Topological Quillen Localization and Homotopy Pro-Nilpotent Structured Ring Spectra

Dissertation

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Abstract

The aim of this paper is two-fold: (i) to establish the associated TQ-local homotopy theory for algebras over a spectral operad $\mathcal{O}$ as a left Bousfield localization of the usual model structure on $\mathcal{O}$-algebras, which itself is almost never left proper, in general, and (ii) to show that every homotopy pro-nilpotent structured ring spectrum is TQ-local. Here, TQ is short for topological Quillen homology, which is weakly equivalent to $\mathcal{O}$-algebra stabilization. As an application, we simultaneously extend the previously known connected and nilpotent TQ-Whitehead theorems to a homotopy pro-nilpotent TQ-Whitehead theorem. We also compare TQ-localization with TQ-completion and show that TQ-local $\mathcal{O}$-algebras that are TQ-good are TQ-complete. Finally, we show that every (-1)-connected $\mathcal{O}$-algebra with a principally refined Postnikov tower is TQ-local, provided that $\mathcal{O}$ is (-1)-connected.

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Chapter 1: Introduction

Homotopy groups and stable homotopy groups of spaces are central invariants in algebraic topology.

Homotopy groups π_* are very powerful but difficult to compute in practice. For instance, they can completely classify topological spaces (under mild technical assumptions) up to homotopy equivalence. However, even the homotopy groups of the 2-sphere $\pi_n(S^2)$ are not entirely known.

Stable homotopy groups π_*^s are comparatively easier to work with, at the expense of losing certain information. However, if we are working with nice spaces, nothing will be lost by working stably: A map between nilpotent spaces is a homotopy equivalence if and only if it is a stable equivalence.

Spectra naturally arise in lots of different areas of geometry and topology. One can think of spectra as "chain complexes of spaces". Analogous to ordinary chain complexes, there is a well-behaved smash product of spectra [20, 36, 43, 56]. This enables us to do "homotopical algebra" with spectra. For example, we can work with ring spectra which are commutative up to coherent homotopy.

Structured ring spectra are spectra with extra algebraic structure encoded by the action of an operad $\mathcal{O}$ [22, 39, 44]. For a fixed operad $\mathcal{O}$, denote by $\mathsf{Alg}_{\mathcal{O}}$ the category of $\mathcal{O}$-algebras. For each $\mathcal{O}$-algebra X, there is an analogous notion of homotopy groups

$\pi_i(X) := \operatorname{colim}_n \pi_{i+n}(X_n)$ defined as homotopy groups of the underlying spectrum [54, 55]. A map $f : X \to Y$ of $\mathcal{O}$-algebras is called a weak equivalence if it induces π_* isomorphisms. For technical reasons we work with reduced operads $\mathcal{O}$; such $\mathcal{O}$-algebras are non-unital.

Topological Quillen (TQ) homology [2, 4, 24, 31, 38, 41] for $\mathcal{O}$-algebras naturally arises as the topological analog of Quillen homology of ordinary commutative algebras [1, 48]; see also [23, 28]. It turns out that under appropriate connectivity conditions, TQ is 1-excisive and agrees to order 1 with the identity functor on $\mathcal{O}$-algebras [13, 32, 40] in the sense of Goodwillie calculus [29]. Hence TQ-homology is weakly equivalent to stabilization $\Omega^\infty\Sigma^\infty$ in $\mathcal{O}$-algebras [3, 32, 46, 52, 53].

When TQ-homology is iterated, built into a cosimplicial TQ-resolution of the form

$$X \longrightarrow \mathsf{TQ}X \rightrightarrows \mathsf{TQ}^2X \mathrel{\substack{\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow}} \mathsf{TQ}^3X \ \cdots$$

and then glued all together with a homotopy limit, it gives the TQ-completion $c\colon X \to X^\wedge_{\mathsf{TQ}}$ [8, 32], analogous to Bousfield-Kan [10] completion for spaces. This construction can be viewed as a method to extract "the part of an $\mathcal{O}$-algebra that TQ-homology detects".

It is proved in [15] that the TQ-completion $c\colon X \to X^\wedge_{\mathsf{TQ}}$ is a weak equivalence if X is 0-connected—in other words, 0-connected $\mathcal{O}$-algebras are already TQ-complete; here, $\mathcal{O}$ is assumed to be levelwise (-1)-connected. We also know from [32] that every 0-connected $\mathcal{O}$-algebra is the homotopy limit of a tower of nilpotent $\mathcal{O}$-algebras, in other words, X is homotopy pro-nilpotent; here we say that X is nilpotent if all the n-ary and higher operations $\mathcal{O}[t] \wedge X^{\wedge t} \to X$ of X are trivial for each n large enough (Definition 3.4.1 and 4.0.1).

What about the larger class, for instance, of homotopy pro-nilpotent $\mathcal{O}$-algebras—are the TQ-completion maps always weak equivalences?

In this paper, we will show the comparison map is a weak equivalence for all homotopy pro-nilpotent $\mathcal{O}$-algebras, provided that one replaces "TQ-completion" with "TQ-localization".

Our first step is to construct the TQ-localization as a "better" model than TQ-completion for "the part of an $\mathcal{O}$-algebra that TQ-homology detects". TQ-completion is known to only be "the right model" when the $\mathcal{O}$-algebra X is TQ-good (i.e., when the comparison map from X to its TQ-completion is a TQ-equivalence) analogous to the situation for spaces [10], but homotopy pro-nilpotent $\mathcal{O}$-algebras are not expected to be TQ-good in general. However, TQ-localization always gives "the right model" for the part of the $\mathcal{O}$-algebra X that TQ-homology sees (at the expense of a much larger construction); just like Bousfield's localization construction [9] for pointed spaces.

We say an $\mathcal{O}$-algebra X is TQ-local if every TQ-homology equivalence $f\colon A \to B$ between cofibrant objects induces equivalence of mapping spaces $\mathbf{Hom}(B, X) \to \mathbf{Hom}(A, X)$. Intuitively, X is called TQ-local if mapping into X can not tell the difference between TQ-homology equivalent objects.

Following the arguments in [9, 27, 37], in light of the cellular ideas in [35], we build TQ-localization by establishing the TQ-local homotopy theory (Theorem 4.3.14) for $\mathcal{O}$-algebras (without any connectivity assumptions).

Here a potential wrinkle is the well-known failure of $\mathcal{O}$-algebras to be left proper in general (e.g., associative ring spectra are not left proper, also see [49, 2.10]); we show that exploiting an observation in [25, 26] enables the desired TQ-localization to be constructed by localizing with respect to a particular set of TQ-equivalences. Here the upshot is that: if X is a cofibrant $\mathcal{O}$-algebra, then its weak TQ-fibrant replacement $l\colon X \to L_{\mathsf{TQ}}X$ is the TQ-localization of X (Theorem 4.3.17). By construction, the comparison map $l\colon X \to L_{\mathsf{TQ}}X$ is a cofibration that is also a TQ-equivalence such that $L_{\mathsf{TQ}}(X)$ is TQ-local.

Another useful observation is that the homotopy limit of a small diagram of TQ-local $\mathcal{O}$-algebras is TQ-local (Proposition 5.1.6). By leveraging the TQ-local homotopy theory of $\mathcal{O}$-algebras with the fact, proved in [14], that M-nilpotent $\mathcal{O}$-algebras are $\mathsf{TQ}|_{\mathrm{Nil}_M}$-complete, we get the following result (to be proved in Chapter 5).

Theorem 1.0.1 (Homotopy pro-nilpotent TQ-localization theorem)**.** *Let X be a fibrant $\mathcal{O}$-algebra.*

(a) If X is nilpotent, then X is TQ-local.

(b) If X is homotopy pro-nilpotent, then X is TQ-local.

(c) If X is the homotopy limit of any small diagram of nilpotent $\mathcal{O}$-algebras, then X is TQ-local.

(d) If X is connected and $\mathcal{O}, \mathcal{R}$ are (-1)-connected, then X is TQ-local.

As an application, we obtain the following homotopy pro-nilpotent TQ-Whitehead theorem that simultaneously extends the previously known connected and nilpotent TQ-Whitehead theorems.

Theorem 1.0.2 (Homotopy pro-nilpotent TQ-Whitehead theorem)**.** *A map $X \to Y$ between homotopy pro-nilpotent $\mathcal{O}$-algebras is a weak equivalence if and only if it is a TQ-homology equivalence; more generally, this remains true if X, Y are homotopy limits of small diagrams of nilpotent $\mathcal{O}$-algebras.*

We also prove that TQ-completion always factors through TQ-localization and show that TQ-local $\mathcal{O}$-algebras that are TQ-good are already TQ-complete (Theorem 5.2.1).

The rest of this thesis is organized as follows.

Chapter 2 will review the basic ideas and construction regarding stabilization of topological spaces.

Chapter 3 will review the relevant notations and constructions of structured ring spectra, in particular, we introduce Topological Quillen homology of structured ring spectra. We will also discuss some previously known TQ-Whitehead theorems

Chapter 4 gives the construction of TQ-local homotopy theory as well as TQ-localization maps.

Chapter 5 proves Theorem 1.0.1 and Theorem 1.0.2. We also provide a comparison of TQ-completion and TQ-localization, followed by a discussion on principally refined Postnikov towers.

Chapter 2: Whitehead theorem for spaces with respect to stabilization

2.1 Stabilization of spaces

Homotopy groups of spaces are central invariants in algebraic topology. They are extremely useful and powerful, but also quite difficult to compute in practice. Today we still don't completely know the homotopy groups of the 2-sphere.

One aspect of the difficulty is that when we consider (reduced) suspension $X \to \Sigma X$ of a n-connected pointed space X, Freudenthal suspension theorem tells us the comparison map

$$\pi_i(X) \to \pi_{i+1}(\Sigma X)$$

is an isomorphism for $i \leq 2n$; is an surjection for $i = 2n+1$; but there is no general result for larger i.

It suggests we could only find simple patterns of homotopy groups at lower levels. However, for arbitrary i, once we iterate the suspension

$$\pi_i(X) \to \pi_{i+1}(\Sigma X) \to \pi_{i+2}(\Sigma^2 X) \cdots$$

the connectivity will increase along the way. Freudenthal suspension theorem tells us the sequence will eventually stabilize, meaning that the comparison maps will eventually become all isomorphism. Motivated by this phenomenon, we study stable homotopy groups of spaces.

Definition 2.1.1. Let X be a pointed connected topological space. We define its i-th stable homotopy group as

$$\pi_i^s(X) := \operatorname*{colim}_{n\geq 0} \pi_{i+n}(\Sigma^n X) = \operatorname*{colim}_{n\geq 0} \pi_n(\Omega^n \Sigma^n X)$$

where Ω stands for taking based loop space.

Using the unit $\eta : id \to \Omega\Sigma$ of the (Σ, Ω)-adjunction, we can also define space-level stabilization.

Definition 2.1.2. Let X be a pointed connected topological space. We define stabilization of X as the homotopy colimit of the sequence

$$QX := \operatorname*{hocolim}_{n\geq 0}(\Omega\Sigma X \to \Omega^2\Sigma^2 X \to \Omega^3\Sigma^3 X \to \cdots)$$

where the structure map $\Omega^n\Sigma^n X \to \Omega^{n+1}\Sigma^{n+1} X$ is defined as $\Omega^n \eta \Sigma^n$ for each $n \geq 1$.

Then there is a natural stabilization map $X \to QX$. Applying π_* to this map we get $\pi_*(X) \to \pi_*(QX) \cong \pi_*^s(X)$. Freudenthal suspension theorem tells us the stabilization map $X \to QX$ is $(2n+1)$-connected when X is n-connected.

Stable homotopy groups only remember information of a space that remains after sufficiently many applications of the suspension functor. Also the Hurewicz type result above suggests the stable homotopy groups only remember the lower level unstable homotopy groups depending on the connectivity of the space, hence losing much information. However, as we will see in Theorem 2.3.2, if we are working with nice spaces, then we will be able to "recover" a space from its stabilization using Bousfield-Kan type completion.

2.2 Bousfield-Kan completion

Carlsson [11] shows we can iterate the stabilization map to form a Bousfield-Kan [10] type completion with respect to stabilization Q.

In more detail, we construct cosimplicial resolution by iterating the stabilization map

$$X \longrightarrow QX \rightrightarrows Q^2X \mathrel{\substack{\rightarrow\\ \rightarrow\\ \rightarrow}} Q^3X \ \cdots$$

Taking homotopy limit gives the natural Q-completion map

$$c\colon\ X \to X^{\wedge}_{Q}$$

Proposition 2.2.1. *A map $f : X \to Y$ is a Q-equivalence (π^s_*-isomorphism) if and only if the induced map $f^{\wedge}_{Q} : X^{\wedge}_{Q} \to Y^{\wedge}_{Q}$ is a weak equivalence.*

Proof. The "if" direction is proved using retract argument and the "only if" direction is because holim_{Δ} preserves weak equivalences. See [10, I.5]. □

One can think of Q-completion as "constructing an approximation of X out of stabilization". We will see in Theorem 2.3.2 that the Q-completion map $c\colon\ X \to X^{\wedge}_{Q}$ is a weak equivalence for any nilpotent space X. In particular, if X is 1-connected (simply connected), we can provide some intuition why we expect the approximation map to be a weak equivalence. The following argument appears in the work of Blomquist-Harper [7], following the idea of Dundas [18].

Let X be a 1-connected space. Consider the stabilization map $X \to QX$. We have seen the stabilization map is 3-connected, hence QX is also 1-connected. We want to find better approximation of X, but simply applying stabilization map to QX wouldn't help. Our approach here is to consider the map $X \to QX$ as an object and apply stabilization to this map. We can show the resulting "2-dimensional stabilization map" is "4-connected".

More precisely, consider the cube

$$\begin{tikzcd} X \arrow[r] \arrow[d] & QX \arrow[d] \\ QX \arrow[r] & QQX \end{tikzcd} \tag{2.1}$$

as described above. Blomquist-Harper [7] proved the 2-cube (2.1) is 4-cartesian. Let T_2X be the homotopy limit of the punctured cube (without the initial object)

$$\begin{tikzcd} & QX \arrow[d] \\ QX \arrow[r] & QQX \end{tikzcd}$$

Then the 2-cube (2.1) is 4-cartesian means the map $X \to T_2X$ is 4-connected. Hence T_2X is a "one-level better approximation" of X compared to QX.

Using similar idea, we can apply stabilization map to the 2-cube (2.1) to get a 3-cube. Let T_3X be the homotopy limit of the punctured 3-cube (without the initial object). One can show the 3-cube is 5-cartesian, meaning the map $X \to T_3X$ is 5-connected. In this way, we build $T_4X, T_5X, T_6X, \cdots$ as better and better approximations of X.

Let $T_\infty X$ be the homotopy limit of such approximations (note there are naturally induced structure maps $T_nX \to T_{n-1}X$)

$$T_\infty X := \operatorname*{holim}_n T_nX$$

Then $X \to T_\infty X$ is a weak equivalence. It is proved in Blomquist-Harper [7] that $T_\infty X$ is weakly equivalent to $X^\wedge_Q$. Hence $c\colon X \to X^\wedge_Q$ is a weak equivalence for 1-connected space X.

