Volatility of European Options

A Thesis

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Vasudha Khandelwal,

Graduate Program in Department of Mathematics

The Ohio State University

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Master's Examination Committee:

Dr Chunsheng Ban, Advisor

Dr Bo Guan

Mr Daniel Heyer

Abstract

European options are the most fundamental financial derivatives which are extensively traded in the current world. They form as an underlying for various exotic derivatives. The objective of this thesis is to understand pricing models of European puts and calls - in a way that is consistent with the market quoted prices. Three different models are considered for it, Dupire model - deterministic local volatility model; Heston model - simple mean-reverting stochastic volatility model; and SABR model - complex non-mean-reverting stochastic volatility model. Further, their relation to market quoted Black-Scholes implied volatility is explored. Advantages and disadvantages of each model are discussed when applying it to the options on S&P500.

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Finally, I wish to thank my parents for their support and encouragement throughout my study.

Vita

June 8, 1994	Born - Rajasthan, India
Jul 2012 - May 2016	Bachelor of Technology in Mathematics and Computing, Indian Institute of Technology Guwahati, India

Fields of Study

Major Field: Mathematics

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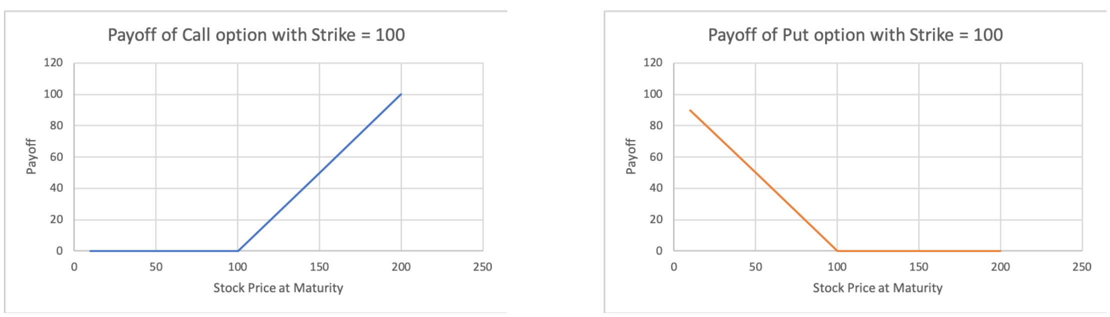
Chapter 1: Introduction

European options are the most fundamental financial derivatives defined on an underlying stock. They are of two types, call and put. The call option gives the buyer the right to buy a stock at a fixed price called strike price at the time of maturity whereas the put option gives the buyer the right to sell the stock at the strike price at the time of maturity. Fig-1.1 shows their payoff at the time of maturity.



(a) Call Option (b) Put Option

Figure 1.1: *Payoff of European Options*

Different models are used to price and analyze these options which have varying assumptions for the volatility of the underlying stochastic process, from being constant to stochastic. The objective of this thesis is to understand and implement some of these models for options on S&P500 using historical data. Three particular models are considered here, namely, Dupire model - deterministic local volatility model; Heston model - simple mean-reverting stochastic volatility model; and SABR model - complex non-mean-reverting stochastic volatility model. These models are analyzed here with two particular intent - firstly to reproduce observed market prices and secondly to forecast future prices of the options.

The most commonly used model is the Black-Scholes model which assumes that stock price S_t follows the geometric Brownian motion with constant volatility.

$$dS_t = \mu S_t \, dt + \sigma S_t \, dW_t \tag{1.1}$$

where W_t is Brownian motion, μ is the drift rate and σ represents the constant volatility. The drift rate can be thought of as the annualized change in the expected value of the stock price. Under the no dividend assumption this comes out to be the risk-neutral rate. The market price of the option is quoted in terms of the constant Black-Scholes implied volatility i.e. that value of volatility which when plugged in the Black-Scholes equation of the option price will return a theoretical value equal to the current market price of the option. Comparison of implied volatilities can further reveal information about corresponding stocks, however, this model cannot be exclusively used to price options since the implied volatility cannot be directly obtained from any market observable factor. Despite the popularity of Black-Scholes model, the fact that it uses different implied volatility for different strikes and maturities for a fixed stock defined as the volatility smile Fig-1.2 invalidates its assumption of constant volatility. This also results in unstable vega hedges[1]. Constantly changing the

[1]Vega gives the sensitivity of the option price to the volatility. Vega hedging strategy involves constructing a portfolio which is vega-neutral in other words it is insensitive to change in volatility.

volatility assumption changes the hedge ratios in an uncontrolled way. Hence to address this problem, further models were explored which don't have constant volatility assumption.

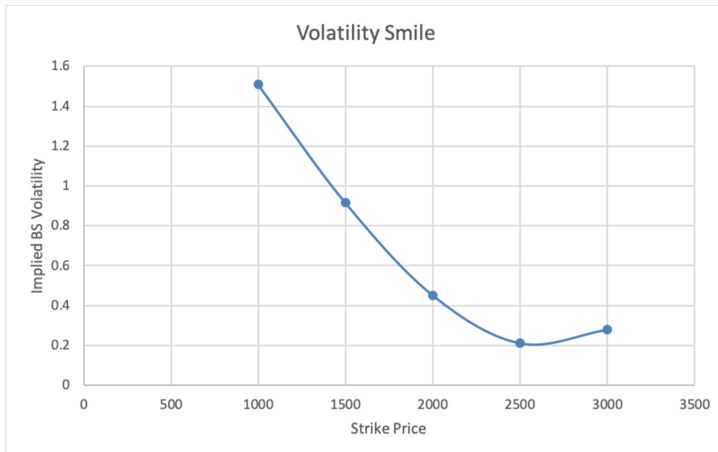


Figure 1.2: *Implied Black-Scholes Volatility for the call option on S&P500 with varying strikes as of date 2 January, 2019 maturing on 18 January, 2019.*

Local volatility models are fundamental models developed as state-dependent coefficients i.e. volatility is a function of strike price and time left to maturity. It can be thought of as a generalization of the Black-Scholes model. These coefficients can then be used directly to calculate the price of the options. This concept was first developed by Bruno Dupire [3] and Emanuel Derman and Iraj Kani [2] in 1994. They observed that there exist a unique diffusion process which is consistent with the risk-neutral densities[2] obtained from the market prices of European options. Under the Dupire

[2]A risk-neutral measure is a probability measure where the price of the stock is exactly equal to the discounted expectation of the stock price under this measure. It is possible only in arbitrage-free market.

model, the stock price has following stochastic process

$$dS_t = \mu(t)\ S_t\ dt + \sigma(t, S_t)\ S_t\ dW_t \tag{1.2}$$

where W_t is a Brownian motion and $\mu(t)$ is the risk-neutral drift of stock price process. Volatility represented by $\sigma(t, S_t)$ is a deterministic function. The Dupire model fits the volatility smile well. However, being deterministic function it proves to be inefficient in cases when the derivative is dependent on the random nature of the volatility itself.

