Operational and quantum K-theory of toric varieties

Dissertation

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By

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Abstract

This thesis contains two different projects on toric geometry. The first project describes the operational K-theory introduced by D. Anderson and S. Payne in [3] for toric varieties, via the introduction of a ring of Grothendieck weights. We prove several properties of Grothendieck weights, which combinatorially characterize them in low dimensions. The second project introduces generalized Rogers-Szegő polynomials, which depend on the data of a smooth lattice polytope P. For P an interval these specialize to the polynomials studied in [39]. We prove a q-series identity for these functions involving certain q-hypergeometric functions introduced in [25] and separately in [32]. The identity is a q-deformation of the well-known identity of Brion [9] in Ehrhart theory, and is proved via equivariant K-theory on quasimap spaces. We finish by proving some combinatorial properties of generalized Rogers-Szegő polynomials.

To the memory of Ajita Shah

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Chapter 1

Introduction

This thesis contains the results of two mostly disjoint projects on different aspects of the K-theory of toric varieties. In the first, consisting of material from [37] we consider the operational K-theory op$K^\circ(X)$ introduced by D. Anderson and S. Payne in [3]. This theory has close ties to the Grothendieck group of coherent sheaves, and chief among the results from the first project of this thesis is a presentation of latter, for complete toric varieties of dimension up to three. Besides this, we provide a combinatorial formula for products of classes in op$K^\circ(X)$, as well as formulae for maps between operational K-theory and other invariants. As a corollary we deduce a result about vector bundles on toric surfaces.

The second part consists of work from [4], which is joint with Dave Anderson. We take our inspiration from the well-known identity by M. Brion on lattice point generating functions of polytopes, which was first proved using equivariant K-theory on the associated toric variety. We find, by calculating a limit of holomorphic Euler characteristics of certain line bundles on quasimap spaces, a q-analogue of the lattice point generating function of a polytope. We study the behavior of these functions and prove a q-analogue of Brion's identity for these functions. We remark that in certain cases these functions coincide with q-Whittaker functions as introduced in [20] and linked to quasimap spaces of flag varieties in [8].

Both of the following chapters will heavily use the notation of toric varieties and K-theory. Here we establish our conventions and recall some basic propositions that we will need to refer to.

1.1 Fans from toric varieties

Let T be an algebraic torus over $\mathbb{C}$. For the purposes of this thesis, a toric variety is a normal complex algebraic variety with a T-action and a dense orbit. This elementary assumption leads to a rich combinatorial theory which we outline below. By modding out by a generic stabilizer, we can assume that T embeds as the (open) dense orbit. All varieties from here onwards will be toric varieties unless otherwise stated. By and large we follow the conventions of [13] and [15].

Let $M = \mathrm{Hom}_{alg.gp.}(\mathrm{T}, \mathbb{C}^*)$ be the character lattice of T, and $N = \mathrm{Hom}_{alg.gp.}(\mathbb{C}^*, \mathrm{T})$ the cocharacter lattice. There is the natural pairing $\langle \ , \ \rangle : M \times N \to Hom_{alg.gp.}(\mathbb{C}^*, \mathbb{C}^*) \cong \mathbb{Z}$ given by composition of maps, that makes M and N dual to each other. We let $M_{\mathbb{R}}$ and $N_{\mathbb{R}}$ denote $M \otimes_{\mathbb{Z}} \mathbb{R}$ and $N \otimes_{\mathbb{Z}} \mathbb{R}$ respectively.

Definition 1.1.1. A *cone* in a real vector space is a convex subset closed under scaling by positive real numbers. A cone is *rational polyhedral* if it is the intersection of finitely many closed half-spaces defined over $\mathbb{Q}$. A cone is *strongly convex* if it does not contain any non-trivial linear subspaces of the ambient vector space. If C is a cone and F is a linear form such that $F(C) \geqslant 0$, then $\{F = 0\}$ is a *supporting hyperplane* of C, and $\{F = 0\} \cap C$ is a *face* of C.

We say a convex set C is a *cone with vertex* v if the translation $C - v$ is a cone in the sense defined above.

Given a toric variety X, we obtain a set of strongly convex rational polyhedral cones in

$N_{\mathbb{R}}$ in the following manner. Since a cocharacter $n = n(t)$ is a function from $\mathbb{C}^*$ to T, we can define the set N° as the subset of $n(t) \in N$ such that $\lim_{t\to 0} n(t)$ is well-defined in X. We can partition N° into equivalence classes N°_α, defined by the condition that $n(t), n'(t)$ are equivalent when $lim_{t\to 0} n(t) = lim_{t\to 0} n'(t)$.

Now, it is standard notation to let greek letters refer to cones, so let α itself denote the closure of the cone generated by N°_α. Given X, we denote the set of all cones obtained in this way by Δ. The set Δ is an instance of a *polyhedral fan*.

Definition 1.1.2. A polyhedral fan Δ is a finite set of polyhedral cones in a real vector space such that

- for α in Δ, any face of α is also in Δ, and
- for α, β in Δ, the intersection $\alpha \cap \beta$ is a face of both α and β.

One may also start with a strongly convex rational polyhedral fan Δ (meaning a polyhedral fan with strongly convex rational cones) in $N_{\mathbb{R}}$, and obtain a toric variety: given a rational polyhedral cone α, the dual cone $\alpha^\vee$ in $M_{\mathbb{R}}$ is the subset of all m such that $m(\alpha) \geqslant 0$. The lattice points $M \cap \alpha^\vee$ form a finitely-generated semigroup S_α, and $U_\alpha = \operatorname{Spec} \mathbb{C}[S_\alpha]$ is an affine variety. As α varies in Δ, the affine varieties U_α glue together to form a toric variety X. The procedure described here to obtain a toric variety from a fan is inverse to the one described above to obtain a fan from a toric variety.

Inside U_α, there is a T-invariant closed orbit O_α. The closure of O_α in X is a T-invariant subvariety denoted by $V(\alpha)$. In fact, $V(\alpha)$ is also a toric variety, and if we denote the stabilizer of O_α by T_{O_α}, we may naturally identify the dense torus of $V(\alpha)$ with the quotient $T_\alpha = T/T_{O_\alpha}$. The character lattice for T_α is $M_\alpha = \alpha^\perp \subset M$. We denote the $\mathbb{Z}$-span of lattice points in α by N^α, and denote the quotient by $N_\alpha = N/N^\alpha \cong M_\alpha^\vee$. The lattice N_α can be identified with the lattice of one-parameter subgroups of T_α. We use $\langle , \rangle$ to denote the pairing between M_α and N_α. We denote by $\Delta(k)$ the set of all k-dimensional cones in Δ. If

α is a face of β, we write $\alpha \prec \beta$. If $\alpha \precneqq \beta$ and α is maximal among cones contained in β, we write $\alpha \precdot \beta$. When $\alpha \prec \beta$, the image of β in N_α is a cone which we denote by $\overline{\beta}$. If $\alpha \precdot \beta$, $\overline{\beta}$ is a ray, whose primitive generator we denote by $v_{\beta,\alpha}$, or v_β if $\alpha = \{0\}$.

Proposition 1.1.3. *There is a correspondence between strongly convex rational polyhedral fans in $N_\mathbb{R}$ and toric varieties which compactify T. For such a fan Δ and toric variety X, there is a containment-reversing bijection between cones in Δ and T-invariant subvarieties of X. The k-dimensional cone α corresponds to the codimension-k T-invariant subvariety $V(\alpha)$.*

For further details, see the excellent references on toric varieties [13] and [15]. We establish the not-very-standard convention here that we exclusively use ρ to denote 1-dimensional cones, and σ to denote maximal cones.

We will also need the notion of multiplicity. The Hilbert-Samuel multiplicity of a variety X along a subvariety Y is a measure of how singular X is along Y. If A is the local ring of X along Y and $\mathscr{M}$ the maximal ideal, then $A \supset \mathscr{M} \supset \mathscr{M}^2 \supset \dots$ is a filtration. The leading term of the Hilbert polynomial of the associated graded module is a polynomial in t of degree $d = \dim(X) - \dim(Y)$. Then, $d!$ times the leading term is the *Hilbert-Samuel multiplicity*, see [16, Section 4.3].

If β is a simplicial cone with extremal rays generated by $v_1, \dots, v_k$, its multiplicity is $\mathrm{mult}(\beta) := [N^\beta : \mathbb{Z}v_1 + \dots + \mathbb{Z}v_k]$. If $\alpha \prec \beta$, let $\mathrm{mult}_\alpha(\beta)$ denote the multiplicity of $\overline{\beta}$ in N_α, so $\mathrm{mult}(\beta) = \mathrm{mult}_{\{0\}}(\beta)$. Geometrically, $\mathrm{mult}_\alpha(\beta)$ is the Hilbert-Samuel multiplicity of $U_\beta \cap V(\alpha)$ along $V(\beta)$. In the appendix, we show how to write a relative multiplicity $\mathrm{mult}_\alpha(\beta)$ in terms of usual multiplicities.

1.2 Polytopes and ample line bundles

Given a polytope P in $M_{\mathbb{R}}$, let Δ denote its corresponding *inward normal fan* in $N_{\mathbb{R}}$. The fan Δ consists of cones C_F for each face F of P. The cone C_F is defined as the set of $n \in N_{\mathbb{R}}$ satisfying $\langle n, m_P - m_F \rangle \geqslant 0$, for m_P and m_F arbitrary elements of P and F respectively. If P has rational vertices, its inward normal fan is rational as well, and if P is full-dimensional, the cones in its normal fan are strongly convex.

Unless specified otherwise, a "polytope in M" is a full-dimensional polytope in $M_{\mathbb{R}}$ with vertices in M. Given a polytope P in M, the corresponding normal fan Δ is rational, and we can define a toric variety X. The polytope P also induces a T-invariant divisor D_P on X: if F is a facet of P, then C_F is a ray in Δ. Let v_F be the primitive element of N in C_F, and let a_F be the smallest number such that $a_F + \langle v_F, m \rangle \geqslant 0$ for m any element of P. Then set $D_P = \sum_F a_F [V(C_F)]$, where the sum is over all facets of P.

From this divisor we obtain a corresponding invertible sheaf $\mathscr{O}(D_P)$ on X. Viewed as a subsheaf of the sheaf of rational functions on X, $\mathscr{O}(D_P)$ is generated on the affine open U_{C_F} by the rational function corresponding to any $m \in F$. The following proposition addresses the sheaf cohomology groups $H^i(X, \mathscr{O}(D_P))$.

Proposition 1.2.1. *The invertible sheaf $\mathscr{O}(D_P)$ is ample, and $H^0(X, \mathscr{O}(D_P))$ has a basis over $\mathbb{C}$ corresponding to $m \in P$. For $i > 0$, $H^i(X, \mathscr{O}(D_P))$ vanishes.*

The proposition is a consequence of basic theorems about sheaf cohomology of toric divisors. See e.g. [13, Chapter 9]

1.3 K-theory

The Grothendieck ring of vector bundles on an algebraic variety X is denoted by $K^\circ(X)$. It is generated as an abelian group by isomorphism classes of vector bundles, modulo relations

$[F] = [E] + [G]$ for exact sequences $0 \to E \to F \to G \to 0$. The product of classes $[E]$ and $[F]$ is the class of the tensor product $[E \otimes F]$. For an arbitrary morphism of algebraic varieties $f : Y \to X$ and a vector bundle E on X, there is the pullback vector bundle f^*E on Y, which induces a map $f^* : K^\circ(X) \to K^\circ(Y)$.

There is also the Grothendieck ring of perfect complexes $K^\circ_{perf}(X)$, which is generated by complexes locally quasi-isomorphic to a finite complex of vector bundles, but we will not use this.

The Grothendieck group of coherent sheaves on X is denoted by $K_\circ(X)$, and defined in the same manner as $K^\circ(X)$, replacing "vector bundle" with "coherent sheaf." For $f : Y \to X$ proper and $\mathscr{G}$ coherent on Y, there is a pushforward $f_*[\mathscr{G}] = \sum_i (-1)^i [R^i f_* \mathscr{G}]$. For $f : Y \to X$ flat and $\mathscr{F}$ coherent on X there is a pullback $f^*[\mathscr{F}] = [f^*\mathscr{F}]$.

Though for coherent sheaves $\mathscr{E}$ and $\mathscr{F}$ the tensor product $\mathscr{E} \otimes \mathscr{F}$ is coherent, a naively defined product is not well-defined on equivalence classes of coherent sheaves. In the next chapter we discuss this further.

If X is smooth, then $K^\circ(X)$ and $K_\circ(X)$ are isomorphic as groups (see, e.g. [44, Chapter II, Theorem 8.2], but otherwise this is not true. For example, it is an easy exercise that for X the nodal rational curve over $\mathbb{C}$, the Picard group of X is uncountable. The same is then true for $K^\circ(X)$, which has a surjection onto $\mathrm{Pic}(X)$ via the map given by determinant bundles, e.g. [44, Chapter I]). On the other hand, the Grothendieck group $K_\circ(X)$ is finitely generated (by the pushforwards of structure sheaves of T-invariant subvarieties, for example).

Chapter 2

Operational K-theory

The Grothendieck group of coherent sheaves may be defined for any algebraic variety X. When X is smooth, the group $K_\circ(X)$ has a product: for two sheaves $\mathscr{F}$ and $\mathscr{G}$ in X, the product of classes $[\mathscr{F}]$ and $[\mathscr{G}]$ in $K_\circ(X)$ is the sum $\sum_i (-1)^i [\mathscr{T}or_{\mathscr{O}_X}(\mathscr{F}, \mathscr{G})]$. Unfortunately, such a product on $K_\circ(X)$ cannot be defined when X is singular, as there is no longer any reason for this sum to be finite. Instead, one may consider one of many different Grothendieck rings which agree with $K_\circ(X)$ when X is smooth: though the ring of vector bundles $K^\circ(X)$, and the ring of perfect complexes $K^\circ_{perf}(X)$ are natural to define geometrically, they can be uncountably generated [25] even for projective simplicial toric varieties. The operational K-theory ring op$K^\circ(X)$ in contrast is finitely generated for projective toric varieties as a consequence of [3, Theorem 1.3]. It is also an $\mathbb{A}^1-$homotopy invariant, like the homotopy K-theory of [43], but unlike $K^\circ(X)$ and $K^\circ_{perf}(X)$. Thus, there is reason to expect it should be more tractable to describe. We give a short introduction to operational K-theory in section 2.1 and refer to [3] for details.

We fix a complete toric variety X over $\mathbb{C}$, and Δ its polyhedral fan. By [3, Theorem 1.6], the torus-equivariant operational K-theory op$K^\circ_T(X)$ can be described by the ring of piecewise exponential functions on Δ, but this theory does not necessarily surject onto

$\mathrm{op}K^\circ(X)$ (see Example 2.5.5), so the question of describing the latter remains. Describing $\mathrm{op}K^\circ(X)$ is the focus of this chapter.

Here is one application we provide: it is already known that $\mathrm{op}K_T^\circ(X) \to \mathrm{op}K^\circ(X)$ is not always surjective for complete toric varieties. In [2, Theorem 1.7], the authors demonstrate that there is a nonsimplicial toric 3-fold for which the rank of $\mathrm{op}K_T^\circ(X)_{\mathbb{Q}}$ is strictly smaller than the rank of $\mathrm{op}K^\circ(X)_{\mathbb{Q}}$, by using an operational Riemann-Roch theorem. However, this approach cannot work to show non-surjectivity if the image of $\mathrm{op}K_T^\circ(X)$ has finite index inside $\mathrm{op}K^\circ(X)$ (for example, when X is simplicial). In Example 2.5.5, we use our description of $\mathrm{op}K^\circ(X)$ and of the map $\mathrm{op}K_T^\circ(X) \to \mathrm{op}K^\circ(X)$ to demonstrate that there is a toric surface such that the image of $\mathrm{op}K_T^\circ(X)$ has finite index inside $\mathrm{op}K^\circ(X)$. This is the first example of nonsurjectivity where the toric variety X is simplicial. We remark that in both cases the non-surjectivity of operational groups implies that the forgetful map of vector bundles $K_T^\circ(X) \to K^\circ(X)$ is also not surjective, as noted in [2].

We would also like to mention some reasons external to K-theory that motivate why one might want to study $\mathrm{op}K^\circ(X)$. The K-theoretic story addressed here echoes work on the Chow groups $A_*(X)$. In [18], Fulton and Sturmfels showed that when X is complete the operational Chow ring $A^*(X)$ is isomorphic to a ring of balanced $\mathbb{Z}$-valued functions on Δ called *Minkowski weights*, with a displacement rule for calculating products. This is now well-known in the context of tropical geometry, in which Minkowski weights appear as an instance of weighted, balanced polyhedral complexes, and the displacement rule is a special case of the intersection product for tropical cycles [1]. If one desires a K-theoretic analogue to the methods of tropical geometry, a first step would be to determine the K-theoretic analogue of Minkowski weights. Additionally, a description of $\mathrm{op}K^\circ(X)$ for projective simplicial toric varieties has a straightforward interpretation in terms of Ehrhart theory, see Proposition 2.1.4.

To motivate our results on $\mathrm{op}K^\circ(X)$, note that one way to frame the main theorem of [18] is that they *define* Minkowski weights as the set of $\mathbb{Z}$-valued functions on Δ which satisfy a balancing condition, and *prove* that (1) this ring of functions is isomorphic to $A^*(X)$, as a group and (2) the product on Minkowski weights which is compatible with that of $A^*(X)$ can be calculated by a displacement rule.

In contrast, we start by defining a ring of $\mathbb{Z}$-valued function on Δ which describes $\mathrm{op}K^\circ(X)$ tautologically. We call the elements of this ring Grothendieck weights, denoted by $\mathrm{GW}(\Delta)$, because they can naturally be identified with linear forms on the Grothendieck group $K_\circ(X)$. On the other hand, what sort of balancing conditions elements of $\mathrm{GW}(\Delta)$ should satisfy will not be clear to us a priori. Nonetheless, we provide several results in this direction, which in lower dimensions lead to a combinatorial balancing-condition characterization along the lines of Minkowski weights.

In higher dimensions, the problem becomes more difficult, primarily because our approach to characterizing Grothendieck weights relies on finding expressions for Todd classes of toric varieties in terms of T-invariant subvarieties. Though there is an extensive literature on how to do this [7, 10, 31, 35, 36], the coefficients of such an expression depend on various choices made in rewriting self-intersections of toric divisors D_i in terms of square-free monomials.

We continue our study of the ring of Grothendieck weights by addressing how to compute products. Let $c, d : \Delta \to \mathbb{Z}$ be Grothendieck weights, and let $\delta : X \to X \times X$ be the diagonal map. By applying [3, Proposition 6.4], the natural map $K_\circ(X) \otimes K_\circ(X) \to K_\circ(X \times X)$ is an isomorphism. Since the classes of pushforwards of structure sheaves $[\mathscr{O}_{V(\alpha)}]$ generate $K_\circ(X)$, any element of $K_\circ(X \times X)$ has an expression $\sum_{\alpha,\beta\in\Delta} m_{\alpha,\beta}[\mathscr{O}_{V(\alpha)}] \otimes [\mathscr{O}_{V(\beta)}]$. Then, our Theorem 2.4.2 is a K-theoretic analogue of [17, Theorem 4], which reduces the problem of calculating products to calculating the coefficients $m_{\alpha,\beta}$. In the context of Chow groups and Minkowski weights, an elegant method to calculate such coefficients was provided in [18, Theorem 4.2], via a displacement rule. In K-theory we explain one approach to doing this in subsection 2.4.5.

After our results on the structure of $\mathrm{GW}(\Delta)$, we move to the maps it has to and from other well-known fan-based invariants in section 2.5. There is a map from Minkowski weights to GW(Δ) that corresponds to the operational Riemann-Roch transformation as described in [2], we describe how to calculate the map from an expression for the Todd class. Additionally, we describe the forgetful map from $\mathrm{op}K_T^\circ(X) \to \mathrm{op}K^\circ(X)$ referenced above, via an explicit formula for the map from piecewise exponential functions on Δ to GW(Δ). Our approach here follows the results of Katz and Payne in [27].

