Thermodynamic formalism, statistical properties and multifractal analysis of non-uniformly hyperbolic systems

Dissertation

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Abstract

In this thesis, we concentrate on the study of different properties of non-uniformly hyperbolic systems. From §1 to §3, we introduce our main results, preliminaries and main techniques. The core of the thesis is in §4 - §6, where we illustrate the detailed proofs of our main results.

The first result is on thermodynamic formalism. In §4, we work with a C^∞ non-uniformly hyperbolic diffeomorphism on the 2-torus, known as the Katok map. We prove for a Hölder continuous potential with one additional condition, or geometric t-potential φ_t with $t < 1$, the equilibrium state exists and is unique. Motivated by the 'Orbit Decomposition' technique we used in the derivation of the thermodynamic formalism, we also obtain a weak version of Gibbs property and the level-2 large deviation principle for the equilibrium state from above.

In §5, we prove an asymptotic version of the Central Limit Theorem for the unique measure of maximal entropy of the geodesic flows on rank-one non-positively curved manifold with Hölder observables. We generalize an approach of Denker, Senti and Zhang from the discrete case with uniform expansiveness and specification to the continuous flow where only partial specification holds, and simplify the condition so that only Lindeberg condition and a weak positive variance condition are required. Moreover, we show that the Lindeberg condition follows from a strong positive variance condition which parallels the one used in the classic study of Central Limit Theorem in dynamics. We also extend our results to dynamical arrays of Hölder observables, and to weighted periodic orbit measures which converge to a unique equilibrium state.

Finally, in §6, we investigate how results in thermodynamic formalism of non-uniformly hyperbolic systems can benefit the study of dimension theory and multifractal analysis in those cases. The main example we study here is the topological entropy and Hausdorff dimension of Lyapunov level sets in the case of the geodesic flow on rank-one surfaces with no focal points.

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Finally, let me acknowledge that the work in §4 was published in the journal Ergodic Theory & Dynamical Systems. The work in §5, which was published in the journal Communications in Mathematical Physics, is a joint work with Daniel Thompson, with the one in §6 being a joint work with Kiho Park. With both coauthors we have created countless meaningful memories which I will never forget for my lifetime.

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CHAPTER 1

Introduction

This thesis is devoted to the study of equilibrium states in non-uniformly hyperbolic systems. In a discrete dynamical system (X, T), equilibrium states refer to those Borel probability measures that are invariant under the action of T and maximize the free energy associate to certain potential $\varphi \in C(X)$, which is defined as $E_\mu(f) = h_\mu(f) + \int \varphi d\mu$, where $h_\mu(f)$ is the measure-theoretic entropy of μ. This characterization of free energy also parallels the classic mechanical energy in physics as entropy gives the global estimate on expansion under f with respect to μ, thus being considered as the kinetic energy, and potential function literally quantifies the total interaction between points, thus generating the potential energy. This natural inner connection motivates the study of thermodynamic formalism in dynamics, whose foundation was established by Ruelle [**45**], Sinai [**47**] and Bowen [**7**] who first developed the theory on the existence and uniqueness of equilibrium states. These results enable researchers to deeply understand the global behavior of orbits as well as their statistical properties, e.g. Bernoulli property, exponential decay of correlations and central limit theorem. Moreover, the tools enlightened by their work also lead to further applications including Gibbs property, large deviations and dynamical zeta functions [**43, 52**].

Thermodynamic formalism turns to be most useful when the orbit complexity grows exponentially. In particular, this is the case when the system possesses some degree of hyperbolicity. If the map is uniformly hyperbolic and topologically transitive, it is known that any Hölder continuous potential has an unique equilibrium state satisfying all the statistical properties from above as expected. In particular, when the potential function is the geometric potential, which measures rate of expansion in the unstable direction, the respective equilibrium state is a certain 'physical measure' known as the SRB measure (which stands for Sinai-Ruelle-Bowen).

At the same time, the uniform hyperbolicity condition could be too restrictive. For instance, for a smooth system, after slowing down the trajectories or adding some non-linear perturbations, it is still common to encounter some overall hyperbolic behaviors, but without uniformity in expansion or contraction. These phenomena naturally generalize to the conception of non-uniform hyperbolicity, which possesses asymptotic expansion or contraction, whose rate depends on the orbit in a way such that no uniform bound is admitted. Nevertheless, as for non-uniformly hyperbolic systems, a general theory on thermodynamic formalism is far away from being complete. In this thesis, we will study thermodynamic formalism of certain important non-uniformly hyperbolic examples in both discrete and continuous case, then investigate applying such formalism and the technique to obtain statistical properties of the equilibrium states as well as dimensional estimate over Lyapunov level sets. These three seemingly distinct objects are in fact closely related and can be studied together, due to the orbit decomposition technique applied here; see §3.2 for a full exposition of the technique. In short, this construction makes the equilibrium state and the essential part of the orbits satisfy certain properties that directly contribute to the establishment of statistical properties and dimensional estimates. Details are revealed through §4 to §6.

1.1. What is a non-uniformly hyperbolic system?

To provide the readers with enough motivation, we describe briefly in this section the historic development on non-uniformly hyperbolic systems by presenting some key ideas and important results. As mentioned above, the notion of non-uniform hyperbolicity originates from the more classical uniformly hyperbolic systems. Roughly speaking, the generalization is done in a way such that stepwise hyperbolicity is weakened so that almost every point admits long-term hyperbolicity with respect some invariant measure. This is explicitly defined by the notion of Lyapunov exponent (see §2.1.8 for definition), which is first developed in the study of stability of differential systems by Lyapunov [**31**] and Perron [**36**]. The work of Pesin [**37**] and Oseledets [**33**] developed these ideas in the context of non-uniformly hyperbolic dynamical systems.

In the mean time, it turns out that non-uniform hyperbolicity is also general enough in a way such that any compact smooth manifold with dimension at least two would support such a system which is smooth, volume preserving, and satisfies Bernoulli property (see [**19**]). Then it is natural to conjecture that whether such system with appropriate regularity (e.g. $C^{1+\alpha}$) is generic in some sense, which still remains one of the biggest open problem in the field of smooth dynamics.

On the other hand, non-uniformly hyperbolic systems still give sufficient structure to develop a powerful dynamical theory. That is to say, properties of uniformly hyperbolic systems, such as positive entropy, stable manifold theorem, absolute continuity of holomony map between leaves, spectral decomposition, as well as mixing property on each ergodic component, are still satisfied by non-uniformly hyperbolic systems almost everywhere (with respect to a prefixed positive smooth volume). The results above are based on the work of Pesin [**37**], also known as Pesin theory nowadays. Being the foundation of the non-uniformly hyperbolic dynamics on smooth manifolds, Pesin theory also has a wide extension to other settings, including billiards and other physical models [**27**], infinite dimensional space such as Hilbert space [**46**], and random mappings [**30**].

Finally, it is worth noticing that Pesin theory can not reveal its true power when we look for the uniqueness of equilibrium states, simply because we are concerned about the family of all invariant probability measures, instead of a single fixed one. Therefore, the study of thermodynamic formalism relies mostly on different techniques. During the past few decades, there has been a variety of techniques with distinct spirit that are able to show their respective strength in different settings, including but not limited to transfer operator [**11**], inducing scheme [**41**], Patterson-Sullivan measure [**35**], etc. In our case, as mentioned above, we apply orbit decomposition technique, which is based on Bowen's specification property and has been applied in a wide range of examples beyond uniform hyperbolicity, e.g. [**15**], [**16**], [**9**]. Full details concerning this technique will be diclosed in §3.2.

1.2. Thermodynamic formalism of the Katok Map

The first example we look at is the Katok map, which is a C^∞ toral automorphism in dimension 2 and generated by a slow-down of the trajectories of a uniformly hyperbolic toral automorphism in a small neighborhood near the fixed point (see §2.2 for full details). As in the case of its one dimensional model example of the Manneville-Pomeau map on S^1 ($x \to x + x^{1+\alpha} \mod 1$ for some $\alpha \in (0,1)$), non-uniform hyperbolicity of the Katok map is caused by the existence of the neutral fixed point. As mentioned in the preceeding paragraph, the technique we apply here is called 'Orbit Decomposition', which is first introduced in [**17**]. The main idea is to generalize the dynamical properties for the map and regularity conditions for potential from [**7**] in a way that they will hold on an 'essential collection of orbit segments' which dominates in global complexity of orbits (reflected by topological pressure; see §2.1.2). The potentials we observe on are Hölder continuous potentials φ satisfying a gap condition $\varphi(0) = P(\delta_0) < P(\varphi)$, where δ_0 is the Dirac measure at the origin, and geometric-t potentials defined by

$$\varphi_t = t\varphi^{geo},$$

with

$$\varphi^{geo} := -\log |D\widetilde{G}|_{E^u(x)}|$$

being the standard geometric potential, where $E^u(x)$ is the unstable distribution of $D\widetilde{G}$ at $x \in \mathbb{T}^2$ and $\widetilde{G}$ is the Katok map. We will prove that such potentials satisfy the required regularity property on an appropriate collection of orbit segments that dominate in pressure, thus make the orbit decomposition technique applicable. Notice that the Katok map is topologically conjugate to the original linear toral automorphism via a homeomorphism, thus has specification property and expansiveness. By the main theorem in [**7**], it has a unique measure of maximal entropy. Nevertheless, as the conjugacy map is not Hölder continuous, thermodynamic formalism of the Katok map is far from trivial. Meanwhile, the unstable distribution E^u is also not Hölder continuous, so we need to deal with these two families of potentials from above separately.

Before we state our theorem, we need one final remark on the notations. To define the Katok map, we need two parameters α and r_0. Roughly speaking, α is the exponential slow-down rate of the perturbation function near the origin, and r_0 is the radius of the perturbed region. Full details can be found in §2.2.

Our main result concerning thermodynamic formalism of the Katok map is stated as follows

Theorem A. *For the Katok map whose defining α and r_0 are sufficiently small, if the potential φ satisfies one of the following conditions*

(1) φ is Hölder continuous and satisfies $\varphi(\underline{0}) < P(\varphi)$ with $\underline{0}$ being the origin,

(2) φ is the geometric-t potential for $t \in (-\infty, 1)$,

then it has a unique equilibrium state.

The orbit decomposition technique has one additional advantage in constructing the equilibrium state as a non-uniform Gibbs measure. In our case where the map satisfies global specification, we obtain a global weak Gibbs property for the equilibrium states derived in Theorem A, which is enough to conclude its large deviation principle. The following result addresses the ideas from above and is proved in §4.5.

THEOREM B. *The equilibrium states derived in Theorem A have the level-2 large deviations principle.*

1.3. Central Limit Theorem in non-positive curvature

The essential collection of orbit segments constructed in the orbit decomposition technique can be readily used in the study of thermodynamic formalism of non-uniformly hyperbolic systems in geometric settings, as well as to show the statistical behaviors of the equilibrium states. In probability, Central Limit Theorem describes the behavior of the distribution of normalized sum of i.i.d random variables being asymptotically normal. In dynamical systems, it can be shown that systems with uniform hyperbolicity exhibit such stochastic behavior, see for example [**44**]. A natural idea is to extend above results to those systems where hyperbolicity can not be demonstrated globally, which is the main topic of the second part of this thesis. We consider the case of the geodesic flow on a compact non-positively curved rank-one Riemannian manifold, and estimate the limit distribution of ergodic sums (integrals) with respect to the measure of maximal entropy. We will show that certain approximations of such measure by closed geodesics obey an asymptotic version of Lindeberg Central Limit Theorem.

The Lindeberg condition is a criteria from classic probability which says that the variance of any individual random variable is negligible compared to the total variance. In many situations, this condition serves as an equivalent condition to the Central Limit Theorem when the random variables being added are only independent but not identically distributed. This idea was recently explored by Denker, Senti and Zhang [**18**] in the setting of discrete dynamics where uniform specification is available.

In our result, we extend their idea to the case of non-uniformly hyperbolic flows. The target measure we study is the measure of maximal entropy μ_{KBM} (known as the Knieper-Bowen-Margulis measure). We construct a sequence of certain measures distributed on regular closed geodesics that converge to μ_{KBM} in weak*-topology. In particular, the closed geodesics selected are those with enough hyperbolicty and obtained by Orbit Decomposition technique. We will then establish the Central Limit theorem for the time average of Hölder continuous observables with respect to the sequence of measures constructed from above.

In order to state our main theorem, we briefly introduce some notations. Consider M being a compact smooth connected rank-one Riemannian manifold with non-positive curvature. Denote the action of geodesic flow on the unit tangent bundle T^1M by $G = \{g_t\}_{t\in\mathbb{R}}$. For each $l \in \mathbb{N}$ and $\eta > 0$, denote $\mathrm{Per}_R^{\geq\eta}(T_l - \delta_l, T_l]$ by the set of regular closed geodesics whose period is in $(T_l - \delta_l, T_l]$ and 'hyperbolic strength' is at least η, where $\delta_l \downarrow 0$ and $T_l \uparrow \infty$. Define the measure m_l by choosing one point on each such geodesic (named E_l) and distributing mass equally. We also define the measure ν_l by 'lifting' m_l via a gluing process on $E_l^{k_l}$ using specification, where $k_l \uparrow \infty$. Write S_l for the total length of the orbit segment specified this way. A simple version of our main result is stated as follows.

Theorem C. *Suppose the observable f is Hölder continuous and satisfies the following weak positive variance condition*

$$\liminf_{l\to\infty} \int (F(\cdot, T_l) - \int F(\cdot, T_l)dm_l)^2 dm_l > 0,$$

where $F(v,t) = \int_0^t f(g_s(v))ds$, then the sequence of measures (ν_l) constructed as above converges to μ_{KBM} and satisfies the following asymptotic Central Limit Theorem

$$\lim_{l\to\infty} \nu_l \left(\left\{ v : \frac{F(v,S_l) - S_l \int f d\nu_l}{\sigma_{\nu_l}(F(\cdot, S_l))} \leq a \right\} \right) = N(a) \text{ for all } a \in \mathbb{R},$$

where N is the cumulative distribution function of the standard normal distribution $\mathcal{N}(0,1)$, if and only if a Lindeberg-type condition holds for (ν_l).

Details on the choice of parameters and the statement of Lindeberg condition can be found in §2.5 and §5. Meanwhile, we can show that the Lindeberg condition actually holds under a stronger assumption on variance. The following result is proved in §5.3.

Theorem D. *Under the above setting, if instead f satisfies the following strong positive variance condition*

$$\liminf_{l\to\infty} \int \left(\frac{F(\cdot,T_l) - \int F(\cdot,T_l)dm_l}{\sqrt{T_l}}\right)^2 dm_l > 0, \tag{1.3.1}$$

then the Lindeberg condition holds for (ν_l).

In fact, our results also extends to a larger class of equilibrium states whose uniqueness are justified in [**9**]. Meanwhile, due to the nature of Lindeberg Central Limit Theorem, our technique also applies to dynamical arrays with distinct observables. See §5.4 for more details.

1.4. Multifractal analysis on surfaces without focal points

The other strength of thermodynamic formalism lies in its connection with dimension theory. In particular, when the potential function is the geometric-t potential, we can use the properties of equilibrium states to obtain multifractal information of the Lyapunov regular sets of the system by studying their Hausdorff dimension and topological entropy. In the third part, we focus on the case of the geodesic flow on surfaces with no focal points. Our main object is the so-called *Lyapunov regular sets*, which is defined as

$$\mathcal{L}(\beta) := \{v \in T^1S \colon v \text{ is Lyapunov regular and } \chi(v) = \beta\}.$$

Here we say $v \in T^1S$ is Lyapunov regular if $\chi(v)$ exists; see §2.1.8 for detailed explanations. As described above, we will concentrate on the estimate of the topological entropy and Hausdorff dimension of $\mathcal{L}(\beta)$, which are denoted by $h(\mathcal{L}(\beta))$ and $\dim_H(\mathcal{L}(\beta))$ respectively. We rely on the thermodynamic formalism of geometric-t potentials, which are denoted by φ_t and defined for all $t \in \mathbb{R}$ and $v \in T^1S$ as follows

$$\varphi_t(v) = t\varphi^{geo}(v) \quad \text{with} \quad \varphi^{geo}(v) := -\lim_{t\to 0}\frac{1}{t}\log\left\|dg_t|_{E^u_v}\right\|.$$

where E^u_v is the unstable distribution at v (see §2.3.1). It was shown in [**13**] that φ_t has a unique equilibrium state for all $t < 1$. In particular, this shows that the pressure function

$$\mathcal{P}(t) := P(\varphi_t),$$

where $P(\varphi_t)$ is the topological pressure of φ_t, is C^1 except for $t < 1$ (see §6.1 for details). As $\mathcal{P}$ is also convex and non-increasing, we can define

$$\alpha_1 := \lim_{t \to -\infty} D^+\mathcal{P}(t) \quad \text{and} \quad \alpha_2 := D^-\mathcal{P}(1).$$

At $t = 1$, there exists a phase transition. Based on this observation, we will discuss $\mathcal{L}(\beta)$ in two separated cases concerning the domain of β. The first case $\beta \in (-\alpha_1, -\alpha_2)$ corresponds to the time before phase transition. In this case, we can explicitly evaluate $h(\beta)$ by applying results on thermodynamic formalism of φ_t. The second case $\beta \in [-\alpha_2, 0)$ corresponds to the time at phase transition, where uniqueness of equilibrium states fails at $t = 1$. In this case, we will follow the technique in [**10**] and construct an increasingly nested sequence of basic sets $\{\Lambda_i\}_{i \in \mathbb{N}}$ (see §6.2 for definition) that exhaust hyperbolicity of the whole system. For any such β, we show that $\mathcal{L}(\beta)$ will eventually intersect with Λ_i for all i that is sufficiently large. As the multifractal information on the basic sets is well understood, we can establish an effective lower bound for $h(\mathcal{L}(\beta))$ and $\dim_H(\mathcal{L}(\beta))$ using such information of Λ_i. The construction of $\{\Lambda_i\}_{i \in \mathbb{N}}$ relies on the hyperbolic indicator function λ_T introduced in §2.3.2 and will be proceeded explicitly in §6.3.

Summarizing the above two cases, our main result is stated as follows

Theorem E. *For the geodesic flow on a compact Riemannian surface without focal points, we have*

(1) The Lyapunov level set $\mathcal{L}(-\alpha)$ is non-empty if and only if $\alpha \in [\alpha_1, 0]$.

(2) For $\alpha \in (\alpha_1, 0)$, we have

$$h(\mathcal{L}(-\alpha)) = \mathcal{E}(\alpha),$$

and

$$\dim_H \mathcal{L}(-\alpha) \geq 1 + 2 \cdot \frac{\mathcal{E}(\alpha)}{-\alpha},$$

where $\mathcal{E}(\alpha)$ is the Legendre transform of $\mathcal{P}$ at α; see §2.4.3 for details.

CHAPTER 2

Preliminaries

2.1. Conceptions in dynamical systems

In this section, we introduce the definitions of objects that help us describe basic quantities and settings in dynamical systems. In general, we will limit our discussion to compact metric space. There are certain places, e.g. section 2.4, where we introduce quantities that are not restricted to the compact setting. In those cases, non-compactness will be specified.

2.1.1. Background setting. Let (X,d) be a compact metric space and $f : X \to X$ be a continuous map. Such a pair (X,f) is called a (discrete) dynamical system. Throughout the thesis f will be a diffeomorphism. Denote by $\mathcal{B}(X)$ the Borel algebra on X and $\mathcal{M}(X)$ the set of probability Borel measure on X. We call a measure $\mu \in \mathcal{M}(X)$ to be f-invariant if for any $E \in \mathcal{B}(X)$, we have $\mu(f^{-1}(E)) = \mu(E)$. Denote by $\mathcal{M}(f)$ the set of f-invariant probability measure on X. In addition, if $\mu \in \mathcal{M}(f)$ has the property that for any $E = f^{-1}(E)$, $\mu(E) = 1$ or 0, then it is called ergodic. Denote by $\mathcal{M}_e(f)$ the set of ergodic measures under f.

For any continuous real-valued function $\varphi \in C(X)$, we write

$$S_n(\varphi) = S_n^f(\varphi) = \sum_{k=0}^{n-1} \varphi(f^k x).$$

Given $n \in \mathbb{N}$ and $x, y \in X$, we define Bowen metric as

$$d_n(x,y) = \max_{0 \le k \le n-1} d(f^k(x), f^k(y)).$$

Definition 2.1.1 (Bowen Ball). Given $\epsilon > 0$, $x \in X$ and $n \in \mathbb{N}$, the Bowen ball of order n at center x with radius ϵ is defined as

$$B_n(x,\epsilon) = \{y \in X : d_n(x,y) < \epsilon\}.$$

Definition 2.1.2. Let $Z \subset X$. We call a set $Y \subset Z$ to be (n, ϵ)-spanning for Z if for any $x \in Z$, there is a $y \in Y$ such that $d_n(x, y) \leq \epsilon$. We also call a set $Y' \subset Z$ to be (n, ϵ)-separated if for any $y_1, y_2 \in Y'$, $y_1 \neq y_2$, we have $d_n(x, y) > \epsilon$.

We refer to §7 in [**49**] for the basic properties of spanning and separated sets.

More generally, instead of a continuous map f, we consider the case of flow, which is a family of continuous self maps $\{g_t\}_{t\in\mathbb{R}}$ on X satisfying $g_t \circ g_s = g_{t+s}$ for all $t, s \in \mathbb{R}$, and continuous in the time parameter. In this case, given a continuous function $h \in C(X)$ and any $t \in \mathbb{R}$, we write

$$H(x, t) = \int_0^t h(g_s(x))ds.$$

Meanwhile, for any $t_1, t_2 \in \mathbb{R}$, we write

$$H(x, [t_1, t_2]) = \int_{t_1}^{t_2} h(g_s(x))ds.$$

Throughout the paper, we use this notation convention whenever we use other lower case letters for an observable in the flow case. For example, given an observable k, we write $K(x, t) := \int_0^t k(g_s(x))ds$. Definition for Bowen metric, Bowen ball, (n, ϵ)-spanning and (n, ϵ)-separated set can be directly adapted to the flow case.

2.1.2. Topological pressure. For a dynamical system, topological entropy describes the exponential rate of increasing in the number of orbit segments that are separated enough. Topological entropy generalizes topological entropy in distributing weight to each orbit segments. We follow §9 in [**49**] to give an explicit definition of the topological pressure. Suppose $Y \subset X$ and $\delta > 0$. For each $n \in \mathbb{N}$, we write

$$\Lambda_n^{sep}(Y, \varphi, \delta; f) = \sup\left\{\sum_{x\in E} e^{S_n\varphi(x)} : E \subset Y \text{ is an } (\delta, n)\text{-separated set}\right\}.$$

The pressure of φ on Y at scale $\delta > 0$ is defined as

$$P(Y, \varphi, \delta; f) = \limsup_{n\to\infty} \frac{1}{n} \log \Lambda_n^{sep}(Y, \varphi, \delta; f),$$

and the pressure of φ on Y is

$$P(Y, \varphi; f) = \lim_{\delta\to 0} P(Y, \varphi, \delta; f).$$

In particular, when $Y = X$, we write $P(Y, \varphi, \delta; f)$ as $P(\varphi, \delta)$ and $P(X, \varphi; f)$ as $P(\varphi)$. We will sometimes omit f in all of the notations above if we focus on one setting.

In our case, we also need to consider the pressure of a collection of orbit segments. We follow the definition from [**17**]. For each $\mathscr{D} \subset X \times \mathbb{N}$, we interpret it as a collection of finite orbit segments and write $\mathscr{D}_n = \{x \in X : (x, n) \in \mathscr{D}\}$. At this time the partition sum is written as follows

$$\Lambda_n^{sep}(\mathscr{D}, \varphi, \delta; f) = \sup \left\{ \sum_{x \in E} e^{S_n \varphi(x)} : E \subset \mathscr{D}_n \text{ and is an } (\delta, n)\text{-separated set} \right\},$$

which allows us to define $P(\mathscr{D}, \varphi; f)$ in the same way as above.

In particular, when $\varphi = 0$, the above process gives the definition of topological entropy. We write $P(Y, 0; f)$ as $h(Y, f)$, $h(X, f)$ as $h(f)$ and $P(\mathscr{D}, 0; f)$ as $h(\mathscr{D}, f)$.

In a similar way we can define the pressure of a flow by considering continuous $t \to \infty$ instead of discrete n. Notations are inherited from the discrete case and details can be found in [**9**].

We will also need the following variation of the definition in pressure, which appears in [**17**]. Given a fixed scale $\epsilon > 0$, we define

$$\Phi_\epsilon(x, n) := \sup_{y \in B_n(x, \epsilon)} \sum_{k=0}^{n-1} \varphi(f^k y).$$

From the above definition we see immediately that $\Phi_0(x, n) = \sum_{k=0}^{n-1} \varphi(f^k x)$.

For $\mathscr{D} \subset X \times \mathbb{N}$, we write

$$\Lambda_n^{sep}(\mathscr{D}, \varphi, \delta, \epsilon; f) = \sup \left\{ \sum_{x \in E} e^{\Phi_\epsilon(x, n)} : E \subset \mathscr{D}_n \text{ and is an } (\delta, n)\text{-separated set} \right\}.$$

The pressure of φ on $\mathscr{D}$ at scale δ, ϵ is given by

$$P(\mathscr{D}, \varphi, \delta, \epsilon; f) = \limsup_{n \to \infty} \frac{1}{n} \log \Lambda_n^{sep}(\mathscr{D}, \varphi, \delta, \epsilon; f).$$

Again, when $\mathscr{D}$ is the entire $X \times \mathbb{N}$, we simply write $P(\varphi, \delta, \epsilon)$. We also remark that $\Phi_\epsilon(x, n)$ is often used in defining topological pressure as a dimension quantity in multifractal analysis.

2.1.3. Measure theoretic entropy. We call a collection of finite disjoint measurable subsets of X whose union is the full X a (finite) partition of X. Given a measure $\mu \in \mathcal{M}(f)$ and a partition

ξ of X, we define

$$H_\mu(\xi) := -\sum_{A\in\xi}\mu(A)\log(\mu(A)),$$

and

$$h_\mu(f,\xi) := \lim_{n\to\infty}\frac{1}{n}\log H_\mu(\bigvee_{i=0}^{n-1} f^{-i}\xi).$$

Notice that the above limit always exists (see §4 in [**49**]). We define the measure theoretic entropy of (X,f) with respect to μ as

$$h_\mu(f) = \sup\{h_\mu(f,\xi) : \xi \text{ is a finite partition of } X\}.$$

In the case of flow $G = \{g_t\}_{t\in\mathbb{R}}$, for a G-invariant measure ν, the measure theoretic entropy of (X,G) with respect to ν is

$$h_\nu(G) := h_\nu(g_1).$$

2.1.4. Variational principle and equilibrium states. For a compact dynamical system and continuous φ, the variational principle from §9 in [**49**] says that

$$P(\varphi) = \sup_{\mu\in\mathcal{M}(f)}\left\{h_\mu(f) + \int\varphi d\mu\right\} = \sup_{\mu\in\mathcal{M}_e(f)}\left\{h_\mu(f) + \int\varphi d\mu\right\}. \tag{2.1.1}$$

Definition 2.1.3. A measure $\mu\in\mathcal{M}(f)$ achieving the supremum in (2.1.1) is called an equilibrium state (ES). If $\varphi = 0$, such a measure is called measure of maximal entropy (MME).

The study of existence and uniqueness of equilibrium states has always been one of the main topics in the research of smooth dynamics. When the entropy map $\mu\to h_\mu(f)$ is upper semi-continuous, by weak*-compactness of $\mathcal{M}(X)$, we know the equilibrium state always exists. This happens in the case where the map is expansive (see §2.1.5), or more generally, where the map is entropy-expansive (see [**4**] for details). The uniqueness is the main target we focus on in our work. It is known that in the case of topological mixing Markov shift and uniformly hyperbolic system, all Hölder continuous potentials possess a unique equilibrium state. This is further generalized in [**7**]. Nevertheless, in non-uniformly hyperbolic systems, very few examples are known to exhibit such a property. We shall have more discussion on this later in §3.

2.1.5. Expansiveness. Expansiveness is the property of no orbits being able to stay permanently close within certain distance. We first look at the discrete case and give the most classic definition.

Definition 2.1.4 (expansiveness). Given $\epsilon > 0$, a system (X, f) is called expansive at scale ϵ if for any $x, y \in X$ satisfying $d(f^i(x), f^i(y)) < \epsilon$ for all $i \in \mathbb{Z}$, we have $x = y$. (X, f) is called expansive if it is expansive at some scale.

It is not hard to see that if (X, f) is expansive at scale ϵ, then it is expansive at all scales smaller than ϵ.

The definition of an expansive flow is much more complicated. One of the most commonly used definition is given in [**8**], which is preserved under homeomorphic conjugation and makes the system possess only countably many periodic orbits. As we work with flows that are not expansive, we will not give a definition here. Instead, we give a much weaker replacement which asks that the measures with large free energy observe expansive behavior.

We start with defining so-called non-expansive set at some certain scale. Given $x \in X$ and $\epsilon > 0$, we write $\Gamma_\epsilon(x) := \{y \in X \mid d(g_t(x), g_t(y)) \leq \epsilon \text{ for all } t \in \mathbb{R}\}$. In the discrete case, we simply define the above set using all $n \in \mathbb{N}$ and Definition 2.1.4 just refers to $\Gamma_\epsilon(x) = \{x\}$ for all $x \in X$. Since $\Gamma_\epsilon(x) = \bigcap_{t \geq 0} g_{-t}\bar{B}_{[-2t,2t]}(g_t(x), \epsilon)$, we know $\Gamma_\epsilon(x)$ is compact. As we wish to capture expansiveness using a measure, we would like the measure to have zero measure over the non-expansive set. This gives rise to the following definition

Definition 2.1.5. For a discrete dynamical system, we write the set of non-expansive points at scale ϵ as $\mathrm{NE}(\epsilon) := \{x \in X : \Gamma_\epsilon(x) \neq \{x\}\}$. For a flow, $\mathrm{NE}(\epsilon) := \{x \in X : \Gamma_\epsilon(x) \not\subseteq f_{[-s,s]}(x) \text{ for any } s \geq 0\}\}$. An invariant (under f or g_t) Borel probability measure is called almost expansive at scale ϵ if $\mu(\mathrm{NE}(\epsilon)) = 0$.

To see whether the set of non-expansive points at some scale is negligible regarding pressure, we need the following quantity. This is known as the pressure of obstructions to expansivity in [**17**]. We only give a definition in the case of flow, which can be easily adapted to the discrete case.

Definition 2.1.6. Given a potential $\varphi \in C(X)$ and $\epsilon > 0$, the pressure of obstructions to expansivity at scale ϵ is given as

$$\begin{aligned} P^{\perp}_{\exp}(\varphi, \epsilon) &= \sup_{\mu \in \mathcal{M}_e(g_t)} \left\{ h_\mu(g_1) + \int \varphi d\mu : \mu(\mathrm{NE}(\epsilon)) > 0 \right\} \\ &= \sup_{\mu \in \mathcal{M}_e(g_t)} \left\{ h_\mu(g_1) + \int \varphi d\mu : \mu(\mathrm{NE}(\epsilon)) = 1 \right\}. \end{aligned}$$

From the definition we notice that if $P_\mu(\varphi) > P^{\perp}_{\exp}(\varphi, \epsilon)$ and μ is g_t-invariant and ergodic, then μ is almost expansive at scale ϵ. Choosing μ among the ergodic measures is crucial as otherwise a convex combination of a non-expansive measure and any other g_t-invariant measure will make the supremum equal to $P(\varphi)$.

2.1.6. Specification. Specification is a strengthened version of topological transitivity which describes the property that different Bowen balls can be connected by an orbit segment with any pre-fixed gaps that are bounded from below uniformly. The following is the definition of specification in the case of flows.

Definition 2.1.7 (Specification for flows). A collection of orbit segments $\mathscr{D} \subset X \times \mathbb{R}^+$ is said to have specification at scale ϵ if the following holds: given $\epsilon > 0$, there exists $\tau = \tau(\epsilon)$ such that for every $(x_1, t_1), \ldots, (x_N, t_N) \in \mathscr{D}$ and every collection of times $\tau_1, \ldots, \tau_{N-1}$ with $\tau_i \geq \tau$ for all i, there exists a point $y \in X$ such that for $s_0 = \tau_0 = 0$ and $s_j = \sum_{i=1}^{j} t_i + \sum_{i=0}^{j-1} \tau_i$, we have

$$f_{s_{j-1}+\tau_{j-1}}(y) \in B_{t_j}(x_j, \rho)$$

for every $j \in \{1, \ldots, N\}$.

The above definition can also be adapted to the discrete case easily by making all the time integer-valued. In either case, we say $\mathscr{D}$ satisfies specification if it satisfies specification at some scale. In contrary to what we have seen in the definition of expansiveness, if $\mathscr{D}$ has specification at some scale ϵ, it then has specification at all scale greater than ϵ. Meanwhile, sometimes we are only interested in gluing orbit segments that are long enough. In these situations, we consider the following weak version of specification (also stated in the flow version).

Definition 2.1.8 (Tail specification). A collection of orbit segments $\mathscr{D} \subset X \times \mathbb{R}^+$ has tail specification at scale ϵ if there is some $T_0 \in \mathbb{R}^+$ such that $\mathscr{D}_{\geq T_0} := \{(x,t) \in \mathscr{D} | t \geq T_0\}$ has specification.

2.1.7. The Bowen property. The Bowen property describes the property of the total variation of potential within any Bowen ball being uniformly bounded from above. This was first defined for maps by Bowen in [**4**], then extended to flows by Franco in [**22**]. Here we follow [**17**] in defining Bowen property over a collection of orbit segments $\mathscr{D}$.

Definition 2.1.9 (Bowen property). In the discrete case, given $\mathscr{D} \subset X \times \mathbb{N}$ and $\epsilon > 0$, we say a potential function φ satisfies the Bowen property at scale ϵ if the following holds

$$\sup\{|S_n\varphi(x) - S_n\varphi(y)| : (x,n) \in \mathscr{D}, d_n(x,y) < \epsilon\} < \infty.$$

For the continuous case, given $\mathscr{D} \subset X \times \mathbb{R}^+$ and $\epsilon > 0$, the Bowen property is defined via

$$\sup\{|\int_0^t \varphi(g_s(x))ds - \int_0^t \varphi(g_s(y))ds| : (x,t) \in \mathscr{D}, d_t(x,y) < \epsilon\} < \infty.$$

In either case, $\mathscr{D}$ is said to have Bowen property if it has Bowen property at some scale ϵ.

One can see that if we define $\mathscr{D}$ as the full orbit collection ($X \times \mathbb{N}$ or $X \times \mathbb{R}^+$), the above definition matches those original ones in [**4**] and [**22**]. This dynamical regularity property of potential function is essential in the proof of uniqueness result for equilibrium states. In uniformly hyperbolic systems, it can be proved that every Hölder continuous potential has the Bowen property. This no longer holds in the non-uniformly hyperbolic systems, even for one-dimensional systems. For example, one can easily see that in the case of the Manneville-Pomeau Map, which is defined on $[0,1]$ as $f : x \to x + x^{1+\alpha} \mod 1$ with $\alpha \in (0,1)$, its geometric potential $-\log f'$ is Hölder continuous, while it can be proved that this potential does not have Bowen property due to the neutrality of f at 0. To make up for that, our strategy is to ask for the Bowen property to hold on some collection of orbit segments that is essential in pressure estimating. Details will again be revealed in §3.

2.1.8. Lyapunov exponents. In this section we give a brief introduction on Lyapunov exponents. In dynamical systems, Lyapunov exponents quantifies the long-term growth rate along the orbit. Essentially, it describes the exponential rate of expansion for the vectors under the action of flow or diffeomorphism. In the continuous case $(X, \mathcal{F})$, given $p \in X$ and $v \in T_pX$, the forward Lyapunov exponent of v is defined as

$$\chi^+(v) := \limsup_{t\to\infty} \frac{1}{t} \log ||Dg_t(v)||. \tag{2.1.2}$$

Similarly, the backward Lyapunov exponent of v is written as

$$\chi^-(v) := \limsup_{t\to-\infty} \frac{1}{|t|} \log ||Dg_t(v)||. \tag{2.1.3}$$

We also briefly describe here the regularity of Lyapunov exponents, referring to §1, 2 in [**3**] for readers who are interested in the details. Roughly speaking, χ^+ is forward Lyapunov regular at p if the filtrations of χ^+ (determined by the value of χ^+ on T_pX) and its dual in the cotangent bundle are 'well adapted' to each other. This property is essential in showing that $\chi^+(v)$ being negative implies stablity of solution initially conditioned on v under perturbation in the forward direction for the corresponding differential equation. Similarly we have 'backward Lyapunov regular' for χ^- to describe the stability of solution in the backward direction. Moreover, we say the pair of Lyapunov exponents (χ^+, χ^-) is regular at p if χ^+ (resp. χ^-) is forward (resp. backward) regular at p and their filtrations comply. Since we will only consider the case where $\dim(X) = 2$ in the study of Lyapunov exponents, regularity is significantly simplified and can be presented as follows

Definition 2.1.10. If $\dim(X) = 2$, the forward Lyapunov exponent χ^+ is forward regular at $p \in X$ if for every $v \in T_pX$, (2.1.2) can be replaced by

$$\chi^+(v) := \lim_{t\to\infty} \frac{1}{t} \log ||Dg_t(v)||. \tag{2.1.4}$$

Similarly, χ^- is backward regular at $p \in X$ if for every $v \in T_pX$, (2.1.3) can be replaced by

$$\chi^-(v) := \lim_{t\to-\infty} \frac{1}{|t|} \log ||Dg_t(v)||. \tag{2.1.5}$$

The pair (χ^+, χ^-) is called Lyapunov regular at p if $\chi^+(v) = -\chi^-(v)$. In this case, we call

$$\chi(v) := \chi^+(v) = -\chi^-(v)$$

the *Lyapunov exponent* of v and say the point p is *Lyapunov regular*.

We will study the level sets for Lyapunov exponent. That is, for $\beta \in \mathbb{R}$, we define

$$\mathcal{L}(\beta) := \{v \colon v \text{ is Lyapunov regular and } \chi(v) = \beta\}. \tag{2.1.6}$$

We remark that in general, having (2.1.4) or (2.1.5) is not enough to deduce forward or backward regularity and having forward and backward regularity can not guarantee the regularity of the Lyapunov exponent. Details can be found in §1.3 and §1.5 in [**3**].

Notice that χ is $\mathcal{F}$-invariant. Therefore, if μ is an ergodic Borel probability measure, we know the Lyapunov exponents are constant μ-a.e. We call an ergodic Borel probability measure a hyperbolic measure if none of these Lyapunov exponents is zero.

All the above can be easily adapted to the discrete case and we omit the details. We end this section by giving the following famous theorem, which states that regularity is typical from the measure theoretic point of view.

THEOREM 2.1.11 (Multiplicative Ergodic Theorem). *If $\mathcal{F}$ (resp. f) is a C^1 flow (resp. diffeomorphism) on a compact smooth Riemannian manifold X, then the set of Lyapunov regular points has full measure with respect to any $\mathcal{F}$ (resp. f)-invariant measure.*

2.2. The Katok Map and its properties

In this section, we define the Katok map and show its basic properties. The Katok map is a C^∞ diffeomorphism on $\mathbb{T}^2$ which preserves Lebesgue measure and is non-uniformly hyperbolic. Katok [**24**] originally constructed the map to verify the existence of C^∞ area-preserving Bernoulli diffeomorphisms of $\mathbb{D}^2$ that are sufficiently flat near $\partial\mathbb{D}^2$, which furthermore gives rise to a Bernoulli diffeomorphism that preserves any given smooth measure on any compact connected smooth surface. We organize this part following [**50**] and will state other properties of the Katok map that are needed for the study of its thermodynamic formalism in §4.

2.2.1. Construction of the Katok map.

First we consider the automorphism of $\mathbb{T}^2$ given by $A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$, which is locally the time-one map generated by the local flow of the following differential system:

$$\frac{ds_1}{dt} = s_1 \log \lambda, \quad \frac{ds_2}{dt} = -s_2 \log \lambda.$$

where (s_1, s_2) is the coordinate representation in the eigendirections of A and $\lambda > 1$ equals the greater eigenvalue of A. We slow down the trajectories of the flow in a neighborhood of origin as follows: Choose a number $0 < \alpha < 1$ and a function $\psi : [0,1] \to [0,1]$ satisfying:

(1) ψ is C^∞ everywhere except for the origin.

(2) $\psi(0) = 0$ and $\psi(r_0^2) = 1$ for some $0 < r_0 < 1$ and r_0 is close to 0.

(3) $\psi'(x) \geq 0$ and is non-increasing.

(4) $\psi(u) = (u/r_0^2)^\alpha$ for $0 \leq u \leq \frac{r_0^2}{2}$.

where r_0 is very small. Let $D_r = \{(s_1, s_2) : s_1^2 + s_2^2 \leq r^2\}$. We also define $r_1 = r_0 \log \lambda$. Now the trajectories are slowed down in D_{r_1} at the rate of ψ, which induces the following differential system:

$$\frac{ds_1}{dt} = s_1 \psi(s_1^2 + s_2^2) \log \lambda, \quad \frac{ds_2}{dt} = -s_2 \psi(s_1^2 + s_2^2) \log \lambda.$$

Denote the time-one map of the local flow generated by this differential system by g. From the choice of r_1 and the assumption that r_0 is small, one could easily see that the domain of g contains D_{r_1}. Moreover, f_A and g coincide in some neighborhood of ∂D_{r_1}. Therefore, the following map

$$G(x) = \begin{cases} A(x) & \text{if } x \in \mathbb{T}^2 \setminus D_{r_1} \\ g(x) & \text{if } x \in D_{r_1} \end{cases}$$

defines a homeomorphism of 2-torus which is C^∞ everywhere except for the origin. One can verify that $G(x)$ preserves probability measure $d\nu = \kappa_0^{-1} \kappa dm$, where κ is defined by

$$\kappa(s_1, s_2) := \begin{cases} (\psi(s_1^2 + s_2^2))^{-1} & \text{if } (s_1, s_2) \in D_{r_0} \\ 1 & \text{elsewhere} \end{cases}$$

and κ_0 is the normalizing constant.

