Equidistribution in homogeneous spaces and Diophantine Approximation

Dissertation

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Abstract

In this dissertation, we study the dynamical behavior of translates of submanifolds in homogeneous spaces, and deduce its applications to Diophantine approximation. The study of equidistribution problems in homogeneous dynamics are significant for its usefulness in applications to number theory and geometry.

This work is comprised of two main parts. In the first part, we study expanding translates of an analytic curve by an algebraic diagonal flow on the homogeneous space G/Γ of a semisimple algebraic group G. We define two families of algebraic subvarieties of the associated partial flag variety G/P, which give the obstructions to non-divergence and equidistribution. We apply this to prove that for Lebesgue almost every point on an analytic curve in the space of $m \times n$ real matrices whose image is not contained in any subvariety coming from these two families, the Dirichlet's approximation theorem cannot be improved.

In the second part, we restrict our discussion to the special linear group, but consider the more general class of differentiable submanifolds. Given a nondegenerate differentiable submanifold of the space of unimodular lattices, we prove that the translates of shrinking balls around a generic point under a diagonal flow get equidistributed with respect to the Haar measure. This implies non-improvability

of Dirichlet's approximation theorem for almost every point on a nondegenerate differentiable submanifold of $\mathbb{R}^n$, answering a question of Davenport and Schmidt in 1969.

This dissertation is lovingly dedicated to my wife, Yushu Hu. Her support, encouragement, and constant love have sustained me during my graduate study in the Ohio State University.

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Chapter 1: Introduction

1.1 Background and motivation

Homogeneous dynamics studies the dynamics on spaces with transitive Lie group actions. It has rich applications to various areas in number theory, including Diophantine approximation and the distribution of integral and rational points on varieties.

In Diophantine approximation, there was a longstanding problem, conjectured by Oppenheim in 1929. It concerns representations of numbers by real quadratic forms in several variables. The seminal work of Margulis in 1987 confirms Oppenheim's conjecture, where he used methods arising from ergodic theory and the study of discrete subgroups of semisimple Lie groups.

Theorem 1.1.1 (Margulis, 1987)**.** *Let Q be an indefinite quadratic form in $n \geq 3$ variables. Suppose that Q is not a multiple of any rational quadratic form, then $Q(\mathbb{Z}^n)$ is dense in $\mathbb{R}$.*

The idea to deduce Oppenheim's conjecture from a statement in homogeneous dynamics is due to Raghunathan, who observed in the 1970s that the conjecture for $n = 3$ is equivalant to the following property of group orbit in the space of lattices.

Theorem 1.1.2 (Margulis, 1987)**.** *Any relatively compact orbit of* $\mathrm{SO}(2,1)$ *in* $\mathrm{SL}_3(\mathbb{R})/\mathrm{SL}_3(\mathbb{Z})$ *is compact.*

The action of certain unipotent subgroups of $\mathrm{SO}(2,1)$ plays a central role in the proof. After Margulis's breakthrough, in a series of papers [27, 28, 29] Ratner proved Raghunathan's conjecture on measure rigidity and orbit closure for unipotent flows on homogeneous spaces.

Theorem 1.1.3 (Ratner, 1991)**.** *Let G be a real Lie group, and Γ a lattice, that is, a discrete subgroup of G of finite covolume. Let u_t be a one-parameter subgroup of G consisting of unipotent elements. Then the closure of every u_t-orbit $u_t x$ on G/Γ is homogeneous, i.e. a closed orbit Hx of a connected closed subgroup H of G. Moreover, the unipotent trajectory gets equidistributed in the orbit closure, i.e.*

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T f(u_t x)dt = \int_{Hx} f\,d\mu, \quad \forall f \in C_c(G/\Gamma)$$

where $d\mu$ is the H-invariant probability measure supported on Hx.