Local volatility can be looked as an average over all instantaneous volatilities in the stochastic volatility world. Stochastic volatility models while fitting the volatility smile assumes more realistic dynamics for volatility. Most stock prices have fat-tailed daily return distribution Fig-1.3 suggesting volatility clustering and justifying mean reversion of volatility. The QQ plot in Fig-1.4 depicts how different the tail distribution of daily returns behaves as compared to the normal distribution. This stochastic realization of volatility provides more stable delta[3] and vega hedges. One of the most popular stochastic volatility models is the Heston model [6] developed in 1993. The underlying diffusion process in the Heston model is

$$\begin{aligned} dS_t &= \mu_t S_t\ dt + \sqrt{\sigma_t} S_t\ dW_1 \\ d\sigma_t &= -\lambda(\sigma_t - \bar{\sigma})\ dt + \eta\sqrt{\sigma_t}\ dW_2 \\ dW_1\ dW_2 &= \rho\ dt \end{aligned} \tag{1.3}$$

where μ_t is the drift-term, λ is the speed of mean reversion of volatility σ_t, $\bar{\sigma}$ represents the long-term mean of volatility and η is the volatility of volatility. W_1 and W_2 are Brownian motion with correlation ρ. Over here the volatility is a stationary stochastic function of the stock price with correlated chi-square increments. Calibrating the

[3]Delta is the derivative of the option value with respect to the stock price i.e. the sensitivity to stock price. Delta hedging strategy involves constructing a portfolio which is delta-neutral in other words it is insensitive to change in stock price.

Heston model is an easy process but it is computationally expensive as it involves solving the characteristic functions using Fourier transformations which appears in the option price formula of Heston model. Hence, other stochastic models were explored which are computationally more efficient.

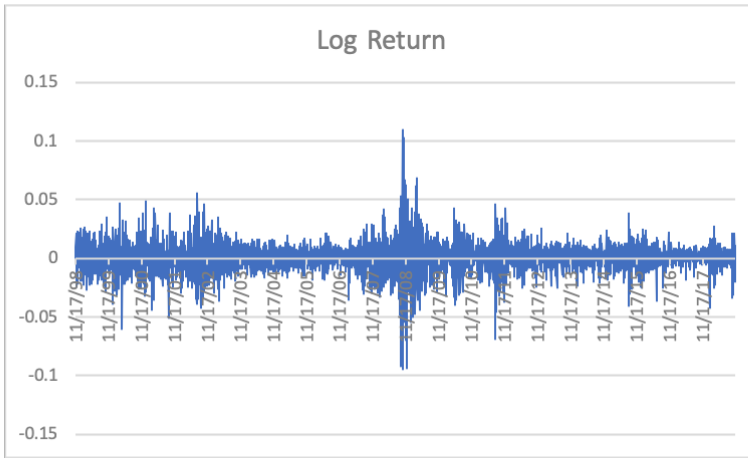


Figure 1.3: *Daily log return of S&P500 for last 20 years*

SABR also known as stochastic alpha beta rho model developed in 2002 by *Hagan et al* [5] is a stochastic volatility model based on the following equations

$$\begin{aligned} dF_t &= \sigma_t F_t^{\beta}\, dW_1 \\ d\sigma_t &= \alpha \sigma_t\, dW_2 \\ dW_1\, dW_2 &= \rho\, dt \end{aligned} \tag{1.4}$$

where F_t is the forward price of the stock and σ_t is the volatility. W_1 and W_2 are Brownian motions which have correlation ρ. The α, β, ρ represent the parameters of this model. Over here the volatility is a non-stationary stochastic function of the stock price with correlated power increments. The popularity of this model is

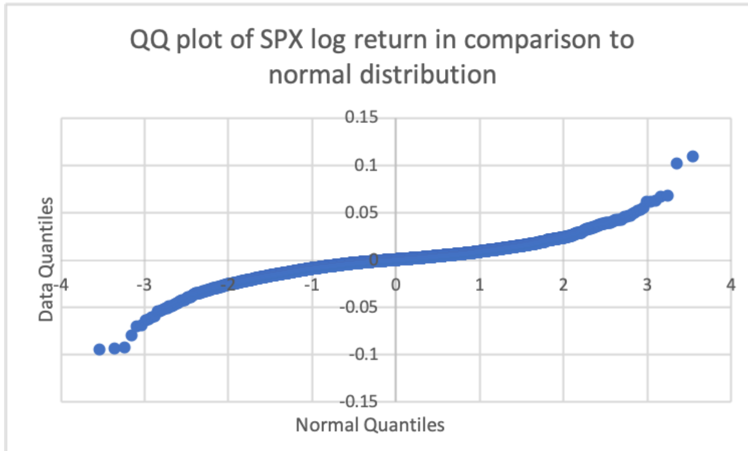


Figure 1.4: *QQ plot of daily log return of S&P500 for last 20 years with respect to standard normal distribution*

associated to the fact that it provides a closed-form solution for Black-Scholes implied volatility which makes it computationally inexpensive and it is also known to give good estimates of vanna[4] and volga[5] risks. Notice that unlike the previous models, SABR model is based on the forward price of the stock, so instead of using Black-Scholes model, its equivalent Black's model is used.

1.1 Black-Scholes Model

Assume that the stock price S_t follows a geometric Brownian motion

$$dS_t = \mu S_t \, dt + \sigma S_t \, dW_t \tag{1.5}$$

where W_t is Brownian motion, μ is the drift rate and σ represents the constant volatility. The above process is based on Black-Scholes definition of constant volatility.

[4]Vanna is the partial derivative of delta with respect to the volatility.

[5]Volga is the second order derivative of option price with respect to the volatility.

The drift rate can be thought of as the annualized change in the expected value of the stock price. Under the no dividend assumption this comes out to be the risk-neutral rate.