In the literature, there are other descriptions of the K-theory of toric varieties, and we address briefly how they relate to the work present in this thesis. In the smooth case the equivariant operational K-theory ring can be identified with the ring of T-equivariant vector bundles $K_T^\circ(X)$. In this context Vezzosi and Vistoli show in [41] that $K_T^\circ(X)$ and also higher K-groups can be descibed via the Stanley-Reisner ring. The non-equivariant theory can then be described as a quotient of the Stanley-Reisner ring. To calculate a map to GW(Δ), one can simply choose a representative in $K_T^\circ(X)$ and compute its localization to obtain a piecewise exponential function. Then, the map to GW(Δ) can be computed via the description we provide.

2.1 Definitions of operational K-theory and GW(Δ)

A class c in $\mathrm{op}K^\circ(X)$ is a collection $(c_f)_f$ of endomorphisms of $K_\circ(Y)$ for each $f : Y \to X$. The collection $(c_f)_f$ must be compatible, in the sense that the maps must commute with proper pushforwards, flat pullbacks, and Gysin homomorphisms. Addition and multiplication are defined coordinate-wise, meaning

$$(c_f)_f + (d_f)_f = (c_f + d_f)_f, \text{ and}$$
$$(c_f)_f \cdot (d_f)_f = (c_f \circ d_f)_f.$$

For further details, we refer the reader to [3, Section 4]. Amazingly, the product is commutative if X admits a resolution of singularities (via the Kimura sequence [3, Proposition 5.4]).

Since we have assumed X is complete, there is the following theorem, which is a special case of [3, Theorem 6.1] that we need:

Theorem 2.1.1. *The natural map from* $\mathrm{op}K^{\circ}(X)$ *to* $K_{\circ}(X)^{\vee}$ *sending* $(c_f)_f$ *to* $\chi(c_{Id}(-))$ *is an isomorphism.*

We make the following definition,

Definition 2.1.2. Let the set of *Grothendieck weights* on Δ, denoted by $\mathrm{GW}(\Delta)$, be all $\mathbb{Z}$-valued functions on Δ of the form ϕ_f, where the value of ϕ_f on α is determined by

$$\phi_f(\alpha) = f([\mathscr{O}_{V(\alpha)}]),$$

where f is a linear form on $K_{\circ}(X)$.

In other words, elements of $\mathrm{GW}(\Delta)$ are obtained from elements of $K_{\circ}(X)^{\vee}$ by recording the value of a form on the classes $[\mathscr{O}_{V(\alpha)}]$. Since these classes generate $K_{\circ}(X)$, it follows that $\mathrm{GW}(\Delta)$ is isomorphic to both $K_{\circ}(X)^{\vee}$ and $\mathrm{op}K^{\circ}(X)$. The interesting question is then how to characterize which functions on Δ are Grothendieck weights, or alternatively, finding generating sets for the relations that hold between the classes $[\mathscr{O}_{V(\alpha)}]$ modulo torsion. To be precise:

Definition 2.1.3. Let $\mathrm{Rel}_{K_{\circ}(X)}$ be the kernel of the map from $\mathbb{Z}^{\Delta} \to K_{\circ}(X)$ that sends e_α to $[\mathscr{O}_{V(\alpha)}]$, and $\mathrm{Rel}_{K_{\circ}(X)_{\mathbb{Q}}}$ the same kernel defined over $\mathbb{Q}$.

To characterize which functions on Δ are Grothendieck weights is equivalent to finding a generating set for $\mathrm{Rel}_{K_{\circ}(X)_{\mathbb{Q}}}$. First, we explain how one can equivalently define $\mathrm{Rel}_{K_{\circ}(X)_{\mathbb{Q}}}$ via Ehrhart theory. For $\alpha \in \Delta$, let the corresponding face of P be denoted by F_α. Also, we recall that the Ehrhart polynomial $\mathrm{Ehr}_P(t)$ of P is the polynomial determined by $\mathrm{Ehr}_P(t_0) = |t_0 P \cap M|$.

Proposition 2.1.4. *Let Δ be a projective simplicial fan and let X be the corresponding toric variety. Let $\phi_P : \mathbb{Q}^\Delta \to \mathbb{Q}[t]$ be the map sending the tuple $(a_\alpha)_\alpha$ to $\sum_{\alpha\in\Delta} a_\alpha \operatorname{Ehr}_{F_\alpha}(t)$. Then*

$$\operatorname{Rel}_{K_\circ(X)_\mathbb{Q}} = \bigcap_{\substack{P \text{ normal} \\ \text{to } \Delta}} \ker(\phi_P).$$

Proof. The "$\subset$" direction is a translation of the well-known theorem of vanishing of higher cohomology for ample line bundles on toric varieties: let $\sum_{\alpha\in\Delta} a_\alpha[\mathscr{O}_{V(\alpha)}] = 0$ in $K_\circ(X)_\mathbb{Q}$, and for P any lattice polytope with normal fan Δ let D_P denote the associated divisor. For each $t_0 \geqslant 0$, we also have the divisor D_{t_0P}. Then

$$0 = \chi([\mathscr{O}(D_{t_0P})] \cdot (\sum_{\alpha\in\Delta} a_\alpha[\mathscr{O}_{V(\alpha)}])) = \sum_{\alpha\in\Delta} a_\alpha \chi([\mathscr{O}(D_{t_0P})|_{V(\alpha)}]) = \sum_{\alpha\in\Delta} a_\alpha \operatorname{Ehr}_{F_\alpha}(t_0),$$

where the last equality follows from Proposition 1.2.1. Thus the polynomial $\sum_{\alpha\in\Delta} a_\alpha \operatorname{Ehr}_{F_\alpha}(t)$ has infinitely many roots, and so it must be 0.

Now we verify the other direction: Suppose, for any polytope P, that $\sum_{\alpha\in\Delta} a_\alpha \operatorname{Ehr}_{F_\alpha}(t) = 0$. Translating to geometry, this states (when $t = 1$) that

$$0 = \sum_{\alpha\in\Delta} a_\alpha \chi(\mathscr{O}(D_P)|_{V(\alpha)}) = \chi\left(\left(\sum_{\alpha\in\Delta} a_\alpha[\mathscr{O}_{V(\alpha)}]\right) \cdot [\mathscr{O}(D_P)]\right).$$

Then, the result follows from the next lemma. ☐

Lemma 2.1.5. *Suppose Δ is a simplicial projective fan and X the corresponding toric variety. If for $x \in K_\circ(X)$, we have $\chi(x \cdot [\mathscr{O}(D_P)]) = 0$ for all P with normal fan Δ, then $x = 0$ in $K_\circ(X)_\mathbb{Q}$.*

Proof. We write "deg" for the projection map from $A^*(X)$ to $A^0(X)$. When X is projective it is well-known (e.g. [28, Theorem 1.4.23]) that ample divisors generate the Néron-Severi space of divisors modulo numerical equivalence $NS(X)_\mathbb{Q}$ (which is the same as $A_{n-1}(X)_\mathbb{Q}$ if

X is toric). Additionally, on a complete toric variety $A^*(X) \cong A_*(X)^\vee$ via $c \to \deg(c \cap -)$, by [17, Theorem 3]. Thus, if for all $c \in A^*(X)$, $\deg(c \cap y) = 0$, then y must be zero. Since X is simplicial, $A^*(X)_\mathbb{Q}$ is generated as an algebra by Chern classes of T-equivariant divisors, which are in turn additively generated by T-equivariant ample line bundles. Thus, $A^*(X)_\mathbb{Q}$ is generated over $\mathbb{Q}$ by 1 and monomials in $c_1(\mathscr{O}(D_P))$ for P normal to Δ. Equivalently, $A^*(X)_\mathbb{Q}$ is generated by 1 and $1 + c_1(\mathscr{O}(D_P)) = \mathrm{ch}(\mathscr{O}(D_P))$ for $\mathrm{ch} : K^\circ(X) \to A^*(X)_\mathbb{Q}$ the Chern character map. Since ch is a ring homomorphism, $\mathrm{ch}(\mathscr{O}(D_P))\,\mathrm{ch}(\mathscr{O}(D_Q)) = \mathrm{ch}(\mathscr{O}(D_P) \otimes \mathscr{O}(D_Q))$. But $\mathscr{O}(D_P) \otimes \mathscr{O}(D_Q) = \mathscr{O}(D_{P+Q})$, for $P + Q$ the Minkowski sum of P and Q. Thus, y is zero if $\deg(y) = 0$ and

$$\deg(\mathrm{ch}(\mathscr{O}(D_P)) \cap y) = 0,$$

for all P. By Riemann-Roch for algebraic schemes as in [16, Chapter 18],

$$\deg(\mathrm{ch}(\mathscr{O}(D_P)) \cap y) = \deg(\tau_X(\mathscr{O}(D_P) \otimes \tau_X^{-1}(y))) = \chi(\mathscr{O}(D_P) \otimes \tau_X^{-1}(y)).$$

Since τ_X is an isomorphism between $K_\circ(X)_\mathbb{Q}$ and $A_*(X)_\mathbb{Q}$, the lemma and proposition are proved. □

2.2 Some properties of Grothendieck weights and low-dimensional toric varieties

In this section we start by proving some properties of Grothendieck weights. For fans of dimension at most 3, these properties will be enough to characterize Grothendieck weights. In constrast to the next section, the main theorem will not require substantial choices for computing Todd classes. Recall that the Riemann-Roch transformation is a map $\tau_X : K_\circ(X) \to A_*(X)_\mathbb{Q}$, which becomes an isomorphism after tensoring $K_\circ(X)$ with $\mathbb{Q}$. The map commutes with proper pushforwards, and for a vector bundle E, there is the equality $\tau_X([E]) = \mathrm{ch}(E) \cdot td(X)$. For details regarding the Todd class $td(X)$ for singular varieties and this version of the Riemann-Roch theorem, see [16, Chapter 18].

First, we require some lemmas. The first one follows from pushing forward the Todd class of a resolution (see, e.g. [14]).

Lemma 2.2.1. *Let α be in $\Delta(k)$, and $\tau_X : K_\circ(X) \to A_*(X)_\mathbb{Q}$ the Riemann-Roch transformation. Then,*

$$\tau_X(\mathscr{O}_{V(\alpha)}) = [V(\alpha)] + \left(\sum_{\alpha \lessdot \beta} \frac{1}{2}[V(\beta)]\right) + c,$$

where $c \in (A_0(X) \oplus \ldots \oplus A_{n-k-2}(X))_\mathbb{Q}$.

Lemma 2.2.2. *Let $\sum_\alpha a_\alpha e_\alpha$ be an element of $\mathrm{Rel}_{K_\circ(X)}$. Then, $\sum_\alpha a_\alpha = 0$.*

Proof. Let $\pi : X \to pt$. Since $\sum_\alpha a_\alpha[\mathscr{O}_{V(\alpha)}] = 0$ in $K_\circ(X)$, the expression $\pi_*(\sum_\alpha a_\alpha[\mathscr{O}_{V(\alpha)}]) = 0$ as well. Since π_* agrees with the Euler characteristic and $V(\alpha)$ is toric, we know $\pi_*[\mathscr{O}_{V(\alpha)}] = 1$, so the lemma follows. □

To proceed, we explain our basic strategy. If we choose a suitable lift of τ_X to an endomorphism of $\mathbb{Q}^\Delta$, we obtain an isomorphism of exact sequences

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Rel}_{K_\circ(X)_\mathbb{Q}} & \longrightarrow & \mathbb{Q}^\Delta & \longrightarrow & K_\circ(X)_\mathbb{Q} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{Rel}_{A_*(X)_\mathbb{Q}} & \longrightarrow & \mathbb{Q}^\Delta & \longrightarrow & A_*(X)_\mathbb{Q} & \longrightarrow & 0,
\end{array}$$

and we can obtain a set of generators of $\mathrm{Rel}_{K_\circ(X)_\mathbb{Q}}$ as the inverse images of generators of $\mathrm{Rel}_{A_*(X)_\mathbb{Q}}$. By Lemma 2.2.1, we can fix a lift $\tau_\dagger$ of τ_X which maps e_α to $e_\alpha + \sum_{\alpha \lessdot \beta} \frac{1}{2} e_\beta + \ldots$, leaving unspecified the coefficients of e_γ for γ containing α as a face of codimension $\geqslant 2$.

Since the Chow groups are graded, $\mathrm{Rel}_{A_*(X)_\mathbb{Q}}$ splits as a direct sum $\oplus_i \mathrm{Rel}_{A_i(X)_\mathbb{Q}}$, and a restatement of [18, Proposition 2.1(b)] is that $\mathrm{Rel}_{A_i(X)}$ is generated by

$$\sum_{\alpha \prec \beta} \langle u, n_{\beta,\alpha} \rangle e_\beta,$$

as α varies among cones of codimension $i+1$ and u varies in $M(\alpha)$ (so e.g. $\mathrm{Rel}_{A_n(X)}$ is trivial).

Now, let us demonstrate some propositions about Grothendieck weights.

Proposition 2.2.3. *A Grothendieck weight is constant on maximal cones.*

Proof. Though this can be shown directly from geometry, one method of proof that is more in line with the next few propositions is to use the lift $\tau_\dagger$ of the Riemann-Roch transformation that we have chosen. We know by [18, Proposition 2.1(b)] cited above that $\mathrm{Rel}_{A_0(X)}$ is generated by $e_\sigma - e_{\sigma'}$, and its inverse image in $\mathrm{Rel}_{K_\circ(X)_\mathbb{Q}}$ is simply $e_\sigma - e_{\sigma'}$ again. This imposes that $g(\sigma) = g(\sigma')$ for any Grothendieck weight. □

The second proposition is about relations between codimension 1 cones:

Proposition 2.2.4. *Let g be a Grothendieck weight, $\sigma \in \Delta(n)$ any maximal cone, and $\alpha \in \Delta(n-2)$. Then*

$$\sum_{\alpha \prec \beta} (g(\beta) - g(\sigma)) v_{\beta,\alpha} = 0.$$

Proof. $\mathrm{Rel}_{A_1(X)}$ contains $\sum_{\alpha \prec \beta} \langle u, v_{\beta,\alpha} \rangle e_\beta$ for each u in $M(\alpha)$. The inverse image of e_β with respect to $\tau_\dagger$ is $e_\beta - \frac{1}{2} e_{\sigma_1} - \frac{1}{2} e_{\sigma_2}$ for σ_1, σ_2 the two maximal cones that contain β. Since $e_\sigma - e_{\sigma_i}$ is in $\mathrm{Rel}_{K_\circ(X)}$, we can see that $\sum_{\alpha \prec \beta} \langle u, v_{\beta,\alpha} \rangle (e_\beta - e_\sigma)$ is also contained in $\mathrm{Rel}_{K_\circ(X)}$. This implies that $\sum_{\alpha \prec \beta} \langle u, v_{\beta,\alpha} \rangle (g(\beta) - g(\sigma)) = 0$ for each $u \in M(\alpha)$, which is equivalent to the Proposition. □

The third one, in the same pattern, is about relations between codimension 2 cones in Δ.

Proposition 2.2.5. *Let g be a Grothendieck weight, $\sigma \in \Delta(n)$ any maximal cone, and $\alpha \in \Delta(n-3)$. Then*

$$\sum_{\alpha \prec \beta} \left(g(\beta) - \sum_{\beta \prec \gamma} \frac{g(\gamma)}{2} \right) v_{\beta,\alpha} = g(\sigma) \left(\sum_{\alpha \prec \beta} (1 - \sum_{\beta \prec \gamma} \frac{1}{2}) v_{\beta,\alpha} \right).$$

Proof. The element $\sum_{\alpha \prec \beta} \langle u, n_{\beta,\alpha} \rangle e_\beta$ is in $\mathrm{Rel}_{A_2(X)}$, so the inverse image in $\mathrm{Rel}_{K_\circ(X)_\mathbb{Q}}$ has the form

$$\sum_\rho \langle u, n_\rho \rangle (e_\rho - (\sum_{\rho \prec \beta} \frac{1}{2} e_\beta)) + \sum_{\sigma \text{ maximal}} a_\sigma e_\sigma$$

for some coefficients a_σ. We can change these generators by multiples of $e_\sigma - e_{\sigma'}$ to obtain that the following is in $\mathrm{Rel}_{K_\circ(X)_\mathbb{Q}}$,

$$\sum_\rho \langle u, n_\rho \rangle (e_\rho - (\sum_{\rho \prec \beta} \frac{1}{2} e_\beta)) + (\sum_{\sigma \text{ maximal}} a_\sigma) e_{\sigma_0}$$

for any chosen maximal cone σ_0. Lemma 2.2.2 implies that

$$\sum_{\sigma \text{ maximal}} a_\sigma = - \sum_\rho \langle u, n_\rho \rangle (1 - (\sum_{\rho \prec \beta} \frac{1}{2})).$$

The presence of this element in $\mathrm{Rel}_{K_\circ(X)}$ implies that a Grothendieck weight g must satisfy

$$\sum_\rho \langle u, n_\rho \rangle (g(\rho) - (\sum_{\rho \prec \beta} \frac{1}{2} g(\beta))) = g(\sigma) \sum_\rho \langle u, n_\rho \rangle (1 - (\sum_{\rho \prec \beta} \frac{1}{2})),$$

which is equivalent to the condition in the proposition. $\square$

In fact, these conditions are enough to characterize Grothendieck weights in low dimensions.

Theorem 2.2.6. *1. A $\mathbb{Z}$-valued function on the fan of $\mathbb{P}^1$ is a Grothendieck weight if and only if it is constant on maximal cones.*

2. A $\mathbb{Z}$-valued function g on Δ the fan of a toric surface is a Grothendieck weight if and only if it is constant on maximal cones, and

$$\sum_{\rho \in \Delta(1)} (g(\rho) - g(\sigma))\, v_\rho = 0,$$

for σ any maximal cone.

3. A $\mathbb{Z}$-valued function g on Δ the fan of a toric threefold is a Grothendieck weight if and only if it is constant on maximal cones and, still writing σ for any maximal cone:

(a) $\sum_{\rho \prec \beta} (g(\beta) - g(\sigma))\, v_{\beta,\rho} = 0$ for any fixed ray ρ, and

(b) $\sum_{\rho \in \Delta(1)} \left(g(\rho) - \sum_{\rho \prec \alpha} \frac{g(\alpha)}{2} \right) v_\rho = g(\sigma) \left(\sum_{\rho \in \Delta(1)} (1 - \sum_{\rho \prec \alpha} \frac{1}{2}) v_\rho \right)$.

Proof. In this range, $\mathrm{Rel}_{A_*(X)} \cong \mathrm{Rel}_{A_0(X)} \oplus \mathrm{Rel}_{A_1(X)} \oplus \mathrm{Rel}_{A_2(X)}$ (the last two factors may be trivial) so the inverse images under $\tau_\dagger$ of the generators of $\mathrm{Rel}_{A_i(X)}$ generate $\mathrm{Rel}_{K_\circ(X)_\mathbb{Q}}$. $\square$

2.3 Balancing on simplicial fans

Since we have defined $\mathrm{GW}(\Delta)$ so that it describes $\mathrm{op}K^\circ(X)$, we now describe different aspects of $\mathrm{GW}(\Delta)$. First, we focus on the following, which is equivalent to the question of finding explicit balancing condition characterizations of $\mathrm{GW}(\Delta)$:

Question 2.3.1. Given Δ of arbitrary dimension, how can we calculate explicit generating sets for $\mathrm{Rel}_{K_\circ(X)_\mathbb{Q}}$?

Citing the results of [18] again, we know generating sets for $\mathrm{Rel}_{A_*(X)}$. By using the Riemann-Roch transformation we can turn generators for $\mathrm{Rel}_{A_*(X)}$ into ones for $\mathrm{Rel}_{K_\circ(X)_\mathbb{Q}}$, but the primary difficulty that must be addressed is how to choose a formula for the Todd class in terms of T-invariant subvarieties (this is intimately related to Danilov's problem, see [14, Section 11] and [7, 31, 36]). Though a canonical expression for $\tau_X(\mathscr{O}_X)$ as a polynomial in toric divisors was found by Pommersheim in [35], one must make choices to write non-squarefree monomials in terms of T-invariant subvarieties.