Theorem 1.1.4 (Ratner, 1991)**.** *Let G be a real Lie group, and Γ a lattice. Let u_t be a one-parameter subgroup of G consisting of unipotent elements. Then every ergodic u_t-invariant probability measure μ on G/Γ is homogeneous. More precisely, there exists a connected closed subgroup H of G such that the measure μ is the H-invariant probability measure supported on a closed orbit of H.*

Ratner's measure rigidity results, along with the linearization technique developed by Dani, Margulis, Mozes, Shah and others, have far-reaching influence on homogeneous dynamics and number theory. To illustrate, we introduce the following counting problem which was solved by Eskin, Mozes and Shah [11] in 1996. Let $p(\lambda)$ be a monic polynomial of degree $n \geq 2$ with integer coefficients and irreducible over $\mathbb{Q}$. Let $\mathrm{M}_n(\mathbb{Z})$ denote the set of $n \times n$ integer matrices, and put

$$V_p(\mathbb{Z}) = \{A \in \mathrm{M}_n(\mathbb{Z}) \colon \det(\lambda I - A) = p(\lambda)\}.$$

Hence $V_p(\mathbb{Z})$ is the set of integral matrices with characteristic polynomial $p(\lambda)$. Consider the Euclidean norm on $n \times n$ real matrices, and let $N(T, V_p)$ denote the number of elements of $V_p(\mathbb{Z})$ with norm less than T.

Theorem 1.1.5 (Eskin, Mozes and Shah, 1996)**.** *Suppose further that $p(\lambda)$ splits over $\mathbb{R}$, and for a root α of $p(\lambda)$ the ring of algebraic integers in $\mathbb{Q}(\alpha)$ is $\mathbb{Z}(\alpha)$. Then, asymptotically as $T \to \infty$,*

$$N(T, V_p) \sim \frac{2^{n-1} h R \omega_n}{\sqrt{D} \cdot \prod_{k=2}^{n} \Lambda(k/2)} T^{n(n-1)/2}$$

where h is the class number of $\mathbb{Z}[\alpha]$, R is the regulator of $\mathbb{Q}(\alpha)$, D is the discriminant of $p(\lambda)$, ω_n is the volume of the unit ball in $\mathbb{R}^{n(n-1)/2}$, and $\Lambda(s) = \pi^{-s}\Gamma(s)\zeta(2s)$.

This result extends the previous results obtained by Duke-Rudnick-Sarnak [9] and Eskin-McMullen [10], which relate counting problems to the limit distributions of translates of reductive group orbits in homogeneous spaces.

1.2 Main results

In this section, we state our main results in this dissertation.

In Chapter 2, we study limit distributions of translates of a real analytic curve on homogeneous space, and give an application to matrix Diophantine approximation. Let $G = \mathbf{G}(\mathbb{R})$ be a semisimple connected real algebraic group of non-compact type, and let L be a Lie group containing G. Let Λ be a lattice in L. Let $\{a(t)\}_{t \in \mathbb{R}^\times}$ be a multiplicative one-parameter subgroup of G with non-trivial projection on each simple factor of G. There is a parabolic subgroup $P = P(a)$ of G associated with $a(t)$:

$$P := \{g \in G \colon \lim_{t \to \infty} a(t) g a(t)^{-1} \text{ exists in } G\}. \tag{1.2.1}$$

Suppose we have a bounded piece of an analytic curve on G given by $\phi : I = [a, b] \to G$, and we fix a point x_0 on L/Λ such that the orbit Gx_0 is dense in L/Λ. Let λ_ϕ denote the parametric measure on L/Λ. If we expect the translated measures to get equidistributed, it is necessary that there is no escape of mass to infinity.

Inspired by the work of Aka, Breuillard, Rosenzweig and de Saxcé, we define the notion of *unstable Schubert varieties* (see Definition 2.2.1) with respect to $a(t)$ for general partial flag variety G/P, which naturally generalizes the notion of constraining pencils in [1]. This enables us to describe obstructions to non-divergence in general case.

Now we project our curve ϕ onto G/P. Consider

$$\begin{aligned} \widetilde{\phi}\colon [a, b] &\longrightarrow G/P \\ s &\longmapsto \phi(s)^{-1}P. \end{aligned} \tag{1.2.2}$$

We are taking inverse here simply because we would like to quotient P on the right side.

We are ready to state our first main theorem on non-escape of mass.

Theorem 1.2.1. *Let $\phi : I = [a, b] \to G$ be an analytic curve such that the image of $\widetilde{\phi}$ is not contained in any unstable Schubert variety of G/P with respect to $a(t)$. Then for any $\epsilon > 0$, there exists a compact subset K of L/Λ such that for any $t > 1$, we have*

$$\frac{1}{b-a} \left|\{s \in [a, b] \colon a(t)\phi(s)x_0 \in K\}\right| > 1 - \epsilon. \tag{1.2.3}$$

We now turn to equidistribution.

