

We now put a different measure $\mu'_{\mathcal{T}_1}$ on configurations that are closed outside $\mathcal{T}_1$. Let F_2 be the set of i-cells of $\mathcal{T}_1$ contained in $\mathcal{T}_2$ and let F_1 be the rest of the i-cells of $\mathcal{T}_1$. We Let η_1 and η_2 be the open cells of F_1 and F_2 respectively. We set $\mu'_{\mathcal{T}_1}|_{\mathcal{T}_2} = \mu_{\mathcal{T}_2}|_{\mathcal{T}_2}$. We then let $\mu'_{\mathcal{T}_1}|_{\mathcal{T}_1 \setminus \mathcal{T}_2}$ be independent Bernoulli plaquette percolation with probability p/q and declare all other plaquettes closed. More explicitly,

$$\mu'_{\mathcal{T}_1}(\omega) := \frac{1}{Z} p^{\eta_1(\omega)} (1-p)^{|F_1| - \eta_1(\omega)} q^{\mathbf{b}_{i-1}\left(P_{\omega|\mathcal{T}_1}\right)} \left(\frac{p}{q}\right)^{\eta_2} \left(1 - \frac{p}{q}\right)^{|F_2| - \eta_2} .$$

We think of this as doing Bernoulli percolation on F_1 with parameter p/q and then a random-cluster percolation on F_2. For $\sigma \in F_1$ and a configuration ξ on $\mathcal{T}_1$, we clearly have

$$\begin{aligned} &\mu'_{\mathcal{T}_1}(\omega(\sigma) = 1 \mid \omega(\tau) = \xi(\tau) \text{ for all } \tau \in \mathcal{T}_1 \setminus \sigma) \\ &\qquad \leq \mu_{\mathcal{T}_1}(\omega(\sigma) = 1 \mid \omega(\tau) = \xi(\tau) \text{ for all } \tau \in \mathcal{T}_1 \setminus \sigma) . \end{aligned}$$

By Lemma 3.2.9, for $\sigma \in F_2$ we also have

$$\begin{aligned} &\mu'_{\mathcal{T}_1}(\omega(\sigma) = 1 \mid \omega(\tau) = \xi(\tau) \text{ for all } \tau \in \mathcal{T}_1 \setminus \sigma) \\ &\qquad \leq \mu_{\mathcal{T}_1}(\omega(\sigma) = 1 \mid \omega(\tau) = \xi(\tau) \text{ for all } \tau \in \mathcal{T}_1 \setminus \sigma) . \end{aligned}$$

The again applying Theorem 1.4.2, we have

$$\mu_{\mathcal{T}_1} \geq_{\mathrm{st}} \mu'_{\mathcal{T}_1} .$$

We now perform a splitting of the state of a plaquette into several Bernoulli variables similar to one found in Proposition 2.6.1. We adapt some of the definitions used there. Let S be the set of i-faces of $\mathcal{T}_1$ that intersect, but are not contained in $\mathcal{T}_2$. For an i-face v of $\mathcal{T}_2$, let $J(v)$ be the set of all perpendicular i-faces that meet v at an $i-1$ face. Then v, $v+\boldsymbol{e}_1$, and $J(v)$ are the i-faces of an $(i+1)$-face $w(v)$. Also, for a perpendicular i-face u of S, let $K(u) = \{v : u \in J(v)\}$. Let p_S satisfy

$$\frac{p}{q} = 1 - (1-p_S)^{2(d-i)} \ .$$

For all pairs (v,u) where $v \in \mathcal{T}_2$ and $u \in J(v)$, let $\kappa(v,u)$ be independent $\operatorname{Ber}(p_S)$ random variables. Then by construction, Bernoulli percolation with parameter p/q on S is equivalent to setting each cell $v \in S$ to be open if and only if $\sum_{u \in J(v)} \kappa(v,u) > 0$. Now given Bernoulli p/q percolation on F_1, let

$$H = \{v \in \mathcal{T}_2 : \kappa(v,u) = 1 \text{ for each } u \in J(v) \text{ and } v + \boldsymbol{e}_1 \text{ is open}\} \ .$$

Let $p' = p + \frac{p}{q}(1-p)p_S^{2i}$ and take $\mu_{\mathcal{T}_2,p'}$ to be the random-cluster model on $\mathcal{T}_2$ with parameter p' instead of p. By construction, for any plaquette $\sigma \in H$, each giant cycle in P_ω is homologous to a giant cycle in P_{ω_σ}. Therefore we have

$$\mu'_{\mathcal{T}_1} \geq_G \mu_{\mathcal{T}_2,p'} \ ,$$

and so

$$\mu_{\mathcal{T}_0} \geq_G \mu_{\mathcal{T}_2,p'} \ ,$$

Now as in Chapter 2 we may take p such that $p < p_u(q,i,d-1) < p'$. Then for each N we have

$$\mu_{p,q,i,d,N}(A) \geq \mu_{p',q,i,d-1,N}(A) \ ,$$

so

$$\liminf \mu_{p,q,i,d,N}(A) \geq \liminf \mu_{p',q,i,d-1,N}(A) \geq 1/2\,,$$

and thus

$$p_u(q,i,d) \leq p < p_u(q,i,d-1)\ .$$

Combining this with Proposition 3.2.4 also gives $p_u(q,i,d-1) < p_u(q,i+1,d)$.

