HELPING STUDENTS AFFECTED WITH MATHEMATICS DISORDERS LEARN MATHEMATICS

A Thesis

Presented in Partial Fulfillment of the Requirements for the Degree Master of Mathematical Sciences in the Graduate School of the Ohio State University

By

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2020

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Abstract

This research focusses on the various forms of mathematics learning disorders that afflict many students from an early age, often adversely affecting not only their academic achievement, but their lives, and some of the tools and methods that are available to help overcome these afflictions. In closing, a case study illustrates how Learning Progressions may be used as a tool to help an adult, affected by math learning disorders most of her life overcome her math anxiety and start to enjoy learning mathematics.

This work is dedicated to my wife, Lisa, without whom none of this would have been possible.

Acknowledgments

My sincerest gratitude goes to my advisor Rodica Costin, and my co-advisor Michael Battista whose brilliant insights and advice were invaluable throughout this research project. I am especially thankful to my co-advisor Michael Battista, whose courses on mathematical learning progressions and conceptual and procedural knowledge inspired this project and provide a promising direction for future research. I would also like to thank my office mates of Math Tower 350 and 200, whose constant friendship and humor have made this program unforgettable: Erin Gray, Enrique Gomez, Nathan Bushman, and Joe Antonides.

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Chapter 1 - Introduction

Many children with mathematics learning disorder have a persistent deficit in the ability to store number combinations in or retrieve them from long-term memory (Geary, 2005). It has been demonstrated that mathematics learning difficulties are also often tied to gender, meaning girls are more likely to be affected than boys, via the bias of educational methods and stereotype threats (Boaler, 2006).

Children who exhibit mathematics learning difficulties include the students who perform at a level of performance often four to five years behind the level of performance of same-aged peers that are not afflicted with mathematics learning difficulties, resulting in the low average range. Thus, eight and nine-year old students with learning disabilities perform at about the first-grade level on calculations and application, while 16-year old students with learning disabilities score at about the fifth-grade level (Carnine, 1997).

Studies show that students with learning disabilities experience greater difficulty than their same-aged peers with both computation and word problems. This situation exists despite the fac that the bulk of instructional time is spent learning procedures for doing mathematics computation and simple word problems (Goldman, 1997).

Multiple studies show that methods exist that can help these students overcome their mathematics learning difficulties. Learning progressions, scaffolding and explicit instruction are some of the supportive methods investigated as part of this research. Exploring how Learning Progressions can be used is illustrated by a study performed with an adult female afflicted with a life-long math learning disorder.

The emphasis on *all* students needing to be able to solve problems, reason, and take charge of their learning should be cause for celebration, because students with learning disabilities are not being excluded from opportunities to develop the life skills needed for success (Goldman, 1997).

Chapter 2 – Different types of Math Learning Difficulties

Research shows that mathematics learning difficulties manifest in different forms. The researcher Kate Gardner refers to math difficulties as Math Learning Disabilities (MLD) in her 1998 Article by the same name, or dyscalculia, which with MLD may be used synonymously of one another (Mazzocco, 2011). In their article, *Early Identification and Interventions for Students with Learning Difficulties*, Russel Gersten, Nancy Jordan, and Jonathan Flojo use the term mathematics difficulties rather than mathematics disabilities.

Geary et al. (2000) found that for many children, mathematics difficulties are not stable over time, identifying a group of "variable" children who showed mathematics difficulties on a standardized test in first grade but not in second grade, noting that it is likely that some of these children outgrew their developmental delays, whereas others could have been misidentified to begin with (Gersten, 2005).

Another aspect of MLD resides in the fact that many children have trouble bridging informal math knowledge to formal school math. To build these connections takes time, experiences, and carefully guided instruction. The use of structured, concrete materials is important to securing these links, not only in the early elementary grades, but also during concept development stages of higher-level math.

An extremely handicapping, though less common math disability derives from significant visual-spatial-motor disorganization. The formation of foundation math concepts is impaired in this small subgroup of students. The organizational and social problems that accompany this math disability are also in need of long-term appropriate remedial attention in order to support successful life adjustment in adulthood (Garnett, 1998).

Additionally, recent evidence points to an often-ignored factor that may shape how well students are able to benefit from learning opportunities: math anxiety – the fear of, or apprehension about, math (Foley, 2017).

Chapter 3 – Different Causes for Math Learning Difficulties

One possible origin for MLD, according to Mazzocco et al. in their 2011 research, could be a deficit in the Approximate Number System (ANS) that supports nonverbal numerical representations which appears in early development and is universally shared among humans, and emerges without explicit instruction. This is a student's ability to rapidly estimate a certain quantity, either in terms of relationship, such as 7 is greater than 5, or as estimating the relative size of groups of numbers, dots, or in the use of the unmarked number line (Battista, 2020).

Note: See Appendix A of this document for more information pertaining to Approximate Number System (ANS).

Another cause for MLD could be the mismatch between the students' learning characteristics and the design of instruction materials and practices, meaning the poor fit between the design of mathematics instructional materials and the students' learning characteristics, such as memory skills, strategy acquisition and application, vocabulary, and language coding. Design features in instructional materials that lead to this mismatch include too rapid rate of introduction of new concepts, insufficiently supported explanations and activities, and insufficient practice and review. This certainly would aggravate any anxiety the student experiences when facing mathematical problems, affecting the student as he or she contemplates signing up for future math courses, because when concepts are introduced rapidly with minimal explanations and sparse practice and review, students with learning disabilities may be overwhelmed by memorization, strategies, vocabulary, and language coding; leading to a steadily growing gap in mathematics attainment between general and special education students (Carnine 1997).

Another potential cause of difficulty may reside with students' confusion about the conventions of written math notation that are sustained by the practice of using workbooks and ditto pages filled with problems to be solved. In these formats, students learn to act as problem answerers rather than demonstrators of math ideas, such as when students interpret the equal sign to mean compute. Students who show particular difficulty ordering math symbols in the conventional vertical, horizontal, and multistep algorithms need much experience translating from one form to another (Garnett, 1998).

In the next chapter we will see how girls and women can be affected by MLD because of the way mathematics is instructed, and the fact that our society has been male dominated for so long. From her research, Jo Boaler (2006) concludes that the emphasis on knowledge-transmission creates a system of mathematics education in which students are taught to mimic experts.

Slowness or inability to master an isolated skill may prevent a learner from being presented with an opportunity to fully engage in a complete task. This has two negative consequences for learners: They never get to see the complete task and understand where the skills fit in, and they are prevented from attempting the complete task because it is assumed that they do not have the incrementally developed skills to perform the task. This very often happens in school tasks. The research literature on writing by students with learning disabilities is a good example (Goldman, 1997).

Even when declarative and procedural knowledge are in place, students with learning disabilities often fail to apply that knowledge in meaningful ways when confronted with problem situations. When confronted with a problem, students with disabilities typically do not use the mathematical knowledge they have acquired to solve problems unless they are explicitly informed about the relationship between the knowledge and the problem (Goldman, 1997).

Studies show that students with learning disabilities experience greater difficulty than their same-aged peers with both computation and word problems. This situation exists in spite of the bulk of instructional time being spent learning procedures for doing mathematics computation and simple word problems (Goldman, 1997).

3.1 – Gender Related Math Learning Difficulties

According to the National Center for Education Statistics, during the 2017 to 2018 school year, 17 percent of male students ages 6 to 21 received special services under the Individuals With Disabilities Act (IDEA), compared to 9 percent of female students benefiting from these services (Haddad, 2019). Is this discrepancy warranted? Do more boys get special education support than girls because more boys actually *have* more Learning Disabilities? Or is that they're *perceived* by teachers and other education professionals to have more LDs? There's research to suggest the latter: According to Understood.org, studies have found that based on scientific criteria, there is no gender gap when it comes to learning problems. It's just that teachers recommend twice as many boys as girls for LD support (Haddad, 2019).

But are there still gender differences in math? It depends on which math outcomes we look at. At both elementary and secondary levels, boys and girls score similarly on many state tests, and girls get relatively good grades in math classes. However, some gender differences in math attitudes and skills appear during elementary school, and ultimately, boys are much more likely than girls to pursue careers in some key math-intensive fields, such as engineering and computer science (Ganley, 2016). In recent years, concerns about boys and reading have taken some attention away from girls and math, as girls have higher reading achievement than boys in early elementary school. However, it is important to consider that research shows that reading gender gaps narrow during the elementary grades, whereas gender gaps in math grow during early elementary school. In general, gender differences in math performance are small, which is important to keep in mind. Gender differences on math tests tend to be more pronounced when the content of the assessment is less related to the material that is taught in school (for example, on the SAT-Math as opposed to a math test in school). In addition, researchers consistently find that gender gaps are larger among higher-performing students, which may partially explain why we see gender gaps in math-related careers, as these are often pursued by the highest-performing students (Ganley, 2016).