Theorem 1.2.2. *Let $\phi\colon I = [a, b] \to G$ be an analytic curve such that the following two conditions hold:*

(a) *The image of* $\widetilde{\phi}$ *is not contained in any unstable Schubert variety of* G/P *with respect to* $a(t)$*;*

(b) *For any* $g \in G$ *and any proper algebraic subgroup* F *of* L *containing* $\{a(t)\}$ *such that* Fgx_0 *is closed and admits a finite* F*-invariant measure, the image of* ϕ *is not contained in* $P(F \cap G)g$*.*

Then for any $f \in C_c(L/\Lambda)$*, we have*

$$\lim_{t \to \infty} \frac{1}{b-a} \int_a^b f(a(t)\phi(s)x_0)\,\mathrm{d}s = \int_{L/\Lambda} f\,\mathrm{d}\mu_{L/\Lambda}, \tag{1.2.4}$$

where $\mu_{L/\Lambda}$ *is the* L*-invariant probability measure on* L/Λ*.*

One can even require $F \cap G$ to be reductive if we replace the family of unstable Schubert varieties with the slightly larger family of *weakly unstable Schubert varieties* (see Definition 2.2.1).

Theorem 1.2.3. *Let* $\phi \colon I = [a,b] \to G$ *be an analytic curve such that the following two conditions hold:*

(A) *The image of* $\widetilde{\phi}$ *is not contained in any weakly unstable Schubert variety of* G/P *with respect to* $a(t)$*;*

(B) *For any* $g \in G$ *and any proper algebraic subgroup* F *of* L *containing* $\{a(t)\}$ *such that* Fgx_0 *is closed and admits a finite* F*-invariant measure and that* $F \cap G$ *is reductive, the image of* ϕ *is not contained in* $P(F \cap G)g$*.*

Then for any $f \in C_c(L/\Lambda)$*, we have*

$$\lim_{t \to \infty} \frac{1}{b-a} \int_a^b f(a(t)\phi(s)x_0)\,\mathrm{d}s = \int_{L/\Lambda} f\,\mathrm{d}\mu_{L/\Lambda}, \tag{1.2.5}$$

where $\mu_{L/\Lambda}$ *is the* L*-invariant probability measure on* L/Λ*.*

If a reductive subgroup H contains $\{a(t)\}$, then $P_H = P \cap H$ is a parabolic subgroup of H associated with $a(t)$, and HP/P is homeomorphic to H/P_H. Hence we give the following definition.

Definition 1.2.4 (Partial flag subvariety)**.** A *partial flag subvariety of G/P with respect to $a(t)$* is a subvariety of the form gHP/P, where g is an element in G, and H is a reductive subgroup of G containing $\{a(t)\}$.

Now we give an application of the above result to Diophantine approximation on real matrices. In 1842, Dirichlet proved a theorem on simultaneous approximation of a matrix of real numbers (DT): Given any two positive integers m and n, a matrix $\Psi \in M_{m\times n}(\mathbb{R})$, and $N > 0$, there exist integral vectors $\mathbf{p} \in \mathbb{Z}^n \backslash \{\mathbf{0}\}$ and $\mathbf{q} \in \mathbb{Z}^m$ such that

$$\|\mathbf{p}\| \leq N^m \quad \text{and} \quad \|\Psi\mathbf{p} - \mathbf{q}\| \leq N^{-n}, \tag{1.2.6}$$

where $\|\cdot\|$ denotes the supremum norm, that is, $\|x\| = \max_{1\leq i\leq k}|x_i|$ for any $\mathbf{x} = (x_1, x_2, \cdots, x_k) \in \mathbb{R}^k$.

Given $0 < \mu < 1$. After Davenport and Schmidt [7], we say that $\Psi \in \mathrm{M}_{m\times n}(\mathbb{R})$ is DT_μ-improvable if for all sufficiently large $N > 0$, there exists nonzero integer vectors $\mathbf{p} \in \mathbb{Z}^n$ and $\mathbf{q} \in \mathbb{Z}^m$ such that

$$\|\mathbf{p}\| \leq \mu N^m \quad \text{and} \quad \|\Psi\mathbf{p} - \mathbf{q}\| \leq \mu N^{-n}. \tag{1.2.7}$$

We say that Ψ is not DT-improvable, if for any $0 < \mu < 1$, Ψ is not DT_μ-improvable.