Furthermore, G is perturbed to an area-preserving C^∞ diffeomorphism via a coordinate change. Define ϕ in D_{r_0} as

$$\phi(s_1,s_2)=\frac{1}{\sqrt{\kappa_0(s_1^2+s_2^2)}}\Big(\int_0^{s_1^2+s_2^2}\frac{du}{\psi(u)}\Big)^{\frac{1}{2}}(s_1,s_2), \tag{2.2.1}$$

and set ϕ identity outside D_{r_0}. It is not hard to show that ϕ transfers the measure ν into area (the Lebesgue measure on $\mathbb{T}^2$), and the map $\widetilde{G}:=\phi\circ G\circ\phi^{-1}$ is thus area-preserving. Moreover, one can check $\widetilde{G}$ is a C^∞ diffeomorphism on 2-torus. This resulting $\widetilde{G}$ is what we call the Katok map.

2.2.2. Basic properties. We add a comment on the property of ϕ. Observe that ϕ is in fact a scalar product of identity at each point and also a map between circles centered at the origin. Moreover, by writing $\phi(s_1,s_2)$ as $\frac{1}{\sqrt{\kappa_0}}(\int_0^{r^2}\frac{du}{\psi(u)})^{\frac{1}{2}}(\frac{s_1}{\sqrt{s_1^2+s_2^2}},\frac{s_2}{\sqrt{s_1^2+s_2^2}})$ with $r^2:=s_1^2+s_2^2$ and differentiating in r, together with property (2) of ψ and standard geometric argument, we conclude that there is a constant $C=C(\alpha,r_0)$ such that $\frac{d(\phi(s_1,s_2),\phi(s_1',s_2'))}{d((s_1,s_2),(s_1',s_2'))}\geq\frac{C}{\sqrt{\kappa_0}}$ for all $(s_1,s_2),(s_1',s_2')\in\mathbb{T}^2$ such that $(s_1,s_2)\neq(s_1',s_2')$. Since ϕ is invertible, respectively we have

$$\frac{d(\phi^{-1}(s_1,s_2),\phi^{-1}(s_1',s_2'))}{d((s_1,s_2),(s_1',s_2'))}\leq\frac{\sqrt{\kappa_0}}{C}. \tag{2.2.2}$$

This property will be useful when we deduce the regularity of geometric potential of $\widetilde{G}$ from the regularity of geometric potential of G in §4.4.

PROPOSITION 2.2.1. *Here we have some useful properties of the Katok map [24]:*

(1) The Katok map is topologically conjugate to f_A via a homeomorphism h, i.e. $\widetilde{G}=h\circ f_A\circ h^{-1}$. In fact, it is in the C^1 closure of Anosov diffeomorphisms, which means it is a C^1 limit of a sequence of Anosov diffeomorphisms.

(2) For every $x\in\mathbb{T}^2$, it admits two transverse invariant continuous stable and unstable distributions $\widetilde{E}^s(x)$ and $\widetilde{E}^u(x)$ that integrate to continuous, uniformly transverse and invariant foliations $\widetilde{W}^s(x)$ and $\widetilde{W}^u(x)$ with smooth leaves. Moreover, they are the images of stable and unstable eigendirections of f_A under h.

(3) Almost every x with respect to area m has two non-zero Lyapunov exponents, one positive in the direction of $\widetilde{E}^u(x)$ and the other negative in the direction of $\widetilde{E}^s(x)$. The only ergodic measure with zero Lyapunov exponents is δ_0, the point measure at the origin.

(4) It is ergodic with respect to m.

In Proposition 3.1, properties (1) and (2) hold for G with h replaced by $\psi^{-1} \circ h$ and properties (3) and (4) hold for G with respect to ν. The distributions and foliations for G respectively are written as $E^{s,u}$ and $W^{s,u}$. Therefore, the following functions are well-defined and continuous.

Definition 2.2.2 (geometric potential). The geometric potential of $\widetilde{G}$ and G are defined as

$$\varphi^{geo}(x) = \varphi^{geo}_{\widetilde{G}}(x) := -\log |D\widetilde{G}|_{\widetilde{E}^u(x)}|,$$

and

$$\varphi^{geo}_{G}(x) := -\log |DG|_{E^u(x)}|.$$

Since $\widetilde{G}$ is conjugate to G via a homeomorphism that is C^∞ everywhere except at the origin, the dynamical properties of G are inherited by $\widetilde{G}$. For example, both specification and expansiveness are preserved under conjugacy. The only place where the properties of G and $\widetilde{G}$ need to be distinguished is the regularity of φ^{geo} and φ^{geo}_G, since these are essentially two different potentials. Therefore, we want to analyze them separately. The idea will be to first prove regularity of φ^{geo}_G, then use the property of ϕ and the conjugacy between G and $\widetilde{G}$ to obtain the one for φ^{geo}. Details will be displayed in §4.3.

2.3. Geometric background

In this section we recall the basic background of the geometric objects that are studied in this thesis. Specifically, we focus on the geodesic flow in the setting of rank-one non-positively curved Riemannian manifold, as well as surface with no focal points. We will briefly introduce the necessary geometric definitions and properties here, and then show their dynamical properties with more details in §5.1.

Throughout this section M denotes a compact C^∞ boundaryless Riemannian manifold equipped with a Riemannian metric g. We focus on the unit tangent bundle T^1M under the action of the geodesic flow. For each $v \in T^1M$, we have a unique constant speed geodesic γ_v such that $\dot{\gamma}_v(0) = v$. The geodesic flow $\mathcal{F} = \{g_t\}_{t\in\mathbb{R}}$ acts on T^1M by $g_t(v) = \dot{\gamma}_v(t)$. Notice that T^1M is $\mathcal{F}$-invariant and compact. We recall some well-known classical results of the geodesic flow on T^1M.

2.3.1. Jacobi Field and invariant distributions. We begin with the definition of Jacobi Field, which is the variation field of geodesic variation.

DEFINITION 2.3.1. A vector field along the geodesic γ is called *Jacobi Field* if it satisfies

$$J''(t) + R(J(t), \dot{\gamma}(t))\dot{\gamma}(t) = 0, \tag{2.3.1}$$

where $'$ above represents the covariant derivative along $\dot{\gamma}(t)$ and R is the curvature tensor on M.

Given a Jacobi Field $J(t)$ along a geodesic γ, if there is a time $t_0 \in \mathbb{R}$ such that both $J(t_0)$ and $J'(t_0)$ are orthogonal to $\dot{\gamma}(t_0)$, we know from (2.3.1) that both $J(t)$ and $J'(t)$ will be permanently orthogonal to $\dot{\gamma}(t)$. Such a Jacobi Field is called an *orthogonal Jacobi Field*. Write the space of all Jacobi Fields along γ as $\mathcal{J}(\gamma)$ and the one for all orthogonal Jacobi Fields along γ as $\mathcal{J}^{\perp}(\gamma)$.

Since we focus on the action of $\mathcal{F}$ on T^1M, we need a metric on the double tangent space TT^1M, which will be characterized by using Jacobi Field. For each $v \in T^1M$, there exists a direct sum decomposition $T_vT^1M = H(v) \oplus V(v)$, where $H(v)$ and $V(v)$ are horizontal and vertical subspaces equipped with metric induced by the Riemannian metric on M. In fact, $H(v)$ and $V(v)$ can be identified respectively as the kernel and its orthogonal complement of the map $d\pi : T_v(T^1M) \to T_{\pi(v)}M$, where π is the natural projection from the tangent bundle to the base point. Moreover, for each $v \in T^1M$ there is a natural isomorphism between T_vT^1M and $\mathcal{J}(\gamma_v)$ which maps ξ to J_ξ. In particular, the orthogonal component in T_vT^1M to the flow direction corresponds to $\mathcal{J}^{\perp}(\gamma_v)$. The Sasaki metric on T_vT^1M, which is induced by g on $H(v)$ and $V(v)$ and defined by making $H(v)$ and $V(v)$ orthogonal, gives the following useful property

$$||dg_t(\xi)||^2 = ||J_\xi(t)||^2 + ||J'_\xi(t)||^2 \tag{2.3.2}$$

for all $t \in \mathbb{R}$.

Sasaki metric induces a natural distance function on T^1M, which is denoted by d_S. There is another distance function d_{KM} on T^1M used by Knieper in [**28**] (and proposed by Manning in [**32**]), which is defined as

$$d_{KM}(v, w) := \max\{\gamma_v(t), \gamma_w(t) \mid t \in [0, 1]\}.$$

We call d_{KM} the *Knieper Manning metric* on T^1M. It is not hard to observe that d_{KM} and d_S are equivalent. We will typically work with d_{KM} in this thesis and write it simply as d.

An orthogonal Jacobi Field along a geodesic γ is called *stable* if $||J(t)|| < \infty$ for $t \geq 0$, and *unstable* if $||J(t)|| < \infty$ for $t \leq 0$. Denote by $\mathcal{J}^s(\gamma)$ and $\mathcal{J}^u(\gamma)$ the stable and unstable Jacobi Fields. We write the corresponding stable and unstable subbundles in T^1TM at a given vector $v \in T^1M$ as

$$E_v^s := \{\xi \in T_vT^1M : J_\xi \in \mathcal{J}^s(\gamma_v)\},$$

and

$$E_v^u := \{\xi \in T_vT^1M : J_\xi \in \mathcal{J}^u(\gamma_v)\}.$$

We also denote by E_v^c the subbundle corresponding to the flow direction at v and write $E^{cs} := E^c \oplus E^s$, $E^{cu} := E^c \oplus E^u$.

The stable and unstable subbundles enable us to give the following definition in terms of the Riemannian manifold M.

DEFINITION 2.3.2 (rank). The rank of a vector $v \in T^1M$ is given as $1 + \dim(E_v^s \cap E_v^u)$. The rank of M is $\min\{\text{rank}(v) : v \in T^1M\}$.

DEFINITION 2.3.3. The singular set (written as Sing) is the collection of vectors in T^1M whose rank is greater than 1. The regular set (written as Reg) is the complement of Sing, which refers to the collections of vectors in T^1M with rank being 1.

From the above definitions we immediately see that M is rank-one iff $\text{Reg} \neq \emptyset$.

We summarize the properties of stable and unstable subbundles in the case where M is rank-one and non-positively curved. Details on the first seven properties from below can be found in [**21**]. Reference for the other properties are specified after the statements.

PROPOSITION 2.3.4. *Let M be an n-dimensional compact connected C^∞ boundaryless rank-one non-positively curved Riemannian manifold. Then we have*

(1) $\dim(E_v^s) = \dim(E_v^u) = n - 1$, $\dim(E_v^c) = 1$ *for any $v \in T^1M$.*

(2) E^s, E^u, E^c are invariant under the action of geodesic flow $\mathcal{F}$. That is to say, $dg_t(E_v^s) = E_{g_t(v)}^s$ for all $v \in T^1M$ and $t \in \mathbb{R}$, and same holds for E^c and E^u.

(3) All these subbundles depend continuously on v.

(4) E^u and E^s are both orthogonal to E^c.

(5) E^u_v and E^u_s intersect non-trivially iff $v \in Sing$. When $v \in Reg$, they intersect transversally at zero vector in $T_v(T^1M)$.

(6) The subbundles E^s, E^u, E^{cs} and E^{cu} are all integrable and integrate to foliations W^s, W^u, W^{cs} and W^{cu} respectively.

(7) Define stable and unstable horosphere *at $v \in T^1M$ as the footprint of the respective foliations on M, i.e. $H^s(v) = \pi(W^s(v))$ and $H^u(v) = \pi(W^u(v))$. All vectors in $W^s(v)$ (resp. $W^u(v)$) are orthogonal to $H^s(v)$ (resp. $H^u(v)$) and point towards the same side as v.*

*(8) Both stable and unstable leaves are minimal. That is to say, for any $v \in T^1M$, $W^s(v)$ and $W^u(v)$ are both dense in T^1M (see Theorem 3.7 in [**1**]).*

*(9) The geodesic flow $\mathcal{F}$ is topologically mixing. That is to say, for any open subsets $U, V \subset T^1M$ (with topology induced by the Sasaki metric), there exists $T > 0$ such that for all $|t| > T$, we have $U \cap g_{-t}(V) \neq \emptyset$ (proved via minimality; see Theorem 3.5 in [**1**]).*

*(10) The geodesic flow $\mathcal{F}$ satisfies Flat Strip Theorem. That is, any bi-asymptotic geodesics γ_1, γ_2 i.e. the distance between $\gamma_1(t)$ and $\gamma_2(t)$ are uniformly bounded from above for all $t \in \mathbb{R}$, bound a flat strip. This only happens when both γ_1 and γ_2 have their tangent vectors lie in Sing (see Proposition 1.11.4 in [**20**]).*

We will also deal with the case where M is in a different setting: instead of being non-positively curved, we consider the manifold that contains no focal points, which is defined as follows

DEFINITION 2.3.5. A Riemannian manifold M is said to have no focal points if for any non-trivial Jacobi field $J(t)$ along any geodesic γ with $J(0) = 0$, its length $||J(t)||$ is strictly increasing.

Similar to the non-positively curved case, geodesic flow on manifold without focal points satisfies most of the properties displayed in Proposition 2.3.4. Details can be found in Proposition 3.5 in [**13**]

PROPOSITION 2.3.6. *Let M be an n-dimensional compact connected C^∞ boundaryless rank-one Riemannian manifold with no focal points. Then we have*

(1) Properties (1)-(10) *from Proposition 2.3.4 are all preserved. In particular, for any* $v \in T^1M$*, both* $W^s(v)$ *and* $W^u(v)$ *are minimal. This is obtained by showing these leaves are minimal at a certain vector then generalizing to all vectors in* T^1M*.*

(2) When $n = 2$*, i.e. when* M *is a surface, there is a useful characterization of the singular set, given by*

$$\mathit{Sing} = \{v \in T^1M : K(\pi(g_t(v))) = 0 \text{ for all } t \in \mathbb{R}\}. \tag{2.3.3}$$

where K *refers to the curvature.*

We conclude this subsection by introducing the intrinsic metric on the leaves. Suppose $v_0 \in T^1M$ and v_1, v_2 both belong to $W^s(v_0)$. Then we follow (2.11) in [**9**] and define

$$d^s(v_1, v_2) := \inf\{l(\pi\gamma) \mid \gamma : [0,1] \to W^s(v_0), \gamma(0) = v_1, \gamma(1) = v_2\}, \tag{2.3.4}$$

where l is the length of the curve in M and the infimum is taken over all curves connecting v_1 and v_2 in $W^s(v_0)$. Given $\delta > 0$, the local stable leaf of v_0 with size δ is defined as

$$W^s_\delta(v_0) := \{v \in W^s(v_0) : d^s(v_0, v) \le \delta\}. \tag{2.3.5}$$

Similarly we have d^u and $W^u_\delta(v_0)$. We also have a local representation of the metric on W^{cs}. Suppose $v_1, v_2 \in W^{cs}(v_0)$ and t is the unique value such that $g_t(v_1) \in W^s(v_2)$. We have

$$d^{cs}(v_1, v_2) := |t| + d^s(g_t(v_1), v_2),$$

which can be extended to the whole $W^{cs}(v_0)$. Similarly we can define d^{cu}, W^{cs}_δ and W^{cu}_δ.

2.3.2. H-Jacobi Fields and λ function. In this section we introduce an important index function $\lambda : T^1M \to [0, \infty)$, based on which (or its variation form) we can measure the hyperbolicity of the geodesic flow $\mathcal{F}$ on T^1M.

We first give the definition of a special family of Jacobi Fields. Given a unit speed geodesic γ with $\gamma(0) = p \in M$, let $H \subset M$ be a hyperplane orthogonal to $\dot\gamma(0)$ at p. Denote by $\mathcal{J}_H(\gamma)$ the collection of *H-Jacobi Fields*, which refers to those Jacobi Fields obtained by the geodesic variations of γ through geodesics that are orthogonal to H. Notice that for any given γ and H, $\mathcal{J}_H(\gamma)$ is

$(n-1)$-dimensional. Recall H^s (resp. H^u) stands for the stable (resp. unstable) horospheres and we assume its dependence on γ and p. It is easy to observe that $\mathcal{J}_{H^s}(\gamma) = \mathcal{J}^s(\gamma)$ (resp. $\mathcal{J}_{H^u}(\gamma) = \mathcal{J}^u(\gamma)$).

Define $\mathcal{U} = \mathcal{U}_p(H) : T_pH \to T_pH$ as $\mathcal{U}(v) = \nabla_v N$, where N is the unit vector fields orthogonal to H that point towards the same side as $\dot{\gamma}(0)$. Observe that this also determines the second fundamental form of H.

It is shown in Lemma 2.9 of [**9**] that we have the following useful estimate

LEMMA 2.3.7. *If J is an orthogonal Jacobi Field along γ at $t = 0$, then $J'(0) = \mathcal{U}(J(0))$.*

It follows from Lemma 2.3.7 that if we write λ_H as the minimal eigenvalue of $\mathcal{U}$, for every H-Jacobi Field J, we have

$$\langle J, J\rangle'(0) = 2\langle J, \mathcal{U}J\rangle(0) \geq 2\lambda_H \langle J, J\rangle(0),$$

which implies that $(\log ||J||^2)'(0) \geq 2\lambda_H$. In particular

$$(\log ||J||)'(0) \geq \lambda_H. \tag{2.3.6}$$

Observe that we can replace 0 by any time t in the above inequality. Now given $v \in T^1M$, associate to H^s and H^u we can define two linear maps $\mathcal{U}_v^s$ and $\mathcal{U}_v^u$ as above. It is not hard to see that $\mathcal{U}_v^s$ is negative semi-definite, $\mathcal{U}_v^u$ is positive semi-definite, they satisfy $\mathcal{U}_v^s = -\mathcal{U}_{-v}^u$ and both are continuous in v.

We are now ready to define the λ function on T^1M.

DEFINITION 2.3.8. For $v \in T^1M$, define $\lambda^u(v) := \lambda_{H^u}(v)$, which is the minimal eigenvalue of $\mathcal{U}_v^u$. Define $\lambda^s(v) = \lambda^u(-v)$ and $\lambda(v) := \min\{\lambda^s(v), \lambda^u(v)\}$.

From (2.3.6), we immediately see the following (Lemma 2.11 in [**9**])

LEMMA 2.3.9. *Given $v \in T^1M$, let J^u be any unstable Jacobi Field along γ_v and J^s be any stable ones. Then we have*

$$||J^u(t)|| \geq e^{\int_0^t \lambda^u(g_s(v))ds}||J^u(0)|| \qquad \textit{and} \qquad ||J^s(t)|| \leq e^{-\int_0^t \lambda^s(g_s(v))ds}||J^s(0)||.$$

Lemma 2.3.9 is what motivates us to understand λ as an analogy to the hyperbolic strength of the flow. Meanwhile, when dealing with a manifold with no focal points, as λ being 0 does not necessarily imply the vanishing of hyperbolicity in either direction, we need the following integral on λ from [**13**].

DEFINITION 2.3.10. For any $v \in T^1M$ and $T > 0$, we define $\lambda_T(v)$ as

$$\lambda_T(v) := \int_{-T}^{T} \lambda(g_s(v))ds.$$

We also give several definitions on the collections of vectors whose λ is bounded from below by a positive number.

For any $\eta > 0$, we write

$$\mathrm{Reg}(\eta) := \{v \in T^1M : \lambda(v) \geq \eta\}, \tag{2.3.7}$$

and

$$\mathrm{Reg}_T(\eta) := \{v \in T^1M : \lambda_T(v) \geq \eta\}. \tag{2.3.8}$$

To end this section, we introduce some properties of λ in the case where M is a surface with no focal points, which are used in §6.

PROPOSITION 2.3.11. *In the above setting, we have*

(1) $\lambda|_{\mathrm{Sing}} \equiv 0$.

(2) If $\lambda(v) = 0$, *then* $K(\pi v) = 0$.

(3) If $\lambda(g_t v) = 0$ *for all* $t \in \mathbb{R}$, *then* $v \in \mathrm{Sing}$.

PROOF. Since M is 2-dimensional, when $v \in \mathrm{Sing}$, $E^s_v = E^u_v$. In particular, the norm of its unstable Jacobi field $||J^u(t)||$ is bounded for all $t \in \mathbb{R}$ thus a constant by flat strip theorem. This shows $\lambda^u(v) = 0$. Similarly $\lambda^s(v) = 0$ and the first statement is concluded.

For the second statement, suppose $v \in T^1M$ satisfies $\lambda(v) = 0$. Without loss of generality we assume $\lambda^s(v) = 0$. Then the stable Jacobi Field $J^s(t)$ along γ_v satisfies $(J^s)'(0) = 0$. Since the stable Jacobi Field is one-dimensional in this case, we also use J^s to denote its norm. Then $(J^s)'$ is a non-positive function, which shows that $(J^s)''(0) = 0$. As a result, by Jacobi Equation, we have $K(\pi v) = -(J^s)''(0)/J^s(0) = 0$. The last statement follows easily from (2) and Prop 2.3.6 (2). $\square$

2.4. Dimension Theory in dynamical systems

In this section we will define those dimensional quantities which are frequently studied in dynamical systems. In particular, we will give a dimensional definition of topological pressure for non-compact sets and compare them with which defined in §2.1. We will also state some theorems that are helpful in multifractal analysis of level sets of Lyapunov exponents in our settings.

Throughout the section we assume (X, d) is a compact metric space (and study subsets of X which are not necessarily compact), denote by $\mathcal{F}$ the action of flow, f the diffeomorphism on X, φ the continuous potential, $S_n(\varphi)$ the n-th ergodic sum on X and $B(x, r)$ the ball of radius r centered at $x \in X$.

2.4.1. Hausdorff dimension. The most well-known quantity in dimension theory is Hausdorff dimension, which is a fractal dimension that generalizes the dimension of real vector space. It is particularly useful in distinguishing sets with zero Lebesgue measure. The definition is given in the following steps.

Suppose (X, d) is a compact metric space. For given $Z \subset X$, $s \in \mathbb{R}$ and $\delta > 0$, we write

$$H(Z, s, \delta) := \inf\{\sum_i r_i^s : Z \subset \bigcup_i B(x_i, r_i), x_i \in X, r_i \leq \delta\}.$$

Observe that $H(Z, s, \delta)$ is non-decreasing as $\delta \downarrow 0$, so we can define $H(Z, s) := \lim_{\delta \downarrow 0} H(Z, s, \delta)$. Meanwhile, from the definition of $H(Z, s, \delta)$, if there is some s_1 such that $H(Z, s_1) \in [0, \infty)$, then for any $s_2 > s_1$, $H(Z, s_2) = 0$. Moreover, if s_1' is such that $H(Z, s_1') = \infty$, then for any $s_2' < s_1'$, $H(Z, s_2') = \infty$. Therefore, there is one unique $s_0 \in [-\infty, \infty]$ such that $H(Z, s) = \infty$ for all $s < s_0$ and $H(Z, s) = 0$ for all $s > s_0$. The value of this s_0 is known as the Hausdorff dimension of Z.

Definition 2.4.1. The Hausdorff dimension of Z is defined as

$$\dim_H(Z) := \inf\{s : H(Z, s) = 0\} = \sup\{s : H(Z, s) = \infty\}.$$

2.4.2. Topological pressure. In this section we introduce the topological pressure, being a dimensional characteristic, of sets that are not necessarily invariant or compact. This definition is due to [**39**], which is a generalization of Bowen's definition of topological entropy for non-compact sets [**6**]. Close to the spirit of how we define Hausdorff dimension, given $Z \subset X$, $s \in \mathbb{R}$ and $\epsilon > 0$, we define the following

$$M(Z, s, \epsilon, N, \varphi) := \inf_{\Gamma} \sum_{B_n(x_i,\epsilon)} \exp(-n_i s + \sup_{x \in B_n(x_i,\epsilon)} S_{n_i}\varphi(x)),$$

where the infimum is taking over Γ, whose elements are countable coverings of Z using Bowen balls with radius ϵ and degree n being greater than or equal to N. As before we define

$$m(Z, s, \epsilon, \varphi) := \lim_{N \to \infty} M(Z, s, \epsilon, N, \varphi).$$

Following the same argument as in §2.4.1, the following is well-defined.

$$P_Z(\varphi, \epsilon) := \inf\{s : m(Z, s, \epsilon, \varphi) = 0\} = \sup\{s : m(Z, s, \epsilon, \varphi) = \infty\}. \tag{2.4.1}$$

By using compactness of X and applying Lebesgue Number Lemma (see §11 in [**38**] for details), the following quantity is well-defined

Definition 2.4.2. The dimensional topological pressure of φ on Z is defined by

$$P_Z(\varphi) = \lim_{\epsilon \downarrow 0} P_Z(\varphi, \epsilon).$$

In particular, when $\varphi = 0$, $h(Z) = P_Z(0)$ is the dimensional topological entropy of Z.

We should not confuse the above definition with the one defined in §2.1. In our situation, the definition for topological pressure in §2.1 is only used for X (which is compact by itself) and some collection of orbit segments in X. The evaluation on topological pressure for a subset of X only appears in the multifractal analysis part, where we use Definition 2.4.2 to characterize the topological pressure (entropy) of the level sets of Lyapunov exponents. In fact, by Theorem 11.5 in [**38**], these two definitions are equivalent for any compact invariant set.

2.4.3. Results in multifractal analysis. In this section we give some results from previous work in the field of dimension theory in dynamical system that are used in this thesis.

Consider a discrete dynamical system (X, f) satisfying specification at all scales. Given potential function $\varphi \in C(X)$ and $\alpha \in \mathbb{R}$, we can define

$$K(\alpha) := \{x \in X : \lim_{n\to\infty} \frac{S_n\varphi(x)}{n} = \alpha\}.$$

By specification, for any $\alpha_1 < \alpha_2$ such that $K(\alpha_1)$, $K(\alpha_2)$ are both non-empty, if we take α_0 to be any convex combination of α_1 and α_2, $K(\alpha_0)$ is also non-empty. Therefore, we know the set of α that makes $K(\alpha)$ non-empty is an interval.

Now we fix φ and consider $h(\varphi, \alpha) := h(K(\alpha))$ as a function of α. Define $\mathcal{P}(t) := P(t\varphi)$. When φ is continuous and non-positive, $\mathcal{P}(t)$ is non-increasing, Lipschitz continuous and convex. We will study $h(\varphi, \alpha)$ by the Legendre transform of $\mathcal{P}(t)$, which is written as

$$\mathcal{E}(\alpha) := \inf_{t\in\mathbb{R}}(\mathcal{P}(t) - t\alpha). \tag{2.4.2}$$

We have the following preliminary result from [**14**], which shows that

LEMMA 2.4.3. *$\mathcal{P}$ is the Legendre transform of h, i.e.*

$$\mathcal{P}(t) = \sup_{\alpha\in\mathbb{R}}(h(\varphi, \alpha) + t\alpha).$$

From above we immediately see that $h(\varphi, \alpha) \le \mathcal{E}(\alpha)$.

When we conduct multifractal analysis on certain sets, we want to bound its dimension from below by which of some reference subsets, for which the dimension is known. In our cases, the reference sets will be hyperbolic sets. The following theorem is useful in the discrete setting.

THEOREM 2.4.4 (Main Theorem in [**51**]). *Let (X, f) be a C^2 diffeomorphism on compact surface and μ be an ergodic Borel probability measure with Lyapunov exponents being $\lambda_1 > 0 > \lambda_2$. Then*

$$\dim(\mu) = h_\mu(f)(\frac{1}{\lambda_1} - \frac{1}{\lambda_2}),$$

where $\dim(\mu) := \inf\{\dim_H(Z) : \mu(Z) = 1\}$.

In particular, when μ is hyperbolic, the above theorem has the following natural corollary

Corollary 2.4.5. *If $X = \mathbb{T}^2$, f preserves the Lebesgue measure on X, then for any hyperbolic measure μ, we have*

$$\dim(\mu) = \frac{2h_\mu(f)}{\chi(\mu)},$$

where $\chi(\mu)$ is the positive Lyapunov exponent of μ.

A similar result in the case of hyperbolic flows can be derived using thermodynamic formalism of hyperbolic set and Corollary 15 in [**2**]. We will state the result when we get more involved in the specific problem in §6.2.

2.5. Central Limit Theorem

The classic Central Limit Theorem (abbreviated as CLT) claims that in some situations, when independent random variables are added, their normalized sum converges in distribution towards the standard normal distribution, regardless of the original distribution of those variables. In dynamical systems, the sum of random variables are replaced by ergodic sums/integrals of observables with certain regularity, e.g. Hölder continuity. One of the most famous and widely used type of CLT in probability is CLT of Lindeberg type, which is what our argument is based on.

To introduce the Lindeberg CLT, we first give a few definitions and notations. Let (Ω, ν) be a probability space, $h : \Omega \to \mathbb{R}$ be an observable (not necessarily continuous) and c be a non-negative constant. We have

Definition 2.5.1. Let $Z(c) = Z(c, h, \nu) = \{x : |h - \int h d\nu| > c\}$. The *Lindeberg* function is

$$L_\nu(h, c) := \int (h - \int h d\nu)^2 \mathbb{1}_{Z(c)}(v) d\nu(v).$$

Recall that given a function $f : \Omega \to \mathbb{R}$, the variance $\sigma_\nu(f)$ is defined by

$$\sigma_\nu^2(f) = \int \left(f - \int f d\nu\right)^2 d\nu = \int f^2 d\nu - \left(\int f d\nu\right)^2.$$

Theorem 2.5.2 (Lindeberg CLT for random arrays). *Let Ω be a sample space which is equipped with a sequence of Borel probability measures $\{\nu_i\}_{i\in\mathbb{N}}$. Consider a triangle array of random variables $\{X_{i,k}\}_{i\in\mathbb{N},1\le k\le m_i}$ where $m_i \uparrow \infty$ in i and $\{X_{i,1}, X_{i,2}, ..., X_{i,m_i}\}$ are independent for each $i \in \mathbb{N}$. Assume $\mathbb{E}_{\nu_i}(X_{i,k}) = \mu_{i,k}$ and $\sigma_{\nu_i}(X_{i,k}) = \sigma_{i,k}^2 < \infty$. Let $Y_{i,k} = X_{i,k} - \mu_{i,k}$, $s_i^2 = \sum_{k=1}^{m_i} \sigma_{i,k}^2$.*

Suppose for every $\epsilon > 0$, the following Lindeberg condition is satisfied

$$\lim_{i\to\infty} \frac{1}{s_i^2} \sum_{k=1}^{n_i} \mathbb{E}_{\nu_i}(Y_{i,k}^2 \chi\{|Y_{i,k}| \geq \epsilon s_i\}) = 0, \tag{2.5.1}$$

then

$$\frac{\sum_{k=1}^{m_i} Y_{i,k}}{s_i} \to N(0,1) \text{ in distribution,} \tag{2.5.2}$$

i.e. for any $a \in \mathbb{R}$, $\lim_{i\to\infty} \nu_i(\frac{\sum_{k=1}^{m_i} Y_{i,k}}{s_i} \leq a) = N(a)$, where $N(a) := \int_{-\infty}^{a} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$ is the cumulative distribution function of standard normal distribution.

Moreover, if the following holds

$$\lim_{i\to\infty} \frac{1}{s_i^2} \max_{1\leq k\leq m_i} \sigma_{i,k}^2 = 0, \tag{2.5.3}$$

then we also have (2.5.2) *imply* (2.5.1).

CHAPTER 3

Main Technique

3.1. Local product structure and cone argument

In the (locally maximal) uniformly hyperbolic case, the uniform expansion in the unstable direction and uniform contraction in the stable direction can be combined together to derive some very useful properties on connecting orbit segments, e.g. shadowing lemma, which roughly says orbit segments that are close consecutively can be actually approximated by one single orbit segment. This is achieved by so-called product structure, which reflects the transversal nature of stable and unstable leaves. In general, though the product structure is likely to hold globally in the universal cover, it can only be expected to hold locally in the ambient manifold. As a result, we call this structure as local product structure. In §2, we have already described the existence of stable/unstable leaves in the cases of the Katok map and geodesic flow on non-positively curved manifold (and manifold with no focal points). Therefore, we are able to define their local product structure respectively.

Definition 3.1.1. For the Katok map, given $\kappa \geq 1$ and $\delta > 0$, the leaves W^s and W^u are said to have local product structure with constant κ at scale δ if the following holds: For any $x, y \in \mathbb{T}^2$ with $d(x, y) < \delta$, there exists a unique $z \in W^u_{\kappa\delta}(x) \cap W^s_{\kappa\delta}(y)$, where $W^u_{\kappa\delta}(x)$ and $W^s_{\kappa\delta}(y)$ refer to the local unstable leaf of x and stable leaf of y with radius $\kappa\delta$. Such z is also written as $[x, y]$.

In the continuous case, we follow Definition 4.2 in [**9**] and define it as follows

Definition 3.1.2. Given $v \in T^1M$, $\kappa \geq 1$ and $\delta > 0$, the foliations W^u and W^{cs} have local product structure with constant κ in a δ-neighborhood of v if for every $\epsilon \in (0, \delta)$ and all $v_1, v_2 \in B(v, \epsilon)$, there exists a unique $v_0 \in W^u_{\kappa\epsilon}(v_1) \cap W^{cs}_{\kappa\epsilon}(v_2)$, which is again denoted by $[v_1, v_2]$.

To prove the local product structure, we apply the cone argument, which is to show the tangent vectors to the stable/unstable leaves are contained in separate invariant cones whose angles are strictly less than $\frac{\pi}{4}$. The cone argument is widely used in smooth dynamics. In fact, the existence and smoothness of such stable/unstable foliations in either the classic uniformly hyperbolic cases or the non-uniformly hyperbolic cases in Pesin Theory are contents of Stable Manifold Theorem (of different types), whose proof heavily relies on the cone argument. We will omit the details here and refer to [**26**] and [**3**] for further reading.

We follow §3 in [**50**] and show the invariant cone argument is indeed applicable. Given $x \in \mathbb{T}^2$, denote by $F^1(x), F^2(x)$ the corresponding expanding and contracting eigenspaces in $T_x\mathbb{T}^2$, which can be regarded as copies of F^1, F^2, the eigenspaces of A corresponding to λ and λ^{-1}. We want to show there exists some $\beta \in (0,1)$ independent of x such that the cone families $C_\beta(F^1(x), F^2(x)) := \{x_1 + x_2 : x_1 \in F^1(x), x_2 \in F^2(x), \frac{|x_1|}{|x_2|} \leq \beta\}$ (resp. $C_\beta(F^2(x), F^1(x))$) are invariant under dG (resp. dG^{-1}).

LEMMA 3.1.3. *There exists $\beta \in (0,1)$ such that for all $x \in \mathbb{T}^2$, we have*

$$dG(C_\beta(F^1(x), F^2(x))) \subset C_\beta(F^1(G(x)), F^2(G(x))),$$

and

$$dG^{-1}(C_\beta(F^2(x), F^1(x))) \subset C_\beta(F^2(G^{-1}(x)), F^1(G^{-1}(x))),$$

where $F^1(x), F^2(x)$ are defined as above. Moreover, β only depends on α (the exponent of fucntion ψ, see §2.2.1*) and $\beta \to 0$ when $\alpha \to 0$.*

PROOF. The case where $\beta = 1$ is proved by Katok in [**24**]. We will follow the main idea of his proof by observing on the variation equation and making further refinement.

Recall from §2.2.1 that the underlying generating differential system for the flow is

$$\frac{ds_1}{dt} = s_1\psi(s_1^2 + s_2^2)\log\lambda, \quad \frac{ds_2}{dt} = -s_2\psi(s_1^2 + s_2^2)\log\lambda.$$

We follow the first few steps of Proposition 4.1 of [**24**] by considering the linear part of the differential system, also known as the variation equation. By a standard partial differentiation, we have for each (ξ_1, ξ_2) in the tangent space the following equations

$$\frac{d\xi_1}{dt} = \log\lambda(\xi_1(2s_1^2\psi'(s_1^2 + s_2^2) + \psi(s_1^2 + s_2^2)) + 2s_1s_2\xi_2\psi'(s_1^2 + s_2^2)),$$

$$\frac{d\xi_2}{dt} = -\log\lambda(\xi_1 s_1 s_2 \psi'(s_1^2+s_2^2) + \xi_2(2s_2^2\psi'(s_1^2+s_2^2) + \psi(s_1^2+s_2^2))).$$

In order to study the evolution of the angle in time t, we define $\eta := \frac{\xi_2}{\xi_1}$ and combine the above two equations together to get

$$\frac{d\eta}{dt} = -2\log\lambda(\eta(\psi(s_1^2+s_2^2) + (s_1^2+s_2^2)\psi'(s_1^2+s_2^2)) + (\eta^2+1)s_1 s_2 \psi'(s_1^2+s_2^2)). \tag{3.1.1}$$

Now we consider the above system in two cases where $0 < s_1^2 + s_2^2 \leq \frac{r_0^2}{2}$ and $\frac{r_0^2}{2} < s_1^2 + s_2^2 \leq r_0^2$. When $0 < s_1^2 + s_2^2 \leq \frac{r_0^2}{2}$, we know from definition that $\psi(x) = (\frac{x}{r_0^2})^\alpha$. After some elementary computations, we have $(s_1^2+s_2^2)\psi'(s_1^2+s_2^2) = \alpha\psi(s_1^2+s_2^2)$ for $0 < s_1^2+s_2^2 \leq \frac{r_0^2}{2}$.

Otherwise, when $\frac{r_0^2}{2} \leq s_1^2+s_2^2 \leq r_0^2$, by applying property (3) of ψ in §2.2.1, we have:

$$\frac{\psi'(s_1^2+s_2^2)}{\psi(s_1^2+s_2^2)} \leq \frac{\psi'(\frac{r_0^2}{2})}{\psi(\frac{r_0^2}{2})} = \frac{2\alpha}{r_0^2} \leq \frac{2\alpha}{s_1^2+s_2^2}.$$

Therefore, for every $0 < x \leq r_0^2$, we have $\frac{\psi'(x)}{\psi(x)} \leq \frac{2\alpha}{x}$. Plugging this into (3.1.1), we are able to derive the following inequality:

$$\frac{d\eta}{dt} \geq -2\log\lambda(\psi'(s_1^2+s_2^2)((s_1^2+s_2^2)(1+\frac{1}{2\alpha})\eta + s_1 s_2(1+\eta^2))). \tag{3.1.2}$$

When $s_1 s_2 = 0$, any cone with arbitrarily small angle is preserved since by (3.1.1), η is increasing when $\eta < 0$ and decreasing when $\eta > 0$. Therefore, we only need to analyze the case where $s_1, s_2 > 0$ because of symmetry. Observe from (3.1.1) that $\frac{d\eta}{dt} < 0$ when $\eta \geq 0$, so we focus on the case where $\eta < 0$. By defining $k := \frac{s_1 s_2}{s_1^2+s_2^2}$ and going through some elementary quadratic analysis, we can see that $\frac{d\eta}{dt} \geq 0$ when $\eta \in [\frac{-(2\alpha+1)-\sqrt{(2\alpha+1)^2-16k^2\alpha^2}}{4k\alpha}, \frac{-(2\alpha+1)+\sqrt{(2\alpha+1)^2-16k^2\alpha^2}}{4k\alpha}]$. As $0 < k \leq \frac{1}{2}$, we have the collection of all possible slope of the invariant cones with such k to be $\bigcap_{k\in(0,\frac{1}{2}]}[\frac{(2\alpha+1)-\sqrt{(2\alpha+1)^2-16k^2\alpha^2}}{4k\alpha}, \frac{(2\alpha+1)+\sqrt{(2\alpha+1)^2-16k^2\alpha^2}}{4k\alpha}]$. Meanwhile, since $\frac{(2\alpha+1)-\sqrt{(2\alpha+1)^2-16k^2\alpha^2}}{4k\alpha}$ is monotonically increasing in k, by plugging in $k = \frac{1}{2}$, we obtain $\beta := \frac{2\alpha}{2\alpha+1+\sqrt{4\alpha+1}}$ as an appropriate choice on the slope of the invariant cone. This also gives a rather straightforward description on the relation between α and β. $\square$

To show local product structure of W^s and W^u for G, we first need the following result on the global product structure over Euclidean space (see Lemma 3.6 in [**15**]). We need this as $\mathbb{R}^2$ is the universal cover of $\mathbb{T}^2$.