In her 2006 article *In Gender and Education,* Jo Boaler asserts that the long held belief in Western Society that girls are not as good as boys in mathematics and science continues despite the increased performance of girls who achieve at the same or higher levels than boys in many countries. Unfortunately, this translates into girls having lower self-images in mathematics as well as beliefs that mathematics is more suited for males, resulting in fewer women and girls being interested in pursuing studies and a career in the mathematical field, in turn leading to a situation where girls and women are systematically discouraged from entering mathematical and scientific fields.

So which issues are girls and women facing in the mathematics classroom that result in the low participation of girls and women in the field of mathematical sciences? Jo Boaler (2006) hypothesizes that the 1980 Benbow and Stanley report significantly lowered mother's expectations of their daughters' potential with mathematics, and that this could be one reason. Indeed, in their 1980 report, Benbow and Stanley favor the hypothesis that sex differences in achievement in, and attitude toward, mathematics result from superior male mathematics ability, which may in turn be related to greater male ability in spatial tasks. They also assert that less well-developed mathematical reasoning ability contributes to girls taking fewer mathematics courses and achieving less than boys. These claims by Benbow and Stanley can only be accepted if one considers that Western culture has been dominated by men for most, if not all, of its history. The prevailing notion that girls are innately inferior in mathematics is certainly a stumbling block in the efforts to provide girls with equal access to advanced mathematical fields despite the fact that studies of brain processing do not support the idea that girls and women are genetically inferior to boys and men in their mathematical potential (Boaler, 2006).

Heather Mendick produced a line of work exploring the tensions faced by young women who study mathematics. Mendick found strongly gendered perceptions of mathematics that influenced young women’s decisions to continue in the discipline (Boaler, 2006). The 1980 Benbow and Stanley’s report states that a substantial sex difference in mathematical reasoning ability score on the mathematics test of the Scholastic Aptitude Test in favor of boys was found in a study of 9927 intellectually gifted junior high school students.

Benbow and Stanley (1980) further found that girls excel in computation, while boys excel on tasks requiring mathematical reasoning ability and that the greatest disparity between the girls and boys is in the upper ranges of mathematical reasoning ability. But this could also be tied to stereotype threat. Stereotype threat is the situation in which members of a group are, or feel themselves to be, at risk of confirming a negative stereotype about their group (Carey, 2016). In the domain of mathematics anxiety, this usually refers to females being reminded of the stereotype that males are better at mathematics than females (Dowker, 2016).

Researchers have studied the differences between boys and girls in categories such as anxiety, fear of success, confidence, self-concept, motivation and perceptions of the usefulness of mathematics and research has found that girls differentially experience and engage in mathematics classes, and that the achievement of boys in mathematics classes was influenced more by prior achievement and value of the subject than girls, who were influenced more by their perceptions of the difficulty of the subject and the idea that it is a subject where males dominate (Boaler, 2006).

Factors that influence gender related math learning difficulties

In her 2006 research, Jo Boaler found that researchers need to consider many different factors that influence the performance and participation of girls and boys. Issues such as under-confidence, sexism in teaching environments, subject distortions and different cognitive preferences may all play a part in low performance and participation.

Boaler established that ideas about female inadequacy are perpetuated, and that tendencies such as lack of confidence, anxiety and failure attributions are generally presented as properties of girls, rather than as responses coproduced by particular working environments. This results in the proposal of interventions aimed to change the girls so that they become less anxious, more confident – *essentially* more masculine. In such programs, the responsibility for change is laid at the feet of the girls, and problems with mathematical pedagogy and practice, and with the broader social system, are not addressed. These include the achievement and participatory patterns that are explained through comparisons with boys where boys' achievements, participation and behaviors are implicitly positioned as "normative" and the benchmark against which girls are understood.

Exploring the teaching environments and curricular materials area of research, Boaler found that the possibility is quite real that sex stereotyping in textbooks or sex-based discrimination by teachers affects the performance and participation of girls and women, thus the 'problem' that requires intervention is no longer the girls but the textbooks and teacher-student interactions that girls are exposed to in their mathematics classes. Typical findings in this area of research show that sex-differences prevail in the interactions of students and teachers that generally favors boys, with boys receiving more attention, reinforcement and positive feedback from mathematics teachers (Boaler, 2006).

In closing, Boaler points out that boys' achievement has been attributed to something within – the nature of their intellect – but their failure has been attributed to something external – a pedagogy, methods, texts, teachers. The full significance of this becomes clear when the subject of the discourse is girls, for in their case their failure is attributed to something within – usually the nature of their intellect – and their success to something external: methods, teachers or particular conditions (Boaler, 2006).

Possible remedies to correct the factors that influence gender related math learning difficulties

The research that increased the awareness of teachers about gendered interactions in classrooms in the 1980's and later years most certainly increased the access girls now have to mathematics in classrooms (Boaler, 2006).

In the mathematics classroom, this translated into advocating for a refocusing of the equity lens away from individuals and categories of people, and onto a system that co-produces difference. This refocusing involved departing from the essentialism of categories evident in claims that girls are 'maladaptive' or conceptually lacking, and committing to careful explorations of the circumstances that produced differences between groups, so rather than try to change the girls, much might be gained by trying to change the teaching environments in which students are working – environments that produce motivational patterns that are unproductive for learning, and in so doing, the tendencies of the girls may be viewed as highly adaptive (Boaler, 2006).

In their 2016 article *What Can We Do about Gender Differences in Math*, Colleen Ganley and Susan Lubienski state that interventions to support girls in math should begin earlier and be part of the regular classroom. Given that gender differences in math achievement, and especially attitudes, develop early, programs targeting gender disparities in middle and high school might be too late, and that we should encourage the development of girls' spatial and problem-solving skills. Teachers should pose problems that cannot be solved in routine ways and encourage diverse methods of problem solving. Throughout the year, teachers should attempt to notice and praise girls when they find novel ways to solve problems, as opposed to relying on procedures they have been taught (even if those procedures produce the expected, right answer). Teachers can also find ways to connect spatial thinking to mathematics content and encourage both girls and boys to tap into this type of thinking (Ganley, 2016).

Girls' perceptions of themselves as mathematicians need to be improved. Given that girls' math anxiety and lack of confidence are critical predictors of later attitudes, achievement, and career choices, elementary school teachers should strive to enhance girls' mathematical confidence. Female teachers can begin by displaying such confidence in the classroom themselves, modeling a spirit of exploration and curiosity in the face of mathematical uncertainty. Additionally, given research suggesting that girls gravitate toward professions that "help people," teachers should expose their students to ways that math-intensive careers (e.g., engineering) can, indeed be helpful to others (Ganley, 2016).

It appears that listening to what girls have to say regarding their approach to learning mathematics would be beneficial to helping make the change in the classroom and the teaching practices, both in the way the mathematics curriculum is delivered and what girls bring to mathematics.

3.2 – Math Anxiety

Students affected by math anxiety (MA) are not limited to the United States: Data from the Program for International Student Assessment (PISA), which tests 15-year-olds' academic achievement worldwide, shows that math anxiety is negatively related to math performance both within and across countries, and students reporting higher levels of math anxiety display lower levels of math performance than their peers who report lower levels of math anxiety (Foley, 2017).

In their 2019 Article entitled *Understanding Mathematics Anxiety*, Szücs, McLellan, and Douker describe math anxiety as a debilitating emotional reaction to mathematics that is increasingly recognized in psychology and education. But Carey notes that some believe math anxiety is just another name for 'bad at math'.

Carey, Hill, Devine, and Szücs (2016) report that adults with math anxiety are less likely to enroll into college or university courses involving mathematics, and that even in young children, task-avoidant behaviors have been found to reduce mathematics performance (p 3). Dowker, Sarkar and Looi (2016) report that most studies suggest that severe math anxiety is uncommon in young children, though some researchers have found significant math anxiety even among early primary school children. Through informal polls performed at the beginning of each semester's course, this author's experience as a Graduate Teaching Associate at the Ohio State University confirms the fact that most of his Pre-College and College Algebra students suffer from math anxiety (at the beginning of each semester between 54 and 58 out of 60 students were found to be afraid of, or try to avoid math).

Possible causes for math anxiety could be pre-existing difficulties in mathematical cognition, that might cause or increase math anxiety, and social factors, such as exposure to teachers who themselves suffer from math anxiety (Dowker, 2016). If people who are anxious about math are charged with teaching others mathematics — as is often the case for elementary school teachers — teachers' anxieties could have consequences for their students' math achievement. Interestingly, elementary education majors are largely female and have the highest levels of math anxiety of any college major (Beilock, 2010).