2.3 Q-Whitehead theorem for nilpotent spaces

We first recall the definition for nilpotent spaces.

Definition 2.3.1. A pointed connected space X is called *nilpotent* if it has a principally refined Postnikov tower.

In more detail, by principally refined Postnikov tower we mean a Postnikov tower

$$\begin{tikzcd} & \vdots \arrow[d] \\ & P_2X \arrow[d] \\ & P_1X \arrow[d] \\ X \arrow[r] \arrow[ur] \arrow[uur] & P_0X \simeq * \end{tikzcd}$$

for X such that each structure map $P_nX \to P_{n-1}X$ ($n \geq 1$) can be factored as a finite composition of maps

$$P_nX = M_{t_n} \to \cdots \to M_2 \to M_1 \to M_0 = P_{n-1}X$$

such that, for each $t_n \geq i \geq 1$, the map $M_i \to M_{i-1}$ fits into a homotopy pullback diagram of the form

$$\begin{tikzcd} M_i \arrow[r] \arrow[d] & *^{\mathrm{fat}} \arrow[d] \\ M_{i-1} \arrow[r] & K(G_i, n+1) \end{tikzcd}$$

where $*^{\mathrm{fat}}$ is a contractible space and $K(G_i, n+1)$ is an Eilenberg-MacLane space.

Intuitively, a connected space is nilpotent if it can be constructed as certain "twisted product" of Eilenberg-MacLane spaces. One can think of a principally refined Postnikov tower as the dual notion of a CW structure.

One can show a connected space X is nilpotent if and only if $\pi_1 X$ is a nilpotent group and $\pi_1 X$ acts nilpotently on higher homotopy groups. See, for example, May-Ponto [45].

As important examples, the class of nilpotent spaces contains all simply connected spaces and connected loop spaces. It also satisfies other nice closure properties. For example, if a connected space F is the fiber of some fibration $E \to B$ where E, B are both nilpotent, then F is also nilpotent. Hence the class of nilpotent spaces is a well behaved, yet general enough, class of spaces to work with.

Carlsson [11] proved the following result.

Theorem 2.3.2. *For any nilpotent space X, the Q-completion map $c\colon X \to X^\wedge_Q$ is a weak equivalence.*

An easy corollary is the Q-Whitehead theorem for nilpotent spaces.

Theorem 2.3.3. *Let $f : X \to Y$ be a map of nilpotent spaces. Then f is a weak equivalence if and only if f is a Q-equivalence.*

Proof. In the following commutative square

$$\begin{tikzcd} X \arrow[r, "\sim"] \arrow[d, "\alpha"] & X^\wedge_Q \arrow[d, "\beta"] \\ Y \arrow[r, "\sim"] & Y^\wedge_Q \end{tikzcd}$$

both horizontal maps are weak equivalences by Theorem 2.3.2. Hence α is a weak equivalence if and only if β is a weak equivalence, if and only if f is a Q-equivalence following Proposition 2.2.1. $\square$

Chapter 3: TQ-homology of structured ring spectra and previously known TQ-Whitehead theorems

3.1 Structured ring spectra

Spectra are used everywhere in modern algebraic topology. One can think of spectra as the topological analog of chain complex of abelian groups. Point-set level models for the symmetric monoidal smash product of spectra were desired for a long time. The first satisfying model was given by Elmendorf-Kriz-Mandell-May in terms of S-modules [20]. After that, several other models were also constructed using, for example, symmetric spectra [36, 55] and orthogonal spectra [43]. One important consequence of these models is that the point-set level well behaved smash product $\wedge$ enables us to do algebra with spectra.

For definiteness, here we choose to work with symmetric spectra. However, all of our main results should also hold in other models of spectra.

Definition 3.1.1. A ring spectrum (resp. commutative ring spectrum) $\mathcal{R}$ is just a monoid (resp. commutative monoid) in the symmetric monoidal category of symmetric spectra. For a commutative ring spectrum $\mathcal{R}$, we let $\mathsf{Mod}_{\mathcal{R}}$ be the category of $\mathcal{R}$-modules (in the category of symmetric spectra).

Operads are gadgets to describe algebraic structures more general than rings and modules. An operad $\mathcal{O}$ in $\mathsf{Mod}_{\mathcal{R}}$ consists of a sequence of $\mathcal{R}$-modules $\mathcal{O}[n]$ $(n \geq 0)$ satisfying certain conditions so that we could use $\mathcal{O}[n]$ to describe n-ary multiplications of $\mathcal{O}$-algebras. In particular, an $\mathcal{O}$-algebra X is equipped with a sequence of compatible maps

$$\mathcal{O}[n] \wedge_{\Sigma n} X^{\wedge n} \to X$$

For those readers who are unfamiliar with operads, some good places to learn about operads include [39, 44, 50].

In the rest of this paper, we fix an (arbitrary) operad $\mathcal{O}$ and work in the category $\mathsf{Alg}_{\mathcal{O}}$ of $\mathcal{O}$-algebras in $\mathsf{Mod}_{\mathcal{R}}$, where $\mathcal{R}$ is a commutative monoid in the category of symmetric spectra.

Recall that when we discuss stabilization of spaces in Chapter 2, we always assume our spaces are pointed. In order to draw the analogy, we work with reduced operads. In other words, we assume $\mathcal{O}[0] = *$ to be trivial so that $\mathcal{O}$-algebras are non-unital.

For technical reasons, we also assume the natural maps $\mathcal{R} \to \mathcal{O}[1]$ and $* \to \mathcal{O}[n]$ are flat stable cofibrations [32] in $\mathcal{R}$-modules for each $n \geq 0$; see, for instance, [15, 2.1, 6.12].

Unless otherwise specified, our results will hold for any operad $\mathcal{O}$ satisfying these assumptions. In particular, our results could specialize to the study of non-unital associative ring spectra, non-unital commutative ring spectra, and non-unital E_n ring spectra.

3.2 Topological Quillen homology

Now we are ready to define Topological Quillen (TQ) homology for $\mathcal{O}$-algebras. Unless otherwise specified, when we talk about (co)fibrations of $\mathcal{O}$-algebras, we are always working with the positive flat stable model structure on $\mathsf{Alg}_{\mathcal{O}}$ [32, 55].

TQ-homology and its relative form, topological Andre-Quillen (TAQ) homology, first introduced in [2] for commutative ring spectra (see also [3, 4, 24, 38]), are defined as derived indecomposables of $\mathcal{O}$-algebras analogous to Quillen homology of commutative algebras [1, 48]; see also [23, 28].

More precisely, let $\tau_1\mathcal{O}$ be the operad where $\tau_1\mathcal{O}[1] = \mathcal{O}[1]$ and $\tau_1\mathcal{O}[k] = *$ for $k \neq 1$. Factoring the canonical truncation map $\mathcal{O} \to \tau_1\mathcal{O}$ in the category of operads as $\mathcal{O} \to J \to \tau_1\mathcal{O}$, a cofibration followed by a weak equivalence (see [32]), we get the corresponding change of operads adjunction

$$\mathsf{Alg}_{\mathcal{O}} \underset{U}{\overset{Q}{\rightleftarrows}} \mathsf{Alg}_J \tag{3.1}$$

with left adjoint on top, where $Q(X) = J \circ_{\mathcal{O}} (X)$ and U denotes the forgetful functor (or less concisely, the "forget along the map $\mathcal{O} \to J$ functor").

Definition 3.2.1. Let X be an $\mathcal{O}$-algebra. The *Topological Quillen homology* (or TQ-homology, for short) of X is

$$\mathsf{TQ}(X) := \mathsf{R}U(\mathsf{L}Q(X))$$

the $\mathcal{O}$-algebra defined via the indicated composite of total right and left derived functors. If X is cofibrant, then $\mathsf{TQ}(X) \simeq UQ(X)$ and the unit of the (Q, U) adjunction in (3.1) is the TQ-Hurewicz map $X \to UQX$ of the form $X \to \mathsf{TQ}(X)$.

TQ-homology has been shown to enjoy properties analogous to the ordinary homology of spaces; see, for instance, [2, 3, 13, 15, 32, 40]. Furthermore, it turns out that TQ-homology is weakly equivalent to stabilization $\Omega^\infty\Sigma^\infty$ in the category of $\mathcal{O}$-algebras, under connectivity assumptions; see, for instance, [3, 40, 46, 52, 53]; a simple proof using Goodwillie's functor calculus [29] is given in [32, 1.14].

3.3 TQ-Whitehead theorem for 0-connected objects

Analogous to the construction for spaces, one can define TQ-completion of $\mathcal{O}$-algebras. Let Z be a cofibrant $\mathcal{O}$-algebra and consider the cosimplicial resolution of Z with respect to TQ-homology of the form

$$Z \longrightarrow (UQ)Z \rightrightarrows (UQ)^2 Z \rightrightarrows\!\!\!\rightarrow (UQ)^3 Z \cdots \tag{3.2}$$

in $\mathsf{Alg}_{\mathcal{O}}$, denoted $Z \to \mathbf{C}(Z)$, with coface maps obtained by iterating the TQ-Hurewicz map $\mathrm{id} \to UQ$ (Definition 3.2.1) and codegeneracy maps built from the counit map of the adjunction (Q, U) in the usual way. Taking the homotopy limit (over Δ) gives the TQ-completion map [15, 32] of the form

$$Z \to Z^\wedge_{\mathsf{TQ}} = \operatorname{holim}_\Delta \mathbf{C}(Z) \simeq \operatorname{holim}_\Delta \widetilde{\mathbf{C}(Z)} \tag{3.3}$$

in $\mathsf{Alg}_{\mathcal{O}}$, where $\widetilde{\mathbf{C}(Z)}$ denotes any functorial fibrant replacement functor $\widetilde{(-)}$ on $\mathsf{Alg}_{\mathcal{O}}$ (obtained, for instance, by running the small object argument with respect to the generating acyclic cofibrations in $\mathsf{Alg}_{\mathcal{O}}$) applied levelwise to the cosimplicial $\mathcal{O}$-algebra $\mathbf{C}(Z)$.

Theorem 3.3.1 (Ching-Harper [15])**.** *If Z is a 0-connected $\mathcal{O}$-algebra and $\mathcal{O}, \mathcal{R}$ are (-1)-connected, then the natural map $Z \to Z^\wedge_{\mathsf{TQ}}$ is a weak equivalence.*

Consequently, we can recover the previously known 0-connected TQ-Whitehead theorem.

Theorem 3.3.2. *A map $X \to Y$ between 0-connected $\mathcal{O}$-algebras is a weak equivalence if and only if it is a TQ-homology equivalence. Here, we are assuming $\mathcal{O}, \mathcal{R}$ are (-1)-connected.*

Proof. This is proved in the exact manner as Theorem 2.3.3. □

3.4 TQ-Whitehead theorem for nilpotent objects

Could we generalize 0-connected TQ-Whitehead theorem? In particular, could we get rid of connectivity assumptions? As an attempt in this direction, Ching-Harper studied nilpotent $\mathcal{O}$-algebras [14], which are $\mathcal{O}$-algebras with "truncated" ring structures.

Definition 3.4.1. Let X be an $\mathcal{O}$-algebra and $M \geq 2$. We say that X is *M-nilpotent* if all the M-ary and higher operations $\mathcal{O}[t] \wedge X^{\wedge t} \to X$ of X are trivial (i.e., if these maps factor through the trivial $\mathcal{R}$-module $*$ for each $t \geq M$). An $\mathcal{O}$-algebra is *nilpotent* if it is M-nilpotent for some $M \geq 2$.

For example, if X is a non-unital commutative ring spectra, then X/X^n is n-nilpotent.

Ching-Harper studied a different TQ-completion construction which is specifically defined for nilpotent $\mathcal{O}$-algebras [14].

For each $n \geq 1$, let $\tau_n\mathcal{O}$ be the operad associated to $\mathcal{O}$ where

$$(\tau_n\mathcal{O})[t] := \begin{cases} \mathcal{O}[t] & \text{for } t \leq n \\ * & \text{otherwise} \end{cases}$$

and consider the associated commutative diagram of operad maps

$$\begin{tikzcd} \mathcal{O} \arrow[r] \arrow[dr, bend right] & J_n \arrow[r] \arrow[d, "\sim"] & J_1 \arrow[r, equal] \arrow[d, "\sim"] & J \\ & \tau_n\mathcal{O} \arrow[r] & \tau_1\mathcal{O} & \end{tikzcd}$$

where the upper horizontal maps are cofibrations of operads, the left-hand and bottom horizontal maps are the natural truncations, and the vertical maps are weak equivalences of operads; for notational simplicity, here we take $J = J_1$. The corresponding change of operad adjunctions have the form

$$\mathsf{Alg}_{\mathcal{O}} \underset{V_n}{\overset{R_n}{\rightleftarrows}} \mathsf{Alg}_{J_n} \underset{U_n}{\overset{Q_n}{\rightleftarrows}} \mathsf{Alg}_J \qquad \mathsf{Alg}_{\mathcal{O}} \underset{U}{\overset{Q}{\rightleftarrows}} \mathsf{Alg}_J$$

with left adjoints on top, where $R_n = J_n \circ_{\mathcal{O}} (-)$, $Q_n = J \circ_{J_n} (-)$, $Q = J \circ_{\mathcal{O}} (-)$, and V_n, U_n, U denote the indicated forgetful functors; in particular, the adjunction on the right, which coincides with adjunction (3.1), is the composite of the adjunctions on the left.

Let $n \geq 1$ and define $M := n+1$. Note every M-nilpotent $\mathcal{O}$-algebra is naturally weakly equivalent to a cofibrant J_n-algebra (regarded as an $\mathcal{O}$-algebra via forgetful functor) [14]. Now we recall the $\mathsf{TQ}|_{\mathrm{Nil}_M}$-completion construction for cofibrant J_n-algebras.

Let X be a cofibrant J_n-algebra and consider the cosimplicial resolution

$$\begin{tikzcd} X \arrow[r] & (U_nQ_n)X \arrow[r, shift left] \arrow[r, shift right] & (U_nQ_n)^2X \arrow[l] \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & (U_nQ_n)^3X \arrow[l, shift left] \arrow[l, shift right] & \cdots \end{tikzcd}$$

in Alg_{J_n}, denoted $X \to \mathbf{N}(X)$, with coface maps obtained by iterating the unit map $\mathrm{id} \to U_nQ_n$ of the adjunction (Q_n, U_n) and codegeneracy maps built from the counit map in the usual way. Applying the forgetful functor V_n gives the diagram

$V_nX \to V_n\mathbf{N}(X)$ of the form

$$V_nX \longrightarrow V_n(U_nQ_n)X \rightleftarrows V_n(U_nQ_n)^2X \Rrightarrow V_n(U_nQ_n)^3X \cdots$$

in $\mathsf{Alg}_{\mathcal{O}}$. Taking the homotopy limit (over Δ) gives the $\mathsf{TQ}|_{\mathrm{Nil}_M}$-completion map

$$V_nX \to X^{\wedge}_{\mathsf{TQ}|_{\mathrm{Nil}_M}} = \operatorname{holim}_\Delta V_n\mathbf{N}(X) \simeq \operatorname{holim}_\Delta \widetilde{V_n\mathbf{N}(X)} \tag{3.4}$$

in $\mathsf{Alg}_{\mathcal{O}}$, where $\widetilde{V_n\mathbf{N}(X)}$ denotes any functorial fibrant replacement functor $\widetilde{(-)}$ on $\mathsf{Alg}_{\mathcal{O}}$ applied levelwise to the cosimplicial $\mathcal{O}$-algebra $V_n\mathbf{N}(X)$.