LEMMA 3.1.4. *Suppose $\beta \in (0,1)$ and F^1, F^2 are orthogonal linear subspaces in $\mathbb{R}^d$. Let W^1, W^2 be any foliations of $F^1 \oplus F^2$ with C^1 leaves satisfying $T_x W^1(x) \subset C_\beta(F^1, F^2)$ and $T_x W^2(x) \subset C_\beta(F^2, F^1)$. Then, for every $x, y \in F^1 \oplus F^2$, $W^1(x) \cap W^2(y)$ consists of a single point. Moreover,*

$$\max\{d_{W^1}(x,z), d_{W^2}(y,z)\} \leq \frac{1+\beta}{1-\beta} d(x,y). \tag{3.1.3}$$

Lemma 3.1.3 and 3.1.4 imply that the lift of W^s and W^u to $\mathbb{R}^2$, written as $W^{s,L}$ and $W^{u,L}$, have a product structure with constant $\gamma = \gamma(\beta) := \frac{1+\beta}{1-\beta}$. To prove our main result, We only need to show that product structure on the universal cover can be 'projected down' to the ambient space, with the constant being unchanged. We sketch the proof as follows. Given some $\epsilon > 0$ that is sufficiently small and $x, y \in \mathbb{T}^2$ such that $d(x,y) < \epsilon$. We can lift x and y to x_L, y_L in $\mathbb{R}^2$ such that $d_L(x_L, y_L) < \epsilon$ (where d_L refers to the lifted metric on $\mathbb{R}^2$). By Lemma 3.1.4 we know $W^{u,L}_{\gamma\epsilon}(x_L) \cap W^{s,L}_{\gamma\epsilon}(y_L)$ consists of a unique point z_L. By projecting z_L down to $\mathbb{T}^2$ we get $z \in W^u_{\gamma\epsilon}(x) \cap W^s_{\gamma\epsilon}(y)$. To show z is the unique intersection, suppose there is another $z' \in \mathbb{T}^2$ such that it is also in $W^u_{\gamma\epsilon}(x) \cap W^s_{\gamma\epsilon}(y)$, then there is a curve $C : [0,1] \to \mathbb{T}^2$ that first goes from z to z' along $W^u_{\gamma\epsilon}(x)$ then from z' to z along $W^s_{\gamma\epsilon}(y)$. Lift C to C_L in $\mathbb{R}^2$. Observe that $C_L(0) \neq C_L(1)$ (otherwise it will contradict the uniqueness of intersection from Lemma 3.1.4). Since they project to the same point, the length of C_L is large, while we know the length of C is at most $4\gamma\epsilon$. Since γ is close to 1 when β is small and ϵ is small, we reach a contradiction.

Therefore, we have shown that

PROPOSITION 3.1.5 (local product structure). *When α, $\epsilon > 0$ are sufficiently small, the leaves W^s, W^u of G have local product structure at scale $500\lambda\epsilon$ with a constant only depending on α.*

Here the constant 500 is chosen so that $500\lambda\epsilon$ is large enough to cover all the scales for local product structure used in the verification of all the properties of the Katok map. Meanwhile, we should keep in mind that $500\lambda\epsilon$ is still much smaller than 1.

Following the same spirit (with necessary adaptions to flows), the local product structure for W^u and W^{cs} of the the geodesic flow are shown in [**9**] for the rank-one non-positively curved manifold, and in [**13**] for the manifold with no focal points. We cite their results here respectively as follows.

In the case where the manifold is rank-one non-positively curved, we have (Lemma 4.4 in [**9**])

LEMMA 3.1.6. *For every $\eta > 0$, there exist $\delta > 0$ and $\kappa \geq 1$ such that for every $v \in Reg(\eta)$, W^u and W^{cs} have local product structure with constant κ in a δ-neighborhood of v. Moreover, for $v_1, v_2 \in B(v, \delta)$, we have*

$$d^u(v_1, [v_1, v_2]) \leq \kappa d(v_1, v_2),$$

$$d^s(v_2, [v_1, v_2]) \leq \kappa d(v_1, v_2).$$

Meanwhile, when the manifold is rank-one and has no focal points, we have the following result (Lemma 5.2 in [**13**])

LEMMA 3.1.7. *For every $\eta > 0$, there exist $\delta > 0$ and $\kappa \geq 1$ such that $Reg_T(\eta)$ has local product structure with constant κ in a δ-neighborhood. Similar to Lemma 3.1.6, for $v \in Reg_T(\eta)$ and $v_1, v_2 \in B(v, \delta)$, we have*

$$d^u(v_1, [v_1, v_2]) \leq \kappa d(v_1, v_2),$$

$$d^s(v_2, [v_1, v_2]) \leq \kappa d(v_1, v_2).$$

3.2. Orbit decomposition technique

We will display here the main technique we apply to study the thermodynamic formalism of non-uniformly hyperbolic maps/flows, which is called orbit decomposition. It is first compeletely introduced in [**17**] as a generalization of Bowen's famous criteria in [**7**].

3.2.1. Introduction. Recall in [**7**], Bowen showed that given an homeomorphism f on a compact metric space X, if f is expansive and satisfies specification, then any potential satisfying Bowen's property will have a unique equilibrium state. The idea of the orbit decomposition technique is to find an essential collection of orbit segments that satisfies (a weaker version of) three properties from above and dominates in pressure.

We will show the orbit decomposition construction in the discrete case. The continuous case is similar and is not used in this thesis directly, so we omit the details and refer to §3 in [**17**]

Consider a compact metric space X and $f : X \to X$ which is $C^{1+\alpha}$. A decomposition for a pair (X, f) consists of three collections $\mathscr{P}, \mathscr{G}, \mathscr{S} \subset X \times \mathbb{N}$ and three functions $p, g, s : X \times \mathbb{N} \to \mathbb{N}$ such that for every $(x, n) \in X \times \mathbb{N}$, the values $p = p(x, n)$, $g = g(x, n)$, $s = s(x, n)$ satisfy $n = p + g + s$ and

$$(x, p) \in \mathscr{P}, \quad (f^p(x), g) \in \mathscr{G}, \quad (f^{p+q}(x), s) \in \mathscr{S}.$$

Meanwhile, for each $M \in \mathbb{N}$, denote by $\mathscr{G}^M$ the set of orbit segments (x, n) such that $p \leq M$, $s \leq M$. We also assume that $(x, 0)$ is contained in all of three collections, which indicates that some elements of the decomposition can be made empty. We will apply the following theorem (Theorem 5.6 in [**17**]) to obtain the uniqueness of equilibrium state from the orbit decomposition construction.

THEOREM 3.2.1. *Let X, f be as above and φ be a continuous potential function on X. Suppose there is an $\epsilon > 0$ such that $P^{\perp}_{\exp}(\varphi, 100\epsilon) < P(\varphi)$ and (X, f) admits a decomposition $(\mathscr{P}, \mathscr{G}, \mathscr{S})$ with the following properties:*

(1) For each $M \geq 0$, $\mathscr{G}^M$ has tail specification at scale ϵ.

(2) φ has the Bowen property at scale 100ϵ on $\mathscr{G}$.

(3) $P(\mathscr{P} \cup \mathscr{S}, \varphi, \epsilon, 100\epsilon) < P(\varphi)$.

Then there is an unique equilibrium state for φ.

The choice on the constant 100ϵ has no specific meanings, while we do want to control expansiveness and regularity in a much larger scale than which of specification as we apply specification multiple times to 'push out' the orbit segments. In particular, 100ϵ provides an upper boundary for the scale under which all the estimates will be safe.

The transition time for $\mathscr{G}^M$ is allowed to be dependent on the choice of M. Specification at all scales for $\mathscr{G}$ also implies specification at all scales for $\mathscr{G}^M$ for any M due to a simple argument in modulus of continuity (see [**17**] for detail). This is the case for the Katok map, as it has specification at any small scale due to its conjugacy to linear automorphism. Nevertheless, the Katok map does not inherit thermodynamic formalism from the original hyperbolic model as the conjugacy homeomorphism is not Hölder continuous.

3.2.2. Orbit decomposition for the Katok map. In this section we construct the orbit decomposition for the Katok map. As mentioned above, the Katok map satisfies specification at all scales. Therefore, we can make $\mathscr{G}$ big as long as Bowen property for the given potential holds on it. Once $\mathscr{G}$ is big enough, the pressure supported on $\mathscr{P}\cup\mathscr{S}$ will be smaller than the total pressure. We follow the structure of §4 in [**50**] to organize this section.

Consider the following set of orbit segments:

$$\mathscr{G}(r) = \{(x,n) : \frac{1}{i}S_i\chi(x) \geq r \text{ and } \frac{1}{i}S_i\chi(G^{n-i}(x)) \geq r \text{ for all } 0 \leq i \leq n\},$$

where χ is the characteristic function for $\mathbb{T}^2 \setminus D_{100\gamma\epsilon+r_1}$ and $r \in (0,1]$ is a parameter. In practice, we only consider the case where r is small. The idea behind such choice of $\mathscr{G}$ is that orbit segments that start and end far away from the origin and spend enough time outside the perturbed area would show high regularity for the chosen family of potential function.

Respectively we choose

$$\mathscr{P}(r) = \mathscr{S}(r) = \{(x,n) \in \mathbb{T}^2 \times \mathbb{N} : \frac{1}{n}S_n\chi(x) < r\}.$$

The case where $n = 0$ shall not cause any ambiguity as we can always assume $\mathbb{T}^2 \times \{0\}$ to be contained in all three collections by adapting the definition. We will see later that the appropriate choice of r will make Theorem 3.2.1 applicable to $(\mathscr{P}(r), \mathscr{G}(r), \mathscr{S}(r))$. Before verifying those properties, we must first show that they actually form an orbit decomposition as described in §3.2.1.

PROPOSITION 3.2.2. *For every $0 < r \leq 1$, the collections $(\mathscr{P}(r), \mathscr{G}(r), \mathscr{S}(r))$ form an orbit decomposition for G.*

PROOF. Given any $(x,n) \in \mathbb{T}^2 \times \mathbb{N}$, we take the largest integer $0 \leq i = i(x,n) \leq n$ such that $S_i\chi(x) < ir$, then the largest integer $0 \leq k = k(x,n) \leq n-i$ such that $S_k\chi(G^{n-k}(x)) < kr$. Meanwhile, if $S_j\chi(x) \geq jr$ for all $0 \leq j \leq n$, we just take $i = 0$. The same works for k. From the way i and k are defined, we immediately get that $\frac{1}{l}S_l\chi(G^i(x)) \geq r$ for $0 \leq l \leq n-i$ and $\frac{1}{m}S_m\chi(G^{n-k-m}(x)) \geq r$ for $0 \leq m \leq n-k$. Therefore, the following is evident

$$(x,i) \in \mathscr{P}(r), \quad (G^ix, n-i-k) \in \mathscr{G}(r), \quad (G^{n-k}x, k) \in \mathscr{S}(r).$$

The proof is now concluded by taking $p(x, n) = i$, $g(x, n) = n - i - k$, and $s(x, n) = k$. $\square$

3.2.3. Orbit decomposition for the geodesic flow. In this section we construct the orbit decomposition for the geodesic flow in both cases as they are very similar. The idea parallels the discrete case as we still want the essential collection of orbit segments to occupy the majority of hyperbolicity of the system. As one could possibly imagine, the main difference in the construction lies in the hyperbolic strength indicator we use. For non-positively curved manifold, we use function λ from Definition 2.3.8 to measure hyperbolicity, while for the manifold with no focal points, the hyperbolic measurement becomes function λ_T from Definition 2.3.10.

We first look at M which is non-positively curved. For any $\eta > 0$, we define

$$\mathscr{G}(\eta) := \Big\{(v,t) : \int_0^\tau \lambda(g_s(v))ds \geq \eta\tau, \int_0^\tau \lambda(g_{-s}g_t(v))ds \geq \eta\tau \text{ for all } \tau \in [0,t]\Big\}, \tag{3.2.1}$$

and

$$\mathscr{P}(\eta) = \mathscr{S}(\eta) = \mathscr{B}(\eta) := \Big\{(v,t) : \int_0^t \lambda(g_s(v))ds < \eta t\Big\}. \tag{3.2.2}$$

As in the proof of Proposition 3.2.2, it can be shown that $(\mathscr{P}(\eta), \mathscr{G}(\eta), \mathscr{S}(\eta))$ is an orbit decomposition for every $\eta > 0$.

Similarly, in the case where M has no focal points, for any $\eta > 0$ and $T > 0$ we define

$$\mathscr{G}_T(\eta) := \Big\{(v,t) : \int_0^\tau \lambda_T(g_s(v))ds \geq \eta\tau, \int_0^\tau \lambda_T(g_{-s}g_t(v))ds \geq \eta\tau \text{ for all } \tau \in [0,t]\Big\}, \tag{3.2.3}$$

and

$$\mathscr{P}_T(\eta) = \mathscr{S}_T(\eta) = \mathscr{B}_T(\eta) := \Big\{(v,t) : \int_0^t \lambda_T(g_s(v))ds < \eta t\Big\}. \tag{3.2.4}$$

Accordingly, $(\mathscr{P}_T(\eta), \mathscr{G}_T(\eta), \mathscr{S}_T(\eta))$ is an orbit decomposition for every $\eta > 0$ and $T > 0$.

REMARK 3.2.3. Both types of decomposition belong to the class of λ-decomposition later defined by Call and Thompson [**12**].

CHAPTER 4

Thermodynamic formalism of the Katok Map

We will focus on the study of thermodynamic formalism of the Katok map in this chapter. As mentioned before, the main technique we apply here is orbit decomposition technique introduced in §3.2. We will formulate our results following the structure of [**50**] by showing all the properties of G and using them to establish what we need for $\widetilde{G}$.

Before we move forward, let us clarify the scales and parameters used in our case. Recall α is the exponent in the perturbation function ψ in §2.2.1, $\beta = \frac{2\alpha}{2\alpha+1+\sqrt{4\alpha+1}}$ is the angle of the invariant cone derived in Lemma 3.1.3, r_0 is the radius of the perturbed ball and $\epsilon > r_0$ is a scale compatible with dynamical properties (e.g. expansiveness, Bowen property) we will verify in this chapter. We need β to be small enough so that $\frac{1+\beta}{1-\beta}$ in (3.1.3) is small, therefore α must be small. ϵ needs to be small for the obvious reason. Meanwhile, we need r_0 to be small so that $500\lambda\epsilon$ covers all the scales containing r_0 and r_1 in this chapter and both scale and constant for local product structure are consistent. Finally, we comment that the choice of ϵ is independent of the size of the gap $P(\varphi) - \varphi(\underline{0})$.

4.1. Expansivity

We know expansiveness is preserved under (homeomorphic) conjugacy. Therefore, by Proposition 2.2.1 (1) and the well-known fact of f_A being expansive, we know G is expansive. In this section we prove that G is expansive at scale 100ϵ. This can not be derived directly from definition. In fact, since the conjugacy map h is not Hölder, we can not directly say that G is expansive at a given small scale. Meanwhile, at this moment we can not just shrink the size of ϵ casually since it is greater than r_0, while r_0 is an intrinsic parameter of the Katok map that is fixed from the very beginning. Therefore, a rigorous proof is needed here. We also remark that once some $\epsilon > r_0$ is found to satisfy Theorem 3.2.1, we can make ϵ arbitrarily small without breaking any dynamical properties at scale ϵ.

Before moving to the proof of expansiveness at scale 100ϵ, we first display a lemma that will be used both in this section and the follow-up statistical estimates in §4.5.

LEMMA 4.1.1. *Suppose $x, y \in \mathbb{T}^2$ and $y \in B_n(x, 100\epsilon)$ for all $n \geq 1$, then there exists a unique $z \in \mathbb{T}^2$ satisfying $G^i(z) \in W^s_{100\gamma\epsilon}(G^i(x)) \cap W^u_{100\gamma\epsilon}(G^i(y))$ for all $0 \leq i \leq n-1$.*

PROOF. Recall from Proposition 3.1.5 that we have local product structure at $500\lambda\epsilon$ with constant $\gamma = \frac{1+\beta}{1-\beta}$. Fix any $x \in \mathbb{T}^2$ and $y \in B_n(x, 100\epsilon)$. As $d(G^i(x), G^i(y)) \leq 100\epsilon$ for all $0 \leq i \leq n-1$, by Proposition 3.1.5 we know there are $z_i \in \mathbb{T}^2$ such that $z_i = W^s_{100\gamma\epsilon}(G^i(x)) \cap W^u_{100\gamma\epsilon}(G^i(y))$ for all $0 \leq i \leq n-1$. Meanwhile, by a standard geometric argument on the invariant cone with angle β, we have $G(z_i) = W^s_{100\lambda(1+\beta)\gamma\epsilon}(G^{i+1}(x)) \cap W^u_{100\lambda(1+\beta)\gamma\epsilon}(G^{i+1}(y))$. By applying local product structure at scale $100\lambda(1+\beta)\gamma\epsilon$ of G and using uniqueness, we immediately have $G(z_i) = z_{i+1}$ for all $0 \leq i \leq n-2$, thus $G^i(z_0) = z_i$ for all $0 \leq i \leq n-1$. It follows that z_0 is the desired z. □

PROPOSITION 4.1.2. *G is expansive at scale 100ϵ. In particular, $P^{\perp}_{\exp}(\varphi, 100\epsilon) < P(\varphi)$.*

PROOF. Suppose there are $x, y \in \mathbb{T}^2$ satisfying $d(G^k(x), G^k(y)) < 100\epsilon$ for all $k \in \mathbb{Z}$. Our goal is to show that x and y have to be the same. We first take care of the positive k by applying Lemma 4.1.1 to $B_n(x, 100\epsilon)$ for each $n > 0$. This provides us with the existence of some $z \in \mathbb{T}^2$ such that $G^i(z) = W^s_{100\gamma\epsilon}(G^i(x)) \cap W^u_{100\gamma\epsilon}(G^i(y))$ for all $i > 0$.

Observe that $d(G^i(x), G^i(z)) \leq 100\gamma\epsilon$ for each $i > 0$, which implies that $d(G^i(y), G^i(z)) \leq 100(1+\gamma)\epsilon$ for all $i > 0$. Meanwhile, by noticing that $G^i(y)$ and $G^i(z)$ are on the same unstable local leaf for all $i > 0$ and applying Lemma 3.7 from [**15**], we have $d^u(G^i(y), G^i(z)) \leq \gamma d(G^i(y), G^i(z)) \leq 100(1+\gamma)\gamma\epsilon$ for all $i > 0$, which would only happen when $z = y$. In other words, we have $y \in W^s_{100\gamma\epsilon}(x)$.

Now applying Lemma 4.1.1 in to backwards iteration and repeating the above argument on $i < 0$, it is not hard to show that $y \in W^u_{100\gamma\epsilon}(x)$. As a result, y belongs to $W^u_{100\gamma\epsilon}(x) \cap W^s_{100\gamma\epsilon}(x)$, which is just x by itself. □

4.2. Pressure Gap

In this section we will verify property (3) in Theorem 3.2.1. Recall in §3.2.2 we construct a family of decompositions $(\mathscr{P}(r), \mathscr{G}(r), \mathscr{P}(r))$ with parameter $r \in (0, 1]$. We want to show that if the potential function φ satisfies $\varphi(\underline{0}) < P(\varphi)$ (equivalent, δ_0 is not an equilibrium state of φ), then we can find some small $r' > 0$ such that $P(\mathscr{P}(r'), \varphi, \epsilon, 100\epsilon) < P(\varphi)$ is satisfied.

Our strategy is stated as follows. First we show the existence of some constant $r' > 0$ such that $P(\mathscr{P}(r'), \varphi) < P(\varphi)$. By definition in §2.1.2 we have $(\mathscr{P}(r'), \varphi, \epsilon) \leq P(\mathscr{P}(r'), \varphi) < P(\varphi)$ for free. Finally, we will show that $P(\mathscr{P}(r'), \varphi, \epsilon) = P(\mathscr{P}(r'), \varphi, \epsilon, 100\epsilon)$. This will conclude the desired property on pressure gap of φ.

4.2.1. General estimates. We first state a pretty classic and useful result regarding the pressure on the set of orbit segments. For each $(x, n) \in \mathbb{T}^2 \times \mathbb{N}$, define the corresponding empirical measure as $\delta_{x,n} := \frac{1}{n}\sum_{i=0}^{n-1} \delta_{G^i(x)}$, where $\delta_{G^i(x)}$ is the Dirac measure at $G^i(x)$. Given $\mathscr{D} \subset \mathbb{T}^2 \times \mathbb{N}$, for every $n \in \mathbb{N}$, we define the convex hull of the empirical measures on $\mathscr{D}$ as follows

$$\mathscr{M}_n(\mathscr{D}) := \left\{ \sum_{i=1}^{k} a_i \delta_{x_i,n} : a_i \geq 0, \sum_{i=1}^{k} a_i = 1, x_i \in \mathscr{D}_n \right\}.$$

We denote by $\mathscr{M}^*(\mathscr{D})$ the collection of weak* limit of elements in $\mathscr{M}_n(\mathscr{D})$ when $n \to \infty$. Since $\mathscr{M}(\mathbb{T}^2)$ is weak* compact, we know $\mathscr{M}^*(\mathscr{D})$ is non-empty whenever $P(\mathscr{D}, \varphi) > -\infty$.

As in the standard proof of variational principle for pressure (see for example [**49**, Theorem 9.10] or [**9**, Proposition 5.1]), we have

Proposition 4.2.1. $P(\mathscr{D}, \varphi) \leq \sup_{\mu \in \mathscr{M}^*(\mathscr{D})} P_\mu(\varphi)$.

4.2.2. Pressure gap estimate. In this section we will find the appropriate r' such that $P(\mathscr{P}(r'), \varphi) < P(\varphi)$. Recall that χ is the characteristic function for $\mathbb{T}^2 \setminus D_{100\gamma\epsilon + r_1}$. For $\mu_n \in \mathscr{M}_n(\mathscr{P}(r))$, we observe that $\int \chi d\mu_n < r$ by definition of $\mathscr{P}(r)$ and χ. Therefore, if we denote by $\mathscr{M}_\chi(r)$ the set of G-invariant Borel probability measures μ satisfying $\int \chi d\mu \leq r$ for each $0 < r \leq 1$, we will have $\mathscr{M}_n(\mathscr{P}(r)) \subset \mathscr{M}_\chi(r)$ for all $n \in \mathbb{N}$ and $0 < r \leq 1$. Moreover, we claim that this inclusion holds true in the limit case, as indicated by the following lemma.

Lemma 4.2.2. $\mathscr{M}^*(\mathscr{P}(r)) \subset \mathscr{M}_\chi(r)$.

Since elements in $\mathscr{M}^*(\mathscr{P}(r))$ are the weak* limits of those in $\mathscr{M}_n(\mathscr{P}(r))$ with $n \to \infty$, to prove Lemma 4.2.2, it suffices to show the weak*-compactness of $\mathscr{M}_\chi(r)$, which is stated as follows.

Lemma 4.2.3. *$\mathscr{M}_\chi(r)$ is weak*-compact for all* $0 < r \leq 1$.

PROOF OF LEMMA 4.2.3. We begin with a sequence $\{\mu_n\}_{n\geq 1}$ in $\mathscr{M}_\chi(r)$. From weak* compactness of $\mathscr{M}(\mathbb{T}^2)$, we know there is a subsequence $\{\mu_{n_k}\}_{k\geq 1}$ that converges to some $\mu \in \mathscr{M}(\mathbb{T}^2)$. To prove the lemma, it suffices to show that $\int \chi d\mu \leq r$. Since χ is the characteristic function for an open set as we define D_r to be the closed balls, it is lower-semi continuous. Then $\int \chi d\mu \leq \liminf_{k\to\infty} \int \chi d\mu_{n_k} \leq r$ by remarks preceding [**49**, Theorem 6.5], which concludes the proof. □

From Lemma 4.2.2 we immediately have

$$\sup_{\mu\in\mathscr{M}^*(\mathscr{P}(r))} P_\mu(\varphi) \leq \sup_{\mu\in\mathscr{M}_\chi(r)} P_\mu(\varphi). \tag{4.2.1}$$

Observe that $\mathscr{M}_\chi(r)$ is non-decreasing in r and $\mathscr{M}_\chi(0) = \bigcap_{r>0} \mathscr{M}_\chi(r)$. Meanwhile, if $\mu \in \mathscr{M}_\chi(0)$, we have $\mu(\mathbb{T}^2 \setminus D_{100\gamma\epsilon+r_1}) = 0$. However, we also know that $\bigcup_{k=-\infty}^{+\infty} G^k(\mathbb{T}^2 \setminus D_{100\gamma\epsilon+r_1}) = \mathbb{T}^2 \setminus \{\underline{0}\}$. Since μ is G-invariant, it must be equal to δ_0, thus $\mathscr{M}_\chi(0) = \delta_0$.

Meanwhile, since G is expansive, the entropy function $\mu \to h_\mu(\varphi)$ is upper semi-continuous by [**49**, Theorem 8.2], so is the pressure function $\mu \to P_\mu(\varphi)$ (by continuity of φ). Therefore, for any $\epsilon' > 0$, there is some open neighborhood $U = U(\epsilon')$ of δ_0 in the weak* topology of $\mathscr{M}(\mathbb{T}^2)$ such that $P_\mu(\varphi) < P_{\delta_0}(\varphi) + \epsilon' = \varphi(\underline{0}) + \epsilon'$ for every $\mu \in U$. By Lemma 4.2.3, U must contain $\mathscr{M}_\chi(r')$ for some $r' > 0$. Since $\varphi(\underline{0}) < P(\varphi)$, we can fix any $\epsilon' \in (0, P(\varphi) - \varphi(\underline{0}))$ and get $r' = r'(\epsilon') > 0$ respectively such that $\sup_{\mu\in\mathscr{M}_\chi(r')} P_\mu(\varphi) \leq \sup_{\mu\in U} P_\mu(\varphi) \leq \varphi(\underline{0}) + \epsilon' < P(\varphi)$. Together with (4.2.1), we have shown that $P(\mathscr{P}(r'), \varphi) < P(\varphi)$ for the above r' we use. In conclusion, we have proved the following result.

PROPOSITION 4.2.4. *For a continuous φ satisfying $\varphi(\underline{0}) < P(\varphi)$, there exists a small $r' > 0$ such that $P(\mathscr{P}(r'), \varphi) < P(\varphi)$.*

4.2.3. Two-scale estimate. As stated in the introductory part, the next move is to show

$$P(\mathscr{P}(r'), \varphi, \epsilon) = P(\mathscr{P}(r'), \varphi, \epsilon, 100\epsilon).$$

We define the following n-th variation term of φ at scale 100ϵ, which will be used both in this section and the large deviation estimate in §4.5.

Definition 4.2.5. $\zeta(n) = \zeta(n, \varphi, 100\epsilon) := \sup_{x\in\mathbb{T}^2, y\in B_n(x,100\epsilon)} |S_n\varphi(y) - S_n\varphi(x)|$.

Since

$$\Lambda_n^{sep}(\mathscr{P}(r'), \varphi, \epsilon; G) \le \Lambda_n^{sep}(\mathscr{P}(r'), \varphi, \epsilon, 100\epsilon; G) \le \Lambda_n^{sep}(\mathscr{P}(r'), \varphi, \epsilon; G)e^{\zeta(n)},$$

we have

$$\begin{aligned} P(\mathscr{P}(r'), \varphi, \epsilon, 100\epsilon) &= \limsup_{n\to\infty} \frac{1}{n} \log \Lambda_n^{sep}(\mathscr{P}(r'), \varphi, \epsilon, 100\epsilon; G) \\ &\le \limsup_{n\to\infty} \frac{1}{n} \log \Lambda_n^{sep}(\mathscr{P}(r'), \varphi, \epsilon; G) + \limsup_{n\to\infty} \frac{1}{n}\zeta(n) \\ &= P(\mathscr{P}(r'), \varphi, \epsilon) + \limsup_{n\to\infty} \frac{1}{n}\zeta(n). \end{aligned} \tag{4.2.2}$$

Therefore, to eliminate 100ϵ from the two-scale estimate, it suffices to show that

Lemma 4.2.6. $\limsup_{n\to\infty} \frac{1}{n}\zeta(n) = 0$.

proof of Lemma 4.2.6. We will evaluate $\zeta(n)$ in terms of the total variations along stable and unstable directions. The key tool is the local product structure at $500\lambda\epsilon$ derived in Proposition 3.1.5. For any $x \in \mathbb{T}^2$ and $y \in B_n(x, 100\epsilon)$, by Lemma 4.1.1 there exists $z \in \mathbb{T}^2$ such that $G^i(z) = W^s_{100\gamma\epsilon}(G^i(x)) \cap W^u_{100\gamma\epsilon}(G^i(y))$ for any $0 \le i \le n-1$. Then we have

$$\begin{aligned} \zeta(n) &= \sup_{x\in\mathbb{T}^2, y\in B_n(x,100\epsilon)} |S_n\varphi(y) - S_n\varphi(x)| \\ &\le \sup_{x\in\mathbb{T}^2, y\in B_n(x,100\epsilon)} (|S_n\varphi(x) - S_n\varphi(z)| + |S_n\varphi(z) - S_n\varphi(y)|) \\ &\le \sup_{x\in\mathbb{T}^2, z\in W^s_{100\gamma\epsilon}(x)} |S_n\varphi(x) - S_n\varphi(z)| + \\ &\quad \sup_{y\in\mathbb{T}^2, G^{n-1}(z)\in W^u_{100\gamma\epsilon}(G^{n-1}(y))} |S_n\varphi(z) - S_n\varphi(y)|. \end{aligned} \tag{4.2.3}$$

From the above decomposition, the proof of Lemma 4.2.6 becomes proving the following result.

Lemma 4.2.7. *Write* $\zeta^s(n) := \sup_{x\in\mathbb{T}^2, z\in W^s_{100\gamma\epsilon}(x)} |S_n\varphi(x) - S_n\varphi(z)|$. *We have*

$$\limsup_{n\to\infty} \frac{1}{n}\zeta^s(n) = 0. \tag{4.2.4}$$

In the unstable case, write $\zeta^u(n) := \sup_{y\in\mathbb{T}^2, G^{n-1}(z)\in W^u_{100\gamma\epsilon}(G^{n-1}(y))} |S_n\varphi(z) - S_n\varphi(y)|$. *We have*

$$\limsup_{n\to\infty} \frac{1}{n}\zeta^u(n) = 0. \tag{4.2.5}$$

PROOF OF LEMMA 4.2.7. To prove (4.2.4), for any $x \in \mathbb{T}^2$ and $n \geq 1$, we define

$$d^s_n(x) := \max\{d(G^{n-1}(x), G^{n-1}(z)), z \in W^s_{100\gamma\epsilon}(x), d_s(x,z) = 100\gamma\epsilon\}.$$

Since W^s is one dimensional, the maximum makes sense here. Notice that $\{d^s_n(x)\}_{n\geq 1}$ consists of a sequence of continuous functions that converges pointwise to 0 monotonically. As $\mathbb{T}^2$ is compact, the convergence of $d^s_n(x)$ to 0 is uniform by Dini's Theorem.

Fix any small $\epsilon' > 0$. It suffices to show that there exists $N = N(\epsilon') \in \mathbb{N}$ such that $\frac{1}{n}\zeta^s(n) < \epsilon'$ for all $n > N$. Since φ is uniformly continuous on $\mathbb{T}^2$, there exists $\delta_0 > 0$ such that $|\varphi(x) - \varphi(y)| < \frac{\epsilon'}{2}$ for all $x, y \in \mathbb{T}^2$ with $d(x,y) < \delta_0$. Meanwhile, by uniform convergence of d^s_n, there is some $m_0 \in \mathbb{N}$ such that $d^s_n(x) < \delta_0$ for all $n > m_0$. By separating $|S_n\varphi(x) - S_n\varphi(z)|$ into two sums that proceed and follow m_0, we have $\zeta^s(n) < 2m_0||\varphi|| + \frac{(n-m_0)\epsilon'}{2}$, where $||\varphi|| := \sup_{x\in\mathbb{T}^2}\varphi(x)$. Now it is not hard to see that we can choose some N large enough such that $\frac{1}{n}\zeta^s(n) < \epsilon'$ for all $n > N$. By shrinking the size of ϵ', (4.2.4) is proved.

To prove (4.2.5), similar to d^s_n, we define $d^u_n(x)$ by

$$d^u_n(x) := \max\{d(x,z), z \in f^{-(n-1)}W^u_{100\gamma\epsilon}(G^{n-1}(x)), d_u(G^{n-1}(x), G^{n-1}(z)) = 100\gamma\epsilon\}.$$

Parallel to the proof above, we have $d^u_n(x)$ converge uniformly to 0 and use it to prove the existence of $M = M(\epsilon_0) \in \mathbb{N}$ which satisfies $\frac{1}{n}\zeta^u(n) < \epsilon_0$ for all $n > M$. This ends the proof of (4.2.5), thus the proof of Lemma 4.2.7. ☐

By applying Lemma 4.2.7 to (4.2.3), we complete the proof of Lemma 4.2.6. ☐

Finally, we apply the result of Lemma 4.2.6 to (4.2.3) and conclude that

$$P(\mathscr{P}(r'), \varphi, \epsilon, 100\epsilon) = P(\mathscr{P}(r'), \varphi, \epsilon) \leq P(\mathscr{P}(r'), \varphi) < P(\varphi). \tag{4.2.6}$$

which is exactly what we want.

4.3. Bowen Property

In this section we discover the Bowen property of certain potential functions over $(\mathbb{T}^2, G)$. We mainly focus on two families of potentials: Hölder potential and geometric-t potentials (recall from Definition 2.2.2 that it is defined as $\varphi_t^G(x) = t\varphi_G^{geo}(x) = -t\log|DG|_{E^u(x)}|$). In general, Hölder continuity and Bowen property are not necessarily related. In the case of uniformly hyperbolic maps, it can be shown that all Hölder continuous functions have Bowen property. This is not the case for non-uniformly hyperbolic maps, even in one-dimension cases, e.g. the Manneville-Pomeau map defined in §1.1, where the geometric potential is obviously Hölder, while it can be shown that it does not satisfy Bowen property. In the case of the Katok map, we will verify the regularity condition only for the essential collection of orbit segments $\mathscr{G}(r)$ as required by Theorem 3.2.1 (2).

4.3.1. General estimates. We will begin with a general estimate on the uniform expansion/contraction along unstable/stable local leaves of points in $\mathscr{G}(r)$.

Lemma 4.3.1. *If $(x,n) \in \mathscr{G}(r)$ and $y \in W^s_{100\gamma\epsilon}(x)$, then we have*

$$d^s(G^i(x), G^i(y)) \le (\lambda(1-\beta))^{-ir} d^s(x,y) \text{ for all } 0 \le i \le n-1. \tag{4.3.1}$$

Similarly, when $(x,n) \in \mathscr{G}(r)$ and $f^{n-1}(y) \in W^u_{100\gamma\epsilon}(f^{n-1}(x))$ and $0 \le j \le n-1$, we have

$$d^u(G^j(x), G^j(y)) \le (\lambda(1-\beta))^{-(n-1-j)r} d^u(f^{n-1}(x), f^{n-1}(y)). \tag{4.3.2}$$

Proof. For (x,n) and y as above and any point $z \in W^s_{100\gamma\epsilon}(x)$ lying between x and y, if $\chi(G^i(x)) = 1$, then from $d(G^i(x), G^i(z)) \le 100\gamma\epsilon$ and the definition of χ, we know $G^i(z)$ is outside the perturbed area. This shows that $\|DG|_{E^s(z)}\| \le (\lambda(1-\beta))^{-1}$. Moreover, DG is non-expanding along stable distributions. Consequently, we have $|DG^i|_{E^s(z)}| \le (\lambda(1-\beta))^{-ir}$ by definition of $\mathscr{G}(r)$. This proves (4.3.1). (4.3.2) is proved similarly by iterating backwards. □

4.3.2. Regularity for Hölder continuous potentials. In this section we focus on Hölder continuous φ. By definition, there exist constants $K > 0$ and $\alpha' \in (0,1)$ such that $|\varphi(x) - \varphi(y)| \le Kd(x,y)^{\alpha'}$ for all $x, y \in \mathbb{T}^2$. We want to show that φ has Bowen property at scale 100ϵ on $\mathscr{G}(r)$ for any $0 < r < 1$. We will estimate the total variation along the orbit by evaluating the distance at each time, which is provided by the following lemma.

Lemma 4.3.2. *If $(x,n)\in\mathscr{G}(r)$ and $y\in B_n(x,100\epsilon)$, then $d(G^k(x),G^k(y))\le 100\gamma\epsilon((\lambda(1-\beta))^{-kr}+(\lambda(1-\beta))^{-(n-k-1)r})$ for all $0\le k\le n-1$.*

Proof. By Lemma 4.1.1, there exists $z\in\mathbb{T}^2$ such that $G^k(z)=W^s_{100\gamma\epsilon}(G^k(x))\cap W^u_{100\gamma\epsilon}(G^k(y))$ for $0\le k\le n-1$. Lemma 4.3.1 then shows that $d(G^k(x),G^k(z))\le 100\gamma\epsilon(\lambda(1-\beta))^{-kr}$. Meanwhile, by pushing the essential collection of orbit segments further from the perturbed area if necessary, we can assume the local unstable leaf connecting $G^k(y)$ and $G^k(z)$ does not intersect the perturbed area for all $0\le k\le n-1$. Then the second part of Lemma 4.3.1 shows that $d(G^k(x),G^k(z))\le 100\gamma\epsilon(\lambda(1-\beta))^{-(n-1-k)r}$, and concludes the proof of Lemma 4.3.2. □

Now Lemma 4.3.2 enables us to show

Proposition 4.3.3. *For any $0<r<1$, φ has Bowen property at scale 100ϵ on $\mathscr{G}(r)$.*

Proof. Fix any $r\in(0,1)$. Suppose $(x,n)\in\mathscr{G}(r)$ and $y\in B_n(x,100\epsilon)$. By Hölder continuity of φ and Proposition 4.3.3, we have

$$\begin{aligned}
|S_n\varphi(x)-S_n\varphi(y)| &\le K\sum_{k=0}^{n-1} d(G^k(x),G^k(y))^{\alpha'}\\
&\le K(100\gamma\epsilon)^{\alpha'}\sum_{k=0}^{n-1}((\lambda(1-\beta))^{-kr}+(\lambda(1-\beta))^{-(n-k-1)r})^{\alpha'}.\\
&\le K(100\gamma\epsilon)^{\alpha'}\sum_{k=0}^{n-1}(2(\max\{(\lambda(1-\beta))^{-kr},(\lambda(1-\beta))^{-(n-k-1)r}\}))^{\alpha'}\\
&= K(200\gamma\epsilon)^{\alpha'}\sum_{k=0}^{n-1}(\max\{(\lambda(1-\beta))^{-kr},(\lambda(1-\beta))^{-(n-k-1)r}\})^{\alpha'}\\
&\le K(200\gamma\epsilon)^{\alpha'}\sum_{k=0}^{\infty}2(\lambda(1-\beta))^{-r\alpha'}=K_0<\infty,
\end{aligned}$$

where the last term converges as β is chosen small so that $\lambda(1-\beta)>1$. This concludes the proof of Proposition 4.3.3. □

4.3.3. Regularity for Geometric-t potengtial. In this section we will explore Bowen property of geometric-t potential. It is well-known that for $C^{1+\alpha}$ uniformly hyperbolic maps, the unstable distributions are Hölder continuous. Meanwhile, as $\log(x)$ is Lipschitz when x is uniformly bounded away from 0 (which is the case there), we immediately see that the geometric potential for a uniformly hyperbolic map is Hölder continuous.

However, we can not apply this argument to most of the non-uniformly hyperbolic maps. In our case, by approximating the Katok map by a sequence of Anosov diffeomorphisms in C^1-topology and applying the standard cone argument from Theorem 2.3.2 in [**3**], we can show that the respective Hölder exponent explodes to 0. Therefore, Hölder continuity of the distribution fails in the case of the Katok map and the verification of the Bowen property for geometric-t potential can not rely on results of §4.3.2.

We follow the proof from §6.2 in [**50**] (see also the appendix in [**12**]). The strategy is to split the variation along stable and unstable directions using local product structure, then apply coordinate transform and make use of the non-uniform expansion rate in E^u over E^s.

Proposition 4.3.4. *$\varphi_G^{geo}(x)$ satisfies Bowen property at scale 100ϵ on $\mathscr{G}(r)$.*

Proof. Consider $E^u : x \to E^u(x)$ as a map from $\mathbb{T}^2$ to G^1, where G^1 is the one-dimensional Grassmannian bundle over $\mathbb{T}^2$. Denote by $\psi' \in C(G^1)$ the function that sends $E \in G^1$ to $-\log|DG(x)|_E|$. Then naturally $\varphi_G^{geo}(x)$ can be identified as $\psi' \circ E^u$. Meanwhile, it is shown in [**15**, Lemma A.1] that the map ψ' is Hölder continuous with exponent α if G is $C^{1+\alpha}$. This is done by identifying G^1 with $\mathbb{T}^2 \times \mathrm{Gr}(1, \mathbb{R}^2)$ and writing out ψ' as a composition of Lipschitz and smooth functions. Since G is smooth in our case, ψ' is Hölder continuous. Therefore, Proposition 4.3.4 can be proved by obtaining a distance estimate that is similar to the one in Lemma 4.3.2 , then following the proof of Proposition 4.3.3.

To be more precise, what we want is the following:

Proposition 4.3.5. *For any $0 < r < 1$, there exist constants $C = C(r) \in \mathbb{R}$ and $\theta = \theta(r) \in (0, 1)$ such that for every $(x, n) \in \mathscr{G}(r), y \in B_{100\epsilon}(x, n)$ and $0 \le k \le n - 1$, we have*

$$d_{G_r}(E^u(G^k(x)), E^u(G^k(y))) \le C(\theta^k + \theta^{n-1-k}),$$

where d_{G_r} is the natural metric on $Gr(1, \mathbb{R}^2)$ defined by $d_{G_r}(E, E') = d_H(E \cap S^1, E' \cap S^1)$, with d_H being the Hausdorff metric on $S^1 \subset \mathbb{R}^2$.