When Δ is simplicial, Pommersheim and Thomas introduced in [36] certain rational numbers t_ρ^α for each cone α and ray ρ such that $\rho \subset \alpha$, which depend on the choice of a generic complete flag $F_\bullet$ in $N_\mathbb{Q}$. These rational numbers will help us to write non-squarefree monomials in terms of T-invariant subvarieties.

Definition 2.3.2. Let $F_\bullet$ be a generic complete flag in $N_\mathbb{Q}$, so F_i is an i-dimensional subspace of $N_\mathbb{Q}$. Given $\alpha \in \Delta(k)$, and i from 1 to k, let v_{ρ_i} be the primitive element of the ray ρ_i of α. Then by genericity, $F_{n-k+1} \cap \mathbb{Q} \cdot \alpha$ is 1-dimensional, and so it determines a vector (unique up to scaling):

$$0 \neq \sum_{i=1}^{k} t_{\rho_i}^\alpha v_{\rho_i} \in F_{n-k+1} \cap \mathbb{Q} \cdot \alpha,$$

We only consider generic $F_\bullet$ such that all t_ρ^α are non-zero.

We use these t_ρ^α in applying the following result from [36], which calculates an explicit formula for monomials in the T-invariant divisors $[V(\rho)]$ as a $\mathbb{Q}$-linear combination of classes of subvarieties $[V(\alpha)]$. For a cone β, let P_β be the set of rays in β. Let S be some set of rays in Δ. For $\rho \in S$, let a_ρ be some positive integers and let l denote the sum, $\sum_{\rho \in S} a_\rho$. Then a restatement of [36, Theorem 3] is:

Proposition 2.3.3.

$$\prod_{\rho \in S} [V(\rho)]^{a_\rho} = \sum_{\substack{\beta \in \Delta(l) \ s.t. \\ \beta \text{ contains all } \rho \in S}} \frac{\prod_{\rho \in S} (t_\rho^\beta)^{a_\rho}}{\operatorname{mult}(\beta) \prod_{\rho \subset \beta} t_\rho^\beta} [V(\beta)].$$

Nominally, one would need to make more choices for every α in Δ to obtain an expression for $\tau_X(\mathscr{O}_{V(\alpha)})$. However, we make the following observation, which avoids this. For $\alpha \in \Delta$

and a generic flag $F_\bullet$ in N, the images $\overline{F_1} \subset \ldots \subset \overline{F_{n-\dim(\alpha)}}$ in N_α form a generic flag. Thus for β a cone containing α and ρ a ray in β not contained in α, there are also numbers $t_{\overline{\rho}}^{\overline{\beta}}$. We relate these to t_ρ^β in the next proposition.

Proposition 2.3.4. *Let $\alpha \prec \beta$ be simplicial cones in a fan Δ. Then for ρ in β not contained in α, $t_{\overline{\rho}}^{\overline{\beta}} = \frac{\mathrm{mult}(\alpha+\rho)}{\mathrm{mult}(\alpha)} t_\rho^\beta$.*

Proof. The unique vector in $\overline{F_{n-k+1}} \cap \mathbb{Q} \cdot \overline{\beta}$ is the image of the unique vector in $F_{n-k+1} \cap \mathbb{Q} \cdot \beta$, which in explicit terms is

$$\overline{\sum_{\rho \subset \beta} t_\rho^\beta v_\rho} = \sum_{\substack{\rho \subset \beta \\ \rho \not\subset \alpha}} t_\rho^\beta \overline{v_\rho}.$$

But the image of a primitive generator of a ray v_ρ is not necessarily primitive, i.e. $\overline{v_\rho} = b_\rho v_{\overline{\rho}}$ for b_ρ a positive integer. In fact, $b_\rho = [\mathbb{Z} v_{\overline{\rho}} : \mathbb{Z} \overline{v_\rho}]$ of subgroups of N_α. If $\pi_\alpha : N \to N_\alpha$ is the quotient map, then $\pi_\alpha^{-1}(\mathbb{Z} v_{\overline{\rho}}) = N^{\alpha+\rho}$, and $\pi_\alpha^{-1}(\mathbb{Z}\overline{v_\rho}) = N^\alpha + \mathbb{Z} v_\rho$. Thus $[\mathbb{Z} v_{\overline{\rho}} : \mathbb{Z}\overline{v_\rho}] = [N^{\alpha+\rho} : N^\alpha + \mathbb{Z} v_\rho]$. Then we can decompose $\mathrm{mult}(\alpha+\rho)$ as a product:

$$\begin{aligned} \mathrm{mult}(\alpha+\rho) &= [N^{\alpha+\rho} : \mathbb{Z} v_1 + \ldots + \mathbb{Z} v_k + \mathbb{Z} v_\rho] \\ &= [N^{\alpha+\rho} : N^\alpha + \mathbb{Z} v_\rho][N^\alpha + \mathbb{Z} v_\rho : \mathbb{Z} v_1 + \ldots + \mathbb{Z} v_k + \mathbb{Z} v_\rho]. \end{aligned}$$

But $[N^\alpha + \mathbb{Z} v_\rho : \mathbb{Z} v_1 + \ldots + \mathbb{Z} v_k + \mathbb{Z} v_\rho] = [N^\alpha : \mathbb{Z} v_1 + \ldots \mathbb{Z} v_k] = \mathrm{mult}(\alpha)$. Thus $[N^{\alpha+\rho} : N^\alpha + \mathbb{Z} v_\rho] = \frac{\mathrm{mult}(\alpha+\rho)}{\mathrm{mult}(\alpha)} = b_\rho$. Thus, we have

$$\sum_{\substack{\rho \subset \beta \\ \rho \not\subset \alpha}} t_\rho^\beta \overline{v_\rho} = \sum_{\substack{\rho \subset \beta \\ \rho \not\subset \alpha}} t_\rho^\beta \frac{\mathrm{mult}(\alpha+\rho)}{\mathrm{mult}(\alpha)} v_{\overline{\rho}}.$$

Since the primitive generators of the rays in $\overline{\beta}$ are the $v_{\overline{\rho}}$'s, we are done by the definition of $t_{\overline{\rho}}^{\overline{\beta}}$. ∎

Now, we use these propositions to write $\tau_X(\mathscr{O}_{V(\alpha)})$ explicitly. For each cone $\alpha \in \Delta$ Brion and Vergne defined the (finite) subgroup $G_\alpha \subset (\mathbb{C}^*)^{\dim \alpha}$ to be the kernel of the map

$(\mathbb{C}^*)^{\dim \alpha} \to T$ given by

$$(c_\rho)_\rho \mapsto \prod_{\rho \subset \alpha} v_\rho(c_\rho).$$

For nested cones $\alpha \prec \beta$, we use the notation G_β^α for the analogous subgroup defined from the data of $\overline{\beta}$ in the quotient fan. Namely:

Definition 2.3.5. Let G_β^α be the kernel of the map $(\mathbb{C}^*)^{\dim \beta - \dim \alpha} \to T_\alpha$ given by

$$(c_\rho)_{\substack{\rho \subset \beta \\ \rho \not\subset \alpha}} \mapsto \prod_{\substack{\rho \subset \beta \\ \rho \not\subset \alpha}} v_{\overline{\rho}}(c_\rho).$$

Let k be the number of rays in the quotient fan Δ_α. We define G_{Δ_α} to be the union inside $(\mathbb{C}^*)^k$ of G_β^α over all β containing α. For a ray ρ in β not contained in α, we denote by a_ρ^α the character $G_\beta^\alpha \to \mathbb{C}^*$ given by projection.

Using these numbers, we have the following proposition. Treating $t)i$ as a variable with degree 1 will refer to the degree 0 coefficient of a formal Laurent series $\psi(t_1, \ldots, t_k)$ by $\psi(t_1, \ldots, t_k)_{[0]}$, i.e. if

$$\psi = \left(\frac{1}{1 - e^{-t}}\right)\left(\frac{1}{1 - e^{-s}}\right) = \left(\frac{1}{t} + \frac{1}{2} + \frac{t}{12} + \mathcal{O}(t^2)\right)\left(\frac{1}{s} + \frac{1}{2} + \frac{s}{12} + \mathcal{O}(s^2)\right),$$

then

$$\psi_{[0]} = \frac{1}{4} + \frac{1}{12}\left(\frac{t}{s} + \frac{s}{t}\right).$$

Proposition 2.3.6. *For X a complete simplicial toric variety, the Riemann-Roch transformation has the form*

$$\tau_X([\mathscr{O}_{V(\alpha)}]) = \sum_{\alpha \prec \beta} \sum_{g \in G_\beta^\alpha} \left(\prod_{\substack{\rho \subset \beta \\ \rho \not\subset \alpha}} \frac{\operatorname{mult}(\alpha + \rho) / \operatorname{mult}(\alpha)}{1 - a_\rho^\alpha(g) e^{-mult(\alpha+\rho) t_\rho^\beta}} \right)_{[0]} \frac{[V(\beta)]}{\operatorname{mult}(\beta)}.$$

Proof. In section 4.2 of [10], the authors provide a formula for the Todd class of a complete simplicial toric variety. Applied to $V(\alpha)$, this gives:

$$\tau_{V(\alpha)}([\mathscr{O}_{V(\alpha)}]) = \sum_{g \in G_{\Delta_\alpha}} \prod_{\rho \in \Delta_\alpha(1)} \frac{[V(\rho)]}{1 - a_\rho^\alpha(g) e^{-[V(\rho)]}}.$$

Since the Todd class commutes with proper pushfoward, we have

$$\tau_X([\mathscr{O}_{V(\alpha)}]) = \sum_{g \in G_{\Delta_\alpha}} \prod_{\rho \in \Delta_\alpha(1)} i_* \frac{[V(\rho)]}{1 - a_\rho^\alpha(g) e^{-[V(\rho)]}}.$$

Applying Proposition 2.3.3, we have

$$\begin{aligned} i_* \prod_{\rho \in \Delta_\alpha(1)} \frac{[V(\rho)]}{1 - a_\rho^\alpha(g) e^{-[V(\rho)]}} &= \sum_{\alpha < \beta} \left(\prod_{\substack{\rho \subset \beta \\ \rho \not\subset \alpha}} \frac{1}{1 - a_\rho^\alpha(g) e^{-t_\rho^{\overline{\beta}}}} \right)_{[0]} \frac{i_*[V(\overline{\beta})]}{\mathrm{mult}_\alpha(\beta)}, \\ &= \sum_{\alpha < \beta} \left(\prod_{\substack{\rho \subset \beta \\ \rho \not\subset \alpha}} \frac{1}{1 - a_\rho^\alpha(g) e^{-t_\rho^{\overline{\beta}}}} \right)_{[0]} \frac{[V(\beta)]}{\mathrm{mult}_\alpha(\beta)}. \end{aligned}$$

Using the formula for $t_\rho^{\overline{\beta}}$ from the previous proposition, and the formula for $\mathrm{mult}_\alpha(\beta)$ from the appendix, we obtain that the above is equal to

$$\sum_{\alpha < \beta} \left(\prod_{\substack{\rho \subset \beta \\ \rho \not\subset \alpha}} \frac{1}{1 - a_\rho^\alpha(g) e^{-\frac{\mathrm{mult}(\alpha+\rho)}{\mathrm{mult}(\alpha)} t_\rho^\beta}} \right)_{[0]} \frac{[V(\beta)]}{\mathrm{mult}(\beta) \prod_{\substack{\rho \subset \beta \\ \rho \not\subset \alpha}} \frac{\mathrm{mult}(\alpha)}{\mathrm{mult}(\alpha+\rho)}}.$$

Due to the "degree 0" imposition, the $\mathrm{mult}(\alpha)$ factor in the exponent $e^{-\frac{\mathrm{mult}(\alpha+\rho)}{\mathrm{mult}(\alpha)} t_\rho^\beta}$ can be cancelled. Summing over $g \in G_\beta^\alpha$ gives the proposition. $\square$

Example 2.3.7. We use this proposition to calculate the t_ρ^α and Riemann-Roch matrix for a weighted projective space $X := \mathbb{P}(1, 1, 2, 3)$. Recall that the fan of X has rays $\rho_1 = (1, 0, 0), \rho_2 = (0, 1, 0), \rho_3 = (0, 0, 1)$, and $\rho_4 = (-1, -2, -3)$. The maximal cones are those generated by 3-element subsets of $\{\rho_1, \rho_2, \rho_3, \rho_4\}$. If our flag in $\mathbb{Q}^3$ is given by

$$\{0\} \subsetneq \operatorname{span}\{(a,b,c)\} \subsetneq \operatorname{span}\{(a,b,c),(d,e,f)\} \subsetneq \mathbb{Q}^3,$$

where a,b,c,d,e,f are some numbers so that (d,e,f) is not a multiple of (a,b,c), then we have expressions for t_ρ^α written in Table 2.1.

Table 2.1: Example 2.3.7

Cone (α)	Ray (ρ)	t_ρ^σ
σ_{123}	ρ_1	a
	ρ_2	b
	ρ_3	c
σ_{124}	ρ_1	$a - c/3$
	ρ_2	$b - 2c/3$
	ρ_4	$-c/3$
σ_{134}	ρ_1	$a - b/2$
	ρ_3	$c - 3b/2$
	ρ_4	$-b/2$
σ_{234}	ρ_2	$b - 2a$
	ρ_3	$c - 3a$
	ρ_4	$-a$
α_{12}	ρ_1	$af - cd$
	ρ_2	$bf - ce$
α_{13}	ρ_1	$ae - bd$
	ρ_3	$ce - bf$
α_{14}	ρ_1	$3(ae-bd) + 2(cd-af) + (bf-ce)$
	ρ_4	$bf - ce$
α_{23}	ρ_2	$bd - ae$
	ρ_3	$af - cd$
α_{24}	ρ_2	$-(3(ae-bd) + 2(cd-af) + (bf-ce))$
	ρ_4	$af - cd$
α_{34}	ρ_3	$3(ae-bd) + 2(cd-af) + (bf-ce)$
	ρ_4	$ae - bd$
ρ	ρ	1

After calculating these t_ρ^α, one can write the Todd class of each subvariety in a uniform way with rational functions in t_ρ^σ as coefficients. The column vector corresponding to the image of $[\mathscr{O}_X]$ is on the last page. For the flag specified by $(a, b, c) = (2, 3, 5)$, $(d, e, f) = (3, 5, 7)$, one obtains the Riemann-Roch matrix seen in Table 2.2.

Table 2.2: The Riemann-Roch matrix of Example 2.3.7

$_{[V(-)]}\backslash^{[\mathscr{O}_{V(-)}]}$	X	ρ_1	ρ_2	ρ_3	ρ_4	α_{12}	α_{13}	α_{14}	α_{23}	α_{24}	α_{34}	σ_{123}	σ_{124}	σ_{134}	σ_{234}
X	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ρ_1	1/2	1	0	0	0	0	0	0	0	0	0	0	0	0	0
ρ_2	1/2	0	1	0	0	0	0	0	0	0	0	0	0	0	0
ρ_3	1/2	0	0	1	0	0	0	0	0	0	0	0	0	0	0
ρ_4	1/2	0	0	0	1	0	0	0	0	0	0	0	0	0	0
α_{12}	29/48	1/2	1/2	0	0	1	0	0	0	0	0	0	0	0	0
α_{13}	29/48	1/2	0	1/2	0	0	1	0	0	0	0	0	0	0	0
α_{14}	-5/48	1/2	0	0	1/2	0	0	1	0	0	0	0	0	0	0
α_{23}	5/12	0	1/2	1/2	0	0	0	0	1	0	0	0	0	0	0
α_{24}	5/12	0	1/2	0	1/2	0	0	0	0	1	0	0	0	0	0
α_{34}	5/12	0	0	1/2	1/2	0	0	0	0	0	1	0	0	0	0
σ_{123}	31/72	79/180	59/120	3/2	0	1/2	1/2	0	1/2	0	0	1	0	0	0
σ_{124}	1/8	41/60	-11/60	0	1/12	1/2	0	1/2	0	1/2	0	0	1	0	0
σ_{134}	1/36	1/9	0	1/9	1/3	1/2	1/2	0	0	1/2	0	0	0	1	0
σ_{234}	5/12	0	11/24	11/24	5/12	0	0	0	1/2	1/2	1/2	0	0	0	1

Definition 2.3.8. Let $\mu_\alpha(\beta)$ be $\sum_{g \in G_\beta^\alpha} \left(\prod_{\substack{\rho \subset \beta \\ \rho \not\subset \alpha}} \frac{\mathrm{mult}(\alpha+\rho)/\mathrm{mult}(\alpha)}{1 - a_\rho^\alpha(g) e^{-mult(\alpha+\rho) t_\rho^\beta}} \right)_{[0]}$. Then, the matrix $(\mu_\alpha(\beta))_{\beta,\alpha}$ defines an isomorphism from $\mathbb{Q}^\Delta$ to itself lifting the Riemann-Roch isomorphism from $K_\circ(X)_\mathbb{Q}$ to $A_*(X)_\mathbb{Q}$. Let $\nu_\alpha(\beta)$ refer to the (β, α)-th entry of the inverse of $(\mu_\alpha(\beta))$, so $\tau_X^{-1}([V(\alpha)]) = \sum_{\alpha < \beta} \nu_\alpha(\beta) [\mathscr{O}_{V(\beta)}]$. In particular, $\nu_\alpha(\alpha) = 1$, and for Δ smooth and the t_ρ^α as defined in 2.3.2 we can write:

$$\nu_\alpha(\beta) = \sum_{\substack{\alpha_0 \precneqq \alpha_1 \precneqq \ldots \precneqq \alpha_k \\ \alpha_0 = \alpha, \alpha_k = \beta}} (-1)^k \left(\prod_{l=1}^{k} \left(\prod_{\rho \in \alpha_l \smallsetminus \alpha_{l-1}} \frac{1}{1 - e^{-t_\rho^{\alpha_l}}} \right)_{[0]} \right).$$

Finally, we can prove:

Theorem 2.3.9. *For Δ a complete simplicial fan, a function $g : \Delta \to \mathbb{Z}$ is a Grothendieck weight if and only if it satisfies*

$$\sum_{\alpha \lessdot \beta} \langle u, v_{\beta,\alpha} \rangle \sum_{\beta < \gamma} \nu_\beta(\gamma) g(\gamma) = 0,$$

for all $\alpha \in \Delta, u \in M(\alpha)$.

Proof. The result [18, Proposition 2.1] says that the expressions $\sum_{\alpha \lessdot \beta} \langle u, v_{\beta,\alpha} \rangle e_\beta$ generate the kernel of the map $\mathbb{Z}^\Delta \to A_*(X)$ sending e_α to $[V(\alpha)]$. Denote this kernel by $\mathrm{Rel}_{A_*(X)_\mathbb{Q}}$.

Since $\tau_X([\mathscr{O}_{V(\alpha)}]) = \sum_{\alpha < \beta} \mu_\alpha(\beta)[V(\beta)]$, we have an isomorphism of exact sequences

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Rel}_{K(X)_\mathbb{Q}} & \longrightarrow & \mathbb{Q}^\Delta & \longrightarrow & K_\circ(X)_\mathbb{Q} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{Rel}_{A_*(X)_\mathbb{Q}} & \longrightarrow & \mathbb{Q}^\Delta & \longrightarrow & A_*(X)_\mathbb{Q} & \longrightarrow & 0,
\end{array}$$

where the map on the right is τ_X, and the middle and left maps are given by sending e_α to $\sum_{\alpha < \beta} \mu_\alpha(\beta) e_\beta$. The inverse image of $\sum_{\alpha \lessdot \beta} \langle u, v_{\beta,\alpha} \rangle e_\beta \in \mathrm{Rel}_{A_*(X)_\mathbb{Q}}$ is $\sum_{\alpha \lessdot \beta} \langle u, v_{\beta,\alpha} \rangle \sum_{\beta < \gamma} \nu_\beta(\gamma) e_\gamma$, so such relations generate $\mathrm{Rel}_{K_\circ(X)_\mathbb{Q}}$, and appropriate multiples of these relations generate a finite index subgroup of $\mathrm{Rel}_{K_\circ(X)}$. Dually, $K_\circ(X)^\vee$ must then consist of linear forms sending such expressions to 0, which implies that the relations in the theorem statement characterize Grothendieck weights. □

The following lemma explains how one can approach non-simplicial fans.