Theorem 3.4.2 (Ching-Harper [14])**.** *Let X be a cofibrant J_n-algebra, then the induced $\mathsf{TQ}|_{\mathrm{Nil}_M}$-completion map $V_nX \simeq X^{\wedge}_{\mathsf{TQ}|_{\mathrm{Nil}_M}}$ is a weak equivalence.*

Consequently, we get the nilpotent TQ-Whitehead theorem.

Theorem 3.4.3. *A map $X \to Y$ between nilpotent $\mathcal{O}$-algebras is a weak equivalence if and only if it is a TQ-homology equivalence.*

Proof. Without loss of generality, we assume X is j-nilpotent and Y is k-nilpotent where $j \leq k$. Then X is also k-nilpotent. Note that up to weak equivalence, k-nilpotent $\mathcal{O}$-algebras can naturally be replaced by cofibrant J_{k-1}-algebras [14]. Hence without loss of generality, we may assume $X \to Y$ is a map between cofibrant J_{k-1}-algebras. The rest of the proof is analogous to the proof of Theorem 2.3.3. $\square$

Chapter 4: TQ-localization of structured ring spectra

In Chapter 3, we have reviewed the result of Ching-Harper [15] (see Theorem 3.3.1) that the TQ-completion map $X \to X^{\wedge}_{\mathsf{TQ}}$ is a weak equivalence for 0-connected X; here, $\mathcal{O}, \mathcal{R}$ are assumed to be (-1)-connected.

Francis-Gaitsgory [21, 3.4.5] conjectured that (i) the TQ-completion map should be a weak equivalence for a larger class of $\mathcal{O}$-algebras called homotopy pro-nilpotent $\mathcal{O}$-algebras.

Definition 4.0.1. An $\mathcal{O}$-algebra is *homotopy pro-nilpotent* if it is weakly equivalent to the homotopy limit of a tower of nilpotent $\mathcal{O}$-algebras (Definition 3.4.1).

It is worth pointing out that homotopy pro-nilpotent $\mathcal{O}$-algebras need not be nilpotent; the following describes a large class of such $\mathcal{O}$-algebras.

Proposition 4.0.2. *If X is a 0-connected $\mathcal{O}$-algebra and $\mathcal{O}, \mathcal{R}$ are (-1)-connected, then X is homotopy pro-nilpotent.*

Proof. This is proved in Harper-Hess [32, 1.12] by showing that the homotopy completion tower of X converges strongly to X. □

Our main result, Theorem 1.0.1, is that conjecture (i) is true in general, provided that in the comparison map we replace "TQ-completion" with "TQ-localization". The intuition is that TQ-localization provides a "better" model than TQ-completion for "the part of an $\mathcal{O}$-algebra that TQ-homology detects". TQ-completion is known to only be "the right model" when the $\mathcal{O}$-algebra X is TQ-good (i.e., when the comparison map from X to its TQ-completion is a TQ-equivalence) analogous to the situation for spaces [10], but homotopy pro-nilpotent $\mathcal{O}$-algebras are not expected to be TQ-good in general. However, TQ-localization always gives "the right model" for the part of the $\mathcal{O}$-algebra X that TQ-homology sees (at the expense of a much larger construction); just like Bousfield's localization construction [9] for pointed spaces.

In this Chapter, we will establish the TQ-local homotopy theory for $\mathcal{O}$-algebras, where the upshot is that: if X is a cofibrant $\mathcal{O}$-algebra, then its weak TQ-fibrant replacement $X \to L_{\mathsf{TQ}}(X)$ is the TQ-localization of X.

In Chapter 5, we will prove homotopy pro-nilpotent $\mathcal{O}$-algebras are TQ-local by leverage the TQ-local homotopy theory with the fact, proved in [14], that M-nilpotent $\mathcal{O}$-algebras are $\mathsf{TQ}|_{\mathrm{Nil}_M}$-complete.

4.1 Detecting TQ-local $\mathcal{O}$-algebras

Definition 4.1.1. A map $i\colon A \to B$ of $\mathcal{O}$-algebras is a *strong cofibration* if it is a cofibration between cofibrant objects in $\mathsf{Alg}_{\mathcal{O}}$.

Definition 4.1.2. Let X be an $\mathcal{O}$-algebra. We say that X is TQ-*local* if (i) X is fibrant in $\mathsf{Alg}_{\mathcal{O}}$ and (ii) every strong cofibration $A \to B$ that induces a weak equivalence $\mathsf{TQ}(A) \simeq \mathsf{TQ}(B)$ on TQ-homology, induces a weak equivalence

$$\mathbf{Hom}(A, X) \xleftarrow{\simeq} \mathbf{Hom}(B, X) \tag{4.1}$$

on mapping spaces in sSet.

Remark 4.1.3. The intuition here is that the derived space of maps into a TQ-local $\mathcal{O}$-algebra cannot distinguish between TQ-equivalent $\mathcal{O}$-algebras (Proposition 4.1.8), up to weak equivalence.

It is not difficult to show the following TQ-Whitehead theorem for TQ-local $\mathcal{O}$-algebras.

Proposition 4.1.4 (TQ-local Whitehead theorem)**.** *A map $X \to Y$ between TQ-local $\mathcal{O}$-algebras is a weak equivalence if and only if it is a TQ-homology equivalence.*

Proof. This follows from the definition of TQ-local $\mathcal{O}$-algebras; see, for instance, Hirschhorn [35, 3.2.13]. □

Evaluating the map (4.1) at level 0 gives a surjection

$$\mathrm{hom}(A, X) \leftarrow \mathrm{hom}(B, X)$$

of sets, since acyclic fibrations in sSet are necessarily levelwise surjections. This suggests that TQ-local $\mathcal{O}$-algebras X might be detected by a right lifting property and motivates the following classes of maps (Proposition 4.1.13); compare with Bousfield [9].

Definition 4.1.5 (TQ-local homotopy theory: Classes of maps)**.** A map $f\colon X \to Y$ of $\mathcal{O}$-algebras is

(i) a TQ-*equivalence* if it induces a weak equivalence $\mathsf{TQ}(X) \simeq \mathsf{TQ}(Y)$

(ii) a TQ-*cofibration* if it is a cofibration in $\mathsf{Alg}_{\mathcal{O}}$

(iii) a TQ-*fibration* if it has the right lifting property with respect to every cofibration that is a TQ-equivalence

(iv) a *weak* TQ-*fibration* (or TQ-*injective fibration; see Jardine [37]*) if it has the right lifting property with respect to every strong cofibration that is a TQ-equivalence

A cofibration (resp. strong cofibration) is called TQ-*acyclic* if it is also a TQ equivalence. Similarly, a TQ-fibration (resp. weak TQ-fibration) is called TQ-*acyclic* if it is also a TQ-equivalence.

Remark 4.1.6. The additional class of maps (iv) naturally arises in the TQ-local homotopy theory established below (Theorem 4.3.14); this is a consequence of the fact that the model structure on $\mathsf{Alg}_{\mathcal{O}}$ is almost never left proper, in general (e.g., associative ring spectra are not left proper); see, for instance, [49, 2.10]. In the very few special cases where it happens that $\mathsf{Alg}_{\mathcal{O}}$ is left proper (e.g., commutative ring spectra are left proper), then the class of weak TQ-fibrations will be identical to the class of TQ-fibrations.

Proposition 4.1.7. *The following implications are satisfied*

$$\begin{aligned} \textit{strong cofibration} &\implies \textit{cofibration} \\ \textit{weak equivalence} &\implies \mathsf{TQ}\textit{-equivalence} \\ \mathsf{TQ}\textit{-fibration} &\implies \textit{weak } \mathsf{TQ}\textit{-fibration} \implies \textit{fibration} \end{aligned}$$

for maps of $\mathcal{O}$*-algebras.*

Proof. The first implication is immediate and the second is because TQ preserves weak equivalences, by construction. The third implication is because the class of TQ-acyclic cofibrations contains the class of TQ-acyclic strong cofibrations. For the last implication, recall that a map is a fibration in $\mathsf{Alg}_{\mathcal{O}}$ if and only if it has the right lifting property with respect to the set of generating acyclic cofibrations. Since the generating acyclic cofibrations have cofibrant domains [57], they are contained in the class of strong cofibrations that are weak equivalences, and hence they are contained in the class of TQ-acyclic strong cofibrations. It follows immediately that every weak TQ-fibration is a fibration. $\square$

Proposition 4.1.8. *Let X be a fibrant $\mathcal{O}$-algebra. Then X is TQ-local if and only if every map $f\colon A \to B$ between cofibrant $\mathcal{O}$-algebras that is a TQ-equivalence induces a weak equivalence* (4.1) *on mapping spaces.*

Proof. It suffices to verify the "only if" direction. Consider any map $f\colon A \to B$ between cofibrant $\mathcal{O}$-algebras that is a TQ-equivalence. Factor f as a cofibration i followed by an acyclic fibration p in $\mathsf{Alg}_{\mathcal{O}}$. Since f is a TQ-equivalence and p is a weak equivalence, it follows that i is a TQ-equivalence. The left-hand commutative diagram induces

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i"] & B \\ B' \arrow[ur, bend right, "p"'] & \end{tikzcd} \qquad \begin{tikzcd} \mathbf{Hom}(A,X) & \mathbf{Hom}(B,X) \arrow[l, "(*)"'] \arrow[dl, bend left, "(\#)"] \\ \mathbf{Hom}(B',X) \arrow[u, "(**)"] & \end{tikzcd}$$

the right-hand commutative diagram. Since p is a weak equivalence between cofibrant objects and X is fibrant, we know that $(\#)$ is a weak equivalence, hence $(*)$ is a weak equivalence if and only if $(**)$ is a weak equivalence. Since i is a strong cofibration, by construction, this completes the proof. $\square$

Proposition 4.1.9. *Consider any map $f\colon X \to Y$ of $\mathcal{O}$-algebras. Then the following are equivalent:*

(i) f is a weak TQ-fibration and TQ-equivalence

(ii) f is a TQ-fibration and TQ-equivalence

(iii) f is a fibration and weak equivalence

Proof. We know that (iii) $\Rightarrow$ (ii) because weak equivalences are TQ-equivalences (Proposition 4.1.7) and acyclic fibrations have the right lifting property with respect to cofibrations. Note that (ii) $\Rightarrow$ (i) by Proposition 4.1.7, hence it suffices to verify the implication (i) $\Rightarrow$ (iii). Suppose f is a weak TQ-fibration and TQ-equivalence; let's verify that f is an acyclic fibration. Since every generating cofibration for $\mathsf{Alg}_{\mathcal{O}}$ is a strong cofibration, it suffices to verify that f has the right lifting property with respect to strong cofibrations. Let $i\colon A \to B$ be a strong cofibration. We want to verify that the left-hand solid commutative diagram

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, "i"'] & X \arrow[d, "f"] \\ B \arrow[r, "h"'] & Y \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "g'"] \arrow[d, "i"'] & \tilde{X} \arrow[r, "g''"] \arrow[dr, "f'"'] & X \arrow[d, "f"] \\ B \arrow[ur, dotted, "\xi"] \arrow[rr, "h"'] & & Y \end{tikzcd}$$

in $\mathsf{Alg}_{\mathcal{O}}$ has a lift. We factor g as a cofibration followed by an acyclic fibration $A \xrightarrow{g'} \tilde{X} \xrightarrow{g''} X$ in $\mathsf{Alg}_{\mathcal{O}}$. It follows easily that the composite $f' := fg''$ is a weak TQ-fibration and TQ-equivalence with cofibrant domain. To verify that the desired lift ξ exists, it is enough to check that f' is an acyclic fibration.

We factor f' as a cofibration followed by an acyclic fibration $\tilde{X} \xrightarrow{j} \tilde{Y} \xrightarrow{p} Y$ in $\mathsf{Alg}_{\mathcal{O}}$, and since f', p are TQ-equivalences, it follows that j is a TQ-equivalence. Hence j is a TQ-acyclic strong cofibration and the left-hand solid commutative diagram

$$\begin{tikzcd} \tilde{X} \arrow[r, equal] \arrow[d, "j"'] & \tilde{X} \arrow[d, "f'"] \\ \tilde{Y} \arrow[ur, dotted, "\eta"] \arrow[r, "p"'] & Y \end{tikzcd} \qquad \begin{tikzcd} \tilde{X} \arrow[r, "j"] \arrow[d, "f'"'] & \tilde{Y} \arrow[r, dotted, "\eta"] \arrow[d, "p"] & \tilde{X} \arrow[d, "f'"] \\ Y \arrow[r, equal] & Y \arrow[r, equal] & Y \end{tikzcd} \tag{4.2}$$

has a lift η. It follows that the right-hand diagram commutes with upper horizontal composite the identity map; in particular, f' is a retract of p. Therefore f' is an acyclic fibration which completes the proof. $\square$

The following is proved, for instance, in [15, 7.6].

Proposition 4.1.10. *If A is a cofibrant $\mathcal{O}$-algebra and $K \in \mathsf{sSet}$, then there are isomorphisms $Q(A\dot{\otimes}K) \cong Q(A)\dot{\otimes}K$ in Alg_J (Definition 3.2.1), natural in A, K.*

Proposition 4.1.11. *If $j\colon A \to B$ is a strong cofibration in $\mathsf{Alg}_{\mathcal{O}}$ and $i\colon K \to L$ is a cofibration in sSet, then the pushout corner map*

$$A\dot{\otimes}L \amalg_{A\dot{\otimes}K} B\dot{\otimes}K \to B\dot{\otimes}L$$

in $\mathsf{Alg}_{\mathcal{O}}$ is a strong cofibration that is a TQ-equivalence if j is a TQ-equivalence.

Proof. We know that the pushout corner map is a strong cofibration by the simplicial model structure on $\mathsf{Alg}_{\mathcal{O}}$ (see, for instance, [32]), hence it suffices to verify that Q applied to this map is a weak equivalence. Since Q is a left Quillen functor, it follows that the pushout corner map

$$Q(A)\dot{\otimes}L \amalg_{Q(A)\dot{\otimes}K} Q(B)\dot{\otimes}K \to Q(B)\dot{\otimes}L$$

is a cofibration that is a weak equivalence if $Q(A) \to Q(B)$ is a weak equivalence, and Proposition 4.1.10 completes the proof. $\square$

Proposition 4.1.12. *If $j\colon A \to B$ is a strong cofibration and $p\colon X \to Y$ is a weak TQ-fibration of $\mathcal{O}$-algebras, then the pullback corner map*

$$\mathbf{Hom}(B,X) \to \mathbf{Hom}(A,X) \times_{\mathbf{Hom}(A,Y)} \mathbf{Hom}(B,Y) \tag{4.3}$$

in sSet is a fibration that is an acyclic fibration if either j or p is a TQ-equivalence.