PROOF OF PROPOSITION 4.3.5. Fix any $r \in (0,1)$, choose any $(x,n) \in \mathscr{G}(r)$ and $y \in B_{100\epsilon}(x,n)$. As mentioned in above, by applying Lemma 4.1.1, we get $z \in \mathbb{T}^2$ such that $G^k(z) = W^s_{100\gamma\epsilon}(G^k(x)) \cap W^u_{100\gamma\epsilon}(G^k(y))$ for $0 \le k \le n-1$. The idea is to estimate $d_{G_r}(E^u(G^k(x)), E^u(G^k(y)))$ along stable and unstable leaves, which are represented by $d_{G_r}(E^u(G^k(x)), E^u(G^k(z)))$ and $d_{G_r}(E^u(G^k(z)), E^u(G^k(y)))$ respectively. Since E^u integrates to W^u and is continuous on a compact local unstable leaf, there exists some constant C_1 such that

$$d_{G_r}(E^u(G^k(z)), E^u(G^k(y))) \le C_1 d(G^k(z), G^k(y)) \le 100 C_1 \gamma\epsilon (\lambda(1-\beta))^{-(n-k-1)r},$$

where the last inequality follows from Lemma 4.3.1. Therefore, to complete the proof of Proposition 4.3.5, it suffices to show

$$d_{G_r}(E^u(G^k(z)), E^u(G^k(x))) \le C'\theta'^k \tag{4.3.3}$$

for some $C' > 0$ and $\theta' \in (0,1)$.

We will start with a coordinate transform. For any $(x,n) \in \mathscr{G}(r)$, $z \in W^s_{100\gamma\epsilon}(x)$ and $0 \le k \le n-1$, we define $(e^i_{z,k})_{i=1}^2$ to be an orthonormal basis for $T_{G^k(z)}\mathbb{T}^2$ such that $e^1_{z,k}$ spans $E^s(G^k(z))$. In fact, we can choose $(e^i_{z,k})_{i=1}^2$ in a way such that the map $z \to e^i_{z,k}$ is K-Lipschitz on $W^s_{100\gamma\epsilon}(x)$ for every k, i, where K is independent of the choices on x, n, i and k. This is because we can choose such K for any open neighborhood of x and compactness of $\mathbb{T}^2$ will make K independent of x. Meanwhile, as we focus on the local stable leaves of orbit segments in $\mathscr{G}(r)$ where an overall exponential contraction in d^s under G exists, K is independent of the other parameters as well.

Now we can compute $d_{G_r}(E^u(G^k(x)), E^u(G^k(z)))$ using their respective coordinate representations in $e^i_{z,k}$ and $e^i_{x,k}$. Let $\pi_{z,k} : T_{G^k(z)}\mathbb{T}^2 \to \mathbb{R}^2$ be the coordinate representation in the basis of $e^i_{z,k}$. We also denote by $A^z_k : \mathbb{R}^2 \to \mathbb{R}^2$ the coordinate representation induced by $DG_{G^k(z)}$, in other words, $A^z_k \circ \pi_{z,k} = \pi_{z,k+1} \circ DG_{G^k(z)}$.

Now we fix (x,n) and z as above and write $E^x_k = \pi_{x,k} E^u(G^k(x))$. The goal is to show that $d_{G_r}(E^z_k, E^x_k) \le C\theta^k$ for some appropriate $\theta \in (0,1)$ that does not depend on x, z, n or k. From now on we focus on the dynamics of A^z_k and A^x_k. First notice that $A^z_k(Z) = Z$, where $Z = \mathbb{R} \times \{0\} \subset \mathbb{R}^2$. Denote by Ω the set of subspaces $E \subset \mathbb{R}^2$ such that $Z \oplus E = \mathbb{R}^2$. Obviously $E^z_k \in \Omega$ for all $z \in W^s_{100\gamma\epsilon}(x)$. Therefore, in order to measure the distance $d_{G_r}(E^z_k, E^x_k)$, we can restrict our discussion to distance over Ω. Now for $E \in \Omega$, define $L_k^E : E_k^x \to Z$ as the linear map whose graph is E. By applying standard trigonometric computation (see [**15**, Appendix] for details) we get $\sin(d_{G_r}(E_k^x, E)) \leq \|L_k^E\|$. Since $\sin(d_{G_r}(E_k^x, E_k^z))$ effectively approximate the value of $d_{G_r}(E_k^x, E_k^z)$ when $d_{G_r}(E_k^x, E_k^z)$ is sufficiently small, to verify (4.3.3), it suffices to show exponential decay of $\|L_k^{E_k^z}\|$ in k and the rate and constant are independent on the choice of x, n and z (and is only dependent on r).

Denote by $P : E_{k+1}^x \to A_k^z E_k^x$ the projection along Z. It is shown in [**15**, Lemma A.4] that

$$L_{k+1}^{E_{k+1}^z} + \mathrm{Id} = (A_k^z|_Z \circ L_k^{E_k^z} \circ A_k^z|_{E_k^x}^{-1} + \mathrm{Id}) \circ P, \tag{4.3.4}$$

which implies that

$$\|L_{k+1}^{E_{k+1}^z}\| \leq \|A_k^z|_Z\| \cdot \|A_k^z|_{E_k^x}^{-1}\| \cdot \|P\| \cdot \|L_k^{E_k^z}\| + \|P - \mathrm{Id}\|. \tag{4.3.5}$$

To study the limit behavior of $\|L_k^{E_k^z}\|$, we know from (4.3.5) that estimates on $\|P\|$ and $\|A_k^z|_Z\| \cdot \|A_k^z|_{E_k^x}^{-1}\|$ are indispensable. From Hölder continuity of DG, Lipschitz continuity of $e_{z,k}^i$ and the fact of z being in $W_{100\gamma\epsilon}^s(x)$, we have constants $C > 0$ and $\alpha_0 \in (0,1)$ that are independent of x, z, n, i, k and satisfy $\|A_k^z - A_k^x\| \leq C(100\gamma\epsilon)^{\alpha_0}(\lambda(1-\beta))^{-r\alpha_0}$, which shows that

$$d_{G_r}(E_{k+1}^x, A_k^z E_k^x) = d_{G_r}(A_k^x E_k^x, A_k^z E_k^x) \leq C'(100\gamma\epsilon)^{\alpha_0}(\lambda(1-\beta))^{-r\alpha_0} \tag{4.3.6}$$

for another constant C' that is also independent of x, z, n, i, k. We comment that α_0 is just the Hölder exponent of DG. To study $\|P - \mathrm{Id}\|$, by taking any $v \in E_{k+1}^x$, we have a triangle formed by v, Pv and $Pv - v$. Then $\frac{\|Pv-v\|}{\|v\|} = \frac{\sin\theta_1}{\sin\theta_2}$, where θ_1 is the angle between v and Pv, θ_2 is the angle between Pv and $Pv - v$. Since $Pv \in A_k^z E_k^x$ and $Pv - v \in Z$, we know $\sin\theta_1 \leq C''(100\gamma\epsilon)^{\alpha_0}(\lambda(1-\beta))^{-rk\alpha_0}$ for some constant C'' by (4.3.6) and θ_2 is uniformly bounded away from 0 by continuity of P and compactness of $\mathbb{T}^2$. Therefore there exists another constant C''' that is again independent of x, z, n, i, k such that $\frac{\|Pv-v\|}{\|v\|} \leq C'''(100\gamma\epsilon)^{\alpha_0}(\lambda(1-\beta))^{-rk\alpha_0}$. This shows that

$$\|P - \mathrm{Id}\| \leq C'''(100\gamma\epsilon)^{\alpha_0}(\lambda(1-\beta))^{-rk\alpha_0}. \tag{4.3.7}$$

Now write $\|L_k^{E_k^z}\|$ as D_k, $\|A_k^z|_Z\| \cdot \|A_k^z|_{E_k^x}^{-1}\|$ as P_k, $C'''(100\gamma\epsilon)^{\alpha_0}$ as Q and $(\lambda(1-\beta))^{-r\alpha_0}$ as u. By plugging (4.3.7) into (4.3.5), we have

$$D_{k+1} \leq P_k(1+Qu^k)D_k + Qu^k. \tag{4.3.8}$$

To study the inequality above, we notice there exists a constant $\lambda_0 \in (0,1)$ such that $P_k \leq \lambda_0$ when $\chi(G^k(x)) = 1$. Meanwhile, $P_i \leq 1$ for all i. Therefore, for any $(x,n) \in \mathscr{G}(r)$ and $z \in W^s_{100\gamma\epsilon}(x)$, we have

$$\prod_{i=0}^{j} P_i \leq {\lambda_0}^{(j+1)r} \text{ for all } 0 \leq j \leq n-1. \tag{4.3.9}$$

Inequality (4.3.9) is the key to make Bowen property still hold even with the presence of neutral fixed points. Write $C_k := \frac{D_k}{\nu^k}$, where $0 < \nu < 1$ is a constant close to 1 that will be determined later. Now (4.4.4) becomes

$$C_{k+1} \leq \frac{P_k}{\nu}(1+Qu^k)C_k + Q\frac{u^k}{\nu^{k+1}}. \tag{4.3.10}$$

It suffices to show C_k is bounded from above for a suitable choice of ν that is not related to the choice of (x,n). We know $C_0 = D_0$ is finite by compactness of $\mathbb{T}^2$ and continuity of the unstable distribution. Suppose $B > 0$ is an upper bound. We define a sequence $\{F_k\}_{k\in\mathbb{N}}$ satisfying $F_0 = B$ and the following relation

$$F_{k+1} = \begin{cases} \frac{1}{\nu}(1+Qu^k)F_k + Q\frac{u^k}{\nu^{k+1}} & \text{if } \chi(G^k(x)) = 0, \\ \frac{\lambda_0}{\nu}(1+Qu^k)F_k + Q\frac{u^k}{\nu^{k+1}} & \text{if } \chi(G^k(x)) = 1. \end{cases}$$

By the property of P_k, λ_0 and the construction of $\{F_k\}$, we know $C_k \leq F_k$ for all k. So instead of C_k, we focus on the uniform upper bound of F_k.

We will need some assumptions on ν for future use. First assume $\frac{u^{\frac{r}{2}}}{\nu} < 1$ and $\frac{{\lambda_0}^{\frac{r}{2}}}{\nu} < 1$. Fix such a ν. Then we choose two constants $\zeta > \frac{1}{\nu}$ and $\frac{\lambda_0}{\nu} < \eta < 1$ such that $u < \nu\eta$ and $\zeta^{1-\frac{r}{2}}\eta^{\frac{r}{2}} < 1$. Such choices on ζ and η are possible since $(\frac{u}{\nu})^{\frac{r}{2}}(\frac{1}{\nu})^{1-\frac{r}{2}} < 1$ and $(\frac{\lambda_0}{\nu})^{\frac{r}{2}}(\frac{1}{\nu})^{1-\frac{r}{2}} < 1$.

We want to set up an further uniform upper bound for the coefficient in front of F_k in the representation of F_{k+1} Choose $N \in \mathbb{N}$ large enough so that when $k \geq N$, $\frac{1}{\nu}(1+Qu^k) < \zeta$ and

$\frac{\lambda_0}{\nu}(1+Qu^k) < \eta$. Now for all possible $(x,n) \in \mathscr{G}(r)$ with $n < N$, by compactness of $\mathbb{T}^2$ and finite choice on n, $F_k = F_k(x,n)$ is uniformly bounded from above by some $M > 0$ for any $0 \le k \le n$. This motivates the construction of the new sequence $\{H_k\}_{k \ge N}$, where $H_N = M$ and the following is satisfied

$$H_{k+1} = \begin{cases} \zeta H_k + \frac{Q}{\nu}(\frac{u}{\nu})^k & \text{if } \chi(G^k(x)) = 0, \\ \eta H_k + \frac{Q}{\nu}(\frac{u}{\nu})^k & \text{if } \chi(G^k(x)) = 1. \end{cases}$$

Again the main problem is shifted to the proof on H_k being uniformly bounded from above. Consider large enough k such that $k > \frac{2N}{r}$. In this case $\sum_{i=N}^{k} \chi(F^i(x)) > kr - N > \frac{rk}{2}$. The following lemma shows that H_k are in fact uniformly bounded from above for all such k.

LEMMA 4.3.6. *There exists a constant $M' > 0$ such that $H_k \le M'$ for all $k > \frac{2N}{r}$, and the choice on M' is independent of x, n, z, k.*

PROOF OF LEMMA 4.3.6. By writing $a_k = a_k(x,n) := \zeta(1-\chi(F^k(x))) + \eta\chi(F^k(x))$ for $k \ge N$, we can simplify the representation of H_{k+1} for such k as

$$H_{k+1} = a_k H_k + \frac{Q}{\nu}(\frac{u}{\nu})^k. \tag{4.3.11}$$

In fact, by iterating (4.3.11) on k , we are able to derive the explicit form of H_k for $k > N$ as follows

$$H_k = (\prod_{i=N}^{k-1} a_i)M + \frac{Q}{\nu}\sum_{j=N}^{k-1}((\frac{u}{\nu})^j \cdot \prod_{s=j+1}^{k-1} a_s). \tag{4.3.12}$$

By using $a_k \ge \eta$ and $u < \nu\eta$, we have

$$\begin{aligned} H_k &= (\prod_{i=N}^{k-1} a_i)M + \frac{Q}{\nu}(\frac{u}{\nu})^N \cdot (\prod_{s=N+1}^{k-1} a_s) \cdot \sum_{l=0}^{k-N-1} (\frac{u}{\nu})^l (\prod_{s=N+1}^{N+l} a_s)^{-1} \\ &\le (\prod_{i=N}^{k-1} a_i)M + \frac{Q}{\nu}(\frac{u}{\nu})^N \cdot (\prod_{s=N+1}^{k-1} a_s) \cdot \sum_{l=0}^{k-N-1} (\frac{u}{\nu} \cdot \frac{1}{\eta})^l \\ &\le (\zeta^{1-\frac{r}{2}}\eta^{\frac{r}{2}})^{k-N-1}(M + \frac{Q}{\nu}(\frac{u}{\nu})^N \cdot \frac{1}{\nu} \cdot \sum_{l=0}^{\infty}(\frac{u}{\nu\eta})^l) \\ &< M + \frac{Q}{\nu}(\frac{u}{\nu})^N \cdot \frac{1}{\nu} \cdot \sum_{l=0}^{\infty}(\frac{u}{\nu\eta})^l < \infty \end{aligned} \tag{4.3.13}$$

for all $k > \frac{2N}{r}$. Here we can replace $\prod_{i=N+1}^{k-1} a_i$ by $(\zeta^{1-\frac{r}{2}}\eta^{\frac{r}{2}})^{k-N-1}$ since $\sum_{i=N}^{k} \chi(F^i(x)) > \frac{rk}{2}$. This ends the proof of Lemma 4.3.6. $\square$

Lemma 4.3.6 shows that H_k is uniformly bounded for $k > \frac{2N}{r}$, therefore for all k. Following the argument preceding to Lemma 4.3.6, we have an uniform upper bound for F_k thus for C_k. This ends the proof of Proposition 4.3.5. $\square$

Finally, by applying Proposition 4.3.5 and following exactly the same argument used in Proposition 4.3.3, we conclude the proof of Proposition 4.3.4. $\square$

4.4. Proof of Main Theorem

With the help of the results obtained in §4.1, §4.2 and §4.3, now we are ready to apply Theorem 3.2.1 to prove our main thermodynamic results for the Katok map.

4.4.1. Proof of Theorem A : φ is Hölder. We begin with the case where the potential function φ is Hölder continuous. First consider Hölder continuous φ satisfying the gap condition $P(\varphi; G) - \varphi(\underline{0}) > 0$. We will study the equilibrium state of φ for G. Recall from §3.2.2 that the collections $(\mathscr{P}(r), \mathscr{G}(r), \mathscr{S}(r))$ form an orbit decomposition of G for all $r \in (0,1)$. We want to show that the conditions in Theorem 3.2.1 hold when r is properly chosen. Specification, thus tail specification, clearly holds at scale ϵ. Obstruction to expansivity is satisfied due to Proposition 4.1.2. Meanwhile, (4.2.6) shows that the pressure gap condition holds for some $r' \in (0,1)$ (at the desired scale). Finally, Bowen property holds for φ along orbit segments in $\mathscr{G}(r)$ for all r at scale 100ϵ by Proposition 4.3.3. As a result, by taking $(\mathscr{P}(r'), \mathscr{G}(r'), \mathscr{S}(r'))$ as the wanted orbit decomposition, all four conditions are satisfied and Theorem 3.2.1 can be applied.

Following the same spirit we can apply the above argument to study $\widetilde{G}$. Suppose φ is Hölder continuous and satisfies the gap condition $P(\varphi, \widetilde{G}) - \varphi(\underline{0}) > 0$. Since $\widetilde{G} = \phi \circ G \circ \phi^{-1}$, for each $r \in (0,1)$, $\mathscr{P}(r)$ and $\mathscr{G}(r)$ naturally induce two collection of orbit segments under ϕ, written as

$$\mathscr{P}'(r) = \mathscr{S}'(r) := \{(x,n) \in \mathbb{T}^2 \times \mathbb{N} : (\phi^{-1}(x), n) \in \mathscr{P}(r)\},$$

and

$$\mathscr{G}'(r) := \{(x,n) \in \mathbb{T}^2 \times \mathbb{N} : (\phi^{-1}(x), n) \in \mathscr{G}(r)\}.$$

Since ϕ is identity outside D_{r_1}, we can easily show that

$$\mathscr{G}'(r) = \{(x,n) : \frac{1}{i} S_i^{\widetilde{G}} \chi(x) \geq r \text{ and } \frac{1}{i} S_i^{\widetilde{G}} \chi(\widetilde{G}^{n-i}(x)) \geq r \text{ for all } 0 \leq i \leq n\}$$

and

$$\mathscr{P}'(r) = \mathscr{S}'(r) = \{(x,n) \in \mathbb{T}^2 \times \mathbb{N} : \frac{1}{n} S_n^{\widetilde{G}} \chi(x) < r\},$$

where $S_i^{\widetilde{G}} \chi(x) := \sum_{j=0}^{i-1} \chi(\widetilde{G}^j(x))$.

Following the proof of Proposition 3.2.2, the collection of orbit segments $(\mathscr{P}'(r), \mathscr{G}'(r), \mathscr{P}'(r))$ form an orbit decomposition for $\widetilde{G}$. As in the case of G, we need to check those conditions in Theorem 3.2.1 hold at the correct scale. Specification is again for free by its homeomorphic conjugacy to the linear toral automorphism f_A. Expansiveness of $\widetilde{G}$ at scale $\frac{C\epsilon}{\sqrt{\kappa_0}}$ can be obtained by using expansiveness of G at scale ϵ and (2.2.2) (where C and κ_0 are from §2.2.2). Meanwhile, by following exactly the same process as in §4.2 and using the gap condition $P(\varphi, \widetilde{G}) - \varphi(\underline{0}) > 0$, we can find $\widetilde{r} = \widetilde{r}(\varphi) \in (0,1)$ and some $\epsilon_1 > 0$ such that $P(\mathscr{P}'(\widetilde{r}), \varphi, \epsilon_1, 100\epsilon_1; \widetilde{G}) < P(\varphi; \widetilde{G})$. Here we need ϵ_1 other than the original ϵ due to the difference of scale and constant of local product structure between G and $\widetilde{G}$. Finally, the same process as in §4.2 shows that φ has Bowen property along orbit segments in $\mathscr{G}'(r)$ for all $r \in (0,1)$ at some scale $100\epsilon_2$. Denote $\min\{\frac{C\epsilon}{\sqrt{\kappa_0}}, \epsilon_1, \epsilon_2\}$ by $\widetilde{\epsilon}$. Since expansiveness, pressure gap and Bowen property will continue to hold when the scale is decreased, all four conditions in Theorem 3.2.1 continue to hold for the orbit decomposition $(\mathscr{P}'(\widetilde{r}), \mathscr{G}'(\widetilde{r}), \mathscr{P}'(\widetilde{r}))$ with scale being $\widetilde{\epsilon}$ and Theorem A is verified for the case of Hölder continuous φ.

4.4.2. Proof of Theorem A : φ is the geometric-t potential. To prove the case where φ is the geometric-t potential, notice that when G and $\widetilde{G}$ are conjugate, φ_G^{geo} and φ^{geo} are different functions. Therefore, we need to verify Bowen property of φ^{geo} on the proper essential collection of orbit segments. We will consider an appropriate orbit decomposition for $\widetilde{G}$ and use Bowen property of φ_G^{geo} on $\mathscr{G}(r)$ to induce Bowen property of φ^{geo} on the respective essential collection of orbit segments. Meanwhile, we also need to deal with the pressure gap, which is $t\varphi^{geo}(\underline{0}) < P(t\varphi^{geo}; \widetilde{G})$ in this case. We will show this gap condition holds for all $t < 1$.

As in the previous section, we use $(\mathscr{P}'(r), \mathscr{G}'(r), \mathscr{P}'(r))$ as the orbit decomposition for $\widetilde{G}$. We will first deduce the regularity property of φ^{geo} on $\mathscr{G}'(r)$ from which of φ_G^{geo} on $\mathscr{G}(r)$. Recall that

$\widetilde{G} = \phi \circ G \circ \phi^{-1}$ and $D\phi(E^u(x)) = \widetilde{E}^u(\phi(x))$. For all $i \geq 0$ we have

$$
\begin{aligned}
&\varphi^{geo}(\widetilde{G}^i(x)) \\
&= -\log|D\widetilde{G}|_{\widetilde{E}^u(\widetilde{G}^i(x))}| \\
&= -\log|D(\phi \circ G \circ \phi^{-1})|_{\widetilde{E}^u(\widetilde{G}^i(x))}| \\
&= -\log|D\phi|_{D(G\circ\phi^{-1})\widetilde{E}^u(\widetilde{G}^i(x))}| - \log|DG|_{D\phi^{-1}\widetilde{E}^u(\widetilde{G}^i(x))}| - \log|D\phi^{-1}|_{\widetilde{E}^u(\widetilde{G}^i(x))}| \\
&= -\log|D\phi|_{D(G\circ\phi^{-1})\widetilde{E}^u(\widetilde{G}^i(x))}| - \varphi_G^{geo}(G^i(\phi^{-1}(x))) - \log|D\phi^{-1}|_{\widetilde{E}^u(\widetilde{G}^i(x))}| \\
&= -\log|D\phi|_{DG(E^u(G^i(\phi^{-1}(x))))}| - \varphi_G^{geo}(G^i(\phi^{-1}(x))) - \log|D\phi^{-1}|_{\widetilde{E}^u(\widetilde{G}^i(x))}| \\
&= -\log|D\phi|_{E^u(G^{i+1}(\phi^{-1}(x)))}| - \varphi_G^{geo}(G^i(\phi^{-1}(x))) - \log|D\phi^{-1}|_{\widetilde{E}^u(\widetilde{G}^i(x))}|.
\end{aligned}
\tag{4.4.1}
$$

Meanwhile, observe that

$$
\begin{aligned}
0 &= -\log|D(\phi\circ\phi^{-1})|_{\widetilde{E}^u(\widetilde{G}^i(x))}| \\
&= -\log|D\phi|_{D\phi^{-1}\widetilde{E}^u(\widetilde{G}^i(x))}| - \log|D\phi^{-1}|_{\widetilde{E}^u(\widetilde{G}^i(x))}| \\
&= -\log|D\phi|_{E^u(G^i(\phi^{-1}(x)))}| - \log|D\phi^{-1}|_{\widetilde{E}^u(\widetilde{G}^i(x))}|.
\end{aligned}
\tag{4.4.2}
$$

By plugging (4.4.2) into (4.4.1) we have

$$
\varphi^{geo}(\widetilde{G}^i(x)) = \log|D\phi^{-1}|_{\widetilde{E}^u(\widetilde{G}^{i+1}(x))}| - \varphi_G^{geo}(G^i(\phi^{-1}(x))) - \log|D\phi^{-1}|_{\widetilde{E}^u(\widetilde{G}^i(x))}|.
\tag{4.4.3}
$$

Now fix $r \in (0,1]$, choose $(x,n) \in \mathscr{G}'(r)$ and $y \in B_n^{\widetilde{G}}(x, \frac{100C\epsilon}{\kappa_0})$. With the help of (4.4.3), we have

$$
\begin{aligned}
&S_n^{\widetilde{G}}\varphi^{geo}(x) - S_n^{\widetilde{G}}\varphi^{geo}(y) \\
&= \sum_{i=0}^{n-1}(\varphi^{geo}(\widetilde{G}^i(x)) - \varphi^{geo}(\widetilde{G}^i(y))) \\
&= \sum_{i=0}^{n-1}(\log|D\phi^{-1}|_{\widetilde{E}^u(\widetilde{G}^{i+1}(x))}| - \varphi_G^{geo}(G^i(\phi^{-1}(x))) - \log|D\phi^{-1}|_{\widetilde{E}^u(\widetilde{G}^i(x))}| \\
&- (\log|D\phi^{-1}|_{\widetilde{E}^u(\widetilde{G}^{i+1}(y))}| - \varphi_G^{geo}(G^i(\phi^{-1}(y))) - \log|D\phi^{-1}|_{\widetilde{E}^u(\widetilde{G}^i(y))}|)) \\
&= \log|D\phi^{-1}|_{\widetilde{E}^u(\widetilde{G}^n(x))}| - \log|D\phi^{-1}|_{\widetilde{E}^u(\widetilde{G}^n(y))}| - \log|D\phi^{-1}|_{\widetilde{E}^u(x)}| + \log|D\phi^{-1}|_{\widetilde{E}^u(y)}| \\
&+ \sum_{i=0}^{n-1}(\varphi_G^{geo}(G^i(\phi^{-1}(y))) - \varphi_G^{geo}(G^i(\phi^{-1}(x)))).
\end{aligned}
\tag{4.4.4}
$$

Let us observe on the last line in (4.4.4). By definition of $\mathscr{G}'(r)$, we know both x and $\widetilde{G}^n(x)$ belong to $\mathbb{T}^2 \setminus D_{100\gamma\epsilon+r_1}$. Therefore, both y and $\widetilde{G}^n(y)$ are in $\mathbb{T}^2 \setminus D_{r_1}$. In particular, as ϕ^{-1} is identity in $\mathbb{T}^2\setminus D_{r_1}$, we know $\log|D\phi^{-1}|_{\widetilde{E}^u(\widetilde{G}^n(x))}|-\log|D\phi^{-1}|_{\widetilde{E}^u(\widetilde{G}^n(y))}|-\log|D\phi^{-1}|_{\widetilde{E}^u(x)}|+\log|D\phi^{-1}|_{\widetilde{E}^u(y)}| = 0$ and we only need to concentrate on $\sum_{i=0}^{n-1}(\varphi_G^{geo}(G^i(\phi^{-1}(y)))-\varphi_G^{geo}(G^i(\phi^{-1}(x))))$. Since $(\phi^{-1}(x), n) \in \mathscr{G}(r)$, we only need to show that $d(G^i(\phi^{-1}(x)), G^i(\phi^{-1}(y))) < 100\epsilon$ holds to apply the result in Proposition 4.3.4. By 2.2.2, we know $d(G^i(\phi^{-1}(x)), G^i(\phi^{-1}(y))) = d(\phi^{-1}\widetilde{G}^i(x), \phi^{-1}\widetilde{G}^i(y)) \le \frac{\kappa_0}{C} d(\widetilde{G}^i(x), \widetilde{G}^i(y)) < \frac{100C\kappa_0\epsilon}{C\kappa_0} = 100\epsilon$. This concludes the Bowen property of φ^{geo} for $\widetilde{G}$ on $\mathscr{G}'(r)$ for any $0 < r \le 1$ at scale $\frac{100\epsilon}{\kappa_0}$.

The only thing left to check is that $t\varphi^{geo}(\underline{0}) < P(t\varphi^{geo}; \widetilde{G})$ holds for all $t < 1$. Since the Lebesgue measure m on $\mathbb{T}^2$ is $\widetilde{G}$-invariant, ergodic and has all Lyapunov exponents to be non-zero, it is an SRB measure. Therefore, we have

$$h_m(\widetilde{G}) = \lambda^+(m) = -\int \varphi^{geo} dm,$$

where λ^+ is the positive Lyapunov exponent with respect to m. Meanwhile, as $-\int \varphi^{geo} dm > 0$, we have

$$P(t\varphi^{geo}; \widetilde{G}) \ge P(t\varphi^{geo}, m; \widetilde{G}) = h_m(\widetilde{G}) + t\int \varphi^{geo} dm = (1-t)\int \varphi^{geo} dm > 0.$$

As a result, we have $P(t\varphi^{geo}; \widetilde{G}) > 0 = P(t\varphi^{geo}, \delta_0)$ for all $t < 1$. This concludes the proof of Theorem A when $\varphi = \varphi_t$ for $t < 1$.

4.5. Statistical properties of equilibrium states

In this section we will exhibit a series of statistical properties of equilibrium states derived in Theorem A. We will also briefly survey on a result of Pesin, Senti and Zhang, which is based on studying thermodynamic formalism of the Katok map using Young's Tower. This provides some nice statistical properties due to the symbolic nature.

4.5.1. Preliminary on Gibbs property. Gibbs property is a property that makes the measure of Bowen balls 'uniformly comparable' regardless of their center and degree. To be more precise, for a discrete dynamical system, given $\varphi \in C(X)$, $\delta > 0$ and $\mathscr{D} \subset X \times \mathbb{N}$, we say an invariant measure μ has Gibbs property at scale δ over $\mathscr{D}$ with respect to φ if there exists a constant

$Q = Q(\delta, \mathscr{D}, \varphi) > 1$ such that for every $(x, n) \in \mathscr{D}$, the following estimate regarding the size of Bowen balls holds

$$Q^{-1} \leq \frac{\mu(B_n(x,\delta))}{e^{-nP(\varphi)+S_n\varphi(x)}} \leq Q. \tag{4.5.1}$$

We can also define Gibbs property from one side. That is to say, if the left (resp. right) inequality holds in (4.5.1), then we say μ has lower (resp. upper) Gibbs property at scale δ over $\mathscr{D}$ with respect to φ. Sometimes we will also omit the reliance on φ when the potential is fixed.

The approach of obtaining Gibbs property is motivated by [**17**]. For the equilibrium state derived from Theorem 3.2.1, it was shown in [**17**] that such measure satisfies a non-uniform version of upper Gibbs property at a scale that is compatible with ϵ over $X \times \mathbb{N}$ and lower Gibbs property at the same scale over $\mathscr{G}^M$. In the case of the Katok map, as specification is satisfied for all orbit segments, one would hope that lower Gibbs property will also hold on $X \times \mathbb{N}$.

Before stating our result, we first introduce the basic setups. Throughout this section we will just work with G as the property for $\widetilde{G}$ can be obtained in the same way. We will also fix φ so that it is either a geometric-t potential with $t < 1$ or a Hölder continuous potential with the gap condition $P(\varphi) - \varphi(\underline{0}) > 0$. The measure μ we focus on is just the unique equilibrium state for φ over G that is derived in Theorem A. We also fix a proper $r \in (0, 1)$ so that $(\mathscr{P}(r), \mathscr{G}(r), \mathscr{P}(r))$ satisfies all properties in Theorem 3.2.1. The constant $\epsilon > 0$ is still small, while the restriction on ϵ being greater than r_0 is removed in this section, as expansiveness, pressure gap and Bowen property automatically hold at smaller scales, while specification in our case holds at all scales.

For each $n \in \mathbb{N}$, we denote by $E_n \subset X$ a maximizing $(n, 5\epsilon)$-separated set for $\Lambda(X, n, 5\epsilon)$. Consider the following measures

$$\nu_n := \frac{\sum_{x\in E_n} e^{S_n\varphi(x)}\delta_x}{\sum_{x\in E_n} e^{S_n\varphi(x)}},$$

and

$$\mu_n := \frac{1}{n}\sum_{i=0}^{n-1}(G^i)_*\nu_n.$$

Similar to §4.2, by following the spirit of second half of the proof of the variational principle in [**49**] and using the fact that ϵ is small enough (in particular, 100ϵ is an expansive constant), we know any weak* limit of $\{\mu_n\}$ is an equilibrium state. As we have shown that the equilibrium state is unique in this case, we know $\{\mu_n\}$ converges to μ in weak*-topology. We will use $\{\mu_n\}$ to approximate μ throughout the proof of Gibbs property.

4.5.2. Global weak Gibbs property. The lower Gibbs property for μ in our case is the following version over all orbit segments, which is weak in a way that the Gibbs constant is decaying subexponentially in n.

PROPOSITION 4.5.1. *There is a constant $Q = Q(\epsilon) > 0$ such that for every $(x, n) \in X \times \mathbb{N}$, we have*

$$\mu(B_n(x, 6\epsilon)) \geq Qe^{-\zeta(n)}e^{-nP(\varphi)+S_n\varphi(x)},$$

where $\zeta(n)$ is defined as in Definition 4.2.5.

PROOF. Since Bowen balls are closed and we are looking for a lower bound, we can estimate $\mu(B_n(x, 6\epsilon))$ using $\nu_s(G^{-k}(B_n(x, 6\epsilon)))$ for any $(x, n) \in X \times \mathbb{N}$ with $s \gg k + n$. We will follow the spirit in the proof of [**17**, Lemma 4.16] and do the adaptions accordingly. Following [**17**, Proposition 4.10], there are constants $T, L > 0$ such that the following holds

$$\Lambda(\mathscr{G}, 12\epsilon, m) > e^{-L}e^{mP(\varphi)} \tag{4.5.2}$$

for all $m \geq T$. In other words, for every such m, there exists an $(m, 12\epsilon)$-separated set $E'_m \subset \mathscr{G}_m$ satisfying

$$\sum_{x \in E'_m} e^{S_m\varphi(x)} \geq e^{-L}e^{mP(\varphi)}. \tag{4.5.3}$$

We will apply specification of G at scale ϵ to study $\nu_s(G^{-k}(B_n(x, 6\epsilon)))$ with $s \gg T$. Denote by $\tau = \tau(\epsilon)$ the transition time of specification at scale ϵ. Fix time s and k such that $k > T + \tau$, $s - k - n > T + \tau$ and $s \gg T$. As in [**17**], we will construct a gluing map π that maps $E'_{k-\tau} \times E'_{s-k-n-\tau}$ to E_s as follows. Given $u = (u_1, u_2) \in E'_{k-\tau} \times E'_{s-k-n-\tau}$, we know from specification of G at scale ϵ that there exists a $y = y(u)$ such that $y \in B_{k-\tau}(u_1, \epsilon)$, $G^k(y) \in B_n(x, \epsilon)$ and $G^{k+n+\tau}(y) \in B_{s-k-n-\tau}(u_2, \epsilon)$. Meanwhile, since E_s is $(s, 5\epsilon)$-spanning, we can find a point $\pi(u) \in E_s$ such that $d_s(\pi(u), y(u)) < 5\epsilon$ (notice that when the choice of such point in E_s is not unique, we just choose any point satisfying the above condition as $\pi(u)$). By definition we immediately observe that $\pi(u) \in G^{-k}(B_n(x, 6\epsilon))$. It is also not hard to see that the map π is injective. In fact, if we choose $u' = (u_1', u_2')$, $u'' = (u_1'', u_2'')$ from $E'_{k-\tau} \times E'_{s-k-n-\tau}$ that are different, as $E'_{k-\tau}$ and $E'_{s-k-n-\tau}$ are $(k-\tau, 12\epsilon)$-separated and $(s-k-n-\tau, 12\epsilon)$-separated respectively, we know $d_s(\pi(u'), \pi(u'')) > 12\epsilon - 2(5\epsilon + \epsilon) = 0$.

Since both $E'_{k-\tau}$ and $E'_{s-k-n-\tau}$ are chosen from $\mathscr{G}(r)$, where φ satisfies Bowen property at scale 100ϵ by §4.3, we have

$$\Phi_0(\pi(u), s) \geq \Phi_0(u_1, k-\tau) + \Phi_0(x, n) + \Phi_0(u_2, s-k-n-\tau) - 4\tau|\varphi| - 2K - \zeta(n), \tag{4.5.4}$$

where $|\varphi| := \sup\{|\varphi(x)| : x \in \mathbb{T}^2\}$ and K is the constant in Bowen property. Meanwhile, by using condition (2) and (3) of Theorem 3.2.1 and applying [**17**, Lemma 4.11], we know there exists a constant $C > 0$ that is independent of s and satisfies $\sum_{z \in E_s} e^{\Phi_0(z,s)} \leq Ce^{sP(\varphi)}$ for all s. Therefore, we have the following lower estimate for $\nu_s(G^{-k}(B_n(x, 6\epsilon)))$ as

$$\begin{aligned}
&\nu_s(G^{-k}(B_n(x, 6\epsilon))) \geq C^{-1}e^{-sP(\varphi)} \sum_{u \in E'_{k-\tau} \times E'_{s-k-n-\tau}} e^{\Phi_0(\pi(u),s)} \\
&\geq C^{-1}e^{-sP(\varphi)}e^{-\zeta(n)-4\tau|\varphi|-2K}\Big(\sum_{u_1 \in E'_{k-\tau}} e^{\Phi_0(u_1,k-\tau)}\Big)\Big(\sum_{u_2 \in E'_{s-k-n-\tau}} e^{\Phi_0(u_2,s-k-n-\tau)}\Big)e^{\Phi_0(x,n)} \\
&\geq C^{-1}e^{-sP(\varphi)}e^{-\zeta(n)-4\tau|\varphi|-2K}(e^{-L}e^{(k-\tau)P(\varphi)})(e^{-L}e^{(s-k-n-\tau)P(\varphi)})e^{\Phi_0(x,n)} \\
&= (C^{-1}e^{-2K}e^{-2L}e^{-4\tau|\varphi|}e^{-2\tau P(\varphi)})(e^{-\zeta(n)}e^{-nP(\varphi)+\Phi_0(x,n)}) \\
&= C_1e^{-\zeta(n)}e^{-nP(\varphi)+\Phi_0(x,n)},
\end{aligned} \tag{4.5.5}$$

where the first inequality follows from $\pi(u) \in G^{-k}(B_n(x, 6\epsilon))$, the injectivity of π and the choice of C, the second one follows from (4.5.4), the third one follows from (4.5.3) and constant C_1 in the last equality is just $C^{-1}e^{-2K}e^{-2L}e^{-4\tau|\varphi|}e^{-2\tau P(\varphi)}$. Moreover, C_1 is independent of s or k and only dependent on ϵ. Therefore, by taking s to be large enough and summing over k, we have

$$\mu_s((B_n(x, 6\epsilon))) = \frac{1}{s}\sum_{k=0}^{s-1}((G^k)_*\nu_s)(B_n(x, 6\epsilon)) \geq \frac{C_1}{2}e^{-\zeta(n)}e^{-nP(\varphi)+\Phi_0(x,n)},$$

where $\frac{1}{2}$ in the last expression follows from the fact that (4.5.5) only holds for k such that $k > T + \tau$ and $s - k - n > T + \tau$, as well as s is sufficiently large. By making s go to ∞, we conclude the proof of Proposition 4.5.1.

Meanwhile, since [**17**, Proposition 4.21] has already provided us with a version of upper Gibbs property for μ on $X \times \mathbb{N}$, instead of repeating steps from above, we only need to slightly modify the original proof. More precisely, we know from [**17**, Proposition 4.21] that there exists a constant $Q' = Q'(\epsilon)$ such that for every $(x, n) \in X \times \mathbb{N}$, we have

$$\mu(B_n(x, 6\epsilon)) \leq Q' e^{-nP(\varphi)+\Phi_{6\epsilon}(x,n)}, \tag{4.5.6}$$

where $\Phi_{6\epsilon}(x, n)$ is as in §2.1.2. We also know from definition of $\zeta(n)$ that $\Phi_{6\epsilon}(x, n) \leq \Phi_0(x, n)+\zeta(n)$. Therefore, we can rewrite (4.5.6) as

$$\mu(B_n(x, 6\epsilon)) \leq Q' e^{-nP(\varphi)+\Phi_0(x,n)+\zeta(n)}, \tag{4.5.7}$$

which is exactly the global weak upper Gibbs property that we are looking for. □

4.5.3. Large Deviation Principle. One of the most important applications for Gibbs property of a measure is to obtain its large deviation principle. In this section we will show that the weak Gibbs property for μ we have in §4.5.2 is sufficient to draw the same conclusion. In dynamics, large deviation principle describes the exponential decay of measure of points whose space average differ from time average by a certain distance.

DEFINITION 4.5.2. Let μ and φ be as above. μ is said to satisfy upper large deviations principle if for any $\delta > 0$ and any continuous $\widetilde{f} : \mathbb{T}^2 \to \mathbb{R}$, we have

$$\limsup_{n\to\infty} \frac{1}{n} \log \mu \left\{ x : |\frac{S_n \widetilde{f}(x)}{n} - \int \widetilde{f} d\mu| \geq \delta \right\} \leq -q(\delta), \tag{4.5.8}$$

where $q(\delta)$ is called the rate function and defined as

$$q(\delta) := P(\varphi) - \sup \left\{ h_\nu(\widetilde{G}) + \int \varphi d\nu : \nu \in \mathscr{M}_{\widetilde{G}}(\mathbb{T}^2), \quad |\int \widetilde{f} d\mu - \int \widetilde{f} d\nu| \geq \delta \right\},$$

or $q(\delta) = \infty$ when no such measure exists.

We can also define the lower large deviations principle in a similar way by take the place of limsup by liminf, $\geq \delta$ by $> \delta$ and $\leq$ by $\geq$ in (4.5.8). When both upper and lower large deviations hold for some fixed $\widetilde{f}$, the statement above is called level-1 large deviations priciple (with respect to the observable $\widetilde{f}$). If both of them hold for all $\widetilde{f}$, the statement above is equivalent to level-2 large deviations principle by using the fact of $C(X)$ being separable.

To prove Theorem B, we need the following definitions. First, for a dynamical system (X, f), we say it has the property of entropy density if for any $\eta > 0$ and invariant Borel probability measure μ', there exists an ergodic measure ν' such that $D(\mu', \nu') < \eta$ and $|h_{\mu'}(f) - h_{\nu'}(f)| < \eta$, where D is a metric over $\mathscr{M}(X)$ that is compatible with the weak* topology. Since $\widetilde{G}$ satisfies specification, it also satisfies the entropy density property. Meanwhile, as mentioned in §4.5.1, we notice that after proving uniqueness and existence of the equilibrium measure μ, we can indeed ignore the lower bound of ϵ. Therefore, by applying 4.2.6 to Proposition 4.5.1, we have

$$\lim_{\epsilon\to 0}\liminf_{n\to\infty}\inf_{x\in\mathbb{T}^2}\Big(\frac{1}{n}\log(\mu(B_n(x,\epsilon))) + \int (P(\varphi)-\varphi)d\delta_{x,n}\Big) \geq 0, \tag{4.5.9}$$

where $\delta_{x,n} = \frac{1}{n}\sum_{i=0}^{n-1}\delta_{G^i(x)}$.