Lemma 2.3.10. *Let Δ be an arbitrary fan and Δ' a smooth refinement. Then $g : \Delta \to \mathbb{Z}$ is a Grothendieck weight if and only if the function on Δ' determined by $\alpha' \to g(\alpha')$ for α the smallest cone in Δ containing α' is a Grothendieck weight on Δ'.*

Proof. Let X' and X be the corresponding toric varieties. Since Δ' is a subdivision of Δ, we have an induced T-equivariant birational morphism $\phi : X' \to X$. If α is the smallest cone in Δ containing α', standard vanishing results for toric varieties imply that $\phi_*([\mathscr{O}_{V(\alpha')}]) = [\mathscr{O}_{V(\alpha)}]$, so if we map $e_{\alpha'} \in \mathbb{Q}^{\Delta'}$ to $e_\alpha \in \mathbb{Q}^{\Delta}$, we have a map of exact sequences:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Rel_{K_0(X')} & \longrightarrow & \mathbb{Q}^{\Delta'} & \longrightarrow & K_\circ(X') & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathrm{Rel}_{K_0(X)} & \longrightarrow & \mathbb{Q}^{\Delta} & \longrightarrow & K_\circ(X) & \longrightarrow & 0. \end{array}$$

All we need to do is to show that the kernel of the middle map surjects onto the kernel of the last map. By the snake lemma, it is enough to show that $\mathrm{Rel}_{K_\circ(X')}$ surjects onto $\mathrm{Rel}_{K_\circ(X)}$. We denote the projection map from $\mathbb{Q}^{\Delta}$ to $\mathbb{Q}^{\Delta(n-l)}$ by π_l. Define a filtration on $\mathrm{Rel}_{K_\circ(X)}$ by $F_k \mathrm{Rel}_{K_\circ(X)} = \bigcap_{l>k} \ker \pi_l$. Then, for example, $F_n \mathrm{Rel}_{K_\circ(X)}$ is just $\mathrm{Rel}_{K_\circ(X)}$, and $F_0 \mathrm{Rel}_{K_\circ(X)}$ are elements of the form $e_{\sigma_1} - e_{\sigma_2}$ for σ_1, σ_2 maximal cones in Δ.

We show by induction that $F_k \mathrm{Rel}_{K_\circ(X')}$ surjects onto $F_k \mathrm{Rel}_{K_\circ(X)}$. For $k = 0$ this is clear, so we assume it is true for some k_0. Let

$$s = \left(\sum_{\alpha \in \Delta(n-k_0-1)} a_\alpha e_\alpha \right) + \left(\sum_{\substack{\alpha \in \Delta(n-l) \\ l \leqslant k_0}} a_\alpha e_\alpha \right),$$

be in $F_{k_0+1} \mathrm{Rel}_{K_\circ(X)}$. Then, in the Chow group $\sum_{\alpha \in \Delta(n-k_0-1)} a_\alpha [V(\alpha)] = 0$.

Let $\mathrm{Rel}_{A_k(X)}$ be the kernel of the map $\mathbb{Q}^{\Delta(n-k)} \to A_k(X)$ sending e_α to $[V(\alpha)]$, and define $\mathrm{Rel}_{A_k(X')}$ similarly. Since ϕ is an envelope (meaning that ϕ is proper and every subvariety of X is the birational image of some subvariety of X', see e.g. [34, Lemma 1]), the natural map from $\mathrm{Rel}_{A_k(X')}$ to $\mathrm{Rel}_{A_k(X)}$ is surjective for any k, so we can find an expression $0 = \sum_{\alpha' \in \Delta'(n-k_0-1)} a_{\alpha'} [V(\alpha')]$ such the sum of $a_{\alpha'}$ for all α' subdividing α is a_α. Then $0 = \sum_{\alpha' \in \Delta'(n-k_0-1)} a_{\alpha'} \tau_{X'}^{-1}([V(\alpha')])$ so there is a relation $r = \sum_{\alpha' \in \Delta'(n-k_0-1)} a_{\alpha'} e_{\alpha'} + \ldots$ which by the Riemann-Roch theorem is in $F_{k_0+1} \mathrm{Rel}_{K_\circ(X)}$ which does not quite map to s. However, s and the image of r only differ by an element of $F_{k_0} \mathrm{Rel}_{K_\circ(X)}$. By our induction hypothesis we are done. □

Remark 2.3.11. We considered using a different set of generators for the Grothendieck group in our definition of Grothendieck weights, e.g. ideal sheaves or canonical sheaves of invariant subvarieties. However, the problem of combinatorially describing the relations between these classes seems equally difficult.

2.4 Products

Grothendieck weights on Δ have a product induced by their isomorphism with $\mathrm{op}K^\circ(X)$. This relies on the following Künneth isomorphism, which is a special case of [3, Proposition 6.4].

Proposition 2.4.1. *The natural map $K_\circ(X) \otimes K_\circ(X) \to K_\circ(X \times X)$ is an isomorphism.*

The product of Grothendieck weights may be computed by calculating a decomposition in $K_\circ(X \times X)$ of the structure sheaf of the diagonal into structure sheaves of $T \times T$-invariant subvarieties. In a precise sense, this is a deformation of the case of Chow groups. Here is the basic theorem, which is a K-theoretic analogue of [17, Theorem 4]:

Theorem 2.4.2. *Let $\delta : X \to X \times X$ be the diagonal map. Given an expression $\delta_*(z) = \sum_i m_i a_i \otimes b_i$ with $m_i \in \mathbb{Q}$, the product of classes f and g in $\mathrm{op}K^\circ(X)$ evaluated on z satisfies:*

$$\chi((f \cdot g)_{Id}(z)) = \sum_i m_i \chi(f_{Id}(a_i))\chi(g_{Id}(b_i)).$$

Proof. To avoid putting an Id subscript under each operational class, we establish the convention for this proof that the Id subscript is implied for all operational classes which appear. For any morphism $\phi : Y \to X$, $f \in \mathrm{op}K^\circ(X)$, and $z \in K_\circ(Y)$, the identity

$$(\phi^* f)(z) = \sum \chi(f(u_i))v_i, \tag{*}$$

holds, where γ_ϕ is the graph of ϕ and $(\gamma_\phi)_*(z) = \sum u_i \otimes v_i \in K_\circ(X) \otimes K_\circ(Y)$. To prove this, let π_1 and π_2 be the projections from $X \times Y$ to X and Y. Then, one has the identities $\pi_2 \circ \gamma_\phi = id_Y$, $\pi_1 \circ \gamma_\phi = \phi$. Also, operational classes satisfy a projection formula, so we have

$$(\phi^* f)(z) = id_{Y*}((\phi^* f)(z)) = \pi_{2*}\gamma_{\phi*}((\gamma_\phi^* \pi_1^* f)(z)) = \pi_{2*}((\pi_1^* f)(\gamma_{\phi*} z)).$$

Substituting our expression for $\gamma_{\phi*} z$ and using the fact that flat pull-back and operational classes commute, we have

$$\pi_{2*}((\pi_1^* f)(\gamma_{\phi*} z)) = \pi_{2*}((\pi_1^* f)(\sum u_i \otimes v_i)) = \pi_{2*}(\sum f(u_i) \otimes v_i),$$

and finally

$$\pi_{2*}(\sum f(u_i) \otimes v_i) = \sum \chi(f(u_i)) v_i,$$

since higher direct images commute with flat pull-back. Then, in the context of the proposition we apply (*) when $\phi = \delta$. Since f, g satisfy $f = \delta^*(id \otimes f)$, we obtain

$$(f \cdot g)(z) = f(g(z)) = f(\sum_i m_i \chi(g(b_i)) a_i) = \sum_i m_i \chi(g(b_i)) f(a_i),$$

to which we apply χ to obtain the proposition. □

Remark 2.4.3. In fact, the proposition and proof as stated are valid for any complete variety which is linear in the sense of [40].

Corollary 2.4.4. *Let f and g be Grothendieck weights and $\delta : X \to X \times X$ be the diagonal map. Given an expression $\delta_*([\mathscr{O}_{V(\alpha)}]) = \sum_{\beta,\gamma} c_{\beta\gamma} [\mathscr{O}_{V(\beta)}] \otimes [\mathscr{O}_{V(\gamma)}]$ with $c_{\beta\gamma} \in \mathbb{Q}$, we have*

$$(f \cdot g)(\alpha) = \sum_{\beta,\gamma} c_{\beta\gamma} f(\beta) g(\gamma).$$

2.4.5 Decomposing diagonals and product formulas

To compute products, the data required is a suitable expression for $\delta_*([\mathscr{O}_{V(\alpha)}])$. Outside of the smooth case where one can use Poincaré duality, we do not know an easy way to do this. Since we have already addressed how to explicitly describe the Riemann-Roch transformation in Section 2.3, we apply it to finding an expression for $\delta_*([\mathscr{O}_{V(\alpha)}])$ in terms of $[\mathscr{O}_{V(\beta)}] \otimes [\mathscr{O}_{V(\gamma)}]$.

Let Δ be a simplicial fan, and suppose that f and g in $\mathrm{GW}(\Delta)$ are given. Their product may be calculated explicitly via the formula in the next theorem. As in the case of Minkowski weights, to undertake the calculation one must choose an auxiliary "displacement" vector in N which we call v. Then for three cones α, β, and γ, we define $m^{\alpha}_{\beta,\gamma}$ in the same manner as [18], by

$$m^{\alpha}_{\beta,\gamma} = \begin{cases} 0 & \text{if } \beta \cap (\gamma + v) = \varnothing, \\ [N : \mathbb{Z} \cdot \beta + \mathbb{Z} \cdot \gamma] & \text{otherwise.} \end{cases}$$

Though it is suppressed in the notation, we emphasize that $m^{\alpha}_{\beta,\gamma}$ depends on the choice of v.

Theorem 2.4.6. *The product weight $h = f \cdot g$ is given by*

$$h(\alpha) = \sum_{\alpha \prec \beta} \mu_\alpha(\beta) \sum_{\substack{\beta \prec \gamma, \epsilon \\ codim(\gamma) + codim(\epsilon) \\ = codim(\beta)}} m^{\beta}_{\gamma,\epsilon} \sum_{\substack{\gamma \prec \zeta \\ \epsilon \prec \eta}} \nu_\gamma(\zeta)\nu_\epsilon(\eta) f(\zeta) g(\eta),$$

where $\mu_\alpha(\beta), \nu_\alpha(\beta)$ were defined in 2.3.8.

Proof. Recall that $\delta : X \to X \times X$ is the diagonal map. Then, we have

$$\begin{aligned} \delta_*([\mathcal{O}_{V(\alpha)}]) &= \delta_*(\tau_X^{-1}(\tau_X([\mathcal{O}_{V(\alpha)}]))) \\ &= \tau_X^{-1}\left(\sum_{\alpha \prec \beta} \mu_\alpha(\beta)(\delta_*([V(\beta)])) \right). \end{aligned}$$

By [18, Theorem 4.2], we may use the $m^{\beta}_{\gamma,\epsilon}$ determined by our generic displacement vector to decompose each $\delta_*([V(\beta)])$, thus obtaining

$$\begin{aligned}
&\tau_X^{-1}\left(\sum_{\alpha\prec\beta}\mu_\alpha(\beta)\delta_*([V(\beta)])\right)\\
=&\tau_X^{-1}\left(\sum_{\alpha\prec\beta}\mu_\alpha(\beta)\sum_{\substack{\beta\prec\gamma,\epsilon\\ \operatorname{codim}(\gamma)+\operatorname{codim}(\epsilon)\\=\operatorname{codim}(\beta)}} m^{\beta}_{\gamma,\epsilon}[V(\gamma)\times V(\epsilon)]\right)\\
=&\sum_{\alpha\prec\beta}\mu_\alpha(\beta)\sum_{\substack{\beta\prec\gamma,\epsilon\\ \operatorname{codim}(\gamma)+\operatorname{codim}(\epsilon)\\=\operatorname{codim}(\beta)}} m^{\beta}_{\gamma,\epsilon}\tau_X^{-1}([V(\gamma)\times V(\epsilon)]).
\end{aligned}$$

But then,

$$\begin{aligned}
\tau^{-1}([V(\gamma)\times V(\epsilon)]) =&\tau^{-1}([V(\gamma)])\tau^{-1}([V(\epsilon)])\\
=&\sum_{\substack{\gamma\prec\zeta\\ \epsilon\prec\eta}}\nu_\gamma(\zeta)\nu_\epsilon(\eta)[\mathscr{O}_{V(\zeta)\times V(\eta)}].
\end{aligned}$$

So, by Corollary 2.4.4 the theorem follows. □

We can use this theorem to show the following proposition and its corollary, which are basic observations about the structure of $\mathrm{GW}(\Delta)$.

Proposition 2.4.7. *Let $\Sigma \subset \Delta$ be fans. The set of Grothendieck weights on Δ that vanish on the complement of Σ forms an ideal in* $\mathrm{GW}(\Delta)$.

Proof. Suppose we have two weights f, g, such that f vanishes on the cones of Δ. Then in general there are some coefficients C such that

$$(f \cdot g)(\alpha) = \sum_{\alpha \prec \beta} \sum_{\substack{\beta \prec \gamma, \epsilon \\ \operatorname{codim}(\gamma)+\operatorname{codim}(\epsilon) \\ =\operatorname{codim}(\beta)}} \sum_{\substack{\gamma \prec \zeta \\ \epsilon \prec \eta}} C_{\substack{\alpha,\beta,\gamma,\epsilon, \\ \zeta,\eta}} f(\zeta) g(\eta).$$

If α is not in Σ, then since $\alpha \prec \zeta$ and Σ is a fan, certainly ζ is not in Σ. Thus $f(\zeta)$ must be 0 for each term in the sum. $\square$

Corollary 2.4.8. *The ring* $\mathrm{GW}(\Delta)$ *is filtered by ideals* I_k *consisting of functions that vanish on cones of codimension less than* k*.*

Example 2.4.9. We calculate the product of the following Grothendieck weights: The first

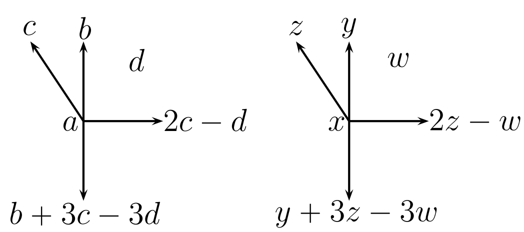


Figure 2.1: Two Grothendieck weights on the fan of a Hirzebruch surface

weight has value d on all maximal cones, and the second has value w. To calculate the product, we need a Riemann-Roch matrix. As detailed in Section 3, we require a complete flag. In two dimensions, this is merely the data of a vector, so we pick the vector $(1, 1)$. The resulting Riemann-Roch matrix is shown in Figure ??.

We must also select a displacement vector v which specifies the values of $m^{\alpha}_{\beta,\gamma}$. If we choose e.g. $v = (5, 1)$, then the non-zero $m^{\alpha}_{\beta,\gamma}$ are $m^{0}_{\rho_1,\rho_4} = 2$, and $m^{0}_{\rho_2,\rho_3} = m^{0}_{0,\sigma_{34}} = m^{0}_{\sigma_{12},0} = m^{\rho_1}_{\rho_1,\sigma_{14}} = m^{\rho_1}_{\sigma_{12},\rho_1} = m^{\rho_2}_{\rho_2,\sigma_{23}} = m^{\rho_2}_{\sigma_{12},\rho_2} = m^{\rho_3}_{\rho_3,\sigma_{34}} = m^{\rho_3}_{\sigma_{23},\rho_3} = m^{\rho_4}_{\rho_4,\sigma_{34}} = m^{\rho_4}_{\sigma_{14},\rho_4} = 1$. The resulting weight is

with a value of $aw - 2bw - 8cw + 9dw + dx + 2cy - 2dy + 2bz + 6cz - 8dz$ on the origin.

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ \frac{5}{12} & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 1 & 0 & 0 & 0 \\ \frac{1}{12} & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 1 & 0 & 0 \\ \frac{41}{60} & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 1 & 0 \\ -\frac{11}{60} & \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Figure 2.2: The matrix for a lift of the Riemann-Roch transformation on the Hirzebruch surface

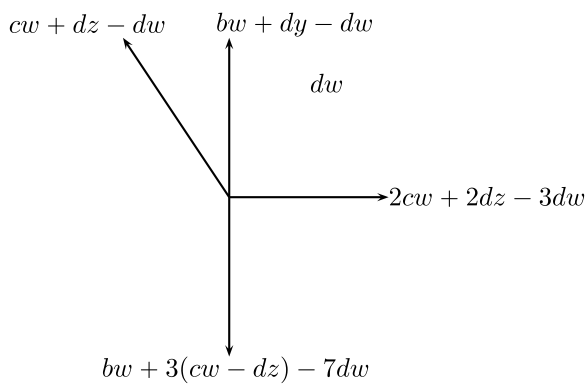


Figure 2.3: The product weight

2.5 Maps to Grothendieck weights

The most straightforward relationship to describe is the map from Minkowski weights to Grothendieck weights:

Proposition 2.5.1. *There is an induced map* $T : \mathrm{MW}^*(\Delta) \to \mathrm{GW}(\Delta)_{\mathbb{Q}}$. *For a simplicial fan this sends a weight* $f \in \mathrm{MW}^k(\Delta)$ *to the function* g, *defined by*

$$g(\alpha) = \sum_{\alpha \prec \beta} \mu_\alpha(\beta) f(\beta).$$

This is an immediate consequence of the formula the Riemann-Roch transformation $\tau_X : K_\circ(X) \to A_*(X)_{\mathbb{Q}}$ given in Proposition 2.3.6, since naturally $\mathrm{MW}^*(\Delta) \cong A_*(X)^\vee$ and $\mathrm{GW}(\Delta) \cong K_\circ(X)^\vee$.

Remark 2.5.2. It is possible to use the previous proposition to algorithmically calculate the inverse image under T of a Grothendieck weight. Let $g \in \mathrm{GW}(\Delta)$ be a Grothendieck weight, and suppose that $g \in I_k$. Then, the function on $\Delta(n-k)$ obtained by restricting g is a Minkowski weight: in the case $g \in I_{k+1}$, g vanishes on cones of dimension $n-k$, so $g|_{\Delta(n-k)}$ is uniformly 0. If $g \in I_k \backslash I_{k+1}$, then $g|_{\Delta(n-k)}$ satisfies the relation in Theorem 2.3.9 which simplifies to the condition that $g|_{\Delta(n-k)} \in \mathrm{MW}^k(\Delta)$. Then, $g - T(g|_{\Delta(n-k)})$ is an element of $\mathrm{GW}(\Delta)_{\mathbb{Q}}$, and is in $(I_{k+1})_{\mathbb{Q}}$. One may repeat this process to obtain that $g - T(g|_{\Delta(n-k)}) - T((g - T(g|_{\Delta(n-k)}))|_{\Delta(n-k-1)}) \in (I_{k+1})_{\mathbb{Q}}$, and so on. After $n-k$ iterates, we obtain an identity of the form $g - T(g|_{\Delta(n-k)}) - T((g - T(g|_{\Delta(n-k)}))|_{\Delta(n-k-1)}) - \ldots = 0$. Applying T^{-1} produces a formula for $T^{-1}(g)$.

Now, for $f \in \mathrm{MW}^k(\Delta)$ we say that an element $g \in \mathrm{GW}(\Delta)$ *lifts* f if $g \in I_k$ and $g|_{\Delta^{(k)}} = f$. Suppose that $f \in \mathrm{MW}^k(\Delta)$. Then for example, $T(f)$ will be contained in $(I_k)_{\mathbb{Q}}$ and will satisfy $T(f)|_{\Delta^{(k)}} = f$, but will not generally be an element of $\mathrm{GW}(\Delta)$. Our next proposition is a sufficient condition for existence of lifts.

Let F_i be the i-th piece of the dimension filtration on the Grothendieck group, meaning that it is generated by coherent sheaves with support of dimension at most i.

Proposition 2.5.3. *Suppose F_k is saturated as a subgroup of $K_\circ(X)$. Then every $f \in$ $\mathrm{MW}^k(\Delta)$ has a lift in* $\mathrm{GW}(\Delta)$.