□

3.3 Comparisons to Potts Lattice Gauge Theory

We now consider the relationship between the i-random-cluster model and the q-state Potts lattice gauge theory. Recall that the latter has the Hamiltonian

$$\hat{H}(f) = \sum_{\sigma \in F_N^i} K\left(W_\sigma(f), 1\right)\ ,$$

or equivalently,

$$\hat{H}(f) = \sum_{\sigma \in F_N^i} K\left(f(\partial\sigma), 0\right)\ ,$$

where $f \in C^{i-1}\left(\mathbb{T}_N^d; \mathbb{Z}(q)\right)$ or $f \in C^{i-1}\left(\mathbb{T}_N^d; \mathbb{Z}_q\right)$ respectively. The probability of a state f in the Potts lattice gauge theory with parameter β is then

$$\frac{1}{Z} e^{-\beta \hat{H}(f)}\ ,$$

where Z is a normalizing constant.

Hiraoka and Shirai [36] showed that there is a coupling similar to the coupling between the classical random cluster model and the Potts model [27, 24].

Proposition 3.3.1 (Hiraoka and Shirai)**.** *Let $q \geq 1$, $p \in [0,1)$, and suppose $p = 1 - e^{-\beta}$. Consider the coupling on $C^{i-1}\left(\mathbb{T}_N^d\right) \times \{0,1\}^{F_N^i}$ defined by*

$$\nu(f,\omega) \propto \prod_{\sigma \in F_N^i} \left[(1-p)\,K(\omega(\sigma),0) + pK(\omega(\sigma),1)\,K\left(\delta^{i-1} f(\sigma),0\right)\right] .$$

Then ν has the following marginals:

- *The first marginal is the q-state Potts model with inverse temperature β given by*

  $$\sum_{\omega \in \{0,1\}^{F_N^i}} \nu(f,\omega) \propto e^{-\beta H(f)} ,$$

  where $H(f) = -\sum_{\sigma \in F_N^i} K\left(\delta^{i-1} f(\sigma),0\right).$

- *The second marginal is the i-random cluster model with parameters p, q given by*

  $$\sum_{f \in C^{i-1}\left(\mathbb{T}_N^d\right)} \nu(f,\omega) \propto p^{\eta(\omega)}\,(1-p)^{\left|F_N^i\right| - \eta(\omega)}\, q^{\mathbf{b}_{i-1}(P_\omega;\mathbb{Z}_q)} .$$

The first marginal of ν is the $(i-1)$-dimensional q-state Potts model with $p = 1 - e^{-\beta}$ and the second marginal is the i-random cluster model.

It will be useful to have the conditional measures from this coupling available, which are also analogous to the conditional measures of the Edwards-Sokal coupling.

Proposition 3.3.2. *Let $p = 1 - e^{-\beta}$. Then the conditional measures of ν are as follows:*

- *Given* $f \in C^{i-1}\left(\mathbb{T}_N^d\right)$, *the conditional measure* $\nu\left(\cdot \mid f\right)$ *is Bernoulli plaquette percolation with probability* p *on the set of plaquettes* σ *that satisfy* $\delta^{i-1}\left(\sigma\right) = 0$.

- *Given* $\omega \in \{0,1\}^{F_N^i}$, *the conditional measure* $\nu\left(\cdot \mid \omega\right)$ *is the uniform measure on* $(i-1)$*-cocycles in* P_ω.

Proof. From the definition of $\nu\left(f,\omega\right)$, under $\nu\left(\cdot \mid s\right)$ a plaquette σ is open with probability p independently of other plaquettes when $\delta^{i-1}\left(\sigma\right) = 0$ and always closed otherwise, giving the first conditional measure. The second conditional measure follows from the observation that $\nu\left(\cdot \mid \omega\right)$ is supported on the $s \in C^{i-1}\left(\mathbb{T}_N^d\right)$ such that $\delta^{i-1}s\left(\sigma\right) = 0$ for each $\sigma \in \omega$, and that each such cochain has the same weight. $\square$

3.3.1 Higher Dimensional Random-Cluster and Potts Models in Infinite Volume

In this section we will first show that, like the classical random cluster model, the i-random-cluster model can be extended to an infinite volume setting. Specifically, we will define the i-random-cluster model on $\mathbb{Z}^d$ via limits of i-random cluster models on finite boxes Λ_n. The proof given in chapter 4 of [33] only requires minor changes to extend to higher dimensions, and we will describe them here. We will then use this to define an infinite volume Potts lattice gauge theory.