Since φ is continuous, following [**42**, Definition 3.2], The inequality (4.5.9) shows that $P(\varphi) - \varphi$ is a lower-energy function for μ. Similarly we have from (4.5.7) and Lemma 4.2.6 that the following holds

$$\lim_{\epsilon\to 0}\limsup_{n\to\infty}\sup_{x\in\mathbb{T}^2}\Big(\frac{1}{n}\log(\mu(B_n(x,\epsilon)) + \int (P(\varphi)-\varphi)d\delta_{x,n}\Big) \leq 0, \tag{4.5.10}$$

which in turns shows that $P(\varphi) - \varphi$ is an upper-energy function for μ, whose definition can be found in [**42**, Definition 3.4].

Now we have collected all the ingredients to prove the level-2 large deviation principle for μ. To do this, it suffices to show both upper and lower large deviation principle hold for μ. As $P(\varphi) - \varphi$ is a lower-energy function for μ and $\widetilde{G}$ satisfies entropy density property, by applying [**42**, Theorem 3.1], we have the lower large deviation principle for μ. Meanwhile, by the result of $P(\varphi) - \varphi$ being an upper-energy function and entropy map being upper semi-continuous, the upper large deviation principle for μ can also be concluded by [**42**, Theorem 3.2]. This ends the proof of Theorem B.

4.5.4. Tower construction and Statistical properties. In this section we briefly describe how to derive thermodynamic formalism of the Katok map via inducing scheme, also known as Young's tower. The study is based on results from [**41**] and [**40**]. The idea is to establish a full shift with countably many symbols that is conjugate to the first recurrence map on some certain region of $\mathbb{T}^2$ (called the base). To be more precise, the base is the image of an element of the Markov partition for f_A (which is far away from the perturbed region) under the conjugacy map h. The diameter of such element can be made arbitrarily small due to the property of Markov partition for basic sets and h being a homeomorphism. As the symbolic dynamics is induced by the first recurrence map, we can construct the alphabet set in a way so that all its symbols consist of first recurrent segments to the base. Furthermore, by Markov property, the shift is in fact a full shift. Then it be shown that the unique equilibrium state for the full shift corresponds to the unique equilibrium states for $\widetilde{G}$ that is 'liftable' on the base when certain conditions are satisfied. In particular, the equilibrium state distributes positive weight to its corresponding base, as well as any open subset in the base (resulted from Gibbs property). Then one can use topological transitivity of $\widetilde{G}$ to easily show that such equilibrium state is not dependent of the choice on the base. Meanwhile, though the set of liftable measures are different given different bases, the collection of the set of all such liftable measures with respect to all different bases far away from the perturbed area run out all of the measures we are interested in. In fact, if a measure is not liftable with respect to any of the bases as above, it has to charge zero measure over all such bases. Therefore, it is supported on a small neighborhood of the origin. However, the only point whose orbit will never escape from the perturbed area in both forward and backward direction is the origin, which shows that the only candidate for such measure is δ_0. As shown in §4.4.2, $P(t\varphi^{geo}) > 0$ for all $t < 1$ and $P_{\delta_0}(\widetilde{G}, t\varphi^{geo}) = 0$ for all $t \in \mathbb{R}$. Therefore, only liftable measures needs consideration in this case.

One of the strength of such Tower construction is that it shows explicitly that there are only two ergodic equilibrium states for φ^{geo}. The first one is among the liftable measures and is in fact the Lebesgue measure m, which is also the SRB measure. The other one is the only candidate among non-liftable ones, which is δ_0 as explained above.

It is also shown in [**40**] that for given r_0 and α (same as what we have in our case) that are small enough, there exists $t_0 = t_0(r_0, \alpha)$ such that for all $t \in (t_0, 1)$, the geometric t-potential $t\varphi^{geo}$ has a unique equilibrium state μ_t for $\widetilde{G}$. Meanwhile, $t_0 \downarrow -\infty$ as r_0 and α decrease to 0.

Finally, the statistical properties for the equilibrium state on the full shift also induces those properties for the unique equilibrium state μ_t when $t \in (t_0, 1)$.

PROPOSITION 4.5.3 (Theorem 3.2 in [**40**]). *When* $t \in (t_0, 1)$*,* μ_t *satisfies Bernoulli property, Central Limit Theorem with respect to a family of observables (including locally Hölder observables) and exponential decay of correlations.*

CHAPTER 5

Asymptotic Central Limit Theorem along closed geodesics for equilibrium measures in non-positive curvature

In this chapter we will focus on the study of our second main result, which is the CLT for the equilibrium measures in the case of geodesic flow on non-positively curved rank-one manifold. To be more precise, we derive an asymptotic version of the Lindeberg type CLT for the measure of maximal extropy (denoted by MME throughout §5) of the geodesic flow in this case and extend the result to the equilibrium states for geometric-t potential and dynamical arrays with different observables. The main idea is enlightened by [**18**] and the main tool we use is specification, which is introduced in Definition 2.1.7. In general, Lindeberg CLT is used in the case where random variables are independent of each other, which is not the case for different terms in the ergodic sum. In order to use it to derive CLT for the equilibrium measure, the basic strategy is to find some 'typical' collection of orbit segments that contains enough information of the target measure, construct a sequence of measures on which the ergodic sums over typical orbit segments are mutually independent and glue these orbit segments together to produce real orbit segments by applying specification. The content of this section is based on the work in [**48**].

Unlike the case of the Katok map, specification in this situation is not global and its application relies heavily on the orbit decomposition construction in §3.2.3. To see how those typical orbit segments are constructed and how specification is put into usage, we will first introduce some important preliminary setup in [**9**]. Throughout the section, we will take M to be a compact connected C^∞ rank-one non-positively curved Riemannian manifold without boundary. We also consider the geodesic flow $\mathcal{F} = \{g_t\}_{t\in\mathbb{R}}$ as an action on T^1M and denote h by its topological entropy.

5.1. Background setups

5.1.1. Preliminaries. We begin with illustrating on the specification property in this case. Recall from (2.3.7) that for any $\eta > 0$, we have $\mathrm{Reg}(\eta) = \{v \in T^1M : \lambda(v) \geq \eta\}$. Denote $\mathscr{C}(\eta)$ by the collection of orbit segments whose starting point and ending point are both in $\mathrm{Reg}(\eta)$. That is,

$$\mathscr{C}(\eta) := \{(v,t) : \lambda(v) \geq \eta,\ \lambda(g_t v) \geq \eta\}.$$

Then, as in Theorem 4.1 in [**9**], we have

THEOREM 5.1.1. *For every $\eta > 0$, $\mathscr{C}(\eta)$ satisfies specification at all scales.*

We will make an appropriate choice on $\mathscr{C}(\eta)$ by fixing some proper $\eta > 0$. To do this, recall from (3.2.2) that $\mathscr{B}(\eta) := \left\{(v,t) : \int_0^t \lambda(g_s(v))ds < \eta t\right\}$. Since λ vanishes on Sing, we immediately see that $\mathrm{Sing} \times [0,\infty) \subset \mathscr{B}(\eta)$ for all $\eta > 0$. Meanwhile, we see from §5 in [**9**] that $h(\mathscr{B}(\eta)) \downarrow h(\mathrm{Sing})$ when $\eta \downarrow 0$. Since $h(\mathrm{Sing}) < h$ (see §8 in [**9**]), we can choose η small enough such that $h(\mathscr{B}(2\eta)) < h' < h$, where $h' := \frac{h + h(\mathrm{Sing})}{2}$. Throughout the rest of this section we will fix such an η.

The target equilibrium measure we study first is the measure of maximal entropy for $(M, \mathcal{F})$, which is unique due to [**9**, Theorem A] (see also [**29**]) and denoted by μ_{KBM} (as it is known as the Knieper-Bowen-Margulis measure).

5.1.2. Counting closed regular geodesics. In this section we will construct the collection of 'typical' orbit segments described as above. Roughly speaking, we choose typical orbit segments to be closed geodesics with enough hyperbolicty along the orbit. We will also give a lower estimate on the cardinality of such closed geodesics. Then this will tell us that there are indeed enough typical orbit segments, with which we can establish a sequence of probability measures that converge to μ_{KBM} weakly. Our goal is to build the asymptotic CLT along this sequence of measures.

We begin with introducing some definitions. For $T > 0$ and $\delta \in (0, T)$, write $\mathrm{Per}_R(T-\delta, T]$ as the set of closed regular geodesics whose length belongs to the interval $(T-\delta, T]$. Meanwhile, for a discrete set E, we denote $\#E$ by its cardinality. Recall from Proposition 6.4 in [**9**], for any $\delta > 0$, there exists $T_\delta > 0$ and $\beta = \beta(\delta)$ such that for all $T > T_\delta$, we have

$$\frac{\beta}{T} e^{Th} \leq \#\mathrm{Per}_R(T-\delta, T] \leq \beta^{-1} e^{Th}. \tag{5.1.1}$$

Moreover, there exists a constant $C > 1$ which is unrelated to δ and satisfies that

$$\frac{e^{-hT_\delta}}{C} \leq \beta \leq Ce^{-hT_\delta}. \tag{5.1.2}$$

To define closed geodesic with enough hyperbolicity along the orbit, for η that is chosen as in §5.1.1, we write

$$\mathrm{Per}_R^{\geq\eta}(T-\delta,T] := \{\gamma \in \mathrm{Per}_R(T-\delta,T] : \int_0^{|\gamma|} \lambda(g_s\gamma(0))ds \geq |\gamma|\eta\}$$

as the collection of elements in $\mathrm{Per}_R(T-\delta,T]$ whose average of λ is at least η. Similarly we can define $\mathrm{Per}_R^{<\eta}(T-\delta,T]$ by replacing $\geq$ from the above definition by $<$. We also fix $\epsilon > 0$ to be small enough so that 4ϵ is an expansive constant for $\mathcal{F}$. Notice that we might shrink the size of ϵ later as it will not affect the choice of other parameters.

We also define $\lambda_{\max} := \max\{\lambda(v) : v \in T^1M\}$ and $\delta' := \frac{\eta}{\lambda_{\max}}$. Now we proceed with the lower estimate of $\#\mathrm{Per}_R^{\geq\eta}(T-\delta,T]$ for properly chosen δ and T.

Lemma 5.1.2. *For any $\delta \in (0,\delta')$, there exists $T_0 = T_0(\delta,\eta) > 0$ such that for all $T > T_0$,*

$$\#\mathrm{Per}_R^{\geq\eta}(T-\delta,T] \geq \frac{\beta}{2T}e^{Th}. \tag{5.1.3}$$

Proof. Since η is chosen so that $h(\mathscr{B}(2\eta)) < h'$, there exists $T_0' = T_0'(\delta) > 0$ such that whenever $T > T_0'$, we have the cardinality of the maximal $(T,4\epsilon)$-separated set for $\mathscr{B}(2\eta)$ to be bounded strictly from above by $e^{Th'}$. Now for any fixed $\delta \in (0,\delta')$, define $T_0(\delta,\eta) := \max\{T_0'(\eta), T_\delta, 1\}$. For $T > T_0$ and any closed regular geodesic $\gamma \in \mathrm{Per}_R^{<\eta}(T-\delta,T]$, we choose a tangent vector $v_\gamma \in T^1M$ to γ at some point. By the choice of δ and δ', we have

$$\int_0^T \lambda(g_s(v_\gamma))ds \leq |\gamma|\eta + \delta\lambda_{\max} < |\gamma|\eta + \delta'\lambda_{\max} < T\eta + \eta < 2T\eta, \tag{5.1.4}$$

which implies that $(v_\gamma, T) \in \mathcal{B}(2\eta)$. Moreover, we know from the choice of 4ϵ that all elements in $\mathrm{Per}_R(T-\delta,T]$ are $(T,4\epsilon)$-separated. In particular, elements in $\mathrm{Per}_R^{<\eta}(T-\delta,T]$ are $(T,4\epsilon)$-separated in $\mathscr{B}(2\eta)$ by (5.1.4). As a result, we know

$$\#\mathrm{Per}_R^{<\eta}(T-\delta,T] < e^{Th'} < \frac{\beta}{2T}e^{Th}. \tag{5.1.5}$$

Finally, combining (5.1.1) and (5.1.4), we have

$$\#\mathrm{Per}_R^{\geq\eta}(T-\delta,T] = \#\mathrm{Per}_R(T-\delta,T] - \#\mathrm{Per}_R^{<\eta}(T-\delta,T] > \frac{\beta}{2T}e^{Th},$$

which concludes the proof of the lemma. □

Throughout the rest of the chapter we will always assume that δ and T satisfy conditions in Lemma 5.1.2. Now we are ready to construct our typical orbit segments. By definition of $\mathrm{Per}_R^{\geq\eta}(T-\delta,T]$, if $\gamma \in \mathrm{Per}_R^{\geq\eta}(T-\delta,T]$, there exists some $t \in [0,T)$ such that $v = \gamma(t) \in \mathrm{Reg}(\eta)$. Since v is periodic, we know that $(v,|\gamma|) \in \mathscr{C}(\eta)$. For such γ, choose $v = v(\gamma)$ as above. It is straightforward to see that we could have many candidates for such v, while we just take one arbitrarily. Write

$$E_\delta(T) = \{v(\gamma) : \gamma \in \mathrm{Per}_R^{\geq\eta}(T-\delta,T]\}.$$

By Lemma 5.1.2, we immediately see that

$$|E_\delta(T)| := \#E_\delta(T) \geq \frac{\beta}{2T}e^{Th}. \tag{5.1.6}$$

Throughout our construction, we will consider $E_\delta(T)$ as the typical collection of orbit segments with appropriate choice on δ and T.

5.1.3. Growth of Variations along $\mathscr{C}(\eta)$. As we are applying specification in our construction, the resulting orbit segment is in a small neighborhood of the original ones. Therefore, we need to control the variation of the Birkhoff sum brought by this shadowing process. To do this, we first define the variation term as follows. Let $\mathscr{C}(\eta)$ be defined as above. For any $h \in C(T^1M)$ and $\delta > 0, T > 0$, write

$$\omega(h,T,\delta,\mathcal{C}) := \sup_{(u,T)\in\mathcal{C}, v\in B_T(u,\delta)} |H(u,T) - H(v,T)|.$$

We have the following result regarding the growth rate of ω in T, which is similar to Lemma 4.2.6.

Lemma 5.1.3. *If δ_0 is sufficiently small, then for any $h \in C(T^1M)$, we have*

$$\lim_{T\to\infty} \frac{\omega(h,T,\delta_0,\mathcal{C}(3\eta/4))}{T} = 0. \tag{5.1.7}$$

PROOF. We choose δ_0 by first choosing $\delta_0' > 0$ such that

(1) $\mathrm{Reg}(\frac{3\eta}{4})$ has local product structure at scale $4\delta_0'$ with constant $\kappa = \kappa(\frac{3\eta}{4}, 4\delta_0') > 1$.

(2) For any $u, v \in T^1M$ with $d(u,v) < \delta_0'$, we have $|\lambda(u) - \lambda(v)| < \frac{\eta}{4}$.

Define $\delta_0 := \delta_0'/\kappa$. Now given $(u,T) \in (3\eta/4)$ and $v \in B_T(u, \delta_0)$, there exists $u_0 \in T^1M$ such that $u_0 \in W^s_{\kappa(\delta_0)}(u) \cap W^{cu}_{\kappa(\delta_0)}(v)$. By definition of distance along W^{cu}, there is some $s \in (-\kappa\delta_0, \kappa\delta_0)$ such that $g_s(u_0) \in W^u_{\kappa(\delta_0)}(v)$. Then we have

$$d(g_T(v), g_{T+s}(u_0)) \leq d(g_T(u), g_T(v)) + d(g_T(u), g_T(u_0)) + d(g_T(u_0), g_{T+s}(u_0)) < 3\kappa\delta_0,$$

which in turns shows that $d^u(g_{T+s}(u_0), g_T(v)) < 3\kappa^2\delta_0$ and $d^{cu}(g_T(u_0), g_T(v)) < 4\kappa^2\delta_0 = 4\kappa\delta_0'$. Therefore, we have $g_T(u_0) \in W^s_{4\kappa(\delta_0')}(g_T(u)) \cap W^{cu}_{4\kappa(\delta_0')}(g_T(v))$, which is unique by our choice on δ_0' and the assumption of $g_T(u) \in \mathrm{Reg}(\frac{3\eta}{4})$. Meanwhile, since $d(g_T(u), g_T(v)) < \delta_0$, we know $g_T(u_0) \in W^s_{\kappa(\delta_0)}(g_T(u)) \cap W^{cu}_{\kappa(\delta_0)}(g_T(v))$. In particular, $g_{T+s}(u_0) \in W^u_{\kappa(\delta_0)}(g_T(v))$. This shows that $g_{t+s}(u_0) \in W^u_{\kappa(\delta_0)}(g_t(v))$. As a result, we know

$$g_t(u_0) \in W^s_{\kappa(\delta_0)}(g_t(u)) \cap W^{cu}_{\kappa(\delta_0)}(g_t(v)) \text{ for all } t \in [0,T]. \tag{5.1.8}$$

Now we can bound the total variation of h between (u,T) and (v,T) from above by which along stable, unstable and central directions as in the proof of Lemma 4.2.6, which is done by $|H(u,T) - H(v,T)| \leq |H(u,T) - H(u_0,T)| + |H(u_0,T) - H(g_s u_0, T)| + |H(g_s u_0, T) - H(v,T)|$. From (2) in the assumption of δ_0', we know $B(\mathrm{Reg}(\frac{3\eta}{4}), \kappa\delta_0) \subset \mathrm{Reg}(\frac{\eta}{2})$ and $B(\mathrm{Reg}(\frac{\eta}{2}), \kappa\delta_0) \subset \mathrm{Reg}(\frac{\eta}{4})$. Therefore, $(v,T) \in \mathscr{C}(\frac{\eta}{2})$. As in Lemma 4.2.7, in order to show (5.1.7), it suffices to show that

$$\lim_{T\to\infty} \frac{\omega_s(h, T; \kappa\delta_0, 3\eta/4)}{T} = 0, \tag{5.1.9}$$

and

$$\lim_{T\to\infty} \frac{\omega_u(h, T; \kappa\delta_0, \eta/2)}{T} = 0, \tag{5.1.10}$$

where

$$\omega_s(h, T; \kappa\delta_0, 3\eta/4) := \sup_{g_T(u)\in \mathrm{Reg}(3\eta/4), v \in W^s_{\kappa\delta_0}(u)} |H(u,T) - H(v,T)|,$$

and

$$\omega_u(h,T;\kappa\delta_0,\eta/2) := \sup_{u\in \mathrm{Reg}(\eta/2), v\in g_{-T}(W^u_{\kappa\delta_0}(g_T(u)))} |(H(u,T)-H(v,T)|.$$

The proofs for (5.1.9) and (5.1.10) are similar to the one of Lemma 4.2.7 by applying [**9**, Lemma 3.10], thus omitted here. We conclude the proof of Lemma 5.1.3. □

Observe that from the definition of ω, (5.1.7) will continue to hold when δ_0 gets smaller. Meanwhile, recall that ϵ can be made arbitrarily small. We will fix ϵ small such that it is a specification scale of $\mathscr{C}(\eta)$ satisfying $0<\epsilon<\delta_0$, then apply the result of Lemma 5.1.3 at scale ϵ.

5.1.4. Construction of measures along closed geodesics. In this section we will approximate μ_{KBM} weakly by constructing a sequence of measures using typical collection of orbit segments, for which we show that asymptotic CLT holds in the next section. Recall that the typical collection of orbit segments we use is $E_\delta(T)$, so we need to make proper choice of δ, T and introduce new parameters if necessary. To do this, we start with constructing the following 4-tuples $((T_l,k_l,\delta_l,C_l))_{l\in\mathbb{N}}$.

HYPOTHESIS 5.1.4. We choose sequences $T_l\in(0,\infty)$, $k_l\in\mathbb{N}$, $\delta_l\in(0,\delta_0)$, and $C_l\in\mathbb{N}$ which satisfy the following relationships (also in the following order):

1) $k_l\uparrow\infty$.
2) $k_l\delta_l^2\downarrow 0$.
3) $T_l>\max\{T_0(\delta_l,\eta),1\}$ for all $l\in\mathbb{N}$, $T_l\uparrow\infty$, $\frac{T_l}{T_0(\delta_l,\eta)}\uparrow\infty$.
4) $\frac{\sqrt{k_l}T_l}{C_l}\downarrow 0$.

where T_0 and δ_0 are from Lemma 5.1.2 and Lemma 5.1.3. For each $l\in\mathbb{N}$, we write

$$E_l := E_{\delta_l}(T_l).$$

Consider the Cartesian product of E_l of order k_l, which is $E_l^{k_l}$. Since $E_l\subset\mathscr{C}(\eta)$, we know that we can apply specification of elements in E_l at scale ϵ. For each l, define a map $\pi_l : E_l^{k_l}\to T^1M$ as follows. Let $\underline{x}=(x_1,x_2,\dots,x_{k_l})\in E_l^{k_l}$. By specification property mentioned above, there is a point $z=\pi_l(\underline{x})$ such that

$$d_{C_lt_i}(g_{(i-1)(C_lT_l+M)}z,x_i)<\epsilon$$

for all $1 \le i \le k_l$, where $M = M(\eta, \epsilon)$ is the transition time in specification for $\mathscr{C}(\eta)$ at scale ϵ. Roughly speaking, z first tracks the closed geodesic (x_1, t_1) for C_l rounds, then tracks (x_2, t_2) for another C_l rounds, etc. This process will end after it tracks (x_{k_l}, t_{k_l}) for C_l times. The transition time at each step is different due to the inconsistency in the length of elements in E_l and dependent on the choice of $\underline{x}$. To be more precise, it is chosen so that the time spent in each loop is uniform and independent of the choice on $\underline{x}$. It is not hard to observe that such transition time is between M and $M + C_l\delta_l$. For each l, the image of π_l is written as

$$P_l = \pi_l(E_l^{k_l}).$$

As E_l is $(T_l, 4\epsilon)$-separated and specification we apply to $\mathscr{C}(\eta)$ is at scale ϵ, we know P_l is $(k_lC_lT_l + (k_l - 1)M, 2\epsilon)$-separated. In particular, π_l is injective and $\#P_l = \#E_l^{k_l}$.

Now we are able to define the desired sequence of measures based on P_l. We first define a measure by uniformly distributing mass over E_l as follows

$$m_l = \frac{1}{\#E_l} \sum_{v \in E_l} \delta_v.$$

We also define μ_l as the self-product of m_l on $E_l^{k_l}$, which is written as

$$\mu_l := \frac{1}{\#E_l^{k_l}} \sum_{\underline{x} \in E_l^{k_l}} \delta_{\underline{x}}.$$

Define $L(v, t)$ as the natural distribution along the orbit segment (v, t). That is to say, for any continuous function ϕ, we have $\int \phi dL(v, t) = \int_0^t \phi(g_s v) ds$. Then we define ν_l on T^1M as follows

$$\nu_l = \frac{1}{\#P_l} \sum_{y \in P_l} \frac{1}{T_l} L(y, T_l) = \frac{1}{\#E_l^{k_l}} \sum_{\underline{x} \in E_l^{k_l}} \frac{1}{T_l} L(\pi_l(\underline{x}), T_l). \tag{5.1.11}$$

We will show that $\{\nu_l\}_{l \in \mathbb{N}}$ is the desired sequence of measures. In particular, we need to show that ν_l converges to μ_{KBM} weakly and satisfies asymptotic CLT. We start with the verification of the first statement as follows.

LEMMA 5.1.5. *Given a sequence of 4-tuples $((T_l, k_l, \delta_l, C_l))_{l \in \mathbb{N}}$ satisfying Hypothesis 5.1.4, we have the corresponding ν_l to converge to μ_{KBM}.*

PROOF. We begin with constructing another sequence of probability measures on T^1M named $\{\mu_l^*\}_{l\in\mathbb{N}}$, which is defined as

$$\mu_l^* = \frac{1}{\#E_l}\sum_{v\in E_l}\frac{1}{T_l}L(v,T_l).$$

We first show that μ_l^* converges to μ_{KBM} weakly. As $\delta_l \downarrow 0$ and $T_l \uparrow \infty$, it is not hard to see that the weak*-limit of μ_l^* is $\mathcal{F}$-invariant. By the choice of T_l, δ_l and Lemma 5.1.2, we know

$$\liminf_{l\to\infty}\frac{1}{T_l}\log\#E_l \geq \lim_{l\to\infty}\frac{1}{T_l}\log(\frac{\beta(\delta_l)}{2T_l}e^{T_lh}) \geq \lim_{l\to\infty}\frac{1}{T_l}\log(\frac{e^{-T_{\delta_l}h}}{2CT_l}e^{T_lh}),$$

where C is the constant from (5.1.2). Meanwhile, by the third assumption in Hypothesis 5.1.4, for any $\epsilon' > 0$, we can choose l' large enough such that $e^{-T_{\delta_l}h} > e^{-\epsilon' T_l h}$ for all $l > l'$. As a result, we have

$$\lim_{l\to\infty}\frac{1}{T_l}\log(\frac{e^{-T_{\delta_l}h}}{2CT_l}e^{T_lh}) \geq \lim_{l\to\infty}\frac{1}{T_l}\log(\frac{e^{(1-\epsilon')T_lh}}{2CT_l}) = (1-\epsilon')h.$$

Since ϵ' is arbitrary, we have

$$\lim_{l\to\infty}\frac{1}{T_l}\log\#E_l = h.$$

As we've seen multiple times in the discrete case, it then follows from the second half of the proof of the variational principle (see [**49**]) that μ_l^* converges weakly to μ_{KBM}. Therefore, in order to prove the statement of the lemma, it suffices to show that for any $f \in C(T^1M)$, we have

$$\lim_{t\to\infty}\int f d\mu_l^* = \lim_{t\to\infty}\int f d\nu_l.$$

To show this, notice that $\int f d\nu_l - \int f d\mu_l^* = \frac{1}{\#E_l}\sum_{x_1\in E_l}V_l(x_1)$, where $V_l(x_1)$ is the variation term defined as

$$V_l(x_1) := \frac{1}{T_l(\#E_l)^{k_l-1}}\sum_{x_2,\ldots,x_{k_l}}(F(\pi_l(x_1,x_2,\ldots,x_{k_l}),T_l) - F(x_1,T_l)).$$

By the choice of δ_0 and δ_l, it is not hard to see that

$$|V_l(x_1)| \leq \frac{\omega(f,T_l,\epsilon,\mathscr{C}(3\eta/4))}{T_l}$$

for every l and $x_1 \in E_l$. As we have assumed that $\epsilon < \delta_0$, by applying Lemma 5.1.3, we have

$$\lim_{l\to\infty} \left| \int f d\mu_l^* - \int f d\nu_l \right| \leq \lim_{l\to\infty} \frac{\omega(f, T_l, \epsilon, \mathcal{C}(3\eta/4))}{T_l} = 0,$$

which concludes the proof of the lemma. □

5.2. Main Theorem

In this section we will state and prove the main theorem of this chapter, which is the asymptotic Lindeberg type CLT of μ_{KBM} and also a detailed version of Theorem C. Throughout the section we will fix a sequence of 4-tuples $((T_l, k_l, \delta_l, C_l))_{l\in\mathbb{N}}$ satisfying Hypothesis 5.1.4. Our main theorem holds for any such sequence of 4-tuples.

5.2.1. Preliminaries and Statement of the Main Theorem.

We begin with a few definitions. Recall that for a continuous function $f \in C(T^1M)$ and $v \in T^1M$, we write $F(v,t) = \int_0^t f(g_s v) ds$. For each $l \in \mathbb{N}$, we will consider the following variance term in the presentation of the CLT statement:

$$\sigma_l^2 := \sigma_{m_l}^2(F(\cdot, T_l)) = \frac{1}{\#E_l} \sum_{x\in E_l} \left(F(x, T_l) - \frac{1}{\#E_l} \sum_{x\in E_l} F(x, T_l) \right)^2. \tag{5.2.1}$$

Define

$$Q_l := \left\lfloor \frac{(T_l - \delta_l)C_l}{T_l} \right\rfloor - 1,$$

which is the 'actual times of rotations' we make on each loop. Roughly speaking, due to the inconsistency of the length of orbit segments in E_l, the shadowing orbit segment resulted from π_l will leave the loop at different time. Meanwhile, every such shadowing orbit segment spent at least $Q_l T_l$ time in each loop. We will show the information we gain along this $Q_l T_l$ period of time is enough to study our CLT. For fixed l and each $p \in [1, k_l]$, write $t_p = (p-1)(C_l T_l + M)$ and define

$$F_p^l := F(g_{t_p} v, Q_l T_l).$$

We also have the following sum of variance, which will be shown later to play in a roll of the actual total variance in CLT of ν_l:

$$s_l^2 = \sum_p \sigma_{\nu_l}^2(F_p^l).$$

Recall $L_\nu(h,c)$ is the Lindeberg function from Definition 2.5.1. Our main theorem is stated as follows.

THEOREM 5.2.1. *Let $(\nu_l)_{l\in\mathbb{N}}$ be as in (5.1.11). Given a Hölder continuous observable $f \in C(T^1M)$, suppose σ_l satisfies*

$$\liminf_{l\to\infty} \sigma_l^2 > 0, \tag{5.2.2}$$

then the Lindeberg condition, which says for all $\gamma > 0$, we have

$$\lim_{l\to\infty} \frac{\sum_{1\le p\le k_l,} L_{\nu_l}(F_p^l, \gamma s_l)}{s_l^2} = 0 \tag{5.2.3}$$

is equivalent to the asymptotic CLT of $(\nu_l)_{l\in\mathbb{N}}$, which says for all $a \in \mathbb{R}$, we have

$$\lim_{l\to\infty} \nu_l\left(\left\{v : \frac{F(v, k_l(C_lT_l + M)) - \int F(\cdot, k_l(C_lT_l + M))d\nu_l}{s_l} \le a\right\}\right) = N(a), \tag{5.2.4}$$

where N above denotes the CDF of the standard normal distribution $\mathcal{N}(0,1)$.

The proof of Theorem 5.2.1 proceeds by verifying the equivalence of conditions between the ones from above and those in the following theorem.

THEOREM 5.2.2. *Let f be as above and recall $\{m_l\}_{l\in\mathbb{N}}$, $\{\mu_l\}_{l\in\mathbb{N}}$ are defined in §5.1.4. We have*

$$\lim_{l\to\infty} \frac{L_{m_l}(F(\cdot, T_l), \gamma\sqrt{k_l}\sigma_l)}{\sigma_l^2} = 0 \tag{5.2.5}$$

to hold for all $\gamma > 0$, if and only if

$$\lim_{l\to\infty} \mu_l\left(\left\{(v_1, \dots, v_{k_l}) : \frac{\sum_{i=1}^{k_l} F(v_i, T_l) - k_l \int F(\cdot, T_l)dm_l}{\sqrt{k_l\sigma_l^2}} \le a\right\}\right) = N(a). \tag{5.2.6}$$

We first sketch the proof of Theorem 5.2.2, which is in fact just a simple application of the classic Lindeberg-Feller CLT for independent random variables. For each l, define a sequence of functions

$\{F_{l,i}\}_{i=1}^{k_l}$ as follows

$$F_{l,i}(x_1,\ldots,x_{k_l}) = F(x_i,T_l).$$

It is straightforward from the above definition that elements in $\{F_{l,i}\}_{i=1}^{k_l}$ are mutually independent for different i as $F_{l,i}$ is only dependent on the i-th component. Write

$$\hat{s}_l^2 = \sum_{i=1}^{k_l} \sigma_{\mu_l}^2(F_{l,i}).$$

Then it follows from Theorem 2.5.2 that the following Lindeberg condition

$$\lim_{l\to\infty} \frac{\sum_{i=1}^{k_l} L_{\mu_l}(F_{l,i},\gamma\hat{s}_l)}{\hat{s}_l^2} = 0 \quad \text{for any } \gamma > 0 \tag{5.2.7}$$

holds if and only if the variance of $\{F_{l,i}\}_{i=1}^{k_l}$ is asymptotically trivial

$$\lim_{l\to\infty} \max_{1\le i\le k_l} \frac{\sigma_{\mu_l}^2(F_{l,i})}{\hat{s}_l^2} = 0 \tag{5.2.8}$$

and CLT holds for $\{\mu_l\}_{l\in\mathbb{N}}$, which means

$$\lim_{l\to\infty} \mu_l\left(\left\{(x_1,\ldots,x_{k_l}) : \frac{\sum_{i=1}^{k_l} F_{l,i} - \int(\sum_{i=1}^{k_l} F_{l,i})d\mu_l}{\hat{s}_l} \le a\right\}\right) = N(a) \tag{5.2.9}$$

holds for all $a \in \mathbb{N}$. Nevertheless, condition (5.2.8) is trivially satisfied as $k_l \uparrow \infty$. This shows that condition (5.2.7) is equivalent to condition (5.2.9). Meanwhile, from the fact of $F_{l,i}$ being i.i.d for each l and the definition of μ_l, it is not hard to see that $\hat{s}_l^2 = k_l\sigma_l^2$ and $L_{\mu_l}(F_{l,i},\gamma\hat{s}_l) = L_{m_l}(F_{l,i},\gamma\hat{s}_l)$. This concludes the proof of Theorem 5.2.2.

We sketch the structure of the rest of §5 as follows. We first give a basic estimate in terms of the total variation term, which will be used throughout the entire proof. Then we verify the equivalence of corresponding conditions in Theorem 5.2.1 and Theorem 5.2.2, thus conclude the result of Theorem 5.2.1. After that, we show that the Lindeberg condition (5.2.3) can be derived from condition (1.3.1) stated in Theorem D. Finally, we extend our result to the cases of dynamical arrays and equilibrium states.

5.2.2. Basic estimates. Suppose f is a Hölder continuous function on T^1M satisfying $|f(x) - f(y)| \le L_0 d(x,y)^\alpha$, where $L_0 > 0$ and $\alpha \in (0,1)$. Then we have

LEMMA 5.2.3. *For any $l \in \mathbb{N}$, $\underline{x} \in E_l^{k_l}$, $t \in [0, T_l]$ and $p \in \{1, \ldots, k_l\}$, there exists some constant $K = K(f)$ such that*

$$|F_p^l(g_t(\pi_l(\underline{x}))) - Q_l F(x_p, T_l)| \leq 2KT_l + (2\kappa\epsilon + 2\delta_l Q_l)\|f\|.$$

PROOF. Fix $l \in \mathbb{N}$. Choose $\underline{x} = (x_1, \ldots, x_{k_l}) \in E_l^{k_l}$ and let $z = \pi_l(\underline{x})$. We also fix any $p \in [1, k_l]$ and $t \in [0, T_l]$. By definition of π_l, we immediately have

$$d_{C_l t(x_p)}(g_{(p-1)(C_l T_l + M)} z, x_p) < \epsilon,$$

where $t(x_p) \in [T_l - \delta_l, T_l]$ is the period of the closed geodesic $(x_p, t(x_p))$. In particular, from the choice on Q_l we know $C_l t(x_p) > (Q_l + 1)T_l$. As a result, we have

$$d_{(Q_l+1)T_l}(g_{(p-1)(C_l T_l + M)} z, x_p) < \epsilon.$$

It then follows from the proof of Lemma 5.1.3 that there exists some $u_p = u_p(\underline{x})$ in T^1M such that

$$g_s(u_p) \in W^s_{\kappa\epsilon}(g_{s+t_p} z \cap W^{cu}_{\kappa\epsilon}(g_s(x_p))$$

for all $s \in [0, (Q_l + 1)T_l]$. Meanwhile, it follows that there exists $s_p = s_p(\underline{x}) \in [-\kappa\epsilon, \kappa\epsilon]$ such that

$$g_{s_p+s}(u_p) \in W^u_{\kappa\epsilon}(g_s(x_p))$$

for all $s \in [0, (Q_l + 1)T_l]$. Once again, we decompose the total variation into variations along different directions. To be more precise, we bound $|F_p^l(g_t z) - Q_l F(x_p, T_l)|$ from above by the sum of the following four terms

(1) $|F_p^l(g_t z) - F(g_t u_p, Q_l T_l)|$,
(2) $|F(g_t u_p, Q_l T_l) - F(g_{t+s_p} u_p, Q_l T_l)|$,
(3) $|F(g_{t+s_p} u_p, Q_l T_l)) - F(g_t x_p, Q_l T_l)|$,
(4) $|F(g_t x_p, Q_l T_l) - Q_l F(x_p, T_l)|$.

We will estimate these terms from above one by one. For (1), for each $q \in [0, Q_l - 1]$, we write

$$F_{p,q}^l(v) := F(g_{t_p} v, [qT_l, (q + 1)T_l]).$$

Observe that $F_p^l = \sum_q F_{p,q}^l$. Meanwhile, we know from the choice on ϵ that for any $u, v \in T^1M$ satisfying $d(u, v) < \delta_0$, we have $|\lambda(u) - \lambda(v)| < \frac{\eta}{4}$. Therefore, By [**9**, Lemma 3.10], we know

$$\begin{aligned}
|F_{p,q}^l(g_t z) - F(g_t u_p, [qT_l, (q+1)T_l])| &\leq \int_{qT_l}^{(q+1)T_l} |f(g_{t+s+t_p} z) - f(g_{t+s} u_p)| ds \\
&\leq L_0 \int_{qT_l}^{(q+1)T_l} d(g_{t+s+t_p} z, g_{t+s} u_p)^\alpha ds \\
&\leq L_0 T_l d(g_{qT_l+t+t_p} z, g_{qT_l+t} u_p)^\alpha \\
&\leq L_0 T_l \kappa \epsilon e^{-\frac{qT_l \eta \alpha}{2}},
\end{aligned}$$

which shows that

$$|F_p^l(g_t z) - F(g_t u_p, Q_l T_l)| \leq \sum_{q=0}^{Q_l - 1} L_0 T_l \kappa \epsilon e^{-\frac{qT_l \eta \alpha}{2}} \leq K T_l,$$

where $K := \frac{L_0 \kappa \epsilon}{1 - e^{-\frac{\alpha \eta}{2}}}$ is a constant that only depends on the choice of f (as ϵ and η are both fixed). This gives the upper estimate on (1).

Similarly, by repeating the above proof along unstable leaves backwards, it is not hard to obtain that KT_l is also an upper bound for (3).

On the other hand, observe that $2||f|||s_p|$ is an upper bound for (2). As $|s_p| < \kappa\epsilon$, we put $2\kappa\epsilon||f||$ as the upper bound. Finally, observe that

$$\left| \int_{qT_l}^{(q+1)T_l} f(g_{t+s} x_p) ds - \int_0^{T_l} f(g_t x_p) dt \right| \leq 2\delta_l ||f||,$$

as $t(x_p) \in [T_l - \delta_l, T_l]$. Therefore, $2\delta_l Q_l ||f||$ is our desired upper bound for (4). By adding all these four upper bounds together, we conclude the statement of Lemma 5.2.3. $\square$

Lemma 5.2.3 immediately implies the following result

LEMMA 5.2.4. *Under the same setting, we have*

$$|\int F_p^l d\nu_l - Q_l \int F(\cdot, T_l) dm_l| \leq 2KT_l + 2\kappa\epsilon \|f\| + 2\delta_l Q_l \|f\|.$$

We introduce some notations for the convenience of formulating the proof of the main theorem. For each l and $x \in E_l$, define

$$D_{m_l}(x) := F(x, T_l) - \int F(\cdot, T_l) dm_l.$$

For $1 \leq p \leq k_l$, $t \in [0, T_l]$ and $\underline{x} \in E_l^{k_l}$, define

$$D_{\nu_l}(\underline{x}, t; p) := F_p^l(g_t(\pi_l(\underline{x}))) - \int F_p^l d\nu_l,$$

and define their difference as

$$\Delta_p^l(\underline{x}, t) := D_{\nu_l}(\underline{x}, t; p) - Q_l D_{m_l}(x_p).$$

Lemma 5.2.3 just states that for each l, any $t \in [0, T_l]$ and $\underline{x} \in E_l^{k_l}$, we have

$$|\Delta_p^l(\underline{x}, t)| \leq 2(2KT_l + 2\kappa\epsilon\|f\| + 2\delta_l Q_l \|f\|). \tag{5.2.10}$$

Meanwhile, the definitions of the three new notations above also imply that

$$D_{\nu_l}(\underline{x}, t; p)^2 = \Delta_p^l(\underline{x}, t)[\Delta_p^l(\underline{x}, t) + 2Q_l D_{m_l}(\underline{x}, p)] + Q_l^2 D_{m_l}(x_p)^2. \tag{5.2.11}$$

We end this section with the following important result on comparison of variance.