Proof. We have the exact sequence

$$0 \to F_k/F_{k-1} \to K_\circ(X)/F_{k-1} \to K_\circ(X)/F_k \to 0.$$

The long exact sequence obtained after applying $(-)^\vee = Hom_{\mathbb{Z}}(-,\mathbb{Z})$ is:

$$
\begin{tikzcd}
0 \arrow[r] & (K_\circ(X)/F_k)^\vee \arrow[r] & (K_\circ(X)/F_{k-1})^\vee \arrow[r] & (F_k/F_{k-1})^\vee \arrow[dll, out=0, in=180] \\
& \mathrm{Ext}^1_{\mathbb{Z}}(K_\circ(X)/F_k,\mathbb{Z}) \arrow[r] & \mathrm{Ext}^1_{\mathbb{Z}}(K_\circ(X)/F_{k-1},\mathbb{Z}) \arrow[r] & \mathrm{Ext}^1_{\mathbb{Z}}(F_k/F_{k-1},\mathbb{Z}).
\end{tikzcd}
$$

Let us consider the first few terms. $(K_\circ(X)/F_{k-1})^\vee$ may be naturally identified with the ideal I_k of Grothendieck weights which vanish on cones of codimension less than k, as defined in Corollary 2.4.8. On the other hand, F_k/F_{k-1} is the k-th piece of graded K-theory, and the map $A_k(X) \to F_k/F_{k-1}$ sending $[V]$ to $[\mathscr{O}_V]$ is an isomorphism after tensoring with $\mathbb{Q}$ (see [16, Chapter 18]). Thus, $(F_k/F_{k-1})^\vee \cong MW^k(\Delta)$. Thus the first few terms in the exact sequence become:

$$0 \to I_{k+1} \to I_k \to \mathrm{MW}^k(\Delta) \to \dots.$$

Suppose that a Minkowski weight $f \in \mathrm{MW}^k(\Delta)$ does not have a lift. Then, I_k cannot surject onto $\mathrm{MW}^k(\Delta)$. Thus, the group $\mathrm{Ext}^1_{\mathbb{Z}}(K_\circ(X)/F_k,\mathbb{Z})$ cannot be trivial. But this group is isomorphic to the torsion subgroup of $K_\circ(X)/F_k$, which is trivial if and only if F_k is saturated as a subgroup of $K_\circ(X)$. $\square$

The ring $\mathrm{GW}(\Delta)$ also admits a map from $\mathrm{PExp}(\Delta)$, which is the ring of continuous functions on Δ that are given on each cone $\alpha \in \Delta$ by an exponential function e^m, for $m \in M$. Anderson and Payne showed that this ring is naturally isomorphic to $\mathrm{op}K_T^\circ(X)$, and thus the map to $\mathrm{GW}(\Delta)$ is induced by the forgetful map $\mathrm{op}K_T^\circ(X) \to \mathrm{op}K^\circ(X)$. We will require K-theoretic equivariant multiplicities $\epsilon_p^K(V(\alpha))$, where $p \in X^T$. These have been recently introduced in [2]. They satisfy $\sum_{p\in X^T} \epsilon_p(V(\alpha))[i_{p*}(\mathscr{O}_p)] = [\mathscr{O}_{V(\alpha)}]$, where i_{p*} is the pushforward in $K_\circ^T$ along the inclusion of the fixed point p.

Theorem 2.5.4. *There is a commuting square*

$$\begin{tikzcd} \mathrm{op}K_T^\circ(X(\Delta)) \arrow[r, "\cong"] \arrow[d, "\text{forgetful}"] & \mathrm{PExp}(\Delta) \arrow[d, "\text{forgetful}"] \\ \mathrm{op}K^\circ(X(\Delta)) \arrow[r, "\cong"] & \mathrm{GW}(\Delta) \end{tikzcd}$$

where the forgetful map from $\mathrm{PExp}(\Delta)$ *to* $\mathrm{GW}(\Delta)$ *sends a piecewise-exponential function* ϕ *to the limit of the function*

$$\alpha \to \sum_{\sigma \in \Delta^{(0)}} \epsilon^K_{V(\sigma)}(V(\alpha)) f|_\sigma,$$

as the argument of ϕ *approaches* $0 \in N$.

Proof. If ϕ is a piecewise exponential function, then via the isomorphism in [3, Theorem 6.1], ϕ corresponds to a $R(T)$-linear function $\phi_{lin} : K^T_\circ(X) \to R(T)$. The function ϕ_{lin} can be written explicitly via the projection formula:

$$\begin{aligned} \phi_{lin}([\mathscr{O}_{V(\alpha)}]) &= \phi_{lin}(\sum_{p \in X^T} \epsilon^K_p(V(\alpha))[i_{p*}(\mathscr{O}_p)]), \\ &= i^*_{X^T}\phi_{lin}(\sum_{p \in X^T} \epsilon^K_p(V(\alpha))[\mathscr{O}_p]), \\ &= \sum_{p \in X^T} \epsilon^K_p(V(\alpha)) i^*_p \phi_{lin}([\mathscr{O}_p]) = \sum_{p \in X^T} \epsilon^K_p(V(\alpha))\phi|_{\sigma_p}, \end{aligned}$$

where σ_p is the maximal cone corresponding to p.

Then, the forgetful map from $\mathrm{op}K_T^\circ(X)$ to $\mathrm{op}K^\circ(X)$ is induced by the projection $X \times T \to X$, meaning it is the pullback from $\mathrm{op}K_T^\circ(X)$ to $\mathrm{op}K_T^\circ(X \times T) \cong \mathrm{op}K^\circ(X)$. Via the identification of $\mathrm{op}K_T^\circ(X)$ with $R(T)$-linear maps from $K^T_\circ(X)$ to $R(T)$, and $\mathrm{op}K^\circ(X)$ with $K_\circ(X)^\vee$, the forgetful map sends $\phi_{lin} : K^T_\circ(X) \to R(T)$ to the linear function on $K_\circ(X)$ sending $[\mathscr{O}_{V(\alpha)}]$ to the equivalence class in $\mathbb{Z}$ of $\psi([\mathscr{O}_{V(\alpha)}])$ (see the appendix of [2] for more details). This is the same as taking the limit as the argument of ϕ approaches $0 \in N$. □

Example 2.5.5. We apply this theorem to the toric variety X with fan Δ in $N = \mathbb{Z}^2$ generated $(\pm 1, \pm 1)$. This example and the following corollary are analogous to [27, Example 4.1 and Theorem 1.5]. In this case, a generating set for $\mathrm{PExp}(\Delta)$ over $R(T)$ is given by the functions in Figure 2.4.

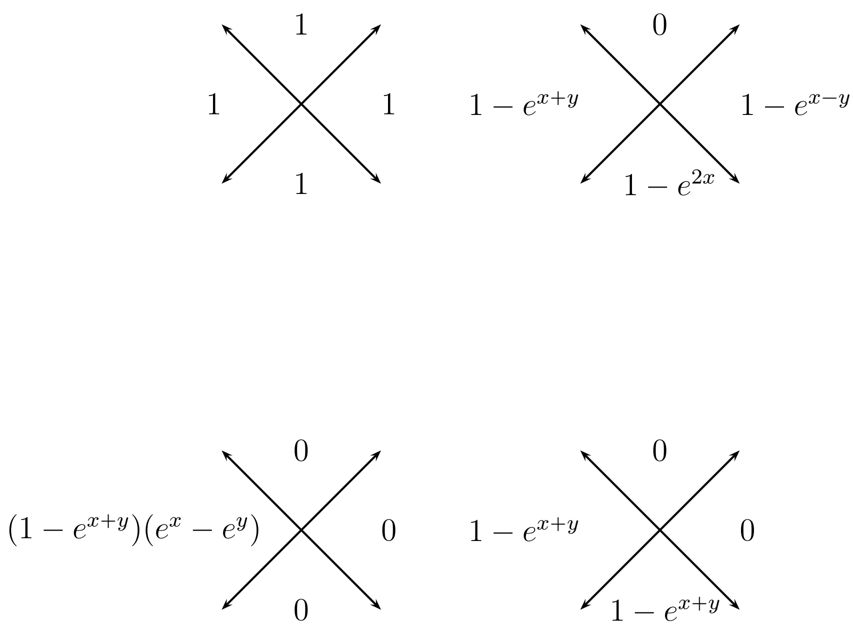


Figure 2.4: Generators for the ring of piecewise exponential functions on a toric surface

We go through the calculation for the piecewise exponential function at the top right: since the equivariant multiplicity of a point is just 1, the value of the Grothendieck weight on any maximal cone is just the value of the piecewise exponential function at 0. For the function we are considering this is 0. For the ray ρ generated by $(1, 1)$, $V(\rho)$ is a $\mathbb{P}^1$, and at the fixed point corresponding to the maximal cone σ generated by $(1, 1)$ and $(1, -1)$ the character on the tangent space is $y - x$, so the equivariant multiplicity is $\frac{1}{1-e^{x-y}}$, by [2, Proposition 6.3]. At the other fixed point of $V(\rho)$, the character is $x - y$, and so the multiplicity is $\frac{1}{1-e^{y-x}}$.

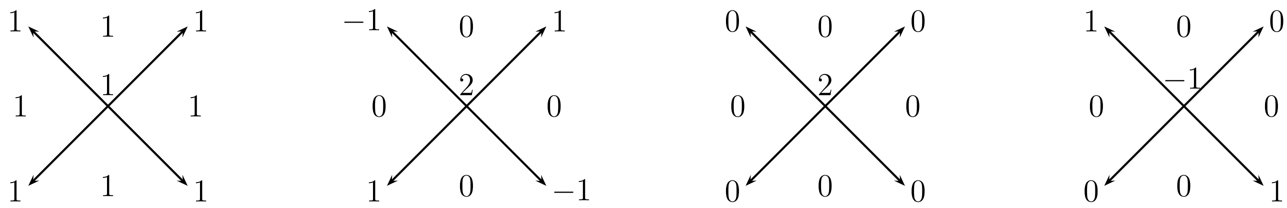


Figure 2.5: Images of those generators

The value of the Grothendieck weight on ρ is then the limit of $\frac{0}{1-e^{y-x}} + \frac{1-e^{x-y}}{1-e^{x-y}}$ as x and y approach 0, which is 1. Similarly, one gets that the value of the Grothendieck weight on the ray $(1,-1)$ is -1. The balancing conditions for Grothendieck weights determine the values on the other rays.

For the cone $\{0\}$, we will require the equivariant multiplicities $\epsilon_p(X)$. Since X is singular at each fixed point, we can compute the equivariant multiplicity at the fixed point p corresponding to σ by resolving, e.g. by adding the ray $(1,0)$, and then summing over the new fixed points which map to p. One gets

$$\epsilon_p(X) = \frac{1}{(1-e^{y})(1-e^{x-y})} + \frac{1}{(1-e^{-y})(1-e^{x+y})} = \frac{1+e^{x}}{(1-e^{x+y})(1-e^{x-y})}.$$

Let the fixed point corresponding to the cone generated by $(-1,-1),(1,-1)$ be q, and the fixed point corresponding to the cone generated by $(-1,1)$ and $(-1,-1)$ be r. Then

$$\begin{aligned}\epsilon_q(X) &= \frac{1+e^{-y}}{(1-e^{x-y})(1-e^{-x-y})},\\ \epsilon_r(X) &= \frac{1+e^{-x}}{(1-e^{-x-y})(1-e^{y-x})}.\end{aligned}$$

By the last theorem, α must be sent to the limit of

$$\frac{(1-e^{x-y})(1+e^{x})}{(1-e^{x+y}))(1-e^{x-y})}+\frac{(1-e^{2x})(1+e^{-y})}{(1-e^{x-y})(1-e^{-x-y})}+\frac{(1-e^{x+y})(1+e^{-x})}{(1-e^{-x-y})(1-e^{y-x})},$$

as the parameters x and y approach 0, which is 2.

In fact, this example shows the following (compare with [27, Theorem 1.5] and [2, Theorem 1.7]):

Corollary 2.5.6. *There exists a complete toric surface with a vector bundle with no finite length resolution by T-equivariant vector bundles.*

Proof. In Example 2.5.5, the $\mathbb{Z}$-linear span of the Grothendieck weights calculated does not include the Grothendieck weight with 1 at the origin and 0 elsewhere, so $\mathrm{PExp}(\Delta)$ does not surject onto $\mathrm{GW}(\Delta)$. Thus, the forgetful map from $\mathrm{op}K_T^\circ(X)$ to $\mathrm{op}K^\circ(X)$ is not surjective. Since vector bundles induce linear forms on coherent sheaves by tensor product followed by pushforward to a point, there is a commutative square:

$$\begin{tikzcd} K_T^\circ(X) \arrow[r] \arrow[d] & \mathrm{op}K_T^\circ(X) \arrow[d] \\ K^\circ(X) \arrow[r] & \mathrm{op}K^\circ(X) \end{tikzcd}.$$

We know from [3, Proposition 7.4] that the bottom map is surjective. Comparing the two ways of traversing the diagram, one sees that the map $K_T^\circ(X) \to K^\circ(X)$ cannot be surjective. This proves the corollary. ∎

Chapter 3

Quantum K-theory and q-series

Our starting point in this chapter is an identity for the lattice point generating function of a polytope P, proved by Brion in [9]. In the simplest case, when P is the interval $[0, n]$, Brion's identity specializes to:

$$1 + x + \ldots + x^n = \frac{1}{1-x} + \frac{x^n}{1-x^{-1}}.$$

Namely, we have on the left-hand side a sum of monomials over points inside P, and on the right a sum of rational functions over vertices of P.

The main result of this chapter is a q-analogue of Brion's identity. Let us introduce some q-series notation: first, we have the q-Pochhammer symbol

$$(a;q)_n = \prod_{i=0}^{n-1}(1 - aq^i),$$

and the q-binomial numbers

$$\binom{n}{k}_q = \frac{(q;q)_n}{(q;q)_k(q;q)_{n-k}}.$$

In fact, there is cancellation between the numerator and denominator of this expression, so the q-binomial numbers turn out to be in $\mathbb{Z}[q]$. We also have the following sum, which is a certain q-hypergeometric series

$$\phi_n(x;q) = \sum_{k=0}^{\infty} \frac{q^{nk}}{(q^{-1};q^{-1})_k (x;q^{-1})_k}.$$

Then, in the case that $P = [0, n]$, our identity specializes to

$$\binom{n}{0}_q + \binom{n}{1}_q x + \ldots + \binom{n}{n-1}_q x^{n-1} + \binom{n}{n}_q x^n = \frac{(q;q)_n}{(q;q)_\infty} \left(\frac{\phi_n(x;q)}{(x;q)_\infty} + x^n \frac{\phi_n(x^{-1};q)}{(x^{-1};q)_\infty} \right).$$

As in the case of Brion's identity, we have a sum over lattice points in P on the left, and a sum over vertices on the right. However, now the coefficients of the monomials x^i on the left are rational functions in q, and the terms in the sum on the right are highly non-algebraic functions. Nonetheless, the identity recovers Brion's identity when q is set to 0.

3.1 Brion's Identity

Let us start out by stating Brion's identity in more generality. Recall (1.1.1) that a cone C in $N_{\mathbb{R}}$ is strongly convex if it does not contain a non-trivial linear subspace of $N_{\mathbb{R}}$.

Definition 3.1.1. The **lattice point generating function** of a cone or polytope P in $\mathbb{R}^n$ is the sum

$$\sigma_P(x) = \sum_{m \in (\mathbb{Z}^n \cap P)} x^m.$$

- If P is a polytope, σ_P is a Laurent polynomial.
- If P is a strongly convex rational polyhedral cone, σ_P has a non-trivial radius of convergence around some point in $\mathbb{C}^n$ and in fact agrees with a unique element of $Frac(R(T))$ (see [9, Section 2]).

Example 3.1.2. Let $C = [0, \infty)$. Then

$$\sigma_C(x) = 1 + x + x^2 + \ldots = \frac{1}{1-x}.$$

Also, for p a vertex of a polytope P, we define the vertex cone K_pP:

Definition 3.1.3. Let K_pP be the convex cone with vertex p, generated by vectors of the form $w - p$ for $w \in P$.

In Figure 3.1, we show P, a labelled vertex p, and the vertex cone K_pP.

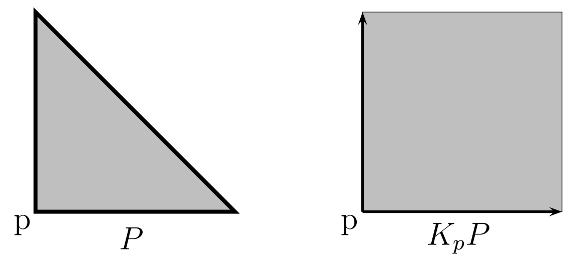


Figure 3.1: A simplex and one of its vertex cones

Let $V(P)$ denote the set of vertices of P. Then, Brion's identity states:

Theorem 3.1.4. *For P a polytope with rational vertices, we have the following identity in $Frac(R(T))$.*

$$\sigma_P(x) = \sum_{p \in V(P)} \sigma_{K_pP}(x).$$

In this generality, the theorem can be found in [6, Chapter 9] where it is proved using combinatorics, but it was originally proved for simple lattice polytopes in [9] using equivariant K-theory. When P is a smooth lattice polytope, we can be a bit more explicit, so we can later compare this directly with our q-analogue. For each vertex p, let $I(p) \subseteq \{1, \ldots, r\}$ be the set of indices i so that v_i is an inward normal vector of a facet containing p. Then, $\{v_i \mid i \in I(p)\}$ is a basis for N. Let $\{u_i(p) \mid i \in I(p)\}$ be the dual basis of M. The vector $u_i(p)$ is the primitive vector along the edge of P containing p which is not contained in the facet defined by v_i. Then

$$\sigma_{K_pP}(x) = \frac{x^p}{(1 - x^{u_1(p)}) \dots (1 - x^{u_n(p)})},$$

so when P is a smooth lattice polytope, Theorem 3.1.4 states that:

$$\sum_{u \in P \cap M} x^u = \sum_{p \in V(P)} \frac{x^p}{(1 - x^{u_1(p)}) \dots (1 - x^{u_n(p)})}. \tag{3.1.5}$$

3.2 The q-analogue

To state our q-analogue in general, recall that M is a lattice of rank n, and $N = \mathrm{Hom}(M, \mathbb{Z})$ is the dual lattice. The corresponding real vector spaces are $M_{\mathbb{R}}$ and $N_{\mathbb{R}}$. We write a polytope $P \subseteq M_{\mathbb{R}}$ as an intersection of half-spaces, so

$$P = \bigcap_{i=1}^{r} \{u \,|\, \langle u, v_i \rangle \geqslant -a_i\}$$

where $v_i \in N$ is the primitive inward normal vector defining the i-th facet of P.

The inward normal vectors $v_1, \dots, v_r$ determine a homomorphism $\mathbb{Z}^r \to N$. Let A be the kernel, and let $\beta \colon A \to \mathbb{Z}^r$ be the inclusion. This induces a *positive* semigroup A_+ inside A defined by $A_+ = \beta^{-1}(\mathbb{Z}^r_{\geqslant 0})$.

We assume that the polytope is *smooth*, so that each vertex is contained in exactly n facets, and the corresponding n primitive normal vectors form a basis for N. Furthermore, we assume the polytope is *radially symmetric*, meaning that $\sum_{i=1}^{r} v_i = 0$. These conditions can be relaxed, but there are well-known families of polytopes which satisfy them—for example, all generalized permutahedra are smooth and radially symmetric.

For each vertex p of a smooth polytope P, let $I(p)$ and $u_i(p)$ be defined as in Section 3.1.

For each $d \in A_+$ and vertex p of P, we define

$$\mathsf{J}_{d,p} = \left(\prod_{i \in I(p)} \frac{1}{(x^{u_i(p)}q^{-1};q^{-1})_{\beta(d)_i}}\right)\left(\prod_{j \notin I(p)} \frac{1}{(q^{-1};q^{-1})_{\beta(d)_j}}\right).$$

Now we can state the main theorem.