In their 2016 Article *The Chicken or the Egg? The Direction of the Relationship Between Mathematics Performance*, Carey, Hill, Devine, and Szücs introduce three theories tied to math anxiety and performance: *Deficit Theory*, that states that poor performance in math will lead to math anxiety, *Debilitating Theory*, that states that math anxiety contributes to poor performance, and the *Reciprocal Theory* that suggests a bidirectional relationship between math anxiety and math performance, in which poor performance can trigger math anxiety, and math anxiety can further reduce performance, in a vicious cycle.

Note that a closely related construct is self-efficacy. Self-efficacy to the topic of mathematics has been defined as an individual's confidence in his or her ability to perform mathematics and is thought to directly impact the choice to engage in, expend effort on, and persist in pursuing mathematics. It is not precisely the same construct as self-rating, as it includes beliefs about the ability to improve in mathematics, and to take control of one's learning, rather than just about one's current performance (Dowker, 2016).

Chapter 4 – Methods to overcome Math Learning Difficulties

Helping mathematics make sense to all students, girls and boys is a worthy goal. In their 1997 research, Susan Goldman and her colleagues postulated that a primary goal of mathematics instruction for youngsters with MLD is the achievement of basic mathematical literacy. It is likely that early interventions for children with MLD may go some way toward preventing a vicious spiral, where MLD cause anxiety, which causes further difficulties with mathematics. Parents and teachers could attempt to model positive attitudes to mathematics and avoid expressing negative ones to children and there could be greater media promotion of mathematics as interesting and important. A promising intervention for mathematics anxiety amounts to "writing out" the negative affect and worry (Dowker, 2016).

Drawing on Jo Boaler's 2006 research, it appears that there exists a progression from uncritical to critical ways of knowing, that a system of 'connected knowing' better represents women while a system of 'separate knowing' better represents men, where elements under the rubric of 'separate knowing' include logic, rigor, abstraction and deduction while 'connected knowing' reflects intuition, creativity, hypothesizing and induction; moreover, research shows that girls often prefer cooperative and discussion based learning environments, rather than individualized or competitive environments. Learning Progressions, as developed by Michael Battista, have been successfully used in many mathematics classrooms as well as in a research study performed on an adult student who has been afflicted with mathematics disorder since the fourth grade.

Kate Garnett (1998) states that it is crucial not to hold students with math learning disabilities back "until they know their facts". Rather, they should be allowed to use a scaffolding such as a pocket-size facts chart in order to proceed to more complex computation, applications, and problem-solving. As the students demonstrate speed and reliability in knowing a number fact, it can be removed from a personal chart.

For knowledge to be useful, students must understand how procedures can function as tools for solving relevant problems (Goldman, 1997).

For example, teachers can provide answered addition problems with a double box next to each for translating these into the two related subtraction problems, as illustrated below.

4 + 3 = 7	7 – 4 =	7 – 3 =

Teachers can also dictate problems (with or without answers) for students to translate into pictorial form, then vertical notation, then horizontal notation. It can be helpful to structure pages with boxes for each of these different forms (Garnett, 1998), an illustration of this type of structure is shown below:

Problem Description	Pictorial Representation
Vertical Notation	Horizontal Notation

Students also can work in pairs translating answered problems into two or more different ways to read them, for example 20 x 56 = 1120 can be read twenty times fifty- six equals one thousand, one hundred and twenty, or twenty multiplied by fifty-six is one thousand, one hundred, twenty. Or, again in pairs, students can be provided with answered problems each on an individual card; they alternate in their demonstration, or proof, of each example using materials (e.g., bundled sticks for carrying problems). To add zest, some of the problems can be answered incorrectly and a goal can be to find the “bad eggs” (Garnett, 1998).

Teachers can help by slowing down the pace of their delivery, maintaining normal timing of phrases, and giving information in discrete segments. Such slowed down “chunking” of verbal information is important when asking questions, giving directions, presenting concepts, and offering explanations. Typically, children with language deficits react to math problems on the page as signals to do something, rather than as meaningful sentences that need to be read for understanding, which is also true for students who have a rote procedural view of mathematics. Therefore, teachers should encourage these students to:

Stop after each answer,

Read aloud the problem and the answer, and

Listen to myself and ask, “Does that make sense?” (Garnett, 1998)

To develop an understanding of math concepts, it may be useful to make repeated use of concrete teaching materials (such as base ten Stem blocks, Cuisenaire rods), with conscientious attention to developing stable verbal renditions of each quantity (e.g., 5), relationship (e.g., 5 is less than 7), and action (e.g., 5 + 2 = 7).

Since understanding visual relationships and organization is difficult for students with math learning disabilities, it is important to anchor verbal constructions in repeated experiences with structured materials that can be felt, seen, and moved around as they are talked about. For example, they may be better able to learn to identify triangles by holding a triangular block and saying to themselves, “a triangle has three sides. When we draw it, it has three connected lines” (Garnett, 1998). Thus, helping students with math learning disabilities develop their habits of verbalizing math examples and procedures can greatly help in removing obstacles to success in mainstream math settings.

In his 1997 *Instructional Design in Mathematics for Students with Learning Disabilities* report, Douglas Carnine established five design principles for improving the quality of instruction for students with learning disabilities: a) big ideas; b) conspicuous strategies; c) efficient use of time; d) clear, explicit instruction on strategies; and e) appropriate practice and review in the context of teaching problem solving. As they link their understandings of the various big ideas of proportion, data analysis, and probability, they are learning to connect mathematical ideas, solve problems, and apply mathematics broadly.

Teach big ideas: Abandon Low-Priority objectives and focus on big ideas. In most cases, mathematics programs attempt to cover exhaustive lists of learning objectives, with little or no attempt to prioritize those objectives on the basis of their relative importance later. For example, basal mathematics texts typically teach 8 to 11 different problem-solving strategies. Each strategy is taught for only one lesson and receives minimal review. If instruction focuses on all the reasonably important big ideas, then not only is understanding enhanced, but new material can be introduced less frequently, which reduces the likelihood of students' being overwhelmed (Carnine, 1997), thus reducing the students' math anxiety.

Teach conspicuous strategies: Provide a series of steps that students follow to achieve some goal. In instruction, such steps are initially presented overtly and explicitly for students, and eventually, as students master strategy, the steps become more covert. Any routine that leads to both the acquisition and utilization of knowledge can be considered a strategy (Carnine, 1997).

Perhaps the major challenge of instruction is to develop "just right" strategies for interventions with those students who do not develop them on their own, including, but not limited to, at-risk students and students with learning disabilities (Carnine, 1997). Applying the use of Learning Progression, developed by Michael Battista, helps the educator estimate the projected learning steps the student will take, then adjusting the theoretical progression to the responses given by the student. These learning progressions are anchored in the student's prior knowledge, the knowledge the student already has, and building on that foundation. For it is not the use of concrete materials that improves mathematical understanding, but rather the explicit construction of links between understood actions on the object and related symbol procedures (Carnine, 1997).

Use time efficiently: This principle addresses a perennial problem in teaching at-risk students and students with learning disabilities: teaching all they need to know (in both quantitative and qualitative terms) without "losing" them by trying to do too much, too quickly, "easing into" complex strategies. There is no reason why instruction on complex strategies comprising many component strategies cannot be spread out over a few days' time – a method of simplifying the communication of complexity without sacrificing crucial inherent complexities (Carnine, 1997).

Clear and explicit instruction on strategies: For practice and review, use a strand organization for lessons. The relatively small numbers of big ideas that are selected for instruction do not have to be presented in their entirety in a single lesson. Each big idea and its component concepts can be taught as a strand. A strand is a portion of several consecutive lessons that is devoted to a particular big idea, and one must keep in mind that students are more easily engaged with the variety that strands offer, because strands make the sequencing of component concepts more manageable. Big ideas in mathematics contain many concepts. Arranging these components concepts in a scope and sequence such that they are taught prior to their integration is made easier when several of them can appear in one lesson (Carnine, 1997).

Appropriate practice and review: Lessons organized around strands make cumulative introduction feasible. In cumulative introduction, after a concept is introduced, it is systematically reviewed and integrated with other related concepts. Cumulative introduction, as an alternative to the traditional spiral introduction, has three important advantages: a) Component concepts of complex strategies can be introduced early, b) practice can be provided on both new and previously introduced concepts until responses are accurate and rapid, and c) distributed practice on some concepts can occur every day, for example, complex proportions/data analysis problems may need to be practiced for several lessons until students can work this type with relative ease (massed practice), then a mixture of simple and complex proportion problems might be presented, to ensure that students know when to apply each strategy. Finally, only one or two problems of both or either type could be presented daily (distributed practice). Distributed practice is easy to schedule when each lesson is designed to incorporate portions of several strands (Carnine, 1997).

