Topological Abel-Jacobi Map for Hypersurfaces in Complex Projective Four-Space

Dissertation

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By

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Abstract

The beginning of the study of the Abel-Jacobi map originates from an attempt to solve the indefinite integral

$$\int \frac{dx}{\sqrt{x^3 + ax^2 + bx + c}}. \tag{1}$$

For a long time, mathematicians were unable to solve.

In the point of view of algebraic geometry, (1) can be expressed as a line integral

$$\int_q^p \frac{dx}{y} \tag{2}$$

on the cubic curve C defined by equation $y^2 = x^3 + ax^2 + bx + c$. Since C is topologically a torus and has nontrivial homology group, the integral (2) depends on the homotopy classes of the paths on C joining q to p. In fact, since the first homology group of C is generated by loops γ and δ, the line integral (2) is only well-defined modulo the periods $n \int_\gamma \frac{dx}{y} + m \int_\delta \frac{dx}{y}$, where $m, n \in \mathbb{Z}$. This gives definition of Abel-Jacobi map for genus one curve.

In general, the Abel-Jacobi map for a genus g curve C_g is to integrate holomorphic 1-forms along unions of paths on C_g modulo periods. In formal language, it is a group homomorphism sending a divisor of degree zero to the Jacobian variety of C_g. The Jacobi inversion theorem says that the Abel-Jacobi map is surjective. Based on this fact in families, Lefschetz showed that any integral $(1, 1)$ class on a smooth projective surface is algebraic.

In higher dimensions, Griffiths (1969) defined an Abel-Jacobi map sending algebraic cycles that are homologously trivial to intermediate Jacobians [30]. However, the Griffiths'

Abel-Jacobi map is rarely surjective. This is one of the difficulties of the Hodge conjecture in higher dimensions.

Generalizing Griffiths' Abel-Jacobi map, Zhao (2015) defined the *topological Abel-Jacobi map*, for a smooth complex projective variety X of odd dimension [63] and it has certain surjectivity property. To define such a map, choose a projective embedding of X. The domain of the topological Abel-Jacobi map is extended to vanishing cycles of smooth hyperplane sections of X, and the target is a "smaller" intermediate Jacobian associated with the middle-dimensional primitive cohomology of X.

In this thesis, we first compare Zhao's definition to an alternative definition of the topological Abel-Jacobi map using extensions of the mixed Hodge structures suggested by Schnell. Second, we study the domain of the topological Abel-Jacobi map and its compactifications. More precisely, by deforming a hyperplane section in the universal family of smooth hyperplane sections of X, the integral vanishing cohomology forms a local system $\mathcal{H}_{van}$, whose étale space has a distinguished component T_v containing a (primitive) vanishing cycle. T_v is our preferred domain for the topological Abel-Jacobi map.

When X is a cubic threefold, a vanishing cycle on a hyperplane section is the difference of two skew lines. We characterize the compactifications of T_v by understanding an irreducible component of the Hilbert scheme of X containing a pair of skew lines. We determined when the topological Abel-Jacobi map extends to the compactifications. Besides, we relate to the ADE singularities on hyperplane sections of X and some Bridgeland moduli spaces.

When X is a hypersurface of $\mathbb{P}^4$ of degree at least four, the H^2 on the hyperplane section has a mixed type, and the vanishing cycles are not algebraic in general. We majorly focus on the topological properties. We showed that the topology of T_v is "complicated enough" to generate the $H^3(X, \mathbb{Q})$ via the topological Abel-Jacobi map. This is related to tube mapping defined by Schnell [51].

To Yimeng (Angela)

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Fields of Study

Major Field: Mathematics

Specialization: Algebraic Geometry

Contents

List of Tables

List of Figures

Figure **Page**

Chapter 1: Introduction

In this chapter, we will outline the major theorems in this thesis. In Section 1.1, we will review the classical Abel-Jacobi map and its generalizations. We introduced two notions of topological Abel-Jacobi maps and proved that the two notions coincide (Theorem 1).

In Section 1.2, we defined the locus of primitive vanishing cycle T_v on hyperplane sections of an odd-dimensional smooth projective variety X, which is this thesis's main object of interest. When X is a smooth hypersurface of $\mathbb{P}^4$, we proved that the monodromy on T_v recovers topological information of X in the case where (Theorem 2).

In Section 1.3, we assume X is a hypersurface of degree 3 in $\mathbb{P}^4$. We first studied the Hilbert scheme of a pair of skew lines on X (Theorem 3). We explored various compactifications of T_v (Theorem 4), and provided interpretations of boundary points of the compactifications.

1.1 Topological Abel-Jacobi Map

1.1.1 Abel-Jacobi Map for Curves

For a compact Riemann surface C, the Abel-Jacobi map is a homomorphism

$$A : Div^0(C) \to JC \tag{1.1}$$

from the group of divisors of degree zero on C to the Jacobian variety $JC = H^0(C, \Omega_C)^\vee / H_1(C, \mathbb{Z})$.

If $D = (f)$ is a principal divisor, then $A(D) = 0$, because f defines a rational family of divisors joining zeros $(f)_0$ and poles $(f)_\infty$, and a compact complex torus has no rational curves.

Abel's theorem says the converse: if $A(D) = 0$, D is principal. As a consequence, (1.1) induces an injection of a group homomorphism

$$Pic^0(C) = Div^0(C)/PDiv^0(C) \to JC. \tag{1.2}$$

The Jacobi inversion theorem implies that this is an isomorphism.

The classical proof of Abel's theorem is to construct a meromorphic 1-form ϕ on C such that (i) ϕ has simple poles along D_{supp}, (ii) the residue of ϕ at $x \in D_{supp}$ is $\frac{n_x}{2\pi i}$ where n_x is the multiplicity of x in D, (iii) ϕ has integral periods against a suitable homology basis of C. The condition (iii) guarantees that the path integral $\int_q^p \phi$ has value differing by an integer when choosing a different path joining q to p. It follows that $f(x) = \exp(2\pi i \int_{x_0}^x \phi)$ is meromorphic on C, where x_0 is a fixed point, and $D = (f)$ guaranteed by (i) and (ii). The construction of ϕ satisfying (i) and (ii) is a consequence of the Riemann-Roch theorem, but the condition (iii) is vare and more difficult to construct. One needs the assumption $A(D) = 0$ and the Reciprocity Law.

1.1.2 A Hodge-Theoretical Interpretation of Abel's Theorem

From a Hodge theoretical perspective, the existence of a meromorphic 1-form ϕ on C satisfying condition (i) and (ii) will define a mixed Hodge structure E_ϕ of weight one fitting into the exact sequence of mixed Hodge structures

$$0 \to H^1(C, \mathbb{Z}) \to E_\phi(\mathbb{Z}) \to \mathbb{Z} \to 0. \tag{1.3}$$

This means $E_\phi(\mathbb{Z})$ is isomorphic to $\mathbb{Z}^{2g} \oplus \mathbb{Z}$ as $\mathbb{Z}$-modules with basis $\{\gamma_i^*, \ldots, \gamma_g^*, \delta_1^*, \ldots, \delta_g^*, D\}$, but the complexification $E_\phi(\mathbb{C}) = E_\phi(\mathbb{Z}) \otimes_\mathbb{Z} \mathbb{C}$ has a $(g+1)$-complex dimensional subspace

$F^1E_\phi(\mathbb{C})$ spanned by $F^1H^1(C,\mathbb{C})$ consisting holomorphic 1-forms on C and the vector

$$(\int_{\gamma_1}\phi,\ldots,\int_{\gamma_g}\phi,\int_{\delta_1}\phi,\ldots,\int_{\delta_g}\phi,1). \tag{1.4}$$

In fact, (1.3) is a subsequence of the exact sequence $0\to H^1(C,\mathbb{Z})\to H^1(U,\mathbb{Z})\xrightarrow{r} H^0(D_{supp},\mathbb{Z})$ where $U=C\setminus D_{supp}$ is the open subspace and r is the residue map. So E_ϕ is a sub-mixed Hodge structure of $H^1(U,\mathbb{Z})$, and the vector (1.4) is the class $[\phi]\in H^1(U,\mathbb{Z})$ of the meromorphic 1-form in terms of the $\mathbb{Z}$-basis.

Now if, in addition, ϕ satisfies the condition (iii), i.e., the γ and δ-periods of the ϕ are integral, Abel's theorem can be interpreted as follows.

Proposition 1.1.1. *(Abel's theorem) if $A(D)=0$, the extension sequence* (1.3) *of mixed Hodge structures splits over $\mathbb{Z}$.*

In general, the Abel-Jacobi map (1.1) of a degree 0 divisor D can be defined via extension sequence (1.3): Pick a meromorphic 1-form ϕ such that $Res(\phi)=D$, then $\tilde{D}_F:=[\phi]$ defines a lifting of D that preserves Hodge filtration (up to a Tate twist). One the othar hand, we can take an integral lifting $\tilde{D}_\mathbb{Z}\in E_\phi(\mathbb{Z})$ of D, then exactness of (1.3), the difference $\tilde{D}_\mathbb{Z}-\tilde{D}_F$ lies in $H^1(C,\mathbb{C})$. Modulo the freedom of choice, the difference $\tilde{D}_\mathbb{Z}-\tilde{D}_F$ is well-defined in $H^1(C,\mathbb{C})/F^1H^1(C,\mathbb{C})+H^1(C,\mathbb{Z})\cong Pic^0(C)$ and is the extension class $[e]$ of the sequence (1.3) according Carlson's theorem. Via the intersection pairing on $H^1(C,\mathbb{Z})$, the torus $Pic^0(C)$ is identified with JC, and the image $[e]$ coincides with the Abel-Jacobi image $A(D)$. Now by Proposition 1.1.1, we have the following interpretation.

The Abel-Jacobi image of a degree-zero divisor D is an obstruction of splitting the extension of mixed Hodge structures (1.3).

For higher dimensions, there is a similar story. Let X be a smooth projective variety, and Z is an algebraic cycle of codimension r in X and is homologous to zero. Griffiths showed that by choosing Γ a topological $(2n-2r+1)$-chain on X with $\partial\Gamma = Z$, then the integral

$$\int_\Gamma \tag{1.5}$$

defines an element in a compact torus associated to the Hodge structure on $H^{2n-2r+1}(X, \mathbb{Z})$ referred as r-th intermediate Jacobi. This map is called the Griffiths' Abel-Jacobi map. It turns out that the image is determined by the mixed Hodge structure on $H^{2r-1}(X \setminus Z_{supp}, \mathbb{Z})$ and the Griffiths' Abel-Jacobi map can also be defined similarly to the curve case.

1.1.3 Topological Abel-Jacobi Map by Zhao

Xiaolei Zhao introduced a new notion called the *topological Abel-Jacobi map* in his thesis [63]. It extends Griffiths' Abel-Jacobi map to the topological cycles. Let X be a smooth projective variety of odd dimension $2n-1$, and let Y be a smooth hyperplane section of X. The topological Abel-Jacobi map is a group homomorphism

$$A : H^{2n-2}_{\text{van}}(Y) \to J_{\text{prim}}(X), \tag{1.6}$$

sending vanishing cycles on Y to the primitive intermediate Jacobian of X at the middle dimension. It agrees with Griffiths' Abel-Jacobi map when vanishing cycles are algebraic. Moreover, the map is real analytic as Y deforms in the locus of the smooth hyperplane sections of X. Here $J_{\text{prim}}(X)$ is the complex torus associated to the primitive cohomology $H^{2n-1}_{\text{prim}}(X) = \ker(i^* : H^{2n-1}(X) \to H^{2n-1}(Y))$ where $i : Y \hookrightarrow X$ is the inclusion map and i^* is the restriction map. $J_{\text{prim}}(X)$ is isogeny to a subtorus of the intermediate Jacobian of X associated to the middle dimensional cohomology $H^{2n-1}(X)$.

Explicitly, the map (1.6) is given by the following: Let ω be a closed differential $(p, 2n-1-p)$-form on X with $p \geq n$, and the class $[\omega]$ lies in $H^{2n-1}_{\text{prim}}(X)$. By $\partial\bar{\partial}$-lemma, there exists a $(p-1, 2n-2-p)$-form σ on Y such that $\bar{\partial}\partial\sigma = \omega_{|Y}$. Let $\alpha \in H^{2n-2}_{\text{van}}(Y, \mathbb{Z})$ whose Poincaré dual is represented by a closed chain γ, and $\Gamma \subseteq X$ such that $\partial\Gamma = \gamma$, then

$$A_\alpha([\omega]) = \int_\Gamma \omega - \int_\gamma \partial\sigma. \tag{1.7}$$

Note that this is just adding a correction term on the Griffiths integral (1.5) to make the the map (1.6) well-defined on topological cycles.

In fact, the sum of the two integrals above is a topological pairing between the relative cohomology class $(\omega, \partial\sigma)$ and relative homology class (Γ, γ) via the Poincaré pairing

$$H^{2n-1}(X, \mathbb{C}) \times H_{2n-1}(X, \mathbb{C}) \to \mathbb{C}. \tag{1.8}$$

1.1.4 Topological Abel-Jacobi and Mixed Hodge Structures

Schnell proposed a different way to construct a topological Abel-Jacobi map using the $\mathbb{R}$-splitting property of the mixed Hodge structure on $H^{2n-1}(X \setminus Y)$ [50]. More explicitly, there is a short exact sequence of mixed Hodge structures

$$0 \to H_0^{2n-1}(X) \to H^{2n-1}(X \setminus Y) \xrightarrow{r} H_{\text{van}}^{2n-2}(Y) \to 0, \tag{1.9}$$

where $H_0^{2n-1}(X)$ is the cokernel of the Gysin homomorphism $H^{2n-3}(Y) \to H^{2n-1}(X)$, and r shifts the weights by $(-1, -1)$.

$H_0^{2n-1}(X)$ and $H_{\text{van}}^{2n-2}(Y)$ inherit pure Hodge structures of weight $2n-1$ and $2n-2$ respectively. $H^{2n-1}(X \setminus Y)$ has a canonical mixed Hodge structure with weight concentrated on the level $2n-1$ and $2n$.

According to Deligne, the mixed Hodge structure $H^{2n-1}(X \setminus Y)$ is $\mathbb{R}$-splitting, meaning that there is a canonical $s_\mathbb{R} : H_{\text{van}}^{2n-2}(Y, \mathbb{R}) \to H^{2n-1}(X \setminus Y, \mathbb{R})$ splitting the exact sequence (1.9) in the sense that such that $r \circ s = Id$ and s shifts weight by $(1, 1)$ after tensoring with $\mathbb{C}$.

Now, let $\alpha \in H_{\text{van}}^{2n-2}(Y, \mathbb{Z})$ and $s_\mathbb{Z}(\alpha) \in H^{2n-1}(X \setminus Y, \mathbb{Z})$ such that $r(s_\mathbb{Z}(\alpha))$, then $s_\mathbb{Z}(\alpha) - s_\mathbb{R}(\alpha) \in H_0^{2n-1}(X, \mathbb{R})$ by exactness of (1.9). Modulo the freedom of choice of $\mathbb{Z}$-lifting $s_\mathbb{Z}(\alpha)$, the difference

$$s_{\mathbb{Z}}(\alpha) - s_{\mathbb{R}}(\alpha) \tag{1.10}$$

is well-defined in the real torus $J_0(X,\mathbb{R}) = H_0^{2n-1}(X,\mathbb{R})/H^{2n-1}(X,\mathbb{Z})$.

Definition 1.1.2. *$\alpha \mapsto s_{\mathbb{Z}}(\alpha) - s_{\mathbb{R}}(\alpha) \in J_0(X,\mathbb{R})$ is called the real topological Abel-Jacobi map.*

Schnell asked

Question 1.1.3. *[50] Is the real topological Abel-Jacobi map* (1.10) *defined by Schnell the same as the topological Abel-Jacobi map* (1.6) *defined by Zhao?*

We need to identify their targets to compare the two topological Abel-Jacobi maps. In fact, there are isomorphisms between real tori

$$\phi : J_0(X,\mathbb{R}) \cong H_{\text{prim}}^{2n-1}(X,\mathbb{R})/H_{2n-1}(X,\mathbb{Z}) \cong F^n H_{\text{prim}}^{2n-1}(X,\mathbb{C})/H_{2n-1}(X,\mathbb{Z}). \tag{1.11}$$

We answered the question affirmatively:

Theorem 1. *For any $\alpha \in H_{\text{van}}^{2n-2}(Y,\mathbb{Z})$, $\phi(s_{\mathbb{Z}}(\alpha) - s_{\mathbb{R}}(\alpha)) = A(\alpha)$. In other words, the real topological Abel-Jacobi map defined by Schnell and Zhao's topological Abel-Jacobi map coincide.*

1.2 Tube Mapping

As Y varies in the open locus $\mathbb{O}^{\text{sm}}$ consisting of smooth hyperplane sections of X, there is a $\mathbb{Z}$-local system $\mathcal{H}_{\text{van}}^{2n-2}$ whose stalk at t is the vanishing cohomology $H_{\text{van}}^{2n-2}(Y,\mathbb{Z})$. Let T denote the underlying analytic space of $\mathcal{H}_{\text{van}}^{2n-2}$, then there is a covering map

$$T \to \mathbb{O}^{\text{sm}}, \tag{1.12}$$

and a real analytic map $f: T \to J_{\text{prim}}X$. It induces the map between fundamental groups:

$$f_*: \pi_1(T, t_0) := \oplus_{i\in I}\pi_1(T_i, \tilde{t}_0^i) \to \pi_1(J_{\text{prim}}X) = H_{2n-1}(X, \mathbb{Z})_{\text{prim}}, \tag{1.13}$$

where I is the index set of connected components of T and $\tilde{t}_0^i \in T_i$ is a base point over $t_0 \in \mathbb{O}^{\text{sm}}$. (1.13) is called *tube mapping.* Schnell showed that

Lemma 1.2.1. *[51] The image of the tube mapping* $\text{Im}(f_*)$ *is cofinite.*

Interpreting differently, note that $\pi_1(T, t_0)$ consists of pairs $(\alpha, \gamma) \in H^{2n-2}_{\text{van}}(Y, \mathbb{Z}) \times \pi_1(\mathbb{O}^{\text{sm}}, t_0)$ such that $\gamma \cdot \alpha = \alpha$, so the trace of flat translate of a class α along a loop γ produces a n-cycle A on X, so what Lemma 1.2.1 means is that the set of "tubes" over a class $\alpha \in H^{2n-2}_{\text{van}}(Y, \mathbb{Z})$ generate a full rank sublattice of the primitive homology $H_{2n-1}(X, \mathbb{Z})_{\text{prim}}$.

T has infinitely many connected components. Among these components, there is a distinguished component T_v, consisting of a *vanishing cycle* δ, the class of the sphere $x_1^2+\cdots+x_n^2 = \varepsilon$ in the neighborhood $z_1^2 + \cdots + z_n^2 = \varepsilon$ of hypersurface Y_ε, and Y_0 has an ordinary node. We're interested in the topology of the component T_v. Topologically, the manifold T_v is a covering space of $\mathbb{O}^{\text{sm}}$ consisting of flat translates of a vanishing cycle α.

Definition. *We call a class* $\alpha \in H^{2n-2}_{\text{van}}(Y, \mathbb{Z})$ *to be a primitive vanishing cycle on* Y *if* α *is a flat translate to* δ*. We call* T_v *the locus of primitive vanishing cycles on (smooth) hyperplane sections of* X*.*

If we restrict the topological Abel-Jacobi to the component T_v. The real analytic map

$$f_v: T_v \to J_{\text{prim}}X, \tag{1.14}$$

induces the tube mapping

$$(f_v)_*: \pi_1(T_v, t_v) \to \pi_1(J_{\text{prim}}X) = H_{2n-1}(X, \mathbb{Z})_{\text{prim}}. \tag{1.15}$$

We showed that

Theorem 2. *When X is a smooth hypersurface of degree ≥ 3 in $\mathbb{P}^4$, then the image* $\mathrm{Im}((f_v)_*)$ *of the tube map* (1.15) *has full rank.*

Equivalently, we showed that the tubes over vanishing cycles generate a full rank sublattice of $H_3(X,\mathbb{Z})$ when X is a smooth hypersurface of $\mathbb{P}^4$.

The proof of this theorem relies on the key fact that the component T_v of primitive vanishing cycles is invariant under monodromy as X varying in the set of smooth hyperplane sections of a smooth projective variety $\mathcal{X}$ of one dimensional higher. Then, a degeneration argument reduces to $\deg(X) = 3$ case.

1.3 Locus of Primitive Vanishing Cycles on Hyperplane Sections of Cubic Threefold

When the hypersurface has degree 3, our X is a cubic threefold. It turns out that a primitive vanishing cycle α on a hyperplane section of X can be written as a difference $\alpha = [L_1] - [L_2]$, where L_1 and L_2 is a pair of skew lines on X.

The set of projective lines on the cubic threefold X is parameterized by a smooth surface F. The intermediate Jacobian $JX = F^2H^3(X,\mathbb{C})^\vee/H_3(X,\mathbb{Z})$ of X is a principally polarized abelian variety of dimension 5, and there is an Abel-Jacobi map

$$\psi : F \times F \to JX,\ (L_1, L_2) \mapsto \int_{L_2}^{L_1}. \tag{1.16}$$

Clemens and Griffiths [19] first studied it, and showed that (1.16) is generically 6-to-1 onto the theta divisor Θ of JX.

The proof of Theorem 2 for the degree 3 case relies on the understanding of the interplay between the topological Abel-Jacobi map (1.14) and the Abel-Jacobi map (1.16).

In fact, by noticing a pair of skew lines on X determines a hyperplane by spanning it, there is an open dense subspace $\mathcal{M}$ of $F \times F$ consisting of pairs of skew lines and that the restriction of (1.16) to $\mathcal{M}$ factors through

$$\mathcal{M} \xrightarrow{e} T_v \xrightarrow{f_v} JX, \tag{1.17}$$

where e is a 6-to-1 cover by sending (L_1, L_2) to the class $[L_1] - [L_2]$. f_v coincides with the topological Abel-Jacobi map defined in (1.6).

But our interest is more than that. The motivation of the rest of the research is to answer the following question.

Question. *Is there a (natural) compactification of T_v so that the topological Abel-Jacobi map* (1.14) *extends? What is the interpretation of boundary points of the compactification?*

In fact, understanding the compactification of the component T_v relies on the understanding of the compactification of $\mathcal{M}$.

1.3.1 Hilbert Scheme of a Pair of Skew Lines

Since $\mathcal{M}$ parameterizes pairs of skew lines, there is a 2-to-1 cover

$$\mathcal{M} \to \mathrm{Hilb}(X),\ (L_1, L_2) \mapsto \mathcal{O}_{L_1 \cup L_2} \tag{1.18}$$

to the Hilbert scheme of X. The image is an open dense subspace of an irreducible component $H(X)$ of $\mathrm{Hilb}(X)$.

Definition. *$H(X)$ is called the Hilbert scheme of a pair of skew lines on X.*

Theorem 3. *$H(X)$ is smooth and is isomorphic to the blowup $\mathrm{Bl}_{\Delta}\mathrm{Sym}^2 F$ of the symmetric product $\mathrm{Sym}^2 F$ along the diagonal.*

The proof is based on the work of Hilbert scheme of a pair of skew lines in projective spaces [16] (the paper studied Hilbert scheme of a pair of codimension two subspaces) and the work on the Abel-Jacobi map on cubic threefolds [19], as well as the characterization of theta divisor [7].

In fact, $H(X)$ parameterizes four types of schemes:

(I): A pair of skew lines.

(II): First order infinitesimal neighborhood of a line in a quadric surface.

(III): Pair of incident line with an embedded point supported at the intersection.

(IV): First order infinitesimal neighborhood of a line in a $\mathbb{P}^2$ together with an embedding point on the line.

Type (I) is reduced; Type (III) is supported on a pair of incident lines, while Type (II) and (IV) are supported on a single line.

The scheme of each of the four types is contained in a hyperplane $\mathbb{P}^3$ and uniquely determines that hyperplane. This describe how $\mathrm{Sym}^2F \setminus \Delta$ parameterizes type (I) and (III) schemes. Since the exceptional divisor of $\mathrm{Bl}_\Delta \mathrm{Sym}^2F$ is a $\mathbb{P}^1$-bundle over $\Delta \cong F$, the proof of Theorem 3 relies on understanding how this $\mathbb{P}^1$-bundle on F parameterizes type (II) and (IV) schemes and how they match up to the normal directions corresponding to points on F.

According to Beauville [7], Θ has a unique singularity 0, and the projective tangent cone of Θ at 0 is isomorphic to the cubic threefold X itself. In particular, the blow-up $\mathrm{Bl}_0(\Theta)$ of Θ at 0 is smooth and the exceptional divisor is isomorphic to X. The Abel-Jacobi map ψ (1.16) extends to the blowup

$$\begin{array}{ccccc} \mathrm{Bl}_{\Delta_F}(F\times F) & \xrightarrow{\tilde{\psi}} & \mathrm{Bl}_0\Theta & \longrightarrow & \mathrm{Bl}_0 JX \\ \downarrow & & \downarrow & & \downarrow \\ F\times F & \xrightarrow{\psi} & \Theta & \longrightarrow & JX \end{array} \tag{1.19}$$

We call $\tilde{\psi}$ the *extended Abel-Jacobi map.*

Lemma 1.3.1. *The restriction of the extended Abel-Jacobi map $\tilde{\psi}$ on the exceptional divisor is identified to*

$$\mathbb{P}T_F \cong \{(t,x) \in F \times X | x \in L_t\} \to X, \tag{1.20}$$

where the second map is given by $(t,x) \mapsto x$ and the first map α is an isomorphism of $\mathbb{P}^1$-bundle which is described by the following: when L_t is a line of the first type (normal bundle is $\mathcal{O}_{L_t} \oplus \mathcal{O}_{L_t}$), $\alpha(L_t, v)$ is the unique point on L_t which is the tangent point of the H_v on X along L_t; When L_t is a line of the second type (normal bundle is $\mathcal{O}_{L_t}(-1) \oplus \mathcal{O}_{L_t}(1)$), $\alpha(L_t, v)$ is the conjugate point of zero locus of v.

Here a conjugate point of p on a line of the second type L is the unique point $q \in L$ such that $T_pX = T_qX$.

As a consequence of Theorem 3, a branched double cover $\widetilde{H(X)}$ of $H(X)$ is identified with $\mathrm{Bl}_{\Delta_F}(F \times F)$, is a compactification of $\mathcal{M}$. $\widetilde{H(X)}$ can be interpreted as a nested Hilbert scheme or simply a "Hilbert scheme" of a pair of ordered skew lines. Also, each type of scheme is contained in a unique hyperplane. Therefore, the fiber of the projection

$$H(X) \to (\mathbb{P}^4)^* \tag{1.21}$$

can be interpreted as Hilbert scheme of a pair of skew lines on a cubic surface.

Proposition 1.3.2. *The fiber of* (1.21) *is finite over H when $X \cap H$ has at worst ADE singularities. The fiber is $\mathrm{Sym}^2 E$ when $X \cap H$ is a cone over elliptic curve E.*

This will help find the fiber of $\mathrm{Bl}_{\Delta_F}(F \times F) \to (\mathbb{P}^4)^*$, which is used in the proof of Theorem 4 and Proposition 1.3.5.

1.3.2 Compactifications of T_v

Let $\mathbb{O}^{\mathrm{sm}}$ denote the open subspace of $(\mathbb{P}^4)^*$ parameterizing smooth hyperplane sections of X. According to (1.17), there is a connected 72-to-1 covering space

$$\pi : T_v \to \mathbb{O}^{\mathrm{sm}}, \tag{1.22}$$

whose fiber at t is identified with the set

$$\{\alpha \in H^2(S_t, \mathbb{Z}) | \alpha \cdot \alpha = -2, \alpha \cdot h = 0\}, \tag{1.23}$$

where h is the hyperplane class and $S_t = X \cap H_t$ is a hyperplane section. Our goal in this section is to understand the compactifications of T_v.

Due to a theorem of Stein [56], T_v admits a unique normal completion $\bar{T}_v$ together with a finite map

$$\bar{\pi} : \bar{T}_v \to (\mathbb{P}^4)^* \tag{1.24}$$

extending π. So the boundary points of $\bar{T}_v$ can be regarded as vanishing cycles "in the limit".

Due to a theorem of Stein [56], T_v admits a unique normal completion $\bar{T}_v$ such that $\bar{\pi} : \bar{T}_v \to (\mathbb{P}^4)^*$ is finite and extends π. So the boundary points of $\bar{T}_v$ can be regarded as vanishing cycles "in the limit" as hyperplane sections become singular.

We answered the first half of the question by relating $\bar{T}_v$ to the theta divisor Θ of the intermediate Jacobian JX of X. We showed that

Theorem 4. *There is a birational morphism* $\mathrm{Bl}_0(\Theta) \to \bar{T}_v$ *contracting finitely many elliptic curves that are in 1-1 correspondence with the Eckardt points on* X.

Here an Eckardt point $p \in X$ is a point where infinitely many lines on X pass through p. (In fact, these lines are parameterized by an elliptic curve E) A smooth cubic threefold has at most finitely many Eckardt points, and a general cubic threefold has no Eckardt point.

Corollary 1.3.3. *When X is general and has no Eckardt points, $\bar{T}_v$ is smooth and isomorphic to* $\mathrm{Bl}_0(\Theta)$.

In [44], the authors find a hyperkähler variety $\mathcal{Z}$ of dimension 8, whose general point parameterizes rational equivalent classes of twisted cubics on a smooth cubic fourfold W. In fact, the space $\bar{T}_v$ that we considered is the restriction of Z to a cubic threefold as a smooth hyperplane section of W. According to [55], $\bar{T}_v$ is a Lagrangian subvariety of $\mathcal{Z}$. So Theorem 4 can be regarded as a characterization of certain Lagrangian subvarieties of a hyperkähler variety.

As for the extension of the topological Abel-Jacobi map, one notes by Corollary 1.3.3, when X is general, the extension is just $\mathrm{Bl}_0(\Theta) \to \Theta \to JX$ (see the diagram (1.19)). However, when X has Eckardt points, the elliptic curves C_i in Theorem 4 are transversal to the exceptional divisor. They are therefore mapped isomorphically onto its image in JX. But on the other hand, C_i is contracted in $\bar{T}_v$, which prevents any possibility of extension to the entire $\bar{T}_v$ over the Eckardt hyperplane. In sum, we have

Proposition 1.3.4. *The topological Abel-Jacobi map* (1.17) $f_v : T_v \to JX$ *extends to* $\bar{T}_v$ *if and only if X has no Eckardt points.*

However, we are looking for an alternative way to compactify the locus of primitive vanishing cycles T_v. We blow up $\mathbb{O} = (\mathbb{P}^4)^*$ at points corresponding to the Eckardt hyperplanes and denote the resulting space as $\tilde{\mathbb{O}}$. Stein's theorem [56] implies again that there exists a unique normal algebraic variety $\tilde{T}_v$ finite over $\tilde{\mathbb{O}}$, and extends (1.22). The space $\tilde{T}_v$ captures the information of the limiting points of the primitive vanishing cycles along a pencil through an Eckardt hyperplane.

To describe the relationship between the compactifications, two compactifications are the same for general cubic threefolds. In general, when X is smooth and has Eckardt points,

$\tilde{T}_v$ is the normalization of the fiber product $\bar{T}_v \times_{\mathbb{O}} \tilde{\mathbb{O}}$. In other words, two spaces agree over $\mathbb{O} \setminus \cup_i \{H_i\}$, where H_i's are Eckardt hyperplanes of X. $\tilde{T}_v$ is obtained by replacing finitely many points from $\bar{T}_v$ with certain divisors.

Proposition 1.3.5. *There is a morphism $\tilde{f}_v : \tilde{T}_v \to \mathrm{Bl}_0 JX$ extending the topological Abel-Jacobi map* (1.17)*.*

We can describe the morphism $\tilde{f}_v$ explicitly by the following: Since $\tilde{T}_v \to \tilde{\mathbb{O}}$ is finite, a point $a \in \tilde{T}_v$ corresponds to a point u on the exceptional $\mathbb{P}^3 \subseteq \tilde{\mathbb{O}}$ over the Eckardt hyperplane $H_0 \in \mathbb{O}$. Then u corresponds to a a pencil $\mathbb{L}_u$ of hyperplanes through the Eckardt hyperplane H_0. Let t be the parameter space on $\mathbb{L}_u$ and $t = 0$ corresponds to H_0, then p lies in (up to normalization) the specialization of the fiber $\bar{\pi}^{-1}(t)$, as $t \to 0$.

From the point of view of the Hilbert scheme, $\bar{\pi}^{-1}(t)$ is a finite quotient of Hilbert scheme of a pair of ordered skew lines $\widetilde{H}(X \cap H_t)$ on the hyperplane section $X \cap H_t$. The extended topological Abel-Jacobi image $\tilde{f}_v(a)$ is determined by the extension of the Abel-Jacobi image of $\widetilde{H}(X) \cong \mathrm{Bl}_{\Delta_F}(F \times F)$ restricted to a hyperplane (and lifted to $\mathrm{Bl}_0 JX$). But this map is described in the first row of the diagram (1.19), except that we restrict to a pencil $\mathbb{L}_u$.

Since $X \cap H_0$ is a cone over an elliptic curve E, the limit scheme of $H(X \cap H_t)$ as $t \to 0$ is a finite subscheme of $\mathrm{Sym}^2 E$ (fiber of (1.21)). Similarly, its natural double cover $\widetilde{H}(X \cap H_t)$ is a finite subscheme $\mathcal{S}$ of $E \times E$.

The Eckardt cone $X \cap H_0$ only contains type (III) and (IV) schemes. Then any $Z \in \mathcal{S}$ corresponds to a type (III) scheme with an order or a type (IV) scheme. When Z is of type (III) with an order, the Abel-Jacobi image of Z is the difference between the two lines as described by (1.16) whose image is off the singularity of Θ. When Z has type (IV), the Abel-Jacobi image is $0 \in JX$, and it lifts to a point on $\mathrm{Bl}_0 JX$ which can be described through (1.20) corresponding to the following: The scheme Z has an embedded point supported at $p \in L$, whose the conjugate point $\bar{p}$ is a unique point on the same line such that $T_pX = T_{\bar{p}}X$ (as described in Lemma 1.3.1). Then the extended Abel-Jacob image of Z is just $\bar{p} \in X \subseteq \mathrm{Bl}_0(\Theta)$ on the exceptional divisor. So if the original $a \in \tilde{T}_v$ lifts to $Z \in \widetilde{H}(X \cap H_0)$ described above, then $\tilde{f}_v(a) = \bar{p}$.

Theorem 4 and Proposition 1.3.5 together answer the first part of the Question 1.3.

1.3.3 Interpretation of Boundary Points

To answer the second half of the Question 1.3, we found some intriguing relationship between the boundary points of $\bar{T}_v$ and the geometry of the singularities of the cubic surfaces S_H arising from the hyperplane sections of X.

First, for a smooth cubic surface S, the set of primitive vanishing cycles on S (1.23) is identified with the root system $R(\mathbb{E}_6)$ of the Lie algebra $\mathbb{E}_6$. So $\pi : T_v \to \mathbb{O}^{\mathrm{sm}}$ (1.22) can be regarded as deforming root system of $\mathbb{E}_6$ that are parameterized by smooth hyperplane sections of a smooth cubic threefold X. Consequently, the monodromy group of π is a subgroup of the Weyl group $W(\mathbb{E}_6)$. According to [54, VI.20] and [17, Theorem 0.1], the monodromy group is the entire Weyl group $W(\mathbb{E}_6)$. From this point of view, we can regard the boundary points of $\bar{T}_v$ can be interpreted as the degeneration of root system $R(\mathbb{E}_6)$.

More precisely, let S_H be a singular cubic surface corresponding to a hyperplane $H \in (\mathbb{P}^4)^* \setminus U$. Due to the classification results of cubic surfaces, S_H has either (i) ADE singularities or (ii) an elliptic singularity.

In case (ii), $\bar{\pi}^{-1}(H)$ is just one point. For case (i), we consider the minimal resolution $\sigma : \tilde{S} \to S$, then $\tilde{S}$ is a weak del Pezzo surface and its root system is isomorphic to $R(\mathbb{E}_6)$ (cf. [22, Cha. 8]). Let W_e be the subgroup of the Weyl groups of $\mathbb{E}_6$ generated by the reflections corresponding to the effective (-2)-curves on $\tilde{S}$ over the singular points of S. $W_e \cong \prod_i W_i$ splits into the direct product of Weyl groups, where W_i is the Weyl group of the Lie algebra corresponding to the singularity p_i on S_H.

Let B be an open neighborhood of $H \in (\mathbb{P}^4)^*$, and let $B^{sm} = B \cap \mathbb{O}^{\mathrm{sm}}$, pick a base point $t' \in B^{sm}$, then we have the monodromy representation

$$\rho : \pi_1(B^{sm}, t') \to W(\mathbb{E}_6). \tag{1.25}$$

Definition 1.3.6. *Call* $\mathrm{Im}(\rho)$ *the local monodromy group of* S_H.

Proposition 1.3.7. *The local monodromy group of* S_H *is isomorphic to* W_e.

W_e acts on the root system $R(\mathbb{E}_6)$ and we denote $R(S_H) := R(\mathbb{E}_6)/W_e$.

Corollary 1.3.8. *If* S_H *has ADE singularities, the fiber* $\bar{\pi}^{-1}(H)$ *of the map* (1.24) *is identified with the orbit space* $R(S_H)$.

The orbit space $R(S_H)$ was originally defined in [44] and is used to parameterize the reduced Hilbert scheme of generalized twisted cubics on S_H.

The fiber $\bar{T}_v \to \mathbb{O}$ over a Eckardt hyperplane H_0 is a single point. Equivalently, the local monodromy group of $X \cap H_0$ is the entire monodromy group $W(\mathbb{E}_6)$. The interpretation of boundary points of $\tilde{T}_v$ over the exceptional $\mathbb{P}^3$ are described below the Proposition 1.3.5.

Chapter 2: Topological Abel-Jacobi Map

In this chapter, we study various notions of Abel-Jacobi maps, including classical Abel-Jacobi map for curves, Griffiths' Abel-Jacobi map in higher dimensions, and two versions of topological Abel-Jacobi maps. Our main theorem is the Theorem 2.5.6, which declares two definitions of the topological Abel-Jacobi maps are the same.

In Section 2.1, we review the construction of the Abel-Jacobi map for curves and some classical results, including Abel's theorem (Theorem 2.1.3) and Jacobi Inversion theorem (Theorem 2.1.8). In Section 2.2, we review the definition of mixed Hodge structures. We provide an alternative construction of the Abel-Jacobi map (Theorem 2.2.13) based on Carlson's theory on extensions of mixed Hodge structures [13]. Consequently, we provide an alternative interpretation of Abel's theorem (Theorem 2.1.7). In Section 2.3, we review Griffiths' construction of the Abel-Jacobi map in higher dimensions for algebraic cycles that are homologously trivial [30].

In Section 2.4, we introduce a notion called the topological Abel-Jacobi map defined by Zhao. It generalizes Griffiths' Abel-Jacobi map to topological cycles [63]. In Section 2.5, Schnell proposed an alternative definition of the topological Abel-Jacobi map. In Section 2.6, we showed that the two definitions on topological Abel-Jacobi maps are the same. In Section 2.7, we review Topologcal Jacobi Inversion theorem (Theorem 2.7.2) proved by Zhao [63].

2.1 Abel's Theorem for Compact Riemann Surfaces

2.1.1 Compact Riemann Surfaces

A smooth projective curve X over the complex field $\mathbb{C}$, more commonly known as a compact Riemann surface, is a real two-dimensional compact oriented surface without boundary. It is topologically classified by its genus, i.e., the number of the "holes", but the tangent space at each point has a preferred J-operator, which rotates each tangent vector by 90-degrees compatible with the orientation of the surface. J defines a complex structure on the compact surface X.

This complex structure also defines a Hodge structure on its first cohomology group $H^1(X,\mathbb{Z})$, namely by complexifying the space, there is a g-dimensional complex subspace $H^{1,0}(X) \subseteq H^1(X,\mathbb{C})$, where $H^{1,0}(X) = H^0(X,\Omega_X)$ is the space of holomorphic 1-forms on X. Moreover, the space $H^{0,1}(X) = \overline{H^{1,0}(X)}$ consists of all anti-holomorphic 1-forms is also a complex subspace of $H^1(X,\mathbb{C})$. The Hodge decomposition theorem says that there is a isomorphism

$$H^1(X,\mathbb{C}) \cong H^{1,0}(X) \oplus H^{0,1}(X)$$

as complex vector spaces.

There is a Jacobian variety $J(X)$ associated to the Hodge structure $H^1(X,\mathbb{Z})$:

$$J(X) = H^{1,0}(X)^\vee / H_1(X,\mathbb{Z}),$$

where the inclusion $H_1(X,\mathbb{Z}) \hookrightarrow H^{1,0}(X)$ is given by sending γ to the linear functional $\omega \mapsto \int_\gamma \omega$ and they are called periods. More specifically, let's choose a sympelctic basis $\gamma_1,\dots,\gamma_g,\delta_1,\dots,\delta_g$ for $H_1(X,\mathbb{Z})$, in other words, a basis such that

$$\gamma_i \cdot \delta_j = \delta_{ij},\ \gamma_i \cdot \gamma_j = 0,\ \delta_i \cdot \delta_j = 0. \tag{2.1}$$

We can normalize and choose a basis $\omega_1, \dots, \omega_g$ for $H^{1,0}(X)$ such that

$$\int_{\gamma_i} \omega_j = \delta_{ij}. \tag{2.2}$$

Then we obtain a $g \times 2g$-matrix

$$\Lambda = [Id_{g\times g}|\Omega] = \begin{bmatrix} 1 & 0 & \cdots & 0 & \int_{\delta_1}\omega_1 & \int_{\delta_2}\omega_1 & \cdots & \int_{\delta_g}\omega_1 \\ 0 & 1 & \cdots & 0 & \int_{\delta_1}\omega_2 & \int_{\delta_2}\omega_2 & & \int_{\delta_g}\omega_2 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 & \int_{\delta_1}\omega_g & \int_{\delta_2}\omega_g & \cdots & \int_{\delta_g}\omega_g \end{bmatrix}. \tag{2.3}$$

Ω is called the period matrix and satisfies the Riemann's bilinear relations:

$$\Omega^t = \Omega, \text{ and } \mathrm{Im}(\Omega) > 0. \tag{2.4}$$

The column vectors of Λ define a full rank lattice of $\mathbb{C}^g$. So the quotient $\mathbb{C}^g/\Lambda$ is a complex torus.

Proposition 2.1.1. *The Jacobian variety $J(X)$ is isomorphic to $\mathbb{C}^g/\Lambda$.*

Let $Div^0(X)$ be the group of the divisors of degree zero on X. Then there is a group homomorphism

$$A : Div^0(X) \to J(X), \tag{2.5}$$

called the *Abel-Jacobi map* obtained by sending a degree-zero divisor $D = \sum_i (p_i - q_i)$ to the point on the torus

$$\Big(\sum_i \int_{q_i}^{p_i} \omega_1, \dots, \sum_i \int_{q_i}^{p_i} \omega_g\Big). \tag{2.6}$$

Then the map is well-defined since by choosing different paths joining q_i to p_i, or permuting the order of the positive and negative part of the divisor, the integral (2.6) differs by an integral combination of $\int_{\gamma_i}$ and $\int_{\delta_i}$.

2.1.2 Abel's Theorem

We can define an equivalence relation on the group $Div(X)$ of divisors on X by $D \sim E$ if and only if $D = E + (f)$, where (f) is the divisor associated to a meromorphic function f on X. Such a divisor is called a principal divisor, and its degree is always zero by Residue Theorem. We denote the set of all principal divisors by $PDiv^0(X)$, which forms a subgroup of $Div^0(X)$. One defines the divisor class group $Div^0(X)/PDiv^0(X)$. It is isomorphic to the group $Pic^0(X)$ of equivalent classes of line bundles of degree zero.

Proposition 2.1.2. *The Abel-Jacobi image of a principal divisor is zero.*

Proof. A principal divisor (f) can be seen as the pullback $f^{-1}(0) - f^{-1}(\infty)$ from a holomorphic map $f : X \to \mathbb{P}^1$. On the other hand, $\psi_t = [x_0 : -tx_0 + x_1]$ defines a $\mathbb{P}^1$-family of morphism from $\mathbb{P}^1 \to \mathbb{P}^1$, which is identity at $t = 0$ and has constant value ∞ at $t = \infty$. So $f_t := \psi_t \circ f$ defines a rational family of degree zero divisors $f_t^{-1}(0) - f_t^{-1}(0) = f^{-1}(t) - f^{-1}(\infty)$ parameterized by $t \in \mathbb{P}^1$. Restrict the Abel-Jacobi map to this family, it defines a homomorphic map $F : \mathbb{P}^1 \to J(X)$. Since $\pi_1(\mathbb{P}^1) = 0$ so F factors through the universal covering map $\mathbb{C}^g \to \mathbb{C}^g/\Lambda = J(X)$. However, any holomorphic map $\tilde{F} : \mathbb{P}^1 \to \mathbb{C}^g$ has to be constant as a result of the maximum modulus principle. It follows that F is a constant map. Finally, at $t = \infty$, it is the zero divisor, whose image under Abel-Jacobi map is zero, so the Abel-Jacobi image of the divisor at $t = 0$ has also to be zero. □

Abel's Theorem claims the converse.

Theorem 2.1.3. *(Abel) If $A(D) = 0$, then D is a principal divisor.*

We will follow the proof in [29, p.232] and [31, Chapter 5]. First, for a meromorphic function f, its divisor can be expressed as $(f) = \sum_i (p_i - q_i)$, or $\sum_k n_j x_k$ with x_k's being distinct points and $\sum_k n_k = 0$. Then the meromorphic 1-form

$$\phi = \frac{1}{2\pi\sqrt{-1}} d\log(f) = \frac{1}{2\pi\sqrt{-1}} \frac{df}{f},$$

satisfying the following properties:

$$\begin{aligned} &(a)\ (\phi)_\infty = \sum_i x_i, \text{namely, } \phi \text{ has simple poles along each } x_i \text{ for all } i; \\ &(b)\ Res_{x_i}(\phi) = \frac{n_i}{2\pi\sqrt{-1}} \text{ for all } i; \\ &(c) \int_{\gamma_j} \phi,\ \int_{\delta_j} \phi \in \mathbb{Z},\ j = 1, \ldots, g. \end{aligned} \tag{2.7}$$

Conversely, we have the following.

Proposition 2.1.4. *let ϕ be any meromorphic 1-form on X satisfying the three properties in* (2.7)*, then by choosing $q \in X$ a point different from x_i for all i,*

$$f(p) = \exp(2\pi\sqrt{-1} \int_q^p \phi) \tag{2.8}$$

is a well-defined meromorphic function on X with $(f) = \sum_i n_i x_i$.

Proof. First f is a well-defined holomorphic function on $X \setminus \cup_i \{x_i\}$: the difference of integral along two paths l, l' joining q to p is a integral combination of $\int_{\gamma_i} \phi$, $\int_{\delta_i} \phi$ and n_i, so is an integer by assumption. Since the $\exp(2\pi\sqrt{-1}n) = 1$ for any integer n, the function f is well-defined holomorphic function on X as long as p is not on the support of $(\phi)_\infty$.

Next, let z be a local coordinate of x_i with $z(x_i) = 0$, and let q_0 be a base point around x_i and $z(q_0) = z_0 \neq 0$, then $\phi = \frac{n_i}{2\pi\sqrt{-1}} \frac{dz}{z} + h(z)dz$, where $h(z)$ is a holomorphic function. Then

$$\begin{aligned} f(z) &= \exp\big(2\pi\sqrt{-1} \int_q^p \phi\big) \\ &= \exp\big(2\pi\sqrt{-1} \int_q^{q_0} (\phi + \int_{q_0}^p \phi)\big) \\ &= \exp\big(2\pi\sqrt{-1}(\int_q^{q_0} \phi + \int_{z_0}^z \frac{n_i}{2\pi\sqrt{-1}} \frac{dz}{z} + \int_{z_0}^z h(z)dz)\big). \end{aligned}$$

Since $\int_{z_0}^{z} \frac{dz}{z} = \ln(z) - \ln(z_0)$, we obtain

$$f(p) = z^{n_i} H(z),$$

where $H(z)$ is a non-vanishing holomorphic function, so f is meromorphic, and $(f) = \sum_i n_i x_i$.

$\square$

Next, we will prove an integral formula called *Reciprocity Law*. We first pick a symplectic basis γ_i, δ_j, $1 \leq i, j \leq g$ for $H_1(X, \mathbb{Z})$ as in (2.1). Realize these cycles as closed loops. Cutting along these loops which are also denoted as γ_i, δ_i, we get a polygon P with $4g$ edges ordered as $\{\gamma_1, \delta_1, \gamma_1^{-1}, \delta_1^{-1}, \ldots, \gamma_g, \delta_g, \gamma_g^{-1}, \delta_g^{-1}\}$. The compact Riemann surface $X = P/\sim$ is obtained by gluing edges γ_i to γ_i^{-1}, and δ_i to δ_i^{-1} with the specified orientation for each $i = 1, \ldots, g$. We have $X \setminus \cup_i(\gamma_i \cup \delta_i)$ is identified with the interior P° of P.

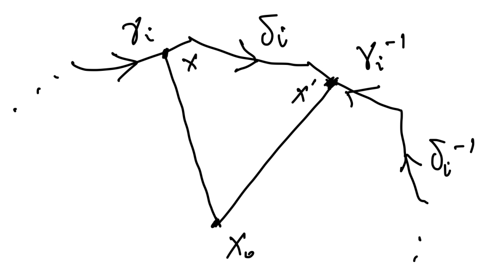


Figure 2.1: Polygon Model of Compact Riemann Surfaces

Fix a base point $x_0 \in P^\circ$. For $\omega \in H^0(X, \Omega_X)$, the integral $u(x) = \int_{x_0}^{x} \omega$ defines a holomorphic function on P^0 and extends continuously to the entire polygon P. Then $du = \omega$. For any $x \in \gamma_i$ and the corresponding point $x' \in \gamma_i^{-1}$ identified on X, by Cauchy integral formula, one has

$$u(x) - u(x') = -\int_{\delta_i} \omega. \tag{2.9}$$

Similarly, for any $x \in \delta_i$ and $x' \in \delta_i^{-1}$ identified on X, one has

$$u(x) - u(x') = \int_{\gamma_i} \omega. \tag{2.10}$$

Let η be a meromorphic 1-form on X. Assume that all poles $\{x_i\}_{i=1}^N$ of η are simple and are contained in $P^\circ \setminus \{x_0\}$. Then by Residue theorem

$$\int_{\partial P} u\eta = 2\pi\sqrt{-1} \sum_{k=1}^{N} Res_{x_k}(\eta) \int_{x_0}^{x_k} \omega. \tag{2.11}$$

On the other hand, by (2.9) and (2.10), one can compute the integral $\int_{\partial\Delta} u\eta$ explicitly:

$$\begin{aligned}\int_{\partial\Delta} u\eta &= \sum_i \int_{\gamma_i + \gamma_i^{-1}} u\eta + \sum_i \int_{\delta_i + \delta_i^{-1}} u\eta \\ &= -\sum_i \int_{\delta_i} \omega \int_{\gamma_i} \eta + \sum_i \int_{\gamma_i} \omega \int_{\delta_i} \eta.\end{aligned}$$

Together with (2.11), we obtain

Proposition 2.1.5. *(Reciprocity Law)*

$$\sum_{i=1}^{g} \Big(\int_{\delta_i} \omega \int_{\gamma_i} \eta + \int_{\gamma_i} \omega \int_{\delta_i} \eta \Big) = 2\pi\sqrt{-1} \sum_{k=1}^{N} Res_{x_k}(\eta) \int_{x_0}^{x_k} \omega. \tag{2.12}$$

Apply to η being a holomorphic 1-form and its conjugation (adopting the proof of Proposition 2.1.5 for $e\bar{t}a$), one obtains the Riemann bilinear relations (2.4).

Now we're ready to prove Abel's Theorem.

Proof of Theorem 2.1.3. According to Proposition 2.1.4, it suffices to construct a meromorphic one form ϕ satisfying the three properties in (2.7).

Step 1. We'll first construct meromorphic 1-form ϕ satisfying condition (a) and (b) in (2.7). This is a Mittag-Leffler type problem, and such ϕ is called a *differential of the third kind.* Suppose $D = \sum_{i=1}^{d} (p_i - q_i)$ for some $d \in \mathbb{Z}_+$. We'll show that for each i, there is a meromorphic one form ϕ_i such that

$$(a)\ (\phi_i)_\infty = p_i + q_i;$$
$$(b)\ Res_{p_i}(\phi_i) = \frac{1}{2\pi\sqrt{-1}}, \quad Res_{q_i}(\phi_i) = -\frac{1}{2\pi\sqrt{-1}}. \tag{2.13}$$

The existence of such ϕ_i is guaranteed by Riemann-Roch theorem: $\dim H^0(X, \Omega_X(p_i+q_i)) = 2g+1-g+\dim H^0(X, \Omega_X(-p_i-q_i)) = g+1$ whenever $p_i \neq q_i$. So $H^0(X, \Omega_X(p_i+q_i))$ contains $H^0(X, \Omega_X)$ as a proper subspace. By choosing a generic $\tilde{\phi}_i \in H^0(X, \Omega_X(p_i+q_i))$, then $Res_{p_i}(\tilde{\phi}_i)$ is nonzero and has opposite sign to $Res_{q_i}(\tilde{\phi}_i)$. Normalizing by a nonzero constant, we obtain ϕ_i satisfying properties (a) and (b) in (2.13).

Then $\phi = \sum_{i=1}^d \phi_i$ is a meromorphic 1-form satisfying condition (a) and (b) in (2.7).

Step 2. Now, the missing part is to include the condition (c) in (2.7). We need to adjust ϕ by adding a holomorphic 1-form to have integral periods. Note that by doing this, the conditions (a) and (b) in (2.7) will be preserved. Also, since $H^{1,0}(X) \cap H^1(X, \mathbb{Z}) = \{0\}$, this adjustment will be unique.