We start by generalizing the notion of boundary conditions. Let Ω be the space of configurations of i-plaquettes in $\mathbb{Z}^d$. For a subset of vertices $V \subset \mathbb{Z}^d$, let F_V^k be the set of k-plaquettes with all vertices contained in V. Then given $\xi \in \Omega$, let

$$\Omega_{\Lambda_n}^{\xi} = \left\{\omega \in \Omega : \omega\left(\sigma\right) = \xi\left(\sigma\right) \text{ for all } \sigma \in F_{\mathbb{Z}^d}^k \setminus F_{\Lambda_{n-1}}^k\right\} .$$

Intuitively, the boundary condition should describe how the states of external plaquettes affect the random cluster measure within Λ_n. In the classical model, this is done by keeping track of which vertices of $\partial\Lambda_n$ are connected externally. In higher dimensions we will need slightly more information, but the idea is the same. Let $P_{\xi,V}$ be the complex consisting of the union of the $(i-1)$-skeleton of $\mathbb{Z}^d$ and the open plaquettes of ξ contained in F_V^i. Let D_n^{i-1} be the $(i-1)$-skeleton of $\partial\Lambda_n$. Then we construct a cubical complex $Q_{\omega,\xi}$ (not necessarily a subcomplex of $\mathbb{Z}^d$) by taking P_{ω,Λ_n} and attaching a cubical complex A_ξ such that

- $A_\xi \cap P_{\omega,\Lambda_n} \subset D_n^{i-1}$.
- The map $\varphi_A : H_{i-1}\left(D_n^{i-1}; F\right) \to H_{i-1}\left(A_\xi; F\right)$ induced by the inclusion $D_n^{i-1} \hookrightarrow A_\xi$ is surjective.
- The kernel of φ_A is the same as the kernel of the map $H_{i-1}\left(D_n^{i-1}; F\right) \to \varphi_P : H_{i-1}\left(P_{\xi,\mathbb{Z}^d\setminus\Lambda_{n-1}}; F\right)$ induced by the inclusion $D_n^{i-1} \hookrightarrow P_{\xi,\mathbb{Z}^d\setminus\Lambda_{n-1}}$.

Such an A_ξ can be constructed by taking P_ξ and filling the $(i-1)$-cycles that are not homologous to cycles in D_n^{i-1}.

Now we can define the i-random-cluster model on Λ_n with boundary condition ξ as follows:

$$\mu^{\xi}_{\Lambda_n,i,p,q}(\omega) = \begin{cases} \frac{1}{Z^{\xi}_{\Lambda_n}} \left[\prod_{\sigma\in F^i_{\Lambda_n}} p^{\omega(\sigma)}(1-p)^{1-\omega(\sigma)}\right] q^{\mathbf{b}_{i-1}\left(Q_{\omega,\xi};F\right)} & \omega\in\Omega^{\xi}_{\Lambda_n} \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 3.3.3. *Let $p\in[0,1]$, $q\geq 1$, and $n\in\mathbb{N}$. Then for every $\xi\in\Omega$, $\mu^{\xi}_{\Lambda_n,p,q}$ is positively associated.*

Proof. The proof is analogous to the proof of Theorem 4.14 of [33]. Consider the i-random-cluster model $\mu_{\Lambda_n\cup A_\xi,i,p,q}$. This satisfies the FKG lattice condition and is thus strongly postively associated. Then since $\mu^{\xi}_{\Lambda_n,i,p,q}$ is $\mu_{\Lambda_n\cup A_\xi,i,p,q}$ conditioned on the plaquettes of A_ξ being open, it follows that $\mu^{\xi}_{\Lambda_n,i,p,q}$ is postively associated. $\square$

As usual, the free and wired boundary conditions are the most important explicit examples. The free boundary measure $\mu^{\mathbf{f}}_{\Lambda_n,i,p,q}$ and the wired boundary measure $\mu^{\mathbf{w}}_{\Lambda_n,i,p,q}$ are obtained by taking ξ to be the all closed and all open configurations respectively. By an analogue of Theorem 4.19 of [33], the weak limits of each of these measures exist.