A risk-neutral portfolio is constructed to derive the value of an option dependent on this underlying stock. Using Ito's Lemma it can be shown that the value V_t of any derivative defined on this stock will satisfy the following stochastic differential equation [7]

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0 \tag{1.6}$$

where r is the annualized risk-free interest rate. The above equation solves out to give following arbitrage-free and risk-neutral closed-form solution for the value of European put P_t and call C_t option

$$\begin{aligned} C_t &= S_t\ N(d_1)\ - Ke^{-r(T-t)}\ N(d_2) \\ P_t &= Ke^{-r(T-t)}\ N(-d_2) - S_t\ N(-d_1) \end{aligned} \tag{1.7}$$

where

$d_1 = \frac{1}{\sigma\sqrt{T-t}}\left[\ln\left(\frac{S_t}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)(T-t)\right]$

$d_2 = d_1 - \sigma\sqrt{T-t}$

$N(.)$ is the cumulative distributive function of standard normal distribution

T is the time of maturity

K is the strike price.

1.2 Organization of the thesis

This thesis has four chapters. Ch-1 gives the objective of this thesis which is to understand several models for pricing European options. It outlines four models namely, the Black-Scholes model, Dupire model, Heston model, and the SABR model. It discusses the underlying assumptions of these models and the advantages-disadvantages of using them. Black-Scholes model is discussed here in detail.

The next chapter, Ch-2 talks about the Dupire model and the Heston model. Dupire model is a simple local volatility model. The derivation of Dupire equation is shown which is used to obtain the local volatility surface using the European option prices for different strikes and maturities. Further, a relation between local volatility and Black-Scholes implied volatility is given and analysis of this model for the call option on S&P500 is done. The next section discusses the Heston model. A closed-form solution using characteristic functions for pricing European options using Heston model is provided. Further, it is implemented for the call option on S&P500 and sensitivity analysis of its various parameters is provided.

The next chapter details the SABR model also known as stochastic alpha beta rho model. Developed by Hagan, Kumar, Lesniewski, and Woodward [5] in 2002, this model is of particular interest because of the ease of its implementation despite having stochastic volatility assumption. A generalized equation along with special cases is provided to calculate the Black-Scholes implied volatility from this model. Further analysis over call option on S&P500 is shown.

An overall summary of the thesis with conclusion is provided in the last chapter with a focus on observations made from analyzing these models on real-world data of call options on S&P500. Advantages and disadvantages of all the above models are discussed.

Chapter 2: Dupire and Heston model

This chapter describes the Dupire local volatility model and the Heston stochastic volatility model. It gives the underlying equations and assumptions on which these models are developed. These models are calibrated so as to reproduce the observed prices in the market and further provide estimates for future prices and the prices which are not quoted in the market. It then relates them to the Black-Scholes implied volatility. Analysis of these models is provided for the call options on S&P 500.

2.1 Dupire Model

Dupire [3] and Derman & Kani [2] were the first ones to realize that under risk-neutral conditions, there exists a unique diffusion process which can be used to obtain the density from the market prices of European options. This state-dependent local volatility function $\sigma(t, S_t)$ gives a deterministic way to obtain option prices. Dupire uses Fokker-Plank equation to derive local volatility whereas Derman & Kani used conditional expectation. Following is the description of Dupire model.

2.1.1 Underlying Equation

Under this model it is assumed that the stock price S_t in risk-neutral condition follows

$$dS_t \; = \mu(t)\; S_t\; dt + \sigma(t, S_t)\; S_t\; dW_t \tag{2.1}$$

where W_t is a Brownian motion and $\mu(t)$ is a time dependent deterministic variable which represents risk-neutral drift of stock price process. $\sigma(t, S_t)$ is a deterministic function of local volatility.

2.1.2 Derivation

The forward price of a call option at $t = 0$ in this model is given by

$$C(S_0, K, T) = \int_K^\infty (S_T - K)\; \phi(T, S_T)\; dS_T \tag{2.2}$$

where $\phi(T, \bullet)$ is the probability density function of stock price S_T at maturity $(t = T)$ which follows the Fokker-Plank equation [4] given by

$$\frac{1}{2}\; \frac{\partial^2}{\partial S_T^2}(\sigma^2 S_T^2 \phi) \;-\; \frac{\partial}{\partial S_T}(\mu S_T \phi) \;=\; \frac{\partial \phi}{\partial T} \tag{2.3}$$

Differentiating Eq-2.2 with respect to strike

$$\frac{\partial^2 C}{\partial K^2} \; = \phi(T, K) \tag{2.4}$$

and with respect to time

$$\begin{aligned}
\frac{\partial C}{\partial T} \; &= \int_K^\infty (S_T - K)\; \left[\frac{\partial}{\partial T}(\phi(T, x))\right]\; dS_T \\
&= \int_K^\infty (S_T - K)\; \left[\frac{1}{2}\; \frac{\partial^2}{\partial S_T^2}(\sigma^2 S_T^2 \phi) \;-\; \frac{\partial}{\partial S_T}(\mu S_T \phi)\;\right]\; dS_T \\
&= \frac{\sigma^2 K^2}{2}\; \phi + \int_K^\infty \mu S_T \phi\; dS_T \\
&= \frac{\sigma^2 K^2}{2}\; \frac{\partial^2 C}{\partial K^2} + \mu(T)\Big(-K\; \frac{\partial C}{\partial K}\Big)
\end{aligned}$$

using integration by parts. Further,

$$\sigma^2(T,K) = \frac{\frac{\partial C}{\partial T} + \mu(T)K\ \frac{\partial C}{\partial K}}{\frac{1}{2}\ K^2\ \frac{\partial^2 C}{\partial K^2}} \tag{2.5}$$

and expressing call option as a function of forward price i.e. $C(F_T, K, T)$ where $F_T = S_0 \exp\{\int_0^T \mu_t\ dt\}$ transforms above to

$$\sigma^2(T,K) = \frac{\frac{\partial C}{\partial T}}{\frac{1}{2}\ K^2\ \frac{\partial^2 C}{\partial K^2}} \tag{2.6}$$

where Dupire assumes zero interest rate and zero dividend yield.