First of all, we adjust ϕ by a holormorphic 1-form so that it has no γ-periods. Recall $\omega_1, \ldots, \omega_g$ is a normalized basis of $H^0(X, \Omega_X)$ with respect to the symplectic basis of $H_1(X, \mathbb{Z})$ by equations (2.2). We define

$$\phi' = \phi - \sum_{i=1}^g (\int_{\gamma_i} \phi)\omega_i.$$

Then we have

$$\int_{\gamma_i} \phi' = 0,\ i = 1, \ldots, g. \tag{2.14}$$

Clearly ϕ' satisfy the same conditions (a) and (b) in (2.7) as ϕ'. We will further adjust ϕ to make its δ-periods integral and ensure the integrality of γ-periods. We can assume that the assumption of Proposition 2.1.5 is satisfied. Namely, all poles of ϕ' are contained in the interior of the $4g$-polygon P and distinct from a prescribed base point x_0 by choosing the symplectic basis suitably. Now apply the Reciprocity Law (2.12) to ϕ' and ω_i, use the identities (2.2) and (2.14) we get

$$\sum_k \int_{q_k}^{p_k} \omega_i = \int_{\delta_i} \phi'. \tag{2.15}$$

Note that the left-hand side of (2.15) is the i-th coordinate of the Abel-Jacobi image lifted to $\mathbb{C}^g$, which by assumption is an integral linear combination

$$a_i + \sum_{j=1}^{g} b_j \int_{\delta_j} \omega_i, \tag{2.16}$$

where $a_1, \ldots, a_g, b_1, \ldots, b_g \in \mathbb{Z}$. Since the period matrix is symmetric (2.4), one has $\int_{\delta_j} \omega_i = \int_{\delta_i} \omega_j$. Combine (2.15) and (2.16), one obtains the identity

$$\int_{\delta_i} \phi' = a_i + \int_{\delta_i} \big(\sum_{j=1}^{g} b_j \omega_j\big), \ i = 1, \ldots, g. \tag{2.17}$$

Now we modify ϕ' by defining

$$\phi'' = \phi' - \sum_{j=1}^{g} b_j \omega_j.$$

It is immediate to check that the periods

$$\int_{\gamma_i} \phi'' = b_i, \text{ and } \int_{\delta_i} \phi'' = a_i$$

are integers for all $i = 1, \ldots, g$, so all the conditions in (2.7) are satisfied for ϕ''. By Proposition 2.1.4, $\exp(2\pi\sqrt{-1}\int_{x_0}^{x} \phi'')$ defines a meromorphic function whose associated divisor is D. $\square$

2.1.3 Mixed Hodge Structure Interpretation

In this section, we'll see another description of Abel's theorem using extensions of certain mixed Hodge structures.

X is again a compact Riemann surface, and $D = \sum_i (p_i - q_i)$ a divisor of degree zero. Let $\mathbb{Z} = \mathbb{Z}[D]$ be the free $\mathbb{Z}$-module formally generated by D. We can consider an exact sequence of $\mathbb{Z}$-modules:

$$0 \to H^1(X, \mathbb{Z}) \to E \xrightarrow{r} \mathbb{Z} \to 0. \tag{2.18}$$

Since $H^1(X,\mathbb{Z})$ is free of rank $2g$ and $Ext^1_{\mathbb{Z}}(\mathbb{Z},\mathbb{Z}^{2g}) = 0$, E is isomorphic to $\mathbb{Z}^{2g+1}$ as $\mathbb{Z}$-modules.

Now we complexify the sequence and consider Hodge filtrations: Since $H^1(X,\mathbb{C})$ has a complex subspace $F^1H^1(X,\mathbb{C}) = H^{1,0}(X)$, we want to define a subspace $F^1E_{\mathbb{C}}$ of $E_{\mathbb{C}}$ such that the sequence

$$0 \to F^1H^1(X,\mathbb{C}) \to F^1E_{\mathbb{C}} \to \mathbb{C} \to 0 \tag{2.19}$$

is exact. This amounts to picking a element $\tilde{D} \in E_{\mathbb{C}}$, such that $r(\tilde{D}) = D$: The span of $\tilde{D}$ with $F^1H^1(X,\mathbb{C})$ will define $F^1E_{\mathbb{C}}$. Conversely, any $g+1$-dimensional subspace of $E_{\mathbb{C}}$ containing $F^1H^1(X,\mathbb{C})$ as a g-dimensional subspace will produce a sequence (2.19). The sequence (2.18) and (2.19) together define a mixed Hodge structure on E.

Putting more concretely, $E \cong \mathbb{Z}^{2g+1}$ is a free abelian group with a basis

$$\{\gamma_1^*, \ldots, \gamma_g^*, \delta_1^*, \ldots, \delta_g^*, D\}. \tag{2.20}$$

There is a g-dimensional subspace of $\mathbb{C}^{2g+1} = \mathbb{Z}^{2g+1} \otimes_{\mathbb{Z}} \mathbb{C}$ whose first $2g$ coefficents in the basis (2.20) above are given by the row vectors v_i of $[Id|\Lambda]$ in (2.3), and the last coefficient is 0. The span of $v_1, \ldots, v_g$ is $F^1H^1(X,\mathbb{C})$.

By Riemann-Roch theorem, there is a meromorphic 1-form ϕ on X with poles of the first order along p_i, q_i's and satisfies the condition (a) and (b) in (2.7). Any two choices of such meromorphic 1-form differ by a holomorphic 1-form. The meromorphic 1-form ϕ defines a class $[\phi]$ in $H^1(U,\mathbb{C})$. Express the class in terms of the basis (2.20), one has

$$[\phi] = c_1\gamma_1^* + \cdots c_g\gamma_g^* + d_1\delta_1^* + \cdots + d_g\delta_g^* + D,$$

where $c_i = \int_{\gamma_i} \phi$ and $d_i = \int_{\delta_i} \phi$ for $i = 1, \ldots, g$.

Now, from the point of view of mixed Hodge structure, the class $[\phi]$ mentioned above defines a $(g+1)$-dimensional $\mathbb{C}$-vector subspace of $\mathbb{C}^{2g+1}$ and fits into the exact sequence (2.19), therefore it follows that

Proposition 2.1.6. *A degree zero divisor $D = \sum_i (p_i - q_i)$ on a compact Riemann surface X defines a extension of mixed Hodge structure*

$$0 \to H^1(X, \mathbb{Z}) \to E_\phi \to \mathbb{Z} \to 0, \tag{2.21}$$

where ϕ is a meromorphic 1-form with poles of first order and satisfies that $Res(\phi) = D$.

In our proof of Theorem 2.1.3 earlier, the condition that $A(D) = 0$ implies the existence of meromorphic 1-form satisfying (2.7). Equivalently, it states that

$$[c_1, \dots, c_g, d_1, \dots, d_g] \in \mathbb{Z}^{2g} \mod \mathbb{C}v_1 \oplus \cdots \oplus \mathbb{C}v_g. \tag{2.22}$$

In fact, there is a unique meromorphic 1-form whose periods c_i, d_i for all $i = 1, \dots, g$ are integers. Our proof in Theorem 2.1.3 is based on finding such meromorphic 1-form explicitly.

The condition (2.22) will imply that the Hodge filtration $F^1E_\phi(\mathbb{C})$ of $E_\phi(\mathbb{C})$ obtained in the sequence (2.21) is defined over $\mathbb{Z}$. Therefore we obtain a Hodge-theoretical statement of Abel's theorem:

Theorem 2.1.7. *(Abel's theorem, second version) If the Abel-Jacobi image of D is zero, then the extension of mixed Hodge structure* (2.21) *is trivial (splits over $\mathbb{Z}$).*

We will provide proof of this theorem in the next section.

2.1.4 Jacobi Inversion Theorem

Theorem 2.1.8. *(Jacobi Inversion) The Abel-Jacobi map* (2.5) *is surjective.*

Theorem 2.1.3 and Theorem 2.1.8 together imply that

Corollary 2.1.9. *The Abel-Jacobi map* (2.5) *induces an isomorphism*

$$\mathcal{A} : Pic^0(X) \cong J(X). \tag{2.23}$$

Note that the isomorphism is taken as groups. However, since we are working in complex field, the exponential sequence $0 \to \mathbb{Z} \to \mathcal{O} \to \mathcal{O}^* \to 0$ induces a long exact sequence which produces an isomorphism $Pic^0(X) = \ker\big(H^1(X, \mathcal{O}^*) \to H^2(X, \mathbb{Z})\big) \cong H^1(X, \mathcal{O})/H^1(X, \mathbb{Z})$. Dolbeault's theorem implies that $H^1(X, \mathcal{O}) \cong H^{0,1}(X)$. So $Pic^0(X) \cong H^{0,1}(X)/H^1(X, \mathbb{Z})$ has complex structure. Therefore (2.23) also preserves complex structures, so is an isomorphism of abelian varieties.

2.2 Abel-Jacobi maps via Mixed Hodge Structures

2.2.1 Mixed Hodge Structures

We first review Deligne's theory of mixed Hodge structures [20].

Definition 2.2.1. *A mixed Hodge structure is a triple $H = (H_{\mathbb{Z}}, W, F)$, where $H_{\mathbb{Z}}$ is a free abelian group, $W_\bullet H_{\mathbb{Z}}$ is an increasing weight filtration and $F^\bullet H_{\mathbb{C}}$ is a decreasing Hodge filtration such that each graded piece $\mathrm{Gr}_i^W H_{\mathbb{C}}$ is a pure Hodge structure of weight i, where $H_{\mathbb{C}} := H_{\mathbb{Z}} \otimes \mathbb{C}$.*

Of course, a pure Hodge structure of weight k (e.g., $H^k(X, \mathbb{Z})$ for a Kähler manifold X) is a mixed Hodge structure with weight concentrated on level k. Here are some nontrivial examples.

Example 2.2.2. *Let U be a smooth variety of dimension n. Then the singular cohomology $H^k(U, \mathbb{Z})$ carries a canonical mixed Hodge structure with weights concentrated on levels between k and $2k$.*

Example 2.2.3. *Let X be a proper variety of dimension n (which can be singular). Then the singular cohomology $H^k(X, \mathbb{Z})$ carries a canonical mixed Hodge structure with weights concentrated on levels between 0 and k.*

It's natural to define morphism of mixed Hodge structures as morphism between free abelian groups that preserve both filtrations. Deligne showed that mixed Hodge structure forms a category compatible with the morphism between algebraic varieties. For example, when X is a surface with an A_1 singularity, and $\tilde{X} \to X$ is the blowup at the singularity. Denote E the exceptional divisor, then the exact sequence of pairs $(\tilde{X}, E)$ can be used to compute mixed Hodge structure on $H^1(X)$.

For smooth quasi-projective U, one can first complete it in projective space and obtain $X = \bar{U}$, then take log resolution $\pi : \tilde{X} \to X$. Up to a suitable base change, $\tilde{X} \setminus \pi^{-1}(U)$ is a simple normal crossing divisor, which allows one to express the weight filtrations on cohomologies on U explicitly via log complex [61, Section 8.4].

Let H be a mixed Hodge structure with w being the highest weight, then its dual space $H^\vee$ admits a dual mixed Hodge structure that has weight filtration defined as

$$W_k H_{\mathbb{Z}}^\vee = \{\phi \in H_{\mathbb{Z}}^\vee | \phi(W_{2w-k-1} H_{\mathbb{Z}}) = 0\}. \tag{2.24}$$

Note that our convention shifts the weight up by $2w$ from the standard definition. This is convenient for geometric purposes.

The Hodge filtration of the dual space $H^\vee$ is defined as

$$F^p H_{\mathbb{C}}^\vee = \{\phi \in H_{\mathbb{C}}^\vee | \phi(F^{w-p+1} H_{\mathbb{C}}) = 0\}. \tag{2.25}$$

2.2.2 Carlson's Extension of Mixed Hodge Structures

In this section, we briefly recall Carlson's theorem on the extension of mixed Hodge structures, [13] and in the next section, we will relate it to Griffiths's Abel-Jacobi map.

Let A, B be two mixed Hodge structure with $B > A$, i.e., $W_n A = A$, and $W_n B = 0$ for some $n \in \mathbb{Z}$. We can consider $Ext^1_{MHS}(B, A)$ consisting the extensions of B by A as MHSs with E_1 and E_2 equivalent if there is a commutative diagram.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A & \longrightarrow & E_1 & \longrightarrow & B & \longrightarrow & 0 \\
 & & \downarrow = & & \downarrow \cong & & \downarrow = & & \\
0 & \longrightarrow & A & \longrightarrow & E_2 & \longrightarrow & B & \longrightarrow & 0
\end{array} \tag{2.26}$$

Notation 2.2.4. *We will use A, B referring to both the mixed Hodge structures and its complex vector spaces. We use $A_{\mathbb{Z}}$, etc., to denote the underlying $\mathbb{Z}$-structure.*

On the other hand, the abelian group $Hom_{\mathbb{Z}}(B_{\mathbb{Z}}, A_{\mathbb{Z}})$ admits a mixed Hodge structure with weight filtration $W_l Hom_{\mathbb{Z}}(B, A) = \{\phi |\ \phi(W_k B) \subset W_{l+k} A, \forall k\}$ and Hodge filtrations given by $F^p Hom_{\mathbb{C}}(B, A) = \{\phi |\ \phi(F^k B) \subset F^{p+k} A, \forall k\}$. In particular $F^0 Hom_{\mathbb{C}}(B, A)$ is the set of complex linear maps preserving Hodge filtrations, and it is "a half" of the filtration. We define the intermediate Jacobian $J^0 Hom(B, A)$ to be the complex torus

$$\frac{Hom_{\mathbb{C}}(B, A)}{F^0 Hom_{\mathbb{C}}(B, A) + Hom_{\mathbb{Z}}(B_{\mathbb{Z}}, A_{\mathbb{Z}})}.$$

Theorem 2.2.5. *(Carlson [13]) With the assumption above, there is a natural isomorphism*

$$Ext^1_{MHS}(B, A) \cong J^0 Hom(B, A). \tag{2.27}$$

Proof. On the one hand, if there is an extension

$$0 \to A \xrightarrow{\alpha} E \xrightarrow{\beta} B \to 0 \tag{2.28}$$

of mixed Hodge structures, one associate a section $s_F : B \to E$ preserving Hodge filtration and a retraction $r_{\mathbb{Z}} : E \to A$ preserving integral structure. Here being a section means $\beta \circ s_F = Id_B$ and a retraction means $r_{\mathbb{Z}} \circ \alpha = Id_A$. The composite $e = r_{\mathbb{Z}} \circ s_F$ defines an element in $Hom(B, A)$. Different choice of s_F differ by an element in $F^0 Hom_{\mathbb{C}}(B, A)$ and different choice of $r_{\mathbb{Z}}$ differ by an element in $Hom_{\mathbb{Z}}(B_{\mathbb{Z}}, A_{\mathbb{Z}})$, so e defines a class $[e]$ in $J^0 Hom(B, A)$.

Conversely, given any representative $e \in Hom(B, A)$ of an element $[e] \in J^0 Hom(B, A)$, we define E_e abstractly as the mixed Hodge structure with underlying $\mathbb{Z}$-structure as $A_{\mathbb{Z}} \oplus B_{\mathbb{Z}}$ and define the Hodge filtration

$$F^p E_e := \{(a, b) \in A \oplus B | b \in F^p B, a - e(b) \in F^p A\}. \tag{2.29}$$

Then the fiber of $F^p E_e \to F^p B$ is the affine space of $e(b) + F^p A$. So if $f \in F^0 Hom_{\mathbb{C}}(B, A)$, $F^p E_{e+f} = F^p E_e$ for all p. If $g \in Hom_{\mathbb{Z}}(B_{\mathbb{Z}}, A_{\mathbb{Z}})$, then take an automorphism σ on $A_{\mathbb{Z}} \oplus B_{\mathbb{Z}}$ such that σ is identity on $A_{\mathbb{Z}}$-summand and sends $(0, b)$ to $(-g(b), b)$. The complexfication $\sigma_{\mathbb{C}}$ will sends $F^p E_{e+g}$ isomorphically onto $F^p E_e$ for all p. So (2.29) defines a mixed Hodge structure E_e which is independent of choice of representative of $e \in [e]$.

Finally, we should show that two identifications are inverse to each other. Suppose we have an extension sequence (2.28). Then choosing section s_F and retraction $r_{\mathbb{Z}}$, and let $e = r_{\mathbb{Z}} \circ s_F$, we have a mixed Hodge structure E_e with Hodge filtration defined by (2.29). Then we claim that the morphism $\phi : E \to E_e$, $u \mapsto (r_{\mathbb{Z}}(u), \beta(u))$ defines an equivalence of extensions of mixed Hodge structures in the sense of (2.26). To see this, first ϕ is an isomorphism over $\mathbb{Z}$ and commutes with α and β. It suffices to show it preserves Hodge filtration: Suppose $u \in F^p E$, then $\phi(u) = (r_{\mathbb{Z}}(u), \beta(u))$. Since $u - s_F(\beta(u)) \in \alpha(F^p A)$ by exactness, we have $r_{\mathbb{Z}}(u) - e(\beta(u)) = r_{\mathbb{Z}}(u - s_F(\beta(u))) \in F^p A$, so $(r_{\mathbb{Z}}(u), \beta(u)) \in F^p E_e$ as desired. The identification from the other direction is straightforward. □

Definition 2.2.6. $[e] = [r_{\mathbb{Z}} \circ s_F] \in J^0 Hom(B, A)$ *is called the extension class of the sequence* (2.28).

Corollary 2.2.7. *An extension of mixed Hodge structure* (2.28) *is trivial if and only if its extension class* $[e]$ *is zero, if and only if* $e = r_{\mathbb{Z}} \circ s_F \in Hom_{\mathbb{Z}}(B_{\mathbb{Z}}, A_{\mathbb{Z}}) + F^0 Hom(B, A)$.

Example 2.2.8. *When* $B = H^{2k-1}(X)$ *for* X *a smooth projective variety, and* $A = \mathbb{Z}(-k+1)$ *is a 1-dimensional pure Hodge structure of weight* $2k-2$ *with type* $(k-1, k-1)$, *Theorem*

2.2.5 implies that the equivalent classes of extensions of B by A is

$$Ext^1_{MHS}(H^{2k-1}(X), \mathbb{Z}(-k+1)) \cong H^{2k-1}(X, \mathbb{C})/F^k H^{2k-1}(X) + H^{2k-1}(X, \mathbb{Z})$$

is an intermediated Jacobian of X.

In particular, when X is a curve, the description above relates the Jacobian variety $J(X)$ to an extension of mixed Hodge structure, which allows us to define the Abel-Jacobi map out of comparing weight and Hodge filtration on $H^1(U)$, where $U \subseteq X$ is an affine open subspace.

2.2.3 Abel-Jacobi Map for Curves via Mixed Hodge Structures

In this section, we'll see how to define the Abel-Jacobi map via extensions of certain mixed Hodge structures for algebraic curves.

Let X be a smooth projective curve, and let $Y = \{p_1, \ldots, p_m\}$ be a finite collection of points. Denote $U = X \setminus Y$ be the complement. There is a long exact sequence of the cohomology of pair (X, U)

$$H^1(X, U) \to H^1(X) \to H^1(U) \to H^2(X, U) \to H^2(X). \tag{2.30}$$

The coefficients can be $\mathbb{Z}, \mathbb{Q}, \mathbb{R}$ or $\mathbb{C}$. All morphisms in (2.30) preserve mixed Hodge structures, so the sequence (2.30) is also exact in the category of mixed Hodge structures. Using the Thom isomorphism $H^k(X, U) \cong H^{k-2}(Y)$, we truncate and obtain a short exact sequence of mixed Hodge structures

$$0 \to H^1(X) \to H^1(U) \xrightarrow{r} H^0_{\text{van}}(Y) \to 0, \tag{2.31}$$

where $H^0_{\text{van}}(Y) = \ker(H^0(Y) \to H^2(X))$ is spanned by $p_i - p_{i+1}$, $1 \leq i \leq m-1$. and r is the residue map, which associate a meromorphic 1-form ϕ on X with poles along Y to the numbers $Res_{p_i}(\phi)$ for each $p_i \in Y$.

$H^1(U)$ carries a canonical mixed Hodge structure by Deligne. It has weight filtrations: $W_1H^1(U,\mathbb{Z}) = H^1(X,\mathbb{Z}) \subseteq W_2H^1(U,\mathbb{Z}) = H^1(U,\mathbb{Z})$ concentrated on level 1 and 2. The residue map r induces isomorphism $\bar{r} : H^1(U,\mathbb{Z})/W_1H^1(U,\mathbb{Z}) \cong \mathbb{Z}$, where degree shift by $(-1,-1)$. The Hodge filtrations are given by $F^1H^1(U,\mathbb{C}) \subseteq F^0H^1(U,\mathbb{C}) = H^1(U,\mathbb{C})$. $F^1H^1(U,\mathbb{C})$ are represented by meromorphic 1-forms with poles along Y.

We'll construct a mixed Hodge structure from the extension sequence (2.31).

For each $\alpha \in H^0_{\text{van}}(Y)$, the exactness of (2.31) in the category of mixed Hodge structures guarantees that then there is a class $\tilde{\alpha}_F \in F^1H^1(U)$ and a class $\tilde{\alpha}_{\mathbb{Z}} \in H^1(U,\mathbb{Z})$, such that $r(\tilde{\alpha}_F) = \alpha$ and $r(\tilde{\alpha}_{\mathbb{Z}}) = \alpha$.

Definition 2.2.9. *We call $\tilde{\alpha}_F$ a F-lift (or a Hodge lift) of α, and $\tilde{\alpha}_{\mathbb{Z}}$ a $\mathbb{Z}$-lift (or integral lift) of α.*

If there is another such F-lift $\tilde{\alpha}'_F$, then by exactness of (2.31) on Hodge filtrations, $\tilde{\alpha}_F - \tilde{\alpha}'_F \in F^1H^1(X)$. Similarly, if there is another $\mathbb{Z}$-lift $\tilde{\alpha}'_{\mathbb{Z}}$, then by exactness of (2.31) in $\mathbb{Z}$-coefficent, $\tilde{\alpha}_{\mathbb{Z}} - \tilde{\alpha}'_{\mathbb{Z}} \in H^1(X,\mathbb{Z})$. It follows that $r(\tilde{\alpha}_{\mathbb{Z}} - \tilde{\alpha}_F) = 0$, so $\tilde{\alpha}_{\mathbb{Z}} - \tilde{\alpha}_F \in H^1(X,\mathbb{C})$. Modulo freedom of choices of lifts, we have

Proposition 2.2.10. *The class $\tilde{\alpha} := \tilde{\alpha}_{\mathbb{Z}} - \tilde{\alpha}_F$ is well-defined in*

$$H^1(X,\mathbb{C})/F^1H^1(X,\mathbb{C}) + H^1(X,\mathbb{Z}). \tag{2.32}$$

Note that $H^1(X,\mathbb{C})/F^1H^1(X,\mathbb{C}) \cong H^{0,1}(X)$, so the torus (2.32) is isomorphic to the Picard torus $Pic^0(X)$. Via the identification (2.23), we can compare the class $\tilde{\alpha}$ to the Abel-Jacobi image $A(\alpha)$.

Lemma 2.2.11. *([25, Lemma 20.5]) Suppose p and q are two distinct points on X. Let l_{pq} be a path joining q to p, and let V a simply-connected open neighborhood of l_{pq} in X, then there is a function f smooth on $X \setminus \{q\}$ and satisfies that* *(i)* $f|_{X\setminus V} \equiv 1$*;*

(ii) f *coincides with* $h_1 z$ *and* $h_2 z^{-1}$ *in an open neighborhood* $V_p \subseteq V$ *of* p *and* $V_q \subseteq V$ *of* q *respectively, where* h_1 *and* h_2 *some non-vanishing holomorphic functions.*

Proof. To prove it, one first consider the case where $(\mathcal{U}, z)$ is a coordinate on X such that $z(\mathcal{U}) \subseteq \mathbb{C}$ is the unite disk and the path l_{pq} lies entirely in $\mathcal{U}$. For simplicity identify $\mathcal{U}$ with the unit disk.

Let $a := z(q)$ and $b := z(p)$. Then there exist $r < 1$ such that $z(l_{pq}) \subsetneq \{|z| < r\}$. The function $\log(\frac{z-b}{z-a})$ has a well-defined branch in $\{r < |z| < 1\}$. Choose a smooth function ψ on $\mathcal{U}$ such that $\psi|_{|z|\leq r} \equiv 1$ and $\psi|_{|z|\geq r'} \equiv 0$, where $r < r' < 1$ and define a $f_0 \in \mathcal{C}^\infty(\mathcal{U} \setminus \{a\})$ by

$$f_0 = \begin{cases} \exp(\psi \log(\frac{z-b}{z-a})), & \text{if } r < |z| < 1, \\ \frac{z-b}{z-a}, & \text{if } |z| \leq r. \end{cases}$$

Since $f_0|_{r'<|Z|<1} = 1$, one can continuously extend f_0 to a function $f \in \mathcal{C}^\infty(X \setminus \{a\})$, by defining it to be 1 on $X \setminus V$.

In the general case, let $c : [0,1] \to X$ be a parameterization of the path l_{pq} with $c(0) = q$ and $c(1) = p$. There exists a partition

$$0 = t_0 < t_1 < \cdots < t_n = 1$$

of the interval $[0,1]$ and coordinate neighborhoods $(\mathcal{U}_j, z_j)$, $j = 1, \ldots, n$, on X with the following properties:

(i) $c([t_{j-1}, t_j]) \subseteq \mathcal{U}_j \subseteq V$,

(ii) $z_j(\mathcal{U}_j) \subseteq \mathbb{C}$ is the unit disk.

Then based on the previous discussion, for each j, one can construct a function f_j satisfying (i) and (ii) with respect to the path $c([t_{j-1}, t_j])$. The product $f := f_1 \cdots f_n$ will satisfy the desired condition. □

Lemma 2.2.12. *Let p and q be two distinct points on a compact Riemann surface X, and let f be a function obtained from Lemma 2.2.11. Then the class represented by differential 1-form $\frac{1}{2\pi\sqrt{-1}}df/f$ is an integral lifting of $p-q$. In other words, $[\frac{1}{2\pi\sqrt{-1}}df/f] \in H^1(U,\mathbb{Z})$ and $r([\frac{1}{2\pi\sqrt{-1}}df/f]) = p-q$ via the residue map in* (2.31).

Proof. By construction of f in Lemma 2.2.11, the condition (i) implies that the 1-form $\frac{1}{2\pi\sqrt{-1}}df/f \equiv 0$ on $X \setminus V$. The condition (ii) implies that $\frac{1}{2\pi\sqrt{-1}}df/f$ has residue 1 and -1 at p and q, respectively, and there is no other poles.

It left to show $[\frac{1}{2\pi\sqrt{-1}}df/f]$ is integral. Since we can choose a symplectic basis $\{\gamma_1,\ldots,\gamma_g,\delta_1,\ldots,\gamma_g\}$ of $H_1(X,\mathbb{Z})$ represented by loops on X that are disjoint from V and Y, choosing c_i be a small loop around p_i with anticlockwise orientation. then $\{\gamma_1,\ldots,\gamma_g,\delta_1,\ldots,\gamma_g,c_1,\ldots,c_{m-1}\}$ is a basis of $H_1(U,\mathbb{Z})$. Assume $p = p_1, q = p_m$, and we can assume c_i is disjoint from V when $i = 2,\ldots,m-1$, then the integral of $\frac{1}{2\pi\sqrt{-1}}df/f$ of all basis elements equal zero except $\frac{1}{2\pi\sqrt{-1}}\int_{c_1} df/f = 1$. So $[\frac{1}{2\pi\sqrt{-1}}df/f] \in H^1(U,\mathbb{Z})$. □

Theorem 2.2.13. *The Abel-Jacobi image $A(\alpha)$ coincides with the class $\mathcal{A}(\tilde{\alpha})$ via the isomorphism* (2.23)

$$\mathcal{A} : H^1(X,\mathbb{C})/F^1H^1(X,\mathbb{C}) + H^1(X,\mathbb{Z}) \cong F^1H^1(X,\mathbb{C})^\vee/H_1(X,\mathbb{Z}), \tag{2.33}$$

which is induced by the intersection pairing on $H^1(X)$.

Proof. Since every class in $H^0_{\mathrm{van}}(Y,\mathbb{Z})$ is an integral linear combination of $p_i - p_j$, we can assume that $\alpha = p - q$, where $p, q \in Y$. Denote $\alpha = p - q$.

By Riemann-Roch theorem, there is a meromorphic 1-form ϕ with first order poles along p and q and such that $Res_p(\phi) = 1$ and $Res_q(\phi) = -1$. In other words, $[\phi] = \tilde{\alpha}_F$ is a F-lift of α. By Lemma 2.2.12, an integral lift $\tilde{\alpha}_{\mathbb{Z}}$ is $[\frac{1}{2\pi\sqrt{-1}}df/f]$.

Now, $\eta := \frac{1}{2\pi\sqrt{-1}}df/f - \phi$ is a smooth closed 1-form on X since the poles cancel out and $[\eta]$ defines the class $\tilde{\alpha} = \tilde{\alpha}_{\mathbb{Z}} - \tilde{\alpha}_F \in H^1(X,\mathbb{C})$.

To prove the theorem, we'll show that for every $\omega \in H^0(X, \Omega_X)$, there is an equality

$$\int_{l_{pq}} \omega = \int_X \eta \wedge \omega. \tag{2.34}$$

Let V' be an open subspace of $X \setminus l_{p,q}$ such that $\{V, V'\}$ is a covering of X and $V \cap V'$ is a regular neighborhood of the loop $\gamma = \partial V$. Fix a point $x_0 \in V \cap V'$, then $u(x) = \int_{x_0}^x \omega$ is a holomorphic function on V. Moreover, the residue theorem implies that $\int_{\partial V} u\phi = u(p) - u(q) = \int_q^p \omega$.

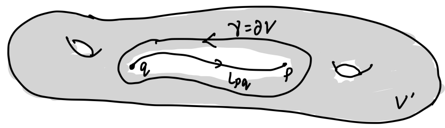


Figure 2.2: V' = shaded area

On the other hand, $g(x) = \int_{x_0}^x (\phi + \eta) = \int_{x_0}^x \frac{1}{2\pi\sqrt{-1}} df/f$ is a well-defined smooth function on V'. The orientation on ∂V is chosen to be anti-clockwise looking from the interior of V. Apply Stokes' theorem on $X \setminus V$ and use the fact that $\partial(X \setminus V) = -\gamma$, we obtain

$$\int_\gamma g\omega = -\int_{X\setminus V} d(g\omega) = -\int_{X\setminus V} (\phi + \eta) \wedge \omega = \int_{X\setminus V} \omega \wedge \eta. \tag{2.35}$$

The last equality is a result of $\phi \wedge \omega = 0$ since ϕ is holomorphic on $X \setminus V$. Then we have identities

$$\begin{aligned} 0 = \int_{\partial V} d(ug) &= \int_{\partial V} g\omega + \int_{\partial V} u(\phi + \eta) \\ &= \int_{X\setminus V} \omega \wedge \eta + \int_q^p \omega + \int_{\partial V} u\eta. \end{aligned}$$

Apply Stokes theorem on $\bar{V}$, we have $\int_{\partial V} u\eta = \int_V \omega \wedge \eta$. Combine with the identity (2.35), we obtain (2.34). (Alternatively, the identify (2.34) can be obtained by [25, Lemma 20.3] together with $\int_X \phi \wedge \omega = 0$ in the sense of improper integral.) □

Corollary 2.2.14. *The Abel-Jacobi image $A(D)$ of a degree zero divisor $D = \sum_i (p_i - q_i)$ is identified via* (2.33) *with the extension class $[e_D]$ of* (2.21)

$$0 \to H^1(X) \to E \to \mathbb{Z} \to 0$$

as a sub exact sequence of (2.31). *In other words, $A(D)$ is identified with the evaluation of the extension class $[e]$ of* (2.31) *at the divisor D.*

Now we are ready to prove the Hodge theoretic version of Abel's theorem introduced in the previous section.

Proof of Theorem 2.1.7. By Corollary 2.2.14, the Abel-Jacobi image of a degree zero divisor $D = \sum_i (p_i - q_i)$ is identified with the extension class $[e]$ of (2.21). So $A(D) = 0$ if and only if $[e] = 0$. □

Similarly, we have a criterion of when is mixed Hodge structure of H^1 on an affine curve is trivial.

Corollary 2.2.15. *The extension class of* (2.31)

$$0 \to H^1(X) \to H^1(X \setminus Y) \xrightarrow{r} H^0_{\mathrm{van}}(Y) \to 0$$

is trivial if and only if the Abel-Jacobi image of $p - q$ is zero for all $p, q \in Y$, where Y is a finite collection of points in X.

2.3 Griffiths' Abel-Jacobi Map

The Abel-Jacobi map for compact Riemann surface is generalized to higher dimensions by Griffiths. Now, let X be a smooth projective variety of dimension n over $\mathbb{C}$. The classical Hodge theorem says that each cohomology $H^k(X, \mathbb{Z})$ carries a Hodge structure of weight k, which means that its complexification admits a Hodge decomposition $H^k(X, \mathbb{C}) \cong H^{k,0} \oplus H^{k-1,1} \oplus \cdots \oplus H^{0,k}$, with $\overline{H^{p,q}} = H^{q,p}$. Denote $F^p H^k(X, \mathbb{C}) = \oplus_{l \geq p} H^{l,k-l}$, then when $k = 2n - 2r + 1$ is an odd number, $H^{2n-2r+1}(X, \mathbb{C})$ is isomorphic to the direct sum of $F^{n-r+1}H^{2n-2r+1}(X, \mathbb{C})$ and its conjugation.

So for each integer $0 < r < n$, there is a compact complex torus $J^{2r-1}(X)$ defined as

$$J^{2r-1}(X) = F^{n-r+1}H^{2n-2r+1}(X, \mathbb{C})^\vee / H_{2n-2r+1}(X, \mathbb{Z}), \tag{2.36}$$

which is called the r-th intermediate Jacobian (or just with the integer omitted in cases where there is no ambiguity, i.e., the middle dimension when $\dim X$ is odd, as we will study later). Also, note that the torus $J^{2r-1}(X)$ is not polarized in general.

Let $\mathscr{Z}^r(X)_{\text{hom}}$ denote the group of algebraic cycles of codimension r on X that are homologous to zero. So there is a $(2n-2r+1)$-chain Γ in X such that $\partial\Gamma = Z$. The integral

$$\int_\Gamma \tag{2.37}$$

defines a linear functional on the space of closed complex differential forms $F^{n-r+1}\mathcal{A}_X^{2n-2r+1} \cap \ker(d)$ with at least $n-r+1$ dz's, where d is the exterior differentiation and $\mathcal{A}_X^k$ is the space of differential k-forms on X. By a fact that the d-exact forms in such space coincides with $d\big(F^{n-r+1}\mathcal{A}_X^{2n-2r}\big)$ [61, Proposition 7.5] (equivalently the strictness of Hodge filtration, or more generally the degeneration of Hodge-de Rham spectral sequence at E_1 page), the integral (2.37) is a well-defined linear functional on $F^{n-r+1}H^{2n-2r+1}(X, \mathbb{C})$. Since another $(2n-2r+1)$-chain Γ' with $\partial\Gamma' = Z$ satisfies that $\Gamma - \Gamma' \in H_{2n-2r+1}(X, \mathbb{Z})$, the map

$$\mathscr{Z}^r(X)_{\text{hom}} \to J^{2r-1}(X), \ Z \mapsto \int_\Gamma \tag{2.38}$$

has an well-defined image in $J^{2n-2r+1}(X)$. Further, as any holomorphic map from a rational curve to compact complex torus is constant, a rational family of cycles will have the same image under the map (2.38). Therefore it factors through the Chow group

$$\mathrm{CH}^r(X)_{\text{hom}} \to J^{2r-1}(X), \tag{2.39}$$

which is called the *Griffiths' Abel-Jacobi map*. It is first introduced by Griffiths on hypersurfaces in $\mathbb{P}^n$ in [30].

Example 2.3.1. *When $r = 1$, the Griffiths' Abel-Jacobi map is more or less the same as the curve case. For example, Abel's Theorem (Theorem 2.1.3) and the Jacobi Inversion Theorem (Theorem 2.1.8) will hold. Equivalently, the isomorphism* (2.23) *in the curve case corresponds to the isomorphism*

$$Pic^0(X) \cong H^{0,1}(X)/H^1(X,\mathbb{Z}) \cong H^{n,n-1}(X)^\vee/H_{2n-1}(X,\mathbb{Z}) \cong J^1(X), \tag{2.40}$$

of abelian varieties (although the polarization is not principal when $n \geq 2$ in general). One reason for that is the divisor class group being isomorphic to the Picard group for a smooth projective variety X. Also, the isomorphism (2.40) *carries the first Chern class $c_1(\mathcal{L})$ of a line bundle $\mathcal{L} = [D]$ defined by a divisor D of degree zero to the Abel-Jacobi image of D (cf. [61, Proposition 12.7]).*

Example 2.3.2. *When $r = n-1$, the intermediate Jacobian $J^{2n-1}(X) = H^{1,0}(X)^\vee/H_1(X,\mathbb{Z})$ is called the Albanese variety of X. The Abel-Jacobi map is given by integrating holomorphic 1-forms against paths joining pair of points as like* (2.6) *in curve case.*

One should note that for higher codimension cycles, i.e. when $r \geq 2$, the intermediate Jacobian $J^{2r-1}(X)$ is in general not polarized. The Griffiths' Abel-Jacobi map (2.38) is in general not surjective (e.g., X is a threefold with $H^{3,0}(X) \neq 0$), nor injective (e.g., Mumford's example on surface of general type [46], where $\mathrm{CH}^0(X)_{\mathrm{hom}}$ is of "infinite dimensional").

For Fano varieties, however, Griffiths' Abel-Jacobi map usually behaves well. For example, Clemens and Griffiths solved the rationality problem of cubic threefold by studying the Abel-Jacobi image of pairs of lines. The next example will be studied in details in Chapter 4.

Example 2.3.3. *[19] Let X be a smooth cubic threefold defined by homogeneous equation $F(x_0, \ldots, x_4)$ of degree three. Let L_1, L_2 be lines on X and Γ a 3-chain with $\partial\Gamma = L_1 - L_2$. The intermediate Jacobian $J^3(X)$ is a principally polarized abelian variety of dimension 5. The space $F^2H^3(X, \mathbb{C}) = H^{2,1}(X)$ has basis $\omega_1, \ldots, \omega_5$, with*

$$\omega_i = Res_X \frac{x_{i-1}\Omega}{F^2},$$

given by Griffiths' residue of rational forms on $\mathbb{P}^4$ with poles along X, where $\Omega = \sum_{i=0}^4 x_i dx_0 \wedge \cdots \wedge \hat{dx_i} \wedge \cdots \wedge dx_4$.

So the image of $L_1 - L_2$ under the Griffiths' Abel-Jacobi map is given by

$$(\int_\Gamma \omega_1, \ldots, \int_\Gamma \omega_5) \in J^3(X).$$

The image of the set of all pairs of lines on X in $J(X)$ coincides with the theta divisor Θ of $J^3(X)$. By studying the branching locus of the Gauss map on Θ, Clemens and Griffiths concluded that X is not rational.

Now, we'll discuss some properties of Griffiths' Abel-Jacobi map for algebraic cycles that vary in a family. Let $\{Z_t\}_{t\in M}$ be a flat family of algebraic cycles of codimension r in X parameterized by a complex manifold M, and Z_t is homologous to zero for all t, then the Griffiths' Abel-Jacobi map (2.38) defines a map

$$M \to J^{2r-1}(X),\ t \mapsto A(Z_t - Z_0),$$

which is actually holomorphic [61, Theorem 12.4, Remark 12.5].

One can also consider a family $\{X_t\}_{t\in M}$ with $X_0 = X$, and a family of cycles $Z_t \subseteq X_t$, Griffiths' Abel-Jacobi map gives a holomorphic section to the intermediate Jacobian bundle $\{J(X_t)\}_{t\in M}$. This leads to the notion of normal functions. By varying cubic threefolds in a general pencil family of hyperplane section of a smooth cubic fourfold ($M = \mathbb{P}^1$), and studying the normal function induced by a Hodge class, Zucker proved the Hodge conjecture for cubic fourfold [64].

2.3.1 Griffiths' Abel-Jacobi Map via Mixed Hodge structures

Let X be a smooth projective variety of dimension n and $Z \in \mathcal{Z}^r(X)_{hom}$, as we have introduced in (2.38), Griffiths' Abel-Jacobi map is defined via integration.

On the other hand, denote $|Z|$ the support of Z, there is an exact sequence on the complement $U = X \setminus |Z|$.

$$0 \to H^{2r-1}(X) \to H^{2r-1}(U) \to H^{2r}_{|Z|}(X) \to H^{2r}(X).$$

The Griffiths' Abel-Jacobi image (2.39) of Z is determined by the extension class of the sequence

$$0 \to H^{2r-1}(X, \mathbb{Z}) \to E \xrightarrow{r} \mathbb{Z} \to 0, \tag{2.41}$$

where E is a sub-mixed Hodge structure of $H^{2r-1}(X \setminus |Z|)$, and $\mathbb{Z}$ the trivial Hodge structure of weight 0 generated by the class $\alpha = [Z]$, and r has type $(-r, -r)$. Then one can choose $\mathbb{Z}$-lift $\tilde{\alpha}_{\mathbb{Z}} \in E_{\mathbb{Z}}$ and F-lift $\tilde{\alpha}_F \in F^r E_{\mathbb{C}} \subseteq F^r H^{2r-1}(X \setminus |Z|, \mathbb{C})$, such that $r(\tilde{\alpha}_{\mathbb{Z}}) = r(\tilde{\alpha}_F) = \alpha$. So the difference $\tilde{\alpha}_{\mathbb{Z}} - \tilde{\alpha}_F \in H^{2r-1}(X, \mathbb{C})$. Modulo the freedom of choices of integral liftings, the class is well-defined in $\mathrm{Ext}^1_{\mathrm{MHS}}(\mathbb{Z}(-r), H^{2r-1}(X, \mathbb{Z})) \cong J^{2r-1}(X)$ using the isomorphism (2.27). This is called Carlson's Abel-Jacobi map.

Proposition 2.3.4. *(cf. [37, Proposition 3.3, 3.4]) Carlson's Abel-Jacobi map coincides with Griffiths' Abel-Jacobi map.*

The proofs are originated from [36] and [24], using duality between Deligne cohomology and Deligne homology. When Z has codimension 1, one also refers to [61, Proposition 12.7] for an equivalent statement and a geometric proof.

2.4 Zhao's Topological Abel-Jacobi Map

In [63], Zhao introduced a notion called the *Topological Abel-Jacobi map* that generalizes Griffiths' Abel-Jacobi map to topological cycles.

Let X be a smooth projective variety of dimension $2n-1$. We choose an embedding $X \hookrightarrow \mathbb{P}^N$ and let Y be a smooth hyperplane section. Denote $i : Y \subseteq X$ the inclusion. The middle dimensional primitive cohomology $H^{2n-1}_{\mathrm{prim}}(X, \mathbb{Z})$ on X is defined as

$$H^{2n-1}_{\mathrm{prim}}(X, \mathbb{Z}) := \ker(i^* : H^{2n-1}(X, \mathbb{Z}) \to H^{2n-1}(Y, \mathbb{Z})). \tag{2.42}$$

There is an associated primitive intermediate Jacobian

$$J_{\mathrm{prim}}(X) := F^n H^{2n-1}_{\mathrm{prim}}(X)^\vee / H_{2n-1}(X, \mathbb{Z})_{\mathrm{prim}}. \tag{2.43}$$

The vanishing cohomology $H^{2n-2}_{\mathrm{van}}(X, \mathbb{Z})$ is defined as the kernel of Gysin homomorphism

$$H^{2n-2}_{\mathrm{van}}(X, \mathbb{Z}) := \ker(i_* : H^{2n-2}(Y, \mathbb{Z}) \to H^{2n}(X, \mathbb{Z})). \tag{2.44}$$

Equivalently, it consists of Poincaré dual classes of the homology classes on Y that is 0 in homology on X. One refers to [62, Chapter 2] for basic properties on primitive and vanishing cohomologies.

Zhao defines a group homomorphism

$$A : H^{2n-2}_{\mathrm{van}}(Y) \to J_{\mathrm{prim}}(X), \tag{2.45}$$

which satisfies two properties [63, Proposition 2.1.1 and 2.2.2]:

(P.1) If the class $\alpha \in H^{2n-2}_{\mathrm{van}}(Y, \mathbb{Z})$ is represented by an algebraic cycle, then A_α agrees with Griffiths' Abel-Jacobi image of α.

(P.2) The map (2.45) varies real analytically as the hyperplane section Y varies in the universal family of smooth hyperplane sections of X.

The map (2.45) is originally defined by sending $\alpha \in H^{2n-2}_{\mathrm{van}}(Y, \mathbb{Z})$ to the linear functional whose value on $[\omega] \in F^n H^{2n-1}_{\mathrm{prim}}(X)$ is

$$A_\alpha([\omega]) = \int_\Gamma \omega - \int_\gamma \tau + \int_Y h_\alpha \wedge \tau + T(\omega), \quad \text{mod periods.} \tag{2.46}$$

Here Γ is a $(2n-1)$-chain on X whose boundary γ is a topological cycle which represents the Poincaré dual of the cohomology class α. τ is a $(2n-2)$-form on Y such that $d\tau = \omega_{|Y}$. Next, h_α is the harmonic representative of α with respect to the Kähler metric on Y induced from X (One can choose the Kähler metric on X to be restriction of Fubini Study metric on $\mathbb{P}^N$). Finally, T is a current on X such that $dT = -\int_Y h_\alpha \wedge (\cdot)$. So T vanishes on d^*-closed forms.

Now, if we take ω_h to be the harmonic representative of the class $[\omega]$, and let $\tau = d^c\sigma_h$ with $\omega_{h|Y} = dd^c\sigma_h$ by the $\partial\bar{\partial}$-lemma. Here $d^c = i(\bar{\partial} - \partial)$ is a real operator and $dd^c = 2i\partial\bar{\partial}$. Then the fouth term of (2.46) vanishes because harmonic forms are d^*-closed. The third term also vanishes because harmonic forms are L^2-orthogonal to $\mathrm{Im}(\partial) \oplus \mathrm{Im}(\bar{\partial})$. Therefore, the right hand side of (2.46) becomes [63, Definition 2.1.2]

$$\int_\Gamma \omega_h - \int_\gamma d^c\sigma_h. \tag{2.47}$$

We prove (P.1) mentioned in the introduction:

Proposition 2.4.1. *[63, Proposition 2.1.1] When α is an algebraic cycle, the topological Abel-Jacobi map* (2.47) *agrees with Griffiths's Abel-Jacobi map.*

Proof. (i) Note that $\int_\gamma d^c\sigma_h = \int_\gamma (d^c\sigma_h - id\sigma_h) = -2i\int_\gamma \partial\sigma_h$. By construction $\partial\sigma_h$ has type $(n, n-2) + (n+1, n-3) + \cdots$, whose integration along an algebraic subvariety Z will vanish. Therefore in (2.47), $A_\alpha([\omega]) = \int_\Gamma \omega_h$, which coincides with Griffiths' definition (2.39). □

As a remark, one notes that if $d(u, v) = (du, u_{|Y} - dv)$ is exact, then $\int_\Gamma du - \int_\gamma (u_{|Y} - dv) = \int_\gamma u - \int_\gamma u_{|Y} = 0$, so an the map (2.47) is a topological pairing

$$H^{2n-1}(X, Y) \times H_{2n-1}(X, Y) \to \mathbb{Z}. \tag{2.48}$$

between relative cohomology class $(\omega_h, d^c\sigma_h)$ and the relative homology class (Γ, γ). In particular, one can choose any closed form ω that defines the same cohomology class as ω_h, and choose $d^c\sigma$ such that $\omega_{|Y} = dd^c\sigma$. So the map is independent of the choice of the Kähler metric of X.

One notes that since d^c is a real operator, (2.49) can be regarded taken values in the real torus $J_{\text{prim}}(X,\mathbb{R}) := H^{2n-1}_{\text{prim}}(X,\mathbb{R})/H_{2n-1}(X,\mathbb{Z})_{\text{prim}}$. So we have a real version of Zhao's topological Abel-Jacobi map

Definition 2.4.2. *For any closed form ω representing the class $[\omega] \in H^{2n-1}_{\text{prim}}(X,\mathbb{R})$, choose σ such that $\omega_{|Y} = dd^c\sigma$. Call*

$$H^{2n-2}_{\text{van}}(Y,\mathbb{Z}) \to J_{\text{prim}}(X,\mathbb{R})$$

$$A_\alpha([\omega]) = \int_\Gamma \omega - \int_\gamma d^c\sigma = \langle(\omega, d^c\sigma), (\Gamma,\gamma)\rangle_{X,Y} \tag{2.49}$$

Zhao's (real) topological Abel-Jacobi map, where $\langle,\rangle_{X,Y}$ is the pairing defined in (2.48)*.*

2.5 Schnell's Construction via Mixed Hodge Structures

Recall that we have introduced mixed Hodge structures in Definition 2.2.1. We'll first introduce a property called $\mathbb{R}$-splitting for certain mixed Hodge structures. Then we'll introduce Schnell's construction of the Topological Abel-Jacobi map.

2.5.1 $\mathbb{R}$-splitting Mixed Hodge Structures

Every mixed Hodge structure H admits a direct sum decomposition

$$H_\mathbb{C} \cong \bigoplus_{p,q} I^{p,q}, \text{ such that } F^pH_\mathbb{C} \cong \bigoplus_{k\geq p,q} I^{k,q}, \ W_kH_\mathbb{C} \cong \bigoplus_{p+q\leq k} I^{p,q}.$$

Moreover, it satisfies a conjugation property

$$\bar{I}^{p,q} \equiv I^{q,p} \mod \bigoplus_{k<p,l<q} I^{k,l}. \tag{2.50}$$

Definition 2.5.1. *A mixed Hodge structure H is called $\mathbb{R}$-splitting if the conjugation property* (2.50) *of Deligne's decomposition is equality, namely*

$$\bar{I}^{p,q} = I^{q,p}, \ \forall p,q. \tag{2.51}$$

Proposition 2.5.2. *Suppose the mixed Hodge structure H has weights concentrated on levels $w-1$ and w consecutively, then H is $\mathbb{R}$-splitting.*

Proof. Let $p+q=w-1$ or w. Then for all $k<p$ and $l<q$, $k+l<w-1$ so $I^{k,l}=0$. So the condition (2.51) trivially holds. □

2.5.2 Curve Again

We will take a moment to review the Abel-Jacobi map in the curve case again. We first note that the extension sequence (2.31)

$$0 \to H^1(X) \to H^1(U) \xrightarrow{r} H^0_{\mathrm{van}}(Y) \to 0$$

is $\mathbb{R}$-splitting, as $H^1(U)$ has weight concentrated on level 1 and 2 consecutively. This means that for each $\alpha \in H^0_{\mathrm{van}}(Y)$, there is a unique F-lift, denoted as $\tilde{\alpha}_{\mathbb{R}}$ that is defined over $\mathbb{R}$. Such class $\tilde{\alpha}_{\mathbb{R}}$ lives in $I^{1,1}_{\mathbb{R}} = I^{1,1} \cap H^1(U,\mathbb{R})$.

Note that the class $\tilde{\alpha}_{\mathbb{Z}} - \tilde{\alpha}_{\mathbb{R}}$ is defined over $\mathbb{R}$, we conclude that

Proposition 2.5.3. *The map*

$$\alpha \mapsto \tilde{\alpha}_{\mathbb{Z}} - \tilde{\alpha}_{\mathbb{R}}$$

factors through the real torus

$$H^1(X,\mathbb{R})/H^1(X,\mathbb{Z}) \xrightarrow{\cong} H^1(X,\mathbb{C})/F^1H^1(X) + H^1(X,\mathbb{Z}).$$

In fact, one can find the class $\tilde{\alpha}_{\mathbb{R}}$ explicitly: First take a meromorphic 1-form ϕ with only first order poles and $Res(\phi) = D$, with $D = \sum_i (p_i - q_i)$ a degree zero divisor as before. Then we can subtract a holomorphic 1-form ω from ϕ, such that the resulting meromorphic 1-form ϕ' has no γ-period. Now express the class $[\phi']$ in the basis (2.20):

$$[\phi'] = d_1\delta^* + \cdots + d_g\delta_g^* + D.$$

Now we need to solve equation

$$\overline{[\phi'] + t_1\omega_1 + \cdots t_g\omega_g} = [\phi'] + t_1\omega_1 + \cdots t_g\omega_g,\ t_1, \ldots, t_g \in \mathbb{C}, \tag{2.52}$$

where $\omega_1, \ldots, \omega_g$ is a normalized basis via conditions in (2.2).

Use the identity $\omega_i = \gamma_i^* + \sum_{j=1}^g \Omega_{ij}\delta_j^*$, where $\Omega = (\Omega_{ij})$ is the period matrix, the equation (2.52) leads to the solutions

$$\begin{bmatrix} t_1 \\ \vdots \\ t_g \end{bmatrix} = -(\mathrm{Im}(\Omega))^{-1} \begin{bmatrix} \mathrm{Im}(d_1) \\ \vdots \\ \mathrm{Im}(d_g) \end{bmatrix}. \tag{2.53}$$

Note that t_i's are real numbers. It follows that $\tilde{\alpha}_{\mathbb{R}} = [\phi'] + \sum_i t_i\omega_i$ is expressed as

$$\tilde{\alpha}_{\mathbb{R}} = \sum_{i=1}^g t_i\gamma_i^* + 2\sum_{i=1}^g \mathrm{Re}(d_i)\delta_i^*. \tag{2.54}$$

A similar argument also works in higher dimensions for Griffiths' Abel-Jacobi maps. So it suffices to consider the classical Abel-Jacobi map in the real torus case.

Now we introduce Schnell's construction on the Topological Abel-Jacobi map.

2.5.3 Schnell's Construction

Schnell [50] defined a topological Abel-Jacobi map using the $\mathbb{R}$-splitting property of the mixed Hodge structure on $H^{2n-1}(X \setminus Y)$ (cf. Corollary 2.6.3).