Proof. Suppose j is a TQ-acyclic strong cofibration and p is a weak TQ-fibration. Consider any cofibration $i\colon K \to L$ in sSet. We want to show that the pullback corner map (4.3) satisfies the right lifting property with respect to i.

$$\begin{tikzcd} K \arrow[r] \arrow[d] & \mathbf{Hom}(B,X) \arrow[d] \\ L \arrow[r] \arrow[ur, dotted] & \mathbf{Hom}(A,X) \times_{\mathbf{Hom}(A,Y)} \mathbf{Hom}(B,Y) \end{tikzcd} \qquad \begin{tikzcd} A\dot{\otimes}L \amalg_{A\dot{\otimes}K} B\dot{\otimes}K \arrow[r] \arrow[d, "(*)"'] & X \arrow[d] \\ B\dot{\otimes}L \arrow[r] \arrow[ur, dotted] & Y \end{tikzcd} \tag{4.4}$$

The left-hand solid commutative diagram has a lift if and only if the corresponding right-hand solid commutative diagram has a lift. Noting that $(*)$ is a TQ-acyclic strong cofibration (Proposition 4.1.11) completes the proof of this case. Suppose j is a strong cofibration and p is a weak TQ-fibration. Consider any acyclic cofibration $i\colon K \to L$ in sSet. We want to show that the pullback corner map (4.3) satisfies the right lifting property with respect to i. The left-hand solid commutative diagram in (4.4) has a lift if and only if the corresponding right-hand solid commutative diagram has a lift. Noting that p is a fibration (Proposition 4.1.7) and $(*)$ is an acyclic cofibration (see, for instance, [32, Section 6]) completes the proof of this case. The case where j is a strong cofibration and p is a TQ-acyclic weak TQ-fibration is similar; this is because p is an acyclic fibration (Proposition 4.1.9). $\square$

Proposition 4.1.13 (Detecting TQ-local $\mathcal{O}$-algebras: Part 1)**.** *Let X be a fibrant $\mathcal{O}$-algebra. Then X is TQ-local if and only if $X \to *$ satisfies the right lifting property with respect to every TQ-acyclic strong cofibration $A \to B$ of $\mathcal{O}$-algebras (i.e., if and only if $X \to *$ is a weak TQ-fibration).*

Proof. Suppose X is TQ-local and let $i\colon A \to B$ be a TQ-acyclic strong cofibration. Let's verify that $X \to *$ satisfies the right lifting property with respect to i. We know that the induced map of simplicial sets (4.1) is an acyclic fibration, hence evaluating the induced map (4.1) at level 0 gives a surjection

$$\hom(A, X) \leftarrow \hom(B, X)$$

of sets, which verifies the desired lift exists. Conversely, consider any TQ-acyclic strong cofibration $A \to B$ of $\mathcal{O}$-algebras. Let's verify that the induced map (4.1) is an acyclic fibration. It suffices to verify the right lifting property with respect to any generating cofibration $\partial\Delta[n] \to \Delta[n]$ in sSet. Consider any left-hand solid commutative diagram of the form

$$\begin{tikzcd} \partial\Delta[n] \arrow[r] \arrow[d] & \mathbf{Hom}(B, X) \arrow[d] \\ \Delta[n] \arrow[r] \arrow[ur, dotted] & \mathbf{Hom}(A, X) \end{tikzcd} \qquad \begin{tikzcd} A\dot{\otimes}\Delta[n] \coprod_{A\dot{\otimes}\partial\Delta[n]} B\dot{\otimes}\partial\Delta[n] \arrow[r] \arrow[d, "(*)"] & X \arrow[d] \\ B\dot{\otimes}\Delta[n] \arrow[r] \arrow[ur, dotted] & * \end{tikzcd}$$

in sSet. Then the left-hand lift exists in sSet if and only if the corresponding right-hand lift exists in $\mathsf{Alg}_{\mathcal{O}}$. The map $(*)$ is a TQ-acyclic strong cofibration by Proposition 4.1.11, hence, by assumption, the lift in the right-hand diagram exists. $\square$

Remark 4.1.14. Since the generating acyclic cofibrations in $\mathsf{Alg}_{\mathcal{O}}$ have cofibrant domains, the fibrancy assumption on X in Proposition 4.1.13 could be dropped; we keep it in, however, to motivate later closely related statements (Propositions 4.2.11 and 4.3.7).

4.2 Cell $\mathcal{O}$-algebras and the subcell lifting property

Suppose we start with an $\mathcal{O}$-algebra A. It may not be cofibrant, so we can run the small object argument with respect to the set of generating cofibrations in $\mathsf{Alg}_{\mathcal{O}}$ for the map $* \to A$. This gives a factorization in $\mathsf{Alg}_{\mathcal{O}}$ as $* \to \tilde{A} \to A$ a cofibration followed by an acyclic fibration. In particular, this construction builds $\tilde{A}$ by attaching cells; we would like to think of $\tilde{A}$ as a "cell $\mathcal{O}$-algebra", and we will want to work with a useful notion of "subcell $\mathcal{O}$-algebra" obtained by only attaching a subset of the cells above. Since every $\mathcal{O}$-algebra can be replaced by such a cell $\mathcal{O}$-algebra, up to weak equivalence, the idea is that this should provide a convenient class of $\mathcal{O}$-algebras to reduce to when constructing the TQ-localization functor; this reduction strategy—to work with cellular objects—is one of the main themes in Hirschhorn [35], and it plays a key role in this paper. The first step is to recall the generating cofibrations for $\mathsf{Alg}_{\mathcal{O}}$ and to make these cellular ideas more precise in the particular context of $\mathcal{O}$-algebras needed for this paper.

Recall from [32, 7.10] that the generating cofibrations for the positive flat stable model structure on $\mathcal{R}$-modules is given by the set of maps of the form

$$\mathcal{R}{\otimes}G^H_m\partial\Delta[k]_+ \xrightarrow{i^{H,k}_m} \mathcal{R}{\otimes}G^H_m\Delta[k]_+ \qquad (m\geq 1,\ k\geq 0,\ H\subset\Sigma_m \text{ subgroup})$$

in $\mathcal{R}$-modules. For ease of notational purposes, it will be convenient to denote this set of maps using the more concise notation

$$S^{H,k}_m \xrightarrow{i^{H,k}_m} D^{H,k}_m \qquad (m\geq 1,\ k\geq 0,\ H\subset\Sigma_m \text{ subgroup})$$

where $S^{H,k}_m$ and $D^{H,k}_m$ are intended to remind the reader of "sphere" and "disk", respectively. In terms of this notation, recall from [32, 7.15] that the generating cofibrations for the positive flat stable model structure on $\mathcal{O}$-algebras is given by the set of maps of the form

$$\mathcal{O}\circ(S^{H,k}_m) \xrightarrow{\mathrm{id}\circ(i^{H,k}_m)} \mathcal{O}\circ(D^{H,k}_m) \qquad (m\geq 1,\ k\geq 0,\ H\subset\Sigma_m \text{ subgroup}) \tag{4.5}$$

in $\mathcal{O}$-algebras.

Definitions 4.2.1–4.2.4 below appear in Hirschhorn [35, 10.5.8, 10.6] in the more general context of cellular model categories; we have tailored the definitions to exactly what is needed for this paper; i.e., in the context of $\mathcal{O}$-algebras.

Definition 4.2.1. A map $\alpha\colon W \to Z$ in $\mathsf{Alg}_{\mathcal{O}}$ is a *relative cell* $\mathcal{O}$*-algebra* if it can be constructed as a transfinite composition of maps of the form

$$W = Z_0 \to Z_1 \to Z_2 \to \cdots \to Z_\infty := \operatorname*{colim}_n Z_n \cong Z$$

such that each map $Z_n \to Z_{n+1}$ is built from a pushout diagram of the form

$$\begin{tikzcd} \coprod_{i\in I_n} \mathcal{O}\circ(S^{H_i,k_i}_{m_i}) \arrow[r, "(*)"] \arrow[d, "\amalg_{i\in I_n}\mathrm{id}\circ(i^{H_i,k_i}_{m_i})"'] & Z_n \arrow[d] \\ \coprod_{i\in I_n} \mathcal{O}\circ(D^{H_i,k_i}_{m_i}) \arrow[r] & Z_{n+1} \end{tikzcd} \tag{4.6}$$

in $\mathsf{Alg}_{\mathcal{O}}$, for each $n \geq 0$. A choice of such a transfinite composition of pushouts is a *presentation* of $\alpha\colon W \to Z$ as a relative cell $\mathcal{O}$-algebra. With respect to such a presentation, the *set of cells* in α is the set $\sqcup_{n\geq 0} I_n$ and the *number of cells* in α is the cardinality of its set of cells; here, $\sqcup$ denotes disjoint union of sets.

Remark 4.2.2. We often drop explicit mention of the choice of presentation of a relative cell $\mathcal{O}$-algebra, for ease of reading purposes, when no confusion can result.

Definition 4.2.3. An $\mathcal{O}$-algebra Z is a *cell* $\mathcal{O}$*-algebra* if $* \to Z$ is a relative cell $\mathcal{O}$-algebra. The *number of cells* in Z, denoted $\#Z$, is the number of cells in $* \to Z$ (with respect to a choice of presentation of $* \to Z$).

Definition 4.2.4. Let Z be a cell $\mathcal{O}$-algebra. A *subcell* $\mathcal{O}$*-algebra* of Z is a cell $\mathcal{O}$-algebra Y built by a subset of cells in Z (with respect to a choice of presentation of

$* \to Z$). More precisely, $Y \subset Z$ is a subcell $\mathcal{O}$-algebra if $* \to Y$ can be constructed as a transfinite composition of maps of the form

$$* = Y_0 \to Y_1 \to Y_2 \to \cdots \to Y_\infty := \operatorname*{colim}_n Y_n \cong Y$$

such that each map $Y_n \to Y_{n+1}$ is built from a pushout diagram of the form

$$\begin{tikzcd}
\coprod_{j\in J_n} \mathcal{O}\circ(S^{H_j,k_j}_{m_j}) \arrow[r, "(**)"] \arrow[d, "\amalg_{j\in J_n}\mathrm{id}\circ(i^{H_j,k_j}_{m_j})"'] & Y_n \arrow[d] \\
\coprod_{j\in J_n} \mathcal{O}\circ(D^{H_j,k_j}_{m_j}) \arrow[r] & Y_{n+1}
\end{tikzcd}$$

in $\mathsf{Alg}_{\mathcal{O}}$, where $J_n \subset I_n$ and the attaching map $(**)$ is the restriction of the corresponding attaching map $(*)$ in (4.6) (taking $W = *$), for each $n \geq 0$.

Definition 4.2.5. Let Z be a cell $\mathcal{O}$-algebra. A subcell $\mathcal{O}$-algebra $Y \subset Z$ is *finite* if $\#Y$ is finite (with respect to a choice of presentation of $* \to Z$); in this case we say that Y has finitely many cells.

Remark 4.2.6. Let Z be a cell $\mathcal{O}$-algebra. A subcell $\mathcal{O}$-algebra $Y \subset Z$ can be described by giving a compatible collection of subsets $J_n \subset I_n$, $n \geq 0$, (with respect to a choice of presentation for $* \to Z$); here, *compatible* means that the corresponding attaching maps are well-defined. It follows that the resulting subcell $\mathcal{O}$-algebra inclusion $Y \subset Z$ can be constructed stage-by-stage

$$\begin{tikzcd}
* = Y_0 \arrow[r] \arrow[d, equal] & Y_1 \arrow[r] \arrow[d] & Y_2 \arrow[r] \arrow[d] & \ldots \arrow[r] & Y_\infty \arrow[r, "\cong"] \arrow[d] & Y \arrow[d] \\
* = Z_0 \arrow[r] & Z_1 \arrow[r] & Z_2 \arrow[r] & \ldots \arrow[r] & Z_\infty \arrow[r, "\cong"] & Z
\end{tikzcd}$$

as the indicated colimit.

Proposition 4.2.7. *Let Z be a cell $\mathcal{O}$-algebra. If $A \subset Z$ and $B \subset Z$ are subcell $\mathcal{O}$-algebras, then there is a pushout diagram of the form*

$$\begin{tikzcd} A \cap B \arrow[r] \arrow[d] & A \arrow[d] \\ B \arrow[r] & A \cup B \end{tikzcd} \tag{4.7}$$

in $\mathsf{Alg}_{\mathcal{O}}$, which is also a pullback diagram, where the indicated arrows are subcell $\mathcal{O}$-algebra inclusions.

Proof. This is proved in Hirschhorn [35, 12.2.2] in a more general context, but here is the basic idea: Consider $* \to Z$ with presentation as in (4.6) (taking $W = *$). Suppose that $S_n \subset I_n$ and $T_n \subset I_n$, $n \geq 0$, correspond to the subcell $\mathcal{O}$-algebras $A \subset Z$ and $B \subset Z$, respectively. Then it follows (by induction on n) that $S_n \cap T_n \subset I_n$ and $S_n \cup T_n \subset I_n$, $n \geq 0$, are compatible collections of subsets and taking $A \cap B \subset Z$ and $A \cup B \subset Z$ to be the corresponding subcell $\mathcal{O}$-algebras, respectively, completes the proof. Here, we are using the fact that every cofibration of $\mathcal{O}$-algebras is, in particular, a monomorphism of underlying symmetric spectra, and hence an effective monomorphism [35, 12.2] of $\mathcal{O}$-algebras. $\square$

The following is proved in [12, I.2.4, I.2.5].

Proposition 4.2.8. *Let M be a model category (see, for instance, [19, 3.3]).*

(a) Consider any pushout diagram of the form

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i"'] & B \arrow[d] \\ C \arrow[r, "g"] & D \end{tikzcd}$$

in M, where A, B, C are cofibrant and i is a cofibration. If f is a weak equivalence, then g is a weak equivalence.

(b) Consider any commutative diagram of the form

$$\begin{tikzcd} A_0 \arrow[d, "\simeq"'] & A_1 \arrow[l] \arrow[r] \arrow[d, "\simeq"'] & A_2 \arrow[d, "\simeq"'] \\ B_0 & B_1 \arrow[l] \arrow[r] & B_2 \end{tikzcd}$$

in M*, where* A_i, B_i *are cofibrant for each* $0 \leq i \leq 2$*, the vertical maps are weak equivalences, and* $A_0 \leftarrow A_1$ *is a cofibration. If either* $B_0 \leftarrow B_1$ *or* $B_1 \rightarrow B_2$ *is a cofibration, then the induced map*

$$A_0 \amalg_{A_1} A_2 \xrightarrow{\simeq} B_0 \amalg_{B_1} B_2$$

is a weak equivalence.