Clearly communicate strategies in an explicit manner: Make strategy explicit and clear. The whole purpose of developing instructional strategies is to analyze and develop expert strategies so that they become clear to nonexpert learners especially since explicit instruction has been shown to be effective for teaching at-risk students (Carnine, 1997).

Another method developed to help students learn mathematics that is applicable to helping students affected by MLD is the use of learning trajectories and learning progressions:

Learning trajectory. A learning trajectory is defined as a hypothetical progression of steps along which a students' mathematical progression is anticipated to take place. While working with the student, it may become necessary to adjust the trajectory based of the students' actual responses to the problems worked.

Learning Progressions. Learning progressions [LP] are descriptions of the successively more sophisticated ways of thinking about a topic that can follow one another as children learn about and investigate a topic (National Research Council, 2007).

Levels. The levels in a learning progression are equivalent to the major milestones a student will reach when working through a mathematical problem.

Sub-Levels. The sub-levels in a learning progression are the smaller steps that help the student go from level to level.

Providing a scaffolded transition to self-directed learning. Instruction can be scaffolded. Scaffolding is that same type of assistance given to students between the introduction of new knowledge and the eventual self-directed application of that knowledge such as scaffolded worksheets (Carnine, 1997).

In their 1997 research, Susan Goldman and her colleagues made the assumption that a primary goal mathematics instruction for youngsters with learning disabilities is the achievement of basic mathematical literacy.

The National Council of Teachers of Mathematics, NCTM (1989), states the mathematics curriculum should engage students in some problems that demand extended effort to solve. Some might be group projects that require students to use available technology and to engage in cooperative problem solving and discussion. The NCTM's suggestions for changes in classroom instructional activities include more emphasis on complex, open-ended problem solving, communication, and reasoning; more connections from mathematics to other subjects and to the world outside the classroom; and greater use of calculators and powerful computer-based tools, such as spreadsheets and graphing programs for exploring relationships - as opposed to having students spend an inordinate amount of time calculating by hand (Goldman, 1997).

If students understand the underlying reasons about "how" and "when" to use a procedure, they will be able to store it as a part of their knowledge network, thus developing links with other pieces of information. Links between conceptual and procedural knowledge can help students select an appropriate procedure because they will understand the rationale for applying the procedure and will be able to identify situations in which it is appropriate to use the procedure (Goldman, 1997).

What kind? Providing support to the students with MLD, who experience a greater degree of difficulty or inability to master an isolated skill, in the form of detailed learning trajectory, will assist in the prevention of them being excluded from the opportunity to engage in the complete task. This will also help them see the complete task and understand where the skills fit in and assist them in their attempt to complete the task.

The difficulty in teaching students how to solve problems can be attributed, in part, to the students' inability to perceive instances in which the knowledge they already possess is useful. The ability to literally "notice" and retrieve useful information appears to be especially problematic for children with learning problems or those who are at risk of school failure and these skills are not developed in traditional word-problem formats (Goldman,1997).

Even when declarative and procedural knowledge are in place, students with learning disabilities often fail to apply that knowledge in meaningful ways when confronted with problem situations. When confronted with a problem, students with disabilities typically do not use the mathematical knowledge they have acquired to solve problems unless they are explicitly informed about the relationship between the knowledge and the problem (Goldman, 1997).

For knowledge to be useful, students must understand how procedures can function as tools for solving relevant problems, and they must be motivated to solve new problems when they can relate to them (Goldman, 1997).

Chapter 5 – Case Study

5.1 – Introduction to Case Study

This case study describes how a Learning Trajectory and associated Learning Progression might be developed, evaluated / validated, modified / improved prior to, during and as a result of interviews, to help an adult student, affected by decades of fear of mathematics and MLD, reason about and make sense of solving linear equations involving simple core-algebraic ideas and concepts, such as the meaning of the equal sign and variables, and be able to solve simple equations with one variable. Having hypothesized a Learning Progression (LP), (See Appendix 2 of this document), for learning the mathematical topic of solving simple equations with one variable, this research investigates how this student might progress along a portion of this Learning Progression. The actual Learning Progression quickly evolved as a result of the student's answers to problems and thought processes.

During this study, the student progressed from Level 0 to Level 3, Sub-Level 3.1, (See Appendix 3 of this document). This Case Study includes the levels of reasoning the student exhibited on each task and the nature of the conceptualizations she seemed to use when working each task. Worthy of note is the fact that the math anxiety she experienced decreased throughout the research period, except for one math anxiety attack that occurred during the third interview.

5.2 – Case Study Description

A total of four interviews took place, each lasting between thirty and thirty-seven minutes. Each interview was recorded and filmed, then transcribed to more easily analyze the content once the interview was completed. All problems presented to the Student were written on 3 ½ by 5 index cards, or Wheatley Algebra Balance operation sheets.

In keeping with the interview guidelines set forth by Battista in his 2018 document *Clinical Interviews and Teaching Experiments: Determining How Students Make Sense of Mathematical Ideas and Learning*, each interview began with the Interviewer explaining to the Student that the purpose of this research was to investigate how she thinks about mathematics. She was asked to think out loud as she worked the problems presented to her; the purpose of all questions asked of her being to better help understand her thinking, not judge if an answer was correct or incorrect.

5.3 – Case Study Interviews and Findings

Stephens and Knuth report that many students struggle to make sense of algebraic equations, whether they are asked to write, interpret, or solve them. It has been suggested that this difficulty stems in part from misconceptions about the meaning of the equal sign. Therefore, the focus of the first interview began with an inquiry into the student's understanding of the equal sign, with the following card.

Problem Statement	Student's Answer
12. Is this an equation? Why?	No, because there is no equal sign. If there is no equal sign, it is not an equation, so that was enough to tell me it was not an equation.

Her answer reflected that she exhibits reasoning at the LP Level 0 (*Student has no concept of the meaning of the equal sign*) and work could begin looking into the student's understanding of equations using the following card:

Problem Statement	Student's Answer
12 = 3 + 9. Is this an equation? Why?	Yes, it is an equation because there is an equal sign. The equation, 12 does equal 3 + 9, and *it is backwards*.

This type of response echoes the research findings reported by Stephens and Knuth (2017), when they stated that while their interviews with sixth-grade students revealed that most students were successful when dealing with number sentences [such as 3 + 9 = 12], they detected some discomfort when the students responded to the less familiar number sentences such as this one. To further explore the student's interpretation of the equation as being *backwards*, the following card was presented to her:

Problem Statement	Student's Answer
3 + 9 = 12. Is this an equation, why?	Yes, it is an equation, just like the previous card, but *it is in the right order*.

The student explained that

"When the 3 + 9 comes before the equal sign, it is telling you what the 3 + 9 equals. It is *telling me* that when I put those two numbers together, I will get 12."

Her statement indicated she was exhibiting reasoning at this Projects' LP Sub-Level 0.1 (*Student understands the equal sign as operational*), an interpretation Stephens and Knuth discovered during their research, namely that students may have an operational view of the equal sign because they see the equal sign as a stimulus to compute and provide an answer rather than a relational view, which implies an understanding of the symbol as denoting the relationship between two equal quantities (2017).

To help the student expand her understanding of the equal sign to include that of a relational view, which is LP Sub-Level 0.2 in this Study, she was asked to look at the two cards [12 = 3 + 9, and 3 + 9 = 12] and share her thoughts about these two equations. After voicing her thoughts, she mentioned that the two sides were in fact the same, and that there was *balance*.

"The equal sign! As long as the two sides equal, they are balanced!"

It appeared the student's process of internalizing the relational view of equations could proceed as she worked on other aspects of equations. It was therefore decided to investigate the student's understanding of variables. In their research, Stephens and Knuth established that students in the middle grades should gain an understanding of the different roles variables play in algebra and learn to express mathematical relationships in succinct ways using variables. Thus, the interview moved to the following card and question:

Problem Statement	Student's Answer
What does the letter mean in $5 + a = 12$? What thought process did you use? Why?	Seven! Because it has to equal 12. So, *a* is equal to 7 so you can do 5 + 7 equals 12!

The student was using the relational aspect of the equal sign to find the value of the variable that ensured balance between the two sides. The exercise was successfully repeated using an *x* rather than an *a*. It seemed the student understood the concept of variable, thus reasoning at LP Sub-Level 1.1 of this study, that the variable could be a label that took the place of the value needed to satisfy the equation. It was time to ascertain the depth of her understanding:

Problem Statement	Student's Answer
What does "*a*" mean in $2a + 1 = 3$?	I don't know, unless it means the "*a*" can only be a 2 in this equation. And it is showing you that the "*a*" is standing in for 2 so you may add "plus 1" to make it be equal to 3.