Denote $U = X \setminus Y$. Then there is a short exact sequence of mixed Hodge structures

$$0 \to H_0^{2n-1}(X) \to H^{2n-1}(U) \xrightarrow{\mathrm{Res}} H_{\mathrm{van}}^{2n-2}(Y) \to 0, \tag{2.55}$$

which comes from the long exact sequence of the pair (X, U), and we set $H_0^{2n-1}(X) = \mathrm{Coker}(i_* : H^{2n-3}(Y) \to H^{2n-1}(X))$.

In the spirit of Carlson's theory, we hope to define a topological Abel-Jacobi map for $\alpha \in H_{\mathrm{van}}^{2n-2}(Y, \mathbb{Z})$ using the extension class $[e]$ of (2.55). First, if $\alpha \in H_{\mathrm{van}}^{n-1,n-1}(Y, \mathbb{Z})$ is algebraic, or in general an integral Hodge class, then just as what we did in (2.41), one can take liftings $\tilde{\alpha}_{\mathbb{Z}} \in H^{2n-1}(U,\mathbb{Z})$ and $\tilde{\alpha}_F \in F^n H^{2n-1}(U,\mathbb{C})$, and take the difference $\tilde{\alpha}_{\mathbb{Z}} - \tilde{\alpha}_F$, whose image is well defined in the the Jacobian $J_0(X) = F^n H_0^{2n-1}(X)^\vee / H_{2n-1}(X,\mathbb{Z})_0$, the complex torus associated to the Hodge structure on $H_0^{2n-1}(X)$.

However, when α has a mixed type, $\tilde{\alpha}_F$ lives in $F^k H^{2n-1}(U,\mathbb{C})$ for some $k < n$. The difference $\tilde{\alpha}_{\mathbb{Z}} - \tilde{\alpha}_F$ is not well defined in the torus $J_0(X)$.

To fix this, we forget the complex structure for a moment. Take a real lifting $\tilde{\alpha}_{\mathbb{R}} \in H^{2n-1}(U,\mathbb{R})$ of α, then $\tilde{\alpha}_{\mathbb{Z}} - \tilde{\alpha}_{\mathbb{R}}$ defines a class in $H_0^{2n-1}(X,\mathbb{R})$. If we choose a different integral lifting $\tilde{\alpha}'_{\mathbb{Z}}$, then the difference $\tilde{\alpha}_{\mathbb{Z}} - \tilde{\alpha}'_{\mathbb{Z}} \in H_0^{2n-1}(X,\mathbb{Z})$. This defines an image in the real primitive intermediate Jacobian

$$J_0(X,\mathbb{R}) := \frac{H_0^{2n-1}(X,\mathbb{R})}{H_0^{2n-1}(X,\mathbb{Z})}.$$

Of course, the freedom of the choices of $\tilde{\alpha}_{\mathbb{R}}$ forms an affine space that is isomorphic to $H_0^{2n-1}(X,\mathbb{R})$. But, the $\mathbb{R}$-splitting property of $H^{2n-1}(U)$ provides a canonical choice. More precisely, there is a canonical section $s_{\mathbb{R}}^U : H_{\text{van}}^{2n-2}(Y,\mathbb{R}) \to H^{2n-1}(U,\mathbb{R})$ such that $\text{Res} \circ s_{\mathbb{R}}^U = \text{Id}$ on $H_{\text{van}}^{2n-2}(Y,\mathbb{R})$, and that

(†) $s_{\mathbb{R}}^U \otimes \mathbb{C}$ is a morphism of $\mathbb{C}$-Hodge structure of type $(1,1)$.

The condition (†) means that $s_{\mathbb{R}}^U \otimes \mathbb{C}$ sends the (p,q)-summand of $H_{\text{van}}^{2n-2}(Y,\mathbb{C})$ isomorphically onto $(p+1,q+1)$-summand of $H^{2n-1}(U,\mathbb{C})$. For example, in curve case, $s_{\mathbb{R}}(p-q) \in I_{\mathbb{R}}^{1,1}$ is represented by a meromorphic 1-form $\phi_{\mathbb{R}}$ with $Res(\phi_{\mathbb{R}}) = p-q$ and all periods $\int_{\gamma_i} \phi_{\mathbb{R}}, \int_{\delta_i} \phi_{\mathbb{R}}$, $i = 1,\dots,g$ are real numbers. It is worked out explicitly in (2.54).

Definition 2.5.4. *[50] Take an integral section $s_{\mathbb{Z}}^U : H_{\text{van}}^{2n-2}(Y,\mathbb{Z}) \to H^{2n-1}(U,\mathbb{Z})$ such that* $\text{Res} \circ s_{\mathbb{Z}}^U = \text{Id}$, *we call the morphism*

$$H_{\text{van}}^{2n-2}(Y,\mathbb{Z}) \to J_0(X,\mathbb{R}),\ \alpha \mapsto s_{\mathbb{Z}}^U(\alpha) - s_{\mathbb{R}}^U(\alpha) \tag{2.56}$$

the Schnell's topological Abel-Jacobi map.

The map (2.56) satisfies the two properties mentioned before. (P.1) is due to the condition (†). For (P.2), one notes that F^p varies holomorphically, versus $\bar{F}^q$ varies anti-holomorphically. Therefore, their intersection varies real analytically, and so does $s_{\mathbb{R}}$.

Under the isomorphism of the real vector space $F^n H^{2n-1}_{\text{prim}}(X,\mathbb{C}) \cong H^{2n-1}_{\text{prim}}(X,\mathbb{R})$, $a \mapsto 2\text{Re}(a)$ and the fact that there is a unimodular pairing $H^{2n-1}_{\text{prim}}(X,\mathbb{Z}) \times H^{2n-1}_{0}(X,\mathbb{Z}) \to \mathbb{Z}$ (cf. Proposition 2.6.4), the (complex) intermediate Jacobian (2.43) is identified with the real Jacobian $J_0(X,\mathbb{R})$ as real torus, so the topological Abel-Jacobi map defined by Zhao (2.45) can be viewed as a morphism to the real torus.

Christian Schnell asked the following question [50]:

Question 2.5.5. *Is Schnell's topological Abel-Jacobi map* (2.56) *the same as Zhao's topological Abel-Jacobi map* (2.45)*?*

We answered this question affirmatively:

Theorem 2.5.6. *Schnell's topological Abel-Jacobi map and Zhao's topological Abel-Jacobi map coincide. The map associates $\alpha \in H^{2n-2}_{\text{van}}(Y,\mathbb{Z})$ with a linear functional*

$$[\omega] \mapsto \int_\Gamma \omega - \int_\gamma d^c\sigma \mod \int_X \lambda_{\mathbb{Z}} \wedge \omega, \text{ where } \lambda_{\mathbb{Z}} \in H^{2n-1}_{\text{prim}}(X,\mathbb{Z}),$$

where ω is a closed real form representing the class $[\omega] \in H^{2n-1}_{\text{prim}}(X,\mathbb{R})$ and σ is a real $(2n-3)$-form on Y such that $\omega_{|Y} = dd^c\sigma$.

2.6 Proofs of the Main Theorem

In this section, we'll show that the canonical mixed Hodge structures on $H^{2n-1}(X,Y)$ and $H^{2n-1}(U)$ are $\mathbb{R}$-splitting. We also review a topological pairing between the two cohomologies, inducing duality on mixed Hodge structures.

We'll also review Zhao's topological Abel-Jacobi map and relate it to Deligne's $\mathbb{R}$-splitting of the sequence

$$0 \to H_0^{2n-2}(Y) \to H^{2n-1}(X,Y) \to H^{2n-1}_{\text{prim}}(X) \to 0. \tag{2.57}$$

The sequence is obtained by the dual of the sequence (2.55) using Poincaré and Lefschetz duality.

The Deligne's $\mathbb{R}$-splitting splitting of (2.57) can be made explicit using relative de Rham cohomology: a class in $H^{2n-1}(X,Y)$ is represented by $(\omega,\tau) \in A_X^{2n-1} \times A_Y^{2n-2}$ such that $d_X\omega = 0$ and $\omega_{|Y} = d_Y\tau$. We found that the Deligne's $\mathbb{R}$-splitting for (2.57) is given by $\omega \mapsto (\omega, d^c\sigma)$, where $\omega|_Y = dd^c\sigma$ as a result of $[\omega]$ being primitive and $\partial\bar{\partial}$-lemma. Finally, the fact that the duality of (2.55) is isomorphic to (2.57) as extensions of mixed Hodge structures allows us to relate the $\mathbb{R}$-splitting of the two sequences.

We will prove the Theorem 2.5.6 later by reducing it to two linear algebra arguments.

As a final remark, the topological Abel-Jacobi map can be defined on hyperplane section Y_0 that has an ordinary double point. This is because $H^{2n-2}(Y_0)$ is still a pure Hodge structure, and the mixed Hodge structure on $H^{2n-1}(X,Y_0)$ is still $\mathbb{R}$-splitting.

2.6.1 Zhao's Topological Abel-Jacobi Map as $\mathbb{R}$-Splitting

Lemma 2.6.1. *The real section*

$$s : H^{2n-1}_{\text{prim}}(X,\mathbb{R}) \to H^{2n-1}(X,Y,\mathbb{R})$$

$$\omega \mapsto (\omega, d^c\sigma)$$

coincides with Deligne's canonical $\mathbb{R}$-splitting section $s_{\mathbb{R}}^{X,Y}$ of the sequence (2.57).

Proof. First of all, we show that the map is well defined. Since d^c is a real operator, the image lies cohomology in real coefficients. Since $\omega_{|Y} = dd^c\sigma$ and ω is closed, $(\omega, d^c\sigma)$ is closed in the mapping cone complex. If there is another σ' such that $dd^c\sigma' = \omega_{|Y}$, then $d^c\sigma - d^c\sigma'$

is a closed form. However, it also lies in $\mathrm{Im}(\partial) \oplus \mathrm{Im}\bar{\partial}$, so it is L^2-orthogonal to the space of harmonic forms, so the cohomology class is zero.

To show the claim, it suffices to show that s sends (p,q) forms into $F^pH^{2n-1}(X,Y,\mathbb{C}) \cap \overline{F^qH^{2n-1}(X,Y,\mathbb{C})}$.

If ω is of (p,q)-type, where $p+q = 2n-1$, then by $\partial\bar{\partial}$-lemma, σ has type $(p-1,q-1)$. We should show that $s(\omega) = (\omega, d^c\sigma)$ defines a class in $F^pH^{2n-1}(X,Y,\mathbb{C}) \cap \overline{F^qH^{2n-1}(X,Y,\mathbb{C})}$. By adding an exact form $(0, id\sigma)$, we see that $(\omega, d^c\sigma) \sim (\omega, 2i\bar{\partial}\sigma)$, so $s(\omega)$ is represented by a closed form in $\bar{F}^qA_X^{2n-1} \times \bar{F}^qA_Y^{2n-2}$, which implies $s(\omega) \in \overline{F^qH^{2n-1}(X,Y,\mathbb{C})}$. Similarly, subtracting the exact form $(0, id\sigma)$, $s(\omega)$ is represented by a closed form in $F^pA_X^{2n-1} \times F^pA_Y^{2n-2}$, which implies $s(\omega) \in \overline{F^qH^{2n-1}(X,Y,\mathbb{C})}$. □

By abusing the notation, we write ω for $[\omega]$ for the rest of the chapter.

Corollary 2.6.2. *Zhao's topological Abel-Jacobi map* (2.49) *can be expressed as*

$$A_\alpha(\omega) = \langle s_{\mathbb{R}}^{X,Y}(\omega), s_{\mathbb{Z}}^U(\alpha)\rangle_{X,Y}^U, \tag{2.58}$$

where $\langle , \rangle_{X,Y}^U$ *is the pairing defined in* (2.61), *and* $s_{\mathbb{Z}}^U : H_{\mathrm{van}}^{2n-2}(Y,\mathbb{Z}) \to H^{2n-1}(U,\mathbb{Z})$ *is a section, i.e.,* $\mathrm{Res} \circ s_{\mathbb{Z}}^U = \mathrm{Id}$.

Proof. Notice that the sequence (2.55) is identified with the homology sequence

$$0 \to H_{2n-1}(X)_0 \to H_{2n-1}(X,Y) \to H_{2n-2}(Y)_{\mathrm{van}} \to 0,$$

via Lefschetz duality and Poincaré duality, so $s_{\mathbb{Z}}^U(\alpha)$ is identified with a relative class (Γ', γ), where γ is a cycle representing the Poincaré dual class of α, and Γ' is a $2n-1$ chain on X such that $\partial\Gamma' = \gamma$. Since $(\Gamma', \gamma) - (\Gamma, \gamma) \in H_{2n-1}(X,\mathbb{Z})_0$, the pairing (2.58) coincides with (2.49) modulo integral periods. □

2.6.2 Some Preparations

Let X be a smooth projective $2n-1$ fold, and Y be a smooth hyperplane section. $i : Y \to X$ is the inclusion map. Denote $U := X \setminus Y$ the complement.

The relative cohomology $H^{2n-1}(X, Y)$ can be defined through de Rham theory: For a smooth manifold M, denote $A_M^\bullet$ the complex of C^∞ forms on M. $H^\bullet(X, Y)$ is the cohomology theory associated to the complex $A_X^\bullet \oplus A_Y^{\bullet-1}$, with $d(\omega, \tau) = (d\omega, \omega_{|Y} - d\tau)$. It is the mapping cone complex of $i^* : A_X^\bullet \to A_Y^\bullet$ (cf. [10, p.78]). $H^\bullet(X, Y)$ admits mixed Hodge structure induced from mapping cone complex [47, p.76]. In particular, its Hodge filtration $F^p H^\bullet(X, Y)$ is the subspace represented by closed forms in $F^p A_X^\bullet \oplus F^p A_Y^{\bullet-1}$.

There is a long exact sequence associated to the pairs (X, Y):

$$\cdots \to H^{2n-2}(X) \xrightarrow{i^*} H^{2n-2}(Y) \to H^{2n-1}(X, Y) \to H^{2n-1}(X) \xrightarrow{i^*} H^{2n-1}(Y) \to \cdots, \quad (2.59)$$

By truncation, we get the short exact sequence (2.57)

$$0 \to H_0^{2n-2}(Y) \to H^{2n-1}(X, Y) \to H_{\text{prim}}^{2n-1}(X) \to 0,$$

where $H_0^{2n-2}(Y, \mathbb{Z}) := \text{coker}(i^* : H^{2n-2}(X, \mathbb{Z}) \to H^{2n-2}(Y, \mathbb{Z}))$ carries pure Hodge structure of weight $2n-2$. So the mixed Hodge structure of $H^{2n-1}(X, Y)$ has weight concentrated on levels $2n-1$ and $2n-2$ consecutively.

The cohomology $H^{2n-1}(U)$ on the complement also admits a canonical mixed Hodge structure due to Deligne. $H^k(U)$ is the k-th hypercohomology of the log complex $\Omega_X^\bullet(\log Y)$. The Hodge filtration $F^p H^k(U, \mathbb{C})$ is the k-th hypercohomology of the subcomplex $\Omega_X^{\geq p}(\log Y)$. As the log complex admits an acyclic resolution using the double complex $\oplus_{p,q} A_X^{p,q}(\log Y)$ (cf. [61, p.212]), $F^p H^k(U, \mathbb{C})$ consists of classes that are represented by closed forms in $\oplus_{l \geq p} A_X^{l,k-l}(\log Y)$. Namely, they are C^∞ k-forms with log pole along Y and have at least p holomorphic differentials.

Its weight filtration is determined by the extension sequence (2.55)

$$0 \to H_0^{2n-1}(X) \to H^{2n-1}(U) \xrightarrow{\text{Res}} H_{\text{van}}^{2n-2}(Y) \to 0,$$

which comes from the long exact sequence associated to the pair (X, U), where we used the Thom isomorphism $H^{2n}(X, U) \cong H^{2n-2}(Y)$. Since residue map Res has type $(-1, -1)$, $H^{2n-1}(U)$ has weights concentrated on levels $2n - 1$ and $2n$.

By Proposition 2.5.2 and the previous discussions, we conclude

Corollary 2.6.3. *Both $H^{2n-1}(X, Y)$ and $H^{2n-1}(U)$ are $\mathbb{R}$-splitting mixed Hodge structures.*

All cohomologies above are defined over $\mathbb{Z}$. There is a natural pairing between the two short exact sequences (2.57) and (2.55):

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H_0^{2n-2}(Y,\mathbb{Z}) & \longrightarrow & H^{2n-1}(X,Y,\mathbb{Z}) & \longrightarrow & H_{\text{prim}}^{2n-1}(X,\mathbb{Z}) & \longrightarrow & 0 \\ & & \times & & \times & & \times & & \\ 0 & \longleftarrow & H_{\text{van}}^{2n-2}(Y,\mathbb{Z}) & \longleftarrow & H^{2n-1}(U,\mathbb{Z}) & \longleftarrow & H_0^{2n-1}(X,\mathbb{Z}) & \longleftarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & \mathbb{Z} & & \mathbb{Z} & & \mathbb{Z} & & \end{array} \tag{2.60}$$

where the pairings on the first and third column are induced from the intersection pairing on $H^{2n-2}(Y, \mathbb{Z})$ and $H^{2n-1}(X, \mathbb{Z})$, respectively. The pairing

$$H^{2n-1}(X, Y, \mathbb{Z}) \times H^{2n-1}(U, \mathbb{Z}) \to \mathbb{Z} \tag{2.61}$$

in the middle of (2.60) can be interpreted as the Poincaré pairing

$$H_c^{2n-1}(U, \mathbb{Z}) \times H^{2n-1}(U, \mathbb{Z}) \to \mathbb{Z},\ ([\psi], [\phi]) \mapsto \int_X \psi \wedge \phi, \tag{2.62}$$

using the isomorphism [47, Corollary B.14]

$$\Psi : H^{2n-1}(X, Y, \mathbb{Z}) \cong H_c^{2n-1}(U, \mathbb{Z}). \tag{2.63}$$

One can also refer to [26, Chapter XII, Theorem 3.1] for the isomorphism Ψ using a different version of relative de Rham cohomology that comes from a complex $\Omega^*(X,Y)$ consisting of C^∞ forms on X that vanish on Y.

In our case, the map $\psi \mapsto (\psi, 0)$ induces the inverse of Ψ. One can show that the Hodge filtration $F^pH_c^{2n-1}(U,\mathbb{C})$ consists of classes that are represented by closed forms in $F^pA_c^{2n-1}(U)$, where $A_c^{2n-1}(U)$ is the space of C^∞ forms with compact support on U, with the standard filtration.

Proposition 2.6.4. *The pairing on each column in* (2.60) *is unimodular. Moreover, the pairing induces isomorphisms between the dual mixed Hodge structures of* (2.57) *and the mixed Hodge structures on* (2.55)*.*

Proof. The argument is standard. The first half of the claim is because the natural pairing between the long exact sequence (2.59) associated with the pair (X,Y) and the pair (X,U) in $\mathbb{Z}$-coefficent is unimodular.

For the second half of the claim, it suffices to show that the dual mixed Hodge structure $H^{2n-1}(X,Y)^\vee$ is isomorphic to $H^{2n-1}(U)$. By definition of dual weight (2.24) and Hodge filtrations (2.25), it is equivalent that the pairing induces an isomorphism

$$W_{2n-1}H^{2n-1}(U) \cong W_{2n-2}H^{2n-1}(X,Y)^\vee \tag{2.64}$$

on weight filtrations, and isomorphisms

$$F^pH^{2n-1}(U,\mathbb{C}) \cong F^{2n-p}H^{2n-1}(X,Y,\mathbb{C})^\vee \tag{2.65}$$

on Hodge filtrations for $0 \leq p \leq 2n-1$.

The isomorphism (2.64) follows from the commutativity of the pairing diagram (2.60). To show the isomorphism (2.65), one concludes that the pairing

$$F^pH^{2n-1}(X,Y,\mathbb{C}) \times F^qH^{2n-1}(U,\mathbb{C}) \to \mathbb{C}$$

is zero for $p+q > 2n-1$ by $F^pH^{2n-1}(X,Y,\mathbb{C}) \cong F^pH_c^{2n-1}(U,\mathbb{C})$ and the corresponding pairing via (2.62) vanishes on the level of cochains by type reason. $\square$

2.6.3 Completion of the Proof

This section is devoted to proving Theorem 2.5.6.

By Proposition 2.6.4, the pairing on the third column of (2.60) is unimodular, so the pairing induces isomorphism $H_0^{2n-1}(X,\mathbb{Z}) \cong H_{\text{prim}}^{2n-1}(X,\mathbb{Z})^\vee \cong H_{2n-1}(X,\mathbb{Z})_{\text{prim}}$, where the last isomorphism is induced by Poincaré pairing. It induces isomorphisms between the real tori

$$J_0(X,\mathbb{R}) = \frac{H_0^{2n-1}(X,\mathbb{R})}{H_0^{2n-1}(X,\mathbb{Z})} \cong \frac{H_{\text{prim}}^{2n-1}(X,\mathbb{R})^\vee}{H_{\text{prim}}^{2n-1}(X,\mathbb{Z})^\vee} \cong \frac{H_{\text{prim}}^{2n-1}(X,\mathbb{R})^\vee}{H_{2n-1}(X,\mathbb{Z})_{\text{prim}}} =: J_{\text{prim}}(X,\mathbb{R}), \tag{2.66}$$

which are equivalent characterizations of the primitive real intermediate Jacobian of X.

To prove that the Schnell's topological Abel-Jacobi map (2.56) coincides with Zhao's topological Abel-Jacobi map (2.49), we need to show for given classes $\alpha \in H_{\text{van}}^{2n-2}(Y,\mathbb{Z})$ and $[\omega] \in H_{\text{prim}}^{2n-1}(X,\mathbb{R})$ the following equality holds modulo periods

$$A_\alpha(\omega) = \int_X \left(s_{\mathbb{Z}}^U(\alpha) - s_{\mathbb{R}}^U(\alpha)\right) \wedge \omega.$$

The right hand side coincides with the pairing $\langle s_{\mathbb{Z}}^U(\alpha) - s_{\mathbb{R}}^U(\alpha), \omega\rangle_X$, where $\langle , \rangle_X$ is the third pairing in (2.60).

By Corollary 2.6.2, this is equivalent to show

Proposition 2.6.5. *For $\alpha \in H_{\text{van}}^{2n-2}(Y,\mathbb{Z})$ and $\omega \in H^{2n-1}(X,\mathbb{R})$,*

$$\langle s_{\mathbb{R}}^{X,Y}(\omega), s_{\mathbb{Z}}^U(\alpha)\rangle_{X,Y}^U = \langle s_{\mathbb{Z}}^U(\alpha) - s_{\mathbb{R}}^U(\alpha), \omega\rangle_X$$

modulo periods.

According to Proposition 2.6.4, the sequence (2.55) is identified with

$$0 \to H_0^{2n-1}(X,\mathbb{Z})^\vee \to H^{2n-1}(X,Y,\mathbb{Z})^\vee \to H_{\text{van}}^{2n-2}(Y,\mathbb{Z})^\vee \to 0 \tag{2.67}$$

as mixed Hodge structures. This will tell us the relationship between Deligne's real splittings of the two sequences.

We will first prove a linear algebra lemma. Suppose E is an $\mathbb{R}$-splitting mixed Hodge structure, which arises from the extension sequence

$$0 \longrightarrow V \xrightarrow{f} E \underset{s}{\overset{g}{\rightleftarrows}} W \longrightarrow 0 \ , \tag{2.68}$$

where V and W are pure Hodge structures of weight $w-1$ and w respectively. f and g are morphisms of mixed Hodge structures of weight 0. Let $s : W \to E$ be the canonical Deligne's real splitting. Let

$$0 \longrightarrow W^\vee \underset{s^*}{\overset{g^*}{\rightleftarrows}} E^\vee \xrightarrow{f^*} V^\vee \longrightarrow 0 \tag{2.69}$$

be the dual sequence inducing the natural mixed Hodge structures, with weight $W^\vee$ and $V^\vee$ being w and $w-1$ and f^* having weight $(-1,-1)$.

Lemma 2.6.6. *Then the dual sequence* (2.69) *is also* $\mathbb{R}$*-splitting, and the corresponding decomposition is*

$$E^\vee \cong g^* W^\vee \oplus \ker(s^*). \tag{2.70}$$

Proof. First, the sequence (2.69) is $\mathbb{R}$-splitting by Proposition 2.5.2.

Next, the splitting (2.70) is defined over $\mathbb{R}$ since pairing (2.68) does. Since $\ker(s^*) \subseteq E^\vee$ is defined over $\mathbb{R}$ and is isomorphic to $V^\vee$ via f^*, it defines a splitting (2.70) of $E^\vee$ over $\mathbb{R}$. To show that it coincides with Deligne's splitting, it suffices to show that the isomorphism of real vector space

$$f^*|_{\ker(s^*)} : \ker(s^*) \to V^\vee(-1) \tag{2.71}$$

induces an isomorphism of the Hodge structure of weight w after tensoring with $\mathbb{C}$, where $V^\vee(-1)$ denotes the Tate twist of $V^\vee$. From now on, all vector spaces are complex. By abusing the notation, we omit the subscript $\mathbb{C}$.

It suffices to show that for each $l \in F^p V^\vee$, its inverse $\tilde{l}$ under (2.71) lies in $F^{p+1}\ker(s^*)$. This is equivalent to show that for each $e \in F^{w-p}E$, $\tilde{l}(e) = 0$.

Via the splitting $E \cong f(V) \oplus s(W)$, $e = e_1 + e_2$, with $e_1 \in f(V)$ and $e_2 \in s(W)$. Since Deligne's splitting s preserves Hodge filtration, $e_2 = s \circ g(e)$ lies in $F^{w-p}E$ as well, so does $e_1 = e - e_2$. Let $v \in V$ be the element such that $f(v) = e_1$, then $v \in F^{w-p}V$ by strictness of f. By definition of $\tilde{l}$, it vanishes on the subspace $s(W)$, so $\tilde{l}(e) = \tilde{l}(e_1) = \tilde{l}(f(v)) = l(v) = 0$. The last equality is due to the assumption on l and the fact that V is a pure Hodge structure of weight $w - 1$. □

By abusing the notation, denote $s_{\mathbb{R}}$ the Deligne's real section and $s_{\mathbb{Z}}$ an integral section for both sequence (2.68) and (2.69). Denote $\langle , \rangle_E$ the natural pairing $E \times E^\vee \to \mathbb{Z}$, and similarly for $\langle , \rangle_W$.

Now, the proof of Proposition 2.6.5 reduces to show

Proposition 2.6.7. *For each $\omega \in W$ and $\alpha \in V^\vee$ there is an equality*

$$\langle s_{\mathbb{R}}(\omega), s_{\mathbb{Z}}(\alpha) \rangle_E = \langle \omega, s_{\mathbb{Z}}(\alpha) - s_{\mathbb{R}}(\alpha) \rangle_W \quad \text{mod periods,} \tag{2.72}$$

Proof. We take an integral basis $v_1, \ldots, v_k, w_1, \ldots, w_n$ of E, where $f^{-1}(v_i)$'s and $g(w_j)$'s form integral basis for V and W respectively. We can find a real basis $u_1, \ldots, u_n$ of $s(W)$, such that

$$u_i = \sum_{j=1}^{k} a_{ij} v_j + w_i$$

for some $a_{ij} \in \mathbb{R}$ and for each $i = 1, \ldots, n$.

On the other hand, $w_1^\vee, \ldots, w_n^\vee, v_1^\vee, \ldots, v_k^\vee$ form a dual basis of $E^\vee$. Moreover, $(g^*)^{-1}(w_j^\vee)$'s form a basis of $W^\vee$ and is the dual basis of $g(w_i)$'s. By Lemma 2.6.6, the decomposition (2.70) is the Deligne's real splitting for $E^\vee$, so we can find a real basis $\rho_1, \ldots, \rho_n$ of the real subspace $\ker(s^*)$ of $E^\vee$, with

$$\rho_j = -\sum_{i=1}^{n} a_{ij} w_i^\vee + v_j^\vee,$$

for $j = 1, \dots, k$.

Finally, to check the equality (2.72), it suffices to check it on basis, so we take $\omega = g(w_i)$ and $\alpha = f^*(v_j^\vee)$, then $s_{\mathbb{R}}(g(w_i)) = u_i$ and $s_{\mathbb{Z}}(f^*(v_j^\vee)) = v_j^\vee + \sum_k n_k w_k^\vee$, with $n_k \in \mathbb{Z}$. Then the left hand side of (2.72) is

$$\langle s_{\mathbb{R}}(\omega), s_{\mathbb{Z}}(\alpha)\rangle_E = \langle u_i, v_j^\vee + \sum_k n_k w_k^\vee \rangle_E = a_{ij} + \text{periods}.$$

On the other hand, $s_{\mathbb{Z}}(f^*(v_j^\vee)) = v_j^\vee + \sum_k m_k w_k^\vee$, with $m_k \in \mathbb{Z}$ and $s_{\mathbb{R}}(f^*(v_j^\vee)) = \rho_j$. So the right hand side of (2.72) is

$$\begin{aligned}\langle \omega, s_{\mathbb{Z}}(\alpha) - s_{\mathbb{R}}(\alpha)\rangle_W &= \langle g(w_i), \sum_{i=1}^{n} (a_{ij} + m_i) w_i^\vee \rangle \\ &= \langle g(w_i), \sum_{i=1}^{n} (a_{ij} + m_i)(g^*)^{-1}(w_i^\vee)\rangle_W \\ &= a_{ij} + \text{periods},\end{aligned}$$

since $\langle g(w_i), (g^*)^{-1}(w_j^\vee)\rangle = \delta_{ij}$. So both sides of the equality (2.72) match up. □

2.7 Topological Jacobi Inversion

Let X be a smooth projective variety of dimension $2n-1$ embedded in a projective space $\mathbb{P}^N$. Let $\mathbb{O}^{\text{sm}}$ be the open subspace of $\mathbb{O} := (\mathbb{P}^N)^*$ parameterizing smooth hyperplane sections of X. Then by varying the hyperplane section Y in $\mathbb{O}^{\text{sm}}$, we get a local system $\mathcal{H}_{\text{van}}^{2n-2}$ whose stalk at each t is identified to the vanishing cohomology $H_{\text{van}}^{2n-2}(Y_t, \mathbb{Z})$. Let T denote the underlying anaytic space of $\mathcal{H}_{\text{van}}^{2n-2}$, then there is a real analytic map

$$AJ : T \to J_{\text{prim}} X. \tag{2.73}$$

Take π_1 of the map. We obtain a group homomorphism

$$\oplus_{i\in I}\, \pi_1(T, t_0^i) \to \pi_1(J_{\text{prim}} X, 0) = H_{2n-1}(X, \mathbb{Z})_{\text{prim}}. \tag{2.74}$$

t_0^i is in the preimage of a fixed point $t_0 \in \mathbb{O}^{\rm sm}$ and is a point on i-th connected component of T with index set I. The left hand side coincides to the set

$$\{(\alpha, \gamma) \in H^{2n-2}_{\rm van}(Y_{t_0}, \mathbb{Z}) \times \pi_1(\mathbb{O}^{\rm sm}, t_0) | \gamma_* \alpha = \alpha\}.$$

(2.74) is called tube map. Schnell showed that

Theorem 2.7.1. *([51]) If $H^{2n-2}_{\rm van}(Y_{t_0}, \mathbb{Z}) \neq 0$, then the image of the tube map* (2.74) *has full rank.*

Based on this result, Zhao proved the so-called Topological Jacobi Inversion Theorem as a generalization of Theorem 2.1.8:

Theorem 2.7.2. *([63]) There is a projective embedding $X \subseteq \mathbb{P}^N$ such that* (2.73) *is surjective. In fact, there is a connected component T_i of T surjects onto $J_{\rm prim}X$.*

Proof. By Schnell's Tube Mapping theorem, the are finitely many pairs (α_i, γ_i) such that the class $\alpha_i \in H^{2n-2}_{\rm van}(Y_{t_0}, \mathbb{Z})$ is fixed by $\gamma_i \in \pi_1(\mathbb{O}^{\rm sm}, t_0)$, $i = 1, \dots, d$, where $d = \dim_{\mathbb{R}}(J_{\rm prim}X)$.

By choosing smooth loops l_i based at t_0^i representing γ_i, it follows that the image of Tl_i, $i = 1, \dots, d$ under the composite of tangent map of (2.73) and projection

$$T(T) \to T(J_{\rm prim}X) \cong J_{\rm prim}X \times H_{2n-1}(X, \mathbb{R})_{\rm prim} \to H_{2n-1}(X, \mathbb{R})_{\rm prim}$$

is surjective, where $T(M)$ is denoted as the tangent bundle of M.

Consequently, one can choose $x_i \in T$ and $v_i \in T_{x_i}T$, such that $AJ_*(v_i)$ generate $H_{2n-1}(X, \mathbb{R})_{\rm prim}$.

Then by choosing a new linear system $\mathcal{L} = \mathcal{O}_X(\cup_i Y_i)$, one can embed X into a larger projective space, with $\tilde{Y} := \cup_i Y_i$ being a hyperplane section. It is shown that the topological Abel-Jacobi map extends to the class $\tilde{\alpha} = \alpha_1 + \cdots + \alpha_d$. The condition $AJ_*(v_i)$ generate $H_{2n-1}(X, \mathbb{R})_{\rm prim}$ implies that the Topological Abel-Jacobi map at $\tilde{\alpha}$ is submersion, so it is submersion in a nearby point $\tilde{\alpha}'$ in $T(\mathcal{L})$ over a smooth hyperplane section. Since $T(\mathcal{L}) \to$

$J_{\mathrm{prim}}X$ is real analytic, the submersion implies that the image contains an open subset U of $J_{\mathrm{prim}}X$. Then use the group structure of $J_{\mathrm{prim}}X$ and $T(\mathcal{L})$, one take a sufficient large m, mU cover the entire torus, so the component $m\tilde{\alpha}'$ will dominate the torus. □

In the end, we propose the following questions based on Theorem 2.7.2.

Question 2.7.3. *How can we determine the component T_i in Zhao's theorem above? Is the argument still true for the primitive vanishing component?*

Question 2.7.4. *Can we prove this result using Schnell's Topological Abel-Jacobi map?*

Chapter 3: Locus of Primitive Vanishing Cycles

In this chapter, we will define the object T_v, called locus of primitive vanishing cycles, associated with a smooth projective variety X with an embedding into $\mathbb{P}^N$. It is of the center of interest for the rest of the thesis.

In Section 3.1, we introduce the definition of primitive vanishing cycles on a hyperplane section of X. When deforming the primitive vanishing cycles in the open subspace $\mathbb{O}^{\mathrm{sm}}$ of $\mathbb{O} := (\mathbb{P}^N)^*$ parameterizing smooth hyperplane sections of X, we obtain the locus of primitive vanishing cycles T_v, which is a covering space of $\mathbb{O}^{\mathrm{sm}}$. Alternatively, T_v is a connected component of T, the étale space of the local system whose stalk is the integral vanishing cohomology of a hyperplane section of X.

In Section 3.2 - 3.5, we study various (partial) compactifications of T (and T_v) with the assumption that $\dim(X)$ is odd. In Section 3.2, we extend $T/\mathbb{O}^{\mathrm{sm}}$ to a branched covering space $T^{\mathrm{dp}}/\mathbb{O}^{\mathrm{dp}}$ accross the locus $\mathbb{O}^{\mathrm{dp}}$ where the hyperplane section has at most one ordinary double point, where the local monodromy on the vanishing cohomology has order two. This construction is based on a theorem on the degree of dual varieties of Landman [42].

In Section 3.3, we introduce a theorem of Stein [56] and Grauert-Remmert [27] (Theorem 3.3.1) which allows us to further extends $T^{\mathrm{dp}}/\mathbb{O}^{\mathrm{dp}}$ to $T^f/\mathbb{O}^f$ where the local monodromy is finite. In Section 3.4, we provide an interpretation of Theorem 3.3.1 in terms of Brieskorn's resolution [11] when X is a threefold and the hyperplane section of interest has at worst ADE type singularities.

In Section 3.5, we introduce Schenll's construction of the full compactification of T using Saito's theory on Hodge modules [53]. Note all (partial) compactifications stated above are compatible. In particular, Schnell's construction takes care of the case where the local monodromy is infinite.

3.1 Primitive Vanishing Cycles and their Deformations

Let X be a smooth projective variety of dimension n embedded in a projective space $\mathbb{P}^N$.

Let $\mathbb{O}^{\mathrm{sm}}$ be the open subspace of $\mathbb{O} := (\mathbb{P}^N)^*$ parameterizing smooth hyperplane sections of X. The complement $X^* := \mathbb{O} \setminus \mathbb{O}^{\mathrm{sm}}$ is a closed subvariety of $(\mathbb{P}^N)^*$ parameterizes singular hyperplane sections of X and is called the *dual variety* of X.

According to a classical result by Lefschetz, a smooth point of X^* corresponds to a hyperplane section that has only one ordinary node. Choose a holomorphic disk $\Delta \subset \mathbb{O}$ such that $\Delta^* = \Delta \setminus \{0\} \subset \mathbb{O}^{\mathrm{sm}}$ and Δ intersects the dual variety X^* transversely at a smooth point. Then $\{X_t\}_{t\in\Delta}$ is a one-parameter family of hyperplane sections of X with X_0 having a single node and X_t smooth for $t \neq 0$. Let $\mathcal{X}_\Delta$ denote the total space, and $B_p \subseteq \mathcal{X}_\Delta$ a small neighborhood of the node $p \in X_0$. When $|t|$ is small enough, the manifold $X_t \bigcap B_p$ is called the Milnor fiber of the family $\mathcal{X}_\Delta \to \Delta$, and it is diffeomorphic to the disk bundle of the tangent bundle TS^{n-1} of a topological $(n-1)$-sphere S^{n-1}. Moreover, the zero section S^{n-1} specializes to the node p as t moves to 0. As a result, the homology class of S^{n-1} is zero in homology of X and defines an element in the vanishing homology

$$H_{n-1}(X_t, \mathbb{Z})_{\mathrm{van}} := \ker(H_{n-1}(X_t, \mathbb{Z}) \to H_{n-1}(X, \mathbb{Z})).$$

So its Poincaré duality defines a class $[S^{n-1}]$ in the vanishing cohomology

$$H^{n-1}_{\mathrm{van}}(X_t, \mathbb{Z}) := \ker(H^{n-1}(X_t, \mathbb{Z}) \to H^{n+1}(X, \mathbb{Z})),$$

via Gysin homomorphism.

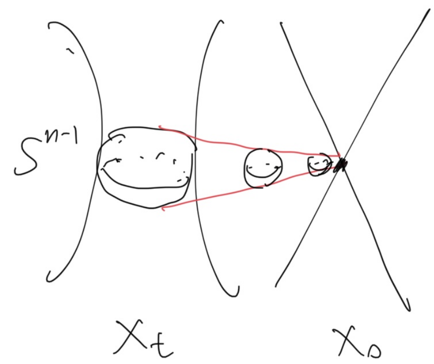


Figure 3.1: Vanishing Cycle

Definition 3.1.1. *The cohomology class $\delta_t = [S^{n-1}] \in H^{n-1}(X_t, \mathbb{Z})$ is called the vanishing cycle of the degeneration $\{X_t\}_{t\in\Delta}$.*

Lemma 3.1.2. *The intersection number $\delta_t \cdot \delta_t$ is 0 when $n-1$ is odd and is ± 2 when $n-1$ is even.*

This is due to computing the Eular class of the tangent bundle TS^{n-1}. The sign is determined by the parity of $n(n-1)/2$ since the orientation changes as the coodinate $(x_1, \ldots, x_{n-1}, y_1, \ldots, y_{n-1})$ on the real manifold TS^{n-1} is changed to complex coordinate $(x_1, y_1 \ldots, x_{n-1}, y_{n-1})$.

Fix a point $t_0 \in \Delta^*$. The local monodromy representation is a group homomorphism

$$\rho_\Delta : \pi_1(\Delta^*, t_0) \to Aut H^{n-1}(X_{t_0}, \mathbb{Z}). \tag{3.1}$$

The Picard-Lefschetz transformation gives a generator of (3.1).

$$f : \alpha \mapsto \alpha + \varepsilon_n(\alpha, \delta)\delta, \tag{3.2}$$

where $\varepsilon_n = \pm 1$ depending on the value of n and $\delta = \delta_{t_0}$ is the vanishing cycle. We call image of ρ_Δ the local monodromy group of the nodal family $\{X_t\}_{t\in\Delta}$.

Lemma 3.1.3. *When $n-1$ is even, then the monodromy group is of an involution. When $n-1$ is odd, then the monodromy group is of infinite order.*

Proof. When $n-1$ is odd, the intersection pairing $(\cdot,\cdot)$ is skew symmetric. By Lemma 3.1.2, $(\delta,\delta)=0$, therefore (3.2) implies

$$f^k(\alpha)=\alpha+k\varepsilon_n(\alpha,\delta)\delta,$$

so the monodromy is infinite.

When $n-1$ is even, the intersection pairing $(\cdot,\cdot)$ is symmetric and is preserved by the monodromy action. So by (3.2), one has

$$(\alpha,\beta)=(f(\alpha),f(\beta))=(\alpha,\beta)+2\varepsilon_n(\alpha,\delta)(\beta,\delta)+(\alpha,\delta)(\beta,\delta)(\delta,\delta),$$

which gives $2\varepsilon_n+(\delta,\delta)=0$. On the other hand, $f(f(\alpha))=f(\alpha+\varepsilon_n(\alpha,\delta)\delta)=\alpha+2\varepsilon_n(\alpha,\delta)\delta+(\alpha,\delta)(\delta,\delta)\delta=\alpha$. So it has order two. □

To define the global monodromy, let

$$\mathcal{X}_{\mathbb{O}^{\mathrm{sm}}}:=\{(x,t)\in X\times\mathbb{O}^{\mathrm{sm}}|x\in X_t\}.$$

Then the projection

$$\pi:\mathcal{X}_{\mathbb{O}^{\mathrm{sm}}}\to\mathbb{O}^{\mathrm{sm}} \tag{3.3}$$

to the second coordinate is the universal family of smooth hyperplane sections of X. π is a submersion, so by Ehresmann's theorem, it is a locally trivial fibration. In other words, if $t\in\mathbb{O}^{\mathrm{sm}}$, then there is an open neighborhood U_t of t in $\mathbb{O}^{\mathrm{sm}}$, such that $\mathcal{X}_{U_t}=\mathcal{X}_{\mathbb{O}^{\mathrm{sm}}}\times_{\mathbb{O}^{\mathrm{sm}}}U_t$ is diffeomorphic to $X_t\times U_t$. So as long as U_t is contractible, $H^k(\mathcal{X}_{U_t},\mathbb{Z})\cong H^k(X_t,\mathbb{Z})$ for each $k\geq 0$. As a consequence, $R^k\pi_*\mathbb{Z}$ is a locally constant sheaf (or a local system) for each $k\geq 0$. Here $R^k\pi_*$ is the k-th derived functor of π.

In other words, if we fix a base point $t_0 \in \mathbb{O}^{\text{sm}}$, then there is a monodromy representation

$$\rho : \pi_1(\mathbb{O}^{\text{sm}}, t_0) \to \text{Aut} H^k(X_{t_0}, \mathbb{Z}),$$

which sends each $[l] \in \pi_1(\mathbb{O}^{\text{sm}}, t_0)$ represented by a loop $l \subseteq \mathbb{O}^{\text{sm}}$ based at t_0 to an automorphism $\rho([l])_* : H^k(X_{t_0}, \mathbb{Z}) \to H^k(X_{t_0}, \mathbb{Z})$ for each $k \geq 0$. This could be done concretely by subdividing l into segments $l = \cup_{i=1}^m l_i$ with each segment l_i joining t_{i-1} to t_i and $t_m = t_0$ and such that each l_i is contained in a contracible open subset. By trivializing π along each l_i, we has diffemorphism $f_i : X_{t_{i-1}} \cong X_{t_i}$. One has diffeomorphism $f : X_{t_0} \cong X_{t_0}$ defined by the compositions $f_m \circ \cdots \circ f_1$. Then it induces automorphism $f_* : H^k(X_{t_0}, \mathbb{Z}) \to H^k(X_{t_0}, \mathbb{Z})$. It turns out that f_* is independent of choice of subdivison of a loop and choice of representative in the homotopy equivalent class $[l]$, and $f_* = \rho([l])$. This is called the *monodromy action of* $[l]$.

If $\alpha \in H^{n-1}_{\text{van}}(X_t, \mathbb{Z})$, then $\rho([l])(\alpha)$ is also contained in $H^{n-1}_{\text{van}}(X_t, \mathbb{Z})$. This is because, for example, the property that Poincaré dual of α is zero in $H_{n-1}(X, \mathbb{Z})$ is preserved in a contractible open neighborhood of t. It follows that the vanishing cohomology is invariant under the monodromy action. We are interested in the monodromy representation

$$\rho_{\text{van}} : \pi_1(\mathbb{O}^{\text{sm}}, t_0) \to \text{Aut} H^{n-1}_{\text{van}}(X_{t_0}, \mathbb{Z}). \tag{3.4}$$

If δ_{t_1} is a vanishing cycle on X_{t_1} with respect to the family $\{X_t\}_{t \in \Delta}$, where Δ is a holomorphic disk transverse to X^* at a smooth point as we defined earlier and $t_1 \in \Delta^*$. Fix $t_0 \in \mathbb{O}^{\text{sm}}$. Take any path $l \subseteq \mathbb{O}^{\text{sm}}$ joining t_1 to t_0. By trivializing the fibration over l, one has diffeomoprhism $X_{t_1} \cong X_{t_0}$, which induces an isomorphism $l_* : H^{n-1}_{\text{van}}(X_{t_1}, \mathbb{Z}) \cong H^{n-1}_{\text{van}}(X_{t_0}, \mathbb{Z})$. Then $l_*(\delta_{t_1})$ is a class in $H^{n-1}_{\text{van}}(X_{t_0}, \mathbb{Z})$.

Definition 3.1.4. *We call $\alpha \in H^{n-1}_{\text{van}}(X_{t_0}, \mathbb{Z})$ a primitive vanishing cycle if it is obtained from the above.*

The following statement is a variant of [62, Proposition 3.23].

Proposition 3.1.5. *Any two primitive vanishing cycles* $\alpha_1, \alpha_2 \in H^{n-1}_{\mathrm{van}}(X_{t_0}, \mathbb{Z})$ *are conjugate to each other by monodromy. In other words, there is a* $[l] \in \pi_1(\mathbb{O}^{\mathrm{sm}}, t_0)$ *such that* $\rho_{\mathrm{van}}([l])(\alpha_1) = \alpha_2$.

Proof. There are holomorphic disks Δ_i in $\mathbb{O}$ intersecting X^* transversely at smooth points for $i = 1, 2$. There are vanishing cycles δ_{t_i} associated to the families $\{X_t\}_{t\in\Delta_i}$ and paths l_i joining t_i to t_0 such that $(l_i)_*(\delta_{t_i}) = \alpha_i$ for $i = 1, 2$. It suffices to show that there is a path $l_{12} \subseteq \mathbb{O}^{\mathrm{sm}}$ joining t_1 to t_2, such that $(l_{12})_*(\delta_1) = \delta_2$, then it follows that $l = l_2 \circ l_{12} \circ l_1^{-1}$ is the desired loop based at t_0 and such that $l_*(\alpha_1) = \alpha_2$.

The existence of l_{12} follows from the fact that the irreducibility of the dual variety X^*. As a consequence, the smooth locus $(X^*)^{\mathrm{sm}}$ of X^* is connected. Suppose Δ_i is transversal to $(X^*)^{\mathrm{sm}}$ at points t_i'. Then one can choose a path $l_{12}' \subseteq (X^*)^{\mathrm{sm}}$ joining t_1' to t_2'. Then define the path l_{12} to be a lift of l_{12}' to a path joining t_1 to t_2 and is contained in the boundary of a tubular neighborhood of $(X^*)^{\mathrm{sm}}$ in $\mathbb{O}^{\mathrm{sm}}$. □

Corollary 3.1.6. *Let* α *be a primitive vanishing cycle on* X_{t_0}*. Then the set* PV_{t_0} *of all primitive vanishing cycles on* X_{t_0} *is the orbit* $Im(\rho_{\mathrm{van}}).\alpha$ *in* $H^{n-1}_{\mathrm{van}}(X_{t_0}, \mathbb{Z})$.

Proposition 3.1.7. *The set of all primitive vanishing cycles in* $H^{n-1}_{\mathrm{van}}(X_{t_0}, \mathbb{Z})$ *generate a subgroup of full rank.*

Proof. Take a Lefschetz pencil, i.e., a general line $\mathbb{L} \subseteq (\mathbb{P}^N)^*$ intersecting X^* transversely at smooth points $s_1, \ldots, s_d$, where $d = \deg(X^*)$, then there are vanishing cycles δ_{t_i} over smooth points t_i around s_i. transport δ_{t_i} to a primitive vanishing cycle $\alpha_i \in H^{n-1}_{\mathrm{van}}(X_{t_0}, \mathbb{Z})$ by a path joining t_i to t_0, then it is follows from a classical theorem that $\{\alpha_1, \ldots, \alpha_d\}$ generates a subgroup of full rank [62, Lemma 2.26]. □

Let T denote the étale space of the local system $\mathcal{H}^{n-1}_{\text{van}}$ on $\mathbb{O}^{\text{sm}}$ whose stalk at t is the vanishing cohomology $H^{n-1}_{\text{van}}(X_t, \mathbb{Z})$, then

$$T \to \mathbb{O}^{\text{sm}} \tag{3.5}$$

is an analytic covering space with infinite sheets and has infinitely many connected components. The monodromy of this covering space encodes the complexity of the local system $\mathcal{H}^{n-1}_{\text{van}}$.

T has a distinguished component T_v, containing a primitive vanishing cycle. By (3.1.5), every two primitive vanishing cycles are conjugate to each other by monodromy action, so T_v is unique and well-defined.

Definition 3.1.8. We call the covering space $T_v \to \mathbb{O}^{\text{sm}}$ *the locus of primitive vanishing cycles* on the hyperplane sections of X.

Our primary goal of this thesis is to understand the topology and geometry of the complex analytic manifold T_v (which is quasi-projective when $T_v \to \mathbb{O}^{\text{sm}}$ is finite) in the case when X has odd dimension.

We should be able to show that $T \to \mathbb{O}^{\text{sm}}$ is of finite cover if and only if $H^{n-1}_{\text{van}}(X_t, \mathbb{Z})$ consists of Hodge classes. (A strengthening argument should show the same result for T_v.) As an example, we should be able to show that when X is a quartic threefold and there is a hyperplane section having an elliptic singularity, the local monodromy group is infinite. Therefore $T \to \mathbb{O}^{\text{sm}}$ has infinite sheets.

We are looking for ways to compactify the covering space T_v. This question often goes hand in hand with compactification of T, since T contains T_v is an irreducible component, and conversely, the monodromy of T_v determines monodromy of T by Proposition 3.1.7.

3.2 Extension Across the Nodal Locus

3.2.1 A Local Argument

Let's first work locally. Let X be a smooth projective variety of odd dimension, and $T_v \to \mathbb{O}^{\text{sm}}$ be the locus of primitive vanishing cycles on hyperplane sections of X. Let Δ be a holomorphic disk in $\mathbb{O}$ transverse to a smooth point of the dual variety X^*. So it corresponds to a family of hyperplane sections $\{X_t\}_{t\in\Delta}$ such that X_t is smooth for $t \neq 0$ and X_0 has an ordinary node.

Denote

$$T(\Delta^*) \to \Delta^* \tag{3.6}$$

the restriction of the covering space $T \to \mathbb{O}^{\text{sm}}$ defined in (3.5).

Claim 3.2.1. *There is an analytic space $T(\Delta) = \overline{T(\Delta^*)}$ and an analytic branched covering $T(\Delta) \to \Delta$ extending* (3.6)*. Moreover, this extension is unique.*

Proof. By assumption, the hyperplane section X_t has even dimension, so by Lemma 3.1.3, the monodromy of $T(\Delta^*) \to \Delta^*$ has order two. So $T(\Delta^*)$ is the disjoint union of sheets $T(\Delta^*) = \coprod_{i\in I} \Delta_i^*$. Let δ be the vanishing cycle and α_i a class corresponding to a point in Δ_i^*. Then each connected covering $\Delta_i^* \to \Delta^*$ is either

(i) 1-to-1, when $(\alpha_i, \delta) = 0$, or

(ii) 2-to-1, when $(\alpha_i, \delta) \neq 0$.

So the disjoint union $\coprod_i \Delta_i \to \Delta$ defines an extension of $T \to \mathbb{O}^{\text{sm}}$. □

3.2.2 Simultaneous Resolution

To provide an interpretation from birational geometry, we consider a one-parameter family of surfaces $\{X_t\}_{t\in\Delta}$ such that X_t is smooth for $t \neq 0$ and X_0 has an ordinary node. Moreover, we assume the total space of the family is smooth. Then $T(\Delta^*)$ is the underlying analytic space of the local system $\mathcal{H}^2_{\text{van}}$ over Δ^*.

The monodromy of $T(\Delta^*) \to \Delta^*$ has order two. Therefore, by taking a double cover $\tilde{\Delta} \to \Delta$ branched at 0 and base change, we get a new family $\{X_s\}_{s\in\tilde{\Delta}}$ with $s^2 = t$. Now the total space has an ordinary double point. We can take a small resolution (in analytic category) and get a smooth family $\{\tilde{X}_s\}_{s\in\tilde{\Delta}}$ of surfaces, where $\tilde{X}_s = X_s$ when $s \neq 0$ and $\tilde{X}_0 \to X_0$ is the minimal resolution of the ordinary node. In other words, we have a commutative diagram

$$\begin{array}{ccc} \tilde{\mathcal{X}} & \xrightarrow{h} & \mathcal{X} \\ \downarrow f & \swarrow g & \\ \tilde{\Delta} & & \end{array}$$

Here f is a smooth morphism whose fiber is $\tilde{X}_s$ and h is a birational map which is isomorphic over $s \neq 0$ and desingularizes the fiber over $s = 0$.