Theorem 3.2.1. *Let P be a smooth, radially symmetric polytope, realized as an intersection of r half-spaces $\{u \,|\, \langle u, v_i\rangle \geqslant -a_i\}$, with notation as above. Then*

$$\sum_{u \in P \cap M} \binom{|a|}{\langle u, v_1\rangle + a_1, \ldots, \langle u, v_r\rangle + a_r}_q x^u = \frac{(q;q)_{|a|}}{(q;q)_\infty^{r-n}} \sum_{p \in V(P)} \frac{x^p \sum_{d \in A_+} q^{\sum a_i \beta(d)_i} \cdot \mathsf{J}_{d,p}}{(x^{u_1(p)};q)_\infty \cdots (x^{u_n(p)};q)_\infty},$$

where $|a| = a_1 + \cdots + a_r$ and $V(P)$ is the set of vertices of P.

The left-hand side belongs to $\mathbb{Z}[q][M]$; i.e., it is a Laurent polynomial in x and a polynomial in q. The right-hand side is a formal power series in both x and q, and it seems (to us) surprising that a $q \to 1$ limit exists, because each term has an essential singularity. The $q \to 0$ limit, on the other hand, recovers Brion's formula in the smooth, radially symmetric case - compare with 3.1.5.

The theorem follows from Theorem 3.6.8, which relaxes the requirement of radial symmetry. We prove this in Section 3.6. As in [9], the basic technique is localization in the equivariant K-theory of a toric variety. Our main observation is that a canonical q-analogue of the lattice point generating function is provided by the *toric quasimap space*, introduced by Morrison-Plesser and Givental in the context of Gromov-Witten theory [21, 32]. These spaces fit together to form an ind-variety contained in the *toric arc scheme* studied by Arkhipov-Kapranov [5]. This infinite-dimensional scheme provides the geometric context for the infinite products appearing in the theorem.

The left-hand side of (3) is the nth *Rogers-Szegő polynomial*. This is a classical family of polynomials, which were shown to be orthogonal with respect to a certain measure on the circle by Szegő in 1926 [39]. Multivariate versions of them play a role in representation theory and combinatorics [11, 26, 42]. In our context, they appear in the case where P is a standard lattice simplex, so the corresponding toric variety is a projective space. In polyhedral geometry, q-analogues of Ehrhart-theoretic quantities have appeared before (e.g. [12]), but our work is in a different direction, see Remark 3.6.9.

3.3 Equivariant K-theory and the Atiyah-Bott formula

Here we review localization in equivariant K-theory and the Atiyah-Bott integration formula for the Euler characteristic that we use in the proof of Theorem 3.6.3. The vocabulary we use for localization is from [2], but in our restricted context the localization and integration theorems that we use are quite old, see [33]. We assume that X be a smooth variety, so that there is a natural isomorphism between the ring of vector bundles $K^\circ(X)$, and the group of coherent sheaves $K_\circ(X)$.

For a group G, the G-equivariant K-theory $K_G^\circ(X)$ is generated by isomorphism classes of G-equivariant vector bundles on X, modulo relations from G-equivariant short exact sequences. As in the case of non-equivariant K-theory, $K_G^\circ(X)$ is a ring with respect to tensor product. If X is a point, then a G-equivariant vector bundle on X is equivalent to the data of a representation, and $K_G^\circ(X)$ is also known as the *representation ring* $R(G)$. For G-equivariant proper morphisms $f : X \to Y$, there is a pushforward $f_* : K_G^\circ(X) \to K_G^\circ(Y)$, and for arbitrary G-equivariant maps $f : X \to Y$, there is a pullback $f^* : K_G^\circ(Y) \to K_G^\circ(X)$. Since for any X, the map $X \to pt$ is G-equivariant, $K_G^\circ(X)$ is always an $R(G)$-module via pullback.

For the sake of simplicity, we impose that $G = T$ is a torus, and further that X has finitely many T-fixed points $y_1, \ldots, y_l$. In particular, this implies that none of the characters which appear in a decomposition of $T_{y_i}X$ are zero in M. Let $Frac(T)$ the fraction field of $R(T)$.

Definition 3.3.1. Let $i : y \hookrightarrow X$ be the inclusion of a smooth isolated fixed point. Then we can define a class

$$\epsilon_y(X) = \frac{1}{i^* i_* [\mathscr{O}_y]},$$

in $Frac(T)$, called the K-theoretic *equivariant multiplicity* of y. (see [2, Section 6] for a more general definition).

We add that this class agrees in our case with the class used in Chapter 2, Section 5.

Let i_k be the inclusion $i_k : y_k \hookrightarrow X$. Then:

Theorem 3.3.2 (Localization)**.** *Let $\phi : \oplus_k K_T^\circ(y_k) \to K_T^\circ(X)$ be the natural map induced by pushforward, and $\psi : K_T^\circ(X) \to \oplus_k K_T^\circ(y_k)$ the natural map induced by pullback. Then over $Frac(T)$, ϕ and ψ are isomorphisms.*

Theorem 3.3.3 (Integration)**.** *Let f be the map from X to a point, and let $[E]$ be the class of a vector bundle on X. Then*

$$f_*[E] = \sum_k [i_k^* E] \cdot \epsilon_{y_k}(X).$$

Proof. The following diagram of proper maps commutes

$$\begin{tikzcd} & X \arrow[dr] & \\ \{y_1, \ldots, y_l\} \arrow[ur] \arrow[rr] & & pt. \end{tikzcd}$$

This induces a commutative diagram between K-theory rings. The pushforward map from $\oplus_k K_T^\circ(y_k) \cong \oplus_k R(T)$ to $K_T^\circ(pt) = R(T)$ is simply addition. Thus, if $\phi^{-1}([E]) = (a_1, \ldots, a_l)$, $f_*[E]$ is simply $a_1 + \ldots + a_l$.

Thus, the theorem follows if we can show that $\phi^{-1}([E]) = ([i_1^* E] \cdot \epsilon_{y_1}(X), \ldots, [i_l^* E] \cdot$

$\epsilon_{y_l}(X))$. But

$$\phi([i_1^*E]\cdot\epsilon_{y_1}(X),\ldots,[i_l^*E]\cdot\epsilon_{y_l}(X)) = \sum_k (i_k)_*(\frac{[i_k^*E]}{(i_k)^*(i_k)_*[\mathscr{O}_{y_k}]}), \tag{3.3.4}$$

which maps to $(i_1^*E,\ldots,i_l^*E)$ under ψ just as $[E]$ does. Since ψ is an isomorphism we are done. $\square$

3.4 The Cox construction

Our main theorem is proved by applying the Atiyah-Bott formula to toric quasimap spaces. These spaces are naturally defined via the Cox construction of a toric variety, which we summarize now. We impose that Δ is a smooth complete fan and enumerate the rays $\Delta(1) = \{\rho_1,\ldots,\rho_r\}$. Each ray ρ_i has a primitive generator $v_i \in N$.

Then, there is an exact sequence

$$0 \to A \xrightarrow{\beta} \mathbb{Z}^r \to N \to 0,$$

where $\mathbb{Z}^r \to N$ sends the ith standard basis vector to the primitive generator of the ith ray, $e_i \mapsto v_i$. The kernel A is isomorphic to $\mathbb{Z}^{r-n}$.

Dualizing, one has an exact sequence

$$0 \to M \to \mathbb{Z}^r \to B \to 0,$$

where $B = \mathrm{Hom}(A,\mathbb{Z})$. Let $G \subseteq (\mathbb{C}^*)^r$ be the subtorus corresponding to the surjection of lattices $\mathbb{Z}^r \to B$, so we have $G \cong (\mathbb{C}^*)^{r-n}$ and $T = (\mathbb{C}^*)^r/G$.

Now, we realize the toric variety $X(\Delta)$ as a GIT quotient of $\mathbb{C}^r$ by G. Each subset $I \subseteq \{1,\ldots,r\}$ determines a coordinate subspace $E_I = \{e_i^* = 0 \,|\, i \in I\} \subseteq \mathbb{C}^r$, as well as a collection of rays. Let $Z(\Delta) \subseteq \mathbb{C}^r$ be the union of those coordinate subspaces E_I such that the corresponding set of rays $\{\rho_i \,|\, i \in I\}$ is not contained in any cone of Δ. Then

$$X(\Delta) = (\mathbb{C}^r \backslash Z(\Delta))/G.$$

Basic facts from toric geometry say that

$$B \cong \operatorname{Pic}(X) \cong H^2(X, \mathbb{Z})$$

and the cone of effective divisors is the image of $\mathbb{Z}^r_{\geqslant 0}$ in B. Dually,

$$A \cong H_2(X, \mathbb{Z}),$$

and the cone of nef curves is the preimage of $\mathbb{Z}^r_{\geqslant 0}$ under the embedding $\beta\colon A \hookrightarrow \mathbb{Z}^r$. This cone is written $A_+ \subseteq A$.

When $X(\Delta)$ is projective, a polytope P which is normal to the fan corresponds to the line bundle $\mathscr{O}(D_P)$ as described in section 1.2. Facets of P correspond to T-invariant divisors of X, and vertices of P correspond to fixed points.

3.5 Quasimap spaces and the toric arc scheme

Next, we review the construction of the toric quasimap space; see [21, 32] for proofs and details. We are interested in parametrizing maps $f\colon \mathbb{P}^1 \to X(\Delta)$ of *degree* d, that is, $f_*[\mathbb{P}^1] = d$ in $A = H_2(X)$. Let $\operatorname{Hom}_d(\mathbb{P}^1, X)$ be the space of such maps. To describe this space, one can lift such maps to $\mathbb{C}^r$, where they can be specified as an r-tuple of univariate polynomials. We consider only degrees d lying in A_+ (which in general is properly contained in the semigroup of all effective curves).

For any r-tuple of nonnegative integers $\delta = (\delta_1, \ldots, \delta_r)$, let

$$\mathbb{C}^r_\delta = \mathbb{C}[t]_{\leqslant \delta_1} \oplus \cdots \oplus \mathbb{C}[t]_{\leqslant \delta_r}$$

where $\mathbb{C}[t]_{\leqslant k} = \{f(t) = f^{(0)} + f^{(1)}t + \cdots + f^{(k)}t^k\}$ is the $(k+1)$-dimensional space of polynomials of degree at most k. (Sometimes it is useful to homogenize, making these bivariate polynomials $F(s,t)$ of degree exactly equal to k.)

The vector space $\mathbb{C}^r_\delta$ has dimension $r+\delta_1+\cdots+\delta_r$, so the torus $(\mathbb{C}^*)^{r+\sum\delta_i}$ acts coordinatewise. The torus $(\mathbb{C}^*)^r$ embeds "diagonally", so that

$$(z_1, \ldots, z_r) \cdot (f_1(t), \ldots, f_r(t)) = (z_1 f_1(t), \ldots, z_r f_r(t)).$$

So the subtorus $G \subseteq (\mathbb{C}^*)^r$ also embeds in $(\mathbb{C}^*)^{r+\sum\delta_i}$. The quasimap space constructed below is a GIT quotient of $\mathbb{C}^r_\delta$ by this action of G.

For any subset $I \subseteq \{1, \ldots, r\}$ let $E'_{I,\delta} \subseteq \mathbb{C}^r_\delta$ be the locus where the corresponding homogeneous polynomials $\{F_i(s,t) \,|\, i \in I\}$ share a common factor. (Dehomogenizing, it is equivalent to require that the $\{f_i \,|\, i \in I\}$ share a common factor, or that at least two of the f_i have degree less than δ_i — since homogenizing produces a common factor of s.)

Let $E_{I,\delta} \subseteq \mathbb{C}^r_\delta$ be the locus where $f_i \equiv 0$ for each $i \in I$, so $E_{I,\delta} \subseteq E'_{I,\delta}$. (Viewing $E_I = \mathbb{C}^{r-\#I}$ as a coordinate subspace and writing $\delta(\hat{I})$ for the subsequence of δ omitting $i \in I$, this is the same as $\mathbb{C}^{r-\#I}_{\delta(\hat{I})}$.)

Let

$$Z'(\Delta)_\delta = \bigcup_I E'_{I,\delta} \quad \text{and} \quad Z(\Delta)_\delta = \bigcup_I E_{I,\delta},$$

both unions over I such that $\{\rho_i \,|\, i \in I\}$ is not contained in a cone of Δ.

Lemma 3.5.1. *The space* $\mathrm{Hom}_d(\mathbb{P}^1, X)$ *is isomorphic to* $(\mathbb{C}^r_{\beta(d)} \backslash Z'(\Delta)_{\beta(d)})/G$.

The space of *(toric) quasimaps* is defined as

$$\mathscr{Q}(\Delta)_d = (\mathbb{C}^r_{\beta(d)} \backslash Z(\Delta)_{\beta(d)})/G.$$

By construction, it contains $\mathrm{Hom}_d(\mathbb{P}^1, X)$ as a dense open subset, since $Z(\Delta)_{\beta(d)} \subseteq Z'(\Delta)_{\beta(d)}$ is closed.

Lemma 3.5.2. *The toric quasimap space $\mathscr{Q}(\Delta)_d$ is a toric variety of dimension $n + \sum \beta(d)_i$, smooth and complete whenever $X(\Delta)$ is.*

Example 3.5.3. If $X(\Delta) = \mathbb{P}^n$, the coordinates on $\mathscr{Q}(\Delta)_d$ may be written in a $(d+1)\times(n+1)$ matrix:

$$\begin{bmatrix} f_0^{(0)} & \cdots & f_n^{(0)} \\ f_0^{(1)} & \cdots & f_n^{(1)} \\ \vdots & & \vdots \\ f_0^{(d)} & \cdots & f_n^{(d)} \end{bmatrix}.$$

(In general, one can write such coordinates in an array where the columns have unequal heights: the ith column has height $\beta(d)_i + 1$.)

Let us write $T_d = (\mathbb{C}^*)^{r+\sum \beta(d)_i}/G$ for the torus acting on $\mathscr{Q}(\Delta)_d$. We are also interested in actions by various subtori. Using $(\mathbb{C}^*)^r \hookrightarrow (\mathbb{C}^*)^{r+\sum \beta(d)_i}$ as above, we have an inclusion of $T = (\mathbb{C}^*)^r/G$ in T_d, so the same torus acting on $X(\Delta)$ also acts on $\mathscr{Q}(\Delta)_d$. On the other hand, there is a loop rotation action of $\mathbb{C}^*$ on $\mathbb{C}^r_{\beta(d)}$ by

$$\zeta \cdot (f_1(t), \ldots, f_r(t)) = (f_1(\zeta^{-1}t), \ldots, f_r(\zeta^{-1}t)),$$

and this descends to an action of $\mathbb{C}^*$ on $\mathscr{Q}(\Delta)_d$. So there is a subtorus $\mathrm{T} = T \times \mathbb{C}^* \subseteq T_d$ acting on $\mathscr{Q}(\Delta)_d$. Localization with respect to T will be the primary tool in proving the main theorem.

Lemma 3.5.4. *The fixed locus $(\mathscr{Q}(\Delta)_d)^{\mathbb{C}^*}$ by the loop rotation action is a union of components $\mathscr{Q}(\Delta)_d^{(d')}$, for each decomposition $d' \leqslant d$ (meaning $d = d' + d''$ is a decomposition as a sum of nef classes). Furthermore, there is a T-equivariant isomorphism $\mathscr{Q}(\Delta)_d^{(d')} \cong X(\Delta)$ for all d'.*

A basic observation the space $\mathscr{Q}(\Delta)_0$ agrees with $X(\Delta)$, since the given constructions become identical. The quasimap spaces $\mathscr{Q}(\Delta)_d$ form a system of closed embeddings for d varying over $A_+ \subset A = H_2(X, \mathbb{Z})$. Namely, given two such curve classes $d' \leqslant d$, there is a closed embedding $\mathscr{Q}(\Delta)_{d'} \hookrightarrow \mathscr{Q}(\Delta)_d$ induced by the inclusion $\mathbb{C}^r_{\beta(d')} \hookrightarrow \mathbb{C}^r_{\beta(d)}$. We refer to the limiting ind-variety as the toric polynomial space $\mathscr{Q}(\Delta)_\infty$. This ind-variety sits inside an even larger space, the toric arc scheme of Arkhipov-Kapranov, see [5]. We review these spaces next.

The toric arc scheme is constructed in nearly the same manner as $\mathscr{Q}(\Delta)_d$. Let

$$\mathbb{C}^r_\infty = \mathbb{C}[[t]] \oplus \ldots \oplus \mathbb{C}[[t]].$$

This space has an infinite-dimensional torus action given by scaling coordinates of the tuple of power series. The torus $(\mathbb{C}^*)^r$ again embeds diagonally, so that

$$(z_1, \ldots, z_r) \cdot (f_1(t), \ldots, f_r(t)) = (z_1 f_1(t), \ldots, z_r f_r(t)),$$

allowing G to act on $\mathbb{C}^r_\infty$. For $I \subset \{1, \ldots, r\}$, let the locus $E_{I,\infty} \subseteq \mathbb{C}^r_\infty$ be defined as the tuples where all the coefficients of f_i vanish for all $i \in I$, and

$$Z(\Delta)_\infty = \bigcup_I E_{I,\infty},$$

where once again the union is over I such that $\{\rho_i | i \in I\}$ is not contained in a cone of Δ.

The Arkhipov-Kapranov toric arc scheme is

$$\Lambda^0 X = (\mathbb{C}^r_\infty \backslash Z(\Delta)_\infty)/G.$$

For us, the most important torus action on $\Lambda^0 X$ is by the product torus

$$\mathrm{T} = \mathbb{C}^* \times (\mathbb{C}^*)^r / G,$$

where the first factor acts by loop rotation. The quasimap space $\mathscr{Q}(\Delta)_d$ embeds as a finite-dimensional subvariety into $\Lambda^0 X$, and the following diagram commutes:

$$\begin{tikzcd} & \mathscr{Q}(\Delta)_{d'} \arrow[dr] & \\ \mathscr{Q}(\Delta)_d \arrow[ur] \arrow[rr] & & \Lambda^0(X). \end{tikzcd}$$

The closed points of the ind-variety $\mathscr{Q}(\Delta)_\infty$ (or equivalently, the union of closed points of $\mathscr{Q}(\Delta)_d$ over all $d \in A_+$) correspond to the tuples of power series inside $\Lambda^0 X$ where only finitely many coefficients of any given power series are non-zero. In other words, the toric polynomial space $\mathscr{Q}(\Delta)_\infty$ embeds into $\Lambda^0 X$ as the subset consisting of power series which are in fact polynomial.

Definition 3.5.5. Let D_i be the T-invariant divisor corresponding to the ray ρ_i in X. For each $k \geqslant 0$ there are divisors D_i^k in $\Lambda^0 X$ and $\mathscr{Q}(\Delta)_\infty$ defined by the vanishing of the k-th coefficient of $f_i(t)$. More generally, for a T-invariant divisor $D = \sum_i a_i D_i$ on X, we let D^0 be the divisor $\sum_i a_i D_i^0$ on $\mathscr{Q}(\Delta)_\infty$ and $\Lambda^0 X$.

Arkhipov and Kapranov observed that $\Lambda^0 X$ admits a family of self-embeddings. Recall β is the inclusion $H_2(X, \mathbb{Z}) \hookrightarrow \mathbb{Z}^r$ defined in Section 3.5. An element d in the semigroup A_+ corresponds to a one-parameter subgroup of G, and by composing with the inclusion $G \hookrightarrow (\mathbb{C}^*)^r$, we can write the image of d in $\mathbb{Z}^r$ explicitly as the cocharacter $(t^{\beta(d)_1}, \ldots, t^{\beta(d)_r})$

of $(\mathbb{C}^*)^r$.

Definition 3.5.6. For $d \in A_+$, let $\epsilon_d : \Lambda^0 X \to \Lambda^0 X$ be the self-embedding

$$(f_1(t), \ldots, f_r(t)) \mapsto (t^{\beta(d)_1} f_1(t), \ldots, t^{\beta(d)_r} f_r(t)).$$

This restricts to a self-embedding on the polynomial space $\mathscr{Q}(\Delta)_\infty$ which we also denote by ϵ_d.

These self-embeddings commute: in fact, $\epsilon_d \circ \epsilon_{d'} = \epsilon_{d+d'}$. They are evidently equivariant with respect to the T-action.

3.6 Integration on quasimap spaces

Now, we study the geometry of the T-action on $\mathscr{Q}(\Delta)_\infty$, leading to the proof of our main theorem. From here on, our toric variety $X = X(\Delta)$ is projective, so Δ is the inward normal fan of a polytope P.