The student was clearly struggling with the fact that 2*a* was 2 multiplied by *a*. Thus, the LP was adjusted accordingly - by inserting a Sub-Level (1.2) - that would address the *student's understanding of the four operations' symbols when used with variables*. The student answered in sequence the following questions:

Problem Statement	Student's answer
"2 + *a*". Do you know what this is?	It is just 2 plus a variable, *a*.
"2 – *a*". Do you know what this is?	Two minus *a*, but it is still not an equation!
"2 ÷ *a*". Do you know what this is?	Two divided by *a*.
"2*a*". Do you know what this is?	I don't know what that is!
"2 x *a*". Do you know what that is?	Yes! It is 2 times *a*!

2*a* = 2 x *a* was written down and it was explained that in math, multiplication is the only operation that does not have a symbol, that adding, subtracting, dividing are all operations and they have symbols, but multiplication operation does not in algebra.

Nevertheless, she still struggled with the meaning of 2*a*:

"2*a* means 2 times *a*? So, (pointing to 2*a* + 1 = 3) this means 2 times 2 equals 4 … plus 1 is 5 … which means this equation is wrong! It is a wrong equation!"

This understanding was re-iterated when she was asked to interpret 3*x* + 2*y*:

"So, it would be 9 plus 4, because here (pointing to 2*a* + 1) we still have 2 times 2, it represents 4 ... 2*a* is 2 times the number in front".

The student was viewing the association of constant and variable as concatenation. And indeed, Stephens and Knuth's 2017 research mentions the conceptual difficulty students experience with variables; that not only are many students generally uncomfortable working with unknown varying quantities, but when confronted with a variable, even if they do interpret it as representing a number, students may feel the need to assign a specific numerical value to it, as in this instance, associating the variable with the number preceding it. She was attempting to make sense of the variables, and she was coming up with her own concepts as she attempted to make sense of the mathematical problem at hand, a fact reported by Battista (2018), namely that students personally construct ideas as they intentionally try to make sense of situations.

The instructional goal was now to have the student correct this misunderstanding. It seemed she was oscillating between what Kieran (1998) refers to as "*arithmetic* and *algebraic* understandings of variables, where for the algebra group, the letter seemed to have meaning only when its value was found, [and] the arithmetic group seemed to view the letter as standing for some unknown number," even if the value she was assigning to it was the result of a faulty concept - concatenation.

Stephens and Knuth stated in their research that equations with operations on both sides, in particular are more likely to elicit relational understandings. Battista also mentioned in a conversation with the researcher of this case study that it is always good to go back and make sure the student *did* understand the concept. Therefore, the next questions were structured around the concept of balance, the relational property of equations. For that purpose, examples of Wheatley's Algebra Balance problems were used to link balance operations to operations on equations, a form of scaffolding intended to help the student visualize, and therefore better conceptualize the process, thus challenging her to explore successively more sophisticated ways of thinking, linked to meaningful assessments to identify the students' mode of reasoning, and instructional tasks to move her to higher levels of sophistication in reasoning (Fonger, 2018).

The second interview began with the student being introduced to a Wheatley algebra balance representing "$4x + 5 = 2x + (5 + 14)$" and she was asked to identify each component of the balance and match it with the corresponding component of the equation, starting with the left-hand side, thus allowing her an opportunity to associate the units in the equation with those represented symbolically in the balance sketch. When the second Wheatley algebra balance card was produced ($4x = 2x + 14$) and the student was asked to describe what happened between the two cards, she answered:

> "They took out the 5's, but I am confused because it should be 4 times (4x) but do not know… what!"

She was confused because $4x$ looks like 4 x, meaning 4 followed by the multiplication symbol, so the interviewer pointed to the simple empty block/boxes, and asked her to write an *x* inside each blank box to highlight the fact that these are the unknowns, and then write the equation that she saw sketched on the balance (See Figure 5.1 below).

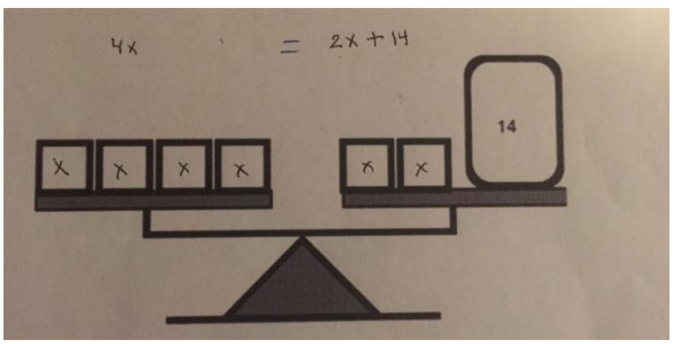

Figure 5.1: Wheatley Algebra Balance Card, $4x = 2x + 14$

The next Wheatley algebra balance card ($2x = 14$) was then presented and she was asked to explain what had occurred between the last two steps.

> "Oh! We subtracted the two *x*'s that were here and here (Pointing to RHS and LHS in the Wheatley algebra balance card shown above), and we kept the other two *x*'s and the "14" … so they are still in balance. (Student writes $2x = 14$ above the balance), because *what we take off was the same amount from each side*."

The student was understanding that balance in an equation is preserved if the same quantity is removed from each side. The next sequential Wheatley algebra balance card was produced, and she was asked what happened between these last two steps.

"We broke the 14 down to two 7's, we still have the 2*x*'s on the LHS."

The student wrote 2 x 7 = 14 above the LHS, again confusing the *x* for the multiplication symbol then, upon her confusion having been brought to her attention, she wrote $2x = 7 + 7$ on the RHS of the balance (See Figure 5.2 below).

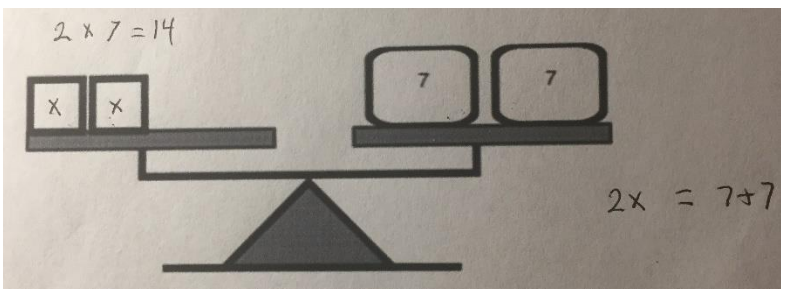

Figure 5.2: Wheatley Algebra Balance Card, $2x = 7 + 7$

The interviewer then produced the final Wheatley algebra balance card in this sequence and asked the student what happened between these final steps.

> "I did that! (Exclaiming with joy and pointing at the line with $2x = 2(7)$), and then you divide by 2 to get $x = 7$. But I still get confused when it is an *x* because I see it as *timezing*. It would be less confusing for me if they were a *b* or an *a*, it would not be as confusing, but when we use an *x* I automatically see that as a times."

The student seemed to be able to associate the symbols on the Wheatley algebra balance cards and write the corresponding equation, but she still had difficulty when dealing with *x* as the variable, confusing it with the multiplication symbol, x. This meant she still struggled when confronted with the concept of variable in this situation. To help her, the interviewer suggested rewriting the original equation with *a*'s, (4a + 5 = 2a + (5 +14)).

> "Now this, I have less trouble with because I am not timezing it automatically in my head and whenever you have it little like that (pointing to the *a* in 4*a*) next to each other like that, that it is *times*."

Following Battista's recommendation to always make sure the student *did* understand, the interviewer produced an index card and asked her to write the original equation and explain the step-by-step method used to solve it, $4a + 5 = 2a + (5 + 14)$.

> "First step was to take away the two 5's and everything else stayed the same, so that would have been ... (Student writes $4a = 2a + 14$), then we took away the 2 boxes here (pointing to LHS) and the 2 boxes here (pointing to the RHS) and that would have been… (Student writes $2a = 14$), and then, we did $2a = 7 + 7$ because there were two sevens, which would have been two times seven. (Student writes 2a = 2x7), then we divided both sides by two, so 2 divided by 14 would be 7, so $a = 7$! Because there is no more timezing because it is just the one number that we want." The students work is shown in Figure 5.3 below.

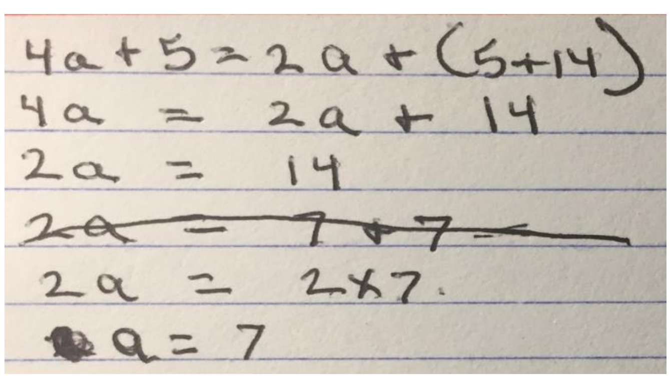

Figure 5.3: Student's Work Solving $4a + 5 = 2a + (5 + 14)$

To not interrupt the flow of the students thought processes, her saying “2 divided by 14 would be 7” was ignored, and she was asked if she liked working with these balance operations?