Then by Ehresmann's theorem, f is topologically trivial, so the local system $R^2 f_* \mathbb{Z}_{\mathrm{van}}$ is trivial and its underlying analytic space $T(\tilde{\Delta})$ is an extension of the double cover $T(\Delta^*) \times_{\Delta^*} \tilde{\Delta}^*$ of $T(\Delta^*)$. By extending the $\mathbb{Z}_2$-quotient to $T(\tilde{\Delta})$, we defines $T(\Delta)$ as $T(\tilde{\Delta})/\mathbb{Z}_2$.

3.2.3 Globalization

Now, denote $\mathbb{O}^{\mathrm{dp}} \subseteq \mathbb{O}$ the locus corresponding to the hyperplane sections, which have at most one ordinary double point. Then by Lefschetz, $\mathbb{O}^{\mathrm{dp}}$ is the complement of the singular locus $Sing(X^*)$ of the dual variety. We claim that

Proposition 3.2.2. *There is an analytic space T^{dp} containing T as open dense subset and a branched covering map $T^{\mathrm{dp}} \to \mathbb{O}^{\mathrm{dp}}$ extending $T \to \mathbb{O}^{\mathrm{sm}}$. Moreover T^{dp} is unique.*

Proof. Take a point $t_0 \in \mathbb{O}^{\mathrm{dp}}$ corresponding to a hyperplane section with one ordinary double point. Then there is an open neighborhood U_{t_0} of t_0 in $\mathbb{O}^{\mathrm{dp}}$ with $U_{t_0} \cong \Delta^{N-1} \times \Delta$, with the polydisk Δ^{N-1} identified with $U_{t_0} \cap \mathbb{O}^{\mathrm{dp}} \setminus \mathbb{O}^{\mathrm{sm}}$. So $U_{t_0} \cong \Delta^{N-1} \times \Delta^*$ and each connected component $T(U_{t_0}^{sm})$ is either a 2-to-1 or is isomorphic to $\Delta^{N-1} \times \Delta^*$ and therefore extends to a branched cover $T(U_{t_0}) \to U_{t_0}$ according to Claim 3.2.1. By uniqueness, we can patch $T(U_{t_0})$ together to obtain T^{dp}. $\square$

3.2.4 Landman's Theorem

An alternative approach to obtaining Proposition 3.2.2 is to globalize the extension of the $\mathbb{Z}_2$-quotient construction at the end of the Section 3.2.2. We need to use a theorem due to Landman, first proved in his unpublished work [42].

Theorem 3.2.3. *(Landman, 76) If X is a smooth odd dimensional projective variety, then X^* has even degree in $(\mathbb{P}^N)^*$.*

Proof of Proposition 3.2.2 based on Theorem 3.2.3.

Since $X^* \subseteq (\mathbb{P}^N)^*$ has even degree, $\mathcal{O}_{(\mathbb{P}^N)^*}(X^*) = \mathcal{L}^{\otimes 2}$ is square of a line bundle $\mathcal{L} \cong \mathcal{O}_{(\mathbb{P}^N)^*}(\frac{\deg(X^*)}{2})$, so by [5, Lemma 17.1], there is a double cover $\sigma : \tilde{\mathbb{O}} \to \mathbb{O}$ branched along X^*. Restrict to $\sigma^{dp} : \tilde{\mathbb{O}}^{dp} \to \mathbb{O}^{\mathrm{dp}}$, the branched locus is smooth. Moreover, by Lemma 3.1.3, the local monodromy around $\mathbb{O}^{\mathrm{dp}} \setminus \mathbb{O}^{\mathrm{sm}} = (X^*)^{\mathrm{sm}}$ has order two. Therefore, the pullback $\tilde{T}$ of T has trivial local monodromy around preimage of $(X^*)^{\mathrm{sm}}$ and extends to a (unbranched) covering space $\tilde{T}^{dp} \to \tilde{\mathbb{O}}^{dp}$.

Now, we have the following diagram

$$\begin{array}{ccc} \tilde{T}^{dp}/\tilde{\mathbb{O}}^{dp} & \overset{\tilde{f}}{\dashrightarrow} & T^{\mathrm{dp}}/\mathbb{O}^{\mathrm{dp}} \\ {\scriptstyle i}\uparrow & & \uparrow \\ \tilde{T}/\tilde{\mathbb{O}}^{sm} & \xrightarrow{f} & T/\mathbb{O}^{\mathrm{sm}} \end{array} \tag{3.7}$$

Here f is the 2-to-1 cover given by fiber product $\tilde{T} = T \times_{\mathbb{O}^{\rm sm}} \tilde{\mathbb{O}}^{sm} \to T$, and i is the natural inclusion. We want to extend the $\mathbb{Z}_2$ quotient f to a morphism $\tilde{f}$ whose image is our desired $T^{\rm dp}$.

This can be done by the following: Let $t_0 \in \mathbb{O}^{\rm dp} \backslash \mathbb{O}^{\rm sm}$ and let U_{t_0} be an open neighborhood of t_0 in $\mathbb{O}^{\rm dp}$. Denote $\pi^{dp} : T^{\rm dp} \to \mathbb{O}^{\rm dp}$ the projection and π^{sm} the restriction to $\mathbb{O}^{\rm sm}$. Let V_i be a connected component of $\pi^{-1}(U_{t_0}^{sm})$. If $V_i \to U_{t_0}^{sm}$ is 1-to-1, then just declare U_{t_0} to be the completion of V_i. When $V_i \to U_{t_0}^{sm}$ is 2-to-1, then $f^{-1}(U_{t_0}^{sm}) \to U_{t_0}^{sm}$ is a disjoint double cover. Take the closure of $f^{-1}(U_{t_0}^{sm})$ in $\tilde{T}^{dp}$ and declare one component to be the completion V_i.

Since both $\mathbb{O}^{\rm dp}$ and $\tilde{T}^{dp}$ are global, the construction above is global and defines an analytic branced covering space $T^{\rm dp}/\mathbb{O}^{\rm dp}$ extending $T/\mathbb{O}^{\rm sm}$. $\square$

3.2.5 Proof of Landman's Theorem.

The rest of this section is devoted to proving Landman's theorem.

An alternative definition of dual variety is defined as a coordinate projection of incidence variety. More precisely, we define the incidence variety $I_X^0 := \{(x, H) \in \mathbb{P}^N \times (\mathbb{P}^N)^*)|x \in X$ smooth, H tangent to X at $x\}$, with the Zariski closure I_X. The dual variety X^* is defined as the image of the second projection $\pi_2(I_X)$. As the linear forms vanishing on a projective n subspace of $\mathbb{P}^N$ is a projective $N-n-1$ subspace of $(\mathbb{P}^N)^*$, it follows that I_X^0 is a locally trivial $N-n-1$ projective bundle over X^{sm}, so I_X^0 is irreducible with dimension $N-1$, and so is the closure I_X. As a result, the dual variety X^* is irreducible with dimension at most $N-1$.

We recall the following classical results:

Proposition 3.2.4. *(1) $X^{**} \cong X$;*

(2) X^* is a hypersurface iff there is a hyperplane section with only one nodal singularity.

Proof. Let U be an analytic neighborhood of a smooth point $x \in X^{sm}$ with coordinates $z_1, ..., z_n$. Now, a hypersurface H tangents to X at x is equivalent to that $H(x) = 0$ and $\partial H/\partial z_i(x) = 0, i = 1, ..., n$, where H is regarded a linear functional on the ambient space restricted to X. Assume that H is a smooth point in X^*, then The cotangent space of I_X at point (x, H) is the linear span of differential of the $n+1$ equations above. In particular, conditions for $(u, K) \in T_xX \times T_H\mathbb{P}^N$ to be in $T_{(x,H)}I_X$ is characterized as

$$K(x) + d_uH(x) = 0$$

$$\partial K/\partial z_i(x) + \partial(d_uH)/\partial z_i = 0, i = 1, .., n$$

As $\partial H/\partial z_i(x) = 0$ is the condition for H to be tangent to X at x, the first equation is equivalent to $K(x) = 0$. Now we consider the dual X^{**} of X^* at point H, which is the hyperplanes $W \subset \mathbb{P}^N$ which contains tangent space T_HX^*. But as the double dual $\mathbb{P}^{N**}$ is canonically identified with $\mathbb{P}^N$ via $W \leftrightarrow x$ iff $W(H) = H(x)$ for all $H \in (\mathbb{P}^N)^*$. Now, $K(x) = 0$ for all $K \in T_HX^*$ along with $H(x) = 0$ shows that x corresponds to a hyperplane tangents to H at X^*, this works for all smooth points $x \in X^{sm}$, and it follows that $I^0_X \subset I^0_{x^*}$. Taking Zariski closure, and by irreducibility as well as $\dim I_X = \dim I_{X^*}$, it follows that $I_X = I_{X^*}$, and therefore $X \cong X^{**}$.

To prove the second argument, X^* has the maximal dimension if and only if the second projection $pr_2 : I_X \to X^*$ is immersion at some smooth point (x, H). This is equivalent to that $\ker pr_{2*} = 0$ at (x, H), which, by the $n+1$ equations above, is equivalent to that $\partial^2 H/\partial z_i \partial z_j(x)$ is nondegenerate, which is the same as saying the hyperplane section X_H has a nodal singularity at x. □

To prove Theorem 3.2.3, we need the following lemma:

Lemma 3.2.5. *Let $r = codim_{(\mathbb{P}^N)^*}X^* - 1 > 0$, then*

(1) At each point $x \in X$, there is a r dimensional projective subspace L of dimension r

such that $x \in \mathbb{P}^r \subset X$; (2) If X is smooth, then for any $H \in (X^*)^{\mathrm{sm}}$, the contact locus Sing$(H \cap X)$ is a projective subspace of dimension r.

Proof. (1) By the reflective argument we have just established, to show (1) is equivalent to showing the same result for X^{**}. But this is an immediate result from the fact that the hyperplanes in $(\mathbb{P}^N)^*$ touching $(X^*)^{\mathrm{sm}}$ at H up to first-order form a r dimensional subspace in $(\mathbb{P}^N)^{**}$, and the union of these projective r space is Zariski dense in X^{**}. Finally, take the Zariski closure, then we have the desired result.

(2) is also direct from the reflectivity theorem, as for $H \in (X^*)^{\mathrm{sm}}$, the set of points $x \in X$, which H tangents at corresponds to the hyperplanes of $(\mathbb{P}^N)^*$ which tangents to X^* at H, but the latter one is obviously a projective r space. □

Proof of Theorem 3.2.3[1]

Step 1. Assume that the dual variety $X^* \subset (\mathbb{P}^N)^*$ is a hypersurface.

Take a generic line $\mathbb{L} \subset (\mathbb{P}^N)^*$ meeting X^* transversely at d points. We want to show that d is even.

Let X_{t_i}, $i = 1, ..., d$ denote the hyperplane section corresponding to intersection points $t_i \in \mathbb{L} \cap X^*$. X_{t_i} has only one ordinary double point. The local analytic equation at the singularity is $x_0^2 + ... + x_n^2 = 0$, which is the limit as $\epsilon \to 0$ of $x_0^2 + ... + x_n^2 = \epsilon$, which is a $(n-1)$-dimensional sphere contained in the nearby smooth fiber X_{t_0}. So X_{t_i} is homotopic equivalent to attaching a cone $D^n \cong C(S^{n-1})$ over S^{n-1} on X_{t_0}. Consequently, $\dim H^{n-1}(X_{t_0}) = \dim H^{n-1}(X_{t_i}) + 1$, and $\dim H^k(X_{t_0}) = \dim H^k(X_{t_i})$ for $k \neq n-1$. Therefore, we have the relation between the euler characteristics of X_{t_0} and X_{t_i} is given by $\chi(X_{t_i}) = \chi(X_{t_0}) - (-1)^{n-1}$.

Let B be the base locus of the pencil. Then there is a fibration $X \setminus B \to \mathbb{L}$ with fiber $X_t \setminus B$. By removing the singular fibers, we get a fiber bundle over $\mathbb{L} \setminus \{t_1, \ldots, t_d\}$. By inclusion-exclusion principle and multiplicativity of the Eular characteristic on a fiber

[1] I would like to thank A. Landman for showing me the proof in an email.

bundle, we have

$$\begin{aligned}\chi(X) &= (2-d)\chi(X_{t_0} - B) + d\chi(X_{t_i} - B) + \chi(B) \\ &= (2-d)(\chi(X_{t_0}) - \chi(B)) + d(\chi(X_{t_i}) - \chi(B)) + \chi(B) \\ &= 2\chi(X_{t_0}) + d(-1)^n - \chi(B) \end{aligned} \tag{3.8}$$

Finally, by Poincaré duality, $\dim H^k(X) = \dim H^{2n-k}(X)$, and as X is assumed to be odd-dimensional, $\chi(X)$ has the same parity with the dimension of its middle dimensional cohomology $H^n(X)$. Again by Poincaré duality, the cup product restricted on which is nondegenerate, and alternating, which implies that $H^n(X)$ is even-dimensional, so $\chi(X)$ is even. Similarly, as B is smooth with two dimensional lower than X, it is still odd-dimensional, and $\chi(B)$ is even. Altogether, modulo 2 in (3.8), we showed that X^* is of even degree.

Step 2. In general, if X^* is not a hypersurface, define $r = \operatorname{codim}_{(\mathbb{P}^N)^*} X^* - 1 > 0$. We take a general hyperplane $H \subset \mathbb{P}^N$ so that H intersects X transversely. Note that in our case, as X is smooth, the incident variety I_X is identified with the projective conormal bundle $\mathbb{P}(N^*_{X|\mathbb{P}^N})$ and $X^* = \pi_2(P(N^*_{X|\mathbb{P}^N}))$ If we denote consider the restricted conormal bundle $D := \mathbb{P}(N^*_{X|\mathbb{P}^N})|_{X_H}$on X_H, clearly $\pi_2(D) \subset X^*$.

We also note that $\pi_2(D)$ is the dual variety of X_H in the hyperplane H and the dual variety X_H^* in $(\mathbb{P}^N)^*$ is the cone over $\pi_2(D)$ with vertex p, where $p \in (\mathbb{P}^N)^*$ is the point corresponding to the hyperplane H.

We claim that $\pi_2(D) = X^*$. If so, X_H^* is one dimensional higher than X^* with the same degree as X^*, therefore one can induct and reduce to the case that X^* is a hypersurface. To prove this, note that the projection $\pi_2 : I_X \to X^*$ has generic fiber $\pi^{-1}(q)$ of dimension $r > 0$. But by Lemma 3.2.5, $\pi_2^{-1}(q)$ is a projective space of dimension $r > 0$, intersecting H

transversely. Therefore $\dim D \cap \pi_2^{-1}(q) = \dim \pi_2^{-1}(q) - 1$. In particular, $D \cap \pi_2^{-1}(q) \neq \emptyset$, so $q \in \pi_2(D)$. As q is picked generically from X^*, it follows that $\pi_2(D) = X^*$. □

So far, we have extended $T \to \mathbb{O}^{\mathrm{sm}}$ to codimension-two locus, based on the fact that the local monodromy around the smooth locus of dual variety has order two. This gives a "partial compactification" of T. We're still interested in a full compactification. We'll provide different ways to compactify T in the next few sections.

3.3 Partial Compactification and Finite Monodromy

Recall that in Proposition 3.2.2, we extend the covering $T \to \mathbb{O}^{\mathrm{sm}}$ across codimension-two to obtain a branched covering $T^{\mathrm{dp}} \to \mathbb{O}^{\mathrm{dp}}$, based on the local monodromy on the boundary smooth divisor has order two. In fact, as long as the local monodromy is finite, the completion of (3.6) exists and is unique. It will follow from a Lemma due to Stein [56] and Grauert-Remmert [27]. Also see [21, p.197].

Theorem 3.3.1. *Let U be a complex manifold, and $f : W \to U$ a finite analytic cover. Assume $\bar{U}$ is a normal analytic space containing U as an open dense subspace. There is a normal analytic space $\bar{W}$ containing W as a dense open subspace, together with finite analytic branched covering map $\bar{f} : \bar{W} \to \bar{U}$, which agrees with f on W. Moreover, when $\bar{U}$ is projective, $\bar{W}$ is also projective.*

We will only give an account for the algebraicity argument. The pushforward $\mathcal{F} = \bar{f}_* \mathcal{O}_W$ defines an analytic coherent sheaf on $\bar{U}$. By Serre's GAGA, the projectivity of $\bar{U}$ implies that $\mathcal{F}$ is an algebraic coherent sheaf. Then by definition of the relative spec construction [33, Exercise II.5.17], $\bar{W}$ is isomorphic to $\mathrm{Spec}_{\mathcal{O}_{\bar{W}}} F$, and therefore is algebraic. □

For each $t_0 \in \mathbb{O}$, pick a suitably small open neighborhood B of t_0. Fix another base point $t' \in B^{\mathrm{sm}} := \mathbb{O}^{\mathrm{sm}} \cap B$, then the monodromy action is a homomorphism

$$\rho_{t_0} : \pi_1(B^{\mathrm{sm}}, t') \to H^{n-1}(X_{t_0}, Z). \tag{3.9}$$

Definition 3.3.2. *We call $G_{t_0} = \mathrm{Im}(\rho_{t_0})$ the (local) monodromy group of the hyperplane section X_{t_0}.*

Corollary 3.3.3. *The assumption is the same as above. We define $\mathbb{O}^f \subseteq \mathbb{O}$ as the open subset consisting of points $t_0 \in \mathbb{O}$, where the local monodromy is finite. Then there is a unique analytic space T^f, together with an analytic branched covering $T^f \to \mathbb{O}^f$ extending* (3.5).

Proof. For each $t_0 \in \mathbb{O}^f$, there is a suitably small open neighborhood B of t_0 such that each connected component of the covering $T(B^{\mathrm{sm}}) \to B^{\mathrm{sm}}$ has a finite sheet. So by Theorem 3.3.1, each connected component extends to a branched covering. By taking the union, one obtains the local completion $T(B^f) \to B^f$ extending $T(B^{\mathrm{sm}}) \to B^{\mathrm{sm}}$, where $B^f = B \cap \mathbb{O}^f$. By uniqueness, we can patch $T(B^f)$ together and obtain T^f. □

Corollary 3.3.4. *Suppose $T \to \mathbb{O}^{\mathrm{sm}}$ has globally finite monodromy (i.e., the image of* (3.4) *is finite), then for each connected component T_i of T, there exists a normal algebraic variety $\bar{T}_i$ compactifying T_i and there is a finite algebraic morphism $\bar{T}_i \to \mathbb{O}$ extending $T_i \to \mathbb{O}^{\mathrm{sm}}$.*

For example, when X is a smooth cubic threefold, the global monodromy is finite. In particular, the compactification $\bar{T}_v$ of the locus of primitive vanishing cycles T_v exists and is a normal algebraic variety. In the next section, we will characterize the space.

3.4 Interpretation for Threefold Case: Brieskorn's Resolution

3.4.1 Simultaneous Resolution

Let $X \to S$ be a flat morphism of analytic spaces. A simultaneous resolution for $X \to S$ is a commutative diagram

where (1) the square on the right is fiber product via a finite surjective map $T \to S$.

(2) $\psi : Y \to X \times_S T$ is proper surjective and ψ induces by $\psi_t : \theta^{-1}(t) \to \chi^{-1}(t)$ is a resolution

$$
\begin{array}{ccccc}
Y & \xrightarrow{\psi} & X \times_S T & \longrightarrow & X \\
 & \searrow^{\theta} & \downarrow{\scriptstyle \chi} & & \downarrow \\
 & & T & \longrightarrow & S
\end{array}
$$

of singularity of $\chi^{-1}(t)$ for all $t \in T$, and

(3) θ is a smooth morphism.

Let $\mathfrak{g}$ be a simple lie algebra of type A, D, E over $\mathbb{C}$, and $\mathfrak{h}$ the Cartan subalgebra, with Weyl group W action. Let G be a simple Lie group whose Lie algebra is $\mathfrak{g}$, then by a theorem of Chevalley, the space $\mathfrak{g}/G$ is isomorphic to $\mathfrak{h}/W$, an affine space of dimension $r = rank(\mathfrak{g})$. Let f be the composite $\mathfrak{g} \to \mathfrak{g}/G \cong \mathfrak{h}/W$, Grothendieck showed that the diagram

$$
\begin{array}{ccc}
G \times_B \mathcal{B} & \longrightarrow & \mathfrak{g} \\
\downarrow & & \downarrow{\scriptstyle f} \\
\mathfrak{h} & \longrightarrow & \mathfrak{h}/W
\end{array}
$$

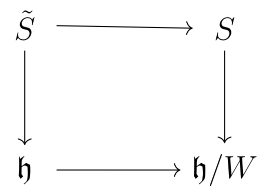
is a simultaneous resolution of f, where $\mathcal{B}$ be the variety parametrizing the set of Borel subalgebra of $\mathfrak{g}$.

Brieskorn [B] used this result to construct a simultaneous resolution of rational double point singularities on surfaces. To be more precise, let x be a subregular nilpotent element on $\mathfrak{g}$, and $S \subset \mathfrak{g}$ be a transversal slice with respect to the subregular locus of $\mathfrak{g}$ containing x, then

Theorem 3.4.1. *(Brieskorn, 1970)*

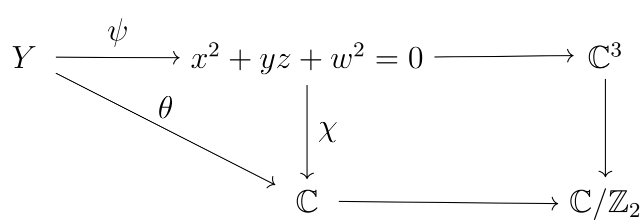
(1) The restriction $f : (S, x) \to (\mathfrak{h}_{\mathbb{C}}/W, \bar{0})$ is a semiuniversal deformation of rational double point singularity.

(2) By restricting to Grothendieck's diagram to the transverse slice, the obtained diagram



is a simultaneous resolution of the universal deformation of the rational double point singularities.

To give the simplest example, for type A_1, the only subregular element is $0 \in \mathfrak{g}$, so the transverse slice is $\mathfrak{g}$ itself. Therefore, the universal deformation of the A_1 singularity is $\mathfrak{g} \cong \mathbb{C}^3 \to \mathbb{C}/\mathbb{Z}_2 \cong \mathbb{C}$, $(x, y, z) \mapsto -x^2 - yz$. As the fiber over 0 is the only singular fiber, the simultaneous resolution is a small resolution of the ordinary double point on threefold



by putting a rational curve with self-intersection -2 along the fiber over the singular point.

3.4.2 Geometric Interpretation

As an application of Brieskorn's theory, we provide a geometric interpretation of a Theorem 3.3.1 in the case when U is an open subset of a polydisk and $W \to U$ comes from the variation of Hodge structure on vanishing cohomology of a smooth family of surfaces on U with at worst ADE singularities over $\bar{U}$. By a global (in fiber direction) version of Brieskorn's theorem, by taking a finite covering and base change, there is a simultaneous resolution of the total space, which implies the certain base change of $T(U)$ will extend to a covering space. By extending the finite quotient, we obtain the desired completion $\bar{W}$.

As an additional remark, Brieskorn's resolution is not unique (although the resulting completion of $T(U)$ is unique), so the "geometric interpretation" will not work when the base is not local.

3.5 General Case: Schnell's Completion and Infinite Monodromy

3.5.1 Infinite Monodromy

Corollary 3.3.4 tells us that when the global monodromy is finite, we can compactify the covering map $T \to \mathbb{O}^{\mathrm{sm}}$. However, the following result will show it is very rare:

Lemma 3.5.1. *Assume $X \subseteq \mathbb{P}^{n+1}$ is a smooth hypersurface of odd dimension. Then $T \to \mathbb{O}^{\mathrm{sm}}$ has finite monodromy if and only if the vanishing cohomology $H^{n-1}_{\mathrm{van}}(X_t, \mathbb{Z})$ is concentrated on Hodge type.*

Proof. The sufficiency is straightforward, since the intersection pairing is definite on the subspace $H^{\frac{n-1}{2},\frac{n-1}{2}}(X_t, \mathbb{C})$. The necessity can be found in [57, p.295]. □

Corollary 3.5.2. *When X is a hypersurface of $\mathbb{P}^4$ with degree at least 4, $T \to \mathbb{O}^{\mathrm{sm}}$ has infinite global monodromy, and*

$$T_v \to \mathbb{O}^{\mathrm{sm}}$$

is a covering space of infinite sheet.

In fact, we can construct a quartic threefold with a hyperplane section having a triple point singularity, and the local monodromy around such a hyperplane section is infinite.

Since this thesis focuses on hypersurfaces of $\mathbb{P}^4$, we are interested in finding an analytic completion of T_v. Schnell provides an answer by [50] using theory on Hodge modules.

3.5.2 Schnell's Completion

For a polarized variation of Hodge structure $(\mathcal{H}, Q)$ of even weight over a quasi-projective variety B_0, as a Zariski open subset of a smooth projective variety B, Schnell [50] constructed a completion of $T_{\mathbb{Z}}$, the étale space of the local system $\mathcal{H}_{\mathbb{Z}}$.

More explicitly, assume that $\mathcal{H}$ has weight $2n$. The data $(\mathcal{H}, Q)$ consists of a $\mathbb{Z}$-local system over B_0, a flat connection ∇ on $\mathcal{H}_{\mathbb{C}} = \mathcal{H} \times_{\mathbb{Z}} \mathcal{O}_{B_0}$, Hodge bundles $F^p\mathcal{H}_{\mathbb{C}}$ and a nondegenerate pairing

$$Q : H_{\mathbb{Q}} \times H_{\mathbb{Q}} \to \mathbb{Q}$$

satisfying the Hodge-Riemann conditions.

Consider $F^n\mathcal{H}$ the associated Hodge bundle, i.e., the subbundle whose fiber at $p \in B_0$ is $F^n\mathcal{H}_p$, the n-th Hodge filtration of the complex vector space $\mathcal{H}_p$. Then it is shown in Lemma 3.1 from [50] that for each connected component T_λ/B_0 of $T_{\mathbb{Z}}/B_0$, the natural mapping

$$T_\lambda \to T(F^n\mathcal{H})$$

$$\alpha \mapsto Q(\alpha, \cdot)$$

is finite, where $T(F^n\mathcal{H})$ is the underlying analytic space of the Hodge bundle.

Moreover, according to Saito's Mixed Hodge Modules theory, there is a Hodge module M underlying a filtered $\mathcal{D}_{B_0}$-module $(\mathcal{M}, F_\bullet\mathcal{M})$ supported on B, as the minimal extension of $(\mathcal{H}, \nabla)$.

Schnell considered the space $T(F_{n-1}\mathcal{M})$ as the analytic spectrum of the $(n-1)$-th filtration of $\mathcal{M}$ and showed that the analytic closure of the image of the composite of

$$\varepsilon : T_\lambda \to T(F^n\mathcal{H}) \to T(F_{n-1}\mathcal{M})$$

is still analytic, therefore it extends to a finite analytic covering by Grauert's theorem, so there is a normal analytic space $\bar{T}_\lambda$ extending T_λ.

Lemma 3.5.3. *[53, Theorem 4.2, 23.1] There is a normal holomorphically convex analytic space $\bar{T}_\lambda$ containing T_λ as an open dense subspace, and a finite holomorphic mapping*

$$\bar{\varepsilon} : \bar{T}_\lambda \to T(F_{n-1}\mathcal{M})$$

with discrete fibers that extends ε.

Schnell defines $\bar{T}_{\mathbb{Z}}$ as the union $\bigcup_\lambda \bar{T}_\lambda$. The closed analytic subscheme $\bar{\varepsilon}^{-1}(0) \subseteq \bar{T}_{\mathbb{Z}}$ is defined to be the *extended locus of Hodge classes.*

In fact, when X is a smooth hypersurface of projective space, the minimal extension $\mathcal{M}$ can be described as Griffiths' residues [52].

Chapter 4: Cubic Threefolds

4.1 Overview

A smooth cubic threefold X is a hypersurface of $\mathbb{P}^4$ defined by a homogeneous polynomial $F(x_0, \ldots, x_4)$ of degree 3. The projective lines in X form a smooth surface F of general type. The intermediate Jacobian JX of X is a principally polarized abelian variety of dimension 5, and there is an Abel-Jacobi map

$$F \times F \to JX,\ (L_1, L_2) \mapsto \int_{L_2}^{L_1}. \tag{4.1}$$

The image of the Abel-Jacobi map is the theta divisor Θ of JX. Beauville showed that there is only one singularity at $0 \in JX$ and the blowup $\mathrm{Bl}_0(\Theta)$ resolves the singularity, and the exceptional divisor K is isomorphic to the cubic threefold X itself.

Let $\mathcal{M} \subseteq F \times F$ be the open subspace of pairs on skew lines, then the restriction of the Abel-Jacobi map to $\mathcal{M}$ factors through the locus of the primitive vanishing cycle T_v:

$$\mathcal{M} \to T_v \subseteq \Theta \setminus \{0\}.$$

So T_v is contained in $\mathrm{Bl}_0(\Theta) \setminus K = \Theta \setminus \{0\}$, so is an open dense subspace of $\mathrm{Bl}_0(\Theta)$.

Theorem 4.1.1. *For a general cubic threefold X, its locus of primitive vanishing cycles on hyperplane sections T_v has a canonical compactification $\bar{T}_v$. Moreover, $\bar{T}_v$ is isomorphic to $\mathrm{Bl}_0(\Theta)$.*

In fact, the Gauss map

$$\Theta \dashrightarrow (\mathbb{P}^4)^*, \ p \mapsto \mathbb{P}(T_p\Theta)$$

extends to a regular map

$$\mathrm{Bl}_0(\Theta) \to (\mathbb{P}^4)^*.$$

Its stein factorization is a composite

$$\mathrm{Bl}_0(\Theta) \xrightarrow{f} W \xrightarrow{g} (\mathbb{P}^4)^*,$$

where W is an an normal algebriac variety, f is birational and g is finite. In fact, we'll show that W is isomorphic to $\bar{T}_v$.

Theorem 4.1.2. *For a smooth cubic threefold X, its locus of primitive vanishing cycles on hyperplane sections T_v has a canonical compactification $\bar{T}_v$. Moreover, there is a birational morphism $\mathrm{Bl}_0(\Theta) \to \bar{T}_v$ which contracts finitely many elliptic curves corresponding to Eckardt points on X.*

When X is general and has no Eckardt point, $\mathrm{Bl}_0(\Theta) \to \bar{T}_v$ is an isomorphism, so Theorem 4.1.1 is a special case of Theorem 4.1.2.

To interpret the boundary points of $\bar{T}_v$, denote

$$\bar{\pi} : \bar{T}_v \to \mathbb{O}$$

the branched covering space extending $\pi : T_v \to \mathbb{O}^{\mathrm{sm}}$. Let $t_0 \in \mathbb{O}$ be a point and $X_0 := X_{t_0}$ the corresponding hyperplane section. Suppose X_0 has at worst ADE singularities, then the minimal resolution

$$\tilde{X}_0 \to X_0$$

has exceptional divisors union of a bunch of (-2) curves determined by the Dynkin diagram of the corresponding ADE type of the singularities. On the other hand, these effective (-2) curves generate a subgroup W_e of the Weyl group $W(\mathbb{E}_6)$. Then we have the following result:

Theorem 4.1.3. *The fiber $\bar{\pi}^{-1}(t_0)$ is identified with the orbit of the group action $W_e \curvearrowright R(\mathbb{E}_6)$.*

Further, since the Abel-Jacobi map (4.1) descends to T_v, we get a holomorphic map

$$T_v \to JX. \tag{4.2}$$

It agrees with the topological Abel-Jacobi map (2.49) that we introduced earlier. So we also call (4.2) topological Abel-Jacobi map.

We can ask if this map extends to the closure $\bar{T}_v$.

Theorem 4.1.4. *The topological Abel-Jacobi map $T_v \to JX$ extends to $\bar{T}_v$ if and only if X has no Eckardt points. On the other hand, we blow up $\mathbb{O}$ at Eckardt hyperplanes and denote the new space as $\tilde{\mathbb{O}}$. By applying Stein's theorem to the completion $\mathbb{O}^{\mathrm{sm}} \subseteq \tilde{\mathbb{O}}$, we get a finite branched covering $\tilde{T}_v \to \tilde{\mathbb{O}}$. Then the topological Abel-Jacobi map* (4.2) *extends to a regular morphism $\tilde{T}_v \to \mathrm{Bl}_0 JX$.*

In Section 4.2, we will review basic facts on cubic surfaces and cubic threefolds. We will prove Theorem 4.1.1 in Section 4.3 and Theorem 4.1.2 in Section 4.4.

In Section 4.5, we'll study the extension of the Abel-Jacobi map from T_v to its completion. In Section 4.6, we will relate the boundary points on $\bar{T}_v$ to the Lie theory of the root system of the minimal resolution of the singular cubic surfaces, in the case where they have at worst ADE singularities (Theorem 4.6.5).

In Section 4.7, we relate to our results to Schnell's construction. In particular, we provide an alternative proof of the tube mapping theorem in the case of cubic threefold.

In Section 4.8, we will discuss some additional results on Eckardt cubic threefolds.

4.2 Preliminaries

4.2.1 Root System on Cubic Surfaces

It's well known that there are 27 lines on a smooth cubic surface S. We'll see in this section how the difference of a pair of skew lines defines a primitive vanishing cycle, and there are 72 primitive vanishing cycles in total.

A cubic surface S can be obtained by blowing up 6 points $p_1, \dots, p_6$ in general position on $\mathbb{P}^2$. In fact, the linear system $\mathfrak{o}$ of cubics through the 6 points has dimension 4. It separates points and tangent vectors on the blowup $Bl_{p_1,\dots,p_6}\mathbb{P}^2$, therefore it induces a projective morphism from $Bl_{p_1,\dots,p_6} \to \mathbb{P}^3$ and is an isomorphism onto the cubic surface S.

Let e_0 be the class of the pullback of a general line on $\mathbb{P}^2$ and $e_1, \dots, e_6$ be the classes of the exceptional divisors, then

$$Pic(S) \cong H^2(S, \mathbb{Z}) \cong I^{1,6}$$

has basis $e_0, e_1, ..., e_6$, and the intersection pairing is given by

$$e_i \cdot e_j = \begin{cases} 1, \text{ if } i = j = 0, \\ -1, \text{ if } i = j \neq 0, \\ 0, \text{ if } i \neq j. \end{cases}$$

The hyperplane class is $h = 3e_0 - e_1 - \cdots - e_6$, which coincides with the anticanonical class $-K_S$ and is represented by the strict transform of a cubic curve passing through the 6 blow-up points.

A line $L \subseteq S$ is a smooth divisor with degree one, and by adjunction formula, $L \cdot L = -1$. So an equivalent definition is

Definition 4.2.1. *Call $\beta \in Pic(S)$ a line on S if it satisfies*

$$\beta^2 = -1, \ \beta \cdot h = 1.$$

Lemma 4.2.2. *The 27 lines on S are:*

(i) 6 exceptional curves E_i, $1 \leq i \leq 6$,

(ii) 15 strict transforms F_{ij} of the lines through p_i and p_j, $1 \leq i < j \leq 6$, and

(iii) 6 strict transforms G_i of conics passing through all points but p_i, for $1 \leq i \leq 6$.

Expressiong in the standard basis, they are e_i, $e_0 - e_i - e_j$, $2e_0 - \sum_{k \neq i} e_k$, for all possible $i \neq j$.

The orthogonal complement $h^{\perp}$ of h in $Pic(S)$ is the $\mathbb{E}_6$-lattice with basis $\alpha_1 = e_0 - e_1 - e_2 - e_3$, $\alpha_i = e_{i-1} - e_i$, $i = 2, \ldots, 6$ and the intersection pairing $(\cdot, \cdot)$ given by

$$\begin{bmatrix} -2 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & -2 \end{bmatrix}. \tag{4.3}$$

It can also be represented in the Dynkin diagram

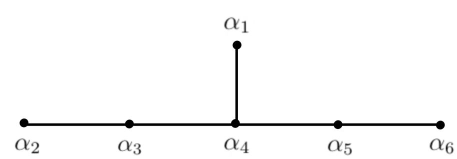


Figure 4.1: $\mathbb{E}_6$ Dynkin Diagram

The automorphism group $\text{Aut}(h^{\perp})$ is isomorphic to the Weyl group $W(\mathbb{E}_6)$ of the $\mathbb{E}_6$ Lie algebra.

Definition 4.2.3. *Call the set*

$$R_S = \{\alpha \in \text{Pic}(S) | \alpha^2 = -2, \alpha \cdot h = 0\}, \tag{4.4}$$

the root system of S. Call $\alpha \in R_S$ a root of S.

Note that (-1) times the matrix (4.3) is the Cartan matrix of root system $R(\mathbb{E}_6)$. One refers to [22, Section 8.2] for the root system of $\mathbb{E}_6$, or [34, Chapter III] for general theory on root systems. The following two lemmas will follow from general theory on the root system of $\mathbb{E}_6$.

Lemma 4.2.4. *The root system R_S of a smooth cubic surface S together with the intersection pairing* (4.3) *is isomorphic to the root system of $\mathbb{E}_6$ with sign of the intersection pairing reversed. Namely,*

$$(R_S, (\cdot,\cdot)) \cong (R(\mathbb{E}_6), -\langle \cdot,\cdot \rangle).$$

Lemma 4.2.5. *(i) R_S consists of 72 roots.*

(ii) The automorphism of $R_S = (R_S, (\cdot,\cdot))$ is the Weyl group $W(\mathbb{E}_6)$ of the Lie algebra $\mathbb{E}_6$. $W(\mathbb{E}_6)$ acts transitively on the set of 72 roots.

Recall that the Weyl group $W(\mathbb{E}_6)$ is generated by reflections $\{r_\alpha\}_{\alpha \in R_S}$, where

$$r_\alpha : \beta \mapsto \beta + (\beta, \alpha)\alpha \tag{4.5}$$

is a reflection with respect to the wall $H_\alpha = \{(\cdot, \alpha) = 0\}$ associated to a root α.

In fact, the action (4.5) has geometric meaning: Consider a family of cubic surfaces $\{S_t\}_{t \in \Delta}$ parameterized by a holomorphic disk Δ such that S_t is smooth when $t \neq 0$ and S_0 has an ordinary double point (equivalently, an A_1 singularity), there is a vanishing cycle α on nearby S_{t_0}. Take a loop l whose class in $\pi_1(\Delta^*, t_0)$ is a generator, then the action (4.5) describe a generator of the monodromy representation

$$\rho_\Delta : \pi_1(\Delta^*, t_0) \to Aut H^2(S_{t_0}, \mathbb{Z})$$

as we considerd in (3.1). Further, (4.5) coincides to the Picard-Lefschetz transformation (3.2).

Lemma 4.2.6. *Let S be a smooth cubic surface. There is one-to-one correspondence between*

$$R_S \leftrightarrow \text{sets of 6 mutually disjoint lines on } S \leftrightarrow \text{sets of planar representation of } S.$$

Proof. For each 6-tuple $(L_1, \ldots, L_6)$ of mutually disjoint lines, there is a unique root α such that $(\alpha, L_i) = 1$ for all $i = 1, \ldots, 6$. Explicitly, α is given by $2v - [L_1] - \cdots - [L_6]$, where $v = \frac{1}{3}(h + [L_1] + \cdots + [L_6])$ can be shown to be a lattice point in $I^{1,6}$ [22, Lemma 9.1.2]. The uniqueness follows from the fact that $[L_1], \ldots, [L_6]$ are linearly independent. This establishes the first correspondence.

For for each 6-tuple $(L_1, \ldots, L_6)$ of mutually disjoint lines, since L_i has self-intersection (-1), blowdown these curves, one obtains $\mathbb{P}^2$. This establishes the second correspondence. □

Proposition 4.2.7. *Every root α on S can be written as the difference $[L_1] - [L_2]$ for a pair of skew lines L_1, L_2 in S in exactly 6 different ways.*

Proof. One can choose a planar representation of S as a blow-up of 6 points on $\mathbb{P}^2$ corresponding to a given root α, so α is expressed as

$$\alpha = 2e_0 - e_1 - \cdots - e_6. \tag{4.6}$$

On they other hand, since $[G_i] = 2e_0 - \sum_{j \neq i} e_j$, one has $\alpha = [G_i] - [E_i]$ for each $i = 1, \ldots, 6$. Its direct to check these are the only way to express α as the class of difference of two lines. □

Now suppose $X \subseteq \mathbb{P}^4$ is a smooth cubic threefold, and S is a smooth hyperplane section. Then the vanishing cohomology $H^2_{\text{van}}(S, \mathbb{Z})$ is identified with a subgroup of $H^2(S, \mathbb{Z})$ that is orthogonal to the hyperplane class h via the intersection pairing on $H^2(S,\mathbb{Z})$. So by Definition 4.4, $R_S \subseteq H^2_{\mathrm{van}}(S,\mathbb{Z})$. In fact, R_S generates a full rank lattice.

Let $\mathbb{O}^{\mathrm{sm}} \subseteq (\mathbb{P}^4)^*$ parameterizes smooth hyperplane sections of X. There is a local system $\mathcal{H}^2_{\mathrm{van}}$ on $\mathbb{O}^{\mathrm{sm}}$ whose stalk at t is isomorphic to the vanishing cohomology $H^2_{\mathrm{van}}(X_t,\mathbb{Z})$. The local system $\mathcal{H}^2_{\mathrm{van}}$ has a realization of analytic covering map $T \to \mathbb{O}^{\mathrm{sm}}$. It has a 72-to-1 sub-covering space

$$q : \mathcal{R} \to \mathbb{O}^{\mathrm{sm}}.$$

Proposition 4.2.8. *q is a connected covering map. As a result, $\mathcal{R}$ is a connected component of T.*

Proof. Consider the set of pairs of skew lines on smooth hyperplane sections

$$\mathcal{M} = \{(L_1, L_2, t) \in F \times F \times \mathbb{O}^{\mathrm{sm}} | L_1, L_2 \subseteq Y_t, L_1 \cap L_2 = \emptyset\}. \tag{4.7}$$

The projection π to the third coordinate is a natural covering map, whose fiber over t consists of pairs of skew lines on X_t.

Then a version of Proposition 4.2.7 in families implies there is a degree-6 covering map $e : \mathcal{M} \to \mathcal{R}$ given by $(L_1, L_2) \mapsto [L_1] - [L_2]$, together with a commutative diagram.

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{e} & \mathcal{R} \\ & \searrow^{\pi} & \downarrow q \\ & & \mathbb{O}^{\mathrm{sm}} \end{array}$$

On the other hand, since any pair of disjoint lines (L_1, L_2) spans a hyperplane in $\mathbb{P}^4$ and determines the hyperplane section Y_t containing both of the lines, there is an inclusion $\mathcal{M} \hookrightarrow F \times F$ as a Zariski open dense subspace. The complement is a divisor, therefore, $\mathcal{M}$ is connected. So as a continuous image of $\mathcal{M}$, $\mathcal{R}$ is connected as well. $\square$

From a different point of view, the connectedness of the covering space $\pi : \mathcal{M} \to \mathbb{O}^{\mathrm{sm}}$ is governed by the monodromy group of 27 lines over $\mathbb{O}^{\mathrm{sm}}$.

Lemma 4.2.9. *([54, VI.20], [17, Theorem 0.1]) Let $\mathcal{X}^{\mathrm{sm}} \to \mathbb{O}^{\mathrm{sm}}$ be the universal family of smooth hyperplane sections of X. The monodromy group permuting the 27 lines is isomorphic to the Weyl group $W(\mathbb{E}_6)$.*

Proof of Proposition 4.2.8 based on Lemma (4.2.9)*.* It suffices to prove the connectivity of π. By Lemma 4.2.9 and Lemma 4.2.6, the monodromy action is transitive on the set of 6 mutually disjoint lines and the stabilizer group is isomorphic to permutation group $\mathfrak{S}_6$ [22, Proposition 9.2.3]. Moreover the stabilizer group acts on the 6 lines $\{L_1, ..., L_6\}$ faithfully: the reflection (4.5) given by the root $[L_i] - [L_j]$ permutes L_i and L_j and fixes other four lines. Therefore, the Weyl group $W(\mathbb{E}_6)$ is transitive on the ordered 6-tuple $(L_1, ..., L_6)$ of mutually disjoint lines. In particular, the monodromy action is transitive on the set of pairs of skew lines (L_1, L_2). So the covering space $\mathcal{M}/\mathbb{O}^{\mathrm{sm}}$ is connected. □

Proposition 4.2.10. *$\mathcal{R} = T_v$ is the same covering space of $\mathbb{O}^{\mathrm{sm}}$. In other words, a primitive vanishing cycle on a smooth hyperplane section of X is a root and vice versa.*

Proof. By Proposition 4.2.8, $\mathcal{R}/\mathbb{O}^{\mathrm{sm}}$ is a connected component of $T/\mathbb{O}^{\mathrm{sm}}$. By definition, $T_v/\mathbb{O}^{\mathrm{sm}}$ is also a connected component, it suffices to show that one root is a primitive vanishing cycle.

Take a one parameter family $\{S_t\}_{t\in\Delta}$ of hyperplane sections of X such that Δ intersects X^* transversely at smooth point. So S_0 has an ordinary node and S_t is smooth for $t \neq 0$. There is a vanishing cycle $\delta \in H^2(S_t, \mathbb{Z})$. We are going to show that δ is a root.

For a smooth cubic surface S, it is isomorphic to blowup of 6 general points $\{p_1, \ldots, p_6\}$ on $\mathbb{P}^2$. When the 6 blowup points move to lie on a conic Q. The linear system $\mathfrak{o}$ of cubics through the 6 points contains a 3-dimensional subspace consisting of cubics that are a union of Q and a line. It follows that $|\mathfrak{o}|$ does not separate the points on the strict transform $\tilde{Q}$, so the projective morphism $Bl_{p_1,\ldots,p_6}\mathbb{P}^2 \to \mathbb{P}^3$ contracts $\tilde{Q}$ to the node of the cubic surface.

Take a double cover of $\tilde{\Delta} \to \Delta$ branched at 0 and base change, the monodromy is eliminated and the new family $\{S_s\}_{s\in\tilde{\Delta}}$ with $t = s^2$ regarded birationally as varying the 6 blowup points $\{p_1(s),\ldots,p_6(s)\}$ on $\mathbb{P}^2$, and when $s = 0$, the 6 points lie on a conic Q.

In other words, there is a commutative diagram

$$\begin{array}{ccc} \tilde{\mathcal{S}} & \xrightarrow{h} & \mathcal{S} \\ \downarrow{\scriptstyle f} & \swarrow{\scriptstyle g} & \\ \tilde{\Delta} & & \end{array}$$

Here $f : \tilde{\mathcal{S}} \to \tilde{\Delta}$ is a smooth morphism whose fiber is $Bl_{p_1(s),\ldots,p_6(s)}\mathbb{P}^2$ and $g : \mathcal{S} \to \tilde{\Delta}$ has fiber S_s. h is a birational map which is isomorphic over $s \neq 0$ and its restriction to $s = 0$ is

$$\tilde{S}_0 \to S_0$$

which desingularizes S_0 and sends $\tilde{Q}$ to the node.

Now denote e_0 the e_i the i-th exceptional divisor for $i = 1,\ldots,6$ as before. By Ehresmann's theorem, these classes are well-defined on the fiber of f over the entire $\tilde{\Delta}$. Since the strict transform $\tilde{Q}$ has class $[\tilde{Q}] = 2e_0 - e_1 - \cdots - e_6$ over $s = 0$. Its deformation to nearby S_s is a vanishing cycle.

As we have seen earlier, the class $2e_0 - e_1 - \cdots - e_6$ is a root, so $S_t = S_{s^2}$ has a primitive vanishing cycle equal to a root. □

In summary, we have shown that when X is a smooth cubic threefold, the locus of vanishing cycles on smooth hyperplane sections of X

$$\pi_v : T_v \to \mathbb{O}^{\mathrm{sm}}, \tag{4.8}$$

is a 72-to-1 covering space, and the fiber over t is identified with the root system on the cubic surface S_t.

We're interested in the topology and geometry of the covering map π_v and its completion.

4.2.2 Lines on Cubic Threefolds

Let X be a smooth cubic threefold. Then its middle dimensional cohomology $H^3(X, \mathbb{Z})$ admits a polarized Hodge structure of weight 3. As we have introduced in (2.36), the intermediate Jacobian

$$JX = F^2H^3(X, \mathbb{C})^\vee / H_3(X, \mathbb{Z})$$

of X is a abelian variety. By the unimodularity of the intersection pairing

$$H^3(X, \mathbb{Z}) \times H^3(X, \mathbb{Z}) \to \mathbb{Z},\ (\omega_1, \omega_2) \mapsto \int_X \omega_1 \wedge \omega_2,$$

JX is principally polarized. By the residue computation in Example 2.3.3, we know the Hodge numbers of $h^{3,0} = 0$ and $H^{2,1} = 5$. So JX is a principally polarized abelian fivefold.

By a classical theorem of Griffiths and Clemens [19], JX is not Jacobian of a genus 5 curve. Moreover, they proved a Torelli theorem for cubic threefold.

Theorem 4.2.11. *[19] The intermediate Jacobian JX as a polarized abelian variety uniquely determines the cubic threefold X.*

Consequently, they showed that a smooth cubic threefold is not rational.

The proof Theorem 4.2.11 relies on studying the Abel-Jacobi map, which we introduced in Example 2.3.3, and will be studied in detail later.

The intermediate Jacobian JX has an ample divisor Θ, such that $h^0(\Theta) = 1$. Such divisor is called a theta divisor. It is unique up to translation.

According to Beauville [7] The theta divisor Θ has a unique singularity, so up to translation, the singularity can be placed at $0 \in JX$. Beauville showed that 0 is a triple point singularity on Θ and the projective tangent cone is isomorphic to the cubic threefold Y. This provides an alternative proof of Theorem 4.2.11.

Let $Gr(2,5)$ be the Grassmannian of projective lines in $\mathbb{P}^4$. Since $X \subseteq \mathbb{P}^4$ is a closed subvariety, the variety $F = \{(x,t) \in X \times Gr(2,5) | x \in L_t\}$ parameterizing projective lines on X is a closed subvariety of $Gr(2,5)$.

Lemma 4.2.12. *F is a smooth surface of the general type.*

F parameterizes two types of lines on X.

Definition 4.2.13. A line $L \subseteq X$ is called to be of first type if the normal bundle $N_{L|X} \cong \mathcal{O}_L \oplus \mathcal{O}_L$; L is called to be of second type if the normal bundle $N_{L|X} \cong \mathcal{O}_L(1) \oplus \mathcal{O}_L(-1)$.

Equivalently, as we will see in the later sections, a line $L \subseteq Y$ is of the first type if and only if there is a smooth quadric surface tangent to Y along L; L is of the second type if and only if there is a projective plane P tangent to X along L. Lines of the first type are dense in F, while the set of lines of the second type forms a divisor of F.

The Albanese variety of F is an abelian variety

$$\mathrm{Alb}(F) = H^{1,0}(F)^{\vee}/H_1(F,\mathbb{Z}).$$

There is an Albanese map

$$alb : F \to Alb(F),\ p \mapsto \int_{p_0}^{p}, \tag{4.9}$$

which turns out to be an embedding.

There is an isomorphism [19] of abelian varieties

$$Alb(F) \cong JX$$

induced by $H_1(F,\mathbb{Z}) \to H_3(X,\mathbb{Z})$ sending one cycle c on F to the "$\mathbb{P}^1$-bundle" over c. This is also called cylindrical homomorphism. It follows that $Alb(F)$ is principally polarized.

For a smooth cubic threefold X and a general point $p \in X$, 6 lines on X pass through p (The intersection $X \cap \mathbb{P}T_pX$ is a cubic surface with an A_1 singularity, and 6 lines pass through the node). However, there might be special points with one-parameter families of lines passing through.

Definition 4.2.14. *Let X be a smooth cubic threefold. An Eckardt point $p \in X$ is a point where infinitely many lines on X pass through p.*

Lemma 4.2.15. *Let X be a smooth cubic threefold. Then there are at most finitely many Eckardt points. $p \in X$ is an Eckardt point if and only if the tangent hyperplane section $T_pX \cap X$ is a cone over a smooth cubic plane curve. Moreover, a general cubic threefold X has no Eckardt points.*

Proof. The first two statements can be derived from Lemma 8.1 in [19]. For the last statement, denote by C the locus in the universal family $\mathbb{P}^{19} = \mathbb{P}(\mathrm{Sym}^3\mathbb{C}^4)$ of cubic surfaces that parameterizes cone over plane cubic curves. Then $\dim C = 12$. Let W be the space of all cubic surfaces in $\mathbb{P}^4$. Since every cubic surface sits in exactly one hyperplane section, then there is a natural projection

$$p : W \to (\mathbb{P}^4)^*,$$

whose fiber is isomorphic to $\mathbb{P}^{19}$. Set $\mathbb{P}^{34} = \mathbb{P}(\mathrm{Sym}^3\mathbb{C}^5)$ to be the space of all cubic hypersurfaces in $\mathbb{P}^4$. Then there is a map

$$f : \mathbb{P}^{34} \times (\mathbb{P}^4)^* \to W$$

$$(X, H) \mapsto X \cap H,$$

by sending a cubic threefold to a hyperplane section. Then f preserves the projection to $(\mathbb{P}^4)^*$. Moreover, f is a fiber bundle over W, and the fiber consists of cubic threefolds containing a fixed cubic surface, which has constant dimension 15. Let $\mathcal{C} \subseteq W$ be the locus of cone over plane cubic curves, then $\operatorname{codim}_W \mathcal{C} = 7$. Therefore the preimage $f^{-1}(\mathcal{C})$ has codimension 7 as well. It follows that its image in $\mathbb{P}^{34}$ under the projection to the first coordinate has codimension at least 3, which completes the proof. $\square$

As a consequence of Lemma 4.2.15, there will be an elliptic curve $E_p \subseteq F$ corresponding to each Eckardt point $p \in X$. In fact, according to [49], all elliptic curves of the F arise in this way. Later we will show that the image of the Abel-Jacobi map of $E \times E$ in JX is isomorphic to E, where the Gauss map takes constant value.

4.2.3 Abel-Jacobi Map

Recall that for a smooth projective variety X. We introduced Griffiths' Abel-Jacobi map (2.39) which sends algebraic cycles of codimension r that are homologous to zero to r-th intermediate Jacobian.