We require some notation. Let $\mathscr{A}$ be an oriented hyperplane arrangement in $M_\mathbb{R}$, meaning an arrangement of primitive vectors $v_1, \ldots, v_r$ and integers $a_1, \ldots, a_r$ such that the hyperplane $H_i \in \mathscr{A}$ is defined by

$$\langle u, v_i \rangle = -a_i. \tag{3.6.1}$$

In particular, each hyperplane determines a positive and negative half-space, by replacing the "=" in (3.6.1) with "⩾" or "⩽."

Given such an arrangement $\mathscr{A}$ and any $u \in M$, we let

$$g_{\mathscr{A},u} = \prod_{i=1}^{r} \frac{1}{(q;q)_{\langle u, v_i \rangle + a_i}}. \tag{3.6.2}$$

For a (smooth) polytope P, we consider the arrangement $\mathscr{A}_P = \{H_1, \ldots, H_r\}$, where H_i are the supporting hyperplanes of P, defined by $\langle u, v_i \rangle = -a_i$. This data determines a line bundle $\mathscr{O}(\sum_i a_i D_i)$ whose global sections are in natural bijection with the lattice points in P. The same data also determines a line bundle $\mathscr{O}(\sum_i a_i D_i^0)$ on $\mathscr{Q}(\Delta)_\infty$ (see Definition 3.5.5).

Theorem 3.6.3. *The equivariant Euler characteristic is given by*

$$\chi_{\mathrm{T}}\left(\mathscr{Q}(\Delta)_\infty, \mathscr{O}\left(\sum_i a_i D_i^0\right)\right) = \sum_{u \in P} g_{\mathscr{A}_P, u}\, x^u.$$

Proof. Since $\mathscr{O}(\sum_i a_i D_i)$ is ample on X, the line bundle $\mathscr{O}(\sum_i a_i D_i^0)|_{\mathscr{Q}(\Delta)_d}$ is nef on $\mathscr{Q}(\Delta)_d$ (this can be checked against T-invariant curves), so Demazure vanishing (Proposition 1.2.1) implies that

$$\chi_{\mathrm{T}}\left(\mathscr{Q}(\Delta)_d,\ \mathscr{O}\left(\sum_i a_i D_i^0\right)|_{\mathscr{Q}(\Delta)_d}\right) = H^0\left(\mathscr{Q}(\Delta)_d,\ \mathscr{O}\left(\sum_i a_i D_i^0\right)|_{\mathscr{Q}(\Delta)_d}\right)$$

as classes in $R(T)$. We want to calculate the limit of this class as a q-series.

Let $f_1, \ldots, f_r$ be coordinates on $\mathbb{C}^r$, so the *Cox ring* of X is the polynomial ring on these variables. By definition, a section of $\mathscr{O}(\sum_i a_i D_i)$ is a rational function ϕ on X such that $\mathrm{div}(\phi) + \sum_i a_i D_i$ is effective. A distinguished basis of T-eigenfunctions for such sections is given by the monomials $\left(\prod_i f_i^{b_i - a_i}\right)$, as the b_i range over nonnegative integers such that $\sum_i (b_i - a_i) D_i = 0$ in $\mathrm{Pic}(X)$.

Similarly, for $\mathscr{Q}(\Delta)_d$ we have Cox ring variables $f_i^{(j)}$, for $1 \leqslant i \leqslant r$ and $0 \leqslant j \leqslant \beta(d)_i$. A basis for sections of $\mathscr{O}(\sum_i a_i D_i^0)|_{\mathscr{Q}(\Delta)_d}$ consists of monomials

$$\left(\prod_i (f_i^{(0)})^{-a_i}\right) \cdot \left(\prod_i \prod_{j=0}^{\beta(d)_i} (f_i^{(j)})^{b_{i,j}}\right)$$

such that the $b_{i,j}$ are non-negative, and $\sum_i \left((\sum_{j=0}^{\beta(d)_i} b_{i,j}) - a_i \right) D_i = 0$ in $\mathrm{Pic}(X)$. We need to compute the characters of these sections.

Pick an element of our distinguished basis of sections of $\mathscr{O}(D)$, that is, a character

x^u corresponding to a lattice point u in P, or equivalently a choice of $b_i \geqslant 0$ satisfying the conditions $\sum(b_i - a_i)e_i = u \in M$. In this notation, recall that $b_i = \langle u, v_i\rangle + a_i$. With these b_i fixed, consider the sections $\left(\prod_i (f_i^{(0)})^{-a_i}\right) \cdot \left(\prod_i \prod_{j=0}^{\beta(d)_i} (f_i^{(j)})^{b_{i,j}}\right)$ such that $\sum_{j=0}^{\beta(d)_i} b_{i,j} = b_i$. The character of $\prod_i f_i^{b_i - a_i}$ is x^u, as is that of $\prod_i (f_i^{(0)})^{b_i - a_i}$. So $\left(\prod_i (f_i^{(0)})^{-a_i}\right) \cdot \left(\prod_i \prod_{j=0}^{\beta(d)_i} (f_i^{(j)})^{b_{i,j}}\right)$ has character $x^u \prod_i \prod_{j=0}^{\beta(d)_i} (q^j)^{b_{i,j}}$. So the coefficient of x^u in the graded character of $H^0\left(\mathscr{Q}(\Delta)_d, \mathscr{O}(D^0)\right)$ is

$$\sum \prod_i \prod_{j=0}^{\beta(d)_i} (q^j)^{b_{i,j}},$$

the sum over $b_{i,j} \geqslant 0$ such that $\sum_{j=0}^{\beta(d)_i} b_{i,j} = \langle u, v_i\rangle + a_i$.

In the limit as $d \to \infty$, the upper bound disappears for the indices j of $b_{i,j}$. The resulting sum is over all choices of weakly increasing sequences $0 \leqslant c_{i,1} \leqslant c_{i,2} \leqslant \cdots \leqslant c_{i,b_i}$ for each i from 1 to r, where the summand is the statistic

$$\prod_{i=1}^{r} \prod_{j=1}^{b_i} q^{c_{i,j}}.$$

Holding all but one weakly increasing sequence fixed, we see that the whole sum must factor into a product over i:

$$\prod_{i=1}^{r} \sum_{0 \leqslant c_{i,1} \leqslant c_{i,2} \leqslant \ldots \leqslant c_{i,b_i}} \prod_{j=1}^{b_i} q^{c_{i,j}}.$$

But $\sum_{0 \leqslant c_{i,1} \leqslant c_{i,2} \leqslant \ldots \leqslant c_{i,b_i}} \prod_{j=1}^{b_i} q^{c_{i,j}} = \frac{1}{(q;q)_{b_i}}$. This proves the theorem. $\square$

In the radially symmetric case, we say more about these functions in Section 3.7. Now, we will describe the T-fixed points of $\mathscr{Q}(\Delta)_\infty$, and decompose the corresponding tangent spaces into characters of T.

Proposition 3.6.4. *The* T*-fixed points of* $\mathscr{Q}(\Delta)_\infty$ *are in bijection with pairs* $p \in X^T$ *and*

$d \in A_+$.

Proof. The locus fixed by loop rotation is easy to determine. By definition, $\mathscr{Q}(\Delta)_\infty$ is a union of $\mathscr{Q}(\Delta)_d$ as d varies over A_+, and the subvariety $\mathscr{Q}(\Delta)_0 \hookrightarrow \mathscr{Q}(\Delta)_\infty$ is certainly $\mathbb{C}^*$-fixed. To exhaust the $\mathbb{C}^*$-fixed components in $\mathscr{Q}(\Delta)_d$ for higher d, it is enough to take the disjoint union of $\epsilon_d(\mathscr{Q}(\Delta)_0)$ over $d \in A_+$, because $\epsilon_d(\mathscr{Q}(\Delta)_0) = \mathscr{Q}(\Delta)_d^{(d)} \subset \mathscr{Q}(\Delta)_d \subset \mathscr{Q}(\Delta)_\infty$. The subvariety $\mathscr{Q}(\Delta)_0$ is T-equivariantly isomorphic to X, so its T-fixed points are in natural bijection with those of X. The same holds for each copy of $\epsilon_d(\mathscr{Q}(\Delta)_0))$, so the fixed points are simply $\epsilon_d(x)$ for $d \in A_+$ and $x \in X^T = \mathscr{Q}(\Delta)_0$. □

Now, recall from the beginning of this chapter that for each vertex p of a smooth polytope P, there is a subset $I(p) \subseteq \{1, \ldots, r\}$ so that the v_i for $i \in I(p)$ are the inward normal vectors of the facets containing p, and $\{u_i(p) \mid i \in I(p)\}$ is the basis of M dual to $\{v_i \mid i \in I(p)\}$. We defined

$$\mathsf{J}_{d,p} = \left(\prod_{i \in I(p)} \frac{1}{(x^{u_i(p)}q^{-1}; q^{-1})_{\beta(d)_i}}\right)\left(\prod_{j \notin I(p)} \frac{1}{(q^{-1}; q^{-1})_{\beta(d)_j}}\right).$$

Proposition 3.6.5. *Let y be the fixed point in X corresponding to a vertex p in P. The equivariant multiplicity at $\epsilon_d(y)$ is equal to*

$$\left(\frac{1}{(q;q)_\infty}\right)^{r-n} \frac{\mathsf{J}_{d,p}}{\prod_{i \in I(p)} (x^{u_i(p)}; q)_\infty}.$$

Proof. Recall that for $x \in X$ a smooth fixed point and a decomposition of T_x^*X into non-vanishing characters $m_1, \ldots, m_n$, the equivariant multiplicity at p is $\prod_i \frac{1}{1-x^{m_i}}$. For an ind-variety, this still holds if the resulting product converges.

The tangent space to $\mathscr{Q}(\Delta)_\infty$ at $\epsilon_d(y)$ splits into tangent and normal directions to the closed embedding $\epsilon_d(\mathscr{Q}(\Delta)_\infty) \hookrightarrow \mathscr{Q}(\Delta)_\infty$, meaning we have

$$T_{\epsilon_d(y)}\mathscr{Q}(\Delta)_\infty = T_{\epsilon_d(y)}\epsilon_d(\mathscr{Q}(\Delta)_\infty) \oplus i^*_{\epsilon_d(y)} N_{\epsilon_d(\mathscr{Q}(\Delta)_\infty)}\mathscr{Q}(\Delta)_\infty, \tag{3.6.6}$$

where $T_{\epsilon_d(y)}\epsilon_d(\mathscr{Q}(\Delta)_\infty)$ is isomorphic as a T-representation to $T_y\mathscr{Q}(\Delta)_\infty$.

First, we decompose $T_y\mathscr{Q}(\Delta)_\infty$:

In the arc scheme, we have coordinates on $(\mathbb{C}^r)^\infty$

$$\begin{bmatrix} f_1^{(0)} & \cdots & f_r^{(0)} \\ f_1^{(1)} & \cdots & f_r^{(1)} \\ \vdots & & \vdots \end{bmatrix},$$

and the point y,

$$\begin{bmatrix} p_1 & \cdots & p_r \\ 0 & \cdots & 0 \\ \vdots & & \vdots \end{bmatrix},$$

is contained in an affine open set in $\Lambda^0 X$ with independent coordinates

- $\frac{f_i^{(k)}}{\prod_{j\notin I(p)}(f_j^{(0)})^{-\langle u_i(p),v_j\rangle}}$, for $i \in I(p)$ and $k \geqslant 0$,
- $\frac{f_i^{(k)}}{f_i^{(0)}}$ for $i \notin I(p)$ and $k \geqslant 1$.

These are T-eigenfunctions which have characters

- $e^{u_i(p)}q^k$, for $i \in I(p)$ and $k \geqslant 0$,
- q^k, for $i \notin I(p)$ and $k \geqslant 1$.

Thus, the tangent space of y in $\mathscr{Q}(\Delta)_\infty$ (which we recall is the subset of $\Lambda^0 X$ where finitely many coordinates are non-zero) splits into a direct sum of the duals of the characters listed above. This contributes

$$\left(\frac{1}{(q;q)_\infty}\right)^{r-n}\frac{1}{\prod_{i\in I(p)}(x^{u_i(p)};q)_\infty}$$

to the equivariant multiplicity at $\epsilon_d(y)$.

Second, we note that $\mathscr{Q}(\Delta)_d \bigcap \epsilon_d(\mathscr{Q}(\Delta)_\infty) = \mathscr{Q}(\Delta)_d^{(d)}$, so there is an identification of the following two bundles:

$$i^*_{\mathscr{Q}(\Delta)_d^{(d)}} N_{\epsilon_d(\mathscr{Q}(\Delta))_\infty} \mathscr{Q}(\Delta)_\infty = N_{\mathscr{Q}(\Delta)_d^{(d)}} \mathscr{Q}(\Delta_d).$$

But this normal bundle on the right-hand side is easily computed:

$$N_{\mathscr{Q}(\Delta)_d^{(d)}} \mathscr{Q}(\Delta_d) = \bigoplus_{i=1}^{r} \bigoplus_{k=1}^{\beta(d)_i} q^{-k} \otimes \mathscr{O}(D_i),$$

where $D_i \subseteq X$ is the ith invariant divisor, and q^{-k} denotes the trivial line bundle on X with character q^{-k}. For $i^*_{\epsilon_d(y)} N_{\epsilon_d(\mathscr{Q}(\Delta)_\infty)} \mathscr{Q}(\Delta)_\infty$, we further restrict this to y. Then, $\mathscr{O}(D_i)$ restricts to $e^{u_i(p)}$ if $i \in I(p)$ or the trivial bundle otherwise. This contributes $\mathsf{J}_{d,p}$ to the equivariant multiplicity at $\epsilon_d(y)$.

□

Remark 3.6.7. Using some standard identifications, together with the fact that all our moduli spaces compactifying maps $\mathbb{P}^1 \to X$ have rational singularities, one can show that $\mathsf{J}_{d,p}$ is a specialization of the *equivariant multiplicity* of the fixed component $\mathscr{Q}(\Delta)_d^{(d)}$—that is, the contribution of this component to the Atiyah-Segal localization formula for $\chi(\mathscr{O}_{\mathscr{Q}(\Delta)_d})$, with respect to the $\mathbb{C}^*$-action on $\mathscr{Q}(\Delta)_d$ which rescales the source curve. This is the interpretation of q in the formula for J: it is the character of this $\mathbb{C}^*$-action. See [22, §2.2, §4.2]. When X is Fano, the equivariant multiplicity referred to above is the d-th term of the K-theoretic J-function in the quantum K-theory of X; see [23, 24].

Now, we are in a position to prove our main theorem:

Theorem 3.6.8. *Let P be a smooth polytope, with notation as above. Then*

$$\sum_{u \in P \cap M} g_{\mathscr{A}_P, u}\, x^u = \frac{1}{(q;q)_\infty^{r-n}} \sum_{p \in V(P)} x^p \sum_{d \in A_+} \frac{q^{\sum a_i \beta(d)_i} \cdot \mathsf{J}_{d,p}}{\prod_{i \in I(p)} (x^{u_i(p)}; q)_\infty},$$

where $g_{\mathcal{A}_P,u}$ is the rational function defined in (3.6.2), and $V(P)$ is the set of vertices of P.

In the radially symmetric case, Theorem 3.2.1 follows immediately by multiplying both sides by $(q;q)_{|a|}$, where $|a| = a_1 + \cdots + a_r$ as before.

Proof. The polytope P determines a toric variety X, a normal fan Δ, and a line bundle $\mathscr{O}(\sum_i a_i D_i)$. The left-hand side is simply $\chi_{\mathrm{T}}(\mathscr{Q}(\Delta)_\infty, \mathscr{O}(\sum_i a_i D_i^0))$, by Theorem 3.6.3.

The right-hand side comes by applying the Atiyah-Bott formula to compute the Euler characteristic of $\mathscr{O}(\sum_i a_i D_i^0)$. This is a sum over the T-fixed points $\epsilon_d(y)$ (where $y \in X^T$ corresponds to $p \in V(P)$) of the product of the T-character of $i^*_{\epsilon_d(y)}\mathscr{O}(\sum_i a_i D_i^0)$ and the equivariant multiplicity at $\epsilon_d(y)$ in $K_{\mathrm{T}}(y)$, after completing with respect to q. Proposition 3.6.5 contributes the factor

$$\frac{\mathsf{J}_{d,p}}{(q;q)_\infty^{r-n} \cdot \prod_{i \in I(p)} (x^{u_i(p)};q)_\infty}$$

present in each term. So it only remains to demonstrate that the restriction of $\mathscr{O}(\sum_i a_i D_i^0)$ to $\epsilon_d(y)$ is x^p. As in the previous proposition, this can be done by picking coordinates around $\epsilon_d(y)$ using the Cox ring. □

Remark 3.6.9. A q-analogue of Ehrhart polynomials was introduced by Chapoton [12]. The q appearing in the rational function on the left of Theorem 3.6.8 is not related to the q appearing in [12]. For him, the q comes from specializing the lattice point generating function of a polytope to a univariate polynomial in a parameter q. For us, when q is set to 0, we recover the usual lattice point generating functions.

3.6.10 Combinatorial interpretation of q-series coefficients

In this section we relate the q-series coefficients (3.6.2) appearing in our main theorem to a q-enumeration of lattice paths. As before, we let P be a lattice polytope in $M_{\mathbb{R}}$, Δ the corresponding inward normal fan in $N_{\mathbb{R}}$. Let $v_1, \ldots, v_r \in N$ be an enumeration of the primitive elements in the rays of Δ. Then, P is defined as the intersection of half-spaces $\{u \,|\, \langle u, v_i \rangle \geqslant -a_i\}$.

Let us define the affine linear map $\phi : M \to N$ by

$$\phi(m) = \sum_i (\langle m, v_i \rangle + a_i) v_i.$$

Remark 3.6.11. If $\sum_i a_i v_i = 0$, the map $\phi : M \to N$ is linear. This induces a fan on M pulled back from the one on N.

Definition 3.6.12. We define a path to be a sequence of vectors $(v_{i_1}, v_{i_2}, \dots, v_{i_k})$ in $\Delta(1)$. For $\mathscr{P}$ a path, let $F(\mathscr{P})$ denote the number of inversions $(v_{i_j}, v_{i_{j'}}$ such that $j < j'$ and $i_j > i_{j'})$ in $\mathscr{P}$, and define the function $f_m(q) \in \mathbb{Z}[q]$ by

$$f_{P,m}(q) = \sum_{\mathscr{P}} q^{F_{\tilde{\rho}}(\mathscr{P})},$$

where the sum is over paths $\mathscr{P}$ with exactly $\langle m, v_i \rangle + a_i$ steps in the direction of v_i.