The following proposition, which is an exercise left to the reader, has been exploited, for instance, in [6, 2.1] and [35, 13.2.1]; it is closely related to the usual induced model structures on over-categories and under-categories; see, for instance, [19, 3.10].

Proposition 4.2.9 (Factorization category of a map)**.** *Let* M *be a model category and* $z\colon A \rightarrow Y$ *a map in* M*. Denote by* $\mathsf{M}(z)$ *the category with objects the factorizations* $\mathbf{X}\colon A \rightarrow X \rightarrow Y$ *of* z *in* M *and morphisms* $\xi\colon \mathbf{X} \rightarrow \mathbf{X}'$ *the commutative diagrams of the form*

$$\begin{tikzcd} \mathbf{X}: \arrow[d, "\xi"] & A \arrow[r] \arrow[d, equal] & X \arrow[r] \arrow[d, "\xi"] & Y \arrow[d, equal] \\ \mathbf{X}': & A \arrow[r] & X' \arrow[r] & Y \end{tikzcd}$$

in M*. Define a map* $\xi\colon \mathbf{X} \rightarrow \mathbf{X}'$ *to be a weak equivalence (resp. fibration, resp. cofibration) if* $\xi\colon X \rightarrow X'$ *is a weak equivalence (resp. fibration, resp. cofibration) in* M*. With these three classes of maps,* $\mathsf{M}(z)$ *inherits a naturally occurring model* *structure from* M*. Since the initial object (resp. terminal object) in* $\mathsf{M}(z)$ *has the form* $A = A \xrightarrow{z} Y$ *(resp.* $A \xrightarrow{z} Y = Y$*), it follows that* $\mathbf{X}$ *is cofibrant (resp. fibrant) if and only if* $A \to X$ *is a cofibration (resp.* $X \to Y$ *is a fibration) in* M*.*

Proof. This appears in [6, 2.1] and is closely related to [19, 3.10] and [47, II.2.8]. □

The following subcell lifting property can be thought of as an $\mathcal{O}$-algebra analog of Hirschhorn [35, 13.2.1] as a key step in establishing localizations in left proper cellular model categories. One technical difficulty with Proposition 4.1.13 for detecting TQ-local $\mathcal{O}$-algebras is that it involves a lifting condition with respect to a collection of maps, instead of a set of maps. Proposition 4.2.10 provides our first reduction towards eventually refining the lifting criterion for TQ-local $\mathcal{O}$-algebras to a set of maps. Even though the left properness assumption in [35, 13.2.1] is almost never satisfied by $\mathcal{O}$-algebras, in general, a key observation, that goes back to the work of Goerss-Hopkins [26, 1.5] on moduli problems, is that the subcell lifting argument only requires an appropriate pushout diagram to be a homotopy pushout diagram—this is ensured (Proposition 4.2.8) by the strong cofibration condition in Proposition 4.2.10.

Proposition 4.2.10 (Subcell lifting property)**.** *Let* $p\colon X \to Y$ *be a fibration of* $\mathcal{O}$*-algebras. Then the following are equivalent:*

(a) The map p *has the right lifting property with respect to every strong cofibration* $A \to B$ *of* $\mathcal{O}$*-algebras that is a* TQ*-equivalence.*

(b) The map p *has the right lifting property with respect to every subcell* $\mathcal{O}$*-algebra inclusion* $A \subset B$ *that is a* TQ*-equivalence.*

Proof. Since every subcell $\mathcal{O}$-algebra inclusion $A \subset B$ is a strong cofibration, the implication (a) $\Rightarrow$ (b) is immediate. Conversely, suppose p has the right lifting property with respect to every subcell $\mathcal{O}$-algebra inclusion that is a TQ-equivalence. Let $i\colon A \to B$ be a strong cofibration of $\mathcal{O}$-algebras that is a TQ-equivalence and consider any solid commutative diagram of the form

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, "i"'] & X \arrow[d, "p"] \\ B \arrow[r, "h"'] \arrow[ur, dotted, "\xi"] & Y \end{tikzcd}$$

in $\mathsf{Alg}_{\mathcal{O}}$. We want to verify that a lift ξ exists. The first step is to get subcell $\mathcal{O}$-algebras into the picture. Running the small object argument with respect to the generating cofibrations in $\mathsf{Alg}_{\mathcal{O}}$, we first functorially factor the map $* \to A$ as a cofibration followed by an acyclic fibration $* \to A' \xrightarrow{a} A$, and then we functorially factor the composite map $A' \to A \to B$ as a cofibration followed by an acyclic fibration $A' \xrightarrow{i'} B' \xrightarrow{b} B$. Putting it all together, we get a commutative diagram of the form

$$\begin{tikzcd} A' \arrow[r, "a"] \arrow[d, "i'"'] & A \arrow[r, "g"] \arrow[d, "i"] & X \arrow[d, "p"] \\ B' \arrow[r, "b"] & B \arrow[r, "h"] & Y \end{tikzcd}$$

where i' is a subcell $\mathcal{O}$-algebra inclusion, by construction. Furthermore, since i is a TQ-equivalence and a, b are weak equivalences, it follows that i' is a TQ-equivalence. Denote by M the pushout of the upper left-hand corner maps i' and a, and consider the induced maps c, d, α of the form

$$\begin{tikzcd} A' \arrow[rr, "a"] \arrow[dd, "i'"'] & & A \arrow[r, "g"] \arrow[dd, "i"] \arrow[dl, bend left, "d"'] & X \arrow[dd, "p"] \\ & M \arrow[urr, dotted, "\xi'"] \arrow[dr, dotted, "\alpha"] & & \\ B' \arrow[rr, "b"] \arrow[ur, bend right, "c"] & & B \arrow[r, "h"] \arrow[uur, dotted, "\xi"'] & Y \end{tikzcd}$$

Since B', A', A are cofibrant and i' is a cofibration, we know that M is a homotopy pushout (Proposition 4.2.8); in particular, since a is a weak equivalence, it follows that c is a weak equivalence. Since c, b are weak equivalences, we know that α is a weak equivalence. By assumption, p has the right lifting property with respect to i', and hence with respect to its pushout d. In particular, a lift ξ' exists such that $\xi' d = g$ and $p\xi' = h\alpha$. It turns out this is enough to conclude that a lift ξ exists such that $\xi i = g$ and $p\xi = h$. Here is why: Consider the factorization category $\mathsf{Alg}_{\mathcal{O}}(pg)$ (Proposition 4.2.9) of the map pg, together with the objects

$$\mathbf{B}:\ A \xrightarrow{i} B \xrightarrow{h} Y, \quad \mathbf{X}:\ A \xrightarrow{g} X \xrightarrow{p} Y, \quad \mathbf{M}:\ A \xrightarrow{d} M \xrightarrow{h\alpha} Y$$

Note that giving the desired lift ξ is the same as giving a map of the form

$$\begin{tikzcd}
\mathbf{X}: & A \arrow[r] \arrow[d, equal] & X \arrow[r] & Y \arrow[d, equal] \\
\mathbf{B}: \arrow[u, dotted, "\xi"'] & A \arrow[r] & B \arrow[r] \arrow[u, dotted, "\xi"'] & Y
\end{tikzcd}$$

in $\mathsf{Alg}_{\mathcal{O}}(pg)$. Also, we know from above that a lift ξ' exists; i.e., we have shown there is a map of the form

$$\begin{tikzcd}
\mathbf{X}: & A \arrow[r] \arrow[d, equal] & X \arrow[r] & Y \arrow[d, equal] \\
\mathbf{M}: \arrow[u, dotted, "\xi'"'] & A \arrow[r] & M \arrow[r] \arrow[u, dotted, "\xi'"'] & Y
\end{tikzcd}$$

in $\mathsf{Alg}_{\mathcal{O}}(pg)$. We also know from above that the map α is a weak equivalence, and hence we have a weak equivalence of the form

$$\begin{tikzcd}
\mathbf{M}: \arrow[d, dotted, "\simeq"', "\alpha"] & A \arrow[r] \arrow[d, equal] & M \arrow[r] \arrow[d, dotted, "\simeq"', "\alpha"] & Y \arrow[d, equal] \\
\mathbf{B}: & A \arrow[r] & B \arrow[r] & Y
\end{tikzcd}$$

in $\mathsf{Alg}_{\mathcal{O}}(pg)$. Since i, d are cofibrations, we know that $\mathbf{B}, \mathbf{M}$ are cofibrant in $\mathsf{Alg}_{\mathcal{O}}(pg)$, and since p is a fibration, we know that $\mathbf{X}$ is fibrant in $\mathsf{Alg}_{\mathcal{O}}(pg)$ (Proposition 4.2.9).

It follows that the weak equivalence $\alpha\colon \mathbf{M} \to \mathbf{B}$ induces an isomorphism

$$[\mathbf{M}, \mathbf{X}] \xleftarrow{\cong} [\mathbf{B}, \mathbf{X}]$$

on homotopy classes of maps in $\mathsf{Alg}_{\mathcal{O}}(pg)$, and since the left-hand side is non-empty, it follows that the right-hand side is also non-empty; in other words, there exists a map $[\xi] \in [\mathbf{B}, \mathbf{X}]$. Hence we have verified there exists a map of the form $\xi\colon \mathbf{B} \to \mathbf{X}$ in $\mathsf{Alg}_{\mathcal{O}}(pg)$; in other words, we have shown that the desired lift ξ exists. This completes the proof of the implication $(b) \Rightarrow (a)$. $\square$

Proposition 4.2.11 (Detecting TQ-local $\mathcal{O}$-algebras: Part 2)**.** *Let X be a fibrant $\mathcal{O}$-algebra. Then X is TQ-local if and only if $X \to *$ satisfies the right lifting property with respect to every subcell $\mathcal{O}$-algebra inclusion $A \subset B$ that is a TQ-equivalence.*

Proof. This follows immediately from Proposition 4.2.10. $\square$

4.3 TQ-local homotopy theory

The purpose of this section is to establish a version of Proposition 4.2.10 (see Proposition 4.3.6), and hence a corresponding version of Proposition 4.2.11 (see Proposition 4.3.7), that includes a bound on how many cells B has. Once this is accomplished, we can run the small object argument to prove the key factorization property (Proposition 4.3.12) needed to establish the associated TQ-local homotopy theory on $\mathcal{O}$-algebras (Theorem 4.3.14) and to construct the associated TQ-localization functor on cofibrant $\mathcal{O}$-algebras as a weak TQ-fibrant (Definition 4.3.15) replacement functor. Our argument can be thought of as an $\mathcal{O}$-algebra analog of the bounded cofibration property in Bousfield [9, 11.2], Goerss-Jardine [27, X.2.13], and Jardine [37, 5.2], mixed together with the subcell inclusion ideas in Hirschhorn [35, 2.3.7].

Proposition 4.3.1. *Let $i\colon A \to B$ be a strong cofibration and consider the pushout diagram of the form*

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d] & B \arrow[d] \\ * \arrow[r] & B/\!\!/A \end{tikzcd} \tag{4.8}$$

in $\mathsf{Alg}_{\mathcal{O}}$. Then there is an associated cofibration sequence of the form

$$Q(A) \to Q(B) \to Q(B/\!\!/A)$$

in Alg_J and corresponding long exact sequence of abelian groups of the form

$$\cdots \mathsf{TQ}_{s+1}(B/\!\!/A) \to \mathsf{TQ}_s(A) \to \mathsf{TQ}_s(B) \to \mathsf{TQ}_s(B/\!\!/A) \to \mathsf{TQ}_{s-1}(A) \to \cdots \tag{4.9}$$

where $\mathsf{TQ}_s(X) := \pi_s\mathsf{TQ}(X)$ denotes the s-th TQ-homology group of an $\mathcal{O}$-algebra X and π_* denotes the derived (or true) homotopy groups of a symmetric spectrum [54, 55].

Proof. This is because Q is a left Quillen functor and hence preserves cofibrations and pushout diagrams. $\square$

Definition 4.3.2. Let κ be a large enough (infinite) regular cardinal such that

$$\kappa > \big|\oplus_{s,m,k} \oplus_H \mathsf{TQ}_s\big(\mathcal{O} \circ (D_m^{H,k}/S_m^{H,k})\big)\big|$$

where the first direct sum is indexed over all $s \in \mathbb{Z},\ m \geq 1,\ k \geq 0$ and the second direct sum is indexed over all subgroups $H \subset \Sigma_m$.

Remark 4.3.3. The significance of this choice of regular cardinal κ arises from the cofiber sequence of the form

$$Q(Z_n) \to Q(Z_{n+1}) \to \coprod_{i \in I_n} Q\big(\mathcal{O} \circ (D_{m_i}^{H_i,k_i}/S_{m_i}^{H_i,k_i})\big)$$

in Alg_J associated to the pushout diagram (4.6).

Proposition 4.3.4. *Let Z be a cell $\mathcal{O}$-algebra with less than κ cells (with respect to a choice of presentation $* \rightarrow Z$). Then*

$$\left|\oplus_s \mathsf{TQ}_s(Z)\right| < \kappa$$

where the direct sum is indexed over all $s \in \mathbb{Z}$.

Proof. Using the presentation notation in (4.6) (taking $W = *$), this follows from Remark 4.3.3, together with Proposition 4.3.1, by induction on n. In more detail: Since $Z_0 = *$ we know that $| \oplus_s \mathsf{TQ}_s(Z_0)| < \kappa$. Let $n \geq 0$ and assume that

$$\left|\oplus_s \mathsf{TQ}_s(Z_n)\right| < \kappa \tag{4.10}$$

We want to show that $\left|\oplus_s \mathsf{TQ}_s(Z_{n+1})\right| < \kappa$. Consider the long exact sequence in TQ-homology groups of the form

$$\cdots \rightarrow \mathsf{TQ}_s(Z_n) \rightarrow \mathsf{TQ}_s(Z_{n+1}) \rightarrow \bigoplus_{i \in I_n} \mathsf{TQ}_s\big(\mathcal{O} \circ (D^{H_i,k_i}_{m_i}/S^{H_i,k_i}_{m_i})\big) \rightarrow \dots \tag{4.11}$$

associated to the cofiber sequence in Remark 4.3.3. It follows easily that

$$\left|\mathsf{TQ}_s(Z_{n+1})\right| \leq \left|\mathsf{TQ}_s(Z_n) \oplus \bigoplus_{i \in I_n} \mathsf{TQ}_s\big(\mathcal{O} \circ (D^{H_i,k_i}_{m_i}/S^{H_i,k_i}_{m_i})\big)\right| < \kappa$$

and hence $\left|\oplus_s \mathsf{TQ}_s(Z_{n+1})\right| < \kappa$. Hence we have verified, by induction on n, that (4.10) is true for every $n \geq 0$; noting that $Z \cong Z_\infty = \operatorname{colim}_n Z_n$ (by definition) completes the proof. □

Proposition 4.3.5 (Bounded subcell property)**.** *Let M be a cell $\mathcal{O}$-algebra and $L \subset M$ a subcell $\mathcal{O}$-algebra. If $L \neq M$ and $L \subset M$ is a TQ-equivalence, then there exists $A \subset M$ subcell $\mathcal{O}$-algebra such that*

(i) A has less than κ cells

(ii) $A \not\subset L$

(iii) $L \subset L \cup A$ *is a* TQ*-equivalence*

Proof. The main idea is to develop a TQ-homology analog for $\mathcal{O}$-algebras of the closely related argument in Bousfield's localization of spaces work [9]; we have benefited from the subsequent elaboration in Goerss-Jardine [27, X.3]. We are effectively replacing arguments in terms of adding on non-degenerate simplices with arguments in terms of adding on subcell $\mathcal{O}$-algebras; this idea to work with cellular structures appears in Hirschhorn [35] assuming left properness; however, the techniques can be made to work without the left properness assumption as indicated below.