Let X be a smooth cubic threefold. We want to study codimension-two algebraic cycles, i.e., curves. The condition of a curve X in X to be homologous to zero is equivalent to its degree $\deg(C) = C \cdot H = 0$, where H is a hyperplane section. Also, instead of studying the entire Chow group of 1-cycles, we'll restrict our attention to the degree-one 1-cycles, i.e., lines.

Since the surface F parameterizes lines on X, the Abel-Jacobi map is given by

$$\psi : F \times F \to JX, \ (p, q) \mapsto \int_{L_q}^{L_p}, \tag{4.10}$$

where the integral is taken over a 3-chain Γ whose boundary is $\partial\Gamma = L_p - L_q$.

It is shown in [19] that ψ is generically 6:1 onto the theta divisor Θ of JX. By definition, when $\phi(L, L) = 0$, so ψ contracts the diagonal Δ_F to the point 0. According to Beauville [7], 0 is the only singularity on Θ and is a triple point.

When Y is general, Δ_F is the only positive dimensional fiber of ψ. We will show in later sections that when Y has Eckardt points, ψ has other positive dimensional fibers as well.

There is a Gauss map

$$\mathcal{G} : \Theta \dashrightarrow (\mathbb{P}^4)^*, \; p \mapsto \mathbb{P}T_p\Theta$$

which associate each smooth point of the theta divisor Θ to its projective tangent hyperplane in $\mathbb{P}T^*(JX) = JX \times (\mathbb{P}^4)^*$ followed by the projection to $(\mathbb{P}^4)^*$.

A key step to showing Theorem 4.2.11 is to understand the composite of $\mathcal{G}$ with the Abel-Jacobi map (4.10).

Lemma 4.2.16. *The composite*

$$\Phi = \mathcal{G} \circ \psi : F \times F \dashrightarrow (\mathbb{P}^4)^*$$

is identified with the map

$$(L_1, L_2) \mapsto \mathrm{Span}(L_1, L_2)$$

associating each pair of skew lines to the hyperplane they spanned.

Beauville [7] showed that ψ induces morphism on the blowup. In other words, there is a commutative diagram

$$\begin{array}{ccc} \mathrm{Bl}_{\Delta_F}(F \times F) & \xrightarrow{\tilde{\psi}} & \mathrm{Bl}_0(\Theta) \\ \downarrow & & \downarrow \\ F \times F & \xrightarrow{\psi} & \Theta \end{array} \tag{4.11}$$

We will provide a modular interpretation of this morphism in the later sections.

To understand the restriction of $\tilde{\psi}$ on the exceptional divisor D of the blowup $\mathrm{Bl}_{\Delta_F}(F \times F)$, we look at the commutative diagram

$$\begin{array}{ccccccccc} \mathbb{P}T_F & \xrightarrow{\cong} & \mathbb{P}N_{\Delta_F}(F\times F) & \hookrightarrow & \mathrm{Bl}_{\Delta_F}(F\times F) & \xrightarrow{\tilde{\psi}} & \mathrm{Bl}_0\Theta & \hookrightarrow & \mathrm{Bl}_0 JX \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ F & \xrightarrow{\cong} & \Delta_F & \hookrightarrow & F\times F & \xrightarrow{\psi} & \Theta & \hookrightarrow & JX \end{array} \tag{4.12}$$

D is isomorphic to the associated projective bundle of the normal bundle $N_{\Delta_F}(F \times F)$. Since $N_{\Delta_F}(F \times F) \cong T_F$, we have an identification

$$D \cong \mathbb{P}N_{\Delta_F}(F \times F) \cong \mathbb{P}T_F.$$

On the other hand, Beauville [7] showed that the exceptional divisor of $\mathrm{Bl}_0(\Theta)$ is isomorphic to the cubic threefold X.

There is geometric interpretation of the map (4.11) on the exceptional divisor D described in [19] Proposition 12.31. Denote $\Gamma \subseteq F \times X$ the incidence variety of pairs (t, x) with $x \in L_t$, then the first projection $\Gamma \to F$ is a natural $\mathbb{P}^1$-bundle.

Lemma 4.2.17. *([19, p.342-p.345], [58], [2]) There is a canonical isomorphism*

$$\alpha : \mathbb{P}T_F \cong \Gamma, \tag{4.13}$$

as $\mathbb{P}^1$-bundle over F.

Corollary 4.2.18. *Via the identification* (4.13)*, the extended Abel-Jacobi map $\tilde{\psi}$ restricted to the exceptional divisor D is identified with to the exceptional divisor*

$$\tilde{\psi}|_E : \mathbb{P}T_F \to Y$$

is isomorphic to

$$\Gamma \to Y, \ (t, y) \mapsto y,$$

by projection to the second coordinate.

One can refer to [35, Chapter 5] for detailed discussion on the related topics about cubic threefolds.

We will describe the map (4.13) explicitly. Over the locus of lines of the first type, α turns out to be equivalent to the following: for a line L_t of first type on Y, the normal bundle is $\mathcal{O}_{L_t} \oplus \mathcal{O}_{L_t}$. Each section $s \in \mathbb{P}H^0(L_t, N_{L_t|Y})$ determines a line in $\mathbb{P}^4$ disjoint from L_t, and the span of the two lines is a hyperplane H_s which is tangent to Y at a unique point p_s on L_t. The map α is defined as

$$\alpha : (t, s) \mapsto p_s.$$

Now as t move in a holomorphic disk Δ with $t = 0$ corresponds to a line of second type, there is a holomorphic vector bundle $\mathcal{V}$ of rank 2 on Δ, whose stalk at t is isomorphic to $H^0(L_t, L_{t|Y})$. The bundle $\mathcal{V}$ can be ontained from the following. Let $I_\Delta = \{(t, x) \in \Delta \times X | x \in L_t\}$ be the incidence variey, where Δ is a holomorphic disk embedded in F. Let π_1 and π_2 be projection to the two coordinates. Then there is exact sequence of coherent sheaves on I_Δ.

$$0 \to T_{I_\Delta/\Delta} \to \pi_2^* T_X \to \mathcal{N} \to 0.$$

Here $T_{I_\Delta/\Delta} = T_{I_\Delta}/\pi_1^* T_\Delta$ is the relative tangent tangent bundle. $\mathcal{N}$ is a rank two vector bundle with

$$\mathcal{N}|_{L_t} \cong N_{L_t|X} \cong \begin{cases} \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}, & t \neq 0, \\ \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1), & t = 0. \end{cases}$$

Then the vector bundle $\mathcal{V}$ is isomorphic to the pushforward $(\pi_1)_*(N)$ and is trivial of rank 2. So $\mathcal{V} \cong \Delta \times \mathbb{C}^2$.

Consider a nonvanishing section $t \mapsto (t, s_t)$ on $\mathcal{V}$. As t goes to 0, the line L_t becomes a line of the second type L_0, and $s_t \in H^0(\mathcal{O}_{L_t} \oplus \mathcal{O}_{L_t})$ becomes a section $s \in \mathcal{O}_{L_0}(1)$, so s_0 has a unique zero $\mathrm{zero}(s_0) \in L$. We will prove in Proposition 5.2.15 that α sends (t_0, s_0) to $\overline{\mathrm{zero}(s_0)}$, the conjugate point of $\mathrm{zero}(s_0)$ under the dual map.

4.3 Compactification of Locus of Primitive Vanishing Cycles: General Cubic Threefold

Now we want to compactify the space T_v. The following argument is a direct consequence of Theorem 3.3.1.

Corollary 4.3.1. *There exists a normal algebraic variety $\bar{T}_v$ together with a finite map*

$$\bar{T}_v \to \mathbb{O}$$

which extends the 72-to-1 covering map $T_v \to \mathbb{O}^{\mathrm{sm}}$. Moreover, $\bar{T}_v$ is unique up to isomorphism.

In this section, we will give proof of Theorem 4.1.1, which is stated below.

Theorem 4.3.2. *For a general cubic threefold X, its locus of primitive vanishing cycles on hyperplane sections T_v has a canonical compactification $\bar{T}_v$. Moreover, $\bar{T}_v$ is isomorphic to* $\mathrm{Bl}_0(\Theta)$.

Proof. According to Theorem 3.3.1, there is a unique normal algebraic variety T_v, together with a finite map

$$\bar{T}_v \to \mathbb{O}$$

which extends the 72-to-1 covering map $T_v \to \mathbb{O}^{\mathrm{sm}}$.

The Abel-Jacobi map $\psi : F \times F \to JX$ is generically 6-to-1, and the image is the theta divisor Θ. If we restrict the Abel-Jacobi map to $\mathcal{M}$, then $\psi|_{\mathcal{M}}$ factors through T_v, because the six differences of two disjoint lines are in the same rational equivalent classes, or by the fact the Abel-Jacobi map agrees with the topological Abel-Jacobi map (2.45) and the six difference of two disjoint lines represent the same vanishing cohomology class (Proposition 4.2.7). Further, we note that these maps preserve projection to $\mathbb{O}^{\mathrm{sm}}$, so we have the following diagram.

$$\begin{array}{ccc} \mathcal{M}/\mathbb{O}^{\mathrm{sm}} & \xrightarrow{e} & T_v/\mathbb{O}^{\mathrm{sm}} \\ & \overset{\psi|_{\mathcal{M}}}{\searrow} & \downarrow j \\ & & \Theta^\circ/\mathbb{O}^{\mathrm{sm}} \end{array}$$

Here Θ° is the image of $\mathcal{M}$, which is dense open in Θ. The evaluation map e sends $(L_1, L_2) \mapsto [L_1] - [L_2]$ to the difference of the classes. Therefore $j : T_v/\mathbb{O}^{\mathrm{sm}} \cong \Theta^\circ/\mathbb{O}^{\mathrm{sm}}$ is an isomorphism.

It follows that $T_v \cong \Theta^\circ \hookrightarrow \mathrm{Bl}_0(\Theta)$ is an open dense subspace. On the other hand, $\mathrm{Bl}_0(\Theta)$ is smooth and the morphism $\mathrm{Bl}_0(\Theta) \to \mathbb{O}$ extends the covering map $T_v \to \mathbb{O}^{\mathrm{sm}}$. By uniqueness of $\bar{T}_v$ from Theorem 3.3.1, it suffices to show that

$$\mathrm{Bl}_0(\Theta) \to \mathbb{O} \tag{4.14}$$

is finite.

To do this, we first note $\mathrm{Bl}_0(\Theta)$ is the Nash blowup of Θ, namely the closure of the graph of Gauss map $\Theta \dashrightarrow \mathbb{O}$.

Second, according to Lemma 4.2.16, if we denote $\overline{\Gamma(\Phi)}$ to be the graph closure of the rational map $\Phi : F \times F \dashrightarrow \mathbb{O}$, $(L_1, L_2) \mapsto \mathrm{Span}(L_1, L_2)$, then there is a commutative diagram

$$\begin{array}{ccccc} \overline{\Gamma(\Phi)} & \xrightarrow{\bar{\psi}} & \mathrm{Bl}_0(\Theta) & & \\ \downarrow & & \downarrow & \overset{\tilde{\mathcal{G}}}{\searrow} & \\ F \times F & \xrightarrow{\psi} & \Theta & \overset{\mathcal{G}}{\dashrightarrow} & \mathbb{O} \end{array} \tag{4.15}$$

So to show (4.14) is finite, it suffices to show that the composite

$$\tilde{\mathcal{G}} \circ \bar{\psi} : \overline{\Gamma(\Phi)} \to \mathbb{O} \tag{4.16}$$

is finite.

By definition, the fiber of (4.16) over $t_0 \in \mathbb{O}$ consists of pairs of lines L_1, L_2 in the hyperplane section X_{t_0} that are limits of pairs of skew lines on the nearby smooth hyperplane sections.

However, by assumption, the cubic threefold X is general, so every hyperplane section of X has at worst ADE singularities and contains finitely many lines. In particular, the fiber of (4.16) is finite, so (4.14) is a finite morphism. □

Remark 4.3.3. *The diagram* (4.15) *is a dominated by the diagram* (4.11)*. In fact, there is a normalization map*

$$\mathcal{N} : \mathrm{Bl}_{\Delta_F}(F \times F) \to \overline{\Gamma(\Phi)}. \tag{4.17}$$

To see this, $\overline{\Gamma(\Phi)}$ *is the projection of graph of* $\tilde{\Phi}$ *under the projection*

$$\mathrm{Bl}_{\Delta_F}(F \times F) \to F \times F \times \mathbb{O}.$$

It is 1-to-1 on the exceptional fibers of lines of the first type, and 2-to-1 over on the exceptional fibers of lines of the second type. Therefore by Zariski's main theorem, (4.17) *is the normalization map.*

In fact, $\overline{\Gamma(\Phi)}$ *is singular over the locus* $\mathcal{C}$ *of lines of the second type, and* $\mathcal{N}^{-1}(\mathcal{C}) \to \mathcal{C}$ *is 2-to-1.*

4.4 Compactification of Locus of Primitive Vanishing Cycles: Eckardt Cubic Threefolds

(4.15) In this section, we aim to prove Theorem 4.1.2, which is stated below.

Theorem 4.4.1. *There is a birational morphism* $\mathrm{Bl}_0(\Theta) \to \bar{T}_v$ *contracting finitely many elliptic curves, which are one-to-one corresponding to the Eckardt points on the cubic threefold* Y*. In particular, when* Y *is general,* $\bar{T}_v$ *is isomorphic to* $\mathrm{Bl}_0(\Theta)$*.*

Different from the case where cubic threefold is general, X is now a smooth cubic threefold and can have Eckardt points. To prove Theorem 4.4.1, we will use the covering map e, the completion of $\mathcal{M}$, together with the Abel-Jacobi map. Recall that we have the following commutative diagram from (4.12).

$$\begin{array}{ccc} Bl_{\Delta_F}(F \times F) & \xrightarrow{\tilde{\psi}} & \mathrm{Bl}_0(\Theta) \\ & \searrow^{\tilde{\pi}} & \downarrow \lambda \\ & & \mathbb{O} \end{array} \tag{4.18}$$

Lemma 4.4.2. *$\tilde{\pi}$ is generically finite. The only positive dimensional fibers are $E_i \times E_i$ over Eckardt hyperplanes u_i.*

To prove Lemma 4.4.2, we will need the notion of Hilbert schemes, which we will study in the next chapter. In fact, $Bl_{\Delta_F}(F \times F)$ is a branched double cover of the Hilbert scheme of a pair of skew lines. By interpreting fibers of $\tilde{\pi}$ as a double cover of Hilbert scheme of a pair of skew lines on a cubic surface, Lemma 4.4.2 will follow from Theorem 5.3.10. Now we turn to an important consequence of this lemma.

Corollary 4.4.3. *The extended Gauss map $\lambda : \mathrm{Bl}_0(\Theta) \to \mathbb{O}$ is generically finite. The only fiber positive dimensional fibers are elliptic curve E_i over Eckardt hyperplane u_i.*

Proof. We look at the restriction of the Abel-Jacobi map $\psi : F \times F \to JX$:

$$E \times E \hookrightarrow F \times F \to JX.$$

The map factors through

$$E \times E \xrightarrow{f} Alb(E) \xrightarrow{g} Alb(F) \cong JX,$$

where $f : (p, q) \mapsto \int_q^p \cdot$. g is inclusion since $E \cong Alb(E)$ and $F \to Alb(F)$ is an embedding.

It follows that $\psi(E \times E) \cong E$ is an elliptic curve, and by commutativity of (4.18), the strict transform of $\psi(E \times E)$ is contracted by λ. $\square$

Notation 4.4.4. *Denote $C_i \subseteq \mathrm{Bl}_0(\Theta)$ be the elliptic curves that are strict transforms of $\psi(E_i \times E_i)$.*

Now we are ready to prove the Theorem 4.4.1.

Proof of Theorem 4.4.1. Similar to the proof of Theorem 4.3.2, there is an inclusion $T_v \hookrightarrow \Theta$ is an open subspace, and the image is disjoint from C_i, we have natural inclusion $i : T_v \hookrightarrow \mathrm{Bl}_0(\Theta)$. Now we take the stein factorization

$$\mathrm{Bl}_0(\Theta) \xrightarrow{\lambda_1} W \xrightarrow{\lambda_2} \mathbb{O},$$

where λ_1 is birational and contracts the curves C_i, while λ_2 is finite. Moreover, W is normal since $\mathrm{Bl}_0(\Theta)$ is normal (p.213 in [28]).

Now, the composition $T_v \hookrightarrow \mathrm{Bl}_0\Theta \to W$ is inclusion and preserves the projection to $\mathbb{O}$. So as branched covering maps, $W/\mathbb{O}$ extends $T_v/\mathbb{O}^{\mathrm{sm}}$. Due to the normality of W and the uniqueness of extension analytic branched covering from Theorem 3.3.1, $W/\mathbb{O} \cong \bar{T}_v/\mathbb{O}$. This proves the theorem. $\square$

We note here that the exceptional divisor K of $\mathrm{Bl}_0(\Theta)$ is isomorphic to X by Beauville. The restriction to K of the projection $\lambda : \mathrm{Bl}_0(\Theta) \to \mathbb{O}$ has target X^*, which is a hypersurface of degree 24. The map is isomorphic to the dual map

$$X \to X^*,$$

which associate each point p to its tangent hyperplane T_pX. This map is finite and birational, and so is the normalization by Zariski's main theorem.

4.5 Extension of the Abel-Jacobi Map

We are interested in whether the Abel-Jacobi map can extend to the completion space.

Proposition 4.5.1. *The Abel-Jacobi map $T_v \to JX$ extends to a morphism $\bar{T}_v \to JX$ if and only if the cubic threefold Y has no Eckardt point.*

Proof. When the cubic threefold Y has no Eckardt point, $\bar{T}_v \cong \mathrm{Bl}_0(\Theta)$. The extension is just the composite

$$\mathrm{Bl}_0(\Theta) \to \Theta \hookrightarrow JX. \tag{4.19}$$

On the other hand, when Y has Eckardt points, the map cannot extend over the point $\lambda(C_i)$, as the image of C_i in JX is an elliptic curve but not a point. □

We will look for a different compactification $\tilde{T}_v$ of T_v and expect the Abel-Jacobi map to extend over.

4.5.1 Stable Reduction.

Let X_0 be a hyperplane section of X with an elliptic singularity. Choose a general pencil of hyperplane sections through X_0 and restrict it to a small holomorphic disk D with $t = 0$ corresponding to X_0. Denote $\mathcal{X} \to D$ the corresponding pullback family of hyperplane sections of X. Then the family is smooth over $D^* = D \setminus \{0\}$.

Denote $\tilde{X}_0$ the blow-up of X_0 at the cone point p. Then $\tilde{X}_0$ is a ruled surface over a planar cubic curve E. We'll consider the stable reduction of the family.

Proposition 4.5.2. *There is a 3-to-1 branched cover $\tilde{D} \to D$ totally ramified at 0, a smooth total space $\tilde{\mathcal{X}}$, and a flat family $\tilde{\mathcal{X}} \to \tilde{D}$ extending the fiber product $\mathcal{X} \times_{D^*} \tilde{D}$. Moreover, the fiber $\tilde{\mathcal{X}}_0$ at 0 is a simple normal crossing divisor $\tilde{X}_0 \cup Z$, where Z is a cubic surface arising as cyclic cover of $\mathbb{P}^2$ along E. Moreover, the normal bundle $N_{\tilde{X}_0|\tilde{\mathcal{X}}}$ restrict to $\mathcal{O}(-1)$ to the ruling of $\tilde{X}_0$.*

Proof. X_0 is a cone over a plane cubic curve E. Take the coordinate such that E is contained in the plane $x_3 = 0$ and the defining equation of E is a homogeneous cubic $F(x_0, x_1, x_2)$. The same equation in $\mathbb{P}^3_{[x_0,x_1,x_2,x_3]}$ defines X_0 and $x_0 = x_1 = x_2 = 0$ is the elliptic singularity p.

The local affine equation of X around the cone point p of X_0 can be expressed as

$$F(x_0, x_1, x_2) + tG(x_0, x_1, x_2, t) = 0, \tag{4.20}$$

where $G(x_0, x_1, x_2, t)$ is homogeneous of degree two. By the generality of the pencil, $G(0, 0, 0, 0) \neq 0$ and the cone point is not in the base locus of the D-family. Then by restricting to a smaller neighorhood, the equation (4.20) defines $\mathcal{X}$ around p.

Let's blowup $\mathcal{X}$ at p. Substitute the equation (4.20) by $x_1 = x_0x_1'$, $x_2 = x_0x_2'$ and $t = x_0t'$, we have

$$x_0^3F(1, x_1', x_2') + x_0t'(c + G_{\geq 1}) = 0, \tag{4.21}$$

where c is a nonzero constant and $G_{\geq 1} = G_{\geq 1}(x_0, x_1', x_2', t')$ has degree at least one. It follows that the fiber of 0 becomes $\tilde{X}_0 \cup 3[\mathbb{P}^2]$, where $\tilde{X}_0$ is the strict transform of X_0 and $\mathbb{P}^2$ is defined by $x_0 = t = 0$. The two components meet transversely along the exceptional curve of $\tilde{X}_0$.

Now take the connected triple cover $\tilde{D} \to D$ branched at 0 and take the fiber product $\mathcal{X} \times_D \tilde{D}$. So we substitute $s^3 = t = x_0t'$ in the local equation (4.21) and obtain

$$x_0^3F(1, x_1', x_2') + s^3(c + G_{\geq 1}) = 0. \tag{4.22}$$

The total space is singular along the preimage $\tilde{\mathbb{P}^2}$ of $\mathbb{P}^2$. However, normalization will smooth the total space. In fact, divide the equation (4.22) by x_0, one has

$$F(1, x_1', x_2') + (\frac{s}{x_0})^3(c + G_{\geq 1}) = 0. \tag{4.23}$$

By restricting to an open subspace such that $c+G_{\geq 1}$ is nonvanishing and becomes a unit, (4.23) defines an integral dependence relation of s/x_0 over certain localization of the ring

$\mathbb{C}[x_0, x_1', x_2', t']$. The normalization $\sigma : \tilde{\mathcal{X}} \to \mathcal{X}$ is done in a local coordiante by introducing the variable $z = s/x_0$ and the new total space $\tilde{\mathcal{X}}$ is locally defined by

$$F(1, x_1', x_2') + z^3(c + G_{\geq 1}) = 0,\ s = x_0 z. \tag{4.24}$$

Note that $x_0 = 0$ is the locus where σ is not isomorphic. So we get a family

$$\tilde{\mathcal{X}} \to \tilde{D}, \tag{4.25}$$

whose fiber at 0 is $\tilde{X}_0 \cup Z$, where Z is triple cover of $\mathbb{P}^2$ branched along E, and $\tilde{X}_0$ meets Z transversely along E.

Since $\mathcal{O}(\tilde{X}_0 + Z) = \mathcal{O}_{\tilde{X}_0}$, one has $\mathcal{O}(\tilde{X}_0 + Z)|_{\tilde{X}_0} = \mathcal{O}_{\tilde{X}_0}$. On the other hand, $\mathcal{O}(Z)|_{\tilde{X}_0} = \mathcal{O}_{\tilde{X}_0}(\tilde{Y}_0 \cap Z) = \mathcal{O}_{\tilde{X}_0}(E_\infty)$, where E_∞ is the section at infinity, i.e., the divisor with $E_\infty^2 = -3$. Therefore

$$N_{\tilde{X}_0|\tilde{\mathcal{X}}} = \mathcal{O}(\tilde{X}_0)|_{\tilde{X}_0} = \mathcal{O}_{\tilde{X}_0}(-E_\infty). \tag{4.26}$$

Since E_∞ intersects each member of the ruling transversely at one point, the normal bundle restricted to the ruling has degree (-1). □

Remark 4.5.3. *In the proof above, we obtained the semistable family* (4.25) *by blowup the original total space, then base change and normalizing. Alternatively, one can base change first and then blow up to obtain the family* (4.25).

Corollary 4.5.4. *The monodromy group of the family $\mathcal{X} \to D$ is $\mathbb{Z}_3$.*

Proof. By Proposition 4.5.2, the normal bundle $N_{\tilde{X}_0|\tilde{\mathcal{X}}}$ restricts to $\mathcal{O}(-1)$ on the ruling of $\tilde{X}_0$. It follows from Nakano's contractibility criterion [8, Theorem 3.2.8] that there exists a complex analytic manifold W and a holomorphic map $\tilde{\mathcal{X}} \to W$ contracting the ruled surface $\tilde{X}_0$ to E and is biholomorphic elsewhere.

Now $W \to \tilde{D}$ is a smooth family, so it is topologically trivial by Ehresmann's theorem, so the monodromy is trivial as well. So the claim follows. □

Remark 4.5.5. *The contraction $\tilde{\mathcal{X}} \to W$ can be chosen to be algebraic. In fact, by a relative version of Mori's Cone theorem [40, Theorem 3.25], it suffices to show that the ruling F on $\tilde{X}_0$ is extremal in the relative Mori cone $\overline{NE}(\tilde{\mathcal{X}}/\tilde{D})$. Assume that*

$$F = aE_\infty + bC, \tag{4.27}$$

where C is an effective curve whose irreducible components are not contained in the ruled surface $\tilde{X}_0$. Then by intersecting both sides of (4.27) with $\tilde{X}_0$ and use (4.26), one obtains

$$-1 = 3a + bC \cdot \tilde{X}_0.$$

Since $C \cdot \tilde{X}_0 \geq 0$, a and b cannot be both positive.

4.5.2 New Completion

Based on what we have seen, when the cubic threefold has an Eckardt point, the Abel-Jacobi map $T_v \to JX$ does not extend to $\bar{T}_v$. One reason is that the local monodromy group at an Eckardt hyperplane is too large so that it acts transitively on the 72 roots. Suggested by the previous proposition, the monodromy group has order 3 in a general pencil through the Eckardt hyperplane, which is much smaller.

From the other point of view, if we specify a one-parameter family of hyperplane sections through the Eckardt hyperplane, the 27 lines on the nearby fiber specialize to 27 lines on the Eckardt cone, so does a vanishing cycle represented by the difference of the classes of two skew lines. This suggests the Abel-Jacobi map extends along this one-dimensional disk.

To change from a 4-dimensional neighborhood to a 1-dimensional neighborhood, we blow up $\mathbb{O} = (\mathbb{P}^4)^*$ at Eckardt hyperplanes and denote the new space to be $\tilde{\mathbb{O}}$. Apply Stein's completion theorem to the finite covering $\pi : T_v \to \mathbb{O}^{\mathrm{sm}}$ with respect to the completion $\mathbb{O}^{\mathrm{sm}} \subseteq \tilde{\mathbb{O}}$, we obtain a normal analytic space $\tilde{T}_v$ together with a finite morphism $\tilde{T}_v \to \tilde{\mathbb{O}}$

extending π. Denote $P \cong \mathbb{P}^3$ a connected component of the exceptional divisor on $\tilde{\mathbb{O}}$, we have the following characterizations of the new completion space $\tilde{T}_v$.

Proposition 4.5.6. *(i) $\tilde{T}_v$ is isomorphic to the normalization of $\bar{T}_v \times_{\mathbb{O}} \tilde{\mathbb{O}}$.*

(ii) $\tilde{T}_v \times_{\tilde{\mathbb{O}}} P \to P$ is finite and has degree $24 \leq d \leq 72$.

Proof. For part (i), note $\bar{T}_v \times_{\mathbb{O}} \tilde{\mathbb{O}} \to \tilde{\mathbb{O}}$ is also finite and extends π, so is its normalization, which has to be isomorphic to $\tilde{T}_v$ by the uniqueness of Stein's completion.

For part (ii), since $\tilde{T}_v \times_{\tilde{\mathbb{O}}} P \to P$ is finite, it suffices to show the general fiber consists 24 points. For a general point v on P and $D \to \tilde{\mathbb{O}}$ a holomorphic disk transverse to P at point v, take the pullback family $\mathcal{X}$ of hyperplane sections of X. By Corollary 4.5.4, the monodromy over $D \setminus \{0\}$ has order 3. So each connected component is either 3-to-1 or 1-to-1 mapping to $D \setminus \{0\}$. It follows that the cardinality of the fiber at v is at least $72/3 = 24$. $\square$

Proposition 4.5.7. *There is a morphism $\tilde{T}_v \to \mathrm{Bl}_0 JX$ extending the topological Abel-Jacobi map $T_v \to JX$.*

Proof. The morphism $\tilde{\pi} : M = \mathrm{Bl}_0(\Theta) \to \mathbb{O}$ has fiber E_i over an Eckardt hyperplane $t_i \in \mathbb{O}$, so $M' := M \setminus \cup_i (E_i) \to \mathbb{O} \setminus \{t_1, \ldots, t_k\}$ is a finite branched covering. Denote $\sigma : \tilde{\mathbb{O}} \to \mathbb{O}$ the blowup at t_i. Then we can take the closure of the pullback of M' in the fiber product

$$\overline{\sigma^{-1}(M')} \subseteq M \times_{\mathbb{O}} \tilde{\mathbb{O}}. \tag{4.28}$$

The projection to the second coordinate $\pi_2 : \overline{\sigma^{-1}(M')} \to \tilde{\mathbb{O}}$ is finite. In fact, let $v \in P_i$ corresponding to a pencil $\mathbb{L}_v \subseteq \mathbb{O}$ of hyperplanes $\{H_t = H_i + tH_v = 0\}$, then the fiber $\pi_2^{-1}(v)$ is the limit of the finitely many points $\tilde{\pi}^{-1}(t)$ as t goes to 0. In other words, the closure of $(\mathbb{L}_v \setminus 0) \times_{\mathbb{O}} M$ in M is a finite cover over an open neighborhood of $0 \in L_v$ and its fiber over $t = 0$ corresponds to $\pi_2^{-1}(v)$.

By Stein's Theorem 3.3.1, the normalization of $\overline{\sigma^{-1}(M')}$ is $\tilde{T}_v$. Now the argument follows from that the composite $\tilde{T}_v \to \overline{\sigma^{-1}(M')} \to \mathrm{Bl}_0(\Theta) \to \Theta$ extends the topological Abel-Jacobi map $T_v \to \Theta$. □

4.6 Interpretation of Boundary Points

For a smooth cubic threefold X, we have shown that there is a unique completion $\bar{T}_v$ of the locus of primitive vanishing cycle T_v and we relate it to the theta divisor of intermediate Jacobian (Theorem 4.1.2).

We have an interpretation of boundary points of $\bar{T}_v$ by the following: The fiber of $X^* = \mathbb{O} \setminus \mathbb{O}^{\mathrm{sm}}$ corresponds to orbits of monodromy action induced by local fundamental groups.

More precisely, Suppose a point $p \in \bar{T}_v$ has the following interpretation. Denote $t_0 \in \mathbb{O}$ the projection of p. Let B be a small ball around t_0. Pick a base point $t' \in B^{\mathrm{sm}} = B \cap \mathbb{O}^{\mathrm{sm}}$. The fundamental group $\pi_1(B^{\mathrm{sm}}, t')$ acts on the root system $R(\mathbb{E}_6) = R_{X_{t'}}$ over t' via monodromy action.

Proposition 4.6.1. *There is a bijection between the fiber of $\bar{T}_v$ at t' and the set of orbits of $\pi_1(B^{\mathrm{sm}}, t') \curvearrowright R(\mathbb{E}_6)$.*

Our goal in this section is to build up the relationship between these monodromy orbits and the minimal resolutions of the singular cubic surfaces corresponding to points in X^* and the related Lie theory of the root systems.

For each $t_0 \in \mathbb{O}$, pick a small neighborhood B around t_0. Fix another base point $t' \in B^{\mathrm{sm}} := \mathbb{O}^{\mathrm{sm}} \cap B$, then the monodromy action permutes the 72 roots on the fiber $(\pi_v)^{-1}(t')$. Since it preserves the intersection pairing and polarization, it preserves the structure of root system. Since $AutH^2(S_{t'}, \mathbb{Z}) = Aut(I_{1,6}) = W(\mathbb{E}_6)$, the monodromy representation as we defined in (3.9) is a homomorphism

$$\rho_{t_0} : \pi_1(B^{\mathrm{sm}}, t') \to W(\mathbb{E}_6). \tag{4.29}$$

Recall that we defined $G_{t_0} = \mathrm{Im}(\rho_{t_0})$ to be the *(local) monodromy group* of the hyperplane section S_{t_0}.

According to the definition of completion of finite analytic cover in Theorem 3.3.1, we have the following monodromy interpretation of the fibers of

$$\pi_v : \bar{T}_v \to \mathbb{O}.$$

Lemma 4.6.2. *The fiber $PV_{t_0} = (\bar{\pi}_v)^{-1}(t_0)$ corresponds to the orbits of the local monodromy action $G_{t_0} \curvearrowright \mathbb{R}(\mathbb{E}_6)$.*

Let S_0 denote the cubic surface as the hyperplane section of X corresponding to $t_0 \in \mathbb{O}$. We would like to relate the monodromy orbits $R(\mathbb{E}_6)/G_{t_0}$ to the root system on its minimal resolution $\tilde{S}_0 \to S_0$.

4.6.1 Minimal Resolution

Let S be a cubic surface with at worst ADE singularities. Let

$$\sigma : \tilde{S} \to S$$

be its minimal resolution, then $\tilde{S}$ is a weak del Pezzo surface of degree 3 and one can still define the root system $R(\tilde{S})$ as

$$\alpha^2 = -2, \ \alpha \cdot K_{\tilde{S}} = 0. \tag{4.30}$$

It is known that $R(\tilde{S})$ is isomorphic to $R(\mathbb{E}_6)$. This can be seen as follows. One can regard $\tilde{S}$ as blowing up six bubble points on $\mathbb{P}^2$ in an almost general position. The six points can be deformed smoothly as they move to a general position along a real one-dimensional path. As the equation (4.30) is topological invariant, the root system on $\tilde{S}$ is defined.

Note that each irreducible component C of the exceptional divisor of σ is a (-2) curve and is orthogonal to the class $K_{\tilde{S}}$ since σ is crepant, so C defines a root and is effective as divisor class. We call such root an *effective root*. The set of all effective roots generates a sub-root system R_e of $R(\tilde{S})$. Since each of the singularity x_i of S corresponds to a bunch of (-2)-curves on $\tilde{S}$ and they generate a sub-root system R_i, R_e is isomorphic to the product $\prod_{i\in I} R_i$, where I is the index set of singularities of S. Each R_i corresponds to a connected sub-diagram of the Dynkin diagram of $\mathbb{E}_6$. One refers to [22], sections 8.1, 8.2, 8.3, and 9.1 for the detailed discussion.

Moreover, the reflections with respect to all of the effective roots define a subgroup $W(R_e)$ of the Weyl group $W(\mathbb{E}_6)$, and $W(R_e)$ is isomorphic to the product $\prod_{i\in I} W_i$, where W_i is the Weyl group generated by the reflections corresponding to the exceptional curves over the singularity x_i. One can consider the action of $W(R_e)$ on $R(\mathbb{E}_6)$. The orbits that are contained in R_e are naturally in bijection with the set of singularities on S. One can also define the maximal/minimal root of an orbit. In particular, the maximal root of the orbit corresponding to x_i equals the cohomology class of the fundamental cycle Z_i at x_i. We refer to [44], section 2.1 for the detailed discussion.

Definition 4.6.3. *We define the $R(S)$ to be the set of the orbit in $R(\mathbb{E}_6)$ under the action by $W(R_e)$. We call $R(S)$ to be the root system on S.*

Note that $R(S)$ is just a set, without any intersection pairing. In fact, the set $R(S) = R(\mathbb{E}_6)/W(R_e)$ is used in [44, Theorem 2.1] to parameterize the connected components of the (reduced) Hilbert scheme of generalized twisted cubics on S.

Our main result is the following.

Proposition 4.6.4. *Assume that the cubic surface S_{t_0} has at worst ADE singularities. Then there is an isomorphism*

$$G_{t_0} \cong W(R_e) \tag{4.31}$$

between the local monodromy group G_{t_0} of S_{t_0} and the subgroup of $W(\mathbb{E}_6)$ generated by the relections of all effective roots in the minimal resolution $\tilde{S}_{t_0}$ of S_{t_0}.

The following theorem will follow from Lemma 4.6.2 and Proposition 4.6.4.

Theorem 4.6.5. *There is a bijection of sets*

$$PV_{t_0} \longleftrightarrow R(S).$$

Example 4.6.6. *Take a one-parameter family $\{S_t\}_{t\in\Delta}$ of cubic surface, where S_0 has an A_1 singularity, and S_t is smooth when $t \neq 0$. Assume that total space is smooth. Then there is a vanishing cycle δ associated to the family. Fix a base point $t' \in \Delta^*$. The monodromy representation $\pi_1(\Delta^*, t') \to R(\mathbb{E}_6)$ is generated by the Picard-Lefschetz transformation*

$$T_\delta : \alpha \mapsto \alpha + (\alpha, \delta)\delta. \tag{4.32}$$

As $\delta^2 = -2$, T_δ has order two, T_δ coincides with the reflection of root system along δ. Therefore, the orbits of monodromy action are naturally identified with the orbits of Weyl group $W(A_1) = \mathbb{Z}_2$ action.

We can compute the cardinality of orbits using the geometry of the nodal cubic surface. The cubic surface S_{t_0} can be represented by blowup of 6 general points on $\mathbb{P}^2$, and as it degenerates to S_0, the 6 points come to lie on a conic. Use the same notation as Proposition 4.2.7, we can choose the vanishing cycle $\delta = 2h - e_1 - \cdots - e_6$. The 72 roots can be expressed as classes

(1) $\pm\delta$, 2 roots;

(2) $\pm(h - e_i - e_j - e_k)$, i, j, k distinct, 40 roots;

(3) $e_i - e_j, i \neq j$, 30 roots.

The roots in classes (1) and (2) have nonzero intersections with δ, while the classes in (3) are orthogonal to δ, so by the Picard-Lefschetz formula, the number of local monodromy orbits is $1 + 20 + 30 = 51$. So $|PV_{t_0}| = |R(S)| = 51$.

In [44] Theorem 2.1, the authors showed that $R(S) = R(\mathbb{E}_6)/W(R_e)$ are in bijection with the connected components of the reduced Hilbert schemes of generalized twisted cubics on S_0. The orbits that contain an effective root correspond to generalized twisted cubics that are not Cohen-Macaulay (whose reduced schemes are planar). Those orbits that don't have any effective roots correspond to the generalized twisted cubics that are arithematic Cohen-Macaulay (whose reduced schemes are not planar). Section 3 of [44], it showed that there is a bijective between the $W(R_e)$-orbit on $R(\tilde{S}) \setminus R_e$ and the linear determinantal representations of cubic surfaces. The cardinality of such orbits is listed in p.102, Table 1. On the other hand, we know that the cardinality of the orbits on R_e is exactly the number of the singularities. So we obtain the cardinality of the root system $R(S)$ by adding up the two numbers.

Corollary 4.6.7. *The cardinality $|R(S)|$, which coincides with the cardinality of the fiber of $\bar{T}' \to \mathbb{O}$ corresponding to the cubic surface S, is listed in the table below.*

R_e	Type	#	R_e	Type	#	R_e	Type	#
$\emptyset$	I	72	$4A_1$	XVI	17	$A_1 + 2A_2$	XVII	9
A_1	II	51	$2A_1 + A_2$	XIII	15	$A_1 + A_4$	XIV	6
$2A_1$	IV	36	$A_1 + A_3$	X	12	A_5	XI	5
A_2	III	31	$2A_2$	IX	14	D_5	XV	3
$3A_1$	VIII	25	A_4	VII	9	$A_1 + A_5$	XIX	3
$A_1 + A_2$	VI	22	D_4	XII	7	$3A_2$	XXI	5
A_3	V	17	$2A_1 + A_3$	XVIII	8	E_6	XX	1

Table 4.1: Numbers of roots on cubic surfaces of given singularity type.

4.6.2 A Local Argument.

We will start to prove Proposition 4.6.4 from this section. We need first to study the local monodromy of a cubic surface around one singularity.

Let $p : \mathcal{X} \to B$ be the family of cubic surfaces over the ball B arising from hyperplane sections on X. Let x_0 be an isolated singularity of S_{t_0}, where S_{t_0} is the hyperplane section $X \cap H_{t_0}$. Take a small ball D_0 in the total space $\mathcal{X}$ around x_0. Then by restricting to $D_0^{\mathrm{sm}} = D_0 \setminus p^{-1}(X^*)$, the morphism

$$p^{\mathrm{sm}} : D_0^{\mathrm{sm}} \to B^{\mathrm{sm}}$$

is a smooth fiber bundle. Let F be a fiber, then there is a monodromy representation

$$\rho_{t_0,x_0} : \pi_1(B^{\mathrm{sm}}, t') \to \mathrm{Aut}H^2(F, \mathbb{Z}). \tag{4.33}$$

We call $\mathrm{Im}(\rho_{t_0,x_0})$ the *(local) monodromy group of S_{t_0} at the singularity x_0.*

Proposition 4.6.8. *Assume x_0 has type ADE. Then the monodromy group $\mathrm{Im}(\rho_{t_0,x_0})$ at x_0 is isomorphic to the Weyl group W_{x_0} corresponding to singularity type of x_0.*

To prove this proposition, we need to use the Milnor fiber theory. One refers to [23] a more detailed survey.

4.6.3 Monodromy Group on Milnor Fiber.

Let $f(x_1, ..., x_n) = 0$ be a hypersurface in $\mathbb{C}^n$ with an isolated singularity at 0, then the Milnor fiber F of f is the $\{f = w\} \cap B^n$ for a ball B^n around origin of small radius and $w \in \mathbb{C}$ with a small magnitude. F has homotopy type of a bouquet of μ spheres of dimension $n - 1$, where μ is the Milnor number of the singularity, which coincides with the dimension of $\mathbb{C}$-vector space $\mathbb{C}[x_1, ..., x_n]/(\frac{\partial f}{\partial x_1}, ..., \frac{\partial f}{\partial x_n})$.

A *deformation* of f is an analytic function

$$g(x_1, ..., x_n, w) : \mathbb{C}^n \times \mathbb{C} \to \mathbb{C}$$

such that $g(x_1, ..., x_n, 0) = f(x_1, ..., x_n)$, and $\tilde{f}(x_1, ..., x_n) = g(x_1, ..., x_n, 1)$ is called a *perturbation* of f. There is a perturbation $\tilde{f}$ of f such that $\tilde{f}$ is a Morse function in the sense that all critical values of $\tilde{f}$ are distinct and all critical points are nondegenerate. There are exactly μ critical points $t_1, ..., t_\mu$ of $\tilde{f}$ and they are contained in a δ-neighborhood D_δ of 0 in $\mathbb{C}$. The Milnor fibers of f and $\tilde{f}$ are diffeomorphic.

We can choose a base point $t' \in D_\delta - \{t_1, ..., t_\mu\}$ and paths $p_i, 1 \leq i \leq \mu$ connecting t' to t_i such that its interior is contained in $D_\delta - \{t_1, ..., t_\mu\}$. We define a loop l_i based at t' where l_i goes around t_i anticlockwise along a small circle centered at t_i and is connected by p_i. The loops $l_1, ..., l_\mu$ generate the fundamental group $\pi_1(D_\delta - \{t_1, ..., t_\mu\}, t')$. The loop l_i induces monodromy action on the cohomology of fiber $H^{n-1}(F, \mathbb{Z})$ given by the Picard-Lefschetz formula

$$T_i : \alpha \mapsto \alpha + (\alpha, \delta_i)\delta_i,$$

where δ_i is the vanishing cycle associated to the critical value t_i. The set of all vanishing cycles $\{\delta_i\}_{i=1}^{\mu}$ generates $H^{n-1}(F, \mathbb{Z})$. When n is odd, $(\delta_i, \delta_i) = \pm 2$, while when n is even, $(\delta_i, \delta_i) = 0$.

Definition 4.6.9. *We define monodromy group of the Milnor fiber of f to be the subgroup of* $\mathrm{Aut}H^{n-1}(F, \mathbb{Z})$ *generated by $T_1, ..., T_\mu$.*

The monodromy group is independent of the choice of perturbation function and the loops $l_1, ..., l_\mu$. Moreover, in the case where $n = 3$ and $f(x_1, x_2, x_3) = 0$ has ADE singularity at origin, the following result is well known.

Lemma 4.6.10. *([3, p.99], [32])) Vanishing cycles $\delta_1, ..., \delta_\mu$ can be naturally chosen to form a basis of the root system of the corresponding ADE type in $H^2(F, \mathbb{Z})$. The monodromy group of f is the Weyl group corresponding to the type of singularity.*

These vanishing cycles are obtained by a sequence of conjugation operations of paths $\{p_i\}_{i=1}^{\mu}$. Such operations are called Gabrielov operations.

Now let S_0 be the cubic surface arising from a hyperplane section of X with an affine chart defined by $f(x_1, x_2, x_3) = 0$ with an isolated singularity at $(0, 0, 0)$ of ADE type. The next result will show that deforming f in the family of hyperplane sections is the "same" as considering the Milnor fiber theory of f.

Lemma 4.6.11. *Choose a linear 2-dimensional hyperplane sections family parameterized by $(\lambda, w) \in \mathbb{C}^2$ with $(0, 0)$ corresponds to $f(x_1, x_2, x_3) = 0$ with an ADE singularity, then there is an $\varepsilon > 0$ such that for all λ, w with $|\lambda|, |w| < \varepsilon$, an affine chart of the total family has analytic equation*

$$f_\lambda(x_1, x_2, x_3) + w = 0, \tag{4.34}$$

where $f_\lambda(x_1, x_2, x_3)$ is the affine equation of the hyperplane section at $(\lambda, 0)$.

Proof. $f(x_1, x_2, x_3) = 0$ is an affine cubic surface with an isolated singularity at $(0, 0, 0)$ of ADE type. Using x_1, x_2, x_3, w as affine coordinates, the cubic threefold X has equation

$$F(x_1, x_2, x_3, w) = f(x_1, x_2, x_3) + wQ(x_1, x_2, x_3) + w^2 L(x_1, x_2, x_3) + w^3\sigma,$$

where Q, L, σ are polynomials of degree $2, 1$, and 0, respectively.

Projecting to the w-coordinate, we obtain a pencil

$$\mathcal{X} \to \mathbb{C} \tag{4.35}$$

of hyperplane sections $f_w(x_1, x_2, x_3) = F(x_1, x_2, x_3, w)$ of X through $f(x_1, x_2, x_3) = 0$. Since the cubic threefold is smooth, $Q(0, 0, 0) \neq 0$. Therefore, the equation of cubic threefold is

$$F(x_1, x_2, x_3, w) = f(x_1, x_2, x_3) + wG(x_1, x_2, x_3, w) = 0,$$

where $G(x_1, x_2, x_3, w)$ is quadratic and is non-vanishing in a small neighborhood D of 0. Therefore, by restricting to D and setting $g = f/G$, we get a family

$$g(x_1, x_2, x_3, w) + w = 0,$$

which is analytically equivalent to the family (4.35) restricted to D.

Now we choose a perturbation of f in the hyperplane section family transversal to the w direction. In other words, we choose a linear function

$$l = ax_1 + bx_2 + cx_3$$

with $a, b, c \in \mathbb{C}$ being general, then

$$f_\lambda(x_1, x_2, x_3) = F(x_1, x_2, x_3, \lambda l),\ \ \lambda \in \mathbb{C}$$

is a pencil of hyperplane sections through f. We consider the two dimensional family spanned by l and w. Then for $(l, w) \in \mathbb{C}^2$, the hyperplane section at $\lambda l + w$ is defined by

$$f_{\lambda,w} = F(x_1, x_2, x_3, \lambda l + w) = f_\lambda(x_1, x_2, x_3) + wG(x_1, x_2, x_3, w + \lambda l) + \lambda l H(x, y, z, w), \quad (4.36)$$

where $H(x, y, z, w) = G(x, y, z, w + \lambda l) - G(x, y, z, \lambda l) = wL(x_1, x_2, x_3) + (2w\lambda l + w^2)\sigma$ is divisible by w.

Therefore, denote $G' = G + \lambda l H/w$, we can express the two dimensional family (4.36) as

$$\frac{f_\lambda(x_1, x_2, x_3)}{G'} + w = 0,$$

in a small neighborhood D^2 of origin. It is analytically equivalent to the family

$$f_\lambda(x_1, x_2, x_3) + w = 0.$$

$\square$

Proof of Proposition 4.6.8. Let Σ_0 be the discriminant locus x_0, namely the locus $\{t \in B | p^{-1}(t) \cap D_0$ is singular$\}$. $\Sigma_0 \subseteq Y^\vee \cap B$ is an irreducible component (when S_0 has only one isolated singularity, they are the same).

Since the complement of the inclusion $B^{\text{sm}} \subseteq B \setminus \Sigma_0$ has real codimension at least two, there is a surjection

$$\pi_1(B^{\text{sm}}, t') \twoheadrightarrow \pi_1(B \setminus \Sigma_0, t'),$$

where t' is a fixed base point. Therefore, one reduces to the case where S_0 has only one singularity and $\Sigma_0 = X^\vee \cap B$.

We choose a general line $\mathbb{L}$ through t' such that $\mathbb{L}$ intersect Σ_0 transversely at smooth points, then $U = B^{\text{sm}} \cap \mathbb{L}$ is an analytic open space. Moreover, by a local version of Zariski's theorem on fundamental groups, there is a surjection

$$\pi_1(U, t') \twoheadrightarrow \pi_1(B^{\text{sm}}, t'). \tag{4.37}$$

Therefore it suffices to show that the monodromy representations generated by the loops in the 1-dimensional open space U is the entire Weyl group.

On the other hand, by Lemma 4.6.11, the hyperplane sections parameterized by U are analytically equivalent to the family

$$f'(x_1, x_2, x_3) + w = 0,$$

where f' is the defining equation of the hyperplane section at t' and is a perturbation of f. Therefore, by Lemma 4.6.10, the monodromy group induced by $\pi_1(U, t')$ is the Weyl group corresponding to the type of x_0. □

4.6.4 Globalization.

Proposition 4.6.12. *Let S_0 be a hyperplane section of X and x_0 be a singular point on S_0 of type ADE. Let F denote the Milnor fiber of x_0, and S_t a nearby general fiber, then there* *is an inclusion $F \hookrightarrow S_t$. The induced map on homology*

$$H_2(F, \mathbb{Z}) \to H_2(S_t, \mathbb{Z})_{\text{van}} \tag{4.38}$$

is injective.

Proof. This is due to Brieskorn's theory [11] and its globalization [4] (also see [40], Theorem 4.43). Using the same notations as we introduced at the beginning of this section, there exists a finite cover $B' \to B$, such that the base-changed total family admits simultaneous resolution in the category of algebraic spaces. In other words, there is a commutative diagram as follows.

$$\begin{array}{ccccc} \mathcal{X}' & \xrightarrow{f} & \mathcal{X} \times_B B' & \longrightarrow & \mathcal{X} \\ & \searrow^{g} & \downarrow & & \downarrow \\ & & B' & \longrightarrow & B \end{array}$$

$\mathcal{X}'$ is a complex analytic manifold, f is bimeromorphic, and g is a proper holomorphic submersion. (The resolution is generally not algebraic since the local gluing data is only analytic.)

$\mathcal{X}' \to B'$ is diffeomorphic to the product $S_t \times B'$ by Ehresmann's theorem, so the Milnor fiber $F = S_t \cap D_0$ is diffeomorphic to an open set U of the central fiber $g^{-1}(0)$. The argument reduces to show that the homology group induced by the inclusion $U \hookrightarrow g^{-1}(0)$ is injective.

$g^{-1}(0)$ is isomorphic to the minimal resolution $\tilde{S}_0$ of S_0. Denote V the exceptional curve in $\tilde{S}_0$ over x_0. Then V is a bunch of μ (-2)-curves, and corresponds to a connected sub-diagram of the Dynkin diagram of $\mathbb{E}_6$. Since the image of U in S_0 is a neighborhood of x_0, U is a regular neighborhood of V. So the induced map $H_2(U, \mathbb{Z}) \to H_2(\tilde{S}_0, \mathbb{Z})$ is injective. $\square$

Remark 4.6.13. *This is false for elliptic singularity, since the Milnor number of such a singularity is 8, while the vanishing homology on S_t has rank 6.*

Corollary 4.6.14. *Via the inclusion* (4.38)*,* $H_2(F,\mathbb{Z})$ *is an irreducible sub-representation of* $H_2(S_t,\mathbb{Z})_{\mathrm{van}}$ *as* $\pi_1(B^{\mathrm{sm}},t')$*-representations. It induces isomorphism between the monodromy group* $\mathrm{Im}(\rho_0)$ *of* x_0 *defined in* (4.33) *and the monodromy group* $\mathrm{Im}(\rho)$ *defined in* (4.29)*.*

Now we're ready to prove our main Proposition in this section.

Proof of Proposition 4.6.4. Denote $x_1, ..., x_k$ the singularities of S_{t_0} with ADE type. Let W_i be the Weyl group corresponding to the type of the singularity x_i, then $W(R_e) = W_1 \times \cdots \times W_k$. We'll show that the local monodromy group is isomorphic to $W_1 \times \cdots \times W_k$ as well.

Let D_i denote a small ball in $\mathcal{X}$ around x_i such that $D_i \cap D_j = \emptyset$ for $i \neq j$. Let $\Sigma_i = \{t \in B | S_t \cap D_i \text{ is singular}\}$ be the discriminant of x_i. Then Σ_i is an irreducible divisor of B and $X^\vee \cap B = \cup_i \Sigma_i(x_i)$. None of the Σ_i contains Σ_j for $i \neq j$, since otherwise the locus will extend to a proper curve, contradicting to the fact that the dual variety $X^\vee$ is smooth in codimension one, and that the smooth locus parameterizes the hyperplane section with one ordinary nodal singularity.

Fix a general point $t' \in B^{\mathrm{sm}}$. We take a general pencil $\mathbb{L}$ in $\mathbb{O}$ through t' intersecting $X^\vee \cap B$ transversely along the smooth locus. So $\mathbb{L}$ intersects each Σ_i transversely at points t_i^j, for $j = 1, ..., \mu_i$, where μ_i is the Milnor number of x_i. None of the t_i^j coincides with $t_{i'}^{j'}$ unless $i = i'$ and $j = j'$. There is a vanishing cycle $\delta_i^j \in H^2(S_{t'},\mathbb{Z})$ associate to t_i^j. The monodromy action T_i^j induced by the simple loop around t_i^j on the 72 roots is given by the Picard-Lefschetz formula (4.32) associated to δ_i^j. Moreover, via the surjectivity

$$\pi_1(\mathbb{L} \cap B, t') \twoheadrightarrow \pi_1(B^{\mathrm{sm}}, t'),$$

the monodromy group defined in (4.29) is generated by T_i^j, $i = 1, ..., k$, $j = 1, ..., \mu_i$. By Proposition 4.6.8 and Corollary 4.6.14, the subgroup generated by $\{T_i^j\}_{j=1}^{\mu_i}$ is the Weyl group W_i, which is also the subgroup generated by the reflections corresponding to the exceptional curves over x_i.