Proposition 3.3.4. *Let $p\in[0,1]$ and $q\geq 1$.*

(a) The limits $\mu^{\mathbf{f}}_{i,p,q} := \lim_{n\to\infty} \mu^{\mathbf{f}}_{\Lambda_n,i,p,q}$ *and* $\mu^{\mathbf{w}}_{i,p,q} := \lim_{n\to\infty} \mu^{\mathbf{w}}_{\Lambda_n,i,p,q}$ *exist.*

(b) $\mu^{\mathbf{f}}_{i,p,q}$ *and* $\mu^{\mathbf{w}}_{i,p,q}$ *are automorphism invariant.*

(c) $\mu^{\mathbf{f}}_{i,p,q}$ *and* $\mu^{\mathbf{w}}_{i,p,q}$ *are positively associated.*

Proof. (a), (b) Both proofs are the same as in Theorem 4.19 of [33]

(c) Both measures are limits of positively associated measures by Lemma 3.3.3, so they are positively associated by Proposition 4.10 of [33]. □

We can use the infinite volume random cluster measures to define corresponding infinite volume Potts models using the coupling given in Proposition 1.6.3.

Lemma 3.3.5. *For any* $\omega \in \Omega$, *there is a basis* $\mathcal{B}$ *of the finitely supported subspace of* $H_{i-1}(P_\omega, F)$ *such that each* $(i-1)$*-cube of* $\mathbb{Z}^d$ *is in the support of finitely many elements of* $\mathcal{B}$.

Proof. Take any basis and put it as the rows of an infinite matrix. Then we can perform row operations to obtain an upper triangular matrix, the rows of which will be the desired basis. □

We will call the minimal such basis with respect to lexicographical order $\mathcal{B}_\omega$

Corollary 3.3.6. *Let $p \in [0,1), q \in \{2,3,\ldots\}$, and $p = 1 - e^{-\beta}$.*

(a) Let ω be distributed according to $\mu^{\mathbf{f}}_{i,p,q}$. Conditional on ω, let $\{A_g : g \in \mathcal{B}_\omega\}$ be i.i.d. $\mathrm{Unif}(\mathbb{Z}_q)$ *random variables. Then the limit $\nu^{\mathbf{f}}_{\beta,q} := \lim_{n\to\infty} \nu^{\mathbf{f}}_{\Lambda_n,\beta,q}$ exists, and the random cocycle*

$$f = \sum_{g \in \mathcal{B}} A_g g$$

is distributed according to $\nu^{\mathbf{f}}_{\beta,q}$.

(b) Let f be distributed according to $\nu^{\mathbf{f}}_{\beta,q}$. Conditional on f, let ω be a random configuration in which each i-plaquette σ is open with probability p if $f(\partial\sigma) = 0$ independent of the states of other plaquettes and closed otherwise. Then ω is distributed according to $\mu^{\mathbf{f}}_{i,p,q}$.

Proof. We proceed similarly to proof of Theorem 4.91 of [33].

(a) By the proof of Theorem 4.19 of [33], there is an increasing set of configurations ω_n such that each ω_n is distributed according to $\mu^{\mathbf{f}}_{\Lambda_n,i,p,q}$ and $\lim_{n\to\infty} \omega_n$ is distributed according to $\mu^{\mathbf{f}}_{i,p,q}$. Moreover, for any i-plaquette σ, $\omega_n(\sigma) = \omega(\sigma)$ for large enough n.

Now let $\left\{A_g : g \in C^{i-1}\left(\mathbb{Z}^d; \mathbb{Z}_q\right)\right\}$ be i.i.d. $\mathrm{Unif}(\mathbb{Z}_q)$ random variables. Then let

$$f_n := \sum_{g \in \mathcal{B}_{\omega_n}} A_g g\,.$$

By the construction of $\mathcal{B}_{\omega_n}$, f_n is distributed according to $\nu^{\mathbf{f}}_{\Lambda_n,\beta,q}$ and is eventually constant on any finite set of $(i-1)$-cubes. Since $\mathcal{B}_{\omega_n} \to \mathcal{B}_\omega$, $\lim_{n\to\infty} f_n$ is distributed as $\nu^{\mathbf{f}}_{\beta,q}$ and we are done.

(b) Let $\{B_\sigma : \sigma \in F^i\}$ be independent $\mathrm{Ber}\,(p)$ random variables. Let f_n be distributed as $\nu^{\mathbf{f}}_{\Lambda_n,\beta,q}$, coupled as before so that f_n is eventually constant on any finite set of $(i-1)$-cubes. Let $\omega_n(\sigma) = B_\sigma K(f_n(\partial\sigma), 1)$. By Proposition 3.3.2, ω_n is distributed according to $\mu^{\mathbf{f}}_{\Lambda_n,i,p,q}$. But then $\omega := \lim_{n\to\infty}$ is distributed according to $\mu^{\mathbf{f}}_{i,p,q} = \mu^{\mathbf{f}}_{\Lambda_n,i,p,q}$. $\square$

3.3.2 Wilson Loops in Potts Lattice Gauge Theory

A straightforward calculation shows for the purposes of Wilson loops, Potts states are equivalent up to coboundaries.