To start, choose any $A_0 \subset M$ subcell $\mathcal{O}$-algebra such that

(i) A_0 has less than κ cells

(ii) $A_0 \not\subset L$

Here is the main idea, which is essentially a small object argument idea: We would like $L \subset L \cup A_0$ to be a TQ-equivalence (i.e., we would like $\mathsf{TQ}_*(L \cup A_0 /\!\!/ L) = 0$), but it might not be. So we do the next best thing. We build $A_1 \supset A_0$ such that when we consider the following pushout diagrams in $\mathsf{Alg}_{\mathcal{O}}$

$$\begin{tikzcd} L \arrow[r] \arrow[d] & L \cup A_0 \arrow[r] \arrow[d] & L \cup A_1 \arrow[d] \\ * \arrow[r] & L \cup A_0 /\!\!/ L \arrow[r, "(\#)"] & L \cup A_1 /\!\!/ L \end{tikzcd}$$

which are also homotopy pushout diagrams in $\mathsf{Alg}_{\mathcal{O}}$, the map $(\#)$ induces

$$\mathsf{TQ}_*(L \cup A_0 /\!\!/ L) \to \mathsf{TQ}_*(L \cup A_1 /\!\!/ L) \tag{4.12}$$

the zero map; in other words, we construct A_1 by killing off elements in the TQ-homology groups $\mathsf{TQ}_*(L \cup A_0 /\!\!/ L)$ by attaching subcell $\mathcal{O}$-algebras to A_0, but in a controlled manner. Since $L \cup A_0 \subset M$ is a subcell $\mathcal{O}$-algebra, it follows that M is weakly equivalent to the filtered homotopy colimit

$$M \cong \operatorname*{colim}_{F_i \subset M}(L \cup A_0 \cup F_i) \simeq \operatorname*{hocolim}_{F_i \subset M}(L \cup A_0 \cup F_i)$$

indexed over all finite $F_i \subset M$ subcell $\mathcal{O}$-algebras and hence

$$0 = \mathsf{TQ}_*(M /\!\!/ L) \cong \operatorname*{colim}_{F_i \subset M} \mathsf{TQ}_*(L \cup A_0 \cup F_i /\!\!/ L)$$

where the left-hand side is trivial by assumption. Hence for each $0 \neq x \in \mathsf{TQ}_*(L \cup A_0 /\!\!/ L)$ there exists a finite $F_x \subset M$ subcell $\mathcal{O}$-algebra such that the induced map

$$\mathsf{TQ}_*(L \cup A_0 /\!\!/ L) \to \mathsf{TQ}_*(L \cup A_0 \cup F_x /\!\!/ L)$$

sends x to zero. Define $A_1 := (A_0 \cup \cup_{x \neq 0} F_x) \subset M$ subcell $\mathcal{O}$-algebra. By construction the induced map (4.12) on TQ-homology groups is the zero map. Furthermore, the pushout diagram in $\mathsf{Alg}_{\mathcal{O}}$

$$\begin{tikzcd}
L \cap A_0 \arrow[r] \arrow[d] & L \arrow[d] \\
A_0 \arrow[r] & L \cup A_0
\end{tikzcd}$$

implies that $L \cup A_0 /\!\!/ L \cong A_0 /\!\!/ L \cap A_0$, hence from the cofiber sequence of the form

$$L \cap A_0 \to A_0 \to L \cup A_0 /\!\!/ L$$

in $\mathsf{Alg}_{\mathcal{O}}$ and its associated long exact sequence in TQ_* it follows that $A_1 \subset M$ subcell $\mathcal{O}$-algebra satisfies

(i) A_1 has less than κ cells

(ii) $A_1 \not\subset L$

Now we repeat the main idea above, but replacing A_0 with A_1: We would like $L \subset L \cup A_1$ to be a TQ-equivalence (i.e., we would like $\mathsf{TQ}_*(L \cup A_1 /\!\!/ L) = 0$), but it might not be. So we do the next best thing. We build $A_2 \supset A_1$ such that the induced map $\mathsf{TQ}_*(L \cup A_1 /\!\!/ L) \rightarrow \mathsf{TQ}_*(L \cup A_2 /\!\!/ L)$ is zero by attaching subcell $\mathcal{O}$-algebras to A_1, but in a controlled manner, ..., and so on: By induction we construct, exactly as above, a sequence of subcell $\mathcal{O}$-algebras

$$A_0 \subset A_1 \subset \cdots \subset A_n \subset A_{n+1} \subset \dots \tag{4.13}$$

satisfying ($n \geq 0$)

(i) A_n has less than κ cells

(ii) $A_n \not\subset L$

(iii) $\mathsf{TQ}_*(L \cup A_n /\!\!/ L) \rightarrow \mathsf{TQ}_*(L \cup A_{n+1} /\!\!/ L)$ is the zero map

Define $A := \cup_n A_n$. Let's verify that $L \subset L \cup A$ is a TQ-equivalence; this is the same as checking that $\mathsf{TQ}_*(L \cup A /\!\!/ L) = 0$. Since (4.13) is a sequence of subcell $\mathcal{O}$-algebras, it follows that $L \cup A$ is weakly equivalent to the filtered homotopy colimit

$$L \cup A \cong \operatorname*{colim}_n (L \cup A_n) \simeq \operatorname*{hocolim}_n (L \cup A_n)$$

and hence

$$\mathsf{TQ}_*(L \cup A /\!\!/ L) \cong \operatorname*{colim}_n \mathsf{TQ}_*(L \cup A_n /\!\!/ L)$$

In particular, each $x \in \mathsf{TQ}_*(L \cup A /\!\!/ L)$ is represented by an element in $\mathsf{TQ}_*(L \cup A_n /\!\!/ L)$ for some n, and hence it is in the image of the composite map

$$\mathsf{TQ}_*(L \cup A_n /\!\!/ L) \rightarrow \mathsf{TQ}_*(L \cup A_{n+1} /\!\!/ L) \rightarrow \mathsf{TQ}_*(L \cup A /\!\!/ L)$$

Since the left-hand map is the zero map by construction, this verifies that $x = 0$. Hence we have verified $L \subset L \cup A$ is a TQ-equivalence, which completes the proof. $\square$

The following is closely related to [9, 11.3], [27, X.2.14], and [37, 5.4], together with the subcell ideas in [35, 2.3.8].

Proposition 4.3.6 (Bounded subcell lifting property)**.** *Let $p\colon X \to Y$ be a fibration of $\mathcal{O}$-algebras. Then the following are equivalent:*

(a) the map p has the right lifting property with respect to every strong cofibration $A \to B$ of $\mathcal{O}$-algebras that is a TQ-equivalence.

(b) the map p has the right lifting property with respect to every subcell $\mathcal{O}$-algebra inclusion $A \subset B$ that is a TQ-equivalence and such that B has less than κ cells (Definition 4.3.2).

Proof. The implication (a) $\Rightarrow$ (b) is immediate. Conversely, suppose p has the right lifting property with respect to every subcell $\mathcal{O}$-algebra inclusion $A \subset B$ that is a TQ-equivalence and such that B has less than κ cells. We want to verify that p satisfies the lifting conditions in (a); by the subcell lifting property, it suffices to verify that p satisfies the lifting conditions in Proposition 4.2.10(b). Let $A \subset B$ be a subcell $\mathcal{O}$-algebra inclusion that is a TQ-equivalence and consider any left-hand solid commutative diagram of the form

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, "\subset"'] & X \arrow[d, "p"] \\ B \arrow[r, "h"'] \arrow[ur, dotted, "\xi"] & Y \end{tikzcd} \qquad \begin{tikzcd} A \arrow[rr, "g"] \arrow[d, "\subset"'] & & X \arrow[d, "p"] \\ A_s \arrow[r, "\subset"'] \arrow[urr, dotted, "\xi_s"] & B \arrow[r, "h"'] & Y \end{tikzcd} \tag{4.14}$$

in $\mathsf{Alg}_{\mathcal{O}}$. We want to verify that a lift ξ exists. The idea is to use a Zorn's lemma argument on an appropriate poset Ω of partial lifts, together with Proposition 4.3.5, following closely [27, X.2.14] and [35, 2.3.8]. Denote by Ω the poset of all pairs (A_s, ξ_s) such that (i) $A_s \subset B$ is a subcell $\mathcal{O}$-algebra inclusion that is a TQ-equivalence and (ii) $\xi_s \colon A_s \to X$ is a map in $\mathsf{Alg}_{\mathcal{O}}$ that makes the right-hand diagram in (4.14) commute (i.e., $\xi_s|A = g$ and $p\xi_s = h|A_s$), where Ω is ordered by the following relation: $(A_s, \xi_s) \leq (A_t, \xi_t)$ if $A_s \subset A_t$ is a subcell $\mathcal{O}$-algebra inclusion and $\xi_t|A_s = \xi_s$. Then by Zorn's lemma, this set Ω has a maximal element (A_m, ξ_m).

We want to show that $A_m = B$. Suppose not. Then $A_m \neq B$ and $A_m \subset B$ is a TQ-equivalence, hence by the bounded subcell property (Proposition 4.3.5) there exists $K \subset B$ subcell $\mathcal{O}$-algebra such that

(i) K has less than κ cells

(ii) $K \not\subset A_m$

(iii) $A_m \subset A_m \cup K$ is a TQ-equivalence

We have a pushout diagram of the left-hand form

$$\begin{tikzcd} A_m \cap K \arrow[r] \arrow[d] & A_m \arrow[d] \\ K \arrow[r] & A_m \cup K \end{tikzcd} \qquad \begin{tikzcd} A_m \cap K \arrow[r] \arrow[d] & A_m \arrow[r, "\xi_m"] & X \arrow[d, "p"] \\ K \arrow[r] \arrow[urr, dotted, "\xi"] & B \arrow[r, "h"'] & Y \end{tikzcd}$$

in $\mathsf{Alg}_{\mathcal{O}}$ where the indicated maps are inclusions, and by assumption on p, the right-hand solid commutative diagram in $\mathsf{Alg}_{\mathcal{O}}$ has a lift ξ. It follows that the induced map $\xi_m \cup \xi$ makes the following diagram

$$\begin{tikzcd} A \arrow[rrr, "g"] \arrow[d] & & & X \arrow[d, "p"] \\ A_m \arrow[r] \arrow[urrr, dotted, "\xi_m"] & A_m \cup K \arrow[r] \arrow[urr, dotted, "\xi_m \cup \xi"'] & B \arrow[r, "h"'] & Y \end{tikzcd}$$

in $\mathsf{Alg}_{\mathcal{O}}$ commute, where the unlabeled arrows are the natural inclusions. In particular, since $K \not\subset A_m$, then $A_m \neq A_m \cup K$, and hence we have constructed an element $(A_m \cup K, \xi_m \cup \xi)$ of the set Ω that is strictly greater than the maximal element (A_m, ξ_m), which is a contradiction. Therefore $A_m = B$ and the desired lift $\xi = \xi_m$ exists, which completes the proof. □

Proposition 4.3.7 (Detecting TQ-local $\mathcal{O}$-algebras: Part 3)**.** *Let X be a fibrant $\mathcal{O}$-algebra. Then X is TQ-local if and only if $X \to *$ satisfies the right lifting property with respect to every subcell $\mathcal{O}$-algebra inclusion $A \subset B$ that is a TQ-equivalence and such that B has less than κ cells (Definition 4.3.2).*

Proof. This follows immediately from Proposition 4.3.6. □

Proposition 4.3.8. *If f is a retract of g and g is a TQ-acyclic strong cofibration, then so is f.*

Proof. This is because strong cofibrations and weak equivalences are closed under retracts and Q is a left Quillen functor. □

Proposition 4.3.9. *Consider any pushout diagram of the form*

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & X \arrow[d, "j"] \\ B \arrow[r] & Y \end{tikzcd} \tag{4.15}$$

in $\mathsf{Alg}_{\mathcal{O}}$. If X is cofibrant and i is a TQ-acyclic strong cofibration, then j is a TQ-acyclic strong cofibration.

Proof. Applying Q to the diagram (4.15) gives a pushout diagram of the form

$$\begin{tikzcd} Q(A) \arrow[r] \arrow[d, "(*)"'] & Q(X) \arrow[d, "(**)"] \\ Q(B) \arrow[r] & Q(Y) \end{tikzcd}$$

in $\mathsf{Alg}_{\mathcal{O}}$. Since $(*)$ is an acyclic cofibration by assumption, it follows that $(**)$ is an acyclic cofibration, which completes the proof. $\square$

Proposition 4.3.10. *The class of* TQ*-acyclic strong cofibrations is (i) closed under all small coproducts and (ii) closed under all (possibly transfinite) compositions.*

Proof. Part (i) is because strong cofibrations are closed under all small coproducts and Q is a left Quillen functor, and part (ii) is because strong cofibrations are closed under all (possibly transfinite) compositions and Q is a left Quillen functor. $\square$

Definition 4.3.11. Denote by I_{TQ} the set of generating cofibrations in $\mathsf{Alg}_{\mathcal{O}}$ and by J_{TQ} the set of generating acyclic cofibrations in $\mathsf{Alg}_{\mathcal{O}}$ union the set of TQ-acyclic strong cofibrations consisting of one representative of each isomorphism class of subcell $\mathcal{O}$-algebra inclusions $A \subset B$ that are TQ-equivalences and such that B has less than κ cells (Definition 4.3.2).

Proposition 4.3.12. *Any map $X \to Y$ of $\mathcal{O}$-algebras with X cofibrant can be factored as $X \to X' \to Y$ a* TQ*-acyclic strong cofibration followed by a weak* TQ*-fibration.*

Proof. We know by [35, 12.4] that the set J_{TQ} permits the small object argument [35, 10.5.15], and running the small object argument for the map $X \to Y$ with respect to J_{TQ} produces a functorial factorization of the form

$$X \xrightarrow{j} X' \xrightarrow{p} Y$$

in $\mathsf{Alg}_{\mathcal{O}}$. We know that j is a TQ-acyclic strong cofibration by Propositions 4.3.9 and 4.3.10. Since J_{TQ} contains the set of generating acyclic cofibrations for $\mathsf{Alg}_{\mathcal{O}}$, we know that p is a fibration of $\mathcal{O}$-algebras, and hence it follows from Proposition 4.3.6 that p is a weak TQ-fibration, which completes the proof. $\square$

Proposition 4.3.13. *Suppose $p\colon X \to Y$ is a map of $\mathcal{O}$-algebras.*

(a) The map p is a weak TQ-fibration if and only if it satisfies the right lifting property with respect to the set of maps J_{TQ} (Definition 4.3.11).