2.1.3 Relation between local volatility and implied volatility

Equating the price of option obtained in Black-Scholes model to that of Dupire model [4] in order to find the relation between Black Scholes implied volatility σ_{BS} and local volatility σ_L gives the following

$$\sigma_L = \frac{\frac{\partial \sigma_{BS}}{\partial T}}{1 - \frac{y}{\sigma_{BS}}\ \frac{\partial \sigma_{BS}}{\partial y}\ + \frac{1}{4}(-\frac{1}{4} - \frac{1}{\sigma_{BS}} + \frac{y^2}{\sigma_{BS}^2})\ (\frac{\partial \sigma_{BS}}{\partial y})^2\ + \frac{1}{2}\ \frac{\partial^2 \sigma_{BS}}{\partial y^2}} \tag{2.7}$$

where $y = \ln(\frac{K}{F_T})$. It can be observed in the above formula that if $y = 0$ i.e. $K = F_T$ then

$$\sigma_L = \frac{\partial \sigma_{BS}}{\partial T} \tag{2.8}$$

i.e

$$\sigma_{BS} = \int_0^T \sigma_L\ dt \tag{2.9}$$

local volatility is the forward Black-Scholes implied volatility.

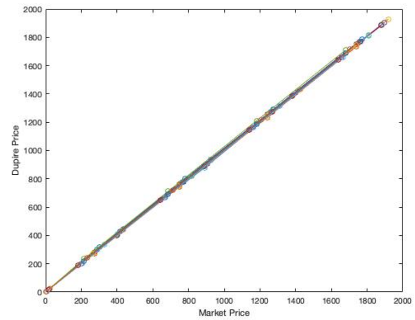
2.1.4 Analysis

The biggest challenge faced in implementing Dupire model is to come up with the interpolation function for implied Black Scholes volatility. The must condition for the interpolated volatility surface is to match exactly the market prices of liquid options. Unlike typical cases, the available option data in the market is sparse and limited i.e. options are generally traded for standard strikes and expiration dates. It is hard to avoid arbitrage in the interpolated volatilities surface when using standard interpolation techniques which is why these interpolated volatilities can't be directly used back in the Black-Scholes model. However in the Dupire model to calibrate the local volatility function, the first and second derivatives of Black-Scholes implied volatility surface with respect to time and strike are required, hence, the smoothness of the surface ensures arbitrage-free local volatility function. Another condition required for the volatility is that it should be non-negative and bounded which puts conditions on the derivatives of implied volatility surface. Hence, finding an appropriate interpolation function which satisfies all the criteria can be tricky.

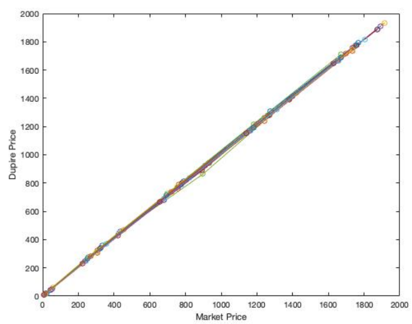
Applying the Dupire model to call option on S&P500 gives out results as shown in Fig-2.1. Here the local volatility surface is obtained from the implied Black-Scholes volatility using Eq-2.7. The option prices are further calculated using an in-built MATLAB function which uses finite difference technique. Comparing it to the Market quoted prices shows the efficiency of Dupire model. It can be observed that Dupire model is able to fit the volatility smile well however it is not an ideal choice to predict future prices of the option as it does not predict the correct behavior of the future volatility and thus produces bad hedge as seen in Table-2.1.

2.2 Heston Model

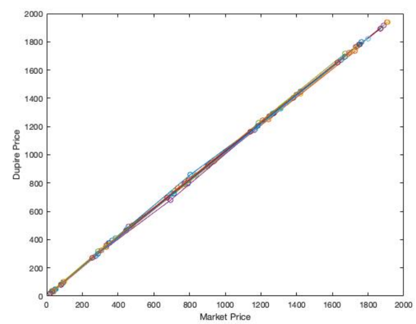
The Heston model [6] is the most popular stochastic volatility model. High-peaked and fat-tailed distributions of stock price returns are characteristics of a mixture of



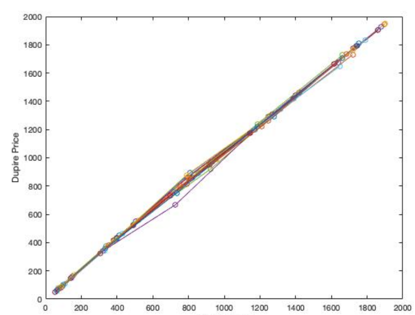
(a) Maturity : 21-Dec-2018



(b) Maturity : 15-Mar-2019



(c) Maturity : 21-Jun-2019



(d) Maturity : 20-Dec-2019

Figure 2.1: *Market quoted prices vs Dupire model price in USD for call option on S&P500 for the dates 1-Oct-2018 to 31-Oct-2018 for strikes* $1000, 1500, 2000, 2500$ *and* 3000*. The Dupire model is calibrated from market implied volatility surface using finite difference technique (optByLocalVolFD function in MATLAB).*

Strike	Vega
1500	0.0043
2000	-0.6375
2500	178.3873
3000	487.3893

Table 2.1: The value of Vega in case of Dupire model for call option on S&P500 maturing on 21-Dec-2018 as of 1-Oct-2018 for different strikes.

distributions with different variances. Hence, suggesting the volatility to be looked at as a random variable instead of deterministic one as is assumed in the local volatility models. Also, the unstable Greeks[6] (vega) in case of Dupire model imply that Markov model based on a single Brownian motion is not a good choice to manage smile risk, a two-factor model can be a better choice. The relatively quiescent and the relatively chaotic periods observed historically in the market supports that volatility is indeed a random function of time.