To any $\Psi \in M_{m\times n}(\mathbb{R})$, we attach an m-dimensional subspace $V_\Psi \subset \mathbb{R}^{m+n}$ which is spanned by the row vectors of the full rank $m \times (m+n)$ matrix

$$\left[I_{m\times m}|\Psi\right]. \tag{1.2.8}$$

Let $\varphi\colon [a,b] \to M_{m\times n}(\mathbb{R})$ be an analytic curve. It induces a curve on $\mathrm{Gr}(m, m+n)$ by

$$\begin{aligned}\Phi\colon [a,b] &\longrightarrow \mathrm{Gr}(m, m+n)\\ s &\longmapsto V_{\varphi(s)}.\end{aligned}$$

We identify $\mathrm{Gr}(m, m+n)$ with G/P, where $G = \mathrm{SL}_{m+n}(\mathbb{R})$ and $P = P(a)$ is the parabolic subgroup associated with $a(t) = \mathrm{diag}(t^n, \cdots, t^n, t^{-m}, \cdots, t^{-m})$. Hence it makes sense to talk about partial flag subvarieties of $\mathrm{Gr}(m, m+n)$. (See Definition 1.2.4.)

Now we are ready for our main theorem on DT-improvability.

Theorem 1.2.5. *Let $\varphi\colon [a,b] \to M_{m\times n}(\mathbb{R})$ be an analytic curve. Suppose that both of the following hold:*

(A) The image of Φ is not contained in any weakly constraining pencil;

(B) The image of Φ is not contained in any proper partial flag subvariety of the Grassmannian variety $\mathrm{Gr}(m, m+n)$ with respect to $a(t)$.

Then for Lebesgue almost every $s \in [a,b]$, $\varphi(s)$ is not DT-improvable.

We note that the case $m = 1$ has been studied by Shah in [35], where it is shown that the translates get equidistributed if the curve is not contained in any proper affine subspace. In this case, both (A) and (B) in the above theorem specialize to proper affine subspaces. Therefore, Theorem 1.2.5 is a natural generalization of Shah's result.

In another direction to generalize Shah's result, we consider differentiable curves instead of real analytic ones. In Chapter 3, we study limit distributions of translates of a shrinking curve on the space $\mathrm{SL}_{n+1}(\mathbb{R})/\mathrm{SL}_{n+1}(\mathbb{Z})$ of unimodular lattices in $\mathbb{R}^n$. It contains joint work with Nimish Shah. We first provide an infinitesimal version for the curve not being contained in any proper affine subspace of $\mathbb{R}^n$.

Definition 1.2.6 (cf. [26, §2])**.** We say that a curve $\zeta : (c,d) \to \mathbb{R}^k$ is *nondegenerate at* $s \in (c,d)$, if $\zeta^{(k-1)}(s)$ exists and the vectors $\zeta^{(0)}(s) := (\zeta(s), \zeta^{(1)}(s), \ldots, \zeta^{(k-1)}(s))$ span $\mathbb{R}^k$.

Let Ω be any open interval in $\mathbb{R}$. We prove the following result on improvability of Dirichlet's theorem.

Theorem 1.2.7. *Let $\psi : \Omega \to \mathbb{R}^n$ be a $(n+1)$-times differentiable map, where Ω is open in $\mathbb{R}^d$. Suppose that $\tilde{\psi} : \Omega \to \mathbb{R}^{n+1}$ given by, $\tilde{\psi}(s) = (1, \psi(s))$ for all $s \in \Omega$, is nondegenerate. Then given an infinite set $\mathcal{N} \subset \mathbb{N}$, for almost every $s \in \Omega$ and any $\lambda \in (0,1)$, $\psi(s)$ is non-improvable along $\mathcal{N}$.*

Later in Chapter 3, we state the above theorem in full generality, covering the case where Ω is of higher dimension.

Chapter 2: Equidistribution of translates of curves and Diophantine approximation on matrices

In this chapter, we study the limit distribution of translates of an analytic curve in the homogeneous space of a real semisimple algebraic group.