LEMMA 5.2.5. *$\lim_{l\to\infty} \frac{\sigma_{\nu_l}^2(F_p^l)}{Q_l^2 \sigma_l^2} = 1$ uniformly in $1 \leq p \leq k_l$. That is to say, for any $\epsilon_0 > 0$, there exists $L \in \mathbb{N}$ such that whenever $l > L$, we have $|\frac{\sigma_{\nu_l}^2(F_p^l)}{Q_l^2 \sigma_l^2} - 1| < \epsilon_0$ for all $1 \leq p \leq k_l$.*

PROOF. First notice that by definition of σ_l^2, we know

$$Q_l^2 \sigma_l^2 = \frac{1}{\# E_l^{k_l}} \sum_{\underline{x} \in E_l^{k_l}} (Q_l D_{m_l}(x_p))^2. \tag{5.2.12}$$

Meanwhile, observe that

$$\sigma_{\nu_l}^2(F_p^l) = \frac{1}{\# E_l^{k_l}} \sum_{\underline{x} \in E_l^{k_l}} \frac{1}{T_l} \int_0^{T_l} (D_{\nu_l}(\underline{x}, t; p))^2 dt. \tag{5.2.13}$$

Therefore, by (5.2.12) and (5.2.13), we know

$$\begin{aligned}
\sigma^2_{\nu_l}(F^l_p) - Q_l^2\sigma_l^2 &= \frac{1}{\#E_l^{k_l}} \sum_{\underline{x}\in E_l^{k_l}} \frac{1}{T_l}\int_0^{T_l} ((D_{\nu_l}(\underline{x},t;p))^2 - Q_l^2 D_{m_l}(x_p)^2)dt \\
&= \frac{1}{\#E_l^{k_l}} \sum_{\underline{x}\in E_l^{k_l}} \frac{1}{T_l}\int_0^{T_l} \Delta_p(\underline{x},t)(\Delta_p(\underline{x},t) + 2Q_l D_{m_l}(x_p))dt \\
&= \int \left(\frac{1}{T_l}\int_0^{T_l} \Delta_p^l(\underline{x},t)^2 dt\right) d\mu_l + \frac{1}{\#E_l^{k_l}} \sum_{\underline{x}\in E_l^{k_l}} \frac{1}{T_l}\int_0^{T_l} 2Q_l\Delta^l_p(\underline{x},t)D_{m_l}(x_p)dt \\
&\le \int \left(\frac{1}{T_l}\int_0^{T_l} \Delta_p^l(\underline{x},t)^2 dt\right) d\mu_l + 2Q_l \sup_{\underline{x},t}\{|\Delta^l_p(\underline{x},t)|\}\int D_{m_l}dm_l \\
&\le \int \left(\frac{1}{T_l}\int_0^{T_l} (\Delta_p^l(\underline{x},t))^2 dt\right) d\mu_l + 2Q_l\sigma_l \sup_{\underline{x},t}\{|\Delta^l_p(\underline{x},t)|\} \\
&\le 4(2KT_l + 2\kappa\epsilon\|f\| + 2\delta_l Q_l\|f\|)^2 + 4Q_l\sigma_l(2KT_l + 2\kappa\epsilon\|f\| + 2\delta_l Q_l\|f\|),
\end{aligned}$$

where the second equality follows from (5.2.11), the inequality on the fifth line follows from Cauchy inequality and the last inequality follows from (5.2.10). Since we have assumed that $\liminf_{l\to\infty}\sigma_l^2 > 0$, by Hypothesis 5.1.4, we have

$$\lim_{l\to\infty}\frac{\sigma^2_{\nu_l}(F^l_p) - Q_l^2\sigma_l^2}{Q_l^2\sigma_l^2} = 0.$$

Moreover, the convergence speed is independent of the choice on $p \in [1, k_l]$ as the upper bound for $\sigma^2_{\nu_l}(F^l_p) - Q_l^2\sigma_l^2$ is only dependent on l. This ends the proof of Lemma 5.2.5. □

The uniform convergence in Lemma 5.2.5 also implies the following result

$$\lim_{l\to\infty}\frac{s_l}{Q_l\sigma_l\sqrt{k_l}} = 1. \tag{5.2.14}$$

Meanwhile, if we write $s_l'^2 := \sigma^2_{\nu_l}(\sum_p F^l_p)$ and $s_l''^2 := \sigma^2_{\nu_l}(F(\cdot, k_l(C_lT_l + M)))$, by following the proof of Lemma 5.2.5, it is not hard to derive the following result

$$\lim_{l\to\infty}\frac{s_l'^2}{s_l^2} = \lim_{l\to\infty}\frac{s_l''^2}{s_l^2} = 1. \tag{5.2.15}$$

Equation (5.2.15) allows us to take the place of s_l in the statement of Theorem 5.2.1 by s_l' or s_l'' freely. Similarly, we can also replace $\int F(\cdot, k_l(C_lT_l + M))d\nu_l$ in (5.2.4) by $k_l(C_lT_l + M)\int f d\nu_l$.

5.2.3. Proof of Main Theorem. Now we start proving Theorem 5.2.1 by verifying the equivalence between (5.2.3) and (5.2.5), as well as (5.2.4) and (5.2.6).

We begin with showing that the gaps created by specification will not affect the asymptotic limit distribution in (5.2.4). As a result, we only need to consider the sum over the essential orbit segments on $(v, k_l(C_lT_l + M))$, which allows us to study the equivalence relation stated above by applying Lemma 5.2.5 directly.

LEMMA 5.2.6. *For each $l \geq 2$ and $v \in T^1M$, we write*

$$A_l(v) := \frac{F(v, k_l(C_lT_l + M)) - \int F(\cdot, k_l(C_lT_l + M))d\nu_l}{s_l},$$

$$B_l(v) := \frac{\sum_{p=1}^{k_l} F_p^l(v) - \int \sum_{p=1}^{k_l} F_p^l d\nu_l}{s_l}.$$

Then A_l converges to B_l in distribution of ν_l when $l \uparrow \infty$. In other words, for any $a > 0$, we have

$$\lim_{l\to\infty} \nu_l(v : |A_l - B_l| > a) = 0.$$

PROOF. Notice that for any $v \in T^1M$, we have

$$\begin{aligned}
&\left|F(v, k_l(C_lT_l + M)) - \sum_p F_p^l\right| \\
&\leq k_l(C_lT_l + M - Q_lT_l)\|f\| \\
&\leq k_l(C_lT_l + M - (\frac{T_l - \delta_l}{T_l}C_l - 2)T_l)\|f\| \\
&= k_l(C_l\delta_l + M + 2T_l)\|f\|,
\end{aligned} \tag{5.2.16}$$

which implies that

$$\left|\int F(\cdot, k_l(C_lT_l + M))d\nu_l - \int \sum_{p=1}^{k_l} F_p^l d\nu_l\right| \leq k_l(C_l\delta_l + M + 2T_l)\|f\|. \tag{5.2.17}$$

By combining (5.2.16) and (5.2.17) together, we have

$$|A_l(v) - B_l(v)| \leq 2k_l(C_l\delta_l + M + 2T_l)\|f\| \tag{5.2.18}$$

for all l and v. Then for any $a > 0$, we have

$$
\begin{aligned}
\lim_{l\to\infty} \nu_l(v : |A_l(v) - B_l(v)| > a) &\le \lim_{l\to\infty} \frac{\int |A_l - B_l| d\nu_l}{a} \\
&\le \lim_{l\to\infty} \frac{2k_l(C_l\delta_l + M + 2T_l)\|f\|}{as_l} \\
&= \lim_{l\to\infty} \frac{2k_l(C_l\delta_l + M + 2T_l)\|f\|}{a\sqrt{k_l Q_l^2 \sigma_l^2}} \\
&= \lim_{l\to\infty} \frac{2k_l\delta_l\|f\|}{a\sqrt{k_l}\sigma_l} + \frac{(2M + 4T_l)\sqrt{k_l}\|f\|}{aQ_l\sigma_l} = 0,
\end{aligned}
\tag{5.2.19}
$$

where the second inequality follows from (5.2.18), the equality on the third line follows from (5.2.14) and the limit being 0 follows from the assumption (5.2.2) and Hypothesis 5.1.4. This shows that A_l converges to B_l in probability regarding ν_l when $l \uparrow \infty$. As classic probability tells us that convergence in probability is stronger than convergence in distribution, we know Lemma 5.2.6 holds. □

Lemma 5.2.6 shows why we can just focus on the distribution of $\sum_{p=1}^{k_l} F_p^l$ under ν_l. Now we proceed the proof of the main result by showing the equivalence of CLT for $\{\mu_l\}_{l\in\mathbb{N}}$ and $\{\nu_l\}_{l\in\mathbb{N}}$. In other words, we want to show the equivalence between (5.2.4) and (5.2.6).

Lemma 5.2.7. (5.2.4) *holds if and only if* (5.2.6) *holds.*

Before moving to the proof, we first introduce a function $Y_p : g_{[0,T_l]}\pi_l(E_l^{k_l}) \to \mathbb{R}$ by

$$Y_p(g_s\pi_l(\underline{x})) = D_{m_l}(x_p),$$

where $D_{m_l}(x_p)$ is as in §5.2.2. We will use Y_p in the next few proofs.

Proof of Lemma 5.2.7. By (5.2.14), Lemma 5.2.6 and continuity of the distribution function of $\mathcal{N}(0,1)$ (which is N), we know (5.2.4) is equivalent to the following

$$\lim_{l\to\infty} \nu_l\left(\left\{g_s(\pi_l(\underline{x})) : \underline{x} \in E_l^{k_l}, s \in [0, T_l], \frac{\sum_{p=1}^{k_l}(F_p^l - \int F_p^l d\nu_l)}{\sqrt{k_l}Q_l\sigma_l} \le a\right\}\right) = N(a). \tag{5.2.20}$$

Meanwhile, by (5.2.10), for any $\underline{x} \in E_l^{k_l}$ and $s \in [0, T_l]$ we have

$$\left| \left(\sum_{p=1}^{k_l} (F_p^l - \int F_p^l d\nu_l) - Q_l \sum_{p=1}^{k_l} Y_p \right) (g_s(\pi_l(\underline{x}))) \right| \leq 2k_l(2KT_l + 2\kappa\epsilon \|f\| + 2\delta_l Q_l \|f\|).$$

Observe from Hypothesis 5.1.4 and (5.2.2) that

$$\lim_{l\to\infty} \frac{2k_l(2KT_l + 2\kappa\epsilon\|f\| + 2\delta_l Q_l \|f\|)}{\sqrt{k_l} Q_l \sigma_l} = 0. \tag{5.2.21}$$

Therefore, for any $b > 0$, we have

$$\lim_{l\to\infty} \nu_l \left(\left\{ g_s \pi_l(x) : \frac{|\sum_{p=1}^{k_l} (F_p^l - \int F_p^l d\nu_l) - Q_l \sum_{p=1}^{k_l} Y_p|}{\sqrt{k_l} Q_l \sigma_l} > b \right\} \right) = 0,$$

which shows that (5.2.20) holds if and only if

$$\lim_{l\to\infty} \nu_l \left(\left\{ g_s(\pi_l(\underline{x})) : \underline{x} \in E_l^{k_l}, s \in [0, T_l], \frac{Q_l \sum_{p=1}^{k_l} Y_p}{\sqrt{k_l} Q_l \sigma_l} \leq a \right\} \right) = N(a) \tag{5.2.22}$$

holds. Since Y_p is invariant in s, it is easy to see that (5.2.22) is equivalent to (5.2.6). This concludes the proof of Lemma (5.2.7). □

Finally we need to prove the equivalence between Lindeberg conditions for $\{\mu_l\}_{l\in\mathbb{N}}$ and $\{\nu_l\}_{l\in\mathbb{N}}$.

LEMMA 5.2.8. *The Lindeberg conditions* (5.2.3) *and* (5.2.5) *are equivalent.*

PROOF. For any $c > 0$, write $Z_l(c) := \{x : |F_p^l - \int F_p^l d\nu_l| > c\}$. Observe that for any $\gamma > 0$, $L_{\nu_l}(F_p^l, \gamma s_l)$ is bounded above by the sum of

$$\frac{1}{E_l^{k_l}} \sum_{\underline{x} \in E_l^{k_l}} \frac{1}{T_l} \int_0^{T_l} \Delta_p^l(\underline{x}, t)[\Delta_p^l(\underline{x}, t) + 2Q_l D_{m_l}(\underline{x}, p)]dt,$$

and

$$\frac{1}{E_l^{k_l}} \sum_{\underline{x} \in E_l^{k_l}} \frac{1}{T_l} \int_0^{T_l} Q_l^2 D_{m_l}(x_p)^2 \mathbb{1}_{Z_l(\gamma s_l)}(g_t \pi_l(\underline{x}))dt,$$

which is just

$$\int (Q_l Y_p)^2 \mathbb{1}_{Z_l(\gamma s_l)} d\nu_l.$$

As shown in the proof of Lemma 5.2.5, the first term from above is equal to $\sigma^2_{\nu_l}(F^l_p) - Q^2_l\sigma^2_l$. Moreover, by (5.2.14), we know $\lim_{l\to\infty} \frac{\sum_{p=1}^{k_l}(\sigma^2_{\nu_l}(F^l_p)-Q^2_l\sigma^2_l)}{s^2_l} = \lim_{l\to\infty}\frac{s^2_l - k_lQ^2_l\sigma^2_l}{s^2_l} = 0$. Therefore, we have

$$\lim_{l\to\infty}\frac{\sum_{p=1}^{k_l} L_{\nu_l}(F^l_p,\gamma s_l)}{s^2_l} \leq \lim_{l\to\infty}\frac{\sum_{p=1}^{k_l}\int (Q_lY_p)^2\mathbb{1}_{Z_l(\gamma s_l)}d\nu_l}{s^2_l}.$$

Since ν_l is supported on $S_l := \{g_s(\pi_l(\underline{x})) : \underline{x} \in E^{k_l}_l, s \in [0, T_l]\}$, we might assume that $Z_l(\gamma s_l)$ is a subset of S_l. Recall that $|F^l_p g_t(\pi_l(\underline{x})) - \int F^l_p d\nu_l| = |D_{\nu_l}(\underline{x}, t)| \leq |\Delta^l_p(\underline{x}, t)| + Q_l|Y_p(g_t(\pi_l(\underline{x}))|$. Therefore, we have

$$Z_l(\gamma s_l) \subset \{g_t(\pi_l(\underline{x})) : |Y_p(g_t(\pi_l(\underline{x}))| \geq Q_l^{-1}(\gamma s_l - |\Delta^l_p(\underline{x}, t)|)\}.$$

Again by (5.2.10), we know $|\Delta^l_p(\underline{x}, t)| \leq \frac{\gamma s_l}{2}$ for all $\underline{x} \in E^{k_l}_l$ and $t \in [0, T_l]$ whenever l is sufficiently large. Therefore, we also have

$$Z_l(\gamma s_l) \subset \{g_t(\pi_l(\underline{x})) : |Y_p(g_t(\pi_l(\underline{x}))| \geq Q_l^{-1}(\gamma s_l/2)\}$$

for such l. By (5.2.14), we also have the following to hold for large enough l

$$Z_l(\gamma s_l) \subset \{g_t(\pi_l(\underline{x})) : |Y_p(g_t(\pi_l(\underline{x}))| \geq \gamma\sqrt{k_l}\sigma_l/4\}.$$

Therefore, for all l sufficiently large, we have

$$\begin{aligned}\int (Q_lY_p)^2\mathbb{1}_{Z_l(\gamma s_l)}d\nu_l &\leq \int (Q_lY_p)^2\mathbb{1}_{\{g_t(\pi_l(\underline{x})):|Y_p(g_t(\pi_l(\underline{x}))|\geq\gamma\sqrt{k_l}\sigma_l/4\}}d\nu_l \\ &= Q^2_l\int D_{m_l}((\underline{x}\to x_p))^2\mathbb{1}_{\{\underline{x}:|D_{m_l}(x_p)|\geq\gamma\sqrt{k_l}\sigma_l/4\}}d\mu_l, \\ &= Q^2_l\int D_{m_l}(x)^2\mathbb{1}_{\{x:|D_{m_l}(x)|\geq\gamma\sqrt{k_l}\sigma_l/4\}}dm_l \\ &= Q^2_lL_{m_l}(F(\cdot, T_l), \gamma\sigma_l\sqrt{k_l}/4).\end{aligned}$$

As a result, by (5.2.14), we have

$$\begin{aligned}\lim_{l\to\infty}\frac{\sum_{p=1}^{k_l}L_{\nu_l}(F_p^l,\gamma s_l)}{s_l^2} &\leq \lim_{l\to\infty}\frac{\sum_{p=1}^{k_l}\int(Q_lY_p)^2\mathbb{1}_{Z_l(\gamma s_l)}d\nu_l}{s_l^2}\\ &\leq \lim_{l\to\infty}\frac{k_lQ_l^2L_{m_l}(F(\cdot,T_l),\gamma\sigma_l\sqrt{k_l}/4)}{s_l^2}\\ &= \lim_{l\to\infty}\frac{L_{m_l}(F(\cdot,T_l),\gamma\sigma_l\sqrt{k_l}/4)}{\sigma_l^2}.\end{aligned}$$

From above, we know (5.2.5) implies (5.2.3). Meanwhile, notice that $L_{\nu_l}(F_p^l,\gamma s_l)$ is bounded below by the sum of

$$-\frac{1}{E_l^{k_l}}\sum_{\underline{x}\in E_l^{k_l}}\frac{1}{T_l}\int_0^{T_l}\Delta_p^l(\underline{x},t)[\Delta_p^l(\underline{x},t)+2Q_lD_{m_l}(\underline{x},p)]dt,$$

and

$$\int(Q_lY_p)^2\mathbb{1}_{Z_l(\gamma s_l)}d\nu.$$

Following the discussion from above, we have

$$\lim_{l\to\infty}\frac{\sum_{p=1}^{k_l}L_{\nu_l}(F_p^l,\gamma s_l)}{s_l^2}\geq\lim_{l\to\infty}\frac{\sum_{p=1}^{k_l}\int(Q_lY_p)^2\mathbb{1}_{Z_l(\gamma s_l)}d\nu_l}{s_l^2}.$$

This time, we use $|F_p^lg_t(\pi_l(\underline{x}))-\int F_p^ld\nu_l|\geq-|\Delta_p^l(\underline{x},t)|+Q_l|Y_p(g_t(\pi_l(\underline{x}))|$, which implies that

$$\{g_t(\pi_l(\underline{x})):|Y_p(g_t(\pi_l(\underline{x}))|\geq Q_l^{-1}(\gamma s_l+|\Delta_p^l(\underline{x},t)|)\}\subset Z_l(\gamma s_l).$$

As $|\Delta_p^l(\underline{x},t)|\leq\gamma s_l$ for all $\underline{x}\in E_l^{k_l}$ and $t\in[0,T_l]$ when l is sufficiently large, we have

$$\{g_t(\pi_l(\underline{x})):|Y_p(g_t(\pi_l(\underline{x}))|\geq 2Q_l^{-1}\gamma s_l\}\subset Z_l(\gamma s_l).$$

By (5.2.14), we have

$$\{g_t(\pi_l(\underline{x})):|Y_p(g_t(\pi_l(\underline{x}))|\geq 4\gamma\sqrt{k_l}\sigma_l\}\subset Z_l(\gamma s_l).$$

As a result, by following the same argument from above, we have

$$\int(Q_lY_p)^2\mathbb{1}_{Z_l(\gamma s_l)}d\nu_l\geq Q_l^2L_{m_l}(F(\cdot,T_l),4\gamma\sigma_l\sqrt{k_l}),$$

which shows that

$$\lim_{l\to\infty} \frac{\sum_{p=1}^{k_l} L_{\nu_l}(F_p^l, \gamma s_l)}{s_l^2} \geq \lim_{l\to\infty} \frac{L_{m_l}(F(\cdot, T_l), 4\gamma\sigma_l\sqrt{k_l})}{\sigma_l^2}.$$

This shows that (5.2.3) implies (5.2.5). Thus these two conditions are equivalent and the proof of Lemma 5.2.8 is concluded. □

Proof of Theorem 5.2.1. Under the assumption of Hypothesis 5.1.4 and (5.2.2), we know (5.2.3) and (5.2.5) are equivalent, as well as (5.2.4) and (5.2.6). Since (5.2.5) and (5.2.6) are equivalent by Theorem 5.2.2, we know (5.2.3) and (5.2.4) are equivalent, which concludes our proof. □

5.3. On the Lindeberg Condition

In this section we comment on the verification of Lindeberg condition (5.2.3) (or (5.2.5)). In classic work, the Lindeberg-type of argument is often applicable when the system has a mixing structure from a probabilistic point of view (see for example condition (I) and (II) in [**23**]). The idea is to divide the ergodic sum (or integral) $S_n f$ (or $S_t f$), where f is the observable, into different segments with gaps in between, in a way such that the length of the gap between adjacent segments increases to infinity, while the essential information is still carried by the segment. Additionally, when some strong integrability condition is added (for example, consider f with finite $2+\delta$ moments and $\sigma^2(S_n f)$ being infinity), the Lindeberg condition will hold automatically.

As for our situation, with $S_t f$ being approximated by the sum over different segments in an intuitively independent way, we pay the price on not having the original μ_{KBM} in the base of the product set. Nevertheless, as each ν_l is weighted over concatenations of k_l segments of (repeated) independent closed geodesics with (approximately) T_l length, one can study the global Lindeberg condition (5.2.3) via the local condition (5.2.5) (which is stated in Lemma 5.2.8). Intuitively if we can make k_l increase very fast and not too slow compared to T_l, eventually the Lindeberg variance contributed by each single segment becomes negligible, thus the local condition (5.2.5) is satisfied.

From now on we strengthen condition (5.2.2) to the following

$$\lim_{l\to\infty} \sigma_l^2 = \infty, \tag{5.3.1}$$

and continue our discussion based on (5.3.1). First notice that under the new assumption on σ_l, we can now weaken the condition $k_l\delta_l^2 \downarrow 0$ in Hypothesis 5.1.4 by

$$\frac{k_l\delta_l^2}{\sigma_l^2} \downarrow 0. \tag{5.3.2}$$

We claim that Theorem 5.2.1 can still be obtained under this new hypothesis. This is because wherever the old condition $k_l\delta_l^2 \downarrow 0$ is applied, we are actually dealing with the limit of $k_l\delta_l^2/\sigma_l^2$; see (5.2.19) in Lemma 5.2.6 and (5.2.21) in Lemma 5.2.7.

With the new assumption (5.3.2), we are able to increase the maximal growth rate of k_l by σ_l^2. Write $k_l = M_l\sigma_l^2/\delta_l^2$. Condition (5.3.2) is now equivalent to $M_l \downarrow 0$. Moreover, since $k_l \uparrow \infty$, M_l can not decrease too fast. Combining these two together we have

$$\frac{\delta_l^2}{\sigma_l^2} \ll M_l \ll 1. \tag{5.3.3}$$

Once the above condition is satisfied, the choice on the sequence $\{M_l\}_{l\in\mathbb{N}}$ is free. Now we look at the Lindeberg condition (5.2.5). For any fixed $\gamma > 0$ and $v \in T^1M$, the indicator function in the integral satisfies

$$\mathbb{1}_{|F(\cdot,T_l)-\int F(\cdot,T_l)dm_l|\geq\gamma\sqrt{k_l}\sigma_l}(v) \leq \mathbb{1}_{2T_l||f||\geq\gamma\sqrt{k_l}\sigma_l}(v) = \mathbb{1}_{K_{\gamma,f}\geq\sqrt{M_l}\sigma_l^2/\delta_l T_l}(v), \tag{5.3.4}$$

where $K_{\gamma,f} := 2||f||\gamma^{-1}$ is a constant. Therefore, if there exists $\{M_l\}_{l\in\mathbb{N}}$ satisfying condition (5.3.3) such that

$$\sqrt{M_l}\sigma_l^2/\delta_l T_l \to \infty, \tag{5.3.5}$$

we can combine (5.3.4) and (5.3.5) and get

$$\begin{aligned}
&\lim_{l\to\infty} \frac{L_{m_l}(F(\cdot,T_l),\gamma\sqrt{k_l}\sigma_l)}{\sigma_l^2} \\
&= \lim_{l\to\infty} \frac{\int(F(\cdot,T_l)-\int F(\cdot,T_l)dm_l)^2\mathbb{1}_{|F(\cdot,T_l)-\int F(\cdot,T_l)dm_l|\geq\gamma\sqrt{k_l}\sigma_l}dm_l}{\sigma_l^2} \\
&\leq \lim_{l\to\infty} \frac{\int(F(\cdot,T_l)-\int F(\cdot,T_l)dm_l)^2\mathbb{1}_{K_{\gamma,f}\geq\sqrt{M_l}\sigma_l^2/\delta_l T_l}dm_l}{\sigma_l^2} \\
&= 0,
\end{aligned} \tag{5.3.6}$$

which verifies Lindeberg condition (5.2.5) (thus (5.2.3)). In particular, this happens when

$$\liminf_{l\to\infty}\frac{\sigma_l^2}{T_l}=\liminf_{l\to\infty}\int(\frac{F(\cdot,T_l)-\int F(\cdot,T_l)dm_l}{\sqrt{T_l}})^2dm_l>0 \tag{5.3.7}$$

as we can just choose $M_l=\delta_l$ (or $k_l=\sigma_l^2/\delta_l$) to make condition (5.3.5) hold. This verifies Theorem D.

Condition (5.3.7) is similar to the classic non-zero variance (which is $\int(\frac{F(\cdot,T_l)-\int F(\cdot,T_l)dP}{\sqrt{T_l}})^2dP$, where P is a probability measure) condition on CLT of dynamics with mixing structure, under which the limit normal distribution is non-degenerate. In those cases, the variance being non-zero (in fact converge to ∞) is equivalent to f not being a coboundary. Here in our cases, we can not draw such a conclusion due to the lack of asymptotic mixing property of measure m_l.

Despite the special case above, one can easily deduce from condition (5.3.3) that there exists M_l such that (5.3.5) holds if and only if

$$\sigma_l^2/\delta_lT_l\to\infty. \tag{5.3.8}$$

To discover when will (5.3.8) hold in general, one first recall from Hypothesis 5.1.4 that the choice on T_l is only determined by δ_l . From condition 3) of the hypothesis we can write $T_l=N_lT_{\delta_l}$ (see the second paragraph in §5.1.2 for the definition of T_{δ_l}) for all l sufficiently large (see the remark after Hypothesis 5.1.4), with $N_l\uparrow\infty$. This leads us to concentrate on how T_{δ_l} is related to δ_l when $\delta_l\downarrow 0$.

To achieve this, we need to extract the geometric meaning of T_{δ_l} from [**9**]. First notice that specification of $\mathcal{C}(3\eta/4)$ and uniform hyperbolicity near the head and tail of elements in $\mathcal{C}(3\eta/4)$ enables us to create different closed geodesics near separated orbit segments following a classic closing lemma argument (see Lemma 4.7 from [**9**]). This is realized by gluing the target orbit segments together with (fixed amount of) reference segments in $\mathcal{C}(3\eta/4)$ so that enough hyperbolicity is reflected along the concatenation. Also notice that the inconsistency (within δ_l) in the length of each closed geodesics from the collection E_l is resulted from the difference along central direction in the local product structure of $\mathcal{C}(3\eta/4)$, which is a constant multiple of specification scale (as η is fixed throughout the paper). Therefore, as suggested by the proof of Lemma 4.7 in [**9**], T_{δ_l} is approximately a constant multiple of $T^0_{\delta_l}$, where $T^0_{\delta_l}$ is the transition time in specification of $\mathcal{C}(3\eta/4)$

at scale δ_l. Furthermore, the proof of Theorem 4.1 and Proposition 4.5 from [**9**] indicates that $T^0_{\delta_l}$ is a constant multiple of $T^u_{\delta_l}$, where $T^u_{\delta_l}$ is defined such that $W^u_{T^u_{\delta_l}}(x)$ is δ_l-dense in T^1M for any $x \in \mathcal{C}(3\eta/4)$ (again we neglect the reliance of T^u on η as η is fixed). Therefore, it suffices to study the relation between $T^u_{\delta_l}$ and δ_l.

Though it is not clear at this point how exactly these two quantities are related to each other in the non-uniformly hyperbolic setting, there are some motivations we can draw from uniformly hyperbolic case. Let us start with a topologically mixing Markov shift (X, σ) with finitely many symbols, equipped with the natural metric (that is, for $\underline{x} = (\cdots, x_{-1}, x_0, x_1, \cdots)$ and $\underline{y} = (\cdots, y_{-1}, y_0, y_1, \cdots)$, $d(\underline{x}, \underline{y}) = 2^{-\inf\{|i|: x_i \neq y_i\}}$. There is no δ_l here as it is discrete, while we can define for each small $\delta > 0$ a $T^u_\delta = T^u_\delta(\sigma)$ such that $W^u_{T^u_\delta}(\underline{x})$ is δ-dense in X for any $\underline{x} \in X$ (here we do not have η term involved as hyperbolicity is uniform). Now take $\delta_n = 1/2^n$, $n \in \mathbb{N}$. By applying the uniform contraction in σ^{-1} (the right shift) along unstable direction at the rate of 2, it is not hard to see that $W^u_{2^n}(\underline{x}) = \{\underline{y} : y_i = x_i \text{ for all } i < -n\}$. Meanwhile, topological mixing implies the existence of $M > 0$ such that for any letters a, b from the alphabet set for X, we have $\sigma^m([a]) \cap [b] \neq \emptyset$ for all $m \geq M$. Combine these two facts from above together, we obtain that $W^u_{2^{n+M}}(\underline{x})$ is δ_n-dense in X for all $\underline{x} \in X$, so $T^u_{\delta_n} = 2^M/\delta_n$. Now we extend this from the shift space to Axiom A flows (M, g_t) on basic sets. By taking a proper collection of differentiable cross-sections (see Definition 2.1 of [**5**]) as symbols in X and transition time as the roof function (say f), we get a conjugacy map π from the suspension flow X_f to the original (M, g_t). Moreover, it can be shown that π is Lipschitz and f is differentiable, in particular bounded uniformly from above and below. Therefore, by applying a classic expansiveness argument (Lemma 1.5 in [**5**]), we derive that for small $\delta > 0$, there exists $T^u_\delta = T^u_\delta(g_t)$ satisfying the similar global denseness property at scale δ and $T^u_\delta = \mathcal{O}(1/\delta)$. As shown in the discussions from the previous paragraph, $T_{\delta_l} = \mathcal{O}(T^u_{\delta_l}) = \mathcal{O}(1/\delta_l)$. Respectively, condition (5.3.7) becomes

$$\sigma_l^2/N_l \to \infty.$$

In general, one shall not expect a non-uniformly hyperbolic system to possess a finite symbolic representation that will contain the global information (i.e. entropy, periodic orbits). Even it does, the corresponding conjugacy map might not be Hölder continuous. In the special case of the Katok map, which is in the C^1-closure of Anosov diffeomorphisms of 2-torus and has a unique MME, it is shown in [**40**] that there exists a countable symbolic representation of the map (inducing scheme) whose symbols are first recurrent orbit segments that originate from and end up in a same prefixed rectangle being far away from the perturbed region. In fact it is shown that the symbolic system is a full shift, so all elements in the corresponding Markov matrix are 1 and mixing time M in the previous paragraph is finite (which is 1). Moreover, since the inducing scheme is induced by the first recurrence map, all the measures with positive entropy are liftable to the shift space (see Theorem 5.1 in [**41**]). As the Katok map is expansive (see Proposition 3.8 from [**50**]), the conjugacy map from the shift space to the Katok map is Lipschitz. Therefore, we can follow what is done in the uniformly hyperbolic case and draw a similar conclusion for the generating flow (in $T^1\mathbb{T}^2$) of the Katok map. It still remains a question on whether non-uniformly hyperbolic systems with such a representation is generic in any topology.

5.4. Extension of Main Results

In this section we extend the result in Theorem 5.2.1 to equilibrium states and dynamical arrays. The idea of the proofs are exactly the same as which of Theorem 5.2.1. We note that these results are not proved in the journal publication [**48**]. We will point out the differences and sketch how the key steps are derived, then omit the others.

5.4.1. Equilibrium States. Consider potential function $\varphi \in C(T^1M)$ which is either Hölder continuous or a geometric-t potential with $t < 1$. In either case, we assume that $P(\mathrm{Sing}, \varphi) < P(\varphi)$. Theorem A in [**9**] then implies that φ has a unique equilibrium state μ_φ for $\mathscr{F}$. We will fix one such φ and study asymptotic CLT of μ_φ.

We choose $\eta' > 0$ such that $P(\mathscr{B}(2\eta)) < \frac{P(\mathrm{Sing},\varphi)+P)}{2} < P$. Due to the existence of non-trivial potential, we will put different weight on different closed geodesics. We make the following definitions. Define $\Lambda^*_{\mathrm{Reg}}(T,\varphi,\delta) := \sum_{\gamma\in \mathrm{Per}_R(T-\delta,T]} e^{\Phi(\gamma)}$ where $\Phi(\gamma) = \int_0^{|\gamma|} \varphi(g_s(v))ds$ with $v := \gamma'(0)$. Similar to the MME case, we write $\Lambda^*_{\mathrm{Reg}}(\geq \eta', T, \varphi, \delta) := \sum_{\gamma\in \mathrm{Per}_R^{\geq\eta}(T-\delta,T]} e^{\Phi(\gamma)}$ and $\Lambda^*_{\mathrm{Reg}}(< \eta', T, \varphi, \delta)$. As in (5.1.1), for any $\delta > 0$, we know from Proposition 6.4 in [**9**] that there exists $\hat{T}_\delta > 0$ and $\hat{\beta} = \hat{\beta}(\delta)$ (which is roughly $e^{-\hat{T}_\delta}$) such that for all $T > \hat{T}_\delta$, we have

$$\frac{\hat{\beta}}{T} e^{TP} \leq \Lambda^*_{\mathrm{Reg}}(T,\varphi,\delta) \leq \hat{\beta}^{-1} e^{TP}. \tag{5.4.1}$$

We keep using the same ϵ and δ' from §5.1.2. Similar to Lemma 5.1.2 we apply (5.4.1) and have the following

LEMMA 5.4.1. *For any $\delta < \delta'$, there exists $\hat{T}_0 = \hat{T}_0(\delta, \eta)$ such that for all $T > \hat{T}_0$, we have*

$$\Lambda^*_{\mathrm{Reg}}(\geq \eta; T, \varphi, \delta) \geq \frac{\hat{\beta}}{2T} e^{TP}. \tag{5.4.2}$$

We still work with the same hypothesis, which is Hypothesis 5.1.4. Therefore, the definitions of E_l and π_l are the same as those in §5.1.4. Due to the nature of equilibrium states, the non-uniform distribution of weight on different orbit segments will end up with different measures from those in §5.1.4. In this case, for each l, we define

$$\hat{m}_l = \frac{1}{F_l} \sum_{v \in E_l} \Phi_l(v) \delta_v,$$

where $\Phi_l(v) := \exp\big(\int_0^{T_l} \varphi(g_s(v))\big)ds$ for $v \in E_l$ and $F_l := \sum_{v \in E_l} \Phi_l(v)$. We also define the self-product of $\hat{m}_l$ on $E_l^{k_l}$ as

$$\hat{\mu}_l = \frac{1}{F_l^{k_l}} \sum_{\underline{x} \in E_l^{k_l}} \prod_{i=1}^{k_l} \Phi_l(x_i) \delta_{\underline{x}}.$$

The induced measure under π_l is

$$\hat{\nu}_l = \frac{1}{F_l^{k_l}} \sum_{\underline{x} \in E_l^{k_l}} \prod_{i=1}^{k_l} \Phi_l(x_i) L(\pi_l(\underline{x}), T_l).$$

Then as in Lemma 5.1.5, by applying Lemma 5.1.3, we have

LEMMA 5.4.2. *Given a sequence of 4-tuples $((T_l, k_l, \delta_l, C_l))_{l \in \mathbb{N}}$ satisfying Hypothesis 5.1.4, we have the corresponding $\hat{\nu}_l$ to converge to μ_φ.*

Write $\hat{\sigma}_l^2 := \sigma^2_{\hat{\nu}_l}(F(\cdot, T_l))$ and $\hat{s}_l^2 = \sum_p \sigma^2_{\hat{\nu}_l}(F_p^l) = \sum_p \sigma^2_{\hat{\nu}_l}(\sum_q F^l_{p,q})$. The main theorem is stated as follows

THEOREM 5.4.3. *Let $(\hat{\nu}_l)_{l \in \mathbb{N}}$ be defined as above. Given a Hölder continuous observable $f \in C(T^1 M)$, if $\hat{\sigma}_l$*

$$\liminf_{l \to \infty} \hat{\sigma}_l^2 > 0. \tag{5.4.3}$$

Then the Lindeberg-type condition, which says that for any any $\gamma > 0$, we have

$$\lim_{l\to\infty} \frac{\sum_{1\le p\le k_l,} L_{\hat{\nu}_l}(F^l_p, \gamma\hat{s}_l)}{\hat{s}^2_l} = 0 \tag{5.4.4}$$

if and only if $\hat{\nu}_l$ satisfies the asymptotic CLT, which says that for any $a \in \mathbb{R}$, we have

$$\lim_{l\to\infty} \hat{\nu}_l(\{v : \frac{F(v, k_l(C_lT_l + M)) - \int F(\cdot, k_l(C_lT_l + M))d\hat{\nu}_l}{\hat{s}_l} \le a\}) = N(a), \tag{5.4.5}$$

where N is the cumulative distribution function of the normal distribution $\mathcal{N}(0,1)$. Conversely, under the hypotheses (5.4.3), (5.4.5) *implies* (5.4.4).

As in the proof of Theorem 5.2.1, the strategy is to verify the equivalence of Lindeberg condition and CLT in Theorem 5.4.3 and those in the following theorem.

Theorem 5.4.4. *The condition*

$$\lim_{l\to\infty} \frac{L_{\hat{m}_l}(F(\cdot, T_l), \gamma\sqrt{k_l}\hat{\sigma}_l)}{\hat{\sigma}^2_l} = 0. \tag{5.4.6}$$

holds if any only if

$$\lim_{l\to\infty} \hat{\mu}_l\left(\left\{(v_1, \ldots, v_{k_l}) : \frac{\sum_{i=1}^{k_l} F(v_i, T_l) - k_l \int F(\cdot, T_l)d\hat{m}_l}{\sqrt{k_l\hat{\sigma}^2_l}} \le a\right\}\right) = N(a). \tag{5.4.7}$$

The verification process relies mainly on the variance comparison result. In this case, we define

$$D_{\hat{m}_l}(x) := F(x, T_l) - \int F(\cdot, T_l)d\hat{m}_l$$

for all $x \in E_l$ and

$$D_{\hat{\nu}_l}(\underline{x}, t; p) := F^l_p(g_t(\pi_l(\underline{x}))) - \int F^l_p d\hat{\nu}_l$$

for all $\underline{x} \in E_l^{k_l}$ and $t \in [0, T_l]$. By writing

$$\hat{\Delta}^l_p(\underline{x}, t) := D_{\hat{\nu}_l}(\underline{x}, t; p) - Q_l D_{\hat{m}_l}(x_p),$$

Similar to (5.2.10), we have the following estimate

$$|\hat{\Delta}^l_p(\underline{x}, t)| \le 2(2KT_l + 2\kappa\epsilon\|f\| + 2\delta_l Q_l\|f\|), \tag{5.4.8}$$

which in turns shows that $\hat{s}_l^2$ satisfies that

$$\lim_{l\to\infty} \frac{\hat{s}_l}{Q_l \hat{\sigma}_l \sqrt{k_l}} = 1. \tag{5.4.9}$$

With the help of (5.4.8) and (5.4.9), we can follow the proof of Lemma 5.2.8 and show that (5.4.4) is equivalent to (5.4.6). Similarly, as in Lemma 5.2.7, we have the equivalence between (5.4.5) and (5.4.7). With assumption (5.4.3) and Hypothesis 5.1.4, we conclude the proof of Theorem 5.4.3.

5.4.2. Dynamical Arrays. One of the main advantages of Lindeberg condition is that we can study the asymptotic distributions with different observables instead of a single one. Meanwhile, even with the assumption of all observables being Hölder continuous, they can have different Hölder exponents and constants, which are not necessarily bounded away from 0 and ∞ respectively. Therefore, we need to adapt the assumptions and calculations accordingly.

Let $(f_l)_{l\in\mathbb{N}}$ be a sequence of Hölder continuous observables satisfying

$$|f_l(x) - f_l(y)| \le L_l d(x,y)^{\alpha_l}$$

with L_l and α_l being the Hölder constant and exponent for each f_l. Since we need to consider the impact of $||f_l||$ brought to the variation term in this case, we need a new assumption on 4-tuples $((T_l, k_l, \delta_l, C_l))_{l\in\mathbb{N}}$.

HYPOTHESIS 5.4.5. We choose sequences $T_l \in (0,\infty)$, $k_l \in \mathbb{N}$, $\delta_l \in (0,\delta_0)$, and $C_l \in \mathbb{N}$ which satisfy the following relationships (also in the following order):

1) $k_l \uparrow \infty$.
2) $k_l \delta_l^2 \max\{||f_l||, 1\} \downarrow 0$.
3) $T_l > \max\{T_0(\delta_l,\eta), 1\}$ for all $l \in \mathbb{N}$, $T_l \uparrow \infty$, $\frac{T_l}{T_0(\delta_l,\eta)} \uparrow \infty$.
4) $\frac{\sqrt{k_l} T_l \max\{|K_l|,1\}}{Q_l} \downarrow 0$ and $\frac{\sqrt{k_l} T_l \max\{||f_l||,1\}}{Q_l} \downarrow 0$.