Proposition 3.3.7. *Fix $\omega \in \{0,1\}^{F_N^i}$. Let $\gamma \in Z_{i-1}(P_\omega; \mathbb{Z}_q)$ and $s \in Z^{i-1}\left(\mathbb{T}_N^d, \mathbb{Z}_q\right)$. Then for any $s' \in Z^{i-1}\left(\mathbb{T}_N^d, \mathbb{Z}_q\right)$ with $[f] = [f'] \in s \in H^{i-1}\left(\mathbb{T}_N^d, \mathbb{Z}_q\right)$, we have $s(\gamma) = s'(\gamma)$.*

Proof. If $i = 1$, $s = s'$ and the result is trivial. Assume $i \geq 2$, let $h \in F_N^{i-2}$, and let $h^* \in C^{i-2}\left(\mathbb{T}_N^d; \mathbb{Z}_q\right)$ be supported on the element of $C_{i-2}\left(\mathbb{T}_N^d; \mathbb{Z}_q\right)$ associated to h. Note that since γ is a cycle, $h^*(\gamma) = 0$. Then since s and s' are cohomologous, we can write

$$s - s' = \sum_{h \in F_N^{i-2}} c_h h^*$$

for some $\{c_h\} \in \mathbb{Z}_q^{F_N^{i-2}}$. This then gives

$$s(\gamma) = s'(\gamma) + \sum_{h \in F_N^{i-2}} c_h h^*(\gamma) = s'(\gamma) .$$

□

We can now compute the distribution of the values of Wilson loops, conditioned on the state of the coupled i-random cluster percolation.

Proposition 3.3.8. *Let* $\gamma \in Z_{i-1}\left(\mathbb{T}_N^d; \mathbb{Z}_q\right)$. *Then*

$$\mathbb{E}_\nu\left(f(\gamma) \mid \omega\right) = \begin{cases} 0 & 0 = [\gamma] \in H_{i-1}\left(P_\omega; \mathbb{Z}_q\right) \\ \mathrm{Unif}\left(\mathbb{Z}_q\right) & 0 \neq [\gamma] \in H_{i-1}\left(P_\omega; \mathbb{Z}_q\right) \end{cases} .$$

Proof. Fix ω and γ. First, assume that $0 = [\gamma] \in H_{i-1}\left(P_\omega; \mathbb{Z}_q\right)$. Then there is a chain $a = \sum_{\sigma \in \omega} c_\sigma \sigma \in H_i\left(P_\omega; \mathbb{Z}_q\right)$ such that $[\gamma] = \partial a$. By Proposition 3.3.2, $f(\partial\sigma) = 0$ for each open plaquette σ in ω. Thus,

$$f(\gamma) = \sum_{\sigma \in \omega} c_\sigma f(\sigma) = 0 .$$

Now assume that $0 \neq [\gamma] \in H_{i-1}\left(P_\omega; \mathbb{Z}_q\right)$. By Proposition 3.3.2, $\nu(f \mid \omega)$ can be sampled by fixing a basis of $H^{i-1}\left(P_\omega\right)$ and taking a random linear combination with independent $\mathrm{Unif}\left(\mathbb{Z}_q\right)$ coefficients. Thus, $f(\gamma)$ is uniformly distributed on an additive subgroup of $\mathbb{Z}_q$. Since the only such subgroups are $\mathbb{Z}_q$ and $\{0\}$, we only need to rule out the latter. By the universal coefficent theorem, there is a dual element $0 \neq [\gamma]* \in H^{i-1}\left(P_\omega; \mathbb{Z}_q\right)$ such that $[\gamma]*([\gamma]) \neq 0$. so $f(\gamma)$ is distributed as $\mathrm{Unif}\left(\mathbb{Z}_q\right)$. □

Recall that when we write W_γ, we are referring to a product involving a cochain with coefficients in $\mathbb{Z}(q)$, the multiplicative group of complex qth roots of unity. $\mathbb{Z}(q)$ is isomorphic to the additive group $\mathbb{Z}_q$ that we have used so far in accordance with topological convention, but we want to emphasize that the expectations in the following corollary are in the former group in accordance with the physical convention.