“Yes, because it allows me to see what my question was there, and I feel better now about the “*x*” (She laughs). I feel good about this exercise. The main block I experienced was working with an *x*. I think it would have been easier if we had started with it being *a*’s. I would like to work a different one without you having to explain things to me. I do think that I have a stumbling block with the division at the end to get the *a*, so I might need to work more to remember to divide it.”

The student then proceeded to successfully work a similar equation and when asked what she did from step-to-step, she said:

“We subtracted equally from each side to keep balance. Can we do another one?”

The interview continued with the student solving two more equations and becoming progressively more comfortable with each step. (Compare student’s self-corrections between Figures 5.4 and 5.5 below).

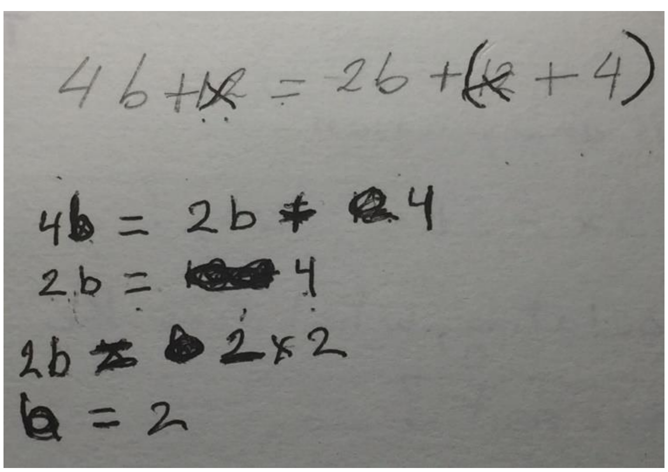

Figure 5.4: Student’s Work Solving $4b + 12 = 2b + (12 + 4)$

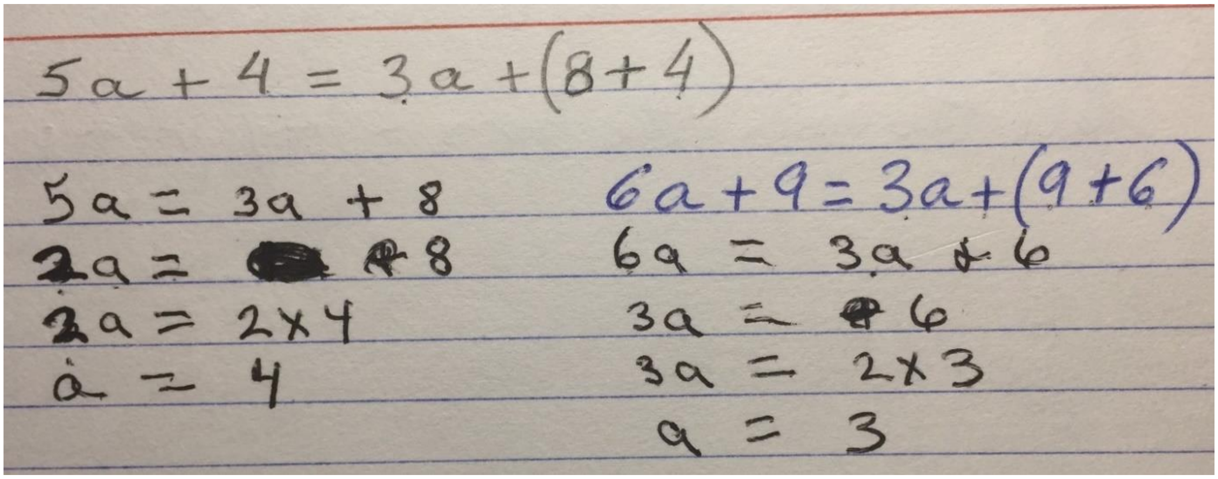

Figure 5.5: Student's Work Solving $5a + 4 = 3a + (8 + 4)$ and $6a + 9 = 3a + (9 + 6)$

Thus, the student made progress in *solving equations with the use of scaffolding (Wheatley Balance Operation Models)*, Sub-Level 2.1 of the Learning Progression. Her confusion when associating the variable *x* with the multiplication symbol, x, was addressed by writing the equations with *a*'s and *b*'s, and it seemed the Wheatley Algebra Balance cards were of help to her, as evidenced by the progress she made, and indeed when asked if they helped, she answered:

> "Yes, it gave my brain something to formulate on … when I work these problems, I have the vision of the boxes in my head".

Having established that the student had the ability to solve simple equations with one variable when assisted by a scaffolding structure, such as the Wheatley Balance Operation Cards, LP Sub-Level 2.1, and in keeping with Battista's recommendation, that it is always good to "go back and make sure the student *did* understand the concept," the study now sought to investigate how the student might perform when presented with Wheatley balance operations that she was unfamiliar with, and investigate if, at any time, the student would display what Kieran refers to as *arithmetic* and *algebraic* understandings of variables. For the algebra group, the letter seemed to be meaningful only when its value was found, the arithmetic group seemed to view the letter as standing for some unknown number (Kieran, 1988).

The third interview began with the interviewer introducing the previously worked $2x = 14$ Wheatley balance operation card as a reminder to the student of the type of problem worked during the previous interview and informed her she would be interpreting and writing down the next balance operations and their associated equations (See Figure 5.6 below).

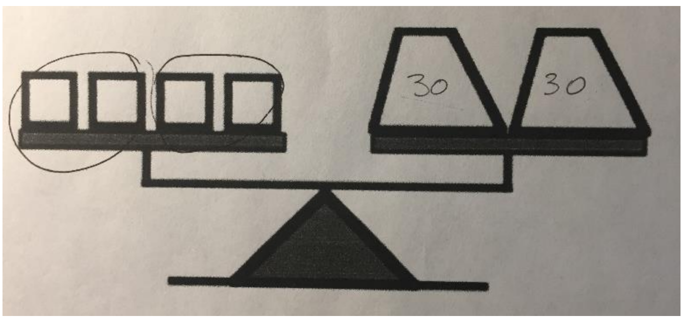

Figure 5.6: Wheatley Algebra Balance Card, $4a = 30 + 30$

She writes $4a = 30 + 30$ … "There is nothing to take away! So, I am trying to figure out what the *a* is."

Upon which she was encouraged to see if she could group terms and proceed from there.

"If you circle two boxes and two boxes, we will get $2a + 2a = 30 + 30$, so if I take away $2a$ and one 30 then it stays balanced. So, I will get $2a = 30$ and half of $2a$ is $1a = 15$, so $1a = 15$. I did it! Can we do another?"

Remembering how the student has suffered from math anxiety most of her life, to hear her request more math came as a very good sign. It seemed she was positively responding to the learning progression and the exercises she was working on. The next card was presented, and she spoke out loud:

"$3b + 13 + 13 + 13 = 1b + 95$ this one is going to be different because there aren't two the same here (pointing to each side of the equation). … The b's are on each side, so I can take $1b$ off so that will be $2b + 13 + 13 + 13 = 95$. … Now I can combine the three 13's into 39. So, this would be $2b + 39 = 95$. So basically, it is 39 plus whatever it is that equals 95, and that is going to be … 95 – 39 = 51, no 56!

When asked what she had done, she replied:

"This is where I always feel I have a block … I still need to figure out what b is…"

She was encouraged to investigate rewriting 95 as 39 plus something, upon which she wrote: 95 = 39 + 56, and replacing 95 with these new values, she wrote $2b + 39 = 39 + 56$.

"Now I can take away the 39's! So that would be $2b = 95$, no! equals 56! So, then it is $2b$ divided by…something… hold on…I don't know … $2b$ divided by 1 is 2 ... No ... (student breaks down in tears) … I feel really stupid!"

The interview was suspended, and the student was given support, compassion, and praise for wanting to understand and progress in math, something she had for so long been afraid of. She then explained what had happened:

"So, it is $2b$ divided by itself… so $2b$ divided by 2 will give $1b$. … I couldn't get 56 divided by 2. I knew that 25 times 2 was 50, but then I got messed up taking the 5 from the 6 which was 1, so then I could not divide the 1… So, I got more and more stressed out and just paralyzed, instead of seeing that I needed to divide the 6 to get 3. It just made me shut down ... It was my brain … the way I feel stupid in math … I still don't know what it means! So, this means that $1\ b = 28$. Can we do another one … to see if I can make it through? Normally when we are done with these, I feel I have learnt something, and this time is like I am just hitting a wall!"