2.2.1 Underlying Equations

The underlying diffusion process is based upon a square-root diffusion, most famously applied in the Cox-Ingersoll-Ross interest rate model [1]

$$\begin{aligned} dS_t &= \mu_t S_t \, dt + \sqrt{\sigma_t} S_t \, dW_1 \\ d\sigma_t &= -\lambda(\sigma_t - \bar{\sigma}) \, dt + \eta \sqrt{\sigma_t} \, dW_2 \\ dW_1 \, dW_2 &= \rho \, dt \end{aligned} \tag{2.10}$$

where μ_t is the drift-term, λ is the speed of mean reversion of volatility σ_t, $\bar{\sigma}$ represents the long-term mean of volatility and η is the volatility of volatility. W_1 and W_2 are Brownian motion with correlation ρ. Over here the volatility σ_t is a stationary

[6]The Greeks in case of finance are defined as the variables depicting sensitivity of the price of derivative to a change in underlying parameters for example delta, vega etc.

stochastic function of the stock price with correlated chi-square increments. In this case the stochastic differential equation for any derivative defined on this underlying stock comes out to be

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma S^2 \frac{\partial^2 V}{\partial S^2} + \rho\eta\sigma S \frac{\partial^2 V}{\partial\sigma\partial S} + \frac{1}{2}\eta^2\sigma \frac{\partial^2 V}{\partial\sigma^2} + rS\frac{\partial V}{\partial S} - rV = \lambda(\sigma_t - \bar{\sigma})\frac{\partial V}{\partial\sigma} \quad (2.11)$$

2.2.2 Closed-Form Solution

Heston solved the above differential equation for European option using characteristic function, it is presented in [4] as

$$C_t = K(e^x \; P_1(x,\sigma,\tau) - P_0(x,\sigma,\tau)) \quad (2.12)$$

where $x := \ln(\frac{F_{t,T}}{K})$, $F_{t,T}$ is the time T forward price of stock and $\tau = T - t$. Then P_1 and P_0 can be solved using inverse Fourier transformation

$$P_j(x,\sigma,\tau) = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \mathrm{Re}\left[\frac{e^{C_j(u,\tau)\bar{\sigma} + D_j(u,\tau)\sigma + iux}}{iu}\right] du \quad (2.13)$$

for

$$C(u,\tau) = \lambda(r_-\tau - \frac{2}{\eta^2}\ln(\frac{1 - ge^{-d\tau}}{1-g}))$$

$$D(u,\tau) = r_-\frac{1 - e^{-d\tau}}{1 - ge^{-d\tau}}$$

$$g = \frac{r_+}{r_-}$$

$$r_\pm = \frac{\beta \pm \sqrt{\beta^2 - 4\alpha\gamma}}{2\gamma} =: \frac{\beta \pm d}{\eta^2}$$

$$\alpha = -\frac{u^2}{2} - \frac{iu}{2} + iju$$

$$\beta = \lambda - \rho\eta j - \rho\eta iu$$

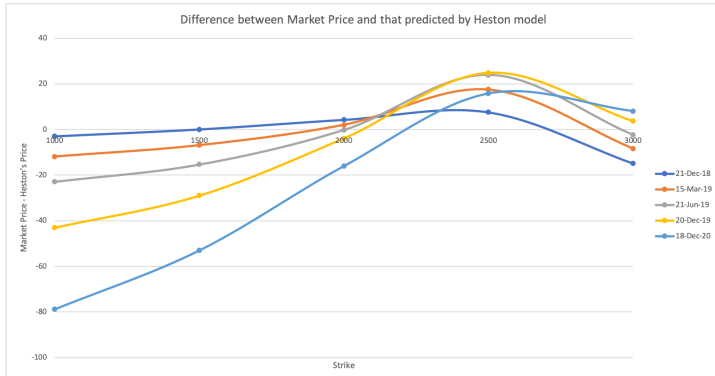
$$\gamma = \frac{\eta^2}{2}$$

Notice that the functions $C(u,\tau)$ and $D(u,\tau)$ are independent of x and σ, thus making the calculation of Greeks direct and hence, the ease of calculating risk parameters.

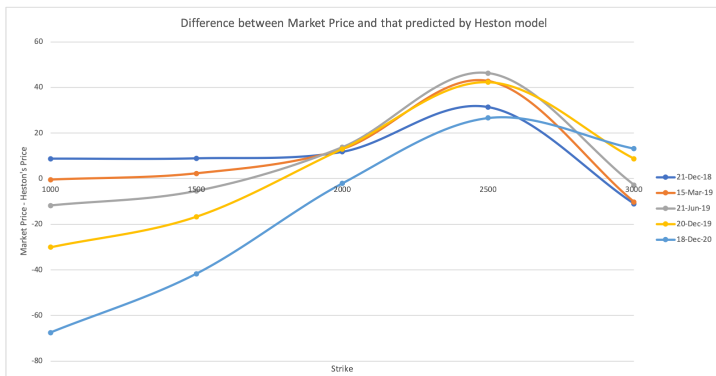
2.2.3 Analysis

Fig-2.2, Fig-2.3 and Fig-2.4 were obtained on applying the Heston model for call option on S&P500. Over here the historical option prices were used to calibrate the parameters using the least squares technique. Fig-2.2 shows three scenarios where the last 5 days, 10 days and 30 days prices were used to estimate the parameters. The graphs depict the difference between market prices and a 1-day forecast of the calibrated Heston model using numerical integration. It is observed that prediction is better in case of small maturity options as compared to long-term maturities. Also, in-the-money options are overvalued by the Heston model and out-of-the-money under-valued.

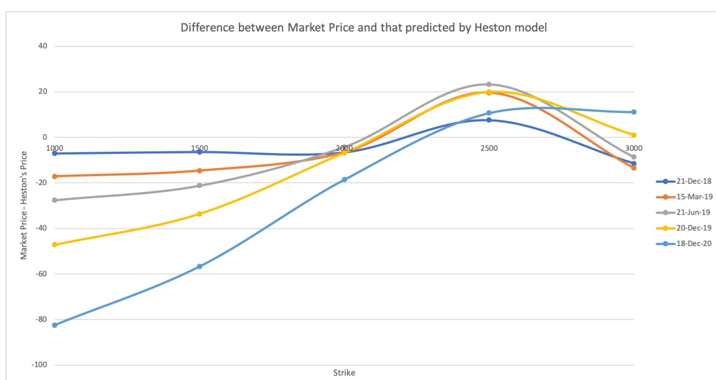
Fig-2.3 gives out the sensitivity analysis of different parameters when calibrated using 30 days prices. It is observed that long-term variance $\bar{\sigma}$ and initial variance σ_0 have a similar impact on the option price. The mean reversion rate λ in a way represents the degree of volatility clustering. ρ represents the correlation between the log-returns and volatility of the asset and it directly affects the skewness of the distribution. Its sign indicates whether the volatility will be right-fat tailed or left-fat tailed. η affects the kurtosis. Fig-2.4 gives out the sensitivity analysis of underlying stock-price.