Proposition 3.6.13. *The coefficient of x^m in $(q;q)_{\sum_i a_i} \cdot \chi_{\mathbb{T}}(\Lambda^0 X, \mathscr{O}(\sum_i a_i D_i))$ is $f_{P,m}(q)$.*

Proof. Both $f_{P,m}(q)$ and the coefficient of x^m in $(q;q)_{\sum_i a_i} \cdot \chi_{\mathbb{T}}(\Lambda^0 X, \mathscr{O}(\sum_i a_E D_i))$ can be identified with the q-multinomial coefficient $\binom{\sum_i a_i}{a_1, a_2, \dots}_q$. $\square$

3.6.14 Jacobi Triple Product

Here we demonstrate that applying the main theorem in a degenerate case produces the Jacobi triple product identity. Recall the q-hypergeometric series we introduced at the beginning of this chapter:

$$\phi_n(x;q) = \sum_{k=0}^{\infty} \frac{q^{nk}}{(q^{-1};q^{-1})_k (x;q^{-1})_k}.$$

Applying Theorem 3.6.3 to the case of the trivial bundle on $\Lambda^0\mathbb{P}^1$. This produces the identity

$$1 = \frac{1}{(q;q)_\infty}\left(\frac{1}{(x;q)_\infty}\phi_0(x;q) + \frac{1}{(x^{-1};q)_\infty}\phi_0(x^{-1};q)\right),$$

or equivalently

$$(q)_\infty(x;q)_\infty(x^{-1};q)_\infty = (x^{-1};q)_\infty\phi_0(x;q) + (x;q)_\infty\phi_0(x^{-1};q).$$

We use the identity $(xq^{-1};q^{-1})_d = (-x)^d q^{-\binom{d+1}{2}}(x^{-1}q;q)_d$ and divide by $(1-x)$ to obtain that the triple product $(q)_\infty(xq;q)_\infty(x^{-1};q)_\infty$ is equal to

$$\sum_{d\geqslant 0}\frac{\prod_{i\geqslant d+1}(1-x^{-1}q^i)}{(q^{-1};q^{-1})_d}q^{\binom{d+1}{2}}(-x)^{-d-1} + \sum_{d\geqslant 0}\frac{\prod_{i\geqslant d+1}(1-xq^i)}{(q^{-1};q^{-1})_d}q^{\binom{d+1}{2}}(-x)^d.$$

Then, the classical Jacobi triple product identity, which states that

$$(q)_\infty(xq;q)_\infty(x^{-1};q)_\infty = \sum_{d\in\mathbb{Z}} q^{d(d+1)/2}(-x)^d,$$

follows from the next proposition:

Proposition 3.6.15.

$$\sum_{d\geqslant 0}\frac{\prod_{i\geqslant d+1}(1-xq^i)}{\prod_{i=1}^d(1-q^{-1})}q^{d(d+1)/2}(-x)^d = \sum_{k\geqslant 0}q^{k(k+1)/2}(-x)^k.$$

Proof. We will require two more routine identities: the q-binomial identity states that

$$\sum_{d=0}^{k}\frac{1}{(q^{-1};q^{-1})_d(q;q)_{k-d}} = 1.$$

We quickly remark that one way to prove this identity is to use the integration formula for $\chi(\mathbb{P}^k,\mathscr{O}_{\mathbb{P}^k})$, where $\mathbb{P}^k$ has the torus action induced by the action on $\mathbb{C}^{k+1}$ with characters

$1, q, \ldots, q^k$.

We also will need the identity

$$e_n(1, q, q^2, \ldots) = \frac{q^{\binom{n}{2}}}{(q;q)_n},$$

see e.g. [38, Section 7.8]. Then, we can calculate:

$$\begin{aligned}\sum_{d \geqslant 0} \frac{\prod_{i \geqslant d+1}(1 - xq^i)}{(q^{-1};q^{-1})_d} q^{d(d+1)/2}(-x)^d &= \sum_{d \geqslant 0} \sum_{n \geqslant 0} \frac{e_n(q^{d+1}, e^{d+2}, \ldots)(-x)^n}{(q^{-1};q^{-1})_d} q^{\binom{d+1}{2}}(-x)^d, \\ &= \sum_{d \geqslant 0} \sum_{n \geqslant 0} \frac{1}{(q^{-1};q^{-1})_d (q;q)_n} q^{n(d+1)} q^{\binom{n}{2}} q^{\binom{d+1}{2}} (-x)^{n+d}.\end{aligned}$$

Using the substitution $k = n + d$, we obtain

$$\sum_{d \geqslant 0} \sum_{n \geqslant 0} \frac{1}{(q^{-1};q^{-1})_d (q;q)_n} q^{n(d+1)} q^{\binom{n}{2}} q^{\binom{d+1}{2}} (-x)^{n+d} = \sum_{k \geqslant 0} \sum_{d=0}^{k} \frac{1}{(q^{-1};q^{-1})_d (q;q)_{k-d}} q^{\binom{k+1}{2}} (-x)^k,$$

whence the proposition (and the Jacobi triple product) follows via the q-binomial identity.

□

3.7 Action of q-difference operators on generalized Rogers-Szegő polynomials

In this section, we study the action of a q-difference operator on the polynomial on the left-hand side of our main theorem. Given a smooth, radially symmetric polytope P, written as an intersection of half-spaces $\{u \,|\, \langle u, v_i \rangle \geqslant -a_i\}$, we define

$$RS_P(x;q) = \sum_{u \in P \cap M} \binom{|a|}{\langle u, v_1 \rangle + a_1, \ldots, \langle u, v_r \rangle + a_r}_q x^u.$$

In a sense to be explained below, these are natural generalizations of the *Rogers-Szegő polynomials* in [39].

The polynomial $RS_P(x;q)$ is translation-invariant: for any $u \in M$, $RS_{P+u}(x;q) = x^u \cdot RS_P(x;q)$. So we may assume one of the vertices of P is the origin, and choose a basis $u_1, \ldots, u_n$ for M so that these are primitive vectors along the edges of P at the origin. Writing $x_1, \ldots, x_n$ for the variables corresponding to this basis, RS_P lies in the polynomial ring $\mathbb{Z}[x_1, \ldots, x_n, q]$.

The i^{th} q-shift operator $T_{i,q}$ acts on a function $f(x_1, \ldots, x_n)$ by

$$T_{i,q}\colon f(x_1, \ldots, x_i, \ldots, x_n) \mapsto f_(x_1, \ldots, qx_i, \ldots, x_n).$$

The associated q-difference operator is

$$D_{i,q}(f) = f - T_{i,q}(f).$$

Let us index the facets of P (of which there are $r > n$) so that for i from 1 to n, the exponent of x_i in x^u is $\langle u, v_i \rangle$. By translating P, we have ensured that the numbers $a_1, \ldots, a_n$ in the defining equations for P are all 0. Let P_i be the polytope where the i-th facet is moved one unit in. More precisely, we replace the defining equation $\{u \,|\, \langle u, v_i \rangle \geqslant 0\}$ with $\{u \,|\, \langle u, v_i \rangle \geqslant 1\}$.

Theorem 3.7.1. *We have*

$$D_{i,q}(RS_P) = (1 - q^{|a|}) RS_{P_i}.$$

Proof. This is a straightforward computation:

$$\begin{aligned} D_{i,q} RS_P(x;q) &= \sum_{u \in P \cap M} \binom{|a|}{\langle u, v_1 \rangle, \ldots, \langle u, v_i \rangle, \ldots, \langle u, v_n \rangle, \ldots, \langle u, v_r \rangle + a_r}_q D_{i,q} x^u, \\ &= \sum_{u \in P \cap M} \binom{|a|}{\langle u, v_1 \rangle, \ldots, \langle u, v_i \rangle, \ldots, \langle u, v_n \rangle, \ldots, \langle u, v_r \rangle + a_r}_q (1 - q^{\langle v_i, u \rangle}) x^u, \\ &= (1 - q^{|a|}) \sum_{u \in P_i \cap M} \binom{|a| - 1}{\langle u, v_1 \rangle, \ldots, \langle u, v_i \rangle - 1, \ldots, \langle u, v_n \rangle, \ldots, \langle u, v_r \rangle + a_r}_q x^u. \end{aligned}$$

The last line is the right side of the equality in the theorem statement. The change of indexing in the last two lines is valid because for $u \in (P \backslash P_i) \cap M$, the inner product $\langle v_i, u \rangle$ vanishes, so $(1 - q^{\langle v_i, u \rangle}) = 0$. $\square$

Now, let us also say what we can about applying the difference operator to the left-hand side of our main theorem. If P is a polytope and p, q are vertices of P, then as in the earlier sections of this thesis, we define the sets $I(p), I(p') \subseteq \{1, \ldots, r\}$ so that the v_i for $i \in I(q)$ are the inward normal vectors of the facets containing p, $\{u_i(p) \mid i \in I(p)\}$ is the basis of M dual to $\{v_i \mid i \in I(p)\}$, and define $u_i(p')$ similarly. Let x_i be the variables corresponding to the basis $u_i(p)$. We can write the p'-th term in the Atiyah-Bott sum as

$$\frac{x^{p'}}{\prod_{i \in I(p')} (x^{u_i(p')}; q)_\infty} \sum_{d \in A_+} \frac{q^{\sum a_i \beta(d)_i}}{\left(\prod_{i \in I(p')} (x^{u_i(p')} q^{-1}; q^{-1})_{\beta(d)_i} \right) \left(\prod_{j \notin I(p')} (q^{-1}; q^{-1})_{\beta(d)_j} \right)}.$$

Then, for some fixed $i_0 \in I(p)$, we can calculate directly

Proposition 3.7.2. *We have*

$$D_{i_0,q}\frac{x^{p'}}{\prod\limits_{i\in I(p')}(x^{u_i(p')};q)_\infty}\sum_{d\in A_+}\frac{q^{\sum a_i\,\beta(d)_i}}{\left(\prod\limits_{i\in I(p')}(x^{u_i(p')}q^{-1};q^{-1})_{\beta(d)_i}\right)\left(\prod\limits_{j\notin I(p')}(q^{-1};q^{-1})_{\beta(d)_j}\right)}$$

$$=\frac{x^{p'}}{\prod\limits_{i\in I(p')}(x^{u_i(p')};q)_\infty}\sum_{d\in A_+}\frac{q^{\sum a_i\,\beta(d)_i}}{\left(\prod\limits_{j\notin I(p')}(q^{-1};q^{-1})_{\beta(d)_j}\right)}$$

$$\cdot\left(\frac{1}{\left(\prod\limits_{i\in I(p')}(x^{u_i(p')}q^{-1};q^{-1})_{\beta(d)_i}\right)}-\frac{q^{\langle p',v_{i_0}\rangle}}{\left(\prod\limits_{i\in I(p')}(x^{u_i(p')}q^{-1};q^{-1})_{\beta(d)_i-\langle u_i(p'),v_{i_0}\rangle}\right)}\right)$$

We now make an extended examination of the case of a lattice simplex. Let $P_{k,n}$ be the convex hull of the origin and $ke_1,\dots,ke_n$, where e_i are the standard basis vectors in $\mathbb{Z}^n$. The inward normal fan Δ has rays $e_1,\dots,e_n$ and $e_0:=-e_1-\cdots-e_n$, and a maximal cone generated by each subset of $\{e_0,\dots,e_n\}$ of size n. The toric variety that corresponds to this fan is $\mathbb{P}^n$, which has divisors $D_0,D_1,\dots,D_n$ given by the vanishing of the corresponding coordinates. The polytope $P_{k,n}$ corresponds to the line bundle $L_{k,n}=\mathscr{O}(kD_0)$.

The toric arc scheme $\Lambda^0\mathbb{P}^n$ is the infinite dimensional projective space with homogeneous coordinates

$$\begin{bmatrix} f_0^{(0)} & \cdots & f_n^{(0)} \\ f_0^{(1)} & \cdots & f_n^{(1)} \\ \vdots & & \vdots \end{bmatrix},$$

and we consider the line bundle

$$L^\Lambda_{k,n}=\mathscr{O}(kD_0^0),$$

where D_0^0 is defined by the vanishing of $f_0^{(0)}$. By Theorem 3.6.3, we calculate that

$$\chi_{\mathrm{T}}(\mathscr{Q}(\Delta)_\infty, L_{k,n}^{\Lambda}|_{\mathscr{Q}(\Delta)_\infty}) = \sum_{i_0+i_1+\ldots+i_n=k} \frac{1}{(q;q)_{i_0}(q;q)_{i_1}\ldots(q;q)_{i_n}} x_1^{i_1}\ldots x_n^{i_n}.$$

This is quite close to the Rogers-Szegő polynomials mentioned at the beginning of this chapter:

Definition 3.7.3. The k-th Rogers-Szegő polynomial in n variables is the sum

$$RS_{k,n} = \sum_{i_0+i_1+\ldots+i_n=k} \binom{k}{i_0, i_1, \ldots, i_n}_q x_1^{i_1}\ldots x_n^{i_n}.$$

Corollary 3.7.4. *We have*

$$(q;q)_k \cdot \chi_{\mathrm{T}}(\mathscr{Q}(\Delta)_\infty, L_{k,n}^{\Lambda}|_{\mathscr{Q}(\Delta)_\infty}) = RS_{k,n}.$$

We must remark here that RS-polynomials are the $t=0$ specialization of one-row Macdonald polynomials (see [26, Section 3]), so the remainder of this section also follows from specializing the operators appearing in the theory of Macdonald polynomials as detailed in [29, Chapter 5] and [30, Chapter VI]. However, we proceed without appealing to that theory.

Theorem 3.7.1 says that

$$\frac{1}{x_i} D_{i,q} RS_{k,n} = (1-q^k) RS_{k-1,n}.$$

From [26], there is a recursion:

Proposition 3.7.5 (Hikami recursion)**.** *Using the convention $RS_{k,n} = 0$ for $k < 0$, we have*

$$RS_{k,n} = \sum_{l=1}^{n+1} (-1)^{l-1} \frac{(q;q)_{k-1}}{(q;q)_{k-l}} e_l(x) RS_{k-l,n}$$

where e_k is the k-th elementary symmetric polynomial in the terms $1, x_1, \dots, x_n$.

Since

$$\frac{(q;q)_{k-1}}{(q;q)_{k-l}} RS_{k-l,n} = \frac{1}{x_i^{l-1}} D_{i,q}^{l-1} RS_{k-1,n},$$

we make the following definitions:

Definition 3.7.6. For $1 \leqslant i \leqslant n$, we set

$$R_i := \sum_{l=1}^{n+1} (-1)^{l-1} \frac{e_l(x)}{x_i^{l-1}} D_{i,q}^{l-1} = \sum_{l=0}^{n} (-1)^l \frac{e_{l+1}(x)}{x_i^l} D_{i,q}^l$$

and

$$L_i := \frac{1}{(1-q)x_i} D_{i,q}.$$

By combining the previous proposition and the Hikami recursion, we obtain the following, which appears in many places including [19] in the case of $n = 1$.

Proposition 3.7.7. *We have*

$$R_i(RS_{k-1,n}) = RS_{k,n} \quad \text{and} \quad L_i(RS_{k,n}) = \frac{1-q^k}{1-q} RS_{k-1,n}.$$

It follows that

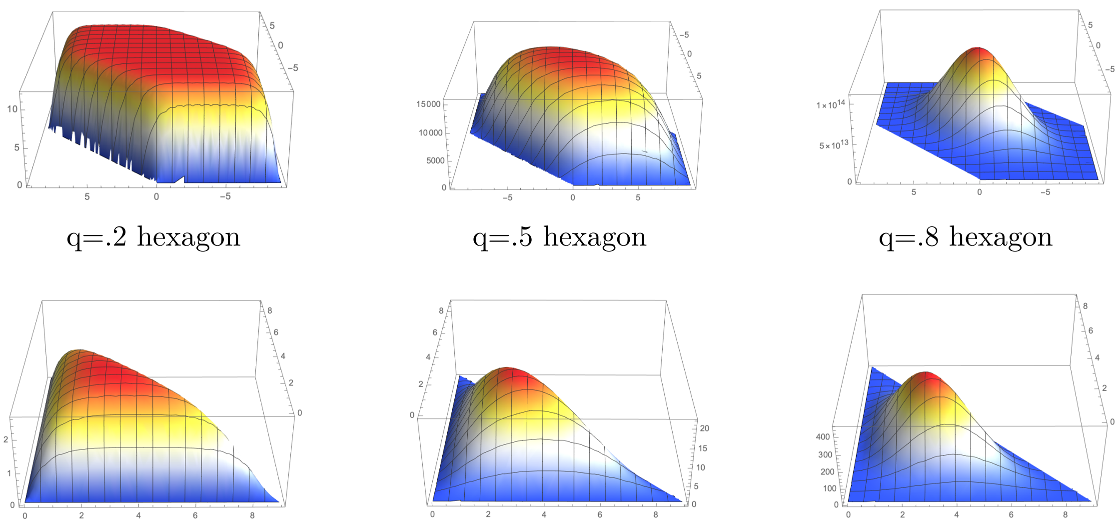
$$[L_i, R_i](RS_{k,n}) = q^k RS_{k,n}.$$

3.8 Measures depending on q

We also wish to draw attention to the measure on the lattice points in P given by the q-series coefficients. Namely, if δ_u is the Dirac measure at u, we consider the measure:

$$\mu_P(x) = \sum_{u \in P \cap M} \binom{|a|}{\langle u, v_1 \rangle + a_1, \dots, \langle u, v_r \rangle + a_r}_q \delta_u(x).$$

For $q = 1$, this measure asymptotically limits to the restriction of a multivariate Gaussian distribution, and for $q = 0$, it specializes to uniform measure on lattice points. The following are some pictures of the associated q-gamma distributions for different values of q. The polytope in the top row is the hexagon (i.e. 2-dimensional permutohedron) with vertices $\pm(9,0), \pm(0,9), \pm(9,9)$. The polytope in the bottom row is the simplex with vertices $(0,0), (9,0)$, and $(0,9)$.



q=.2 hexagon q=.5 hexagon q=.8 hexagon

q=.2 simplex q=.5 simplex q=.8 simplex

Figure 3.2: Measures from generalized RS polynomials

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Appendix A

Multiplicities of cones

Given cones $\alpha \prec \beta$, let $\mathrm{mult}_\alpha(\beta)$ denote the multiplicity of $\overline{\beta}$ in N_α. If $\alpha = \{0\}$ we obtain that $\mathrm{mult}_\alpha(\beta) = \mathrm{mult}(\beta)$ is the usual multiplicity of β. The following lemma describes relative multiplicities of simplicial cones in terms of usual multiplicities. Let α have rays $\rho_1, \ldots, \rho_k$, and β have rays $\rho_1, \ldots, \rho_k, \rho_{k+1}, \ldots, \rho_l$:

Lemma A.0.1. $\mathrm{mult}_\alpha(\beta) = \mathrm{mult}(\beta) \prod_{i=k+1}^{l} \frac{\mathrm{mult}(\alpha)}{\mathrm{mult}(\alpha+\rho_i)}$.

Proof. To simplify notation, we assume that β is a maximal cone. Then we have the following diagram of exact sequences:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \langle v_{\rho_1}, \ldots, v_{\rho_l} \rangle & \longrightarrow & N & \longrightarrow & N/\langle v_{\rho_1}, \ldots, v_{\rho_l} \rangle & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \langle v_{\overline{\rho_{k+1}}}, \ldots, v_{\overline{\rho_l}} \rangle & \longrightarrow & N_\alpha & \longrightarrow & N_\alpha/\langle v_{\overline{\rho_{k+1}}}, \ldots, v_{\overline{\rho_l}} \rangle & \longrightarrow & 0,
\end{array}$$

where the top and bottom quotient groups on the right have cardinality $\mathrm{mult}(\beta)$ and $\mathrm{mult}_\alpha(\beta)$ respectively. We add the kernels and cokernels to the diagram:

$$\begin{tikzcd}
0 \arrow[r] & \langle v_{\rho_1}, \ldots, v_{\rho_k} \rangle \arrow[r, "\cong"] \arrow[d] & \langle v_{\rho_1}, \ldots, v_{\rho_k} \rangle \arrow[r] \arrow[d] & A \arrow[d] & \\
0 \arrow[r] & \langle v_{\rho_1}, \ldots, v_{\rho_l} \rangle \arrow[r] \arrow[d] & N \arrow[r] \arrow[d] & N/\langle v_{\rho_1}, \ldots, v_{\rho_l} \rangle \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \langle v_{\overline{\rho_{k+1}}}, \ldots, v_{\overline{\rho_l}} \rangle \arrow[r] \arrow[d] & N_\alpha \arrow[r] \arrow[d] & N_\alpha/\langle v_{\overline{\rho_{k+1}}}, \ldots, v_{\overline{\rho_l}} \rangle \arrow[r] \arrow[d] & 0 \\
& B \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & 0.
\end{tikzcd}$$

By the snake lemma, the sequence of kernels leading to cokernels is exact, so in fact $A \cong B$. Thus $\mathrm{mult}(\beta) = \mathrm{mult}_\alpha(\beta)|B|$. The cardinality of B on the other hand is also easy to determine: the image of $\langle v_{\rho_1}, \ldots, v_{\rho_l} \rangle$ in $\langle v_{\overline{\rho_{k+1}}}, \ldots, v_{\overline{\rho_l}} \rangle$ is just $\langle \overline{v_{\rho_{k+1}}}, \ldots, \overline{v_{\rho_l}} \rangle$. If $\overline{v_\rho} = b_\rho v_{\overline{\rho}}$, the cardinality of the cokernel (i.e. B) is $\prod_{i=k+1}^{l} b_{\rho_i}$. But in the proof of Proposition 2.3.4, we saw $b_\rho = \frac{\mathrm{mult}(\alpha+\rho)}{\mathrm{mult}(\alpha)}$, which proves the claim. □