(b) The map p is a TQ-acyclic weak TQ-fibration if and only if it satisfies the right lifting property with respect to the set of maps I_{TQ} (Definition 4.3.11).

Proof. Part (a) was verified in the proof of Proposition 4.3.12 and part (b) is because p is an acyclic fibration (Proposition 4.1.9). $\square$

Our main result, Theorem 4.3.14, is that the TQ-local homotopy theory for $\mathcal{O}$-algebras (associated to the classes of maps in Definition 4.1.5) can be established (e.g., as a semi-model structure in the sense of Goerss-Hopkins [24] and Spitzweck [58], that is both cofibrantly generated and simplicial) by localizing with respect to a set of strong cofibrations that are TQ-equivalences; see, for instance, Mandell [42], White [59], and White-Yau [60] where semi-model structures naturally arise in some interesting applications. A closely related (but different) notion of semi-model structure is explored in Fresse [22].

Theorem 4.3.14 (TQ-local homotopy theory: Semi-model structure)**.** *The category $\mathsf{Alg}_{\mathcal{O}}$ with the three distinguished classes of maps (i) TQ-equivalences, (ii) weak TQ-fibrations, and (iii) cofibrations, each closed under composition and containing all isomorphisms, has the structure of a semi-model category in the sense of Goerss-Hopkins [26, 1.1.6]; in more detail:*

(a) The category $\mathsf{Alg}_{\mathcal{O}}$ has all small limits and colimits.

(b) TQ*-equivalences, weak* TQ*-fibrations, and cofibrations are each closed under retracts; weak* TQ*-fibrations and* TQ*-acyclic weak* TQ*-fibrations are each closed under pullbacks.*

(c) If f *and* g *are maps in* $\mathsf{Alg}_{\mathcal{O}}$ *such that* gf *is defined and if two of the three maps* f, g, gf *are* TQ*-equivalences, then so is the third.*

(d) Cofibrations have the left lifting property with respect to TQ*-acyclic weak* TQ*-fibrations, and* TQ*-acyclic cofibrations with* ***cofibrant domains*** *have the left lifting property with respect to weak* TQ*-fibrations.*

(e) Every map can be functorially factored as a cofibration followed by a TQ*-acyclic weak* TQ*-fibration and every map with* ***cofibrant domain*** *can be functorially factored as a* TQ*-acyclic cofibration followed by a weak* TQ*-fibration.*

Furthermore, this semi-model structure is cofibrantly generated in the sense of Goerss-Hopkins [26, 1.1.7] with generating cofibrations the set I_{TQ} *and generating* TQ*-acyclic cofibrations the set* J_{TQ} *(Definition 4.3.11), and it is simplicial in the sense of [26, 1.1.8].*

Proof. Part (a) follows from the usual model structure on $\mathcal{O}$-algebras (see, for instance, [32]). Consider part (b). It is immediate that TQ-equivalences are closed under retracts (since weak equivalences are). We know that cofibrations are closed under retracts (e.g., by the usual model structure on $\mathcal{O}$-algebras). Noting that any right lifting property is closed under retracts and pullbacks, together with Proposition 4.3.13, verifies part (b). Part (c) is because weak equivalences satisfy the two-out-of-three property. Part (d) follows from Proposition 4.1.9 and Definition 4.1.5. The first factorization in part (e) follows from Proposition 4.1.9 by running the small object argument with respect to the set I_{TQ} and the second factorization in part (e) is Proposition 4.3.12 (obtained by running the small object argument with respect to the set J_{TQ}). This semi-model structure is cofibrantly generated in the sense of [26, 1.1.7] by Proposition 4.3.13 and is simplicial in the sense of [26, 1.1.8] by Proposition 4.1.12. □

Definition 4.3.15. An $\mathcal{O}$-algebra X is called TQ-*fibrant* (resp. *weak* TQ-*fibrant*) if $X \to *$ is a TQ-fibration (resp. weak TQ-fibration).

Proposition 4.3.16. *An $\mathcal{O}$-algebra X is* TQ*-local if and only if it is weak* TQ*-fibrant.*

Proof. This follows from Proposition 4.1.13 and Remark 4.1.14. □

Let X be an $\mathcal{O}$-algebra and run the small object argument with respect to the set I_{TQ} for the map $* \to X$; this gives a functorial factorization in $\mathsf{Alg}_{\mathcal{O}}$ as a cofibration followed by an acyclic fibration $* \to \tilde{X} \xrightarrow{\simeq} X$; in particular, $\tilde{X}$ is cofibrant. Now run the small object argument with respect to the set J_{TQ} for the map $\tilde{X} \to *$; this gives a functorial factorization in $\mathsf{Alg}_{\mathcal{O}}$ as $\tilde{X} \to L(\tilde{X}) \to *$ a TQ-acyclic strong cofibration followed by a weak TQ-fibration; in particular, $L(\tilde{X})$ is TQ-local and the natural zigzag $X \simeq \tilde{X} \to L(\tilde{X})$ is a TQ-equivalence. Hence we have verified the following theorem.

Theorem 4.3.17. *If X is an $\mathcal{O}$-algebra, then (i) there is a natural zigzag of* TQ*-equivalences of the form $X \simeq \tilde{X} \to L_{\mathsf{TQ}}(\tilde{X})$ with* TQ*-local codomain, and if furthermore X is cofibrant, then (ii) there is a natural* TQ*-equivalence of the form $X \to L_{\mathsf{TQ}}(X)$ with* TQ*-local codomain.*

Proof. Taking $L_{\mathsf{TQ}}(\tilde{X}) := L(\tilde{X})$ for part (i) and $L_{\mathsf{TQ}}(X) := L(X)$ for part (ii) completes the proof. $\square$

Chapter 5: TQ-Whitehead theorem for homotopy pro-nilpotent structured ring spectra

In Chapter 4, we established the TQ-local homotopy theory for $\mathcal{O}$-algebras, where the upshot is that: if X is a cofibrant $\mathcal{O}$-algebra, then its weak TQ-fibrant replacement $X \to L_{\mathsf{TQ}}(X)$ is the TQ-localization of X. By construction, the comparison map $X \to L_{\mathsf{TQ}}(X)$ is a cofibration that is also a TQ-equivalence such that $L_{\mathsf{TQ}}(X)$ is TQ-local. Intuitively, the TQ-localization $L_{\mathsf{TQ}}(X)$ can be thought of as "the part of X that TQ-homology sees".

In this Chapter we attack the following question: When is the comparison map $X \to L_{\mathsf{TQ}}(X)$ a weak equivalence? In other words, when is an $\mathcal{O}$-algebra X already TQ-local? For instance, we know from [15] that every connected $\mathcal{O}$-algebra is TQ-complete and hence $X \simeq L_{\mathsf{TQ}}(X)$, but we also know from [32] that every connected $\mathcal{O}$-algebra is the homotopy limit of a tower of nilpotent $\mathcal{O}$-algebras and hence X is homotopy pro-nilpotent (Definition 4.0.1); here, $\mathcal{R}, \mathcal{O}$ were assumed to be (-1)-connected.

This leads us to one of the motivations of our work: what amounts to the "first half" of a conjecture of Francis-Gaitsgory [21, 3.4.5] that (i) the natural map comparing X with its TQ-completion should be a weak equivalence for every homotopy pro-nilpotent $\mathcal{O}$-algebra X. Our main result, Theorem 1.0.1, is that (i) is true in general, provided that in the comparison map we replace "TQ-completion" with "TQ-localization". Our strategy of attack is to leverage the TQ-local homotopy theory of $\mathcal{O}$-algebras with the fact, proved in [14], that M-nilpotent $\mathcal{O}$-algebras are $\mathsf{TQ}|_{\mathrm{Nil}_M}$-complete.

In this Chapter, we also compare TQ-localization with TQ-completion and show that TQ-local $\mathcal{O}$-algebras which are TQ-good are already TQ-complete (Theorem 5.2.1). Moreover, we show that $\mathcal{O}$-algebras which admit a principally refined Postnikov tower are TQ-local (Theorem 5.3.3), provided that mild connectivity assumptions are satisfied.

5.1 TQ-local $\mathcal{O}$-algebras and $\mathsf{TQ}|_{\mathrm{Nil}_M}$-resolutions

The purpose of this section is to prove that homotopy pro-nilpotent $\mathcal{O}$-algebras are TQ-local (Theorem 1.0.1) by verifying that $\mathsf{TQ}|_{\mathrm{Nil}_M}$-resolutions of M-nilpotent $\mathcal{O}$-algebras have TQ-local fibrant replacements in $\mathsf{Alg}_{\mathcal{O}}$ (Proposition 5.1.8). Similarly, we will also show that TQ-complete $\mathcal{O}$-algebras are already TQ-local (Proposition 5.1.9).

We start by reviewing some basic properties of TQ-local $\mathcal{O}$-algebras. In particular, we will show in Proposition 5.1.6 that the homotopy limit of a small diagram of TQ-local $\mathcal{O}$-algebras is TQ-local.

Recall from Definition 4.1.2 that an $\mathcal{O}$-algebra X is called TQ-*local* if (i) X is fibrant in $\mathsf{Alg}_{\mathcal{O}}$ and (ii) every TQ-acyclic strong cofibration $A \to B$ induces a weak equivalence

$$\mathbf{Hom}(A, X) \xleftarrow{\simeq} \mathbf{Hom}(B, X)$$

on mapping spaces in sSet. We have seen in Proposition 4.1.13 that a fibrant $\mathcal{O}$-algebra X is TQ-local if and only if $X \to *$ satisfies the right lifting property with respect to every TQ-acyclic strong cofibration $A \to B$ in $\mathsf{Alg}_{\mathcal{O}}$.

Proposition 5.1.1. *Let Y be a fibrant object in Alg_J. Then $UY \in \mathsf{Alg}_{\mathcal{O}}$ is TQ-local.*

Proof. This follows from Proposition 4.1.13 by using the (Q, U) adjunction (3.1). □

Next we observe that the TQ-local property is preserved by weak equivalences between fibrant $\mathcal{O}$-algebras.

Proposition 5.1.2. *Let $X \to Y$ be a weak equivalence between fibrant objects in $\mathsf{Alg}_{\mathcal{O}}$. Then X is TQ-local if and only if Y is TQ-local.*

Proof. Let $A \to B$ be a TQ-acyclic strong cofibration. Consider the commutative diagram of mapping spaces of the form

$$\begin{tikzcd} \mathbf{Hom}(A,X) \arrow[d, "\sim"] & \mathbf{Hom}(B,X) \arrow[l] \arrow[d, "\sim"] \\ \mathbf{Hom}(A,Y) & \mathbf{Hom}(B,Y) \arrow[l] \end{tikzcd}$$

in sSet. Since the vertical maps are weak equivalences, it follows that the top map is a weak equivalence if and only if the bottom map is a weak equivalence. □

This observation generalizes as follows.

Proposition 5.1.3. *Consider any weak equivalence $X \to Y$ in $\mathsf{Alg}_{\mathcal{O}}$. Let X', Y' be fibrant replacements of X, Y, respectively, in $\mathsf{Alg}_{\mathcal{O}}$. Then X' is TQ-local if and only if Y' is TQ-local.*

Proof. By assumption, the comparison map $X \to X'$ is an acyclic cofibration and Y' is fibrant. Then it follows immediately (via lifting) that there exists a map ξ that makes the diagram

$$\begin{tikzcd} X \arrow[r, "\sim"] \arrow[d, "\sim"] & Y \arrow[d, "\sim"] \\ X' \arrow[r, dotted, "\xi"] & Y' \end{tikzcd}$$

in $\mathsf{Alg}_{\mathcal{O}}$ commute. By Proposition 5.1.2, X' is TQ-local if and only if Y' is TQ-local since ξ is a weak equivalence between fibrant objects. $\square$

Proposition 5.1.4. *Let X be an $\mathcal{O}$-algebra and suppose X', X'' are fibrant replacements of X in $\mathsf{Alg}_{\mathcal{O}}$. Then X' is TQ-local if and only if X'' is TQ-local.*

Proof. This follows from Proposition 5.1.3. $\square$

The following generalization of Proposition 5.1.1 will be used in our proof of Propositions 5.1.7 and 5.1.8.

Proposition 5.1.5. *Let Y be a (not necessarily fibrant) object in Alg_J, then every fibrant replacement of UY in $\mathsf{Alg}_{\mathcal{O}}$ is TQ-local.*

Proof. Let $UY \to \widetilde{UY}$ be a fibrant replacement of UY in $\mathsf{Alg}_{\mathcal{O}}$; in particular, $UY \to \widetilde{UY}$ is an acyclic cofibration. Let Y' be a fibrant replacement of Y in Alg_J. Then it follows immediately (via lifting) that there exists a map ξ that makes the diagram

$$\begin{tikzcd} UY \arrow[r, "\sim"] \arrow[d, "\sim"'] & UY' \arrow[d] \\ \widetilde{UY} \arrow[r] \arrow[ur, dotted, "\xi"] & * \end{tikzcd}$$

in $\mathsf{Alg}_{\mathcal{O}}$ commute. Since UY' is TQ-local (Proposition 5.1.1) and ξ is a weak equivalence between fibrant objects, $\widetilde{UY}$ is TQ-local by Proposition 5.1.2. $\square$

Proposition 5.1.6 (Preservation of the TQ-local property: Homotopy limits)**.** *The homotopy limit of a small diagram of* TQ*-local* $\mathcal{O}$*-algebras is* TQ*-local.*

Proof. This follows from the definition of TQ-local $\mathcal{O}$-algebras; see, for instance, Dror Farjoun [16, 1.A.8, 1.G] and Hirschhorn [35, 19.4.4]. Here is another, essentially equivalent, proof: It follows from Proposition 4.1.4 that the homotopy limit in $\mathsf{Alg}_{\mathcal{O}}$ of a small diagram of TQ-local $\mathcal{O}$-algebras is weakly equivalent to its homotopy limit calculated in the TQ-local homotopy theory (Theorem 4.3.14); hence, verifying that the homotopy limit in $\mathsf{Alg}_{\mathcal{O}}$ is TQ-local reduces to the usual fibrancy property of homotopy limits in a homotopy theory (in this case, in the TQ-local homotopy theory); see, for instance, Hirschhorn [35, 18.5.2], together with Ching-Harper [15, 8.9] for a discussion of homotopy limits in the context of $\mathcal{O}$-algebras. $\square$

Proposition 5.1.6 is the main advantage of working with TQ-localization instead of TQ-completion (at the expense of a much larger construction). For instance, consider any pullback diagram of the form

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d, "p"] \\ C \arrow[r] & D \end{tikzcd}$$

in $\mathsf{Alg}_{\mathcal{O}}$. It follows from Proposition 5.1.6 that if B, C, D are TQ-local and p is a fibration, then A is TQ-local. Taking $C = *$, for instance, shows that TQ-local $\mathcal{O}$-algebras play nicely with fibration sequences; this is not expected to be true, in general, if we replace "TQ-local" with "TQ-complete" (but see [51]).