She had been on the right track, breaking down 56 into 50 and 6, and dividing them individually. It was suggested she use a calculator from now on to minimize stress surrounding difficult operations. Learning Progressions in learning division also appeared to be an area that could be worked on in the future with the student. And the fact that she wanted to experience the feeling of having accomplished and learnt something was good news. It seemed she was working through her own productive struggle.

The student successfully worked two additional problems, still stating the value found as "1 *a* ", after which the interview was concluded, estimating enough work had been accomplished during this learning progression session. It should be noted that although the student's work focused on Sub-Level 2.1, she also used tools that are used in Sub-Level 3.1, and that it appeared the perturbation the student experienced towards the end of the session could have originated from her efforts to use the "inverse-operation" method, which is what Kieran (1988) refers to as an *algebraic* understanding of variables, since she was not working the problem by trial and error, which would have been what she refers to as an *arithmetic* understanding of variables.

Given the progress the student had been making, it was determined to investigate how she might progress once the Wheatley Balance Operation model scaffolding had been removed. Additionally, Sub-Level 2.4, *Student understands the meaning of terms inside parentheses*, was introduced in preparation of future work with the student. After a brief re-introduction of the Wheatley Balance Operations to reacquaint the student with the procedures used, the fourth interview began, by building on the progress she had previously made, with the first balance equation: $4x + 31 = 2x + 31 + 26$, and asking her to write the equation that the blocks on the scale represented (See Figure 5.7 below)

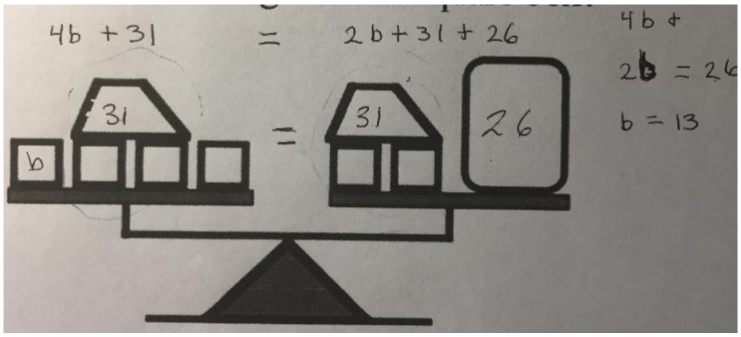

Figure 5.7: Wheatley Algebra Balance Card, $4b + 31 = 2b + 31 + 26$

"And I can make the blocks be anything I want right? … So, I am trying to remember exactly what the format is to write the equations ... and an equation starts with an equal sign (student hesitated to add or multiply the 31 with the 4b, then said) "Plus! It is plus!" and writes $4b + 31 = 2b + 31 + 26$ and exclaims with joy "I did it on my own! … So, first thing we are going to do is take away the 31, so that gives us … and we have 2 here (pointing to the RHS $2b$ and 4 here (pointing to the LHS $4b$) so that would give us $2b$. so it will end up being $2b$ equals 26, … so 2 times what equals 26? So that is 13, so b equals 13. Before we were doing it by steps, by removing first the 31 from each side. Then the next step we would remove $2b$ from each side, so they would still be balanced, but then we would need to find out what equaled the 26 and that … Yeah!"

She student seemed confident in her progression through LP Sub-Level 2.1. She remembered the steps she used during our previous interview, so the internalization process of these procedures seemed to be taking place. She worked the next operation in the Wheatley Balance Operation format without any new difficulty. The next card, $5a + (13 – 3) = 2a + 10 + 3$, introduced the use of parenthesis to the student, in preparation of work with the student that will involve the distributive property - not covered in this research project, but in future interviews.

“I don’t know what the parenthesis means”

It was explained to her that parentheses group the terms they bound into one single term.

“I don’t know what difference that makes…if we are grouping what is inside then we have 10. So, we will have 5*a* + 10 = 2*a* + 10 + 3. But it isn’t balanced anymore … Oh! I can take out the 10 from each side, and I will be left with 5*a* = 2*a* + 3 so then … I can take away the 2 (pointing to 2*a*) which will leave 3*a* = 3, so 1*a* = 1!”

Our student quickly solved the next equation presented to her, shown in Figure 5.8 below:

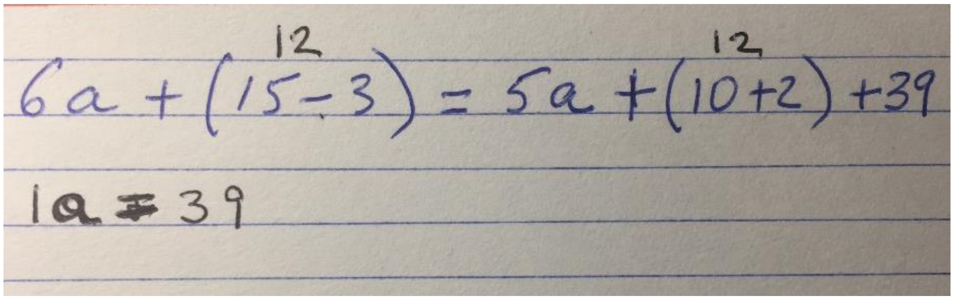

Figure 5.8: Student’s Work Solving 6*a* + (15 - 3) = 5*a* + (10 + 2) + 39

“Ok, 15 - 3 is 12, and 10 + 2 is 12, so … (writes 12 above each term in parenthesis). Take out the 12, take out the 5*a*’s then that leaves 1*a*! so 1*a* plus 39, no! 1*a* = 39. And then ... OK, this … that would mean that 1*a* is 39, there is no more to do! … Yes!”

She began solving the equation [6*a* + (12 - 3) = 2*a* + (6 + 3) + 16] as soon as it was presented to her. Her work is shown in Figure 5.9 below:

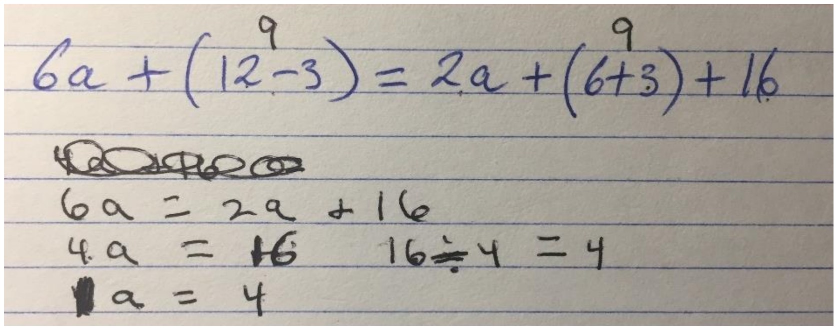

Figure 5.9: Student's Work Solving $6a + (12 - 3) = 2a + (6 + 3) + 16$

"First thing, … my mind goes to the groupings, (writes 9 above each term in parenthesis) so, this … if I take out … this is going to be $4a + 16 =$ … and that is going to be … and I need to divide 16 by … I am getting myself confused … I will start over without skipping steps. (Crosses out what she just wrote and writes) $6a$ … take out the 9… $= 2a + 16$, now take out the 2 (pointing to the HRS $2a$), and that will leave $4a = 16$, so this would be $4a = 4$ because 16 divided by 4 is 4. Oh! No, it's $1a = 4$! I really should be doing the division on both sides of the equal sign, right? … to keep the balance! I should actually be doing two divisions, one on each side. Like I should be doing 4 divided by 4 then this (16) divided by 4, but here (LHS) they will all be 1 on that side (pointing to the LHS), so I should automatically know this is going to be $1a$, and this (pointing to the RHS) is going to be whatever it will be. Can we do another one?"

She was then introduced to the last equation of the Case Study: $10a + (12 - 7) = 5a + (15 - 10) + 25$

"OK, so my first thing is: I am going to go to the groupings and they both equal 5, so I am going to do $10a = 5a + 25$, then I am going to take away this 5 from here (pointing to $5a$ on the RHS), so it is going to be $5a = 25$, then I am going to divide 5 by 5 … is 1, so that is 1*a* and here (pointing to the RHS) equals 5 because 25 divided by 5 is 5. So, I did (Writes) 5 divided by 5 is 1 and 25 divided by 5 is 5, so 1*a* = 5. Yeah!"

She concluded with:

"I feel better knowing about the two sides, because I was not doing that, and I feel like that really hindered me …in keeping the process going. Obviously, I knew this had to be in balance, but I didn't understand that about … (Grabs card with $5a + (13 - 3) = 2a + 10 + 3$ equation) OK, if I am going to divide now, how do I get this (3*a*) and this (3) (pointing to 3a = 3 line in equation), so I just divide it (pointing to the 3 in 3*a*) by itself to get 1. Now, it would be nice to be able to divide 27 by 3 in my head, right?"