Corollary 3.3.9.

$$\mathbb{E}_\nu\left(W_\gamma \mid 0 = [\gamma] \in H_{i-1}\left(P_\omega; \mathbb{Z}_q\right)\right) = 1$$

and

$$\mathbb{E}_\nu\left(W_\gamma \mid 0 \neq [\gamma] \in H_{i-1}\left(P_\omega; \mathbb{Z}_q\right)\right) = 0\,.$$

In particular, if V_γ is the event that γ is null-homologous in P_ω then

$$\mathbb{E}\left(W_\gamma\right) = \mathbb{P}\left(V_\gamma\right),.$$

The previous result is false when q is not prime, as noted in [4]. Also, note that this provides an topological explanation for the phenomenon that Wilson loop expectations are always in $[0,1]$, a phenomenon was was observed in [17].

Corollary 3.3.10. *Let $\gamma, \gamma' \in Z_i\left(\mathbb{T}_N^d; \mathbb{Z}_q\right)$. Then conditioned on*

$$[\gamma] = [\gamma'] \in H_i\left(P_\omega; \mathbb{Z}_q\right)\,,$$

$W_\gamma = W_{\gamma'}$ almost surely. Conversely, conditioned on

$$[\gamma] \neq [\gamma'] \in H_i\left(P_\omega; \mathbb{Z}_q\right)\,,$$

W_γ and $W_{\gamma'}$ are independent.

Combining this with Theorem 3.1.5 gives a relationship between certain Polyakov loops, which are Wilson loops that represent giant $(i-1)$-cycles. First we show that surfaces with giant i-cycles contain such loops.

Lemma 3.3.11. *Let $X \subset \mathbb{T}_N^d$ be a subcomplex. Then for any $l < k \leq d$, if X contains a giant k-cycle, X also contains a giant l-cycle.*

Proof. Fix $l < k \leq d$. Let $\psi : X \hookrightarrow \mathbb{T}_N^d$ be the inclusion map, and for each $0 \leq j \leq d$ let $\psi_{j*} : H_j(X) \hookrightarrow H_j\left(\mathbb{T}_N^d\right)$ be the induced map on homology and let $\psi_j^* : H^j(X) \hookleftarrow H^j\left(\mathbb{T}_N^d\right)$ be the induced map on cohomology. Now for any $\alpha \in H^{k-l}\left(\mathbb{T}_N^d\right)$, the cap product gives us the following commutative diagram:

$$\begin{tikzcd} H_k(X) \arrow[r, "(\cdot)\frown\psi_{k-l}^*(\alpha)"] \arrow[d, "\psi_{k*}"'] & H_l(X) \arrow[d, "\psi_{l*}"'] \\ H^k\left(\mathbb{T}_N^d\right) \arrow[r, "(\cdot)\frown\alpha"] & H^l\left(\mathbb{T}_N^d\right) \end{tikzcd}$$

By assumption, there is $\beta \in H_k(X)$ such that $\psi_{k*}(\beta) \neq 0$. Then by the Künneth formula for homology,

$$H_*\left(\mathbb{T}_N^d; F\right) \simeq \bigotimes_{1\leq j\leq d} H_*\left(S^1\right).$$

In particular, there is an $\alpha \in H^{k-l}\left(\mathbb{T}_N^d\right)$ such that $\psi_{k*}(\beta) \cap \alpha \neq 0$. It then follows from the commutative diagram above that $\psi_{l*}\left(\beta \cap \psi_{k-l}^*(\alpha)\right) \neq 0$, so $\beta \cap \psi_{k-l}^*(\alpha)$ is a giant l-cycle of X.

$\square$

Corollary 3.3.12. *There is a sharp threshold function* $\beta_{\text{surf}} = \beta_{\text{surf}}(q, d, N) \coloneqq -\log(1 - \lambda(q, d, N))$ *for the appearance of a giant surface on which the value of Polyakov loops is constant within homology classes.*

3.3.3 Area and Perimeter Law

Our previous results show that the 2-dimensional random cluster model in $\mathbb{Z}^4$ random cluster model exhibits a sharp phase transition to a "surface–dominated regime" in a global, qualitative sense. For percolation in three dimensions, this phase transition coincides with that of Theorem 1.6.1: a sharp phase transition to "surface–dominated regime" in a quantitative sense. Such a result is unknown for two-dimensional percolation in four dimension, let alone the random cluster model. However, we conjecture that it occurs at the same point as the corresponding transition to a surface-dominated regime in the global sense.

Conjecture 3.3.13. *Let* γ *be a rectangular loop in* $\mathbb{Z}^4$*, and let* V_γ *be the event that* γ *is null-homologous. Then, for the* 2*-dimensional random cluster model in* $\mathbb{Z}^4$*,*

$$\mathbb{P}_p(V_\gamma) \sim \begin{cases} \exp(-C_9(p)\mathrm{Area}(\gamma)) & p < p_{\mathrm{sd}}(q) \\ \exp(-C_{10}(p)\mathrm{Per}(\gamma)) & p > p_{\mathrm{sd}}(q) \end{cases}$$

for some $0 < C_9(p), C_{10}(p) < \infty$.