(a) 5 Day

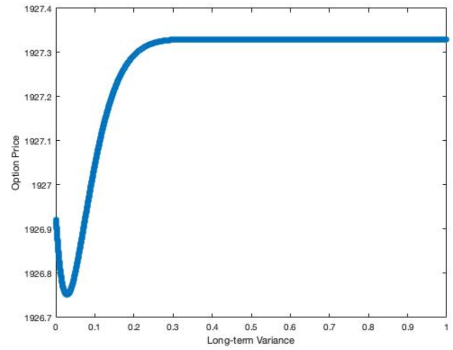


(b) 10 Day

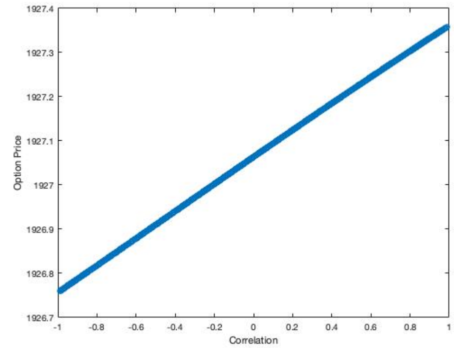


(c) 30 Day

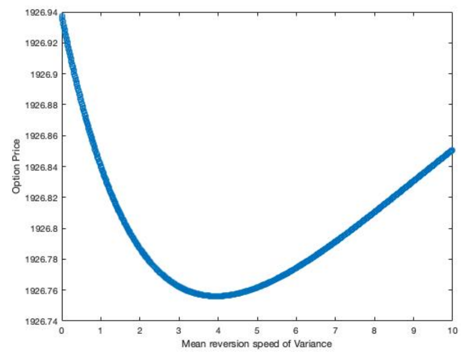
Figure 2.2: *Market quoted prices vs Heston model price for call option on S&P500 for the dates 1-Oct-2018 to 31-Oct-2018 for strikes* $1000, 1500, 2000, 2500$ *and* 3000.



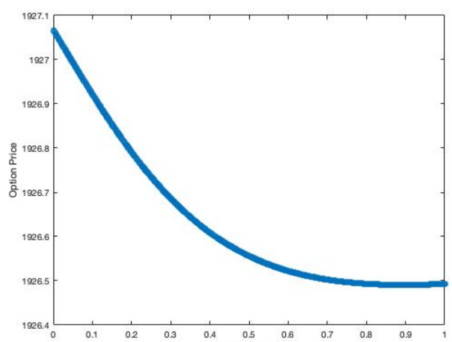
(a) $\bar{\sigma}$: long-term variance level



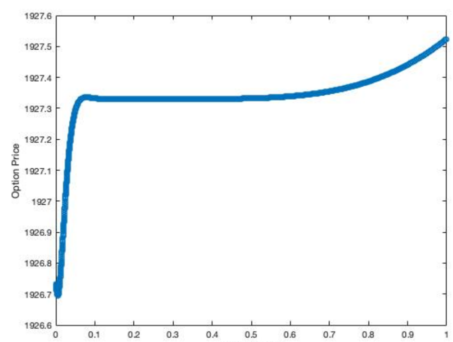
(b) ρ : correlation between Brownian processes Z_1 and Z_2



(c) λ : mean reversion speed for variance



(d) η : volatility of variance



(e) σ_0 : initial variance

Figure 2.3: *Sensitivity analysis of different parameters for Heston model in case of call option on S&P500 with strike* 1000*, maturing on 21-Dec-2018 as of date 1-Oct-2018.*

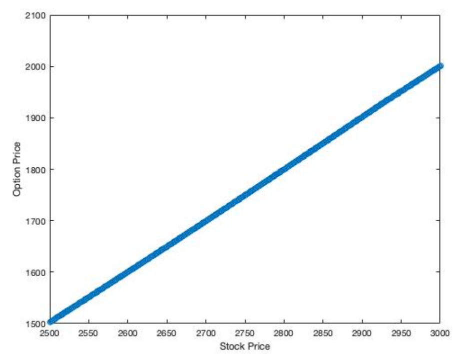


Figure 2.4: *Sensitivity analysis of underlying stock price to the option price for Heston model in case of call option on S&P500 with strike* 1000, *maturing on 21-Dec-2018 as of date 1-Oct-2018.*

Chapter 3: SABR model

Hagan, Kumar, Lesniewski, and Woodward [5] developed a complex stochastic volatility model in 2002 called stochastic alpha beta rho or SABR model. They observed that despite the popularity of the local volatility models in managing smile and skew risks, the dynamic behavior of smiles and skews predicted by them is the opposite of what is observed in the marketplace. Local volatility models claim that underlying stock price and smile have an inverse correlation, however, in reality, they move in the same direction. This contradiction between the model and the marketplace tends to destabilize the delta and vega hedges. In order to address this problem, they came up with the SABR model in which the volatility is non-stationary stochastic function correlated to the stock price. Due to it's non-mean reverting nature, it works well only for short expiration. It captures exact volatility smile for very short expiration.

3.1 Underlying Equation

In the case of the SABR model, the stochastic processes are assumed to have the following dynamics

$$\begin{aligned} dF_t &= \sigma_t F_t^{\beta}\, dW_1 \\ d\sigma_t &= \alpha \sigma_t\, dW_2 \\ dW_1\, dW_2 &= \rho\, dt \end{aligned} \tag{3.1}$$

under the forward measure[7] where F_t is the forward price of the stock and σ_t is the volatility. W_1 and W_2 are Brownian motions which have correlation ρ. The α, β, ρ represent the parameters of this model. Over here the volatility σ_t is a non-stationary stochastic function of the stock price with correlated power increments. The initial condition is given as $F_0 = f$.

SABR model is homogeneous in F_t and σ_t. It accurately predicts dynamics of market implied volatility curves for a single expiration date, however, it may or may not fit the volatility surface with multiple expiration dates especially if they are long-dated. The vega risk in this model can be hedged by selling and buying options on the underlying asset.

3.2 Pricing European Options

This section relates the constant Black-Scholes implied volatility to the SABR model. It gives a general closed-form solution for obtaining the Black-Scholes implied volatility from a given SABR model, which can then be directly used in the Black-Scholes model to obtain the option price. Further specific forms of the general equation for $\beta = 0$ and $\beta = 1$ are given.

[7]Forward measure is a pricing measure absolutely continuous with respect to the risk-neutral measure but it uses bond as numeraire.