Finally, since δ_i^j can be represented by a topological 2-sphere contained in the neighborhood D_i around x_i, so the intersection number

$$(\delta_i^j, \delta_{i'}^{j'}) = 0, \; i \neq i'.$$

Therefore, the monodromy operators T_i^j and $T_{i'}^{j'}$ commute for $i \neq i'$ by Picard-Lefschetz formula. Therefore, the subgroup corresponding to the monodromy group of x_i commutes with the subgroup corresponding to the monodromy group of x_j. It follows that the monodromy group of S_0 is the product $W_1 \times \cdots \times W_k$. □

4.7 Relation to Schnell's Results

4.7.1 $\mathcal{D}$-modules

Recall in Lemma 3.5.3, we introduced Schnell's construction of completion of $T_{\mathbb{Z}}$ for a variation of Hodge structure of even weight.

In the case where the variation of Hodge structure comes from vanishing cohomology of hyperplane sections on a smooth cubic threefold, all H^2 on the general hyperplane section is concentrated in $(1,1)$ part, so $F^2\mathcal{H}$ is trivial and so is $\mathcal{M}$. Therefore the analytic spectrum is nothing but the base space $\mathbb{O}$. So the completion of the locus of primitive vanishing cycle T_v is exactly the $\bar{T}_v$ space we are discussing. In short, as a consequence of Theorem 4.4.1, we have:

Corollary 4.7.1. *When the variation of Hodge structure comes from vanishing cohomology of hyperplane sections on a smooth cubic threefold, the completion space $\bar{T}_{\mathbb{Z}}$ defined in [50] p.10 contains an irreducible component which is biholomorphic to $\bar{T}_v$ space and is obtained by contracting finitely many curves in* $\mathrm{Bl}_0(\Theta)$.

4.7.2 Tube Mapping

In [51], Schnell studied the relationship between the primitive homology $H_n(X,\mathbb{Z})_{\text{prim}}$ of a smooth projective variety $X \subseteq \mathbb{P}^N$ of dimension n and the vanishing homology $H_{n-1}(Y,\mathbb{Z})_{\text{van}}$ of a smooth hyperplane section $Y = X \bigcap H$. Let $\mathbb{O}^{\text{sm}} \subseteq (\mathbb{P}^N)^*$ be the open set of smooth hyperplanes, and $l \subseteq \mathbb{O}^{\text{sm}}$ be a loop based at t, and $\alpha \in H_{n-1}(Y,\mathbb{Z})_{\text{van}}$, if $l_*\alpha = \alpha$, then the trace of α along the loop l is a topological n-chain on X with boundary $\alpha - l_*\alpha = 0$, so it is a n-cycle which is well-defined in the primitive homology. Since the n-cycle is a "tube" on α over the loop l, such map is called *tube mapping.* Schnell proved that

Theorem 4.7.2. *([51]) If $H^{n-1}_{\text{van}}(Y,\mathbb{Z}) \neq 0$, then the tube map*

$$\{([l],\alpha) \in \pi_1(\mathbb{O}^{\text{sm}},t) \times H_{n-1}(Y,\mathbb{Z})_{\text{van}} | l_*\alpha = \alpha\} \to H_n(X,\mathbb{Z})_{\text{prim}}$$

has cofinite image.

Equivalently, the set of tubes on vanishing cohomology classes generates the middle dimensional primitive cohomology on X over $\mathbb{Q}$.

Herb Clemens conjectured that the theorem is still true by restricting the tube map to tubes on a single primitive vanishing cycle α_0. In other words,

Conjecture 4.7.3. *Under the same hypothesis, the image of*

$$\{([l],\alpha_0) \mid [l] \in \pi_1(\mathbb{O}^{\text{sm}},t), l_*\alpha = \alpha\} \to H_n(X,\mathbb{Z})_{\text{prim}} \tag{4.39}$$

is cofinite.

We will prove this conjecture when X is a general cubic 3-fold.

Theorem 4.7.4. *Let X be a smooth cubic threefold, then the conjecture is true.*

Proof. First note that $H_3(X,\mathbb{Z}) = H_3(X,\mathbb{Z})_{\text{prim}}$ due to $H_1(X) = 0$. Second, a primitive vanishing cycle $\alpha_0 \in H_2(X_t,\mathbb{Z})$ is represented by the difference of two lines, so (t,α_0) is a point on T_v. Also, recall that T_v is a finite-sheet covering space of $\mathbb{O}^{\mathrm{sm}}$. A loop $l \subseteq \mathbb{O}^{\mathrm{sm}}$ such that $l_*\alpha_0 = \alpha_0$ based at t corresponds to a loop $\tilde{l} \subseteq T_v$ based at (t, α_0), so by abusing the notation $*$ as the base point, the map (4.39) is the same as

$$\pi_1(T_v, *) \to H_3(X, \mathbb{Z}). \tag{4.40}$$

The following result is proved in [63, p.26] in a more general setting. For the reader's convenience, we provide self-contained proof here.

Proposition 4.7.5. *The map* (4.40) *is the induced map on fundamental groups from the topological Abel-Jacobi map* $\phi : T_v \to JX$*, followed by the isomorphism* $\pi_1(JX, *) \cong H_3(X, \mathbb{Z})$.

Proof. Let $\tilde{l} \subseteq T_v$ be a loop based at (t, α_0), then its Abel-Jacobi image is determined by a family of 3-chains Γ_t indexed by $t \in [0, 1]$ modulo 3-cycles on X, so we can choose Γ_t to be the union $\Gamma_0 \bigcup \Gamma'_t$ where $\Gamma'_t = \bigcup_{s\in[0,t]} \alpha_s$ as trace of primitive vanishing cycles along the path $[0, 1]$. It follows that Γ_1 is a 3-chain such that $\partial\Gamma_1 = \partial\Gamma_0 = \alpha_0$, so the induced map on π_1 sends $\tilde{l}$ to the image of the 3-cycle $\Gamma_1 - \Gamma_0 = \bigcup_{t\in[0,1]} \alpha_t$ in $H_3(X, \mathbb{Z})$. □

Finally, the proof of the theorem follows from the following argument.

Proposition 4.7.6. *The map (4.40) induced by* ϕ *is surjective.*

Proof. First of all, $\phi : T_v \to JX$ factors through the inclusion $T_v \subseteq \mathrm{Bl}_0(\Theta)$. Moreover, $T_v \subseteq \mathrm{Bl}_0(\Theta)$ is a complement of a divisor in a smooth complex manifold, as a smooth loop based can be deformed to be disjoint from a real codimension-two set, there is a surjection $\pi_1(T_v, *) \twoheadrightarrow \pi_1(\mathrm{Bl}_0(\Theta), *)$. Therefore, it suffices to show that $\pi_1(\mathrm{Bl}_0(\Theta), *) \to \pi_1(JX)$ is surjective.

Next, choose $p \in F$ such that its corresponding line L_p is of the second type on X and let D_p be the divisor of lines that are incident to L_p. By Lemma 10.7 of [19], $p \in D_p$, it follows that $\{p\} \times F \setminus D_p$ is disjoint from the diagonal. In particular, let $\sigma : \mathrm{Bl}_{\Delta_F}(F \times F) \to F \times F$

be the blowup map, the restriction of σ^{-1} to the domain of ψ_p is an isomorphism. We define the restricted Abel-Jacobi map

$$\psi_p : \{p\} \times F \setminus D_p \to JX. \tag{4.41}$$

ψ_p lifts to the blowup, so the image of $\pi_1(\mathrm{Bl}_0(\Theta), *) \to \pi_1(JX)$ contains $(\psi_p)_*(\pi_1(\{p\} \times F \setminus D_p, *))$ as a subgroup. Thus it suffices to show that ψ_p induces surjectivity on fundamental groups.

To show this, note that ψ_p factors through the inclusion $\{p\} \times F \setminus D_p \subseteq \{p\} \times F$, which induces a surjective map on the fundamental group for the same reason as in the first paragraph of the proof. Moreover, the map $\{p\} \times F \cong F \to JX$ factors through the Albanese map

$$\begin{array}{ccc} F & \xrightarrow{\psi} & JX \\ \downarrow{\scriptstyle alb} & \nearrow{\scriptstyle \cong} & \\ \mathrm{Alb}(F) & & \end{array} \tag{4.42}$$

together with the isomorphism $\mathrm{Alb}(F) \xrightarrow{\cong} JX$ [19]. It follows that ψ induces an isomorphism between fundamental groups, therefore, so does ψ_p. □

□

4.8 Additional Results on Eckardt Cubic Threefold

4.8.1 Is Intermediate Jacobian a Product

When a cubic threefold X contains an Eckardt point, an elliptic curve $E \subseteq F$ parameterizes lines on X through the Eckardt point. In fact, this corresponds to an elliptic curve $E \subseteq JX$ contained in the Jacobian variety. It can be seen as the Abel-Jacobi image of $E \times E$.

We can ask if JX splits as a product of E and an abelian fourfold. This cannot hold since Beauville showed that the theta divisor Θ of JX has only one singularity, while the theta divisor of a product $E \times A$ is reducible, so the singular locus has dimension 3.

We'll give an alternative proof of this result by showing the polarization of E induced from JX is not principal.

Let $E \subseteq F$ be the elliptic curve corresponding to the lines through an Eckardt point on X. The restriction of the Abel-Jacobi map $\psi : F \times F \to JX$ to $E \times E$ factors through

$$E \times E \to Alb(E) \hookrightarrow Alb(F) \cong JX.$$

Since $Alb(E) \cong E$, this produces the inclusion

$$E \hookrightarrow JX.$$

Proposition 4.8.1. *The restriction to E of the principal polarization on JX is twice the principal polarization on E.*

The proof is given in [14, Lemma 1.12] based on a fact that an Eckardt point p on a cubic threefold X defines an involution $\tau : X \to X$ fixing p and induces an involution $\tau : JX \to JX$ on the intermediate Jacobian. The authors further showed that JX is isogeny to $E \times A$, where A is an abelian fourfold arising from the dual abelian variety of certain Prym variety [14, Theorem 3.11].

We provide an alternative proof of Proposition (4.8.1).

Proof. Consider the commutative diagram

Here $\beta : p - q \mapsto D_p - D_q$ induces the principal polarization. α is induced by the inclusion and γ is given by restriction. Therefore the composite $\gamma \circ \beta \circ \alpha : E \to E$ is induced by the principal polarization of JX. We'll show that this map has degree 2.

$$\begin{array}{ccccccc} Alb(E) & \xrightarrow{\alpha} & Alb(F) & \xrightarrow{\beta} & Pic^0(F) & \xrightarrow{\gamma} & Pic^0(E) \\ \downarrow \cong & & \downarrow \cong & & \downarrow \cong & & \downarrow \cong \\ E & \longrightarrow & JX & \xrightarrow{\cong} & JX & \longrightarrow & E \end{array} \tag{4.43}$$

Note that the divisor D_p has a component E, so we can express D_p as

$$D_p = D'_p + E,$$

where D'_p is the complementary divisor. Similarly, $D_q = D'_q + E$. It follows that

$$\beta \circ \alpha(p - q) = D'_p - D'_q.$$

Composed with γ, we have

$$\gamma \circ \beta \circ \alpha(p - q) = (D'_p - D'_q) \cap E.$$

We'll first do some intersection computation. By adjunction formula,

$$2g(E) - 2 = E.(E + K_F).$$

Use the fact that $K_F \sim 3D_x$, where $x \in F$, and the fact that $D_x \cdot E = 1$, we obtain $E^2 = -3$. It follows that

$$D'_p.E = (D_p - E).E = 1 - (-3) = 4.$$

This implies that the natural projection

$$\pi_p : D'_p \to L_p$$

has degree 4.

On the other hand, L_p is a line of the second type, so when p is general on E, there are 5 distinct lines passing through a given general point x on L_p. The four lines come from the fiber of π_p at x, and the other line is L_p.

Recall that there is a generically 6-to-1 cover from the incidence variety

$$\lambda : I = \{(t, x) : x \in L_t\} \to X$$

and is simply branched over a Zariski dense subset of every component of the divisor of the lines of the second type [19, 10.18].

Regarding the simple ramification locus of 6 lines over L_p, there are two possibilities:

(1) The simple ramification locus is the constant L_p line;

(2) The simple ramification locus is contained in D'_p.

For the second case, there is a section $L_p \to D'_p$. However, the image is a rational curve $C \subseteq D'_p \subseteq F$. This is impossible since the Albanese map $F \to Alb(F)$ is an inclusion.

Therefore case (1) has to be true. When we specialize the 6 lines along L_p towards the cone point, the limit 6 lines are $\pi_p^{-1}(D'_p) + 2L_p$.

Let X' denote the open subspace of X by removing all the Eckardt points. There is a natural map

$$\phi : X' \to JX,$$
$$x \mapsto \int_{\lambda^{-1}(x_0)}^{\lambda^{-1}(x)}.$$

where $x_0 \in X'$ is a fixed point. The map ϕ extends to the blowup $\tilde{X}$ of X at all Eckardt points, therefore, we have a proper morphism

$$\tilde{\phi} : \tilde{X} \to JX.$$

It has to be a constant map since $\pi_1(\tilde{X}) = 0$.

In particular, $\pi_q^{-1}(D'_q) + 2L_q = \pi_p^{-1}(D'_p) + 2L_p$ in JX for any $p, q \in E$.

It follows that $(D'_p - D'_q) \cap E = \pi_p^{-1}(D'_p) - \pi_q^{-1}(D'_q) = 2L_q - 2L_p = 2(L_q - L_p)$.

So the composite $\gamma \circ \beta \circ \alpha : E \to E$ has degree two. $\square$

4.8.2 Triple Lines and Eckardt Points

If a smooth cubic threefold X has an Eckardt point p, then the hyperplane $S = T_pX \cap X$ is a cone over an elliptic curve E and any of the line L in the ruling of the cone is of the second type. So $P \cap X = 2L + L'$, where P is the (unique) plane tangent to X along L.

Definition 4.8.2. *We call L a triple line if $L' = L$, or equivalently, $P \cap X = 3L$.*

Lemma 4.8.3. *([19, Lemma 10.15]) There are at most finitely many triple lines on a smooth cubic threefold.*

Similarly, any smooth cubic threefolds has finitely many Eckardt points (Lemma 4.2.15). In fact, we have an upper bound:

Lemma 4.8.4. *There are at most* 30 *Eckardt points on a smooth cubic threefold.*

Proof. Each Eckardt point $p \in X$ corresponds to a one-parameter family of lines through p parameterized by an elliptic curve E_p. Let $D_2 \subseteq F$ be the divisor of the lines of the second type, then E_p is an irreducible component of D_2.

Choose a general line L on the cubic threefold transversal to all hyperplanes (any line of the first type will do). Let D_L denote the curve of lines incident to L. Let N be the number of Eckardt points. Then $N \leq D_L \cdot D$. On the other hand, by [19, 10.9, 10.21], D_2 is numerically equivalent to $6D_L$. Since $D_L \cdot D_L = 5$, one has $D_L \cdot D = 30$, so $N \leq 30$. □

The number 30 of Eckardt points is reached by Fermat cubic threefold $x_0^3 + x_1^3 + x_2^3 + x_3^3 + x_4^3 = 0$. On the other hand, [49] showed that the 30 elliptic curves intersect mutually at 135 points and [9] showed that these are all the triple lines on a Fermat cubic threefold. In fact, [9] proved the following.

Lemma 4.8.5. *Triple lines correspond to singularities of the divisor of lines of the second type in F.*

As a consequence, the 30 elliptic curves of the Fano surface of lines of Fermat cubic are exactly irreducible components of D_2.

In this section, we'll further explore the relationship between triple lines and Eckardt points on a smooth cubic threefold. First, the following is clear:

Proposition 4.8.6. *If X is a smooth cubic threefold and $L \subseteq X$ is a line through an Eckardt point p, then L is a triple line if and only if the corresponding point on E is a flex point.*

It follows that for each Eckardt point $p \in X$, there are 9 triple lines associated with p. However, since the divisor of lines of the second type may contain other components other than elliptic curves corresponding to Eckardt points, potentially, other types of singularities may occur. One can ask

Question 4.8.7. *Does every triple line come from an Eckardt point?*

We are going to provide a negative answer.

Consider the Clebsch surface defined by the equation

$$G(x_0, \ldots, x_3) = x_2x_0^2 + x_3x_1^2 + x_2^2x_1 + x_3^2x_0.$$

Let X_{Cle} be the cubic threefold arising as a cyclic triple cover of $\mathbb{P}^3$ branched along $G = 0$, i.e., X_{Cle} is defined by

$$F(x_0, \ldots, x_4) = x_2x_0^2 + x_3x_1^2 + x_2^2x_1 + x_3^2x_0 + x_4^3 = 0.$$

Then it contains a line $L = \{x_2 = x_3 = x_4 = 0\}$ of second type. There is a unique plane $P = \{x_2 = x_3 = 0\}$ tangent to X_{Cle} along L. Moreover $P \cap X_{Cle}$ is defined by $x_4^3 = 0$, so L a triple line. However, we're going to show that it does not come from an Eckardt point.

Claim 4.8.8. *L does not pass through any Eckardt point on X_{Cle}.*

Proof. First, any hyperplane section $\{ax_2+bx_3=0\}\cap X_{Cle}$ is a cubic surface which is a cyclic triple cover of plane cubic curve C defined by $ax_2+bx_3=0$ and $x_2x_0^2+x_3x_1^2+x_2^2x_1+x_3^2x_0=0$. So the hyperplane section is an Eckardt cubic surface if and only if the cubic curve is a cone, i.e., three lines intersecting at a single point. We'll show it cannot happen.

For the hyperplane defined by $x_2=0$, the corresponding cubic curve C has equation $x_3x_1^2+x_3^2x_0=x_3(x_1^2+x_3x_0)$, which is not three lines.

For the hyperplane defined by $x_3=\lambda x_2$, the corresponding cubic curve C has equation

$$x_2x_0^2+\lambda x_2x_1^2+x_2^2x_1+\lambda^2x_0x_2^2=x_2(x_0^2+\lambda x_1^2+x_2x_1+\lambda^2x_0x_2).$$

The conic $x_0^2+\lambda x_1^2+x_2x_1+\lambda^2x_0x_2$ has rank 3 when $\lambda\neq 0$ and has rank 2 when $\lambda=0$. C is not three lines in either of the cases. □

In the end, we ask the following questions.

Question 4.8.9. *How to characterize triple lines that are not coming from an Eckardt point?*

Parallel to Lemma 4.8.4, we ask the following question.

Question 4.8.10. *What is the maximal number of triple lines on a smooth cubic threefold?*

Chapter 5: Hilbert Scheme of a Pair of Skew Lines

In this chapter, we study the irreducible component $H(X)$ of the Hilbert scheme of a cubic threefold X that contains a pair of skew lines on X. Our main theorem is to show $H(X)$ is smooth and isomorphic to the blowup of symmetric product $\mathrm{Sym}^2 F$ of Fano surface of lines on X along the diagonal (Theorem 5.2.2). Also, by relating to the singularities of hyperplane sections of X, we characterized the subscheme of $H(X)$ supported on a hyperplane section (Theorem 5.3.10).

In Section 5.1, we review the irreducible component of the Hilbert scheme determined by a pair of skew lines on projective space, which is characterized by Chen, Coskun, and Nollet [16]. In Section 5.2, we prove the Theorem 5.2.2. In Section 5.3, we prove the Theorem 5.3.10. In Section, we provide some relationships of our study to Bridgeland moduli space.

5.1 Hilbert Scheme of a Pair of Skew Lines on Projective Spaces

5.1.1 Hilbert Scheme of a Pair of Skew Lines

Consider a pair (L_1, L_2) of skew lines on $\mathbb{P}^3$. As a closed subvariety, $Z = L_1 \cup L_2$ has Hilbert polynomial $2n+2$. It determines an irreducible component of the Hilbert scheme $\mathrm{Hilb}^{2n+2}(\mathbb{P}^3)$. According to [16], the Hilbert scheme $\mathrm{Hilb}^{2n+2}(\mathbb{P}^3)$ has two irreducible components H_3 and H_3'. A general point in H_3 parameterizes a pair of skew lines. A general point in H_3' parameterizes a smooth conic union an isolated point.

Theorem 5.1.1. *([16], Theorem 1.1) Both H_3 and H_3' are smooth and intersect transversely along the union of the locus of type (III) and type (IV) schemes. Moreover, H_3 is isomorphic to $\mathrm{Bl}_\triangle \mathrm{Sym}^2 Gr(2,4)$, the blowup of the symmetric product of $Gr(2,4)$ along the diagonal. Moreover, the component H_3 parameterizes four types of schemes:*

Type (I): A pair of skew lines;

Type (II): A purely double structure supported on a line;

Type (III): A pair of incident lines with an embedded point determined by the square of the ideal of the intersection point;

Type (IV): A double structure contained in a plane and supported on a line, together with an embedded point determined by the square of the ideal of a point on the line.

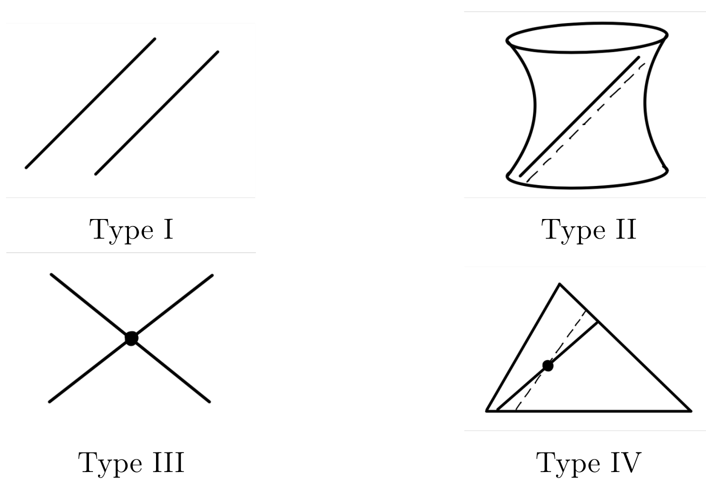


Figure 5.1: Schemes of the Four Types

One can write down the ideals of the schemes of the four types in the projective coordinate x_0, x_1, x_2, x_3. The ideal of a type (I) scheme can be expressed as $(x_0, x_1) \cap (x_2, x_3) = (x_0x_2, x_0x_3, x_1x_2, x_1x_3)$, which is the ideal of two disjoint lines $x_0 = x_1 = 0$ and $x_2 = x_3 = 0$; The type (III) ideal can be written as $(x_0^2, x_0x_1, x_0x_2, x_1x_2) = (x_0, x_1x_2) \cap (x_0, x_1, x_2)^2$, which determines a pair of incident lines $x_0 = x_1 = 0$ and $x_0 = x_2 = 0$ with a spatially embedded point at the intersection point. The type (II) and (IV) ideals are all supported on a single line $x_0 = x_1 = 0$. The corresponding ideal for type (II) scheme is $(x_0^2, x_0x_1, x_1^2, x_0x_2 + x_1x_3)$, while the ideal of the type (IV) scheme is $(x_0^2, x_0x_1, x_1^2, x_0x_2) = (x_0, x_1^2) \cap (x_0, x_1, x_2)^2$. Equivalently, a type (II) scheme corresponds to the first-order neighborhood of a line in a smooth quadric surface, while a type (IV) scheme corresponds to the first-order neighborhood of a line in $\mathbb{P}^2$ union a spatially embedded point on the line.

The flat degenerations of the four types of schemes can be described geometrically as follows.

(I)⇒(II): As two disjoint lines come to coincide linearly;

(I)⇒(III): As two disjoint lines come to intersect at one point, an embedded point occurs at the intersection;

(III)⇒(IV): As two incident lines come to coincide;

(II)⇒(IV): As the smooth quadric surface degenerates to two planes intersecting transversely along a line M. The support line L is contained in one of the planes and normal to the other. The embedded point occurs at the intersection $L \cap M$.

One can interpret Theorem 5.1.1 via the Hilbert-Chow morphism [39, Theorem 6.3]

$$\rho_3 : H_3 \to \mathrm{Sym}^2 Gr(2, 4),$$

which sends type (I) and (III) schemes to their support, and sends type (II) and (IV) schemes to their support with multiplicity two. Denote D the subvariety of $\mathrm{Sym}^2 Gr(2, 4)$ parameterizing pairs of incident lines on $\mathbb{P}^3$. We have the stratification $\Delta \subseteq D \subseteq \mathrm{Sym}^2 Gr(2, 4)$. Then Theorem 5.1.1 states that ρ_3 is the blow-up along Δ. Over Δ is a $\mathbb{P}^3$ bundle consisting of type (II) and type (IV) schemes (with type (IV) scheme forming a smooth quadric surface). The set of type (I) and type (III) scheme over $D \setminus \Delta$ are type (III) schemes and over $\mathrm{Sym}^2 Gr(2, 4) \setminus D$ are type (I) schemes.

There is a similar result in higher dimensional projective spaces [16, Corollary 2.8]. For $m \geq 4$, $\mathrm{Hilb}^{2n+2}(P^m) = H_m \cup H'_m$ consists of two irreducible components. Similar to the $m = 3$ case, a general point of component H_m parameterizes a pair of skew lines and a general point of the component H'_m parameterizes a conic union an isolated point.

H_m still parameterizes the schemes of the four types defined in Theorem 5.1.1. Since every scheme of the four types determines a unique linear subspace $\mathbb{P}^3$ of $\mathbb{P}^m$ containing the scheme (also see [43, Lemma 3.5.3]), there is a morphism

$$\pi : H_m \to Gr(4, m+1)$$

the fiber is isomorphic to H_3.

Lemma 5.1.2. *([16, Corollary 2.8]) H_m is smooth, and is a H_3 bundle via π.*

Therefore, there are two natural morphisms from H_m.

$$\begin{array}{ccc} H_m & \xrightarrow{\quad\pi\quad} & Gr(4, m+1) \\ \downarrow{\scriptstyle \rho_m} & & \\ \mathrm{Sym}^2 Gr(2, m+1) & & \end{array} \tag{5.1}$$

ρ_m is again the Hilbert-Chow morphism [39, Theorem 6.3].

We characterize the Hilbert-Chow morphism as successive blow-ups in the following proposition, which will help prove the main theorem.

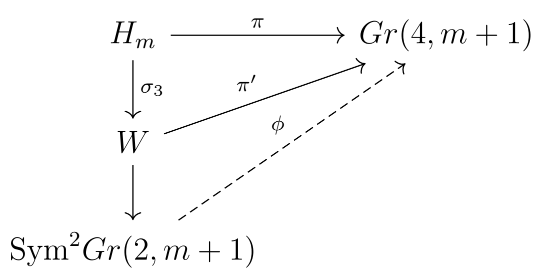
Proposition 5.1.3. *The Hilbert-Chow morphism $H_m \to \mathrm{Sym}^2 Gr(2, m+1)$ factors through*

$$H_m \xrightarrow{\sigma_2} \mathrm{Bl}_\Delta \mathrm{Sym}^2 Gr(2, m+1) \xrightarrow{\sigma_1} \mathrm{Sym}^2 Gr(2, m+1) \tag{5.2}$$

where σ_1 blows up the diagonal, and σ_2 blows up the strict transform of the locus $D \subseteq \mathrm{Sym}^2 Gr(2, m+1)$ parameterizing incident pairs of lines.

Proof. Let I_Δ be the ideal sheaf of the diagonal $\Delta \subseteq \mathrm{Sym}^2 Gr(2, m+1)$, then the pullback $p^* I_\Delta$ is an ideal sheaf of a divisor, which is invertible since H_m is smooth ([16], Corollary 2.8). So by the universal property of the blowup ([33], Proposition II.7.14), the Hilbert-Chow morphism $p : H_m \to \mathrm{Sym}^2 Gr(2, m+1)$ factors through $\mathrm{Bl}_\Delta \mathrm{Sym}^2 Gr(2, m+1)$ as in (5.2), where σ_1 blows up the diagonal, and σ_2 is birational.

Let $D \subseteq \mathrm{Sym}^2 Gr(2, m+1)$ denote the locus of pair of incident lines and $\tilde{D}$ the strict transform, which has codimension $m-2$. For a type (III) supported on a pair of incident lines $L_1 \cup L_2$, the embedded point determines and is uniquely determined by a $\mathbb{P}^3$ containing $L_1 \cup L_2$, and there is a $\mathbb{P}^{m-3}$-family of such hyperplanes. Therefore the general fiber over D (and therefore over $\tilde{D}$) is isomorphic to $\mathbb{P}^{m-3}$. Now $\sigma_2^{-1}(\tilde{D})$ is a divisor. By the same argument, σ_2 factors through $W = \mathrm{Bl}_{\tilde{D}} \mathrm{Bl}_\Delta \mathrm{Sym}^2 Gr(2, m+1)$. We have a commutative diagram.



π' is the morphism induced by the rational map $\phi : (L_1, L_2) \mapsto \mathrm{span}(L_1, L_2)$ and $\pi = \pi' \circ \sigma_3$. The fiber of ϕ is a dense subset of $\mathrm{Sym}^2 Gr(2, 4)$, whose closure in W is isomorphic to $\mathrm{Bl}_\Delta \mathrm{Sym}^2 Gr(2, 4)$, so it is the fiber of π'. On the other hand, we know that the fiber of π is also $H_3 \cong \mathrm{Bl}_\Delta \mathrm{Sym}^2 Gr(2, 4)$, so σ_3 is a bijective birational map. Therefore by Zariski's main theorem, it is an isomorphism. $\square$

Remark 5.1.4. *In fact, according to [48, Theorem A],* $\mathrm{Bl}_{\Delta}\mathrm{Sym}^2 Gr(2, m+1)$ *is isomorphic to the Hilbert scheme of a pair of linear subspace of dimension* $(m-2)$ *in* $\mathbb{P}^m$*. The blowup map* σ_2 *is the morphism considered in [48, Proposition 6.11].*

5.2 Hilbert Scheme of a Pair of Skew Lines On Cubic Threefolds

5.2.1 Main Theorem

In this section, X is a smooth cubic threefold. Recall that we defined (4.7) a space $\mathcal{M}$ consisting of set of triples (L_1, L_2, t) such that X_t is a smooth hyperplane section of X and L_1, L_2 are disjoint lines on X_t. Moreover, there is a natural inclusion $\mathcal{M} \hookrightarrow F \times F$ since a pair of skew lines uniquely determines a hyperplane. There is a 2-to-1 map

$$\mathcal{M} \to \mathrm{Hilb}^{2n+2}(X),\ (L_1, L_2, t) \mapsto \mathcal{O}_{L_1 \cup L_2}. \tag{5.3}$$

The image $H(X)^{\circ}$ is an open dense subspace of an irreducible component $H(X)$ of the Hilbert scheme $\mathrm{Hilb}^{2n+2}(X)$.

Definition 5.2.1. *Call* $H(X)$ *the Hilbert scheme of a pair of skew lines on* X*.*

In this section, we will characterize the space $H(X)$ and study the completion of the 2-to-1 map (5.3).

The vertical maps are Hilbert-Chow morphisms. Since the cubic threefold X is a closed subvariety of $\mathbb{P}^4$, $H(X)$ is naturally a closed scheme of H_4. Recall that F is the Fano surface of lines on cubic threefold, then via the natural inclusion $F \hookrightarrow Gr(2,5)$, there is a following commutative diagram.

Since a general element in $\mathrm{Sym}^2(F)$ and $\mathrm{Sym}^2(Gr(2,5))$ is a pair of skew lines, and only the reduced scheme structure has Hilbert polynomial $2n+2$, therefore both ρ and ρ_4 are birational morphisms. Our goal is to show ρ is identified with the blowup along the diagonal.

Our main theorem in this section is

$$\begin{array}{ccc} H(X) & \hookrightarrow & H_4 \\ \downarrow \rho & & \downarrow \rho_4 \\ \mathrm{Sym}^2(F) & \hookrightarrow & \mathrm{Sym}^2(Gr(2,5)) \end{array} \tag{5.4}$$

Theorem 5.2.2. *$H(X)$ is smooth and isomorphic to $\mathrm{Bl}_{\Delta_F}\mathrm{Sym}^2 F$, the blowup of $\mathrm{Sym}^2 F$ on the diagonal.*

We denote $\mathcal{E}$ the exceptional divisor of $\mathrm{Bl}_{\Delta_F}\mathrm{Sym}^2 F$. Then $\mathcal{E}$ parameterizes type (II) and type (IV) on X.

By lifting to the double cover $F \times F \to \mathrm{Sym}^2 F$, we have a commutative diagram

$$\begin{array}{ccc} \widetilde{H(X)} = \mathrm{Bl}_{\Delta_F}(F \times F) & \xrightarrow{\tilde{g}} & \mathrm{Bl}_{\Delta_F}\mathrm{Sym}^2 F = H(X) \\ \downarrow & & \downarrow \\ F \times F & \xrightarrow{g} & \mathrm{Sym}^2 F \end{array}$$

The morphism on the first row is the double branched along $\mathcal{E}$.

Since the general point of the double cover $\widetilde{H(X)} := \mathrm{Bl}_{\Delta_F}(F \times F)$ parameterizes a pair of ordered lines, it can be regarded as "Hilbert scheme" of a pair of ordered skew lines. In fact, it has a modular interpretation as an irreducible component of the nested Hilbert scheme:

There is a closed subscheme $\mathrm{Hilb}^{2n+2,n+1}(X)$ of $\mathrm{Hilb}^{2n+2}(X) \times \mathrm{Hilb}^{n+1}(X)$ that parameterizes a pair of schemes (Z_1, Z_2) on X with $Z_2 \subseteq Z_1$ and $p_n(Z_1) = 2n+2$ and $p_n(Z_2) = n+1$. Since any scheme with Hilbert polynomial $n+1$ must be a line, an irreducible component $\widetilde{H(X)}$ of $\mathrm{Hilb}^{2n+2,n+1}(X)$ parameterizes a pair (Z, L) with $Z \in H(X)$ and $L \subseteq Z$. By projecting to the first coordinate, $\widetilde{H(X)}$ is a natural double cover of $H(X)$ branched along the locus of type (II) and type (IV) schemes, therefore $\widetilde{H(X)}$ is identified with $\mathrm{Bl}_{\Delta_F}(F \times F)$.

Definition 5.2.3. *Call $\widetilde{H(X)}$ the nested Hilbert scheme of a pair of skew lines on X.*

A direct application of Theorem 3.3.1 leads to

Corollary 5.2.4. *$\widetilde{H(X)}$ is a canonical compactification of $\mathcal{M}$ that extends the 2-to-1 cover $\mathcal{M} \to H(X)^\circ$ considered in (5.3) to the branched double cover $\widetilde{H(X)} \to H(X)$.*

Remark 5.2.5. *It is also a consequence of Theorem 5.2.2 that the Hilbert scheme of a pair of skew lines $H(X)$ is isomorphic to the Hilbert scheme of two points $F^{[2]}$ on the Fano surface of lines F (and $\widetilde{H(X)}$ is isomorphic to the nested Hibert scheme of points $F^{[2,1]}$). It is natural to compare the two family via correspondence $I = \{(t,x) \in F \times X | x \in L_t\}$.*

However, the family parameterized by $F^{[2]}$ only capture the multiplicity on generic point, versus the exceptional locus of $H(X)$ also parameterizes schemes with embedded points and different double structure. Putting differently, via pullback and pushforward of universal family parameterized by $F^{[2]}$ with respect to incidence correspondence on $F^{[2]} \times F \times X$, one gets a family $\mathcal{C}$ of 1-dimensional schemes on $F^{[2]} \cong H(X)$, but the family is not flat. In fact, the family $\mathcal{C}$ forgets the embedded points over the locus of type (III) and type (IV) schemes.

5.2.2 Some Geometric Preparations

Let X be a smooth cubic threefold. We will prove the main theorem (Theorem 5.2.2) in this section. Our strategy is the following. We will show that (i) each pair of incident lines on X supports a unique type (III) scheme. (ii) Given a double structure supported on a single line, there is a $\mathbb{P}^1$-family of double structures of the same type supported on that line. (iii) The $\mathbb{P}^1$-bundle over the Fano surface will match with the exceptional divisor of the blowup $\mathrm{Bl}_{\Delta_F}\mathrm{Sym}^2 F$. This can be globalized and leads to a bijective morphism $\mathrm{Bl}_{\Delta_F}\mathrm{Sym}^2 F \to H(X)$. (iv) Finally, we show it is an isomorphism using smoothness of H_4.

One should note that a line L on the cubic threefold X supports a type (II) scheme if and only if it is a line of the first type, and it supports a type (IV) scheme if and only if it is a line of the second type. This is not the case in the projective space, where every line supports both type (II) and type (IV) schemes.

Definition 5.2.6. *Let $W \hookrightarrow X$ be a closed immersion of schemes. Let $I \subseteq \mathcal{O}_X$ be the ideal sheaf defining W. The* first-order infinitesimal neighborhood *of W in X is defined to be the closed subscheme of X defined by the ideal I^2. Denote such scheme as $Z_{W,X}$.*

When both X and W are smooth, the scheme W' keeps track of the information normal bundle $N_{W|X}$ of W in X. As an example, the first-order infinitesimal neighborhood of a point p in a smooth variety X is the vector space $T_pX \oplus \mathbb{C}$. The notion will be helpful to characterize the schemes of the four types defined in Theorem 5.1.1 in a scheme theoretical way.

Proposition 5.2.7. *Let $m \geq 3$, and a scheme $Z \in H_m$ can be expressed in the following way depending on its type.*

Type (I): $Z = Z_{\text{red}}$, a pair of skew lines;

Type (II): $Z = Z_{L,Q}$, the first-order infinitesimal neighborhood of the line L in a smooth quadric surface Q, where L is the support of Z;

Type (III): $Z = Z_{\text{red}} \cup Z_{p,H}$, where Z_{red} is the union of a pair of lines incident at p, and $Z_{p,H}$ is the first-order infinitesimal neighborhood of p in linear subspace $H \cong \mathbb{P}^3$ of $\mathbb{P}^m$;

Type (IV): $Z = Z_{L,P} \cup Z_{p,H}$, where $Z_{L,P}$ is the first-order infinitesimal neighborhood of the line L in a plane P. L is the support of Z and $p \in L$.

Note that $Z_{L,P}$ has Hilbert polynomial $2n + 1$ and is called a non-reduced conic (see [41] Lemma 2.1.1).

Proposition 5.2.8. *Let X and Y be smooth subvarieties of $\mathbb{P}^m$. Suppose $X \cap Y$ contains a smooth variety W. Denote $Z := Z_{W,X}$ the first-order infinitesimal neighborhood of W in*

X, then Z is contained in Y if and only if X is tangent to Y along W, namely, there is an inclusion $T_X|_W \hookrightarrow T_Y|_W$ of tangent bundles restricted to W.

Proof. The condition $Z \subseteq Y$ is equivalent to surjectivity on the sheaves $\mathcal{O}_Y \twoheadrightarrow \mathcal{O}_X/I^2_{W|X}$, but this map factors through

$$\mathcal{O}_Y \to \mathcal{O}_Y/I^2_{W|Y} \twoheadrightarrow \mathcal{O}_X/I^2_{W|X}.$$

By restricting to an affine chart, we have decomposition $\mathcal{O}_X/I^2_{W|X} \cong \mathcal{O}_W \oplus I_{W|X}/I^2_{W|X}$. The surjectivity on $\mathcal{O}_W$ follows from the assumption $W \subseteq Y$, and the surjectivity

$$I_{W|Y}/I^2_{W|Y} \twoheadrightarrow I_{W|X}/I^2_{W|X}$$

dualizes to the statement that there is an inclusion $N_{W|X} \hookrightarrow N_{W|Y}$, which is equivalent to $T_X|_W \hookrightarrow T_Y|_W$. □

Now we study the elements in $H(X)$. First of all, for type (I) schemes, they have the form $L_1 \cup L_2$ when L_1 and L_2 a pair of disjoint lines on X. For the type (III) scheme, we have the following result.

Lemma 5.2.9. *When L_1 and L_2 intersect at one point p, there is a unique type (III) scheme $Z \in H(X)$ supported on $L_1 \cup L_2$, where the embedded point supported on $p = L_1 \cap L_2$ is the square of the maximal ideal in the tangent hyperplane T_pX.*

Proof. A type (III) subscheme Z of $\mathbb{P}^4$ is a union

$$Z = Z_{\mathrm{red}} \cup Z_{p,H},$$

where $Z_{\mathrm{red}} = L_1 \cup L_2$ is the reduced scheme and $Z_{p,H}$ is the first-order infinitesimal neighborhood of p in a hyperplane $H \in (\mathbb{P}^4)^*$. So if $Z_{\mathrm{red}} = L_1 \cup L_2$ being contained in Y is given, then by Proposition 5.2.8, the condition $Z_{p,H}$ being a subscheme of X is equivalent to $H = T_pX$ being the tangent hyperplane at p. □

Let's give an account of this result using ideals and equations.

First, there is a $\mathbb{P}^1$-family of hyperplanes containing $L_1 \cup L_2$, indicating that there is a $\mathbb{P}^1$-family of type (III) scheme in $\mathbb{P}^4$ supported on $L_1 \cup L_2$. We will show such a scheme is contained in X if and only if it is contained in the tangent hyperplane T_pX.

To see this, we give a coordinate $x_0, ..., x_3$ to the hyperplane T_pX and assume that $p = [1, 0, 0, 0]$, and L_1, L_2 are lines spanned by p and $[0, 1, 0, 0]$, $[0, 0, 1, 0]$ respectively. We need to show the cubic surface $S = X \cap T_pX$ contains the type (III) scheme corresponding to the ideal $(x_3^2, x_3x_1, x_3x_2, x_1x_2)$.

The cubic surface $S = X \cap T_pX$ has equation

$$F_{T_pX} = x_0Q(x_1, x_2, x_3) + C(x_1, x_2, x_3),$$

where Q is a quadric equation and C is a cubic equation. Since F_{T_pX} has to vanishing along L_1 and L_2, Q has no x_1^2, x_2^2 terms and C havs no x_1^3, x_2^3 terms. This implies that $F_{T_pX} \in (x_3^2, x_3x_1, x_3x_2, x_1x_2)$. So the corresponding scheme is contained in S.

Conversely, assume there is a type (III) scheme Z contained in X and is supported on $L_1 \cup L_2$ but is not contained in the tangent hyperplane T_pX. We denote by H the unique hyperplane that contains Z and give the coordinate $x_0, ..., x_3$ to H, so the cubic surface $X \cap H$ is not singular at p and it has equation

$$F_H = x_0^2x_3 + x_0Q_H(x_1, x_2, x_3) + C_H(x_1, x_2, x_3),$$

with again Q_H having no x_1^2, x_2^2 terms and C_H having no x_1^3, x_2^3 terms. Then $F_H \notin (x_3^2, x_3x_1, x_3x_2, x_1x_2)$ since the leading term $x_0^2x_3$ is not in the ideal but $F_H - x_0^2x_3$ is.

Corollary 5.2.10. *The Hilbert-Chow morphism*

$$\rho : H_4(X) \to \mathrm{Sym}^2 F$$

is isomorphic over $\mathrm{Sym}^2 F \setminus \Delta_F$.

Note that a similar result is obtained in [19, Lemma 12.16] using the analytic method.

Proof. ρ sends the set of type (I) and (III) schemes to $\mathrm{Sym}^2 F \setminus \Delta_F$, which is bijective by Lemma 5.2.9, so by Zariski's main theorem, it is an isomorphism. □

Next, we start to discuss the schemes parameterized by $H(X)$ and supported on a single line, that is, the type (II) and type (IV) schemes. It turns out that they correspond to the type of lines on cubic threefold by "dividing by 2". Recall that we have defined in Definition 4.2.13 the lines of the first and second type in X. They are classified by normal bundles.

Definition 5.2.11. A line $L \subseteq X$ is called to be of first type if the normal bundle $N_{L|X} \cong \mathcal{O}_L \oplus \mathcal{O}_L$; L is called to be of second type if the normal bundle $N_{L|X} \cong \mathcal{O}_L(1) \oplus \mathcal{O}_L(-1)$.

All lines in X are in the two classes. The lines of the first type are generic in F, while the set of lines of the second type forms a divisor of F.

Proposition 5.2.12. *Let L be a line on X, then*

(1) L is of the first type if and only if there is a smooth quadric surface in $\mathbb{P}^4$ tangent to X along L. Moreover, all such quadric surfaces are parameterized by L.

(2) L is of the second type if and only if there is a unique plane $P = \mathbb{P}^2$ tangent to X along L.

In fact, part (1) is implicit in [19, Lemma 6.18]. Part (2) comes from [19, 6.6, 6.7]. We will provide a self-contained proof of part (1) here.

Proof. When L is of the first type, $N_{L|X} \cong \mathcal{O}_L \oplus \mathcal{O}_L$ is a trivial rank-two bundle. If we identify the normal bundle $N_{L|\mathbb{P}^4}$ with an Zariski open subspace U of $Gr(2,5)$, then $N_{L|X}$ is identified with a closed subspace of U. Therefore, a nonzero section v is identified with a line L_v disjoint from L and they span a hyperplane H_v. We will show that there is a unique smooth quadric surface Q_v contained in H_v and is tangent to X along L.

Our idea is the following. As the section v is scaled by a factor $\lambda \in \mathbb{C}$, then the set of $L_{\lambda v}$ sweeps out a surface, whose closure should be the quadric surface Q_v that we are looking for. By construction, Q_v is tangent to X along L.

Explicitly, suppose $L = \{x_2 = x_3 = x_4 = 0\}$ and use the local equation of X around L given by

$$x_2 x_0^2 + x_3 x_0 x_1 + x_4 x_1^2 + \text{higher order terms in } x_2, x_3, x_4$$

as in [19, (6.9)], one can determine the local equation of the Fano surface F at L in the Grassmannian $Gr(2,5)$, as in [19, (6.14)]. By linearizing the equation in [19, (6.14)], we find that L_v is the line

$$\lambda[1,0,0,a,-b] + \mu[0,1,-a,b,0], \ [\lambda,\mu] \in \mathbb{P}^1, \tag{5.5}$$

for some $a, b \in \mathbb{C}$ not both zero.

This allows us to determine the equation of the hyperplane H_v:

$$a^2 x_4 + b^2 x_2 + ab x_3 = 0. \tag{5.6}$$

Note that H_v is determined by v up to scaling. In other words, H_v is determined by $[v] \in \mathbb{P}H^0(L, N_{L|X})$.

Note that by (5.5), the line L_{sv} satisfies the extra two equations

$$\begin{cases} sax_1 + x_2 = 0; \\ sbx_0 + x_4 = 0, \end{cases}$$

in addition to (5.6), where $s \in \mathbb{C}^*$ is a scalar. By canceling out the factor s and using (5.6), we find that the quadric equation is

$$\begin{cases} bx_1x_2 + ax_1x_3 + ax_0x_2, & \text{if } b \neq 0; \\ ax_0x_4 + bx_0x_3 + bx_1x_4 = 0, & \text{if } a \neq 0, \end{cases} \tag{5.7}$$

which uniquely determines a smooth quadric surface Q_v tangent to X along L and contained in H_v.

Conversely, if Q is a smooth quadric surface tangent to X along L, then the ruling of Q containing L is a line in $Gr(2,5)$, tangent to F at L, and thus it corresponds to an element in $\mathbb{P}H^0(L, N_{L|X})$.

Finally, note that by the equation (5.6), the hyperplane H_v is the tangent hyperplane $T_{[b,a]}X$ of X at point $[b,a] \in L$. So there is a one-to-one correspondence

$$\mathbb{P}H^0(L, N_{L|X}) \leftrightarrow L \leftrightarrow \text{smooth quadric surfaces tangent to } X \text{ along } L.$$

For part (2), when L of the second type, the image of the dual map $L \to (\mathbb{P}^4)^*,\ x \mapsto T_xX$ along L is a line. So the $\mathbb{P}^2 = \cap_{x\in L} T_xX$ is a plane tangent to X along L. This uniquely characterizes lines of the second type [19, 6.6, 6.7]. □

In [41, Remark 2.1.2 and 2.1.7], the authors showed that only lines of the second type can support the non-reduced conic structures ($Z_{L,P}$ in our notation). We have a similar argument here.

Lemma 5.2.13. *Let $Z \in H(X)$ be a scheme supported on a line L. Then L is of the first type if and only if Z is of type (II); L is of the second type if and only if Z is of type (IV).*

Proof. Let Z be a type (II) subscheme of X supported on a line L. We will show that L cannot be of the second type. Otherwise, let Q be the corresponding smooth quadric surface associated with Z, by Proposition 5.2.8, the normal bundle $N_{L|Q} \cong \mathcal{O}_L$ admits an inclusion to $N_{L|X} \cong \mathcal{O}_L(1) \oplus \mathcal{O}_L(-1)$ as bundles. However, the sheaf map $\mathcal{O}_L \to \mathcal{O}_L(-1)$ is always trivial. Any inclusion of sheaves $\mathcal{O}_L \to \mathcal{O}_L(1)$ has torsion cokernel, so there is no inclusion $N_{L|Q} \hookrightarrow N_{L|X}$ of normal bundles, which is a contradiction.

Let Z be a type (IV) subscheme of X supported on a line L. We will show that L cannot be of the first type. Otherwise, as Z has a closed subscheme $Z_{L,P}$, which is a first-order neighborhood of L in a plane P, X contains $Z_{L,P}$ as a closed subscheme, then by

Proposition 5.2.8, there is a bundle inclusion

$$\mathcal{O}(1) \cong N_{L|P} \hookrightarrow N_{L|X} \cong \mathcal{O}_L \oplus \mathcal{O}_L,$$

which is a contradiction. □

Recall that the dual map of X along L is a map

$$\mathcal{D} : L \to \mathbb{O},\ p \mapsto T_pX. \tag{5.8}$$

When L is of the first type, $\mathcal{D}$ is one-to-one onto a conic. When L is of the second type, $\mathcal{D}$ is two-to-one onto a line. (c.f. [19] Def. 6.6)

Definition 5.2.14. *We call $q \in L$ a conjugate point of $p \in L$ if they have the same image under $\mathcal{D}$, and we denote $q = \bar{p}$.*

Proposition 5.2.15. *The map $\alpha : \mathbb{P}T_F \to \Gamma$ defined in* (4.13) *is described by the following:*

$$\alpha : (t, v) \mapsto \begin{cases} \text{the unique tangent point of } H_v \text{ with } X \text{ on } L, \text{ when } L \text{ is of the first type;} \\ \overline{\text{zero}(v)}, \text{ when } L \text{ is of the second type.} \end{cases}$$

Proof. When L is of the first type, this is implicit on page 342-345 in [19]. Let's make it explicit. By the construction of α for $(s, v) \in \mathbb{P}T_F$, with $L = L_s$ and $v \in H^0(L, N_{L|X})$, we can choose another line $L_{s'}$ which intersects L_s such that the curve $D_{s'}$ of lines incident to $L_{s'}$ coincides with v in $H^0(L, N_{L|X})$ (note s is a smooth point on $D_{s'}$). Use the same notation in [19], $L_{\gamma_{s'}(s)}$ is the third line in the triangle on X determined by L and L_s, then by (12.21), α sends (s, v) to the point $L_s \cap L_{\gamma_{s'}(s)}$. This point, according to Lemma 12.20, is the point whose tangent hyperplane coincides with the limiting hyperplane $\lim_{t \in D_{s'}, t \to 0} \mathrm{Span}(L, L_t)$, which coincides with the hyperplane H_v constructed in (5.6).

However, α is not explicit when L_t is of the second type in [19]. This is due to the dual map along a line of the second type is 2-to-1, i.e., for a general $p \in L_t$, the tangent hyperplane T_pX is also tangent to X at a different point $\bar{p} \in L_t$, so when we take the limit of a tangent point

As the locus of lines of the first type is dense in F, the description on α on the fibers of lines of the first type as above uniquely determines how α acts on the pair (t, v), where L_t is a line of the second type. We can choose a one-parameter family of lines $\{L_t\}_{t\in\Delta}$ with Δ a holomorphic disk and $t \neq 0$ for lines of the first type and $t = 0$ a line of the second type.

There is a rank two vector bundle V on $I_\Delta \cong L \times \Delta$, whose stalk over each $t \in \Delta$ is $\mathcal{O}_L \oplus \mathcal{O}_L$ when $t \neq 0$ and $\mathcal{O}_L(1) \oplus \mathcal{O}_L(-1)$ when $t = 0$. Since the second projection $(\pi_2)_*V$ to Δ is a trivial bundle of rank two, we can choose two linearly sections v_1 and v_2 over Δ. When $t \neq 0$, v_1 and v_2 correspond to two lines $L_{1,t}, L_{2,t}$ in a Zariski open subset of $Gr(2, 5)$ that are disjoint from L and are infinitesimally close to X when scaled by a small number (see proof of Proposition 5.2.12). When $t = 0$, the two lines $L_{1,0}$ and $L_{2,0}$ intersect L at two distinct points, corresponding to linearly independent sections on $\mathcal{O}_L(1)$.

We want to keep track of the map α when $t \to 0$. Similar to (6.9) and (6.10), we can choose a one-parameter family of automorphisms σ_t on $\mathbb{P}^4$, such that $\sigma L_t = L$ and $\sigma_t(Y)$ has local equations

$$x_2x_0^2 + tx_3x_0x_1 + x_4x_1^2 + \text{higher order terms in } x_2, x_3, x_4 \tag{5.9}$$

around L. Now, given a section $v \in (\pi_2)_*V$, similar to the proof of Proposition 5.2.12, we find that the line $L_{v,t}$ is given by

$$\lambda[1, 0, 0, a, -tb] + \mu[0, 1, -ta, b, 0],\ [\lambda, \mu] \in \mathbb{P}^1.$$

Therefore, the hyperplane spanned by L and $L_{v,t}$ is given by $a^2x_4+b^2x_2+tabx_3 = 0$, which has the unique tangent point $[b, a]$ when $t \neq 0$. So $\alpha(0, v)$ should be $[b, a]$ as well by continuity.