The final interview of this Case Study thus concluded with the student having progressed through Sub-Levels 2.4 and 3.1 of her actual LP. She seemed much more at ease with the process of solving by subtracting equal quantities of like terms from each side, and with the Wheatley Balance Operation model scaffolding removed.

5.4 – Case Study Conclusions

This Case Study documents the Student's actual trajectory through a Learning Progression for Solving Simple Equations with one Variable that, guided by the students' progress, evolved from a Hypothetical Learning Progression for Solving Simple Equations with one Variable that had been established, based on the findings of the research of Battista, Fonger, Kieran, Knuth, and Stephens. The student progressed from Level 0 to Sub-Level 3.1. by acquiring the understanding of the concept of the relational property of the equal sign, the fact that in mathematics, multiplication of a constant by a variable has no symbol, the meaning of grouping like terms inside parentheses; solving simple equations with one variable without the help of scaffolding (Wheatley Balance Operation Models), and performing operations on equations that maintained equivalence relations.

Additionally, as the student progressed through each Level and Sub-Level and encountered mathematical concepts and procedures she was unfamiliar with, the math anxiety she felt decreased, with the exception of an anxiety attack that occurred during the third interview, from which she quickly recovered.

Chapter 6 – Conclusion

Mathematics Learning Disorders are often referred to as mathematics learning disabilities, math learning difficulties, or dyscalculia. Although this research focusses on students affected by Mathematics Learning Disorders (MLD) in the United States, such students are not limited to the United States but are distributed wherever math is taught. The disability appears to affect students as soon as they get into middle school if not earlier, and affects girls as well as boys, although either through a bias in the teaching curriculum, or through stereotype threat, the learning disorder induced by math anxiety seems to affect more girl and women students more than boy and men students. Students affected with MLD perform at a level of performance four to five years behind same aged peers who are not affected with MLD.

MLD is often tightly connected to math anxiety since poor performance in math can cause math anxiety, which in turn will further prevent the affected student from succeeding in math, thus creating a vicious cycle, that can be broken thanks to the intervention of an understanding math teacher with tools such as learning progressions and scaffoldings. Noteworthy is the fact that because girl and women students may also inherit anxiety from female teachers who are themselves affected by an aversion to math, girl and women students seem more prone to being affected by math anxiety.

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Appendices

APPENDIX A - Approximate Number System (ANS)

Approximate Number System (ANS) can be explained as "the mental system of approximate number representations that is activated during both non symbolic approximations and symbolic number tasks" (Mazzocco, p 2); as such, students with Math Learning Disabilities have "significantly poorer ANS precision than students in all other achievement groups" (Mazzocco, p 1) which means that these students have difficulty judging which numbers are greater or smaller when presented an array of random numbers, or estimating the number of dots when quickly presented with a figure such as Figure 1 below:

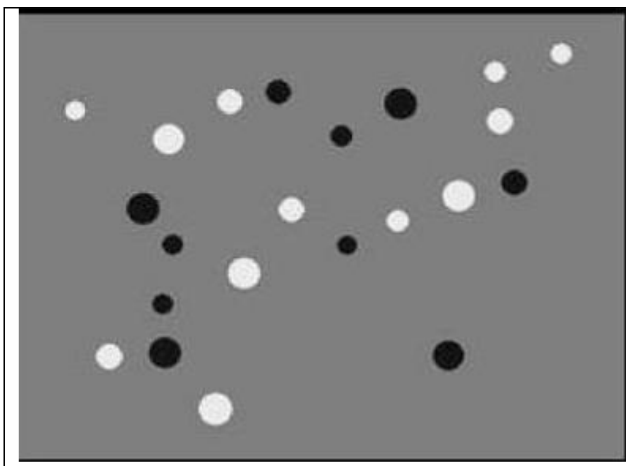	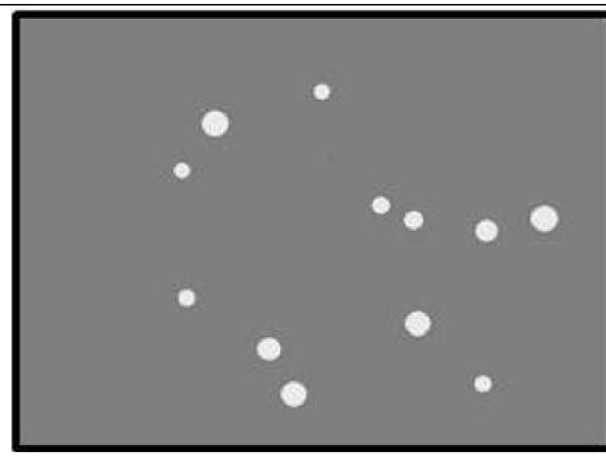
Question: "Are there more yellow dots or more blue dots?"	Question: "How many yellow dots are there?"

Figure 1

(Blue dots appear in black, and yellow dots appear in gray.) (Mazzocco, p 19)

APPENDIX B - Hypothetical Case Study Learning Progression

Hypothetical Learning Progression – Solving Simple Equations with one Variable		
Level	Sub-Level	Description
0		Student has no concept of the meaning of the equal sign
	0.1	Student understands the equal sign as an operation
	0.2	Student understands the equal sign as a relation
1		Student recognizes the meaning of equations but has no concept of Variables
	1.1	Students understands variables as labels
	1.2	Student has an Algebraic understanding of Variables
	1.3	Student has an Arithmetic understanding of variables
2		Student understands the meaning of equations and variables
	2.1	Student solves equations by trial and error (Arithmetic Approach)
	2.2	Student solves equations using the "inverse-Operation" Method (Algebraic Approach)
3		Student can solve simple Equations with one variable on one side of the equal sign using addition & subtraction operations
	3.1	Student can solve simple Equations with one like-term variable on each side of the equal sign using addition & subtraction operations
	3.2	Student can solve simple Equations with two like-term variables on one side of the equal sign using addition & subtraction operations
	3.3	Student can solve simple Equations with two like-term variables on each side of the equal sign using addition & subtraction operations
4		Student can solve simple Equations with multiple like-term variables on each side of the equal sign using addition & subtraction operations
	4.1	Student can solve simple Equations with two like-term variables on one side of the equal sign using addition & subtraction operations
	4.2	Student can solve simple Equations with one like-term variable on each side of the equal sign using multiplication & division operations
	4.3	Student can solve simple Equations with two like-term variables on each side of the equal sign using multiplication & division operations
5		Student can solve simple Equations with multiple like-term variables on each side of the equal sign using all four operations
6		Student recognizes simple equations with one variable but does not understand the distributive property
	6.1	Student cannot solve equations using the distributive property with one variable on one side of the equal sign
	6.2	Student cannot solve equations using the distributive property with two like-term variables on one side of the equal sign
	6.3	Student cannot solve equations using the distributive property with one variable on each side of the equal sign
	6.4	Student cannot solve equations using the distributive property with two like-term variables on each side of the equal sign
	6.5	Student cannot solve equations with two factored expressions with like-term variables on each side of the equal sign
7		Student can solve simple equations with one variable using the distributive property

APPENDIX C - Actual Case Study Learning Progression

Actual Learning Progression – Solving Simple Equations with one Variable		
Level	Sub-Level	Description
0		Student has no concept of the meaning of the equal sign
	0.1	Student understands the equal sign as an operation
	0.2	Student understands the equal sign as a relation
1		Student recognizes the meaning of the equal sign but has no concept of Variables
	1.1	Students understands variables as labels
	1.2	Student Understands the Four Operation Symbols when used with variables
	1.3	Student has an Algebraic understanding of Variables
	1.4	Student has an Arithmetic understanding of Variables
2		Student understands the meaning of equations and variables
	2.1	Student Solves equations with use of scaffolding (Wheatley Balance Operation Models)
	2.2	Student solves equations by trial and error (Arithmetic Approach)
	2.3	Student solves equations using the “inverse-Operation” Method (Algebraic Approach)
	2.4	Student understands the meaning of combining constants inside parentheses
3	2	Student can solve simple Equations with one variable on one side of the equal sign using addition & subtraction operations
	3.1	Student can solve simple Equations with one like-term variable on each side of the equal sign using addition & subtraction operations
	3.2	Student can solve simple Equations with two like-term variables on one side of the equal sign using addition & subtraction operations
	3.3	Student can solve simple Equations with two like-term variables on each side of the equal sign using addition & subtraction operations

Note: During this Research Project, the student interviewed progressed from Level 0 to Sub-Level 3.1 of this Learning Progression.