3.2.1 General Equation

Single perturbation technique is used to obtain prices for European options as given in [5] which is further used to obtain constant Black-Scholes implied volatility under SABR model

$$\sigma_{BS}(K,f) = \frac{\sigma_0}{(fK)^{\frac{1-\beta}{2}}\{1 + \frac{(1-\beta)^2}{24}\ln^2\frac{f}{K} + \frac{(1-\beta)^4}{1920}\ln^4\frac{f}{K} + ...\}} . \frac{z}{x(z)} .$$
$$\left\{1 + \left[\frac{(1-\beta)^2}{24}\frac{\sigma_0^2}{(fK)^{1-\beta}} + \frac{1}{4}\frac{\rho\beta\alpha\sigma_0}{(fK)^{\frac{1-\beta}{2}}} + \frac{2-3\rho^2}{24}\alpha^2\right](T-t) + \dots \right. \quad (3.2)$$

where

$$z = \frac{\alpha}{\sigma_0}\,(fK)^{\frac{1-\beta}{2}}\;\ln f/K$$
$$x(z) = \ln\left(\frac{\sqrt{1-2\rho z+z^2}+z-\rho}{1-\rho}\right)$$

3.2.2 Special Case when $\beta = 1$

In case when $\beta = 1$ the model behaves like stochastic log normal model in which case the Eq-3.2 turns out to be

$$\sigma_{BS}(K,f) = \sigma_0\frac{z}{x(z)}\left\{1 + \left[\frac{1}{4}\,\rho\alpha\sigma_0 + \frac{2-3\rho^2}{24}\alpha^2\right](T-t) + \dots \right.$$

where z and $x(z)$ are same as before with $\beta = 1$.

3.2.3 Special Case when $\beta = 0$

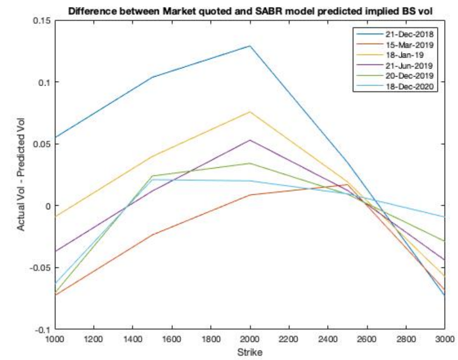
In case when $\beta = 0$ the model behaves like stochastic normal model in which case the Eq-3.2 turns out to be

$$\sigma_{BS}(K,f) = \frac{\sigma_0\ln f/K}{f-K}.\frac{z}{x(z)}.\left\{1 + \left[\frac{1}{24}\,\frac{\sigma_0^2}{fK} + \frac{2-3\rho^2}{24}\alpha^2\right](T-t) + \dots \right.$$

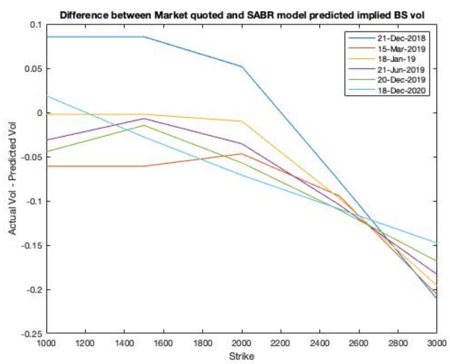
where z and $x(z)$ are same as before with $\beta = 0$.

3.3 Analysis

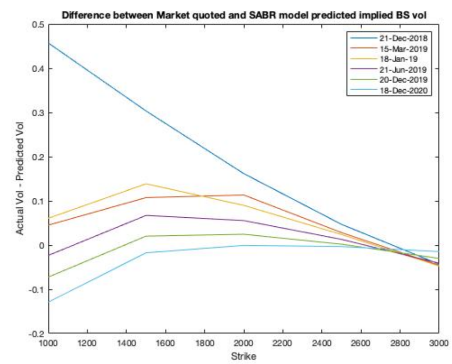
Fig-3.1, Fig-3.2 and Fig-3.3 were obtained on applying the SABR model for call option on S&P500 for fixing different values of β particularly 0, 0.5 and 1 respectively. Over here the historical option prices were used to calibrate the parameters using the least squares technique. Graphs for four different scenarios are shown wherein the last 1 day, 5 days, 10 days and 30 days prices were used to calculate parameters. The graphs depict the difference between market quoted Black-Scholes volatility and a 1-day forecast of the Black-Scholes volatility by the calibrated SABR model.



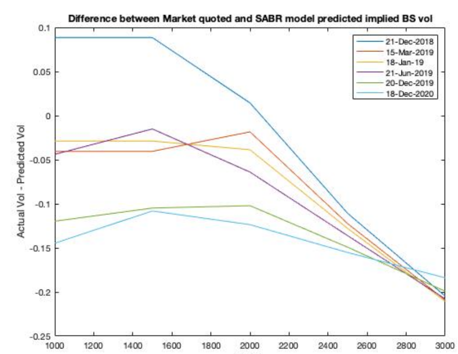
(a) 1 Day



(b) 5 Day

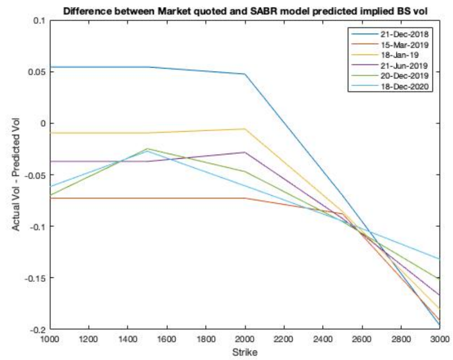


(c) 10 Day

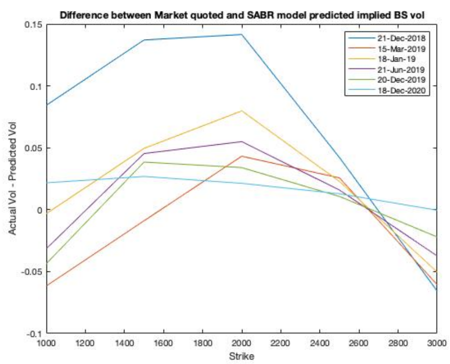


(d) 30 Day

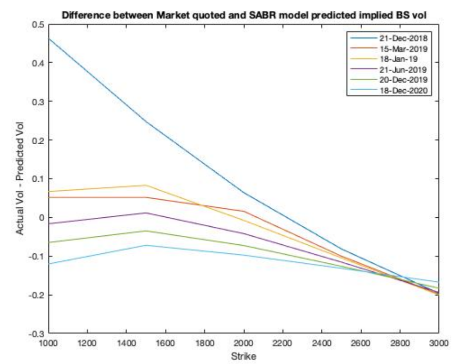
Figure 3.1: *Market quoted prices vs SABR model price for call option on S&P500 for the dates 1-Oct-2018 to 31-Oct-2018 for strikes* $1000, 1500, 2000, 2500$ *and* 3000 *with* $\beta = 0$.