We still focus on μ_{KBM} and use the same sequence $\{\nu_l\}_{l\in\mathbb{N}}$ defined in §5.1.4 to approach μ_{KBM}. The main theorem is stated as follows

THEOREM 5.4.6. *Under Hypothesis* (5.4.5)*, given a sequence of observables* $\{f_l\}_{l\in\mathbb{N}}$ *satisfying*

$$\liminf_{l\to\infty} \sigma_l^2 > 0, \tag{5.4.10}$$

the Lindeberg condition, which says for all $\gamma > 0$, we have

$$\lim_{l\to\infty} \frac{\sum_{1\le p\le k_l,} L_{\nu_l}(F_p^l, \gamma s_l)}{s_l^2} = 0 \tag{5.4.11}$$

is equivalent to the asymptotic CLT of $\{\nu_l\}_{l\in\mathbb{N}}$, which is

$$\lim_{l\to\infty} \nu_l(\{v : \frac{F_l(v, k_l(C_l T_l + M)) - \int F_l(\cdot, k_l(C_l T_l + M))d\nu_l}{s_l} \le a\}) = N(a). \tag{5.4.12}$$

The main idea is still the same as which in the proof of Theorem 5.2.1. In this case, for the upper estimate on the variance, following the proof of Lemma 5.2.4, we have the following to hold for all $\underline{x} \in E_l^{k_l}$, $p \in [1, k_l]$ and $t \in [0, T_l]$

$$|F_p^l(g_t(\pi_l(\underline{x}))) - Q_l F_l(x_p, T_l)| \le 2K_l T_l + (2\kappa\epsilon + 2\delta_l Q_l)\|f_l\|,$$

where $K_l := L_l\kappa\epsilon/(1 - e^{-\frac{\eta\alpha_l}{2}})$. This immediately shows that

$$|\Delta_p^l(\underline{x}, t)| \le 2(2K_l T_l + 2\kappa\epsilon\|f_l\| + 2\delta_l Q_l\|f_l\|), \tag{5.4.13}$$

where $\Delta_p^l(\underline{x}, t)$ is defined in §5.2.2. Following the proof of Lemma 5.2.5, to show

$$\lim_{l\to\infty} \frac{s_l^2}{Q_l^2\sigma_l^2 k_l} = 1, \tag{5.4.14}$$

it suffices to prove that

$$\lim_{l\to\infty} \frac{2(2K_l T_l + 2\kappa\epsilon\|f_l\| + 2\delta_l Q_l\|f_l\|)}{Q_l\sigma_l} = 0,$$

which is true by Hypothesis 5.4.5 and (5.4.10).

Meanwhile, we also need to adapt the proof of Lemma 5.2.6. The only place where Hypothesis 5.2.2 is used is the following

$$\lim_{l\to\infty} \left(\frac{2k_l\delta_l\|f\|}{\sqrt{k_l}\sigma_l} + \frac{(2M + 4T_l)\sqrt{k_l}\|f\|}{Q_l\sigma_l} \right) = 0,$$

which is now replaced by

$$\lim_{l\to\infty} \left(\frac{2k_l\delta_l\|f_l\|}{\sqrt{k_l}\sigma_l} + \frac{(2M + 4T_l)\sqrt{k_l}\|f_l\|}{Q_l\sigma_l} \right) = 0$$

in this case. It is easy to see that this holds true under Hypothesis 5.4.10.

In the verification of equivalence of CLT between (μ_l) and (ν_l), we just need to show that

$$\frac{2k_l(2K_lT_l + \kappa\epsilon\|f_l\| + 2\delta_l Q_l\|f_l\|)}{\sqrt{k_l}Q_l\sigma_l} < b$$

holds true for all $b > 0$ when l is sufficiently large. This follows for the same reason as above.

Finally, as (5.4.14) still holds in this case, the Lindeberg conditions for (μ_l) and (ν_l) are also equivalent. This concludes the proof of Theorem 5.4.6.

CHAPTER 6

Multifractal Analysis of Geodesic Flows on Surfaces With No Focal Points

In this section we conduct multifractal analysis on Lyapunov level sets, i.e. level sets of Lyapunov exponents. The case we focus on is the geodesic flow on surfaces without focal points. In particular, we estimate the Hausdorff dimension and topological entropy of such level sets. The dimension theory is well-known in the Anosov case, i.e. when Sing $= \emptyset$. As mentioned in the introduction, our main strategy is to first apply the result on thermodynamic formalism from [**13**] to study the case prior to phase transition, then follow [**10**] to establish the lower bound for entropy and dimension after phase transition. The content is based on the work in [**34**].

Denote by S a compact, connected, smooth, boundaryless and rank-one surface with no focal points and by $G = \{g_t\}_{t\in\mathbb{R}}$ the geodesic flow acting on T^1S. We also assume that Sing is non-empty. Recall from Definition 2.1.6 that the *Lyapunov level sets* is defined as

$$\mathcal{L}(\beta) := \{v \in T^1S \colon v \text{ is Lyapunov regular and } \chi(v) = \beta\}.$$

We denote the topological entropy (also called entropy spectrum) and Hausdorff dimension of $\mathcal{L}(\beta)$ by $h(\mathcal{L}(\beta))$ and $\dim_H(\mathcal{L}(\beta))$ respectively, where the topological entropy we use here is as in Definition 2.4.2. The main tools we rely on to study the multifractal information are the pressure function and its Legendre transform, which are defined by

$$\mathcal{P}(t) := P(t\varphi^{geo}) \quad \text{and} \quad \mathcal{E}(\alpha) := \inf_{t\in\mathbb{R}} \Big(\mathcal{P}(t) - t\alpha\Big). \tag{6.0.1}$$

§6 is organized as follows. In §6.1, we will introduce some preliminary results on thermodynamic formalism of G and properties of $\mathcal{P}$ and $\mathcal{E}$. In §6.2, we prove Theorem E by assuming our main technical result to hold, which is Proposition 6.2.3. In §6.3, we prove Proposition 6.2.3.

6.1. Preliminaries

We start with a brief survey on thermodynamic formalism of G on T^1S and how that is related to multifractal analysis. Recall that in the flow case, the geometric potential at $v \in T^1S$ is defined by

$$\varphi^{geo}(v) := -\lim_{t\to 0}\frac{1}{t}\log\left\|dg_t|_{E_v^u}\right\|,$$

where E_v^u is the unstable distribution at v as in Proposition 2.3.6. For any $t \in \mathbb{R}$, we call $t\varphi^{geo}(v)$ the geometric-t potential at v. It was shown in [**13**] that

Proposition 6.1.1. *In this case, $t\varphi^{geo}$ has a unique equilibrium state μ_t for G for all $t < 1$.*

With this in hand, we can summarize the properties of $\mathcal{P}$ as follows

Proposition 6.1.2. *The function $\mathcal{P}$ in this case satisfies the following properties*

(1) $\mathcal{P}$ is convex, non-increasing and satisfies $\mathcal{P}(t) = 0$ for all $t \geq 1$.

(2) If $t\varphi^{geo}$ has a unique equilibrium state ν_t, then $\mathcal{P}$ is differentiable at t and the $\mathcal{P}'(t) = \int \varphi^{geo} d\nu_t$. In particular, $\mathcal{P}$ is C^1 except for $t = 1$.

(3) Due to (1) and (2), we can define

$$\alpha_1 := \lim_{t\to-\infty} D^+\mathcal{P}(t).$$

Then for every $\alpha \in [\alpha_1, 0]$, there exists a unique supporting line ℓ_α to $\mathcal{P}$ with slope α.

(4) For $t < 1$, the unique supporting line to $\mathcal{P}$ at $(t, \mathcal{P}(t))$ is

$$\ell_{\alpha_t}(s) := h(\mu_t) + s\alpha_t,$$

where $\alpha_t = \displaystyle\int \varphi^{geo} d\mu_t$. In particular, $\mathcal{E}(\alpha_t) = h(\mu_t)$.

Proof. The first statement follows immediately from the fact of φ^{geo} being non-positive and Sing being non-empty. To prove the second statement, suppose $t\varphi^{geo}$ has a unique equilibrium state ν_t. Consider any sequence of $\{t_n\}_{n\in\mathbb{N}}$ satisfying $t_n \uparrow t$. For each n, take ν_n to be any equilibrium state of $t_n\varphi^{geo}$ (which always exists by upper semi-continuity of the entropy map $\mu \to h_\mu(f)$). By restricting to a subsequence if necessary, we assume that ν_n converges to ν^- in weak*-topology. Then by upper semi-continuity of the entropy map and continuity of $\mathcal{P}$, we know

ν^- is an equilibrium state of $t\varphi^{geo}$. By the uniqueness assumption we know $\nu^- = \nu_t$. Meanwhile, by continuity of φ^{geo}, we know

$$\int \varphi^{geo} d\nu_t = \int \varphi^{geo} d\nu^- = \lim_{n\to\infty} \int \varphi^{geo} d\nu_n \leq D^- \mathcal{P}(t).$$

Similarly we have $\int \varphi^{geo} d\nu_t \geq D^+ \mathcal{P}(t)$. As a result, we have

$$\int \varphi^{geo} d\nu_t = D^+\mathcal{P}(t) = D^-\mathcal{P}(t) = \mathcal{P}'(t).$$

Continuity of $\mathcal{P}'$ follows immediately by continuity of φ^{geo} and upper semi-continuity of the entropy map. (3) and (4) are direct consequence of (1) and (2). □

Besides α_1, we also define

$$\alpha_2 := D^-\mathcal{P}(1).$$

The discrepancy between α_2 and $D^+\mathcal{P}(1) = 0$ causes the occurrence of phase transition at $t = 1$. We will study $\mathcal{L}(\beta)$ in terms of $\beta \in (-\alpha_1, -\alpha_2)$ and $\beta \in [-\alpha_2, 0)$ in the next section.

We also need to consider uniformly hyperbolic subsystems where the multifractal information is well-understood. We consider the collection of basic sets defined as follows

Definition 6.1.3. We call a closed, G-invariant and hyperbolic set $\Lambda \subset T^1S$ *basic set* if it is locally maximal and the action of G on Λ is transitive.

For any given basic set $\Lambda \subset T^1S$, as in (6.0.1), we define

$$\mathcal{P}_\Lambda(t) := P_\Lambda(t\varphi^{geo}) \text{ and } \mathcal{E}_\Lambda(\alpha) := \inf_{t\in\mathbb{R}}(\mathcal{P}_\Lambda(t) - t\alpha).$$

The multifractal properties of basic sets that we will use throughout §6 are summarized in the following proposition.

Proposition 6.1.4. *Let $\Lambda \subset T^1S$ be a basic set. We have*

(1) $\mathcal{P}_\Lambda$ is strictly convex, strictly decreasing and real analytic on $\mathbb{R}$. In particular, both $\alpha_1(\Lambda) := \lim_{t\to-\infty} \mathcal{P}'(t)$ and $\alpha_2(\Lambda) := \lim_{t\to\infty} \mathcal{P}'(t)$ are well-defined.

(2) For any $\alpha \in [\alpha_1(\Lambda), \alpha_2(\Lambda)]$, $\mathcal{P}_\Lambda(t)$ has a unique supporting line $\ell_{\Lambda,\alpha}$ with slope α, which intersects y-axis at $(0, \mathcal{E}(\alpha))$.

(3) *For all* $\alpha \in [\alpha_1(\Lambda), \alpha_2(\Lambda)]$, *we have* $\mathcal{L}(-\alpha) \cap \Lambda \neq \emptyset$. *For other* α, $\mathcal{L}(-\alpha) \cap \Lambda = \emptyset$.

(4) *The derivative of* $\mathcal{P}_\Lambda$ *satisfies* $\mathcal{P}'_\Lambda(t) = -\chi(\mu_t)$, *where* μ_t *is the unique equilibrium state of* $t\varphi^{geo}|_\Lambda$. *Meanwhile, we have*

$$\mathcal{E}_\Lambda(-\chi(\mu_t)) = h(\mu_t).$$

(5) *For every* $\alpha \in (\alpha_1(\Lambda), \alpha_2(\Lambda))$, *we have*

$$\dim_H(\mathcal{L}(-\alpha) \cap \Lambda) = 1 + 2 \cdot \frac{\mathcal{E}_\Lambda(\alpha)}{-\alpha}$$

and

$$h(\mathcal{L}(-\alpha) \cap \Lambda) = \mathcal{E}_\Lambda(\alpha).$$

6.2. Proof of the Main Theorem

We start from a partial statement for the main theorem.

Proposition 6.2.1. (1) $\mathcal{L}(-\alpha) = \emptyset$ *for all* $\alpha < \alpha_1$ *and* $\alpha > 0$.

(2) $\mathcal{L}(-\alpha)$ *is non-empty for* $\alpha \in [\alpha_1, \alpha_2)$.

(3) $h(\mathcal{L}(-\alpha)) = \mathcal{E}(\alpha)$ *for every* $\alpha \in (\alpha_1, \alpha_2)$.

Proof. The first statement follows essentially from the variational principle. We prove for the case where $\alpha < \alpha_1$. The case where $\alpha > 0$ follows from a similar argument. Suppose that $\alpha < \alpha_1$ and $\mathcal{L}(-\alpha) \neq \emptyset$. Let $v \in \mathcal{L}(-\alpha)$. For every $n \in \mathbb{N}$, define $\delta_{v,n}$ to be the Dirac measure along the orbit segment (v, n), i.e. $\int f d\delta_{v,n} = \frac{1}{n}\int_0^n f(g_s(v))ds$ for every $f \in C(T^1S)$. By restricting to a subsequence if necessary, we assume that $\delta_{v,n}$ converges to some measure $\mu \in \mathcal{M}(G)$ in weak*-topology. Since φ^{geo} is continuous, we know

$$\int -\varphi^{geo} d\delta_{v,n} \to \int -\varphi^{geo} d\mu = -\alpha.$$

In particular, the above shows that $P_\mu(t\varphi^{geo}) \geq t\alpha > t\alpha_1$ for all $t < 0$. By definition of α_1, when t is sufficiently small, we have $P_\mu(t\varphi^{geo}) > \mathcal{P}(t)$, contradicting the variational principle.

The second statement follows directly from Proposition 6.1.2. We first look at the case where $\alpha \in (\alpha_1, \alpha_2)$. Since $\mathcal{P}$ is convex, non-increasing and C^1, there exists a unique supporting line to

$\mathcal{P}$ with slope α, called l_α. Moreover, l_α intersects $\mathcal{P}$ at a point $(t_\alpha, \mathcal{P}(t_\alpha))$. By Proposition 6.1.2 (2), we know $\int \varphi^{geo} d\mu_{t_\alpha} = \alpha$. Since μ_{t_α} is the unique equilibrium state of $t_\alpha \varphi^{geo}$, it is ergodic. Therefore, all its generic points (in both directions) belong to $\mathcal{L}(-\alpha)$.

When $\alpha = \alpha_1$, consider a sequence of measures $\{\mu_{t_\alpha}\}$ with $\alpha \downarrow \alpha_1$. As above we assume μ_{t_α} converges to some $\nu \in \mathcal{G}$ in weak*-topology and get $\int \varphi^{geo} d\nu = \alpha_1$. The desired result follows from taking an ergodic decomposition and repeating the argument from above.

Finally, for the last statement, recall from Lemma 2.4.3, for every $t \in \mathbb{R}$, we have

$$\mathcal{P}(t) = \sup_{\alpha \in \mathbb{R}} (h(\mathcal{L}^+(-\alpha)) + t\alpha),$$

where $\mathcal{L}^+(-\alpha)$ is defined as

$$\mathcal{L}^+(-\alpha) := \{v \in T^1 S \colon \chi^+(v) = -\alpha\}.$$

As a result, we have

$$\mathcal{E}(\alpha) = \inf_{t \in \mathbb{R}} (\mathcal{P}(t) - t\alpha) \geq h(\mathcal{L}^+(-\alpha)) \geq h(\mathcal{L}(-\alpha)). \tag{6.2.1}$$

On the other hand, from Proposition 6.1.2 (4), we have

$$\mathcal{E}(\alpha) = h(\mu_{t_\alpha}) = \inf\{h(Z) \colon Z \subset T^1 S,\ \mu_{t_\alpha}(Z) = 1\} \leq h(\mathcal{L}(-\alpha)),$$

which together with (6.2.1) conclude the proof of (3). □

From the proof above we also extract the following result

Lemma 6.2.2. *For all $\alpha \in [\alpha_1, 0]$, we have*

$$\mathcal{E}(\alpha) \geq h(\mathcal{L}(-\alpha)).$$

Now we move forward and study the entropy spectrum of $\mathcal{L}(-\alpha)$ when $\alpha \in [\alpha_2, 0)$ and Hausdorff dimension of all Lyapunov level sets. The main technical result we rely on is the following

Proposition 6.2.3. *There exists an increasingly nested sequence of basic sets $\{\widetilde{\Lambda}_i\}_{i \in \mathbb{N}}$ such that for any basic set $\Lambda \subset T^1 S$, there exists $n \in \mathbb{N}$ such that $\Lambda \subseteq \widetilde{\Lambda}_n$.*

We will use the above result to show the existence of $\mathcal{L}(-\alpha)$ for all $\alpha \in [\alpha_2, 0)$ by showing their intersection with $\widetilde{\Lambda_n}$ is non-empty when n is large enough. We will also show that the pressure functions $\mathcal{P}_n(t) := \mathcal{P}_{\widetilde{\Lambda}_n}(t)$ converges to $\mathcal{P}(t)$ for each t, so as the supporting line ℓ_α^n. This will connect the multifractal information read from $\{\widetilde{\Lambda}_i\}_{i\in\mathbb{N}}$ using Proposition 6.1.4 to which of the entire T^1S. Throughout this section, we assume Proposition 6.2.3 to hold and proceed our proof based on this. We will prove Proposition 6.2.3 in §6.3.

We observe from Proposition 6.1.4 that for any basic set $\Lambda \subset T^1S$, the set of α which makes $\mathcal{L}(-\alpha) \cap \Lambda$ non-empty is a closed interval. Therefore, if a basic set contains points with Lyapunov exponents close to 0 and $-\alpha_1$, it has non-empty intersection with most of the Lyapunov regular sets. To achieve this, we first show that there exist closed geodesics whose Lyapunov exponents are close to 0 and $-\alpha_1$ respectively.

We first build connections between $\chi(v)$ and the Riccati equation along γ_v for Lyapunov regular $v \in T^1S$. Let $J \in \mathcal{J}^\perp(\gamma_v)$. Notice that $u = u(J) := J'/J$ is a real-valued function satisfying the following Riccati equation

$$u'(t) + u(t)^2 + K(\gamma(t)) = 0. \tag{6.2.2}$$

Assume $v \in T^1S$ is Lyapunov regular. Consider $\xi \in E_v^u \subset T_vT^1S$ and write $\xi(t) := Dg^t(\xi)$ for all $t \geq 0$. We have

$$\chi(v) = \lim_{t\to\infty} \frac{1}{t} \log \frac{||\xi(t)||}{||\xi(0)||} = \lim_{t\to\infty} \frac{1}{t} \log ||\xi(t)||.$$

We know from (2.3.2) that

$$||\xi(t)||^2 = ||J_\xi(t)||^2 + ||J_\xi'(t)||^2.$$

Meanwhile, by definition of λ^u, we have

$$||J_\xi'(t)|| = \lambda^u ||J_\xi(t)||.$$

Since λ^u is continuous on the compact manifold T^1S, we have

$$\chi(v) = \lim_{t\to\infty} \frac{1}{t} \log ||J_\xi(t)|| = \lim_{t\to\infty} \frac{1}{t} \int_0^t u(J_\xi)(t)dt. \tag{6.2.3}$$

The above description characterizes how to bound the Lyapunov exponent of a closed geodesic from above by averaging the curvature along the orbit, which is shown in the lemma below:

LEMMA 6.2.4. *For any closed geodesic $(v,t) \in T^1S \times [0,\infty)$, we have*

$$\chi(v) \leq \sqrt{-\frac{1}{t}\int_0^t K(\gamma_v(s))\,ds}. \tag{6.2.4}$$

Before we prove this, we need to show the right hand side of (6.2.4) is well-defined. By writing $u(t) = u(J_\xi(v))(t)$, we have $u(t) = u(0)$. In particular, $\int_0^t u'(s)ds = 0$. Plugging this into (6.2.2) shows us that $\displaystyle\int_0^t u^2(s) + K(\gamma_v(s))ds = 0$. Therefore, we have

$$\int_0^t K(\gamma_v(s))ds = -\int_0^t u^2(s)\,ds \leq 0.$$

PROOF OF LEMMA 6.2.4. The proof follows from the argument above and a simple application of Cauchy's inequality as follows

$$\chi(v) = \lim_{T\to\infty}\frac{1}{T}\int_0^T u(s)ds \leq \lim_{T\to\infty}\sqrt{\frac{1}{T}\int_0^T u^2(s)ds} = \lim_{T\to\infty}\sqrt{\frac{1}{T}\int_0^T (-K(\gamma_v(s)))ds},$$

where the first equality follows from (6.2.3). Since (v,t) is periodic, we immediately see that $\lim_{T\to\infty}\sqrt{\frac{1}{T}\int_0^T(-K(\gamma_v(s)))ds} = \sqrt{-\frac{1}{t}\int_0^t K(\gamma_v(s))\,ds}$, which concludes the proof. □

We also introduce a version of shadowing lemma used in our case as follows

LEMMA 6.2.5 (Shadowing lemma). *Let S be the surface in our case. For any $\eta, \epsilon, \tau > 0$, there exists $\delta > 0$ such that the following holds: if $\{(v_i, t_i)\}_{i\in\mathbb{Z}}$ is any collection of orbit segments such that $v_i, g_{t_i}v_i \in \mathrm{Reg}(\eta)$, $t_i \geq \tau$ and $d(g_{t_i}v_i, v_{i+1}) < \delta$ hold for all $i \in \mathbb{Z}$, then there exists a geodesic γ and a sequence of times $\{T_i\}_{i\in\mathbb{Z}}$ with $T_0 = 0$, $T_i + t_i - \epsilon \leq T_{i+1} \leq T_i + t_i + \epsilon$ satisfying $d(\dot\gamma(t), \dot\gamma_{v_i}(t - T_i)) < \epsilon$ for all $t \in [T_i, T_{i+1}]$ and $i \in \mathbb{Z}$.*

The geodesic γ is unique up to re-parametrization. Moreover, if the orbits being shadowed are periodic, then the shadowing orbit is by itself periodic.

Now we can use Lemma 6.2.4 and 6.2.5 to construct the closed regular geodesics with the desired properties.

Proposition 6.2.6. *Let (S, G) be as in our case, then there exist closed geodesics with Lyapunov exponents arbitrarily close to 0 and $-\alpha_1$ respectively.*

Proof. We start with the construction of closed geodesics with small Lyapunov exponents. For any $\epsilon > 0$, by Proposition 2.3.11 and an easy compactness argument, there exists $\eta = \eta(\epsilon) > 0$ such that

$$\{v \in T^1 S : |K(\pi v)| \geq \epsilon\} \subseteq \mathrm{Reg}(\eta).$$

Choose any $v_0 \in \mathrm{Sing}$. By transitivity of G, we can find a sequence of vectors $\{v_n\}_{n\in\mathbb{N}}$ satisfying the follows

(1) $d(v_n, v_0) \downarrow 0$ when $n \to \infty$.

(2) The orbit of v_n will enter $\{v \in T^1 S : |K(\pi v)| \geq \epsilon\}$ both forwards and backwards for all n. We denote the first forward (resp. backward) entrance time for the orbit of v_n into $\{v \in T^1 S : |K(\pi v)| \geq \epsilon\}$ by t_n (resp. τ_n) for all $n \in \mathbb{N}$. By restricting to a subsequence if necessary, we might assume that $\tau_n \downarrow \infty$ and $t_n \uparrow \infty$ when $n \to \infty$, $g_{\tau_n}(v_n)$ converges to some $v \in T^1 S$ and $g_{t_n}(v_n)$ converges to some $w \in T^1 S$. Notice that

$$|K(\pi(g_t(v_n)))| < \epsilon \text{ for all } t \in (\tau_n, t_n). \tag{6.2.5}$$

Meanwhile, it is not hard to see that $|K(\pi v)| = |K(\pi w)| = \epsilon$. As a result, we know $v, w \in \mathrm{Reg}(\eta)$. Again by topological transitivity of G, we can find some orbit segment (v', t') that originates near w and terminates near v. In particular, both v' and $g_{t'}(v')$ are in $\mathrm{Reg}(\eta/2)$. Therefore, by applying Lemma 6.2.5 to $(g_{\tau_n}(v_n), t_n - \tau_n)$ and (v', t'), we get a geodesic (w_n, T_n) where T_n is roughly $t' + t_n - \tau_n$. By (6.2.5), we know (w_n, T_n) spends most of the time in $\{v \in T^1 S : |K(\pi v)| < 2\epsilon\}$. By applying Lemma 6.2.4, $\chi(w_n) \leq 2\sqrt{\epsilon}$ when n is large enough. By making $\epsilon \downarrow 0$, we finish our construction for the first part.

Now we want to construct a closed regular geodesic whose Lyapunov exponent can be made arbitrarily close to $-\alpha_1$. First recall from Proposition 6.2.1 that $\mathcal{L}(-\alpha_1)$ is non-empty. We choose any $v_0 \in \mathcal{L}(-\alpha_1)$ and focus on the evolution of the solution function $u(t)$ to the Riccati equation (6.2.2) over γ_{v_0}. Since $-\alpha_1 = \chi(v_0) = \lim_{T\to\infty} \frac{1}{T}\int_0^T u(t)\,dt$, we observe that $\liminf_{t\to\pm\infty} K(\gamma_{v_0}(t)) < 0$. In particular, there exists $\epsilon > 0$ such that $g_t(v_0)$ enters $\{v \in T^1 S : K(\pi v) \leq -\epsilon\}$ infinitely many times both forwards and backwards. As in the first half of the proof, we have the existence of $\eta = \eta(\epsilon) > 0$. Therefore, for any $\epsilon_0 > 0$, there exist a positive sequence $\{t_k\}_{k\in\mathbb{N}}$ with $t_k \uparrow \infty$ and a negative sequence $\{t'_k\}_{k\in\mathbb{N}}$ with $t'_k \downarrow \infty$ such that

(1) $g_{t_k}(v_0) \in \mathrm{Reg}(\eta)$ and $g_{t'_k}(v_0) \in \mathrm{Reg}(\eta)$ for all $k \in \mathbb{N}$.

(2) $\frac{1}{t_k - t'_k}\int_{t'_k}^{t_k} -\varphi^{geo}(g_s(v_0))ds > -\alpha_1 - \epsilon_0$ for all $k \in \mathbb{N}$.

By passing to a sequence if necessary, we assume that $g_{t_k}(v_0) \to v$ and $g_{t'_k}(v_0) \to w$. Define $\epsilon' > 0$ such that whenever $d(v, w) < \epsilon'$, $|\varphi^{geo}(v) - \varphi^{geo}(w)| < \epsilon_0$. Now if $v = w$, we apply Lemma 6.2.5 to $\epsilon', \eta, t_1 - t'_1$ and see that $(g_{t'_k}(v_0), t_k - t'_k)$ can be ϵ'-shadowed by some closed geodesic (v_k, T_k) whose period T_k is roughly $t_k - t'_k$ when k is sufficiently large. By condition (2) from above and the choice on ϵ', we know $\chi(v_k) > -\alpha_1 - 3\epsilon_0$. Meanwhile, if $v \neq w$, by transitivity of G, there exists an orbit segment (v', t') such that $t' > t_1 - t'_1$, $d(v', v) < \delta$ and $d(g_{t'}(v'), w) < \delta$, where $\delta = \delta(\epsilon', \eta, t_1 - t'_1)$ is from Lemma 6.2.5. As above we have (w_k, T'_k) whose period T'_k is roughly $t_k - t'_k + t'$ which ϵ'-shadows $(g_{t'_k}(v_0), t_k - t'_k)$ and (v', t') when k is sufficiently large. Now (w_k, T'_k) satisfies $\chi(w_k) > -\alpha_1 - 3\epsilon_0$ for large k. As ϵ_0 can be made arbitrarily small, we conclude our proof. $\square$

We will use the above result to show that the sequence of basic sets $\{\widetilde{\Lambda}_i\}_{i\in\mathbb{N}}$ constructed in Proposition 6.2.3 exhausts hyperbolicity of the system. In other words, we want to show that $\mathcal{P}_n(t)$ converges to $\mathcal{P}(t)$ in every t, so does ℓ^n_α. The proof relies on the classic Katok horseshoe theorem [**25**], which says that if $\epsilon > 0$, then for any hyperbolic measure $\mu \in \mathcal{M}(G)$ and potential $\varphi\colon T^1S \to \mathbb{R}$, there exists a basic set $\Lambda \subseteq T^1S$ such that

$$P_\Lambda(\varphi) > P_\mu(\varphi) - \epsilon.$$

The following result is essential in completing the proof of Theorem E.

Proposition 6.2.7. *The sequence of basic sets $\{\widetilde{\Lambda}_i\}_{i\in\mathbb{N}}$ constructed in Proposition 6.2.3 satisfies the following*

(1) $\mathcal{P}_n(t) \uparrow \mathcal{P}(t)$ when $n \to \infty$ for all $t \in \mathbb{R}$.

(2) $\ell^n_\alpha(t) \uparrow \ell_\alpha(t)$ when $n \to \infty$ for all $t \in \mathbb{R}$ and $\alpha \in (\alpha_1, 0)$.

We comment that in the statement of Proposition 6.2.7 (2), for any $\alpha \in (\alpha_1, 0)$, we only work with n where ℓ_α^n is well-defined. We will show that given any such α, this is always the case whenever n is sufficiently large.

Proof. We separate the cases into $t < 1$ and $t \geq 1$. When $t < 1$, we know $t\varphi^{geo}$ has a unique equilibrium state μ_t for G. Moreover, as $\chi(\mu_t) > 0$, μ_t is hyperbolic. Then by Katok horseshoe theorem, for any $\epsilon > 0$, we know there exists a basic set $\Lambda_t \subset T^1S$ such that $P_{\Lambda_t}(t\varphi^{geo}) > \mathcal{P}(t) - \epsilon$. By Proposition 6.2.3, we have $\mathcal{P}_n(t) > \mathcal{P}(t) - \epsilon$ for all n large enough. This proves the case for $t < 1$.

When $t \geq 1$, we know $\mathcal{P}(t) = 0$ for such t. By Proposition 6.2.6, there exists a closed geodesic (v, t) such that $\chi(v) < \epsilon/t$. Then if we denote the probability measure uniformly distributed on the closed geodesic (v, t) by μ_v, we know μ_v is a hyperbolic measure and $P_{\mu_v}(t\varphi^{geo}) = -t\chi(\mu_v) > -\epsilon$. Therefore, there exists a basic set $\Lambda \subset T^1S$ such that $P_\Lambda(t) > P_{\mu_v}(t\varphi^{geo}) - \epsilon > -2\epsilon$. Again by Proposition 6.2.3, we have $\mathcal{P}_n(t) > \mathcal{P}(t) - 2\epsilon$ for all n large enough. This proves the case for $t \geq 1$, thus ends the proof of the first statement.

The proof of the second statement is identical to the one of Proposition 12 in [**10**], so we omit the proof. $\square$

Now we are ready to complete the proof of Theorem E.

Proof of Theorem E. We begin with proving $\mathcal{L}(-\alpha) \neq \emptyset$ for all $\alpha \in [\alpha_1, 0]$. Because of Proposition 6.2.1, we only need to show the above result for $\alpha \in [\alpha_2, 0]$. Since Sing $\subset \mathcal{L}(0)$ and Sing is non-empty, we know $\mathcal{L}(0) \neq \emptyset$. Meanwhile, for any $\alpha \in [\alpha_2, 0)$, by Proposition 6.2.6 we know there exist closed regular geodesics (v_1, t_1) and (v_2, t_2) such that $0 < \chi(v_1) < -\alpha < \chi(v_2)$. Moreover, the orbits of (v_1, t_1) and (v_2, t_2) are basic sets by themselves. Therefore, Proposition 6.2.3 shows that eventually we can find n large enough such that Λ_n contains both (v_1, t_1) and (v_2, t_2). By Proposition 6.1.4, we know $-\alpha_2(\Lambda_n) \leq \chi(v_1) < -\alpha < \chi(v_2) \leq -\alpha_1(\Lambda_n)$, thus $\mathcal{L}(-\alpha) \cap \Lambda_n \neq \emptyset$. In particular, this implies that $\mathcal{L}(-\alpha) \neq \emptyset$ and ℓ_α^n is well-defined.

Meanwhile, since ℓ_α^n converges to ℓ_α for all $\alpha \in (\alpha_1, 0)$ when $n \to \infty$, we know $\ell_\alpha^n(0)$ converges to $\ell_\alpha(0)$. Therefore, we have

$$\lim_{n\to\infty} \mathcal{E}_n(\alpha) = \mathcal{E}(\alpha) \text{ for all such } \alpha.$$

As a result, we have

$$h(\mathcal{L}(-\alpha)) \geq \lim_{n\to\infty} h(\mathcal{L}(-\alpha) \cap \widetilde{\Lambda}_n) = \lim_{n\to\infty} \mathcal{E}_n(\alpha) = \mathcal{E}(\alpha),$$

and

$$\dim_H(\mathcal{L}(-\alpha)) \geq \lim_{n\to\infty} \dim_H(\mathcal{L}(-\alpha) \cap \widetilde{\Lambda}_n) = \lim_{n\to\infty} 1 + 2 \cdot \frac{\mathcal{E}_n(\alpha)}{-\alpha} = 1 + 2 \cdot \frac{\mathcal{E}(\alpha)}{-\alpha},$$

which are the required lower bounds for $h(\mathcal{L}(-\alpha))$ and $\dim_H(\mathcal{L}(-\alpha))$. This ends the proof of Theorem E. $\square$

6.3. Proof of Proposition 6.2.3

In this section we prove Proposition 6.2.3, which is the main technical construction for the proof of Theorem E. The proof follows from adapting the construction in [**10**] to our setting by using the function λ_T defined in Definition 2.3.10.

Before proceeding to the proof, we first state some preliminary results in terms of hyperbolicity in our case. First notice by continuity of λ and compactness of T^1S, for every $\epsilon > 0$, there exists $\delta = \delta(\epsilon) > 0$ such that

$$d(u, v) < \delta \implies |\lambda(u) - \lambda(v)| < \epsilon. \tag{6.3.1}$$

Setting

$$\widetilde{\lambda}(v) := \max\Big\{\lambda(v) - \epsilon, 0\Big\}, \tag{6.3.2}$$

the following lemma provides control on the $d^{s/u}$-distance using $\widetilde{\lambda}$. We also define

$$\lambda_{\max} := \max_{v \in T^1S} \lambda^u(v) = \max_{v \in T^1S} \lambda^s(v).$$

The following result follows directly from Lemma 2.3.9.

Lemma 6.3.1. *For any $\epsilon > 0$, let $\delta = \delta(\epsilon)$ be as in (6.3.1). Then for any $v \in T^1S$, $w, w' \in W^s_\delta(v)$, and $t \geq 0$, we have*

$$d^s(g_t w, g_t w') \leq d^s(w, w') \cdot \exp\Big(- \int_0^t \widetilde{\lambda}(g_s v)\, ds \Big).$$

In the case where $\epsilon = \eta/4T$ and $\delta = \delta(\eta/4T)$ for some $T, \eta > 0$, there exists $C = C(T, \eta) > 0$ such that the following holds: if $g_s v \in \mathrm{Reg}_T(\eta)$ for all $s \in \mathbb{R}$, then for any $w, w' \in W^s_\delta(v)$ and $t \geq 0$, we have

$$d^s(g_t w, g_t w') \leq C \cdot d^s(w, w') \cdot \exp\Big(- \frac{\eta t}{4T} \Big).$$

Similarly, the analogous statements hold for d^u with relevant modifications.

The following proposition characterizes uniformly hyperbolicity among compact subsets of T^1S.

Proposition 6.3.2. *Any compact G-invariant subset $\Lambda \subseteq T^1S$ is uniformly hyperbolic if and only if $\Lambda \subseteq \mathrm{Reg}$.*

Proof. The forward direction is clear by definition of Sing. We prove the backward direction. Suppose that Λ is a compact subset of Reg. Since $\mathrm{Reg}_T(\eta)$ exhausts Reg, from the compactness of Λ there exists $\eta, T > 0$ such that $\Lambda \subseteq \mathrm{Reg}_T(\eta)$. Now consider any $v \in \Lambda$ and $\xi \in E^u_v$, and let $(J_\xi(t), J'_\xi(t))$ be the corresponding Jacobi field in $\mathcal{J}^u(\gamma_v)$. Setting $\lambda(t) := \lambda^u(g_t(v))$, we know from (2.3.2) and Lemma 2.3.7 that

$$\begin{aligned}
\frac{||dg_t(\xi)||}{||\xi||} = \frac{\sqrt{||J_\xi(t)||^2 + ||J'_\xi(t)||^2}}{\sqrt{||J_\xi(0)||^2 + ||J'_\xi(0)||^2}} &= \frac{\sqrt{(1+\lambda^2(t))||J_\xi(t)||^2}}{\sqrt{(1+\lambda^2(0))||J_\xi(0)||^2}} \\
&= \frac{\sqrt{(1+\lambda^2(t)) \exp\Big(2 \int_0^t \lambda(s) ds\Big)}}{\sqrt{(1+\lambda^2(0))}} \\
&\geq \frac{\exp\Big(\int_0^t \lambda(s) ds\Big)}{\sqrt{1+\lambda^2_{\max}}}.
\end{aligned}$$

Here, we have used $\lambda^u(v) \geq \lambda(v)$ in deducing the inequality. Similar to the proof of Lemma 6.3.1 above, we have

$$\int_0^t \lambda(g_s v)\, ds \geq \frac{1}{2T} \int_0^t \lambda_T(g_s v)\, ds - 2Tk_{\max} \geq \frac{\eta t}{2T} - 2Tk_{\max},$$

where the second inequality is due to the fact that $\Lambda \subset \mathrm{Reg}_T(\eta)$. This shows that the derivative of the geodesic flow expands $\xi \in E^u_v$ exponentially at rate $\eta/2T$. Applying an analogous argument to $\xi \in E^s_v$ shows that Λ is uniformly hyperbolic. $\square$

Now we prove the following key proposition in creating locally maximal sets around any hyperboilc set, which corresponds to [**10**, Proposition 8].

Proposition 6.3.3. *For any closed G-invariant hyperbolic set $\Lambda \subset T^1S$ and any neighborhood U of Λ, there exists a closed G-invariant locally-maximal hyperbolic set $\widetilde{\Lambda}$ such that $\Lambda \subseteq \widetilde{\Lambda} \subseteq U$.*

Proof of Proposition 6.3.3. Let Λ and U be given as in the proposition. We may assume that U is small enough so that every G-invariant compact set contained in U is hyperbolic. By Proposition 6.3.2, there exist $\eta_\Lambda, T_\Lambda > 0$ such that $\Lambda \subset \mathrm{Reg}_{T_\Lambda}(\eta_\Lambda)$. By continuity of λ and the compactness of Λ, we could assume without loss of generality that $\bar{U} \subset \mathrm{Reg}_{T_\Lambda}(\eta_\Lambda)$ by possibly redefining scales η_Λ and T_Λ. Meanwhile, by compactness of $\mathrm{Reg}_{T_\Lambda}(\eta_\Lambda)$, there exist $\delta_{LPS} = \delta(T_\Lambda, \eta_\Lambda) > 0$ and $\kappa := \kappa(T_\Lambda, \eta_\Lambda) > 1$ such that vectors in a δ_{LPS}-neighborhood of any $v \in \mathrm{Reg}_{T_\Lambda}(\eta_\Lambda)$ have the local product structure with constant κ. The strategy of the proof is to use the product structure on the ambient space to construct $\widetilde{\Lambda}$ as the image of a subshift of finite type under a continuous injective map. In this way, $\widetilde{\Lambda}$ inherits the natural local product structure from the shift space and becomes locally maximal.

We begin with the construction of cross section on each $v \in T^1S$ that is orthogonal to the flow direction. The idea follows from the first few paragraphs of [**10**, Lemma 2.4]. Given $v \in T^1S$ and $\delta > 0$, we define $O_\delta(v)$ as a subset of T_vT^1S that consists of all vectors in T_vT^1S that are orthogonal to $(v, 0) \in T_vT^1S$ (which corresponds to the flow direction of γ_v) in the Sasaki metric and have norm bounded from above by δ; that is,

$$O_\delta(v) := \{\xi \in T_vT^1S \colon \langle \xi, (v,0)\rangle_S = 0,\ \|\xi\| \leq \delta\}.$$

We define $D_\delta(v)$ as the image of $O_\delta(v)$ under the exponential map $\exp_v$. We will use $D_\delta(v)$ to construct the above mentioned cross section on each $v \in T^1S$.

Choose $\delta_0 > 0$ such that $\delta_0 < d(\Lambda, \mathrm{Sing})$; this is possible because Λ is a compact subset contained in Reg. It is not hard to see that when δ is sufficiently small, for any $v \in T^1S$, the central-stable (resp. central-unstable) foliations W^{cs} (resp. W^{cu}) induces an one-dimensional stable (resp. unstable) foliation $\mathcal{W}^s$ (resp. $\mathcal{W}^u$) on $D_\delta(v)$, whose leaves are defined by

$$\mathcal{W}^s_v(w) := W^{cs}(w) \cap D_\delta(v), \quad \mathcal{W}^u_v(w) := W^{cu}(w) \cap D_\delta(v).$$

In the cases where we conduct discussions on a single $D_\delta(v)$ that is clear, we will simplify the notation by just writing $\mathcal{W}^{s/u}(w)$.

Meanwhile, for any $\beta \in (0,1)$ that is sufficiently small, by continuity of W^s, there exists $\delta_\beta > 0$ such that for any $\delta \in (0, \delta_\beta)$, $v \in T^1S$, $w \in D_\delta(v)$ and $u_1 \in \mathcal{W}^s_v(w)$, we have

$$d^c(g_t(u_1), u_1) \leq \beta d^s(g_t(u_1), w), \tag{6.3.3}$$

where $t \in \mathbb{R}$ is the unique small time such that $g_t(u_1) \in W^s(w)$. In particular, for such u_1 and w, we have

$$d(u_1, w) \leq d^c(u_1, g_t(u_1)) + d^s(g_t(u_1), w) \leq (1+\beta) d^s(g_t(u_1), w). \tag{6.3.4}$$

Meanwhile, from 3.1.7 we also have

$$\kappa d(u_1, w) \geq d^u(g_t(u_1), w).$$

Consequently, the following holds true

$$\kappa^{-1} d^s(g_t(u_1), w) \leq d(u_1, w) \leq (1+\beta) d^s(g_t(u_1), w). \tag{6.3.5}$$

A similar result holds along the central-unstable direction in an obvious way. From now on we fix some $0 < \beta \ll 1$ that is sufficiently small.