Conditional on this result, we have the following phase transition for expectations of Wilson loop variables.

Corollary 3.3.14. *Suppose $q \geq 1$ is a prime integer, let γ be a rectangular loop in $\mathbb{Z}^4$, and let W_γ be the Wilson loop variable for γ. Also, let ϕ be sampled from the q-state Potts lattice gauge theory and let $\beta_{sd}(q) = \log\left(1+\sqrt{q}\right)$. Then, assuming the previous conjecture,*

$$\mathbb{E}_{p,q}(W_\gamma) \sim \begin{cases} \exp(-C_{11}(p,q)\mathrm{Area}(\gamma)) & \beta > \beta_{sd}(q) \\ \exp(-C_{12}(p,q)\mathrm{Per}(\gamma)) & \beta < \beta_{sd}(q) \end{cases}$$

for some $0 < C_{11}(p,q), C_{12}(p,q) < \infty$.

We can, however, show a partial result by comparison to plaquette percolation. First, we show a stochastic domination result for the plaquette random cluster model which is a direct generalization of the corresponding classical result [27].

Lemma 3.3.15. *The plaquette random cluster model $\mu_{p,q,i,N}$ is stochastically decreasing in $q \geq 1$ for fixed p. On the other hand, if we fix $\hat{p} = \frac{p/q}{1-p+p/q}$ then $\mu_{p,q,i,N}$ is stochastically increasing in $q \geq 1$.*

Proof. This is a consequence of the fact that adding an i-plaquette can only reduce $\mathbf{b}_{i-1}$ by one or leave $\mathbf{b}_{i-1}$ unchanged.

Recall that by Equation 3.12, $\hat{p}$ is the conditional probability that a plaquette is open given that it reduces $\mathbf{b}_{i-1}$ by one given the states of the other plaquettes. First fix p. Since $\hat{p}$ is decreasing in q and the conditional probability that a plaquette is open given that it does not kill an $(i-1)$-cycle is constant in q (namely, it equals p), an application of Theorem 1.4.2 shows that $\mu_{p,q,i,N}$ is stochastically decreasing in q. Now fix $\hat{p}$. Then p is decreasing as a function of q, so again applying Theorem 1.4.2 to the conditional probabilities given by Equation 3.12 gives that $\mu_{p,q,i,N}$ is stochastically increasing in q. $\square$

Next, we need two more results from [1].

Theorem 3.3.16 (Aizenman–Chayes–Chayes–Frölich, Russo [3]). *Let γ be a rectangular loop in $\mathbb{Z}^d$, and let V_γ be the event that γ is null-homologous. There exist constants $\hat{c}_1(p,d), \hat{c}_2(p,d) > 0$ for which*

$$\exp(-\hat{c}_1(p,d)\mathrm{Area}(\gamma) \leq \mathbb{P}(V_\gamma) \leq \exp(-\hat{c}_2(p,d\mathrm{Per}(\gamma))\,. \tag{3.13}$$

Furthermore, There are constants $0 < \tilde{p}_1\,(d) \leq \tilde{p}_2\,(d) < 1$ so that

$$\log\left(\mathbb{P}_p(V_\gamma)\right)\begin{cases}\sim -\hat{c}_1(p,d)\mathrm{Area}(\gamma) & p < \tilde{p}_1\,(d) \\ = \Theta\left(\mathrm{Per}(\gamma)\right)) & p > \tilde{p}_2\,(d)\end{cases}.$$

The proof of the following is identical for the plaquette random cluster model and plaquette percolation, requiring only positive association.

Lemma 3.3.17. *Let $\{\gamma_n\}$ be a sequence of planar, rectangular loops whose dimensions diverge with n. Then, for the plaquette random cluster model $\mu_{p,q,i,\infty}$*

$$\lim_{n\to\infty} \frac{\mathbb{P}\left(V_{\gamma_n}\right)}{\mathrm{Area}(\gamma_n)}$$

converges.