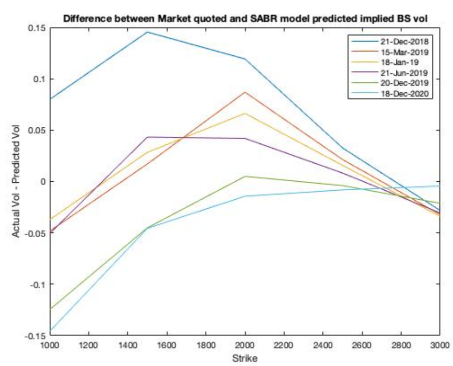
(a) 1 Day



(b) 5 Day

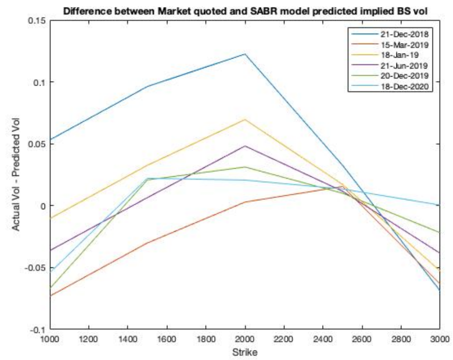


(c) 10 Day

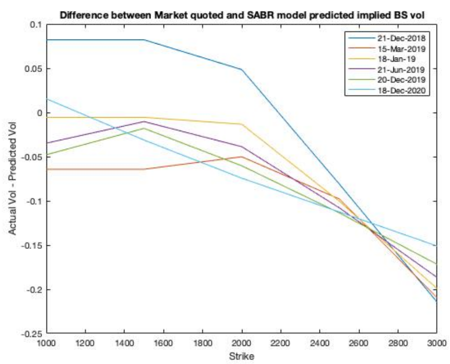


(d) 30 Day

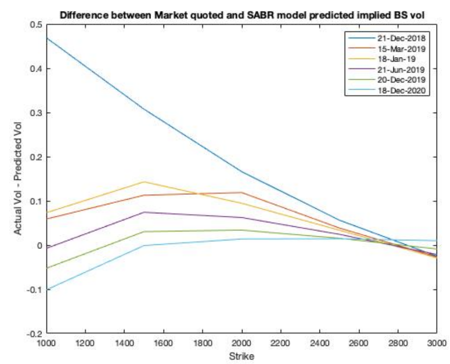
Figure 3.2: *Market quoted prices vs SABR model price for call option on S&P500 for the dates 1-Oct-2018 to 31-Oct-2018 for strikes* $1000, 1500, 2000, 2500$ *and* 3000 *with* β *= 0.5*



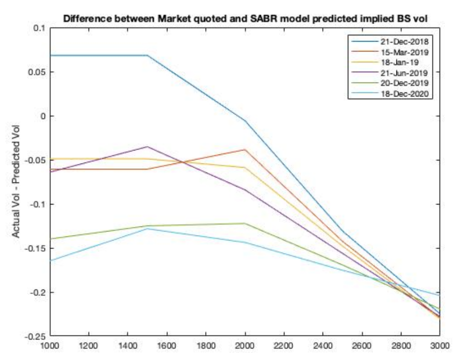
(a) 1 Day



(b) 5 Day



(c) 10 Day



(d) 30 Day

Figure 3.3: *Market quoted prices vs SABR model price for call option on S&P500 for the dates 1-Oct-2018 to 31-Oct-2018 for strikes* $1000, 1500, 2000, 2500$ *and* 3000 *with* $\beta = 1$

Chapter 4: Conclusion

The Black-Scholes model has laid down the foundations of financial engineering. It was the first option pricing model which gained global popularity. The analytical ease of understanding and implementing the model still makes it very useful in the current world. However, its shortcomings are also well-known. Most drawbacks of the Black-Scholes model arise from the fact that its underlying assumptions almost never hold true in the real world. Firstly, as seen in Fig-1.4 the assumption of a normal distribution of log-returns does not hold true. Fat tailed and high peak distributions along with volatility clustering effects corroborates the assumption of non-Gaussian distribution. Secondly, the volatility smile observed in the market directly contradicts the constant volatility assumption.

The Dupire local volatility model follows the same dynamics as the Black-Scholes model, but instead of constant volatility assumption, it uses a deterministic function for volatility. This makes it fit the volatility smile well. The ease of using a deterministic function is challenged by the need of finding an interpolation function which can satisfy all the required conditions. As a result, it provides unstable Greeks making it a bad hedge strategy. Hence, the model isn't effective for forecasting future prices.

The Heston stochastic volatility model, on the other hand, allows non-deterministic volatilities. The dynamics of this model are able to give out European option prices close to the prices observed in the market. It assumes non-log-normal distributions for the stock price. It also incorporates the mean-reverting property of volatility. Semi-closed form solution for European options makes the calibration of this model easier. The model is observed to have high sensitivity to its parameters, making calibration an important process. Heston performs better for at-the-money options. Being a stochastic volatility model makes forecasting future prices a computationally expensive process.

Another stochastic model that was described in Ch-3 was the SABR model. Unlike Heston model, SABR assumes volatility to be non-mean-reverting, making it work well only for short maturities. For given short-term-expiration, it is able to capture the volatility smile well. The closed form solution of Black-Scholes implied volatility in term of the SABR model, makes it computationally very efficient and easy to work with to reproduce as well as forecast option prices.

Overall the chapters in this thesis provide a detailed framework of several models used in option pricing. These different techniques can be used for both, either replicating the market implied volatility surface or the finding a computationally efficient way to predict the future price of options. While Dupire model is able to efficiently reproduce the market prices, the computational ease of SABR model makes it a better choice to forecast future prices.

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