By applying flat strip theorem for manifolds with no focal points, we notice that for any $v_0 \in \mathrm{Sing}$, if $w_0 \in W^{cs}(v_0)$, then $\lambda^+(w_0) = \lambda^+(v_0) = 0$. For the same reason, if $w_0 \in W^{cu}(v_0)$, then $\lambda^-(w_0) = \lambda^-(v_0) = 0$. In particular, this implies that Λ does not intersect with the central-stable and central-unstable leaves $W^{cs/cu}(v)$ of $v \in \mathrm{Sing}$. Therefore, on $D_\delta(v)$ for some $v \in T^1S$, we can partition $D_\delta(v)$ using the induced stable and unstable leaves of vectors in Sing into *su-rectangles* which do not intersect Λ.

As in (6.3.1) and Lemma 6.3.1, let $\delta_\Lambda := \delta(\eta_\Lambda/4T_\Lambda)$ such that

$$d(v,w) < \delta_\Lambda \implies |\lambda(u) - \lambda(v)| \leq \frac{\eta_\Lambda}{4T_\Lambda}.$$

Fix δ such that

$$0 < \delta < \min\{\delta_\Lambda, \delta_0, \delta_\beta, \delta_{LPS}/2\}. \tag{6.3.6}$$

Since both foliations W^s and W^u are minimal under the action of geodesic flow and Sing $\neq \emptyset$, following the proof of [**10**, Proposition 7] we are able to build su-rectangles on $D_\delta(v)$ containing v with arbitrarily small diameter. Choose one of such a su-rectangle for every $v \in$ Reg. These will be our desired cross section at each v and are denoted by C_v. When $v \in$ Sing, we simply put $D_{\delta_0}(v)$ as C_v.

Notice that T^1S is contained in the union $\displaystyle\bigcup_{v\in T^1S} g_{[-\frac{1}{2},\frac{1}{2}]}C_v$ and that $g_{[-\frac{1}{2},\frac{1}{2}]}C_v$ contains an open set containing v for every $v \in T^1S$. Hence, we can choose a finite set $\{v_i\}_{i=1}^n \subset T^1S$ such that T^1S is contained in the union $\displaystyle\bigcup_{1\leq i\leq n} g_{[-\frac{1}{2},\frac{1}{2}]}C_{v_i}$. Writing C_i to denote C_{v_i}, we set

$$C := \bigcup_{1\leq i\leq n} C_i.$$

Let $\tau\colon C \to (0,1]$ be the first return time from C to itself, and we define the first return map accordingly by

$$\mathscr{F}(v) := g_{\tau(v)}(v).$$

By flowing C_v slightly γ_v and potentially increasing the maximal first recurrence time, we may assume that $C_i \cap C_j = \emptyset$ for $i \neq j$. Since C_i is compact and pairwise disjoint, there exists $c_1 > 0$ such that τ is bounded from below by c_1.

Denote by ∂C_i the boundary of C_i for $1 \leq i \leq n$ and ∂C by the union of ∂C_i. Since Λ does not intersect C_v for $v \in$ Sing (by the definition of δ) or ∂C_v for $v \in$ Reg, it does not intersect ∂C. Therefore, for each $w \in \Lambda \cap C_i$ for some $1 \leq i \leq n$, we can again follow the proof of [**10**, Proposition

7] and construct su-rectangles on C_i containing w with arbitrarily small diameter. For any $\alpha > 0$ and w as above, denote such su-rectangle with diameter smaller than α by R_w^α.

From a compactness argument, we can choose a finite subset $\{w_j\}_{1\le j\le m} \subset \Lambda \cap C$ such that

$$\Lambda \cap C \subset \bigcup_{1\le j\le m} R_{w_j}^\alpha.$$

In order to refine $R_{w_j}^\alpha$ so they become mutually disjoint, notice that for $v \in \mathrm{Reg}$, by the construction of ∂C_v, C_v has a natural product structure. That is to say, for any $v_1, v_2 \in C_v$, both

$$[v_1, v_2]_C := \mathcal{W}_{v,\delta}^u(v_1) \cap \mathcal{W}_{v,\delta}^s(v_2)$$

and $[v_2, v_1]_C = \mathcal{W}_{v,\delta}^u(v_2) \cap \mathcal{W}_{v,\delta}^s(v_1)$ are contained in C_v. Therefore, if there exists $1 \le i < j \le m$ such that $R_{w_i}^\alpha \cap R_{w_j}^\alpha \ne \emptyset$, by applying local product structure on vertices, we can extend the edges until they intersect another edge and divide $R_{w_i}^\alpha \cup R_{w_j}^\alpha$ into finitely many su-rectangles in C_v with mutually disjoint interior. Moreover, we can shrink the size of the su-rectangles so that they become disjoint while their union still contains $\Lambda \cap C$. This is due to the positive distance between $\Lambda \cap C$ and the collection of all the (extended) edges of $R_{w_i}^\alpha$. Therefore, we may assume that $\{R_{w_j}^\alpha\}_{j=1}^m$ are mutually disjoint.

Since $d(\Lambda, \partial C) > 0$ and $\tau \le 1$, we can make α small enough such that both $\mathscr{F}$ and τ are smooth on R_j^α for every $1 \le j \le m$.

To summarize, we have fixed $0 < \alpha \ll \delta$ and created

$$\mathcal{R} = \mathcal{R}(\alpha) := \{R_1, R_2, \cdots, R_m\}$$

as a collection of subsets of C satisfying the following properties:

(1) $\{R_j\}_{j=1}^m$ is a collection of mutually disjoint closed su-rectangles in C.

(2) $\Lambda \cap C$ is contained in the union $\displaystyle\bigcup_{j=1}^m R_j$.

(3) Each R_j contains at least one element in $\Lambda \cap C$.

(4) $\mathrm{diam}(R_j) < \alpha$ for every $1 \le j \le m$.

(5) Both $\mathscr{F}$ and τ are smooth on R_j and $\mathscr{F}(R_j)$ is contained in one single C_i for every $1 \le j \le m$.

We will need the following lemma repeately in our argument, whose proof again relies on our construction of $\mathcal{R}$.

Lemma 6.3.4. *For any $v, w \in R_i$ and $\mathscr{F}(v), \mathscr{F}(w) \in R_j$ for some $1 \le i, j \le m$,*

$$\mathscr{F}([v,w]_C) = [\mathscr{F}(v), \mathscr{F}(w)]_C. \tag{6.3.7}$$

Similarly, if $v, w \in R_i$ and $\mathscr{F}^{-1}(v), \mathscr{F}^{-1}(w) \in R_j$, then

$$\mathscr{F}^{-1}([v,w]_C) = [\mathscr{F}^{-1}(v), \mathscr{F}^{-1}(w)]_C. \tag{6.3.8}$$

Proof of Lemma 6.3.4. The proof for (6.3.7) and (6.3.8) are symmetric, so we will just prove (6.3.7).

Since R_i has diameter less than $\alpha \in (0, \delta)$ and contains a vector in $\Lambda \cap C$, it from the choice (6.3.6) of δ that R_i has the local product structure with constant κ. In particular, the intersection $W^u_{\kappa\alpha}(v) \cap W^{cs}_{\kappa\alpha}(w)$ consists of a unique vector (which does not necessarily lie on R_i) and there exists $s_1 \in \mathbb{R}$ such that

$$g_{s_1}([v,w]_C) := W^u_{\kappa\alpha}(v) \cap W^{cs}_{\kappa\alpha}(w).$$

Then $d^u(v, g_{s_1}([v,w]_C)) \le \kappa\alpha$, so by (6.3.3) we have $|s_1| \le \kappa\alpha\beta$. From Lemma 6.3.1, we have

$$d^u(\mathscr{F}(v), g_{\tau(v)+s_1}([v,w]_C)) \le e^{k_{\max}\tau(v)} \cdot d^u(v, g_{s_1}([v,w]_C)) \le e^{k_{\max}}\kappa\alpha,$$

which is less than δ because $0 < \alpha \ll \delta$. Also, $d^{cs}(\mathscr{F}(w), g_{\tau(v)+s_1}([v,w]_C)) \le \kappa\alpha < \delta$ as d^{cs} is non-increasing in forward time. In particular, $g_{\tau(v)+s_1}([v,w]_C)$ is equal to the intersection $W^u_\delta(\mathscr{F}(v)) \cap W^{cs}_\delta(\mathscr{F}(w))$ consisting of a unique vector.

Similarly, there exists $s_2 \in \mathbb{R}$ with $|s_2| \le \kappa\alpha\beta$ such that $g_{s_2}([\mathscr{F}(v), \mathscr{F}(w)]_C)$ is equal to $W^u_{\kappa\alpha}(\mathscr{F}(v)) \cap W^{cs}_{\kappa\alpha}(\mathscr{F}(w))$. In particular, $g_{\tau(v)+s_1}([v,w]_C)$ coincides with $g_{s_2}([\mathscr{F}(v), \mathscr{F}(w)]_C)$:

$$g_{s_2}([\mathscr{F}(v), \mathscr{F}(w)]_C) = W^u_\delta(\mathscr{F}(v)) \cap W^{cs}_\delta(\mathscr{F}(w)) = g_{\tau(v)+s_1}([v,w]_C).$$

As $|s_1 - s_2| \le 2\kappa\alpha\beta$, τ is continuous on each element of $\mathcal{R}$, and $\tau \ge c_1 > 0$, we have $\tau(v) + s_1 - s_2 = \tau([v,w]_C)$ and $\mathscr{F}([v,w]_C) = [\mathscr{F}(v), \mathscr{F}(w)]_C$, as required. $\square$

We use the elements from $\mathcal{R}$ to establish the alphabet in the target shift space. Following [**10**], for $N \geq 1$ we define

$$\mathcal{R}_N := \left\{ \mathcal{D} = \bigcap_{j=-N}^{j=N} \mathscr{F}^{-j} R^j \colon R^j \in \mathcal{R} \text{ and } \mathcal{D} \cap \Lambda \neq \emptyset \right\}$$

as the collection of sets of the form $\bigcap_{j=-N}^{j=N} \mathscr{F}^{-j} R^j$ that contains at least one element in Λ. The following result is important regarding the choice of N.

LEMMA 6.3.5. *For any $\epsilon > 0$, there exists $N_1 = N_1(\epsilon) \in \mathbb{N}$ such that diam$(\delta) < \epsilon$ for every $N > N_1$ and $\delta \in \mathcal{R}_N$.*

PROOF OF LEMMA 6.3.5. Let $\epsilon > 0$ be given. In order to prove the lemma, it suffices to show that there exists $N_1 \in \mathbb{N}$ such that for any $N > N_1$, $\mathcal{D} \in \mathcal{R}_N$, $v \in \mathcal{D} \cap \Lambda$, and $w \in \mathcal{D}$, we have $d(v, w) < \frac{\epsilon}{2}$.

We claim that we only need to show the cases where w is on $\mathcal{W}^{cs}(v)$ or $\mathcal{W}^{cu}(v)$ with the upper bound $\frac{\epsilon}{2}$ replaced by $\frac{\epsilon}{8\kappa+2}$. Indeed, suppose that $d(v, [v, w]_C) < \frac{\epsilon}{8\kappa+2}$ and $d(v, [w, v]_C) < \frac{\epsilon}{8\kappa+2}$. Note from Lemma 6.3.4 that both $[v, w]_C$ and $[w, v]_C$ belong to $\mathcal{D}$ and the triangle inequality gives $d([v, w]_C, [v, w]_C) < \frac{\epsilon}{4\kappa+1}$. Since w coincides with $[[w, v]_C, [v, w]_C]_C$, we obtain $d([w, v]_C, w) < \frac{\kappa\epsilon(1+\beta)}{4\kappa+1} \leq \frac{2\kappa\epsilon}{4\kappa+1}$ from (6.3.5) and $\beta \in (0, 1)$. Combined with $d(v, [w, v]_C) < \frac{\epsilon}{8\kappa+2}$, we get

$$d(v, w) \leq d(v, [w, v]_C) + d([w, v]_C, w) \leq \frac{2\kappa\epsilon}{4\kappa+1} + \frac{\epsilon}{8\kappa+2} = \frac{\epsilon}{2}.$$

We will prove the case when $w \in \mathcal{W}^{cu}_\delta(v) \cap \delta$. The other case when $w \in \mathcal{W}^{cs}_\delta(v) \cap \delta$ can be established analogously. Assume δ is represented as $\bigcap_{j=-N}^{j=N} \mathscr{F}^{-j} R^j$ with $R^j \in \mathcal{R}$. Then both $\mathscr{F}^j(v)$ and $\mathscr{F}^j(w)$ belong to R^j for all $-N \leq j \leq N$. As in the proof of Lemma 6.3.4, for each $-N \leq j \leq N$ there exists $s_j \in \mathbb{R}$ such that

$$g_{s_j}(\mathscr{F}^j(w)) \in W^u_\delta(\mathscr{F}^j(v)).$$

Recalling that $\eta_\Lambda, T_\Lambda > 0$ are chosen so that $\Lambda \subset \mathrm{Reg}_{T_\Lambda}(\eta_\Lambda)$, parallel to (6.3.2) we set

$$\widetilde{\lambda}(v) := \max\Big\{\lambda(v) - \frac{\eta_\Lambda}{4T_\Lambda}, 0\Big\}.$$

From Lemma 6.3.1, we have for each $1 \le j \le N$ that

$$d^u(\mathscr{F}^j(v), g_{s_j}(\mathscr{F}^j(w))) \ge \exp\Big(\int_0^{\tau(\mathscr{F}^{j-1}(v))} \widetilde{\lambda}(g_s(\mathscr{F}^{j-1}(v)))ds\Big) d^u(\mathscr{F}^{j-1}(v), g_{s_{j-1}}(\mathscr{F}^{j-1}(w))).$$

Setting $S_\tau^N(v) := \sum_{j=0}^{N-1} \tau(\mathscr{F}^j(v))$ and denoting the constant from Lemma 6.3.1 by $C = C(T_\Lambda, \eta_\Lambda)$, iterations of the above inequality over all $1 \le j \le N$ gives

$$\begin{aligned} d^u(\mathscr{F}^N(v), g_{s_N}(\mathscr{F}^N(w))) &\ge \exp\Big(\int_0^{S_\tau^N(v)} \widetilde{\lambda}(g_s(v)))ds\Big) d^u(v, g_{s_0}(w)) \\ &\ge C^{-1} \cdot \exp\Big(\frac{\eta N c_1}{4T}\Big) d^u(v, g_{s_0}(w)) \\ &\ge C^{-1}(1+\beta) \cdot \exp\Big(\frac{\eta N c_1}{4T}\Big) d(v, w), \end{aligned}$$

where the second inequality uses $S_\tau^N(v) \ge Nc_1$ from the fact that the first return time τ is bounded below by c_1 and the third inequality is due to (6.3.5). Since both $\mathscr{F}^N(v)$ and $\mathscr{F}^N(w)$ belong to R^N whose diameter less than α, we have

$$d^u(\mathscr{F}^N(v), g_{s_N}(\mathscr{F}^N(w))) \le \kappa d(\mathscr{F}^N(v), \mathscr{F}^N(w)) \le \kappa\alpha.$$

Combining both inequalities implies that $N_1 := \frac{4T}{\eta c_1} \cdot \log\Big(\frac{C(8\kappa+2)\kappa\alpha}{\epsilon(1+\beta)}\Big)$ gives $d(v, w) < \frac{\epsilon}{8\kappa+2}$ as required. $\square$

We will choose large N and use elements in $\mathcal{R}_N$ as the alphabet of the shift space. Given a bi-infinite sequence $(\cdots, a_{-1}, a_0, a_1, \cdots) = (a_i)_{i\in\mathbb{Z}}$ with $a_i \in \mathcal{R}_N$ for all $i \in \mathbb{Z}$, we follow the definition from [**10**] and call it N-*admissible* if for any $i \in \mathbb{Z}$, there exists $u_i \in a_i \cap \Lambda$ such that

$$\mathscr{F}(u_i) \in a_{i+1}.$$

Denote by $\mathcal{A}_N$ the set of all N-admissible sequences. Notice that the $\mathcal{A}_N$ naturally has the local product structure defined by $[a, b] := (\ldots, a_{-2}, a_{-1}, a_0, b_1, b_2, \ldots)$, where $a = (a_i)_{i\in\mathbb{Z}}$ and $b = (b_i)_{i\in\mathbb{Z}}$

with $a_0 = b_0$, and such a product structure will translate to the product structure on our desired locally maximal set $\widetilde{\Lambda}$.

For any $a = (a_i)_{i\in\mathbb{Z}} \in \mathcal{A}_N$ and $\epsilon > 0$, we call $w \in C$ an *ϵ-shadowing* of a if there exists $u_i \in \Lambda \cap a_i \cap \mathscr{F}^{-1}(a_{i+1})$ for each $i \in \mathbb{Z}$ such that

$$d(\mathscr{F}^i(w), u_i) < \epsilon.$$

We will show for any $\epsilon > 0$ sufficiently small, there exists $N_0 \in \mathbb{N}$ such that for any $N > N_0$, every element in $\mathcal{A}_N$ has a unique ϵ-shadowing, and such a shadowing map $\psi : \mathcal{A}_N \to C$ is injective. We first need the following result on the long-term hyperbolicity of $\mathscr{F}$ on $\mathcal{R}_N$.

Lemma 6.3.6. *There exists $N_2 \in \mathbb{N}$ such that for any $N > N_2$, $\delta \in \mathcal{R}_N$, $u \in \mathcal{D}$, and $w \in \mathcal{W}^s_\delta(u) \cap \mathcal{D}$, we have*

$$d(\mathscr{F}^N(u), \mathscr{F}^N(w)) \leq \frac{1}{2} d(u, w).$$

Similarly, for any $w \in \mathcal{W}^{cu}_\delta(u) \cap \delta$, we have

$$d(\mathscr{F}^{-N}(u), \mathscr{F}^{-N}(w)) \leq \frac{1}{2} d(u, w).$$

Proof of Lemma 6.3.6. By shrinking α if necessary and applying the continuity of λ_{T_Λ}, we assume that

$$\lambda_{T_\lambda}(g_t(u)) > \frac{3\eta_\Lambda}{4} \tag{6.3.9}$$

for any $N \geq 1$, $\delta \in \mathcal{R}_N$, $v \in \delta$, and $t \in [S^{-N}_\tau(v), S^{-N}_\tau(v)]$. This can be achieved because α is independent of T_Λ and η_Λ, and τ is continuous on $\mathcal{R}$ and bounded from above by 1.

Choose any $N \in \mathbb{N}$. We only prove the case where $w \in \mathcal{W}^s_\delta(u) \cap \mathcal{D}$. As in Lemma 6.3.4, there exists $s_j \in \mathbb{R}$ for each $-N \leq j \leq N$ with $|s_j| \leq \kappa\alpha\beta$ such that

$$g_{s_j}(\mathscr{F}^j(w)) \in W^s_\delta(\mathscr{F}^j(u)).$$

Then by (6.3.5), it suffices to show that

$$(1+\beta) d^s(\mathscr{F}^N(u), g_{s_N}(\mathscr{F}^N(w))) \leq (2\kappa)^{-1} d^s(u, g_{s_0}(w)).$$

By setting $N_2 := \dfrac{4T}{\eta c_1} \cdot \log(2C\kappa(1+\beta))$ where $C = C(T_\Lambda, \eta_\Lambda)$ is from Lemma 6.3.1, the above inequality follows from

$$d^s(\mathscr{F}^N(u), g_{s_N}(\mathscr{F}^N(w))) \leq C \exp\Big(-\frac{\eta N c_1}{4T}\Big) d^s(u, g_{s_0}(w))$$

obtained as in the proof of Lemma 6.3.5. $\square$

Now we are ready for the construction of shadowing map ψ. Set

$$\delta := d\Big(\Lambda \cap C, C \setminus \bigcup_{j=1}^{m} R_j\Big),$$

which is necessarily positive from the construction of $\mathcal{R}$. Fix $\epsilon \in (0, \delta/6\kappa)$ and $N_0 \in \mathbb{N}$ such that $N_0 > \max\{N_1(\epsilon), N_2\}$.

With such choice of δ, whenever $u \in R_i$ for some i and $v \in C$ satisfies $d(u,v) < \delta$, then v also belongs to R_i. Moreover, from the choice of N_0 and Lemma 6.3.5, whenever we have $u, v \in C$ such that $\mathscr{F}^i(u)$ and $\mathscr{F}^i(v)$ belong to the same element of $\mathcal{R}$ for all $-N_0 \leq i \leq N_0$, then u and v belong to the element of $\mathcal{R}_{N_0}$, and hence $d(u,v) < \epsilon$. We will use these facts repeatedly in what follows.

Lemma 6.3.7. *Any $2N_0$-admissible sequence has a unique 4ϵ-shadowing. Moreover, the shadowing map $\psi : \mathcal{A}_{2N_0} \to C$ is injective.*

Proof of Lemma 6.3.7. Let $a = (a_i)_{i=-\infty}^{\infty}$ be $2N_0$-admissible. For each $i \in \mathbb{Z}$, let $u_i \in a_i \cap \Lambda$ such that $\mathscr{F}(u_i) \in a_{i+1}$ and R^i be the element of $\mathcal{R}$ containing a_i.

We will first show any finite segment of a has a shadowing which is not necessarily unique. Consider $(a_i)_{i=0}^{rN_0}$ for some $r \in \mathbb{N}$, and define

$$v_{N_0} := [\mathscr{F}^{N_0}(u_0), u_{N_0}]_C \in R^{N_0}.$$

By Lemma 6.3.4, $\mathscr{F}^j(v_{N_0})$ is equal to $[\mathscr{F}^{N_0+j}(u_0), \mathscr{F}^j(u_{N_0})]_C$,and hence, belongs to R^{N_0+j} for all $j \in [-2N_0, N_0]$. In particular, $\mathscr{F}^j(v_{N_0})$ belongs to the same element of $\mathcal{R}$, namely R^j, as $\mathscr{F}^{N_0+j}(u_0)$ for all such j. Lemma 6.3.5 then implies that for all $j \in [-N_0, 0]$, we have

$$d(\mathscr{F}^j(v_{N_0}), \mathscr{F}^{N_0+j}(u_0)) < \epsilon.$$

Due to the same reasoning, we have $d(\mathscr{F}^j(u_0), u_j) < \epsilon$ for all $j \in [0, N_0]$, and the triangle inequality gives

$$d(\mathscr{F}^j(v_{N_0}), u_{N_0+j}) < 2\epsilon$$

for all such j.

Meanwhile, by $d(v_{N_0}, u_{N_0}) < \epsilon$, $\mathscr{F}^j(v_{N_0}) \in R^{N_0+j}$ for all $j \in [-N_0, N_0]$ and $N_0 > N_2$, we have

$$d(\mathscr{F}^{N_0}(v_{N_0}), \mathscr{F}^{N_0}(u_{N_0})) < \epsilon/2. \tag{6.3.10}$$

We want to show

LEMMA 6.3.8. $\mathscr{F}^{N_0+j}(v_{N_0}) \in R^{2N_0+j}$ *for all* $j \in [0, N_0]$.

PROOF OF LEMMA 6.3.8. We know both $\mathscr{F}^{N_0}(u_{N_0})$ and $\mathscr{F}^{N_0}(v_{N_0})$ are in R^{2N_0}. As in Lemma 6.3.4, there exists $s_0 \in \mathbb{R}$ with $|s_0| < \kappa\alpha\beta$ such that $g_{s_0}(\mathscr{F}^{N_0}(v_{N_0})) \in W^s_\delta(\mathscr{F}^{N_0}(u_{N_0}))$. From (6.3.10), we have

$$d^s(g_{s_0}(\mathscr{F}^{N_0}(v_{N_0})), \mathscr{F}^{N_0}(u_{N_0})) < \kappa\epsilon/2.$$

The fifth defining property of $\mathcal{R}$ listed above Lemma 6.3.4 shows that both $\mathscr{F}^{N_0+1}(u_{N_0})$ and $\mathscr{F}^{N_0+1}(v_{N_0})$ are in the same C_i for some i, so there exists $s_1 \in \mathbb{R}$ with $|s_1| < \kappa\beta\delta$ such that $g_{s_1}(\mathscr{F}^{N_0+1}(v_{N_0})) \in W^s_\delta(\mathscr{F}^{N_0+1}(u_{N_0}))$. Then we have

$$\begin{aligned} d(\mathscr{F}^{N_0+1}(v_{N_0}), \mathscr{F}^{N_0+1}(u_{N_0})) &\leq (1+\beta) d^s(g_{s_1}(\mathscr{F}^{N_0+1}(v_{N_0})), \mathscr{F}^{N_0+1}(u_{N_0})) \\ &\leq (1+\beta) d^s(g_{s_0}(\mathscr{F}^{N_0}(v_{N_0})), \mathscr{F}^{N_0}(u_{N_0})) \\ &< (1+\beta)\kappa\epsilon/2 \\ &< \Delta. \end{aligned}$$

Therefore, $\mathscr{F}^{N_0+1}(v_{N_0})$ also belongs to R^{2N_0+1} from the defining property of Δ.

Now we can repeat the argument from above by starting from $j = 1$. As d^s is not increasing under g_t for $t \geq 0$, the above technique applies to all $2 \leq j \leq N_0$ to conclude the proof. $\square$

Continuing with the proof of Lemma 6.3.7, Lemma 6.3.5 and 6.3.8 show that $d(\mathscr{F}^j(u_{N_0}), \mathscr{F}^j(v_{N_0})) < \epsilon$ for all $j \in [0, N_0]$, which implies that

$$d(u_{N_0+j}, \mathscr{F}^j(v_{N_0})) < 2\epsilon$$

for all $j \in [0, N_0]$ thus for all $j \in [-N_0, N_0]$.

We follow the spirit of the proof of the classic shadowing lemma and repeat the above process. Define

$$v_{2N_0} := [\mathscr{F}^{N_0}(v_{N_0}), u_{2N_0}]_C.$$

As both $\mathscr{F}^{j-2N_0}(v_{2N_0})$ and $\mathscr{F}^{j-N_0}(v_{N_0})$ are in R^j for $j \in [0, 3N_0]$, we have

$$d(\mathscr{F}^{j-2N_0}(v_{2N_0}), \mathscr{F}^{j-N_0}(v_{N_0})) < \epsilon$$

for $j \in [N_0, 2N_0]$. Taking $j = 2N_0$ and applying Lemma 6.3.6, for $j \in [0, N_0]$ we have

$$d(\mathscr{F}^{j-2N_0}(v_{2N_0}), \mathscr{F}^{j-N_0}(v_{N_0})) < \epsilon/2.$$

Similar to Lemma 6.3.8 we have $\mathscr{F}^{j-2N_0}(v_{2N_0})$ belong to R^j for $j \in [3N_0, 4N_0]$, which implies that

$$d(\mathscr{F}^{j-2N_0}(v_{2N_0}), u_j) < 2\epsilon$$

for all $j \in [2N_0, 3N_0]$.

We will proceed the above process one more time and make the general statement. Define $v_{3N_0} := [\mathscr{F}^{N_0}(v_{2N_0}), u_{3N_0}]_C$. As both $\mathscr{F}^{j-3N_0}(v_{3N_0})$ and $\mathscr{F}^{j-2N_0}(v_{2N_0})$ are in R^j for $j \in [N_0, 4N_0]$, we have

$$d(\mathscr{F}^{j-3N_0}(v_{3N_0}), \mathscr{F}^{j-2N_0}(v_{2N_0})) < \epsilon$$

for $j \in [2N_0, 3N_0]$. Similarly by Lemma 6.3.6 we have

$$d(\mathscr{F}^{j-3N_0}(v_{3N_0}), \mathscr{F}^{j-2N_0}(v_{2N_0})) < \epsilon/2$$

for $j \in [N_0, 2N_0]$. In particular, we have

$$
\begin{aligned}
&d(\mathscr{F}^{-2N_0}(v_{3N_0}), \Lambda) \\
&< d(\mathscr{F}^{-2N_0}(v_{3N_0}), \mathscr{F}^{-N_0}(v_{2N_0})) + d(\mathscr{F}^{-N_0}(v_{2N_0}), v_{N_0}) + d(v_{N_0}, \mathscr{F}^{N_0}(u_{N_0})) \\
&< \frac{\epsilon}{2} + \epsilon + \epsilon < 3\epsilon.
\end{aligned} \tag{6.3.11}
$$

Therefore, following the proof of Lemma 6.3.8 and using $3(1+\beta)\kappa\epsilon < 6\kappa\epsilon < \Delta$, we have $\mathscr{F}^{-3N_0+j}(v_{3N_0}) \in R^j$ for $j \in [0, N_0]$, which in turns shows that

$$
d(\mathscr{F}^{-3N_0+j}(v_{3N_0}), \mathscr{F}^{-2N_0+j}(v_{2N_0})) < \epsilon/4 \text{ for all } j \in [0, N_0]. \tag{6.3.12}
$$

Similar to the case of v_{2N_0} we also have $d(\mathscr{F}^{j-3N_0}(v_{3N_0}), u_j) < 2\epsilon$ for all $j \in [3N_0, 4N_0]$ and $\mathscr{F}^{j-3N_0}(v_{3N_0}) \in R^j$ for all $j \in [4N_0, 5N_0]$.

Now we can generalize the whole process.

LEMMA 6.3.9. *Suppose we have constructed v_{iN_0} for all $i \in [1, k]$ with some $k \in [3, r-1]$. For all such i, we have*

(1) $v_{iN_0} = [\mathscr{F}^{N_0} v_{(i-1)N_0}, u_{iN_0}]_C$

(2) $\mathscr{F}^{j-iN_0}(v_{iN_0}) \in R^j$ *for* $j \in [0, \max\{(i+2)N_0, rN_0\}]$.

(3) $d(\mathscr{F}^{j-iN_0}(v_{iN_0}), \mathscr{F}^{j-(i-1)N_0}(v_{(i-1)N_0})) < 2^{[\frac{j-1-iN_0}{N_0}]+1}\epsilon$ *for* $j \in [0, iN_0]$.

(4) $d(\mathscr{F}^{j-iN_0} v_{iN_0}, u_j) < 2\epsilon + \sum_{l=0}^{i-1} \frac{\epsilon}{2^l} < 4\epsilon$ *for* $j \in [0, iN_0]$.

(5) $d(\mathscr{F}^{j-iN_0} v_{iN_0}, u_j) < 2\epsilon$ *for* $j \in [iN_0, (i+1)N_0]$.

Define $v_{(k+1)N_0} := [\mathscr{F}^{N_0} v_{kN_0}, u_{(k+1)N_0}]_C$.

We know $\mathscr{F}^{j-(k+1)N_0} v_{(k+1)N_0} \in R^j$ for $j \in [(k-1)N_0, (k+2)N_0]$ and

$$
d(\mathscr{F}^{j-(k+1)N_0}(v_{(k+1)N_0}), \mathscr{F}^{j-kN_0}(v_{kN_0})) < \epsilon
$$

for $j \in [kN_0, (k+1)N_0]$. By Lemma 6.3.6 we have $d(\mathscr{F}^{j-(k+1)N_0}(v_{(k+1)N_0}), \mathscr{F}^{j-kN_0}(v_{kN_0})) < \epsilon/2$ for $j \in [(k-1)N_0, kN_0]$. As in (6.3.11) we know

$$
d(\mathscr{F}^{-2N_0}(v_{(k+1)N_0}), \Lambda) < 3\epsilon,
$$

which by the proof of Lemma 6.3.8 shows that $\mathscr{F}^{j-(k+1)N_0}(v_{(k+1)N_0}) \in R^j$ for $j \in [(k-2)N_0, (k-1)N_0]$ and $d(\mathscr{F}^{j-(k+1)N_0}(v_{(k+1)N_0}), \mathscr{F}^{j-kN_0}(v_{kN_0})) < \epsilon/4$ for $j \in [(k-2)N_0, (k-1)N_0]$. By applying property (3) from above, this process can be repeated so that both property (3) and (4) can be realized in the case of $i = k+1$. Property (5) is derived in the same way as in Lemma 6.3.8.

As a result, we are able to construct some $v_0^{rN_0} \in T^1S$ such that $d(\mathscr{F}^i(v_0^{rN_0}), u_i) < 4\epsilon$ for all $i \in [0, rN_0]$.

Now for each $r \in \mathbb{N}$, with given u_i we have a 4ϵ-shadowing of $(a_i)_{i=-r}^{r}$ as v_{-r}^{r}. $\mathscr{F}^r(v_{-r}^{r})$ is a Cauchy sequence on R^0, therefore has a unique limit, say v_0. Apparently v_0 is a 4ϵ-shadowing of $(a_i)_{i=-\infty}^{\infty}$. It is also not hard to see that v_0 does not depend on the choice of $\{u_i\}_{i\in\mathbb{Z}}$. In fact, if we have some $\{u_i'\}_{i\in\mathbb{Z}}$ which ends up with a 4ϵ-shadowing v_0', then both $\mathscr{F}^i(v_0)$ and $\mathscr{F}^i(v_0')$ are in R^i for all $i \in \mathbb{Z}$. By Lemma 6.3.5, $v_0 = v_0'$. The map ψ (which is obviously continuous) and injective as elements in $\mathcal{R}$ are disjoint. This completes the proof of Lemma 6.3.7. □

We are now ready to conclude the proof of Proposition 6.3.3 by constructing $\widetilde{\Lambda}$ as follows:

$$\widetilde{\Lambda} := \bigcup_{t\in\mathbb{R}} g_t(\psi(\mathcal{A}_{2N_0})).$$

We claim that $\widetilde{\Lambda}$ is the desired set satisfying the statements of Proposition 6.3.3. Indeed, every vector in Λ is contained in $\mathcal{A}_{2N_0}$ and mapped to itself by ψ, so $\Lambda \subset \widetilde{\Lambda}$. From its construction, $\widetilde{\Lambda}$ is G-invariant and compact as it is the image of a compact set under the continuous map ψ. Moreover, $\widetilde{\Lambda}$ is uniformly hyperbolic as it contained in U, and it is locally maximal because it inherits the local product structure of $\mathcal{A}_{2N_0}$. This completes the proof of Proposition 6.3.3. □

We also need the following result on gluing basic sets together, which parallels Proposition 9 in [**10**].

Lemma 6.3.10. *Given any basic sets Λ^1 and Λ^2 in T^1S, there is a third basic set Λ that contains both of them.*

Proof. Since both Λ^1 and Λ^2 are basic sets, we are able to find $v_i \in \Lambda^i$ such that both forward and backward G-orbit of v_i are dense in Λ^i with $i \in \{1, 2\}$. As in the proof of Proposition 6.3.2, we might assume $\Lambda^i \subseteq \mathrm{Reg}_T(\eta)$ for $i \in \{1, 2\}$ (Notice that T and η here are unrelated to those used in the proof of Proposition 6.3.3). Therefore, by definition we have $\int_{-T}^{T} \lambda(g_t(v_i))dt \geq \eta$ for $i \in \{1,2\}$. In particular, there is some $t_i \in [-T,T]$ such that $\lambda(g_{t_i}(v_i)) \geq \frac{\eta}{2T}$ for $i \in \{1,2\}$. Fix $\delta > 0$ small such that $B(\mathrm{Reg}(\frac{\eta}{2T}), \delta) \subseteq \mathrm{Reg}(\frac{\eta}{4T})$. Since G is transitive on T^1S, We can find some $v', v'' \in T^1S$ and $t', t'' > 0$ such that $d(v', g_{t_1}(v_1)) < \delta$, $d(g_{t'}(v'), g_{t_2}(v_2)) < \delta$, $d(v'', g_{t_2}(v_2)) < \delta$ and $d(g_{t''}(v''), g_{t_1}(v_1)) < \delta$. By definition of δ we have $v', g_{t'}(v'), v'', g_{t''}(v'') \in \mathrm{Reg}(\frac{\eta}{4T})$. Then by Lemma 6.2.5 we have two orbits w_1 and w_2 as follows. The orbit of w_1 is backward asymptotic to the orbit of v_1, forward asymptotic to the orbit of v_2 and in the middle close to the orbit segment (v', t'). Similarly, the orbit of w_2 is backward asymptotic to the orbit of v_2, forward asymptotic to the orbit of v_1 and in the middle close to the orbit segment (v'', t''). Let us write Λ^3 as the union of Λ^1, Λ^2, the orbit of w_1 and the orbit of w_2. Λ^3 is closed since the orbits of w_1 and w_2 are forward and backward asymptotic to orbits in Λ^1 and Λ^2, which are by themselves compact. Moreover, notice that the orbits of w_1 and w_2 are both in Reg. As $\Lambda^3 \subset \mathrm{Reg}$ is closed and G-invariant, by Proposition 6.3.2 we know Λ^3 is hyperbolic. By Proposition 6.3.3 there is a basic set Λ^4 that contains Λ^3.

Now we will show Λ^3 is in $\Omega_G(\Lambda^4)$, which refers to the non-wandering set of Λ^4 under G. As Λ^4 is a basic set, thus being locally maximal, there is an open neighborhood V of λ_4 such that $\Lambda^4 = \bigcap_{t\in\mathbb{R}} g_t(V)$. Write $\delta_1 := \mathrm{dist}(V^c, \Lambda^3)$. We know the orbit of w_1 is forward asymptotic to the orbit of v_2, which is forward dense in Λ_2. Meanwhile, the orbit of w_2 is backward asymptotic to the orbit of v_2, which is backward dense in Λ_2. Therefore, there are $T_1, T_2 > 0$ such that $d(g_{T_1}(w_1), g_{-T_2}(w_2)) \ll \delta_1$ and both $g_{T_1}(w_1)$ and $g_{-T_2}(w_2)$ belong to $\mathrm{Reg}(\frac{\eta}{4T})$. Similarly we can find $T_1', T_2' > 0$ such that $d(g_{-T_1'}(w_1), g_{T_2'}(w_2)) \ll \delta_1$ and both $g_{-T_1'}(w_1)$ and $, g_{T_2'}(w_2)$ belong to $\mathrm{Reg}(\frac{\eta}{4T})$. We can now shadow $(g_{-T_1'}(w_1, T_1' + T_1))$ and $(g_{-T_2}(w_2, T_2' + T_1))$ using a closed geodesic w_3. Since the orbit of w_3 is always in V, it must be in Λ^4. Therefore, The orbit of w_1 and w_2 are non-wandering for Λ^4 under G. As Λ^1 and Λ^2 are by themselves basic sets, we have $\Lambda^3 \subseteq \Omega_G(\Lambda^4)$.

Meanwhile, the same process as above shows that Λ^3 is transitive under G. Therefore, by spectral decomposition of $\Omega_G(\Lambda^4)$ we can find a basic set that contains Λ^3, which concludes the proof. $\square$

We will now apply Proposition 6.3.3 and Lemma 6.3.10 to construct $\{\widetilde{\Lambda_i}\}_{i\in\mathbb{N}}$ mentioned in Proposition 6.2.3, thus complete its proof.

First define $\{\Lambda_n\}_{n\in\mathbb{N}}$ as $\Lambda_n = \overline{\mathrm{Per}(T^1S \setminus B(\mathrm{Sing}, \frac{1}{10n}))}$, where $\mathrm{Per}(E)$ is the set of closed geodesics whose entire orbit lies in E for $E \subset T^1S$ and $B(\mathrm{Sing}, \delta)$ refers to the set of all the points whose distance to Sing is less than δ for $\delta > 0$. Each Λ_n is closed and G-invariant in Reg, thus being hyperbolic. By Proposition 6.3.3 there exists G-invariant compact hyperbolic locally maximal Λ'_n that contains Λ_n. Let us look at $\Omega_G(\Lambda'_n)$. Observe that $\Omega_G(\Lambda'_n)$ is closed and contains every closed geodesic in Λ_n, therefore contains Λ_n itself. By applying spectral decomposition we can write $\Omega_G(\Lambda'_n) = \bigcup_{j=1}^m \Lambda_n^j$, where $m \geq 1$ and Λ_n^j is a basic set for all $1 \leq j \leq m$. Now we can apply Lemma 6.3.10 at most $m-1$ times to construct $\widetilde{\Lambda_i}$ that we need. Moreover, by applying Lemma 6.3.10 if necessary, we can make $\widetilde{\Lambda}_i$ an increasingly nested sequence.

It remains to show that for any basic set $\Lambda \subset T^1S$, there is $n \in \mathbb{N}$ such that $\Lambda \subseteq \widetilde{\Lambda_n}$. First notice that there exists an increasingly nested sequence of open sets $\{V_n\}_{n\in\mathbb{N}}$ satisfying $\widetilde{\Lambda_n} = \bigcap_{t\in\mathbb{R}} g_t(V_n)$. As Λ is a basic set, it is the closure of all closed geodesics in itself. Therefore, $\Lambda \subset \bigcup_{n\in\mathbb{N}} \Lambda_n \subset \bigcup_{n\in\mathbb{N}} \widetilde{\Lambda_n} \subset \bigcup_{n\in\mathbb{N}} V_n$ and $\{V_n\}_{n\in\mathbb{N}}$ becomes an open cover of Λ, which admits a finite cover. Now we have some $n \in \mathbb{N}$ such that $\Lambda \subset V_n$. Since Λ is G-invariant, it is in $\widetilde{\Lambda_n}$, which ends our construction of $\{\widetilde{\Lambda_i}\}_{i\in\mathbb{N}}$. This concludes the proof of Proposition 6.2.3.

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