As $t \to 0$, the limit hyperplane is $a^2x_4+b^2x_2 = 0$, which is tangent to X at two points $[b, a]$ and $[b, -a]$. On the other hand, the limit line L_0 is given by $x_2 = x_4 = x_3 - ax_0 - bx_1 = 0$, which intersects L at point $[b, -a]$, which is the conjugate point of $\alpha(0, v) = [b, a]$. This finishes the proof. □

5.2.3 Proof of the Main Theorem

Our key proposition is the following.

Proposition 5.2.16. *Let L be a line on the cubic threefold X. Using the notation in* (5.4), *there is an isomorphism*

$$\beta : \rho^{-1}(L, L) \cong L$$

which can be described in the following way. Let $Z \in \rho^{-1}(L, L)$ be a scheme.

(1) When L is of the first type, the scheme Z supported on L has type (II) and is contained in a unique hyperplane H_Z, which is tangent to X at a unique point p_Z along L, we have $\beta(Z) = p_Z$;

(2) When L is of the second type, the scheme Z supported on L has type (IV) and has an embedded point supported at p, we have $\beta(Z) = \bar{p}$.

Moreover, the map is continuous with respect to $L \in F$. In other words, there is an identification of $\mathbb{P}^1$-bundles

$$\beta : \rho^{-1}(\Delta_F) \cong \Gamma \cong \mathbb{P}T_F.$$

Proof. If L is of the first type, then by Lemma 5.2.13, it can only support type (II) schemes. Recall that a type (II) subscheme $Z_{L,Q}$ of $\mathbb{P}^4$ supported on L is a first-order infinitesimal neighborhood of the line L in a smooth quadric surface $Q \subseteq \mathbb{P}^4$. Therefore, if Q is tangent to the cubic threefold X along X, the first-order infinitesimal neighborhood of L in Q is contained in the first-order infinitesimal neighborhood of L in X, so the corresponding type (II) scheme $Z_{L,Q}$ is contained in X. Conversely, every type (II) subscheme of X arises in this way. According to Proposition 5.2.12 (1), there is a $\mathbb{P}^1$-family of such smooth quadric surfaces, so there is a $\mathbb{P}^1$-family of type (II) subschemes of X supported on L, parameterized by

$$L \leftarrow \mathbb{P}H^0(L, N_{L|X}) \rightarrow \rho^{-1}(L),$$

$$\{p \in L | H_s = T_p X\} \leftarrow\!\shortmid s \mapsto Z_{L,Q_s}.$$

Since the dual map along L is one-to-one, each type (II) scheme over L is contained in a unique hyperplane.

If L is of the second type, then by Lemma 5.2.13, it can only support type (IV) schemes. Recall that a type (IV) subscheme Z of $\mathbb{P}^4$ supported on L can be written as

$$Z = Z_{L,P} \cup Z_{x,H},$$

where $Z_{L,P}$ is a first-order infinitesimal neighborhood of L in a plane $P \subseteq \mathbb{P}^4$ and $Z_{x,H}$ is the first-order neighborhood of a point $x \in L$ in a hyperplane that contains P. We are looking for the condition such that Z is a subscheme of X. By Proposition 5.2.12 (2), there is a unique plane P tangent to X along L, so $Z_{L,P}$ is a subscheme of X. By Proposition 5.2.8, to require that $Z_{x,H}$ is a subscheme of X is the same as requiring that $H = T_x Y$ is the tangent hyperplane at x. This, in return, uniquely determines the scheme $Z_{x,H}$ and, therefore, the type (IV) scheme. Finally, note that $H^0(N_{L|X}) = H^0(\mathcal{O}_L(1))$, whose global sections are in bijection to their zeros on L and therefore bijective to the conjugate points on the zeros, so there is a correspondence between the set of type (IV) schemes supported on L:

$$\mathbb{P}H^0(L, N_{L|X}) \rightarrow L \rightarrow \rho^{-1}(L),$$

$$s \mapsto \mathrm{Zero}(s) = x \mapsto Z_{L,P} \cup Z_{\bar{x},T_{\bar{x}}X}.$$

Finally, to prove that the identification constructed above is consistent with degenerating a line of the first type to a line of the second type, we need to show that as the type (II) schemes Z_t degenerate to a type (IV) scheme Z_0 on X, the limit of the tangent points p_t is the conjugate point of the support of the embedded point of Z_0.

We use the coordinates (5.9) and continue the computation similarly to the proof of Proposition 5.2.12. We get the equation of quadric surfaces

$$\begin{cases} h_{a,b}(t) = a^2x_4 + b^2x_2 + tabx_3 = 0; \\ q_{a,b}(t) = bx_1x_2 + tax_1x_3 + ax_0x_2 = 0, \end{cases} \tag{5.10}$$

when $b \neq 0$ (and similarly for $a \neq 0$). The flat limit of the corresponding type (II) schemes is determined by the flat family of ideals

$$(h_{a,b}(t), (x_2, x_3, x_4)^2, q_{a,b}(t)) \Rightarrow (a^2x_4 + b^2x_2, (x_2, x_3, x_4)^2, x_2(bx_1 + ax_0)),$$

where the quadric surface $h_{a,b}(0) = q_{a,b}(0)$ is reducible and is the union of the two planes $x_2 = a^2x_4 + b^2x_2 = 0$ and $bx_1 + ax_0 = a^2x_4 + b^2x_2 = 0$. Therefore the embedded point is supported at the intersection of L with $x_2 = bx_1 + ax_0 = a^2x_4 + b^2x_2 = 0$, which is exactly $[b, -a] \in L$. It is the conjugate point of the limit tangent point. □

Proposition 5.2.16 above tells us that we can globalize the flat family of schemes of the four types over $\mathrm{Bl}_{\Delta_F}\mathrm{Sym}^2F$.

Corollary 5.2.17. *There is a bijective morphism $\delta : \mathrm{Bl}_{\Delta_F}\mathrm{Sym}^2F \to H(X)$.*

Proof. Recall we have shown in Corollary 5.2.10 that the Hilbert-Chow morphism ρ is an isomorphism off the diagonal.

Now we lift $Bl_{\Delta_F}Sym^2F$ to its double cover $Bl_{\Delta_F}(F \times F)$ and the exceptional divisor is identified with

$$E \cong \mathbb{P}N_{\Delta_F}(F \times F) \cong \mathbb{P}T_F.$$

For a one-parameter family of pair of lines $\{L_t\}_{t\in\Delta}$ where Δ is a holomorphic disk and with $L_t \neq L_0$ for $t \neq 0$, the flat limit of $\rho^{-1}(L_0, L_t)$ is a scheme with a double structure supported on L, which corresponds to a section $v \in \mathbb{P}H^0(L, N_{L|X})$. According to the identification $PT_F \cong \Gamma$ as in (4.13) together with Proposition 5.2.16, the correspondence is continuous with respect to moving the one-parameter family and moving the line $L \in F$.

Therefore, $\mathrm{Bl}_{\Delta_F}(F \times F)$ parameterizes flat families of skew lines. By the universal property of Hilbert schemes, there is a morphism $\mathrm{Bl}_{\Delta_F}(F \times F) \to H(X)$ which is two-to-one off the exceptional divisor E, and one-to-one on E, so it descends to 2:1 quotient and induces a bijective morphism $\mathrm{Bl}_{\Delta_F}\mathrm{Sym}^2 F \to H(X)$ as claimed. □

In fact, there is an alternative way to prove the Corollary 5.2.17 without using the geometric characterization of Proposition 5.2.15. It is shorter but needs to invoke the Abel-Jacobi map and Beauville's characterization of the singularity of the theta divisor. We sketch the proof here.

First we have the dominant birational map

$$\delta : \mathrm{Bl}_{\Delta_F}\mathrm{Sym}^2 F \dashrightarrow H(X),$$

by assigning $(L_1, L_2) \mapsto \mathcal{O}_{L_1 \cup L_2}$. We want to show it extends to a morphism.

One considers the rational map

$$\Phi : F \times F \dashrightarrow \mathbb{O},\ (L_1, L_2) \mapsto \mathrm{Span}(L_1, L_2).$$

It factors through the Abel-Jacobi map (4.10)

$$F \times F \to \Theta \dashrightarrow \mathbb{O},$$

where the second map is the Gauss map, which associates each smooth point of the theta divisor Θ to the projective tangent hyperplane at that point.

According to the diagram (4.12), Φ extends to a morphism on $\mathrm{Bl}_{\Delta_F}(F \times F)$ and factors through

$$\tilde{\Phi} : \mathrm{Bl}_{\Delta_F}(F \times F) \to \mathrm{Bl}_0(\Theta) \to \mathbb{O}.$$

$\tilde{\Phi}$ descends to $\mathbb{Z}_2$-quotient (just as Φ does), so it defines a morphism $\tilde{\Phi}' : \mathrm{Bl}_{\Delta_F}\mathrm{Sym}^2 F \to \mathbb{O}$ and provides a continuous way of assigning hyperplanes.

Denote $\pi : H(X) \to \mathbb{O}$ the natural projection. Then we have $\pi \circ \delta = \tilde{\Phi}'$. Using the properties of the dual map (5.8), there are at most two schemes of type (II) or (IV) which is supported on a line and is contained a hyperplane, indicating that the morphism $\overline{\Gamma(\delta)} \to \mathrm{Bl}_{\Delta_F}\mathrm{Sym}^2 F$ from the graph closure $\overline{\Gamma(\delta)}$ of δ is finite, and thus is an isomorphism by Zariski's theorem. It follows that δ is a morphism. Finally, injectivity can be checked on each fiber of $E \to \Delta_F$.

Now we are ready to prove our main theorem.

Proof. (Proof of Theorem 5.2.2) It suffices to show $\delta : \mathrm{Bl}_{\Delta_F}\mathrm{Sym}^2 F \to H(X)$ is an isomorphism. It suffices to show that the bijective morphism, composed with inclusion $i : H(X) \hookrightarrow H_4$ is an immersion, namely, of maximal rank at each point.

By expressing $H_4 \to \mathrm{Sym}^2 Gr(2,5)$ as successive blowups as in Proposition 5.1.3, we have a commutative diagram.

$$\begin{array}{ccc} & & H_4 \\ & \overset{i\circ\delta}{\nearrow} & \downarrow \sigma_2 \\ \mathrm{Bl}_{\Delta_F}\mathrm{Sym}^2 F & \xrightarrow{\phi} & \mathrm{Bl}_{\Delta}\mathrm{Sym}^2 Gr(2,5) \\ \downarrow & & \downarrow \sigma_1 \\ \mathrm{Sym}^2 F & \longrightarrow & \mathrm{Sym}^2 Gr(2,5) \end{array} \tag{5.11}$$

ϕ is the unique map which extends the rational map $\mathrm{Sym}^2 F \dashrightarrow \mathrm{Bl}_{\Delta}\mathrm{Sym}^2 Gr(2,5)$ induced by the universal property of the blowup (Corollary II.7.15, [33]). Also note that the two sides of ϕ are the Hilbert schemes of two points $F^{[2]}$ on F, and $Gr(2,5)^{[2]}$ on $Gr(2,5)$, respectively, so ϕ is identified with the inclusion of smooth varieties

$$F^{[2]} \hookrightarrow Gr(2,5)^{[2]}.$$

It follows that $\sigma_2 \circ i \circ \delta = \phi$ is an immersion, so $i \circ \delta$ has to be an immersion. $\square$

5.3 Hilbert Scheme of a Pair of Skew Lines On Cubic Surfaces

In the last section, we showed the Hilbert scheme of skew lines $H(X)$ on a smooth cubic threefold X is smooth and has two natural morphisms

$$\begin{array}{ccc} H(X) & \xrightarrow{\pi} & \mathbb{O} \\ \downarrow{\scriptstyle \rho} & & \\ \mathrm{Sym}^2 F & & \end{array} \tag{5.12}$$

The morphism π is the composite of the horizontal map of (5.1) and inclusion $H(X) \hookrightarrow H_4$.

In this section, we would like to understand the fiber of π. This is the same as understanding the schemes parameterized by H_4 that are contained in hyperplane sections of the cubic threefold Y.

Definition 5.3.1. *Let $S \subseteq \mathbb{P}^3$ be a cubic surface. Define the Hilbert scheme of skew lines on S to be*

$$H(S) := \mathrm{Hilb}^{2n+2}(S) \cap H_3.$$

When S is smooth, $H(S)$ is reduced and consists of 216 pairs of skew lines. However, when S is singular, $H(S)$ is not reduced. We want to study the cardinality of the reduced scheme $H(S)_{\mathrm{red}}$.

5.3.1 Lines on Cubic Surfaces

For smooth cubic surfaces, there are exactly 27 lines. For cubic surfaces with "mild" singularities, the number of lines is less than 27, and such number depends on the type of singularities as well as how lines pass through these singularities. However, if a cubic surface is "too singular", it contains infinitely many lines. The following result will make this more precise.

Lemma 5.3.2. *Let S be a cubic surface obtained from a hyperplane section of a smooth cubic threefold Y, then S is normal and belongs to one of the following two cases:*

(i) S has at worse rational double points (RDPs);

(ii) S has an elliptic singularity.

In case (i), the cubic surface S contains at most 27 lines. In case (ii), S is isomorphic to cone over a smooth plane cubic curve, therefore containing a one-parameter family of lines.

Proof. First we show that S has to be normal. According to Theorem 9.2.1 in [22], a non-normal cubic surface is either cone over singular cubic curve or projective equivalent to $t_0^2t_2 + t_1^2t_3 = 0$, or $t_2t_0t_1 + t_3t_0^2 + t_1^3 = 0$. In either case, S has to be singular along a line L. Now assume S is the hyperplane section $t_4 = 0$, then X has defining equation

$$F_i(t_0, ..., t_3) + t_4Q(t_0, ..., t_4) = 0,$$

with $F_i(t_0, ..., t_3)$ the defining equation of S_i and $Q(t_0, ..., t_4)$ a homogeneous quadric. Then by taking the partial derivatives and restricting to the line, one finds that X is singular at the intersection between the line L and the quadric surface $Q(t_0, ..., t_3, 0) = 0$, which contradicts that X is smooth. Therefore S is normal and has only isolated singularities.

By the classification theorem of cubic surfaces [12], S either has at worst RDPs (at worst $\mathbb{E}_6$ singularity) or is a cone over a smooth plane cubic curve. One refers to [22], section 9.2.2 for the number of lines on all cubic surfaces with at worst RDPs. □

Then by Lemma 5.3.2 and Lemma 4.2.15, we have

Corollary 5.3.3. *Let X be a smooth cubic threefold, and $S = X \cap H$ be a hyperplane section. Then S has only finitely many lines except when H is a tangent hyperplane T_pX of X at an Eckardt point $p \in X$. In particular, when X is general, all hyperplane sections S of X have only finitely many lines.*

5.3.2 Lines of First and Second Type

From now on, we assume S is a normal cubic surface. Note that S can be embedded to a smooth cubic threefold as a hyperplane section. We want to characterize $H(S)_{\mathrm{red}}$. This requires analyzing how the schemes of the four types are supported on the pair of "skew" lines in different configurations in S.

As long as S contains two skew lines L_1, L_2, it defines a type (I) subscheme of S. For type (III) schemes, we have a direct observation as follows.

Proposition 5.3.4. *Let L_1, L_2 be two lines on S that are incident at one point p, then there is a type (III) subscheme of S supported on $L_1 \cup L_2$ if and only if S is singular at p*

Proof. By Lemma 5.2.9, such a scheme is contained in the tangent hyperplane T_pX, therefore, Z lies in the unique hyperplane section $Y \cap T_pX$, which is singular at the incident point. □

For type (II) and type (IV) schemes contained in S, they affect the local geometry of the support line L inside S. We need to introduce the following concepts.

Definition 5.3.5. Let L be a line in the cubic surface S. Call L to be of the *first type* if there is a smooth quadric surface tangent to S along L. Call L to be of the *second type* if there is a plane $\mathbb{P}^2$ tangent to S along L.

The next proposition will explain the reason to introduce our definition.

Proposition 5.3.6. *Let $L \subseteq S$ be a line, then*

(1) L passes through at least one singularity of S if and only if L is of either the first type or the second type.

(2) L is of the first type (resp. second type) if and only if the torsion-free part of the conormal sheaf $N^*_{L|S} = I_L/I_L^2$ is $\mathcal{O}_L$ (resp. $\mathcal{O}_L(-1)$).

(3) L is of the first type (resp. second type) in S if and only if $S = H \cap X$ is a hyperplane section of a smooth cubic threefold X with H tangent to X at some point on L, and L is of the first type (resp. second type) in X.

Proof. Assume the line is defined by $x_2 = x_3 = 0$, then S has equation

$$F = x_2 Q_0(x_0, x_1) + x_3 Q_1(x_0, x_1) + \text{higher order terms in } x_2, x_3,$$

where Q_0 and Q_1 are homogeneous quadrics in x_0, x_1. The dual map along L is

$$\mathcal{D}|_L = [0, 0, Q_0(x_0, x_1), Q_1(x_0, x_1)].$$

Q_0 and Q_1 cannot both be zero because otherwise S will be singular along the line, violating the normality assumption. So the image of the dual map is either a point or isomorphic to $\mathbb{P}^1$ in $(\mathbb{P}^3)^*$. In the first case, the point in dual space corresponds to a hyperplane $\mathbb{P}^2$ which is tangent to S along L, so L is of the second type. Moreover, Q_0 is parallel to Q_1, so they have nonempty common zero loci, where S will be singular. In the second case, if Q_0 and Q_1 have no common zeros, then S will be smooth along L. Otherwise, Q_0 and Q_1 has a common reducible factor, then by canceling the factor and changing the coordinate, the dual map becomes $\mathcal{D}|_L = [0, 0, x_0, x_1]$, and the quadric surface $x_0 x_2 + x_1 x_3 = 0$ is tangent to S along L, which is a line of the first type.

For part (2), note that the conormal sheaf I_L/I_L^2 on L has rank two at the singularities $L \cap S^{\text{sing}}$ and is locally free of rank one over the smooth locus. By definition, if L is of the first type (resp. second type), and let J_L be the ideal sheaf of L in the smooth quadric surface (resp. the $\mathbb{P}^2$), with J_L/J_L^2 the corresponding conormal bundle which is isomorphic to $\mathcal{O}_L$ (resp. $\mathcal{O}_L(-1)$), there is a short exact sequence

$$0 \to T \to N^*_{L|S} \to J_L/J_L^2 \to 0$$

with T the torsion sheaf of rank one supported on $L \cap S^{\text{sing}}$. The sequence splits and the torsion-free part of $N^*_{L|S}$ is isomorphic to J_L/J_L^2.

For part (3), to build up the relationship with cubic threefold, assume X is a smooth cubic threefold and $L \subseteq X$ is a line of the first type (resp. second type). Let H be a hyperplane tangent to X at any point on L, then the hyperplane section $S = X \cap H$ is singular at the tangent point. Moreover, there is a smooth quadric surface Q (resp. $\mathbb{P}^2$) tangent to X along L (Proposition 5.2.12) and is contained in the hyperplane H. So L is the first type (resp. second type) in the cubic surface S.

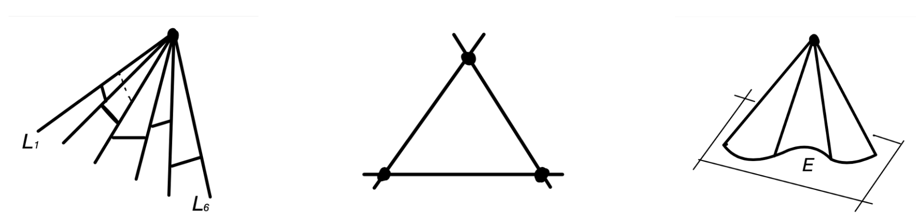
Conversely, we can embed S into a smooth cubic threefold X as a hyperplane section $S = X \cap H$, as long as S is normal and has only isolated singularities. The line L is regarded as a subvariety of both S and X. If L is of the first type (resp. second type) in S, then there is a smooth quadric surface (resp. $\mathbb{P}^2$) tangent to S along L, which will automatically imply that L is of the first (resp. second type) in X, by comparing the dual maps. Finally, the hyperplane H will be tangent to X at some point on L, since otherwise S will be smooth along L. □

As a direct consequence, lines of the first type (resp. second type) on S correspond to type (II) (resp. (IV)) schemes, just as cubic threefolds case in Proposition 5.2.13. Now we provide some explicit examples of how the schemes of the four types are contained in a normal cubic surface. For a smooth cubic surface S has 216 pair of skew lines corresponding to 216 type (I) schemes. Below are some other typical examples.

Example 5.3.7. (A_1 singularity) Let S be a cubic surface with a single singularity of type A_1. Then it has 21 lines, with 15 lines away from the nodes and 6 lines $L_1, ..., L_6$ passing through the node. There are 120 disjoint pairs of lines corresponding to the type I scheme; each L_i supports a unique type II scheme structure, and there are 6 such lines. The union $L_i \cup L_j$ with $i \neq j$ supports a unique type III structure with the embedded point supported at the intersection point, and there are 15 such pairs. $|H(S)_{\text{red}}| < \infty$.

Example 5.3.8. ($3A_2$ singularities) Let S be defined by $xyz = w^3$. Then it has only 3 lines $x = w = 0, y = w = 0, z = w = 0$ with each line passing through two of the three A_2 singularities $[1, 0, 0, 0]$, $[0, 1, 0, 0]$, $[0, 0, 1, 0]$. In this case, each line supports type IV schemes in two different ways. The embedded point can be supported on either of the two singularities that the line passes. S also contains type III schemes supported on every pair of the three lines with the embedded point supported at the intersection point. $|H(S)_{\text{red}}| < \infty$.

Example 5.3.9. (elliptic singularity) Let S be a cone over a smooth cubic curve E. Each line only supports type IV scheme in a unique way: The embedded point is supported at the cone point. Moreover, each pair of the distinct lines intersects at the cone point, so their union supports a type III scheme. $H(S)_{\text{red}} \cong \text{Sym}^2 E$.



A_1 singularity | $3A_2$ singularities | elliptic singularity

Figure 5.2: Examples of Singular Cubic Surfaces

Now we answer the question proposed at the beginning of this section.

Theorem 5.3.10. *Denote $p_1, ..., p_k$ the Eckardt points on X, then the only positive dimensional fibers of*

$$\pi : H(X) \to \mathbb{O}$$

are at Eckardt hyperplane sections H_{p_i}. The reduced structure of such fiber is isomorphic to $\mathrm{Sym}^2 E_i$, where E_i is an elliptic curve. In particular, if X is general and has no Eckardt point, then $H(X) \to \mathbb{O}$ is finite.

Proof. If a hyperplane section S is obtained by intersecting X with the tangent hyperplane T_pX at some Eckardt point, then S is the cone of an elliptic curve E. Then $H(S)_{\mathrm{red}} \cong \mathrm{Sym}^2 E$ as we have seen.

If $S = X \cap H$, where H is not a tangent hyperplane at Eckardt point, then by Corollary 5.3.3, it only has finitely many lines, and thus it has finitely many pairs of lines as well. Now by Proposition 5.3.4 and Proposition 5.3.6, there are finitely many schemes of the four types contained in S. In other words, $H(S)$ is zero-dimensional.

Note that if a hyperplane is tangent to X at an Eckardt point, it cannot be tangent to any other point, so there is a one-to-one correspondence between the positive dimensional fibers of $H(X) \to \mathbb{O}$ and the Eckardt points. □

If a line L on S is away from the singularities of S, then L is rigid in S. However, the occurrence of singularity along the line changes the local geometry of L around S, according to Prop 5.3.6 (1) and (2). One may ask whether the singularities on S that a line L passes determine the type of the line.

Proposition 5.3.11. *(1) If a line $L \subseteq S$ passes through only one singularity of S and the singularity has type A_1, then L is of the first type.*

(2) If L passes through more than one singularity, then L is of the second type.

Proof. For (1), assume the line is given by $x_2 = x_3 = 0$ and the cubic surface has equation

$$x_0 Q(x_1, x_2, x_3) + C(x_1, x_2, x_3) = 0,$$

with the cubic $C = x_1^2(ax_2 + bx_3) + \cdots$ and quadric $Q = x_1(cx_2 + dx_3) + \cdots$ intersecting transversely at 6 points, guaranteeing that $[1, 0, 0, 0]$ is an A_1 singularity. The dual map along L is $\mathcal{D}|_L = [0, 0, cx_0 + ax_1, dx_0 + bx_1]$. The transversality condition implies that the two linear forms are linearly independent, so it corresponds to dual map of a smooth quadric surface along L.

For (2), we regard S as a hyperplane section of a smooth cubic threefold X, then use the fact that the dual map on X along L is 1-to-1 (resp. 2-to-1) when L is of the first type (resp. second type). □

Remark 5.3.12. Based on the examples that we studied and the previous proposition, the line of the first type tends to pass fewer singular points, and the line of the second type tends to pass through more singular points. If L passes through only one singularity, one may wonder whether the type of the singularity that L passes through determines the type of the line. However, this is not the case. There is a normal cubic surface defined by the equation $F = x_0x_1x_2 + x_2x_3^2 + x_3x_1^2$ (with an A_4 at $[1, 0, 0, 0]$ and an A_1 singularity at $[0, 0, 1, 0]$). Both of the lines $x_2 = x_3 = 0$, $x_1 = x_2 = 0$ only pass through the A_4 singularity. However, by computing the dual map along the lines, we conclude that one of the lines is of the first type and the other is of the second type.

Remark 5.3.13. *The relative Hilbert scheme of lines $\mathcal{F}$ of a cubic threefold X is flat over $\mathbb{O} \setminus \{H_1, ..., H_k\}$, where $\{H_1, ..., H_k\}$ is the set of the Eckardt hyperplanes and has length* 27 *on each fiber (also see the introduction of [59] and Example 1.1 (b) of [60]). Classically, the lines for cubic surfaces with RDPs were studied by Cayley [15], and the number "27" is* *interpreted as the number of lines counted with multiplicities, and the multiplicity of a line depends on the type of singularities it passes through.*

Similarly, if we consider the pair of skew lines, again, the relative Hilbert scheme of skew lines $H(\mathcal{S}/\mathbb{O})$ is flat over $\mathbb{O} \setminus \{H_1, ..., H_k\}$. Similarly, the length of the Hilbert schemes on S is the constant number 216, which is the number of pairs of skew lines on a smooth cubic surface.

5.4 A Modular Interpretation

In this section, we discuss some relations between our results and the Bridgeland stable moduli spaces studied in [1] and [6].

In [1], the author studied the moduli space $\mathcal{M}_\sigma(w)$ of Bridgeland stable objects in the Kuznetsov component with Chern character $w = H - \frac{1}{2}H^2 + \frac{1}{3}H^3$ for a smooth cubic threefold X. Let S denote a hyperplane section of X. The moduli space $\mathcal{M}_\sigma(w)$ parameterizes the following two objects:

(1) $\mathcal{O}_S(D)$, a reflexive sheaf or rank 1 associated to certain Weil divisor D on S (when S is general, $D = L_1 - L_2$ for a pair of skew lines L_1, L_2 on S);

(2) $I_{p|S}$, the ideal sheaf of a point in $S = X \cap H$, where H is the tangent hyperplane section at p.

In both cases, the stable object is contained in a unique hyperplane section, so there is a natural projection

$$\mathcal{M}_\sigma(w) \to \mathbb{O}. \tag{5.13}$$

Proposition 5.4.1. *The projection* (5.13) *is generically finite, and its only positive dimensional fibers are elliptic curves that correspond to the Eckardt points on X.*

Proof. It is shown in section 3.3 [1] that the $\mathcal{M}_\sigma(w)$ is isomorphic to the moduli space $\mathcal{M}_G(\kappa)$ of Gieseker-stable sheaves with Chern character $\kappa = (3, -H, \frac{1}{2}H^2, -\frac{1}{6}H^3)$ studied in [6]. According to Lemma 7.5 of [6], there is an isomorphism $\mathcal{M}_G(\kappa) \cong \mathrm{Bl}_0(\Theta)$. Via the isomorphisms, the projection (5.13) coincides with the Gauss map defined in (4.18)

$$\lambda : \mathrm{Bl}_0(\Theta) \to \mathbb{O},$$

because it agrees on the general points, where a pair of skew lines is sent to their spanning hyperplane section. Now the argument follows from Theorem 4.4.1. □

We reinterpret the map that we introduced in (4.11).

$$\tilde{\psi} : \mathrm{Bl}_{\Delta_F}(F \times F) \to \mathrm{Bl}_0(\Theta).$$

In corollary 5.2.4, we introduced a double cover $\widetilde{H(X)}$ of the Hilbert scheme $H(X)$. Informally, we can regard $\widetilde{H(X)}$ as the "Hilbert scheme" that parameterizes universal flat families of ordered skew lines. There is an isomorphism

$$\widetilde{H(X)} \cong \mathrm{Bl}_{\Delta_F}(F \times F).$$

Thus there is an Abel-Jacobi map

$$\widetilde{AJ} : \widetilde{H(X)} \to JX$$

by composing the blowup map and $F \times F \to JX$ as in (4.12).

For the stable moduli space, there is also an Abel-Jacobi map

$$AJ : \mathcal{M}_\sigma(w) \to JX$$

by sending $\mathcal{O}_S(D)$ to $\int_{D_0}^{D}$, for some fixed divisor D_0.

Regarding $\mathcal{M}_\sigma(w)$ as the blowup of the theta divisor, then according to Proposition 2.1 in [1], the exceptional divisor $K \cong X$ parameterizes ideal sheaves $I_{p|S}$ of singular points on hyperplane sections S of X. The complement of K parameterizes coherent sheaves $\mathcal{O}_S(D)$ with D being certain Weil divisor on the hyperplane section S.

Proposition 5.4.2. *The Abel-Jacobi map $\widetilde{AJ}$ factors through the moduli space $\mathcal{M}_\sigma(w)$ up to by adding a constant on the torus JX. In other words, there is the following commutative diagram (up to by adding a constant).*

$$\begin{array}{ccc} \widetilde{H(X)} & \xrightarrow{\Psi} & \mathcal{M}_\sigma(w) \\ & \searrow^{\widetilde{AJ}} & \downarrow{\scriptstyle AJ} \\ & & JX \end{array} \tag{5.14}$$

By restricting Ψ to the exceptional divisors on both sides, we have

$$\Psi|_{\mathbb{P}T_F} : \mathbb{P}T_F \to X$$

by sending a scheme Z of type (II) or (IV) to the ideal sheaf I_p, where p is the unique point determined by Z defined in Proposition 5.2.16*. It is isomorphic to the projection from the incidence variety to X by the identification* (4.13)*.*

Proof. We define a rational map $\Psi : (L_1, L_2) \mapsto \mathcal{O}_S(L_1 - L_2)$. Their Abel Jacobi images differ by a constant due to the presence of D_0. So we have a commutative diagram. Since Ψ

$$\begin{array}{ccc} \widetilde{H(X)} & \overset{\Psi}{\dashrightarrow} & \mathcal{M}_\sigma(w) \\ \| & & \| \\ \mathrm{Bl}_{\Delta_F}(F \times F) & \xrightarrow{\tilde{\psi}} & \mathrm{Bl}_0(\Theta) \end{array}$$

agrees with $\tilde{\psi}$ at general points, Ψ uniquely extends to a morphism. □

Although there is a complete classification of cubic surfaces S, and the Weil divisors on S are pretty much controlled by the divisors on the minimal resolution $\tilde{S}$ of S, it is not explicitly known the expression of the divisor D on singular hyperplane sections of X such that $\mathcal{O}_S(D) \in \mathcal{M}_\sigma(w)$.

As we know the off-diagonal part of $\widetilde{H(X)}$ parameterizes for type (I) and (III) schemes with an order, we conjecture that their supports determine the divisor D.

Conjecture 5.4.3. *If $\mathcal{O}_S(D)$ is a stable object parameterized by $\mathcal{M}_\sigma(w)$, then D has the form $L_1 - L_2$, where L_1, L_2 are two lines on S (can be singular) and lie in one of the two cases:*

(i) L_1, L_2 disjoint (Lines can pass through singularities);

(ii) L_1 and L_2 intersect at one point p, which is a singularity of p.

On the other hand, [6] uses the twisted cubics on X to characterize the moduli space $\mathcal{M}_G(\kappa)$. More precisely, a general point of $\mathcal{M}_G(\kappa)$ parameterizes a coherent sheaf $\mathcal{E}$ that fits into the exact sequence

$$0 \to \mathcal{E} \to \mathcal{O}_Y^{\oplus 3} \to \mathcal{O}_S(C) \to 0,$$

where C is a twisted cubic on hyperplane section S (which is a Weil divisor when S is singular).

Let $\mathcal{C}^\circ$ denote the open locus of smooth twisted cubics in the Hilbert scheme of X. There is a commutative diagram.

$$\begin{array}{ccc} \mathcal{C}^\circ & \longrightarrow & \mathcal{M}_G(\kappa) \\ & \overset{\phi}{\searrow} & \downarrow \\ & & JX \end{array} \tag{5.15}$$

The vertical arrows are Abel-Jacobi maps. For example, by fixing a planar cubic C_0, ϕ is defined by $C \to \int_{C_0}^{C}$, which factors through the moduli space $\mathcal{M}_G(\kappa)$.

Question 5.4.4. *Is there a comparison map of the diagrams* (5.14) *and* (5.15)*? In other words, applying the isomorphism* $\mathcal{M}_\sigma(w) \cong \mathcal{M}_G(\kappa)$ *between the two moduli spaces, is there a rational map* $\widetilde{H(X)} \dashrightarrow \mathcal{C}^\circ$ *commuting with Abel-Jacobi maps?*

Chapter 6: Tube Mapping for Hypersurfaces

6.1 Overview

Recall that in Section 4.7, we have introduced *tube mapping* for a smooth projective variety $X \subseteq \mathbb{P}^N$ of dimension n, which is to fix a smooth hyperplane section Y of X and consider a homomorphism

$$\Phi : \{([l], \alpha) \in \pi_1(\mathbb{O}^{\mathrm{sm}}, t_0) \times H_{n-1}(Y, \mathbb{Z})_{\mathrm{van}} | l_* \alpha = \alpha\} \to H_n(X, \mathbb{Z})_{\mathrm{prim}}, \tag{6.1}$$

by sending $([l], \alpha)$ to the n cycle on X as a tube over α along the loop l.

Schnell showed that

Theorem 6.1.1. *([50]) Φ induces a map on the level of homology groups*

$$\Phi_* : H_1(T, \mathbb{Z}) \to H_1(J_{\mathrm{prim}}, \mathbb{Z}) \cong H_n(X, \mathbb{Z})_{\mathrm{prim}}$$

has a cofinite image.

In the case when n is odd, (6.1) coincides with the induced map between fundamental groups of the topological Abel-Jacobi map

$$f : T \to H_n(X, \mathbb{Z})_{\mathrm{prim}}. \tag{6.2}$$

What we are interested in is the restriction of the topological Abel-Jacobi map (6.2) to the T_v component

$$f_v : T_v \to H_n(X, \mathbb{Z})_{\text{prim}}. \tag{6.3}$$

and its induced map on fundamental groups

$$(f_v)_* : \pi_1(T_v, *) \to H_3(X, \mathbb{Z}). \tag{6.4}$$

Equivalently, $(f_v)_*$ coincides with the tube mapping on a single primitive vanishing cycle $\alpha_0 \in H_{n-1}(Y, \mathbb{Z})_{\text{van}}$:

$$\Phi_v : \{([l], \alpha_0) \mid [l] \in \pi_1(\mathbb{O}^{\text{sm}}, t_0), l_*\alpha = \alpha\} \to H_n(X, \mathbb{Z})_{\text{prim}}. \tag{6.5}$$

In this chapter, we will validate the Conjecture 4.7.3 when X is a smooth hypersurface of $\mathbb{P}^4$. (Note that the statement is trivial for degree 1 and 2.) In other words, we prove that

Theorem 6.1.2. *When X is a smooth hypersurface of $\mathbb{P}^4$, the map*

$$\Phi_v = (f_v)_* : \pi_1(T_v, *) \to H_3(X, \mathbb{Z}) \tag{6.6}$$

has cofinite image.

Note here $H_3(X, \mathbb{Z}) = H_3(X, \mathbb{Z})_{\text{prim}}$ since $H_3(\mathbb{P}^4, \mathbb{Z}) = 0$.

To describe a geometric meaning, we state the following observations:

I. *The vanishing cycle is conjugate to each other* so that we can fix one vanishing cycle $\alpha \in H^{2n-2}(X_{t_0}, \mathbb{Z})_{\text{van}}$.

II. Choose a Lefschetz pencil $\mathbb{L}$ and $U_X \subseteq \mathbb{L}$ the locus where hyperplane sections are smooth, then Zariski's lemma states that the map *the map $\pi_1(U_X, t_0) \to \pi_1(\mathbb{O}^{\text{sm}}, t_0)$ induced by inclusion $U_X \hookrightarrow \mathbb{O}^{\text{sm}}$ is surjective.* So we only need to consider those loops contained in the Lefschetz pencil.

It follows that an equivalent statement of Theorem 6.1.2 will be

Theorem 6.1.3. *Let X be a smooth hypersurfaces in $\mathbb{P}^4$ of degree $d \geq 3$ and X_{t_0} a smooth hyperplane section. Let $\alpha_0 \in H_2(X_{t_0}, \mathbb{Z})_{\text{van}}$ be a primitive vanishing cycle. Let G be the subgroup of $\pi_1(U_X, t_0)$ consisting of $[\gamma]$ satisfying $[\gamma] \cdot \alpha_0 = \alpha_0$, then the map*

$$\Phi'_v : \{\alpha_0\} \times G \to H_3(X, \mathbb{Z}) \tag{6.7}$$

has a cofinite image.

Our strategy is to first reduce from arguing the cofinitenss of the image of (6.7) to arguing image of (6.7) is nonzero. This is based on a key Lemma (Lemma 6.2.1). For the case $d = 3$, we already proved it in Theorem 4.7.4. In fact, based on Lemma 6.2.1, we can provide a more straightforward proof in Section 6.3. It is based on the fact that a vanishing cycle on a cubic surface is represented by the difference of two disjoint lines $[L_1] - [L_2]$ together with the fact that the Abel-Jacobi map on cubic threefolds of a pair of lines is generically 6-to-1 [19].

The general situation relies on the degeneration of the hypersurface of degree d into a union of hyperplanes of degree 3 and degree $d - 3$ meeting transversely. After birational modification on the total space of the family, we obtain a semistable family where the asymptotic Hodge theory is well understood [45]. The proof follows from the analysis of the degeneration of vanishing cycle and its monodromy.

In Section 6.4, we study dual varieties in families, aiming to prove Lemma 6.4.2, which roughly says that one can choose a Lefschetz pencil for a family of hypersurfaces parameterized by a disk with a small radius. In Section 6.5, we do some preparation for proving the Theorem 6.1.3. In Section 6.6, we reduce the theorem to $d = 4$ case and complete the proof.

6.2 Key Lemma

Lemma 6.2.1. *Given a smooth variety $X \subset \mathbb{P}^N$. Assume the tube map* (6.4) *defined on $\pi_1(T_v, *)$ has nonzero image, then the image of the tube map* (6.4) *is cofinite.*

Proof. We can choose $W \subset \mathbb{P}^{N+1}$ a smooth variety containing X as a smooth hyperplane section. Choose a general pencil $\mathbb{L}_W$ of hyperplane sections of W passing through $X = W \bigcap H_{v_0}$ and let U_W be the points corresponding to smooth hyperplane section. Then there is a monodromy action

$$\rho : \pi_1(U_W, v_0) \to \mathrm{Aut} H_n(X, \mathbb{Q})_{\mathrm{van}}.$$

It is well known that that the action ρ is irreducible [62, Theorem 3.27]. On the other hand, one can show that the image of tube mapping is invariant under the monodromy action. In particular, $\mathrm{Im}(\Phi'_v) \otimes \mathbb{Q}$ is a ρ-subrepresentation of $H_n(X, \mathbb{Q})_{\mathrm{van}}$, so the irreduciblity of ρ together with our assumption implies that $\mathrm{Im}(\Phi'_v) \otimes \mathbb{Q}$ has to be the whole $H_n(X, \mathbb{Q})_{\mathrm{van}}$, which implies that $\mathrm{Im}(\Phi'_v) \subseteq H_n(X, \mathbb{Z})_{\mathrm{van}}$ is cofinite.

Lastly, let's show that the image of tube mapping is indeed invariant under the monodromy action. Choose a smooth loop $l \subseteq U_W$ based at v_0, then by restricting to a small segment l_i contained in a small open neighborhood U_i of U_W over which the family $\{W \bigcap H_v\}_{v \in U_W}$ is C^∞ trivial, we can fix a uniform Lefschetz pencil for all $(n-1)$-folds $W_v = W \bigcap H_v$ for $v \in l_i$ and the family U_{W_v} varies smoothly, so the tube map (6.7) is locally trivial. It follows that the image of the Tube map on U_W is a sub-local system of $H_n(W_v, \mathbb{Z})_{\mathrm{van}}$. Finally, as we have explained, the vanishing cycle is conjugate to each other, together with Zariski's lemma (so it doesn't matter the choice of base point and Lefschetz pencil), so this sub-local system U_W has trivial monodromy. □

6.3 Cubic Threefold Again

Recall that we already proved Theorem 6.1.2 for cubic threefold in Theorem 4.7.4. Here we will provide alternative proof.

Proposition 6.3.1. *Theorem 6.1.3 is true for $d = 3$.*

Proof. By Lemma 6.2.1, it suffices to prove the tube mapping (6.7) is nonzero.

By restricting the 72-to-1 cover $T_v \to \mathbb{O}^{\text{sm}}$ (4.8) to the Lefschetz pencil $\mathbb{L}$, we get a 72-to-1 cover $\pi_U : T_v(U_X) \to U_X$ between affine curves. Since the local monodromy has order two, the preimage of a puncture disk $\Delta^* \subseteq U_X$ is a disjoint union of puncture disks Δ_i^*, with $\pi_U : \Delta_i^* \to \Delta^*$ either 2-to-1 or 1-to-1. By filling in the holes, we can compatify $T_v(U_X)$ to a proper smooth curve C over $\mathbb{L}$ branched over $\mathbb{L} \backslash U_X$ (or a consequence of Proposition 3.2.2).

Moreover, the (topological) Abel-Jacobi map $T_v(U_X) \to JX$ extends to $f_{|C} : C \to JX$, which is a finite map [19]. Now the induced map between fundamental groups

$$(f_{|C})_* : \pi_1(C, *) \to \pi_1(JX, 0) = H_3(X, \mathbb{Z}) \tag{6.8}$$

is nonzero, otherwise, $f_{|C}$ factors through the universal cover $\mathbb{C}^5 \to JX$. But any holomorphic map $C \to \mathbb{C}^5$ is constant, due to the maximal modulus principle, so $f_{|C}$ is a constant map, which is a contradiction to the fact that it is finite. □

6.4 Degeneration of Dual Varieties

In this section, we digress to discuss dual varieties in families, with general fiber smooth and the central fiber normal crossing. We would like to explore the concept of the "dual variety in the limit" to prove Lemma 6.4.2.

Let X_d be a degree $d \geq 2$ smooth hypersurface of $\mathbb{P}^{n+1}$ defined by F_d; X_{d_1}, X_{d_2} be smooth hypersurfaces of degree d_1 and d_2 respectively defined by F_{d_1} and F_{d_2}, with $d = d_1 + d_2$.

We require F_d, F_{d_1}, F_{d_2} to be general, so that their common zero locus is a complete intersection. Besides, by Bertini's theorem, $F_s := sF_d + F_{d_1}F_{d_2}$ is smooth for $s \neq 0$ when $|s|$ is small enough. So for such $s \neq 0$, there is a dual map on smooth hypersurface $X_s := \{F_s = 0\}$

$$\mathcal{D}_s : X_s \mapsto (\mathbb{P}^{n+1})^*$$

$$x \mapsto \big(\frac{\partial F_s}{\partial x_0}(x), ..., \frac{\partial F_s}{\partial x_{n+1}}(x)\big), \tag{6.9}$$

with $\frac{\partial F_s}{\partial x_j}(x) = s\frac{\partial F_d}{\partial x_j}(x) + F_{d_2}\frac{\partial F_{d_1}}{\partial x_j}(x) + F_{d_1}\frac{\partial F_{d_2}}{\partial x_j}(x)$, $j = 0, ..., n+1$ by direct computation.

The image $(X_s)^*$ is called the dual variety of X_s and it is well known that it is a hypersurface of degree $m = d(d-1)^n$ in the dual space. So this defines a rational section μ on the sheaf $S^m(V^*) \otimes \mathcal{O}_\Delta$ over Δ which has possibly a pole along $s = 0$ where $V = \mathbb{C}^{n+2}$, but by multiplying by a suitable power of s, we can assume the section μ is regular and $\mu(0) \neq 0$. This will not change the defining hypersurface in projective space, so it defines a hypersurface.

Definition 6.4.1. *Define $(X_0)^*$ to be the projective hypersurface $\{\mu = 0\} \subseteq (\mathbb{P}^{n+1})^*$ and call it the dual variety in the limit associated to the family $sF_d + F_{d_1}F_{d_2}$.*

X_0^* is reducible since it contains dual variety of X_{d_1} and dual variety of X_{d_2}. However, since the dual family $\{X_s^*\}_{s\in\Delta}$ is flat, the degree of X_0^* should equal to degree of X_0^*, but a simple count shows that $d(d-1)^n > d_1(d_1-1)^n + d_2(d_2-1)^n$ so there should be more components in X_0^*. In Appendix A, we find other components components explicitly.

Finally, we will prove the following:

Lemma 6.4.2. *By shrinking Δ to a smaller disk, $\{H_t\}$ is Lefschetz pencil for all X_s with $s \in \Delta$. In other words, we can choose a line $\mathbb{L} \subseteq (\mathbb{P}^{n+1})^*$, such that $\mathbb{L}$ is transverse to all X_s^*. For $s = 0$, this means being transversal to each component of X_0^**

Proof. This argument is based on continuity. First, we choose $\mathbb{L}$ to be transverse to X_0^*, then we show that it is transverse to all X_s up to shrinking to a smaller disk.

When $s \neq 0$, the dual variety X_s^* is an irreducible hypersurface of $(\mathbb{P}^{n+1})^*$, defined by a single homogeneous polynomial $\{G_s = 0\}$ varying continuously with respect to the parameter s.

Then the dual variety in the limit $X_0^* := \{z | G_0(z) = 0\}$ is defined by $G_0 := \lim_{s\to 0} G_s$.

By assumption, $\mathbb{L}$ is disjoint from the singularities $\mathrm{Sing}(X_0^*)$, so there is an open neighborhood $\mathcal{U}$ of $\mathrm{Sing}(X_0^*)$ in $(\mathbb{P}^{n+1})^*$ such that $\mathcal{U} \bigcap \mathbb{L} = \emptyset$, so by continuity, we can choose Δ small enough so that $\mathrm{Sing}(X_s^*) \subseteq \mathcal{U}$ for all $s \in \Delta$. Therefore, $\mathbb{L}$ intersect X_s^* along the smooth locus $(X_s^*)^{sm}$ for each $s \in \Delta$.

Let $\{p_1, ..., p_k\}$ be the set of points of $\mathbb{L} \cap X_0^*$. Let $\sum_{i=0}^{n+1} a_i^j w_i = 0$, $j = 1, ..., n$ be n hyperplanes in $(\mathbb{P}^{n+1})^*$ whose common zero loci is the line $\mathbb{L}$, where $w_0, ..., w_{n+1}$ is the coordinate on the dual space. So by the transversality assumption, the tangent vectors $\big(\partial G_0/\partial w_0(p_i), ..., \partial G_0/\partial w_{n+1}(p_i)\big)$ is not contained in the span of the three hyperplanes. In other words, the matrix $M(s, w)$ is of full rank at $s = 0, w = p_i$, for $i = 1, ..., k$, where $M(s, w)$ is the $(n+1) \times (n+2)$ matrix

$$M(s,w) = \begin{bmatrix} a_0^1 & a_1^1 & \cdots & a_{n+1}^1 \\ \cdots & \cdots & \cdots & \cdots \\ a_0^n & a_1^n & \cdots & a_{n+1}^n \\ \frac{\partial G_s}{\partial w_0}(w) & \frac{\partial G_s}{\partial w_1}(w) & \cdots & \frac{\partial G_s}{\partial w_{n+1}}(w) \end{bmatrix}.$$

Again by continuity, $M(s, w)$ will remain to be of maximal rank for $s \in \Delta$ and $w \in \mathcal{U}_i$ for Δ small open neighborhood of 0, and $\mathcal{U}_i$ small open neighborhood of p_i. Finally, we can shrink Δ so that for each $s \in \Delta$, the intersection $\mathbb{L} \bigcap (X_s^*)^{sm}$ is contained in $\bigcup_i \mathcal{U}_i$. Therefore $\mathbb{P}^1$ is a Lefschetz pencil for all $s \in \Delta$. □

6.5 Deforming of Vanishing Cycles

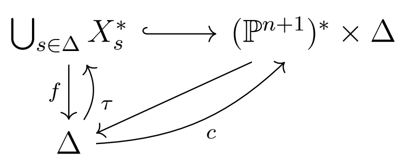
Recall that in the previous section, we associated the family of hypersurfaces of degree d a family of dual varieties

$$f : \bigcup_{s \in \Delta} X_s^* \to \Delta$$

with X_0^* as limit of dual variety nearby, which contains $X_{d_1}^*$ as an irreducible component (where we assume d_1 is the degree at least two factor). Also, we chose a general pencil $\mathbb{L} \subseteq (\mathbb{P}^{n+1})^*$ intersecting transversely to X_s^* for all $s \in \Delta$.

Choose a point $p \in \mathbb{L} \bigcap X_{d_1}^*$, so in particular, p is a smooth point of Y^* and away from other components of X_0^*. By inverse image theorem, up to shrinking to a smaller disk, we can find $\tau(s) \in \mathbb{L}$ varying differentiably with respect to $s \in \Delta$ such that $\tau(s) \in \mathbb{L} \bigcap (X_s^*)^{sm}$ and $\tau(0) = p$.

In other words, τ defines a C^∞ section whose image lies in the smooth part $(X_s^*)^{sm}$ and additionally $\tau(0) \in (X_{d_1}^*)^{sm}$.



This gives a family of hyperplanes $H_{\tau(s)}$ which are tangent to X_s and $H_{\tau(0)}$ is tangent to X_{d_1} and the singularity on the hyperplane section has nondegenerate tangent cone. Choose a constant section c, where $c \in (\mathbb{P}^{n+1})^* \setminus X_0^*$ is close to $\tau(0)$ so that there is a local vanishing cycle $\alpha \in H_{n-1}(X_{d_1} \bigcap H_c, \mathbb{Z})_{\text{van}}$ which specializes to the node as H_c specializes to $H_{\tau(0)}$. Now up to shrinking to a smaller disk of Δ containing 0, there is a vanishing cycle $\alpha_s \in H_{n-1}(X_s \bigcap H_c)_{\text{van}}$ which specializes to the node as H_c specializes to $H_{\tau(s)}$.

The goal of this section is to prove the following (imprecise) statement:

The vanishing cycle is on the hyperplane section H_c is a trivial family over Δ. (6.10)

Note at the same time section τ gives a nodal locus via dual correspondence.

We take $\tau(0) \in X_{d_1} \subseteq \mathbb{P}^{n+1}$ which is not on the base locus of Δ-pencil, i.e., not on $X_{d_1} \bigcap X_d$, so if we take a small polydisk D containing $\tau(0)$ so $\tau(s) \in D$ when s is small.

Also we require D to stay away from base locus, then D can thought as living in the total space $\mathscr{X}$. We can choose affine coordinate $x_1, ..., x_n, t$ where t corresponds to the pencil $\mathbb{L}$. Recall that $F_s = sF_d + F_{d_1}F_{d_2}$ to be the homogeneous polynomial varying in s, so restriction of F_s to a fixed t is the equation of the hyperplane section $X_s \bigcap H_t$. For each $s \in \Delta$, we denote $\tau(s) = (x_1^s, ..., x_n^s, t^s)$ the nodal locus, i.e., the hyperplane section $X_s \cap H_{t-t_s}$ has an ordinary node at $\tau(s)$. Since $\partial F_s/\partial t(\tau(s)) \neq 0$, the implicit function theorem says that there is a smooth function $f_s(x_1, ..., x_n)$, such that

$$F_s(x_1, ..., x_n, f_s(x_1, ..., x_n)) \equiv 0.$$

Moreover f_s is holomorphic function in $x_1, ..., x_n$ and is analytic with respect to the parameter s. There is a power series expansion

$$f_s(x_1, ..., x_n) = Q_s(x_1 - x_1^s, ..., x_n - x_n^s) + \text{higher powers},$$

where Q_s is a nondegenerate quadric form.

Now by a parametric version of the holomorphic Morse lemma, we have

Claim 6.5.1. *There is an analytic change of coordinate $x'_1, ..., x'_n$ such that*

$$f_s(x'_1, ..., x'_n) = x'^2_1 + \cdots + x'^2_n.$$

Moreover, the coordinate change depends analytically with respect to the parameter s.

This implies the following result, which is a precise statement of (6.10):

Corollary 6.5.2. *There is an analytic isomorphism*

$$D \xrightarrow{\cong} \{x'^2_1 + \cdots + x'^2_n = t\} \times \Delta$$

preserving projection to Δ.

Before we end the section, we prove a lemma that will be used later.

Lemma 6.5.3. *In the fixed t-pencil $\mathbb{L}$ in the Lemma 6.4.2, there exists a connected analytic open subset obtained $U \subseteq \mathbb{L}$ by removing finitely many closed disks from $\mathbb{L}$, such that*

(i) c (defined ealier in this section) is contained in U, and

(ii) for all $s \in \Delta^*$, and $t \in U$, $H_t \cap X_s$ is smooth, and

(iii) for each $t \in U$ $H_t \bigcap X_{d_i}$ and $H_t \bigcap X_{d_1} \bigcap X_{d_2}$ is smooth.