Proof of Theorem 3.1.6. Let $p_1(\beta) = 1 - e^{-\beta}$ and $p_2(\beta) = \frac{p_1(\beta)/q}{1-p_1(\beta)+p_1(\beta)/q}$. It follows from Lemma 3.3.15 and Corollary 3.3.9 that

$$\mathbb{P}_{p_1(\beta)}\left(V_\gamma\right) \leq \mathbb{E}_\beta\left(W_\gamma\right) \leq \mathbb{P}_{p_2(\beta)}\left(V_\gamma\right)$$

where the first and third terms are probabilities taken with respect to plaquette percolation, and the expectation is taken with respect to Potts lattice gauge theory. Then the inequalities in Equation 3.1 follow from the corresponding statements in (3.13). In addition, we may set $\beta_1 = -\log\left(1-\hat{p}\right)$ where

$$\hat{p} = \frac{\tilde{p}_1(d)\, q}{\tilde{p}_1(d)(q-1)+1}$$

and $\tilde{p}_1(d)$ is given by Lemma 3.3.16. Finally, we may set $\beta_2 = -\log\left(1-\tilde{p}_2(d)\right)$ and note that the existence of the constant $-C_2(\beta, q)$ follows from Lemma 3.3.17.

□

Chapter 4: Future Directions

Although the idea of higher dimensional percolation has been around for a long time, many fundamental questions remain unanswered. We will give a few possible directions of varying difficulty to pursue.

4.1 Finite Volume Homological Percolation

An immediate followup question to our work in Chapter 2 is whether $\lambda^{\square}(N, i, d)$ and $\lambda_i^{\text{⬡}}(N, d)$ converge as $N \to \infty$ for all i, d. Often one can work with crossing probabilities by using monotonicity, but tori of different sizes do not seem to be easily comparable. A similar and likely more difficult question can be asked in the setting of Chapter 3 as well.

Another question is whether our restrictions on the characteristic of the coefficient field for homology are necessary or not. There seem to be important dependencies when the Potts model is involved (see [4]), but it would be interesting to know whether the representation-theoretic differences cause giant cycles to appear at multiple thresholds.

4.2 Infinite Volume Homological Percolation

This work was originally intended to work towards a generalization of the Harris–Kesten theorem when $d = 2i$, on the whole lattice $\mathbb{Z}^d$ rather than on the torus $\mathbb{T}_N^d$. The first difficulty is one of definitions. The notion of homological percolation that we have studied in this thesis requires nontrivial homology in the ambient space, which is not present in $\mathbb{R}^d$. There are many possible approaches, one example being compactifying $\mathbb{R}^d$ to a torus T^d. A possibly artificial way to do this is as follows:

Let $f : \mathbb{R}^d \to (-1,1)^n$ be defined by

$$f(\mathbf{x}) \coloneqq \left(\frac{x_1}{1+|x_1|}, \frac{x_2}{1+|x_2|}, \ldots, \frac{x_d}{1+|x_d|} \right),$$

where $\mathbf{x} = (x_1, x_2 \ldots, x_d)$. Then let $\mathbb{T}^d$ be the torus obtained by identifying opposite faces of the hypercube $[-1,1]^d$, let $i : (-1,1)^n \hookrightarrow \mathbb{T}^d$ be the inclusion map, and let $f_{\mathbb{T}^d} \coloneqq i \circ f$. Then we can take plaquette percolation P on $\mathbb{Z}^d$ and consider the threshold for homological percolation in $\tilde{P} = \tilde{P}(p,i,d) \coloneqq \overline{f_{\mathbb{T}^d}(P)} \subset \mathbb{T}^d$.

Conjecture 4.2.1. *The limit $\lim_{N\to\infty} \lambda^{\square}(N,i,d)$ exists and is a sharp threshold for the appearance of giant cycles in $\tilde{P}$.*

A second difficulty lies in putting together sheets of plaquettes from smaller pieces. In various proofs of the Harris–Kesten theorem, a key step is to go from crossing squares to crossing long, skinny rectangles—see, for example, Chapter 3 of [12]. We do not currently have a high-dimensional version of the Russo–Seymour–Welsh method, passing from homological "crossings" of high-dimensional cubes to long, skinny boxes.

4.3 Criticality

Critical percolation on graphs has been intensely studied in the past few decades. Many questions asked there have higher dimensional analogues, none of which have been answered to our knowledge. Scaling limits for plaquette percolation would be particularly interesting to explore. Bond percolation in the plane at criticality conjecturally converges to Schramm Loewner Evolution (SLE) under certain conditions. It would be very interesting to find a higher dimensional analogue, either in a torus or in the full lattice. This could be a reasonable question to approach experimentally.

4.4 Lattice Gauge Theory

The coupling between the classical random-cluster model and the Potts model has been highly fruitful, and we would hope that the higher dimensional version will also be useful. Upgrading Theorem 3.1.6 to a sharp threshold would certainly be of interest, and perhaps a better understanding of the infinite volume 2-random-cluster model could help. Since the decay of correlations is a key feature in the phase transition of the Ising model, an analogue for Wilson along the lines of Corollary 3.3.12 would also be interesting.

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