Proposition 5.1.7. *If Z is a cofibrant $\mathcal{O}$-algebra, then the* TQ*-completion* (3.3) $Z^{\wedge}_{\mathsf{TQ}}$ *of Z is the homotopy limit of a small diagram of* TQ*-local $\mathcal{O}$-algebras.*

Proof. We want to show that the Δ-shaped diagram $\widetilde{\mathbf{C}(Z)}$ in (3.3) is objectwise TQ-local; i.e., that $\widetilde{\mathbf{C}(Z)}^s$ is TQ-local for each $s \geq 0$. This follows from Proposition 5.1.5. In more detail: Consider the case $s = 0$. Let $Y := QZ \in \mathsf{Alg}_J$. Then $UY = (UQ)Z$, hence it suffices to verify that $\widetilde{UY}$ is TQ-local which is true by Proposition 5.1.5. Similarly, consider the case $s \geq 1$. Let $Y := Q(UQ)^s Z \in \mathsf{Alg}_J$. Then $UY = (UQ)^{s+1}Z$, hence it suffices to verify that $\widetilde{UY}$ is TQ-local and Proposition 5.1.5 completes the proof. □

Proposition 5.1.8. *If X is a cofibrant J_n-algebra, then the* $\mathsf{TQ}|_{\mathrm{Nil}_M}$*-completion* $X^{\wedge}_{\mathsf{TQ}|_{\mathrm{Nil}_M}}$ *of X is the homotopy limit of a small diagram of* TQ*-local $\mathcal{O}$-algebras.*

Proof. We want to show that the Δ-shaped diagram $\widetilde{V_n\mathbf{N}(X)}$ in (3.4) is objectwise TQ-local; i.e., that $\widetilde{V_n\mathbf{N}(X)}^s$ is TQ-local for each $s \geq 0$. This follows from Proposition 5.1.5. In more detail: Consider the case $s = 0$. Let $Y := Q_nX \in \mathsf{Alg}_J$, then $UY = V_n(U_nQ_n)X$ and hence it suffices to verify that $\widetilde{UY}$ is TQ-local; this is true by Proposition 5.1.5. Similarly, consider the case $s \geq 1$. Let $Y := Q_n(U_nQ_n)^sX \in \mathsf{Alg}_J$. Then $UY = V_n(U_nQ_n)^{s+1}X$ and hence it suffices to verify that $\widetilde{UY}$ is TQ-local; this is true by Proposition 5.1.5 which completes the proof. □

Proposition 5.1.9. *If X is an M-nilpotent $\mathcal{O}$-algebra (resp. Z is an $\mathcal{O}$-algebra) for some $M \geq 2$, then its* $\mathsf{TQ}|_{\mathrm{Nil}_M}$*-completion* $X^{\wedge}_{\mathsf{TQ}|_{\mathrm{Nil}_M}}$ *(resp.* TQ*-completion* $Z^{\wedge}_{\mathsf{TQ}}$*) is* TQ*-local.*

Proof. By Proposition 5.1.6, it suffices to verify that $X^{\wedge}_{\mathsf{TQ}|_{\mathrm{Nil}_M}}$ (resp. $Z^{\wedge}_{\mathsf{TQ}}$) is the homotopy limit of a small diagram of TQ-local $\mathcal{O}$-algebras. Now the result follows from Propositions 5.1.8 and 5.1.7, respectively. $\square$

Proposition 5.1.10. *Let $M \geq 2$.*

(a) If X is an M-nilpotent $\mathcal{O}$-algebra, then the natural map $X \simeq X^{\wedge}_{\mathsf{TQ}|_{\mathrm{Nil}_M}}$ is a weak equivalence.

(b) If Z is a 0-connected $\mathcal{O}$-algebra and $\mathcal{O}, \mathcal{R}$ are (-1)-connected, then the natural map $Z \simeq Z^{\wedge}_{\mathsf{TQ}}$ is a weak equivalence.

Proof. Part (a) is proved in Ching-Harper [14, 2.12] and part (b) is proved in Ching-Harper [15, 1.2]. $\square$

Now we are ready to show Homotopy pro-nilpotent $\mathcal{O}$-algebras are TQ-local.

Proof of Theorem 1.0.1. Part (a) follows from Propositions 5.1.9 and 5.1.10. Part (b) and Part (c) follow from part (a), together with Proposition 5.1.6. Part (d) follows from part (c), together with Proposition 4.0.2; alternately, it follows from Propositions 5.1.9 and 5.1.10. $\square$

Remark 5.1.11. It is worth pointing out (Proposition 5.1.4) that if some fibrant replacement of an $\mathcal{O}$-algebra X is TQ-local, then every fibrant replacement of X is TQ-local.

As an application, we obtain the following homotopy pro-nilpotent TQ-Whitehead theorem that simultaneously extends the previously known 0-connected and nilpotent TQ-Whitehead theorems.

Proof of Theorem 1.0.2. This follows from Theorem 1.0.1, together with Proposition 4.1.4. □

5.2 Comparing TQ-localization with TQ-completion

In this section we discuss the relation between TQ-localization and TQ-completion. Let X be a cofibrant $\mathcal{O}$-algebra. We constructed the TQ-localization map $X \to L_{\mathsf{TQ}}(X)$ by running the small object argument in Theorem 4.3.17. By construction, $L_{\mathsf{TQ}}(X)$ is TQ-local and the TQ-localization map $X \to L_{\mathsf{TQ}}(X)$ is a TQ-acyclic strong cofibration (Definition 4.1.5).

The TQ-completion map $X \to X^{\wedge}_{\mathsf{TQ}}$ can be thought of as an approximation of the TQ-localization map. For instance, we know that the TQ-completion $X^{\wedge}_{\mathsf{TQ}}$ of X is always TQ-local (Proposition 5.1.9).

Theorem 5.2.1 (Recognizing when TQ-local $\mathcal{O}$-algebras are TQ-complete)**.** *Let X be a cofibrant $\mathcal{O}$-algebra. Then the TQ-completion map $c\colon X \to X^{\wedge}_{\mathsf{TQ}}$ factors through the TQ-localization map $l\colon X \to L_{\mathsf{TQ}}(X)$ via a commutative diagram of the form*

$$\begin{tikzcd} X \arrow[r, "c"] \arrow[d, "l"'] & X^{\wedge}_{\mathsf{TQ}} \arrow[d] \\ L_{\mathsf{TQ}}(X) \arrow[r] \arrow[ur, dotted, "\xi"] & * \end{tikzcd}$$

in $\mathsf{Alg}_{\mathcal{O}}$. Furthermore, if X is TQ-local, then the following are equivalent:

(i) The natural map $X \to X^{\wedge}_{\mathsf{TQ}}$ is a TQ-equivalence; i.e., X is TQ-good.

(ii) The natural map $X \simeq X^{\wedge}_{\mathsf{TQ}}$ is a weak equivalence; i.e., X is TQ-complete.

(iii) The comparison map ξ is a weak equivalence.

Proof. This is analogous to the Bousfield-Kan completion of spaces [10]. Since $X^\wedge_{\mathsf{TQ}}$ is TQ-local and $l\colon X \to L_{\mathsf{TQ}}(X)$ is a TQ-acyclic strong cofibration, there exists a lift ξ that makes the diagram commute (Proposition 4.1.13) in $\mathsf{Alg}_{\mathcal{O}}$. Suppose X is TQ-local, then l is a TQ-equivalence between TQ-local objects, hence a weak equivalence by the TQ-local Whitehead theorem (Proposition 4.1.4). Therefore ξ is a weak equivalence if and only if c is a weak equivalence. This verifies (ii) $\Leftrightarrow$ (iii). Since $X, X^\wedge_{\mathsf{TQ}}$ are TQ-local, c is a TQ-equivalence if and only if c is a weak equivalence by the TQ-local Whitehead theorem. This verifies (i) $\Leftrightarrow$ (ii). $\square$

It is worth pointing out the following two propositions.

Proposition 5.2.2. *A map $f\colon X \to Y$ between $\mathcal{O}$-algebras is a TQ-homology equivalence if and only if the induced map $f^\wedge_{\mathsf{TQ}}\colon X^\wedge_{\mathsf{TQ}} \to Y^\wedge_{\mathsf{TQ}}$ is a weak equivalence.*

Proof. This is proved by arguing exactly as in [10, I.5], but here is the basic idea: The "if" direction is proved using retract argument and the "only if" direction is because holim_Δ preserves weak equivalences. $\square$

Proposition 5.2.3. *Let X be an $\mathcal{O}$-algebra, then the following are equivalent:*

(i) X *is* TQ-good.

(ii) $X^\wedge_{\mathsf{TQ}}$ *is* TQ*-complete.*

(iii) $X^\wedge_{\mathsf{TQ}}$ *is* TQ*-good.*

Proof. This follows from exactly the same argument as in [10, I.5]. $\square$

5.3 Postnikov towers and TQ-localization

In this section we assume that $\mathcal{O}, \mathcal{R}$ are (-1)-connected. We show that a (-1)-connected $\mathcal{O}$-algebra is TQ-local if it has a principally refined Postnikov tower.

Proposition 5.3.1. *Let X be a (-1)-connected cofibrant $\mathcal{O}$-algebra. Then there exists a coaugmented tower $\{X\} \to \{X_n\}$ (the Postnikov tower of X) of the form*

$$\begin{tikzcd} & X \arrow[d] \arrow[dr, bend left] \arrow[drr, bend left] \arrow[drrr, bend left] & & & \\ {*} = & X_{-1} & X_0 \arrow[l] & X_1 \arrow[l] & X_2 \arrow[l] & \cdots \arrow[l] \end{tikzcd}$$

in $\mathsf{Alg}_{\mathcal{O}}$ such that for each $n \geq -1$:

(a) X_n is a cofibrant and fibrant $\mathcal{O}$-algebra.

(b) the structure map $X \to X_n$ is $(n+1)$-connected and $\pi_k X_n = *$ for all $k \geq n+1$.

(c) the structure map $X_{n+1} \to X_n$ is a fibration.

Proof. The Postnikov tower can be constructed using small object arguments analogous to the arguments in [13, 17, 20, 55]. In more detail: Let I_n be the set of n-connected generating cofibrations in $\mathsf{Alg}_{\mathcal{O}}$ and let J be the set of generating acyclic cofibrations in $\mathsf{Alg}_{\mathcal{O}}$ (see, for instance, [30]). Start by setting $X_{-1} = *$. For each $n \geq 0$, we inductively run the small object argument with respect to $I_{n+1} \bigcup J$ to factor the map $X \to X_{n-1}$ in $\mathsf{Alg}_{\mathcal{O}}$ as $X \to X_n \to X_{n-1}$. Then $X \to X_n$ is a cofibration, $X_n \to X_{n-1}$ is a fibration and $\pi_k X_n = *$ for all $k \geq n+1$. By assumption, $\mathcal{O}, \mathcal{R}$ are (-1)-connected, hence $X \to X_n$ is $(n+1)$-connected by the small object argument construction. $\square$

Analogous to the definition for spaces, principal Postnikov towers and principally refined Postnikov towers are defined as follows.

Definition 5.3.2. Let X be a (-1)-connected $\mathcal{O}$-algebra. We say that a Postnikov tower $\{X_n\}$ of X is *principal* if for each $n \geq 0$, the structure map $X_n \to X_{n-1}$ fits into a homotopy pullback diagram of the left-hand form

$$\begin{tikzcd} X_n \arrow[r] \arrow[d] & *^{\mathrm{fat}} \arrow[d] \\ X_{n-1} \arrow[r] & K(\pi_n X, n+1) \end{tikzcd} \qquad \begin{tikzcd} M_i \arrow[r] \arrow[d] & *^{\mathrm{fat}} \arrow[d] \\ M_{i-1} \arrow[r] & K(G_i, n+1) \end{tikzcd} \tag{5.1}$$

in $\mathsf{Alg}_{\mathcal{O}}$, where $*^{\mathrm{fat}}$ is an $\mathcal{O}$-algebra that is weakly equivalent to $*$ (i.e., a "fat point" in $\mathsf{Alg}_{\mathcal{O}}$) and $K(\pi_n X, n+1)$ is an object in $\mathsf{Alg}_{\mathcal{O}}$ with $\pi_n X$ as the only nontrivial homotopy group concentrated at level $n+1$.

We say that $\{X_n\}$ is *principally refined* if, for each $n \geq 0$, the structure map $X_n \to X_{n-1}$ can be factored as a finite composite $X_n = M_{t_n} \to \cdots \to M_2 \to M_1 \to M_0 = X_{n-1}$ of maps such that, for each $t_n \geq i \geq 1$, the map $M_i \to M_{i-1}$ fits into a homotopy pullback diagram of the right-hand form (5.1) in $\mathsf{Alg}_{\mathcal{O}}$, where the G_i's are abelian groups and $K(G_i, n+1)$ is an object in $\mathsf{Alg}_{\mathcal{O}}$ with G_i as the only nontrivial homotopy group concentrated at level $n+1$. In particular, every principal Postnikov tower is principally refined.

Theorem 5.3.3. *Let X be a fibrant $\mathcal{O}$-algebra. If $X, \mathcal{O}, \mathcal{R}$ are (-1)-connected and X has a principally refined Postnikov tower, then X is TQ-local.*

Proof. We know that every 0-connected fibrant $\mathcal{O}$-algebra is TQ-local (Theorem 1.0.1), hence, in particular, each Eilenberg-MacLane object $K(G_i, n+1)$ appearing in the principally refined Postnikov tower of X has TQ-local fibrant replacements in $\mathsf{Alg}_{\mathcal{O}}$. By inducting up the principally refined Postnikov tower, it follows that each X_n is TQ-local (Proposition 5.1.6). Since X is the homotopy limit of its Postnikov tower

$\{X_n\}$, which is objectwise TQ-local, it follows that X is TQ-local (Proposition 5.1.6) which completes the proof. $\square$

We provide some examples of (-1)-connected algebras which admit principally refined Postnikov towers.

(i) Let X be a cofibrant 0-connected $\mathcal{O}$-algebra. Analogous to results in [2, 5, 17, 34], one can show that the Postnikov tower of X is principal.

(ii) Consider ΩX for any 0-connected cofibrant $\mathcal{O}$-algebra X. Since the loop functor Ω commutes with homotopy pullbacks in $\mathsf{Alg}_{\mathcal{O}}$, ΩX has a principal Postnikov tower by applying Ω to the principal Postnikov tower of X.

(iii) Consider UY for any (-1)-connected cofibrant J-algebra Y. The category Alg_J is Quillen equivalent to $\mathsf{Alg}_{\tau_1 \mathcal{O}} \cong \mathsf{Mod}_{\mathcal{O}[1]}$ [32, 7.21], hence the homotopy category of Alg_J is stable. Therefore, the Postnikov tower of Y in Alg_J is already principal. Applying forgetful functor U induces principal Postnikov tower for UY in $\mathsf{Alg}_{\mathcal{O}}$.

(iv) One can construct additional examples by pulling back quotient towers along cocellular maps as described in [45, 3.3].

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