Proof. Since the interesection points of the pencil and the dual variety $\mathbb{L} \bigcap X_s^*$ varies continuously, so for each $z_i \in \mathbb{L} \bigcap X_0^*$, there is a small disk $D_{z_i} \subseteq \mathbb{L}$ centered at z_i, such that the intersection of $\bigcup_{s \in \Delta} \mathbb{L} \bigcap X_s^* \subset \bigcup_i D_{z_i}$. □

6.6 Proof of Theorem 6.1.3 for Quartic Threefold

In this section, we will prove Theorem 2 for quartic by degenerating it into a union of the cubic threefold Y and a hyperplane P in $\mathbb{P}^4$, where Y and P intersect transversely. More precisely, let F_X, F_Y and F_P be general homogeneous polynomials of degree 4, 3 and 1 respectively, and the one dimensional of a quartic is

$$\mathscr{X} = \{sF_X + F_Y F_P = 0\} \subset \Delta \times \mathbb{P}^4, \tag{6.11}$$

with $s \in \Delta$ a small disk centered at $0 \in \mathbb{C}$, special fiber $Y \cup P$ and general fiber a smooth quartic 3-fold. By Bertini's theorem, we can choose the disk small enough so that $s = 0$ is the only singular fiber.

X_s is used to denote the quartic threefold given by the equation $\{F_s = F_X + F_Y F_P = 0\}$. Also, by Lemma 6.5.3, we have an open subset $U \subseteq \mathbb{L}$ such that it contains a base point c where the vanishing cycle on the hyperplane section $X_s \bigcap H_c$ deforms trivially as s varies in Δ. Moreover, $H_t \bigcap X_s$ is smooth for all $t \in U$ and $s \in \Delta*$ and all $H_t \bigcap Y$ and $H_t \bigcap P \bigcap Y$ are smooth.

According to Clemens's Degeneration on Kähler Manifolds [18], there is a deformation retract of X_s onto $Y \bigcup P$ which induces diffeomorphism of $Y \setminus (Y \bigcap P)$ into a smooth submanifold X'_s of X_s (and $P \setminus (Y \bigcap P)$ into a smooth submanifold X''_s of X_s and disjoint from X'_s). So to guarantee that the 3-cycles in the image of the tube mapping T can be deformed to nearby quartic, we have to make sure both the vanishing cycle α and those 3-cycles transported along loops γ which fix α are all supported in $Y \setminus (Y \bigcap P)$.

Now, since one can inductively degenerate X_d to X_{d-1} and a smooth hyperplane P intersecting transversely, and by Lemma 6.2.1, the proof of Theorem 6.1.3 reduces to prove

Proposition 6.6.1. *The tube map* (6.7) *of quartic threefold X is nonzero.*

6.6.1 Terminologies

In this section, we will introduce some terminologies on the tube mapping on an open submanifold of a (possibly singular) variety.

Let M be a n-dimensional smooth subvariety of $\mathbb{P}^N$, and let $\mathbb{L} = \mathbb{P}^1$ be a pencil of hyperplanes in $\mathbb{P}^N$ in general position. Denote $U \subseteq \mathbb{L}$ as the set of points corresponding to the hyperplanes that are not tangent to M. Consider the incidence variety

$$\tilde{M} := \{(x,t) \in M \times U | x \in M \bigcap H_t\},$$

together with projections

$$\begin{array}{ccc} \tilde{M} & \xrightarrow{\sigma} & M \\ \downarrow{\scriptstyle \pi} & & \\ U & & \end{array}$$

π is proper submersion, so it is locally trivial thanks to Ehresmann's theorem.

Let $t_0 \in U$ be a fixed base point and $\alpha \in H_{n-1}(M \bigcap H_{t_0}, \mathbb{Z})_{\text{van}}$ a vanishing cohomology on the hyperplane section and $\gamma : [0,1] \to U$ with $\gamma(0) = \gamma(1) = t_0$ be a loop based at t_0 satisfying $[\gamma] \cdot \alpha = \alpha$ in $H_{n-1}(M \bigcap H_{t_0}, \mathbb{Z})$ as monodromy action. So the trace of α along γ defines is an integral n-cycle $A_{\alpha,\gamma}$ on $\tilde{M}$

Definition 6.6.2. *Call $A_{\alpha,\gamma} \in H_n(\tilde{M}, \mathbb{Z})$ the tube mapping associated to the pair (α, γ).*

So $\sigma(A_{\alpha,\gamma})$ defines a primitive class in $H_n(M, \mathbb{Z})_{\text{prim}}$ which is the Tube mapping in the sense of [50].

More generally, we can consider tube mapping supported in an open submanifold if Ehresmann's theorem still holds. More precisely, let $M^\circ \subseteq M$ be an open submanifold and $\tilde{M}^\circ$ the set of pairs (x, t) with $x \in M^\circ \bigcap H_t$. Assume that the restriction of π to $\pi|_U : \tilde{M}^\circ \to U$ is $\mathbb{C}^\infty$ locally trivial, i.e., for each $t \in U$, there is an open subset $V \subseteq U$ and a fiber preserving diffeomorphism

$$\begin{array}{ccc} \pi^{-1}(V) \bigcap \tilde{M}^\circ & \xrightarrow{\cong} & (\pi^{-1}(t) \bigcap \tilde{M}^\circ) \times V \\ \downarrow \pi & \swarrow & \\ V & & \end{array}$$

Assume additionally that the monodromy action identity $[\gamma] \cdot \alpha = \alpha$ holds in $H_{n-1}(M^\circ \bigcap H_{t_0}, \mathbb{Z})$. Then the trace of α along γ is a n-cycle on $\tilde{M}^\circ$.

Definition 6.6.3. *We call $A_{\alpha,\gamma}$ the tube mapping associated to the pair (α, γ) supported in M°.*

In what follows, we will show for cubic 3-fold, and hypersurface 3-fold of higher degree, such open submanifold M° exist (and in fact diffeomorphic to each other), which support a certain amount of 3-cycles arising from such pairs (α, γ).

6.6.2 Proof of Proposition 6.6.1

The proof breaks up into several steps.

Step 1. Vanishing cycle on affine complement.

Let $Y_t = Y \bigcap H_t$ for $t \in \mathbb{L}$, and $V_t = Y_t \setminus (Y \bigcap P)$ the affine complement. Denote $U_Y \subseteq \mathbb{L}$ the set of points where hyperplane sections on Y is smooth. We first claim that

Lemma 6.6.4. *For any $t \in U_Y$, the vanishing homology on cubic surface $H_2(Y_t, \mathbb{Z})_{\mathrm{van}}$ is isomorphic to the image of $H_2(V_t, \mathbb{Z}) \to H_2(Y_t, \mathbb{Z})$ induced by inclusion $V_t \hookrightarrow Y_t$.*

Proof. This is a special case of Prop. 7.3 of [30]. Write $P_t = H_t \bigcap P$ the projective 2-plane. By definition, the vanishing homology $H_2(Y_t, \mathbb{Z})_{\mathrm{van}}$ is the kernel of $H_2(Y_t, \mathbb{Z}) \to H_2(H_2, \mathbb{Z}) = \mathbb{Z}$ induced by inclusion, which is identified with kernel of

$$H_2(Y_t, \mathbb{Z}) \to H_0(Y_t \bigcap P_t, \mathbb{Z}), \ \alpha \mapsto \alpha \cap P_t \tag{6.12}$$

by the intersection pairing on H_t. Now (6.12) fits into the exact sequence

$$H_2(V_t, \mathbb{Z}) \to H_2(Y_t, \mathbb{Z}) \to H_0(Y_t \bigcap P_t, \mathbb{Z}),$$

where the last map factors through Thom isomorphism

$$H_2(Y_t, \mathbb{Z}) \to H_2(Y_t, V_t, \mathbb{Z}) \cong H_0(Y_t \bigcap P_t, \mathbb{Z}).$$

So by exactness, the lemma is proved. □

It follows that one can represent a vanishing cycle α by a cycle supported in the affine complement V_t, and therefore any open subspace of V_t which is deformation equivalent to V_t.

Step 2. 3-cycles away from hyperplane.

Let $\mathcal{U}(P)$ be a tubular open neighborhood of $Y \bigcap P$ in Y and for $t \in U$ denote the $Y'_t := Y_t \backslash \mathcal{U}(P)$ the submanifold with boundary. The following is a consequence of a theorem which will be stated in the Appendix.

Lemma 6.6.5. *The family $\{Y'_t\}_{t \in U}$ is C^∞-locally trivial. Namely, for each $t \in U'_Y$, there is a neighborhood $\mathcal{V}$ of t such that there is a fiber preserving diffeomorphism $\pi^{-1}(\mathcal{V}) \cong V_t \times \mathcal{V}$ preserves inclusion into $Y_t \times \mathcal{V}$.*

This lemma tells us that it makes sense to talk about monodromy of homology on Y'_t over the base U. We are going to show that the monodromy of the vanishing cycle on the open part Recall $U_Y \subseteq \mathbb{L}$ is the set of points where hyperplane sections on Y are smooth, and $U \subseteq \mathbb{L}$ is obtained by finitely many small disks centered at $\mathbb{L} \bigcap X_0^*$, so in particular, $U \subseteq U_Y$. Choose a base point $t_0 \in U$ (in particular, we choose $t_0 = c$). Our main proposition in this section will be

Proposition 6.6.6. *There are finitely many loops $l_1, ..., l_n \in U$ based at t_0 which generate the fundamental group $\pi_1(U_Y, t_0)$. Moreover, for any vanishing cycle $\alpha \in H_2(Y_{t_0}, \mathbb{Z})_{\text{van}}$ supported in Y'_{t_0}, the trace of α transported along any (composite of) $l_i, i = 1, ..., n$ is a 3-cycle in $Y \setminus \mathcal{U}(P)$.*

Proof. Denote $p_1, ..., p_n \in \mathbb{L} \setminus U_Y$ be the points corresponding Y'_{p_i} being homotopic to complement of a smooth cubic curve in a singular cubic surface, and $q_1, ..., q_m \in \mathbb{L}$ be the points corresponding to Y'_{q_j} homotopic to complement of a singular cubic curve in a smooth cubic surface. Now, let the loop l_i based at 0 be defined as a straight line towards p_i, go around anticlockwise, and go back and stay in U. Then loops $l_1, ..., l_n$ is a generating set of 3rd primitive homology on the 3-fold under tube map. Moreover, the closed region bounded by any composite of these loops does not contain the point q_j, so it does not deposit monodromy on 1st homology of the cubic curve. It follows from the exact sequence

$$H_1(C_{t_0},\mathbb{Z}) \to H_2(Y'_{t_0},\mathbb{Z}) \to H_2(Y_{t_0},\mathbb{Z}) \to 0,$$

that the monodromy on the open Y'_{t_0} coincides with the monodromy of the compact cubic surface. In other words, we have a commutative diagram $\square$

$$\begin{array}{ccc} H_2(Y_{t_0},\mathbb{Z}) & \xrightarrow{\gamma_*} & H_2(Y_{t_0},\mathbb{Z}) \\ \uparrow & & \uparrow \\ H_2(Y'_{t_0},\mathbb{Z}) & \xrightarrow{\gamma_*} & H_2(Y'_{t_0},\mathbb{Z}) \end{array}$$

Step 3. Deformation of 3-cycle to nearby quartic.

Based on two steps discussed above, we have

• an analytic open subset $U \subseteq \mathbb{L}$ such that all $t \in U$ corresponds to hyperplane H_t intersecting transversely with X_s for all $s \in \Delta$ (when $s = 0$, this implies H_t is transverse to both Y and P, moreover $H_t \bigcap P$ is plane transverse to the cubic surface $Y \bigcap H_t$);

• a base point $t_0 \in U$ ($t_0 = c$) and a local vanishing cycle $\alpha \in H_2(Y \bigcap H_{t_0},\mathbb{Z})_{\text{van}}$ supported in the open part Y'_{t_0}, and a continous family of local vanishing cycles $\alpha_s \in H_2(X_s \bigcap H_{t_0},\mathbb{Z})_{\text{van}}$;

• a loop $\gamma \subseteq U$ based at t_0 such that the monodromy action $[\gamma]\cdot\alpha = \alpha$ in $H_2(Y'_{t_0},\mathbb{Z})$. So the associated tube mapping class $A_{\alpha,\gamma}$ is supported in $Y \setminus \mathcal{U}(P)$, whose image is nonzero in $H_3(Y,\mathbb{Z})$.

Our goal is to produce a 3-cycle A_s in the nearby fiber X_s which is obtained by the tube mapping of the pair (α_s,γ), and A_s specializes to $A_0 := A_{\alpha,\gamma}$. In this section, we will deal with the construction of A_s. In the next section, we will construct the family of vanishing cycles α_s on the quartic $X_s \bigcap H_{t_0}$.

Consider the total space of the family of quartic threefolds over a small disk.

$$\mathscr{X} = \{(x,s) \in \mathbb{P}^4 \times \Delta | (sF_X + F_Y F_H)(x) = 0\} \to \Delta,\ (x,s) \mapsto s.$$

The total space is singular along $\{s = 0, F_X = 0, F_Y = 0, F_H = 0\}$, which is a smooth curve E of genus 19 in $Y \bigcap P$.

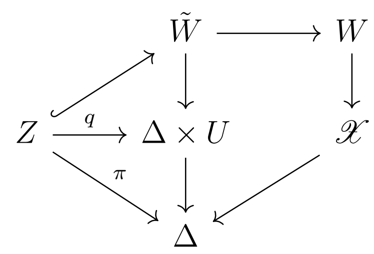
For the reason of Hodge theory, we want a smooth total space carrying the degeneration, and the special fiber should be a normal crossing divisor, so the information of weight filtration will be related to the geometry of the special fiber, so we need to resolve the total space.

The singularity is a nondegenerate node along a transversal hyperplane, so we can produce a small resolution on the total space by blowing up threefold in W which contains E. A good thing for small resolution is that the special fiber will be normal crossing with *two* components (comparatively, blowup along E will produce an extra component), so the weight filtration of the limiting mixed Hodge structure is easy to describe. Since our proof of Proposition 6.6.1 relies heavily on the knowledge of the weight filtration, we prefer a small resolution in this situation.

To produce such a small resolution, for example, we can blow up P in $\mathscr{X}$. As P is a divisor, the blowup does not change the $\mathscr{X}$ outside the singular locus E. In fact, it replaces E with a $\mathbb{P}^1$-bundle over E. We denote the new total space as W, with projection $W \to \Delta$. So fiber over Δ^* stays the same as smooth quartic threefold X_s, while the central fiber is isomorphic to $Y \bigcup \tilde{P}$, where $\tilde{P}$ is the blowup of P along the curve E. So we write $W = \bigcup_{s\in\Delta} \tilde{X}_s$.

Now consider the $\tilde{W} = \{\big((x,s),t\big) \in W \times U | x \in \tilde{X}_s \bigcap H_t\}$ which blows up the base locus on the pencil $\mathbb{P}^1$. So there is a commutative diagram

By taking an open neighborhood $\mathcal{U}$ of $Y \bigcap \tilde{P}$ away from the total space $\tilde{W}$, the map $\tilde{W} \setminus \mathcal{U} \to \Delta \times U$ is a submersion. By composing with projection $\Delta \times U \to \Delta$ and up to



shrinking to a smaller disk of Δ containing origin, the fiber of $\tilde{W} \setminus \mathcal{U} \to \Delta$ has two disjoint components $\tilde{W}'_s$ and $\tilde{W}''_s$. On the special fiber, $\tilde{W}'_0$ (resp. $\tilde{W}''_0$) is blowup along base locus of open submanifold of Y (resp. $\tilde{P}$) away from $Y \bigcap \tilde{P}$. We denote Z the union $\bigcup_{s\in\Delta} \tilde{W}'_s$ and denote q the projection of Z to $\Delta \times U$ and π to Δ, respectively.

Claim 6.6.7. *Both q and π are C^∞-locally trivial.*

Proof. By considering the closure $\bar{Z}$ of Z inside $\tilde{W}$ and extending the two projections to $\bar{Z} \bigcap \pi^{-1}(\Delta)$ are both proper and submersive along the boundary and interior, so it satisfies the assumption of Theorem B.0.1 in the appendix, so both q and π are C^∞-locally trivial. □

Proposition 6.6.8. *Up to shrinking to a smaller disk of Δ, there is a fiber preserving diffeomorphism*

$$\psi : Z \xrightarrow{\cong} Z_0 \times \Delta$$

such that $\pi_{Z_0} \circ \psi(Z_s \bigcap H_t) \subseteq Z_0 \bigcap H_t$ for all $t \in U$. In other words, the ψ is a trivialization which preserves hyperplane sections.

Proof. By shrinking U to a smaller open neighborhood U' whose closure is contained in U, we can find finitely many open covering $U_1, ..., U_N$ of U' and a smaller open disk $\Delta' \subseteq \Delta$ containing origin such that the q map over $\Delta' \times U_i$ is a trivial family. In other words, there is a diagram

where $t_i \in U_i$ and $q^{-1}(U_t) = Z|_{U_t}$.

$$\begin{array}{ccc} Z|_{U_i} & \xrightarrow[\cong]{\psi_i} & (\tilde{W}'_0 \bigcap H_{t_i}) \times \Delta' \times U_i \\ & \searrow^{q} & \downarrow \\ & & \Delta' \times U_i \\ & \searrow^{\pi} & \downarrow \\ & & \Delta' \end{array}$$

Denote ψ_i the trivialization on $Z|_{U_i}$. Note that it preserves hyperplane sections. To construct a trivialization globally on Z, we need to use *partition of unity*. To be more precise, let (x_1, x_2) be a real coordinate on Δ' and $\partial/\partial x_j$, $j = 1, 2$ a constant real vector field on Δ'. Pullback to the product $(\tilde{W}'_0 \bigcap H_t) \times \Delta' \times U_i$ and then pushforward via ψ_i^{-1}. Now we get a vector field v^i_j on $Z|_{U_i}$ whose horizontal part is $\partial/\partial x_j$. Now choose a partition of unity of Z with respect to the open covering $Z|_{U_i}$, we get smooth functions f_i supported in $\{Z|_{U_i}\}$ such that $\sum f_i \equiv 1$. It follows that $v_j := \sum_i f_i v^i_j$ defines a vector field on Z globally with constant horizontal part $\partial/\partial x_j$. Let ϕ_v denote the one parameter group of diffeomorphism generated by a vector field v. This induces a desired fiber preserving diffeomorphism

$$\psi : Z_0 \times \Delta' \cong Z$$

$$(z, ax_1 + bx_2) \mapsto \phi_{av_1 + bv_2}(z)$$

□

Proposition (6.6.8) above allows us to define a family of 3 cycles A_λ on $Z_s \subseteq \tilde{W}_s$ via the following way: Denote $\psi_s = \psi(\cdot, s)$ for $s \in \Delta$ the diffeomorphism. Let α be a vanishing cycle supported on $Z_0 \bigcap H_{t_0}$ and $\gamma(0) = \gamma(1) = t_0$ be a loop on U satisfying $[\gamma] \cdot \alpha = \alpha$ and $A_0 = A$ the 3-cycle as tube mapping associated to the pair (α, γ) supported on the open submanifold $Y \setminus \mathcal{U}(P)$. Define $A_s := \psi_s(A)$ the 3-cycle on $Z_s \subseteq X_s$.

Corollary 6.6.9. *The* 3*-cycles* A_s *on* $Z_s \subseteq X_s$ *are Tube mapping associated to a pair* (α, γ)*.*

Now we are ready to finish the proof of Proposition 6.6.1.

Proof of Proposition 6.6.1. As we have shown above, there is a 3-cycle A_s in the quartic 3-fold X_s for $s \in \Delta$ as tube mapping of a pair (α_s, γ), where $\alpha_s \in H_2(X_s \bigcap H_{t_0}, \mathbb{Z})_{\text{van}}$ and $\gamma \subseteq U$ is a loop based at t_0 which fixes α via monodromy action. So it suffices to show that A_s is not a zero class in $H_3(X_s, \mathbb{Z})$ for some $s \neq 0$.

Recall at the beginning of Step 3, we produce a small resolution on the total space of the quartic family (6.11) and get a family

$$h : W \to \Delta \tag{6.13}$$

with W smooth and general fiber being X_s and special fiber being $Y \bigcup \tilde{P}$.

This is a semistable degeneration, and there is an associated limiting mixed Hodge structure H^3_{lim} with W_3 part contributed by the image of $H^3(Y \bigcup \tilde{P})$.

Another way to describe the $W_3 H^3_{\text{lim}} = H^3(Y \bigcup \tilde{P})$ is by considering the invariant sections on a local system: Denote $h' : W^* \to \Delta^*$ the restriction of (6.13). Then the invariant sections on $j_* R^3 h'_* \underline{\mathbb{Z}}$ are identified with points on $i^* j_* R^3 h'_* \underline{\mathbb{Z}}$, which are precisely $W_3 H^3_{\text{lim}}$.

On the other hand, $H^3(Y \bigcup \tilde{P})$ fits into an exact sequence

$$0 \to H^2(Y \bigcap \tilde{P}, \mathbb{Z})_{\text{van}} \to H^3(Y \bigcup \tilde{P}, \mathbb{Z}) \to H^3(Y, \mathbb{Z})_{\text{prim}} \oplus H^3(\tilde{P}, \mathbb{Z})_{\text{prim}} \to 0.$$

Since by our construction, the 3-cycle A_s specializes to A_0 contained in $Y \setminus \mathcal{U}(P)$ via a family of 3-cycles A_s defined in the Corollary 6.6.9 above, so A_0 is a primitive cohomololgy class on Y and nonzero, in particular $A_0 \in W_3 H^3_{\text{lim}} = i^* j_* R^3 h'_* \underline{\mathbb{Z}}$ according to the exact sequence above, so the 3-cycles A_s defines a section in

$$\eta : \Delta \to j_* R^3 h'_* \underline{\mathbb{Z}}$$

with $\eta(0) \neq 0$. It follows that $\eta(s)$ is not zero for $s \neq 0$ close enough to 0. In particular, for such s, A_s is not a zero class in $H_3(X_s, \mathbb{Z})$. $\square$

Appendix A: Degeneration of Dual Varieties

A.1 Overview

The motivation of this chapter is the following, consider a family of plane conics $C_t = \{xy + tz^2 = 0\} \subset \mathbb{P}^2$ with t varying in a neighborhood of 0. Then for $t \neq 0$, the dual curve is a smooth conic, but when $t = 0$, the conic consists of two lines, whose dual is the set of two points. The dual map on each individual fiber does not need to have the same dimension. However, if we look at the dual curves C_t^* in family, which is $\{4tuv + w^2\} \subset (\mathbb{P}^2)^*$ and is a flat family. Take limit as $t \to 0$, we still get a degree two curve $\{w^2 = 0\}$. We'd like to use the second situation to define the dual curve in the limit.

More generally, let X_d be a degree $d \geq 2$ smooth hypersurface of $\mathbb{P}^{n+1}$ defined by F_d; X_{d_1}, X_{d_2} be smooth hypersurfaces of degree d_1 and d_2 respectively defined by F_{d_1} and F_{d_2}, with $d = d_1 + d_2$.

We require F_d, F_{d_1}, F_{d_2} to be general, so that their common zero locus is a complete intersection. Besides, by Bertini's theorem, $F^s := sF_d + F_{d_1}F_{d_2}$ is smooth for $s \neq 0$ when $|s|$ is small enough. So for such $s \neq 0$, there is a dual map on smooth hypersurface $X^s := \{F^s = 0\}$

$$\mathcal{D}_s : X^s \mapsto (\mathbb{P}^{n+1})^*$$

$$x \mapsto \Big(\frac{\partial F^s}{\partial x_0}(x), ..., \frac{\partial F^s}{\partial x_{n+1}}(x)\Big) \tag{A.1}$$

with $\frac{\partial F^s}{\partial x_j}(x) = s\frac{\partial F_d}{\partial x_j}(x) + F_{d_2}\frac{\partial F_{d_1}}{\partial x_j}(x) + F_{d_1}\frac{\partial F_{d_2}}{\partial x_j}(x)$, $j = 0, ..., n+1$ by direct computation.

The image $(X^s)^*$ is called the dual variety of X^s and it is well known that it is a hypersurface of degree $m = d(d-1)^n$ in the dual space. So this defines a rational section μ on the sheaf $S^m(V^*) \otimes \mathcal{O}_\Delta$ over Δ which has possibly a pole along $s = 0$ where $V = \mathbb{C}^{n+2}$, but by multiplying by a suitable power of s, we can assume the section μ is regular and $\mu(0) \neq 0$. This will not change the defining hypersurface in projective space, so it defines a hypersurface.

Definition A.1.1. *Define $(X^0)^*$ to be the projective hypersurface $\{\mu = 0\} \subseteq (\mathbb{P}^{n+1})^*$ and call it the dual variety in the limit associated to the family $sF_d + F_{d_1}F_{d_2}$.*

The purpose of this section is to understand different connected components of $(X^0)^*$ and its corresponding multiplicities.

Theorem A.1.2. *The dual variety in the limit $(X^0)^*$ associated to the family $\{F^s = 0\}_{s\in\Delta}$ is a reducible hypersurface in $(\mathbb{P}^{n+1})^*$. When $n = 1$, $(X^0)^*$ is a curve with components:*

I. dual variety of X_{d_1} and X_{d_2}, reduced;

II. dual variety of $X_{d_1} \bigcap X_{d_2}$, each component acquiring with multiplicity two.

When $n \geq 2$, $(X^0)^*$ consists of components of type I, II as above, together with

III. dual variety of $X_d \bigcap X_{d_1} \bigcap X_{d_2}$, reduced.

Moreover if $d_i = 1$, then the corresponding component $X^*_{d_i}$ is trivial (dual variety of a hyperplane is a point) and the components in (II) and (III) are cones over dual variety in the hyperplane X_{d_i}.

Example A.1.3. *Consider the family of smooth cubic curves degenerate into a conic Q union a line L intersecting transversely, for an explicit example*

$$F^s(x_0, x_1, x_2) = s(x_0^3 + x_1^3 + x_2^3) + x_0(x_0^2 + x_1^2 + x_2^2) = 0.$$

Let (u, v, w) be the coordinates on dual space $(\mathbb{P}^2)^*$, then the dual variety in the limit $(X^0)^*$ consists of (1) a conic as dual curve of Q; (2) lines $v = \pm iw$ with multiplicity two.

Since the dual curve of a smooth cubic has degree 6, the decomposition reads $6 = 2+2\times 2$.

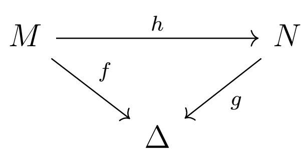
Example A.1.4. *Consider the family* $sF_5+F_3F_2 = 0$ *of quintic surface degenerate to quadric surface union a cubic surface.*

The dual variety in the limit $(X^0)^*$ *consists of (1) dual variety of cubic, which has degree 12; (2) dual variety of quadric, which is again a quadric; (3) dual variety of* $X_3 \cap X_2$*, which has degree 18, with multiplicity two; (4) dual variety of* $X_5 \bigcap X_3 \bigcap X_2$*, which is the set of hyperplanes through the 30 points, namely union of 30 hyperplanes in* $(\mathbb{P}^3)^*$.

Since the dual variety of quintic surface has degree 80, the decomposition reads $80 = 12+2+2\times 18+30$.

A.2 Multiplicity Counting

Assume M, N are complex varieties and are flat over Δ via f, g. h is a regular map making the diagram commutes:



Let Z be a component of $M_0 = f^{-1}(0)$, let's assume that $h(Z)$ is a component of $N_0 = g^{-1}(0)$.

The multiplicity m_Z of Z is the order of vanishing f^*t on Z, where t is the local equation of $0 \in \Delta$. Similarly the multiplicity $n_{h(Z)}$ of $h(Z)$ is the order of vanishing of g^*t on $h(Z)$. Then by $f^* = h^* \circ g^*$, we have the equality

$$n_Z = k \cdot m_{h(Z)}, \tag{A.2}$$

where k is the ramification index of h at the component Z, which can be defined as following: Choose $p \in Z$ a general point, and Δ_p a holomorphic disk in M which intersects Z transversly at p, then the restriction $h|_{\Delta_p}$ is a k-to-1 map onto its image.

So we immediately have the following argument:

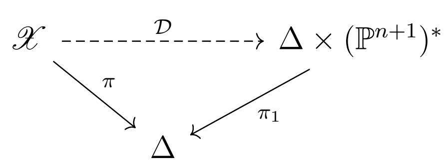
Proposition A.2.1. *If h has ramification index one along Z, then then $n_{h(Z)}$ coincides with m_Z.*

A.3 Proof of Theorem A.1.2

We define the total space of family

$$\mathscr{X} = \{sF_d + F_{d_1}F_{d_2} = 0\} \subset \Delta \times \mathbb{P}^{n+1} \tag{A.3}$$

over the disk Δ with $\pi : \mathscr{X} \to \Delta$ the projection map. So the fiber over s is the hypersurface $\{F^s = 0\}$ and the special fiber $F^0 = F_{d_1}F_{d_2}$ is reducible. The total space $\mathscr{X}$ is singular along $S := \{s = 0, F_d = 0, F_{d_1} = 0, F_{d_2} = 0\}$ since it has local analytic equation $sx + yz = 0$. There is a commutative diagram



where $\mathcal{D} : (s, p) \mapsto (\frac{\partial F^s}{\partial x_0}(p), ..., \frac{\partial F^s}{\partial x_{n+1}}(p))$ is the dual map on each fiber, which is regular outside the locus $C := \{s = 0, F_{d_1} = 0, F_{d_2} = 0\}$.

Identify S (resp. C) with the complete intersection $X_d \bigcap X_{d_1} \bigcap X_{d_2}$ (resp. $X_{d_1} \bigcap X_{d_2}$) in $\mathbb{P}^{n+1}$. We want to reach a diagram as we discussed in the previous section. To do this, we need to resolve the singular locus of $\mathscr{X}$ and the indeterminacy locus of $\mathcal{D}$. So we blowup $\mathscr{X}$

along S and then blowup the strict transform of the indeterminacy locus C to get a smooth total space $\tilde{\mathscr{X}}$ with regular dual map $\tilde{\mathcal{D}}$, and reach a diagram

$$\begin{array}{ccc} \tilde{\mathscr{X}} & & \\ \downarrow \lambda & \searrow^{\tilde{\mathcal{D}}} & \\ \mathscr{X}' & \overset{\mathcal{D}'}{\dashrightarrow} & \Delta \times (\mathbb{P}^{n+1})^* \\ \downarrow \pi' & \swarrow_{\pi_1} & \\ \Delta & & \end{array}$$

where π' is the composite of π and the blowup $\sigma : \mathscr{X}' \to \mathscr{X}$. We first prove that

Proposition A.3.1. *$\tilde{D}$ has multiplicity index one on each component $\tilde{\mathscr{X}}^0$.*

Proof. This is due to both π and π_1 are projections to the first factors Δ, moreover $\mathcal{D}$ is the identity map on this factor. □

Proposition A.3.2. *The special fiber $\tilde{\mathscr{X}}^0 := (\lambda \circ \pi')^{-1}(0)$ has the following irreducible components:*

I. strict transforms of X_{d_1} and X_{d_2}, reduced;

II. the exceptional divisor $\tilde{C}$ over the strict transform of C, multiplicity two;

III. the exceptional divisor $\tilde{S}$ over S, reduced.

Moreover, their image under $\tilde{\mathcal{D}}$ are corresponding dual varieties of type I-III stated in Theorem A.1.2.

Proof. It suffices to prove types II and III. According to Proposition 2, it suffices to show $\tilde{C}$ and $\tilde{S}$ have the corresponding multiplicities, and their image is the corresponding dual varieties of type II and III.

The local analytic equation of $q_0 \in C$ in $\mathscr{X}$ is,

$$u = 0, \ v = 0$$

in the hypersurface $\{sf + uv = 0\} \subseteq \Delta^3_{s,u,v} \times \Delta^{n-1}$, where $f = f(s,u,v)$ is an analytic function with $f(0,0,0) \neq 0$ and the last of $n-1$ variables are free. If $q_0 \in C \setminus S$, then it has a neighborhood unaffected by the first blowup σ, so the multiplicity of $\tilde{C}$ is the same as the multiplicity of the exceptional divisor of blowup of $(0,0,0)$ of $\{sf + uv = 0\} \subseteq \Delta^3_{s,u,v}$, which is *two* by straightforward computation.

Now let's prove the image of $\tilde{C}$ under $\tilde{\mathcal{D}}$ is the dual variety of $X_{d_1} \bigcap X_{d_2}$. The total transform of λ has local analytic equation over a point $q_0 \in C \setminus S$

$$sf + uv = 0 \text{ , rank} \begin{bmatrix} \alpha & \beta & \gamma \\ u & v & s \end{bmatrix} \leq 1, \ (\alpha, \beta, \gamma) \in \mathbb{P}^2.$$

Set $\beta = 1$, so $\alpha v = u$, $s = v\gamma$, $v(\gamma f + \alpha v) = 0$ and the map is

$$\tilde{\mathcal{D}} : (\gamma, v, \alpha) \mapsto \big(\gamma \frac{\partial F_d}{\partial x_j}(q) + \frac{\partial F_{d_1}}{\partial x_j}(q) + \alpha \frac{\partial F_{d_2}}{\partial x_j}(q)\big)_{j=0,\ldots,n+1} \in (\mathbb{P}^{n+1})^*, \tag{A.4}$$

where the local coordinate of q depends on γ, v, α. As v goes to zero, by equation $\gamma f + \alpha v = 0$, γ goes to 0 and we have the dual map on the exceptional divisor

$$\tilde{\mathcal{D}}(q_0) = \big(\frac{\partial F_{d_1}}{\partial x_j}(q_0) + \alpha \frac{\partial F_{d_2}}{\partial x_j}(q_0)\big)_{j=0,\ldots,n+1} \in (\mathbb{P}^{n+1})^*.$$

Similarly on chart $\alpha = 1$. It follows that the image at $q_0 \in X_{d_1} \bigcap X_{d_2} \setminus X \bigcap X_{d_1} \bigcap X_{d_2}$ is the set of linear combination of normal vectors along F_{d_1} and F_{d_2} at q_0, which form a Zariski open dense subset of $(X_{d_1} \bigcap X_{d_2})^*$. This finish the proof of the case II.

In case III (only exist when $n \geq 2$), we choose a point $p_0 \in S$, then S has local analytic equation in $\mathscr{X}$

$$x = 0, \ y = 0, \ z = 0$$

in $\{sx + yz = 0\} \subseteq \Delta^4_{s,x,y,z} \times \Delta^{n-2}$ where the last $n-2$ variables are free. Since the exceptional divisior $\tilde{S} = \sigma^{-1}(S)$ is generically unaffected by the second blowup λ, the multiplicity of $\tilde{S}$ is computed over a general point $p_0 \in S$. Again, this multiplicity coincides with the multiplicity of the exceptional divisor of the blowup of origin of $sx + yz = 0$, which is *one*.

Finally, let's show that the image of $\tilde{S}$ under $\tilde{\mathcal{D}}$ is the dual variety of $X_d \bigcap X_{d_1} \bigcap X_{d_2}$. (By abuse of notation, $\tilde{S}$ is also denoted as its strict transform under the blowup λ.) The total transform of σ has local equation over a general point $p_0 \in S$

$$sx + yz = 0, \ \text{rank} \begin{bmatrix} \alpha & \beta & \gamma & \delta \\ s & x & y & z \end{bmatrix} \leq 1, \ (\alpha, \beta, \gamma, \delta) \in \mathbb{P}^3. \tag{A.5}$$

If we choose an affine chart $\alpha = 1$, then the equation $(A.5)$ becomes $x = s\beta, y = s\gamma, z = s\delta$ and $s^2(\beta + \gamma\delta) = 0$, so by substitution and scaling in projective coordinate, the dual map $(A.1)$ becomes

$$\mathcal{D}' : (s; \gamma, \delta) \mapsto \big(\frac{\partial F_d}{\partial x_j}(p) + \gamma \frac{\partial F_{d_1}}{\partial x_j}(p) + \delta \frac{\partial F_{d_2}}{\partial x_j}(p)\big)_{j=0,\ldots,n+1} \in (\mathbb{P}^{n+1})^* \tag{A.6}$$

around a point $p_0 \in S$ and the local coordinate of p depends on s, γ, δ. So $(A.6)$ implies that the dual map extends to (an Zariski open subset of) $\sigma^{-1}(S)$ by

$$S \times \mathbb{C}^2 \ni (p_0, \gamma, \delta) \mapsto \big(\frac{\partial F_d}{\partial x_j}(p_0) + \gamma \frac{\partial F_{d_1}}{\partial x_j}(p_0) + \delta \frac{\partial F_{d_2}}{\partial x_j}(p_0)\big)_{j=0,\ldots,n+1} \in (\mathbb{P}^{n+1})^*. \tag{A.7}$$

Note the map is well-defined on this chart due to the assumption that S is smooth complete intersection, so three normal directions $\frac{\partial F_d}{\partial x_j}, \frac{\partial F_{d_1}}{\partial x_j}, \frac{\partial F_{d_2}}{\partial x_j}$ are linearly independent. This shows that the image of $\tilde{S}$ under $\tilde{D}$ contains a Zariski dense subset of $(X_d \bigcap X_{d_1} \bigcap X_{d_2})^*$, so it has to be the whole dual variety.

□

Since the dual varieties $\{(X^s)^*\}_{s \in \Delta}$ as we defined in the previous section is a flat family, so in particular each member has the same degree. In what follows, we will show that the sum of degree of the components of three types agrees with degree of the nearby fiber. This proves that $(X^0)^*$ has no other components, therefore will complete the proof of Theorem 1.

Proposition A.3.3. *The following identities holds:*

$$\deg(X_d^*) = \deg(X_{d_1}^*) + \deg(X_{d_2}^*) + 2\deg((X_{d_1} \bigcap X_{d_2})^*), \textit{ if } n = 1;$$
$$\deg(X_d^*) = \deg(X_{d_1}^*) + \deg(X_{d_2}^*) + \deg((X_d \bigcap X_{d_1} \bigcap X_{d_2})^*) + 2\deg((X_{d_1} \bigcap X_{d_2})^*), \textit{ if } n \geq 2. \tag{A.8}$$

Proof. First of all X_d^* has degree $d(d-1)^n$, and $X_{d_i}^*$ has degree $d_i(d_i-1)^n$, $i = 1, 2$. So $n = 1$ case is the consequence of the identity

$$d(d-1) = d_1(d_1-1) + d_2(d_2-1) + 2d_1d_2,$$

which one can readily check by hand.

For $n \geq 2$ case, we need a formula for dual variety of a complete intersection. According to page 362 of Kleiman's *The enumerative theory of singularities* [38], If $Y \subset \mathbb{P}^N$ is a smooth variety of dimension m with dual Y^* being a hypersurface, then $\deg(Y^*)$ coincides with $\int_Y s_m(E)$, with $E = N^*_{Y|\mathbb{P}^N} \otimes \mathcal{O}_Y(1)$ where s_m is the Segree class and $N^*_{Y|\mathbb{P}^N}$ is the conormal bundle. This uses the fact that the dual variety Y^* is the image of $\mathbb{P}_Y(E)$ and $\deg(Y^*)$ is the $(m-1)$-fold intersection of hyperplane class with Y^*. So we have

Lemma A.3.4. *Let Y be a complete intersection of type $(d_1, ..., d_k)$ of dimension n, so $E = \oplus_i \mathcal{O}_Y(1-d_i)$ and $\deg(Y^*)$ coincides with the coefficient of h^n of the polynomial*

$$\prod_{i=1}^{k}(1-(d_i-1)h)^{-1}\prod_{i=1}^{k} d_i.$$

Apply the formula to $X_d \bigcap X_{d_1} \bigcap X_{d_2}$, which has dimension $n-3$, so we get its degree of dual variety

$$\deg((X_d \bigcap X_{d_1} \bigcap X_{d_2})^*) := N^{n-3}_{d,d_1,d_2} = \sum_{\substack{i+j+k=n-3 \\ i,j,k \geq 0}} (d_1-1)^i(d_2-1)^j(d-1)^k d_1 d_2 d.$$

Similarly to the complete intersection $X_{d_1}\bigcap X_{d_2}$ of dimension $n-2$, the degree of its dual variety is

$$\deg((X_{d_1}\bigcap X_{d_2})^*) := N^{n-2}_{d_1,d_2} = \sum_{\substack{i+j=n-2\\ i,j\geq 0}} (d_1-1)^i(d_2-1)^j d_1 d_2.$$

So the proof of Corollary $A.3.3$ reduces to show the equality

$$d(d-1)^n = d_1(d_1-1)^n + d_2(d_2-1)^n + N^{n-2}_{d,d_1,d_2} + 2N^{n-1}_{d_1,d_2} \tag{A.9}$$

We prove by induction on n. The base case is $n=2$, one readily check the following identity holds

$$d(d-1)^2 = d_1(d_1-1)^2 + d_2(d_2-1)^2 + 2(d_1+d_2-2)d_1d_2 + dd_1d_2.$$

We want to show the equality $(A.9)$. By assuming the equality is true for $n-1$ case, so we have

$$\begin{aligned} d(d-1)^n &= d(d-1)^{n-1}(d-1) = [d_1(d_1-1)^{n-1} + d_2(d_2-1)^{n-1} + N^{n-3}_{d,d_1,d_2} + 2N^{n-2}_{d_1,d_2}](d-1) \\ &= d_1(d_1-1)^n + d_2(d_2-1)^n + d_1d_2[(d_1-1)^{n-1} + (d_2-1)^{n-1}] + (d-1)(N^{n-3}_{d,d_1,d_2} + 2N^{n-2}_{d_1,d_2}). \end{aligned}$$

So it reduces to show the equality

$$N^{n-2}_{d,d_1,d_2} + 2N^{n-1}_{d_1,d_2} = (d-1)(N^{n-3}_{d,d_1,d_2} + 2N^{n-2}_{d_1,d_2}) + d_1d_2[(d_1-1)^{n-1} + (d_2-1)^{n-1}],$$

which is a consequence of identity $(d-1)N^{n-3}_{d,d_1,d_2} = N^{n-2}_{d,d_1,d_2} - dN^{n-2}_{d_1,d_2}$ with direct computation. This finishes the proof. □

Appendix B: A Theorem in Differential Topology

Theorem B.0.1. *(Ehresmann's theorem for manifolds with boundary) Let M be a smooth manifold possibly with boundary, and B is a smooth manifold such that $\partial B = \emptyset$. Let $\pi : M \to B$ be a proper smooth map such that the restriction to the interior $\pi|_{M^\circ}$ and restriction to the boundary $\pi|_{\partial M}$ are submersion, then M is locally trivial over B, that is, for each $b \in B$, there exists an open neighborhood U of b such that there is a diffeomorphism*

$$\Psi : \pi^{-1}(U) \cong M_0 \times U$$

such that $\pi = p_2 \circ \Psi$, where $M_0 = \pi^{-1}(b)$ and p_2 is the projection to the second factor.

Bibliography

[1] M. Altavilla, M. Petkovic, and F. Rota. Moduli spaces on the kuznetsov component of fano threefolds of index 2. *Preprint. arXiv:1908.10986 v3*, 2021.

[2] Allen B. Altman and Steven L. Kleiman. Foundations of the theory of Fano schemes. *Compos. Math.*, 34:3–47, 1977.

[3] V. I. Arnold, S. M. Gusein-Zade, and A. N. Varchenko. *Singularities of differentiable maps, Volume 2. Monodromy and asymptotics of integrals. Transl. from the Russian by Hugh Porteous and revised by the authors and James Montaldi.* Mod. Birkhäuser Classics. Boston, MA: Birkhäuser, 2012.

[4] M. Artin. Algebraic construction of Brieskorn's resolutions. *J. Algebra*, 29:330–348, 1974.

[5] Wolf P. Barth, Klaus Hulek, Chris A. M. Peters, and Antonius Van de Ven. *Compact complex surfaces*, volume 4 of *Ergeb. Math. Grenzgeb., 3. Folge.* Berlin: Springer, 2004.

[6] A. Bayer, S. Beentjes, S. Feyzbakhsh, G. Hein, D. Martinelli, F. Rezaee, and B. Schmidt. The desingularization of the theta divisor of a cubic threefold as a moduli space. *Preprint. arXiv:2011.12240*, 2020.

[7] A. Beauville. Les singularités du diviseur Θ de la jacobienne intermédiaire de l'hypersurface cubique dans $\mathbb{P}^4$. In *Algebraic threefolds (Varenna, 1981)*, volume 947 of *Lecture Notes in Math.*, pages 190–208. Springer, Berlin-New York, 1982.

[8] Mauro C. Beltrametti and Andrew J. Sommese. *The adjunction theory of complex projective varieties*, volume 16 of *De Gruyter Expo. Math.* Berlin: de Gruyter, 1995.

[9] G. G. Bockondas and S. Boissiere. Triple lines on a cubic threefold. *Preprint. arXiv:2201.08884 v2*, 2022.

[10] Raoul Bott and Loring W. Tu. *Differential forms in algebraic topology*, volume 82 of *Grad. Texts Math.* Springer, Cham, 1982.

[11] E. Brieskorn. Singular elements of semi-simple algebraic groups. In *Actes du Congrès International des Mathématiciens (Nice, 1970), Tome 2*, pages 279–284. Gauthier-Villars, Paris, 1971.

[12] J. W. Bruce and C. T. C. Wall. On the classification of cubic surfaces. *J. Lond. Math. Soc., II. Ser.*, 19:245–256, 1979.

[13] James A. Carlson. Extensions of mixed Hodge structures. Journees de geometrie algebrique, Angers/France 1979, 107-127 (1980)., 1980.

[14] Sebastian Casalaina-Martin and Zheng Zhang. The moduli space of cubic surface pairs via the intermediate Jacobians of Eckardt cubic threefolds. *J. Lond. Math. Soc., II. Ser.*, 104(1):1–34, 2021.

[15] A. Cayley. A memoir on cubic surfaces. *Philos. Trans. R. Soc. Lond.*, 159:231–326, 1869.

[16] Dawei Chen, Izzet Coskun, and Scott Nollet. Hilbert scheme of a pair of codimension two linear subspaces. *Commun. Algebra*, 39(8):3021–3043, 2011.

[17] Y. Cheng. Hyperplane sections of hypersurfaces. *Preprint. arXiv:2001.10983 v2*, 2020.

[18] C. H. Clemens. Degeneration of Kähler manifolds. *Duke Math. J.*, 44:215–290, 1977.

[19] C. H. Clemens and Ph. A. Griffiths. The intermediate Jacobian of the cubic threefold. Matematika, Moskva 17, No. 1, 3-41 (1973)., 1973.

[20] Pierre Deligne. Théorie de Hodge. II. (Hodge theory. II). *Publ. Math., Inst. Hautes Étud. Sci.*, 40:5–57, 1971.

[21] G. Dethloff and H. Grauert. Seminormal complex spaces. In *Several complex variables VII. Sheaf-theoretical methods in complex analysis*, pages 183–220. Berlin: Springer-Verlag, 1994.

[22] Igor V. Dolgachev. *Classical algebraic geometry. A modern view.* Cambridge: Cambridge University Press, 2012.

[23] Alan H. Durfee. Fifteen characterizations of rational double points and simple critical points. *Enseign. Math. (2)*, 25:132–163, 1979.

[24] Hélène Esnault and Eckart Viehweg. Deligne-Beilinson cohomology. Beilinson's conjectures on special values of L-functions, Meet. Oberwolfach/FRG 1986, Perspect. Math. 4, 43-91 (1988)., 1988.

[25] Otto Forster. *Lectures on Riemann surfaces. Transl. from the German by Bruce Gilligan*, volume 81 of *Grad. Texts Math.* Springer, Cham, 1981.

[26] Claude Godbillon. *Éléments de topologie algébrique.* Paris: Hermann, 1998.

[27] Hans Grauert and Reinhold Remmert. Komplexe Räume. *Math. Ann.*, 136:245–318, 1958.

[28] Hans Grauert and Reinhold Remmert. *Coherent analytic sheaves*, volume 265 of *Grundlehren Math. Wiss.* Springer, Cham, 1984.

[29] Phillip Griffiths and Joseph Harris. *Principles of algebraic geometry.* New York, NY: John Wiley & Sons Ltd., 1994.

[30] Phillip A. Griffiths. On the periods of certain rational integrals. I, II. *Ann. Math. (2)*, 90:460–495, 496–541, 1969.

[31] Phillip A. Griffiths. *Introduction to algebraic curves. Transl. from the Chinese by Kuniko Weltin.*, volume 76 of *Transl. Math. Monogr.* Providence, RI: American Mathematical Society, 1996.

[32] S. M. Gusein-Zade. Dynkin diagrams in singularity theory. In *Lie groups and Lie algebras: E. B. Dynkin's seminar*, pages 33–42. Providence, RI: American Mathematical Society, 1995.

[33] Robin Hartshorne. *Algebraic geometry*, volume 52 of *Grad. Texts Math.* Springer, Cham, 1977.

[34] James E. Humphreys. *Introduction to Lie algebras and representation theory. 3rd printing, rev*, volume 9 of *Grad. Texts Math.* Springer, Cham, 1980.

[35] D. Huybrechts. Geometry of cubic hypersurfaces. `https://www.math.uni-bonn.de/people/huybrech/Notes.pdf`, 2022. Online notes.

[36] Uwe Jannsen. Deligne homology, Hodge-$\mathcal{D}$-conjecture, and motives. Beilinson's conjectures on special values of L-functions, Meet. Oberwolfach/FRG 1986, Perspect. Math. 4, 305-372 (1988)., 1988.

[37] Matt Kerr, James D. Lewis, and Stefan Müller-Stach. The Abel-Jacobi map for higher Chow groups. *Compos. Math.*, 142(2):374–396, 2006.

[38] Steven L. Kleiman. The enumerative theory of singularities. Real and compl. Singul., Proc. Nordic Summer Sch., Symp. Math., Oslo 1976, 297-396 (1977)., 1977.

[39] János Kollár. *Rational curves on algebraic varieties*, volume 32 of *Ergeb. Math. Grenzgeb., 3. Folge.* Berlin: Springer-Verlag, 1995.

[40] János Kollár and Shigefumi Mori. *Birational geometry of algebraic varieties. With the collaboration of C. H. Clemens and A. Corti*, volume 134 of *Camb. Tracts Math.* Cambridge: Cambridge University Press, 1998.

[41] Alexander G. Kuznetsov, Yuri G. Prokhorov, and Constantin A. Shramov. Hilbert schemes of lines and conics and automorphism groups of Fano threefolds. *Jpn. J. Math. (3)*, 13(1):109–185, 2018.

[42] A. Landman. Examples of varieties with small dual varieties. picard-lefschetz theory and dual varieties, 1976. Unpublished notes.

[43] Y.A. Lee. The hilbert schemes of curves in $\mathbb{P}^3$. Bachelor's thesis, Harvard University, 2000.

[44] Christian Lehn, Manfred Lehn, Christoph Sorger, and Duco van Straten. Twisted cubics on cubic fourfolds. *J. Reine Angew. Math.*, 731:87–128, 2017.

[45] David R. Morrison. The Clemens-Schmid exact sequence and applications. Topics in transcendental algebraic geometry, Ann. Math. Stud. 106, 101-119 (1984)., 1984.

[46] D. Mumford. Rational equivalence of 0-cycles on surfaces. Matematika, Moskva 16, No. 2, 3-10 (1972)., 1972.

[47] Chris A. M. Peters and Joseph H. M. Steenbrink. *Mixed Hodge structures*, volume 52 of *Ergeb. Math. Grenzgeb., 3. Folge.* Berlin: Springer, 2008.

[48] Ritvik Ramkumar. The Hilbert scheme of a pair of linear spaces. *Math. Z.*, 300(1):493–540, 2022.

[49] Xavier Roulleau. Elliptic curve configurations on Fano surfaces. *Manuscr. Math.*, 129(3):381–399, 2009.

[50] Christian Schnell. Private communication. An email to Xiaolei Zhao on Mar 12, 2014.

[51] Christian Schnell. Primitive cohomology and the tube mapping. *Math. Z.*, 268(3-4):1069–1089, 2011.

[52] Christian Schnell. Residues and filtered D-modules. *Math. Ann.*, 354(2):727–763, 2012.

[53] Christian Schnell. The extended locus of hodge classes. *Preprint. arXiv:1401.7303*, 2014.

[54] B. Segre. The non-singular cubic surfaces. A new method of investigation with special reference to questions of reality. XI +180 p. Oxford, Clarendon Press. London, Oxford University Press (1942)., 1942.

[55] Evgeny Shinder and Andrey Soldatenkov. On the geometry of the Lehn-Lehn-Sorger-van Straten eightfold. *Kyoto J. Math.*, 57(4):789–806, 2017.

[56] Karl Stein. Analytische Zerlegungen komplexer Räume. *Math. Ann.*, 132:63–93, 1956.

[57] Tomohide Terasoma. Complete intersections with middle Picard number 1 defined over $\mathbb{Q}$. *Math. Z.*, 189:289–296, 1985.

[58] A. N. Tyurin. The geometry of the Fano surface of a nonsingular cubic $\mathrm{F} \subset \mathrm{p}^4$ and Torelli theorems for Fano surfaces and cubics. *Math. USSR, Izv.*, 5:517–546, 1972.

[59] Nikolaos Tziolas. Multiplicities of smooth rational curves on singular local complete intersection Calabi-Yau threefolds. *J. Algebr. Geom.*, 10(3):497–513, 2001.

[60] Nikolaos Tziolas. Infinitesimal extensions of $\mathbb{P}^1$ and their Hilbert schemes. *Manuscr. Math.*, 108(4):461–482, 2002.

[61] Claire Voisin. *Hodge theory and complex algebraic geometry. I. Translated from the French by Leila Schneps*, volume 76 of *Camb. Stud. Adv. Math.* Cambridge: Cambridge University Press, 2002.

[62] Claire Voisin. *Hodge theory and complex algebraic geometry. II. Transl. from the French by Leila Schneps*, volume 77 of *Camb. Stud. Adv. Math.* Cambridge: Cambridge University Press, 2003.

[63] X. Zhao. *Topological Abel-Jacobi Mapping and Jacobi Inversion.* PhD thesis, The University of Michigan, 2015.

[64] Steven Zucker. The Hodge conjecture for cubic fourfolds. *Compos. Math.*, 34:199–209, 1977.