2.1 Background and main results

Some problems in number theory can be recast in the language of homogeneous dynamics. Let G be a Lie group and Γ be a lattice in G, i.e. a discrete subgroup of finite covolume. Take a sequence $\{g_i\}$ in G and a probability measure μ on G/Γ which is supported on a smooth submanifold of G/Γ. The following question was raised by Margulis in [22]:

Basic Question (Margulis)**.** What is the distribution of $g_i\mu$ in G/Γ when g_i tends to infinity in G?

In 1993, Duke, Rudnick and Sarnak [9] studied the case where μ is a finite invariant measure supported on a symmetric subgroup orbit, and applied it to obtain asymptotic estimates for the number of integral points of bounded norm on affine symmetric varieties. At the same time, Eskin and McMullen [10] gave a simpler proof using the mixing property of geodesic flows. It was later generalized by Eskin, Mozes and Shah [11] to the case where μ is a finite invariant measure supported on a reductive group orbit, and applied it to count integral matrices of bounded norm with a given characteristic polynomial. Later Gorodnik and Oh [15] worked in the Adelic setting, and gave an asymptotic formula for the number rational points of bounded height on homogeneous varieties.

In another direction, the dynamical behavior of translates of a submanifold of expanding horospherical subgroups in $\mathrm{SL}_n(\mathbb{R})/\mathrm{SL}_n(\mathbb{Z})$ is closely related to metric Diophantine approximation. In 1998, Kleinbock and Margulis [19] proved extremality of a non-degenerate submanifold in $\mathbb{R}^n$, and their proof was based on quantitative non-divergence of translates of the submanifold by semisimple elements. Their work was later extended from $\mathbb{R}^n$ to the space $M_{m\times n}(\mathbb{R})$ of $m \times n$ real matrices (see e.g. [20, 3, 1]).

While quantitative non-divergence results are useful in the study of extremality, equidistribution results can be applied to study the improvability of Dirichlet's theorem. In 2008, Kleinbock and Weiss [21] first explored improvability in the language of homogeneous dynamics, based on earlier observations by Dani [5] as well as Kleinbock and Margulis [19]. Later Shah [35] obtained a strengthened result for analytic curves in $\mathbb{R}^n$ by showing equidistribution of expanding translates of curves in $\mathrm{SL}_{n+1}(\mathbb{R})/\mathrm{SL}_{n+1}(\mathbb{Z})$ by singular diagonal elements $a(t) = \mathrm{diag}(t^n, t^{-1}, \cdots, t^{-1})$. This work has also been generalized to $m \times n$ matrices in a recent preprint [39] by Shah and Lei Yang, where they considered the case $G = \mathrm{SL}_{m+n}(\mathbb{R})$ and $a(t) = \mathrm{diag}(t^n, \cdots, t^n, t^{-m}, \cdots, t^{-m})$. We shall discuss this subject in more details in Section 2.1.3.

It is also worth considering the case $G = \mathrm{SO}(n,1)$, as there are interesting applications to hyperbolic geometry. See Shah's works [36, 34] and later generalizations by Lei Yang [43, 41]. We shall provide more details in Section 2.1.2.

Motivated by the previous works, we are interested in the following equidistribution problem, which was proposed by Shah in ICM 2010 [37]. Let $G = \mathbf{G}(\mathbb{R})$ be a semisimple connected real algebraic group of non-compact type, and let L be a Lie group containing G. Let Λ be a lattice in L. Let $\{a(t)\}_{t\in\mathbb{R}^\times}$ be a multiplicative one-parameter subgroup of G, i.e. we have a homomorphism of real algebraic group $a\colon \mathbb{G}_m \to \mathbf{G}$. Suppose we have a bounded piece of an analytic curve on G given by $\phi\colon I = [a,b] \to G$, and we fix a point x_0 on L/Λ such that Gx_0 is dense in L/Λ. Let λ_ϕ denote the measure on L/Λ which is the parametric measure supported on the orbit $\phi(I)x_0$, that is, λ_ϕ is the pushforward of the Lebesgue measure. When does $a(t)\lambda_\phi$ converge to the Haar measure on L/Λ with respect to the weak-* topology, as t tends to infinity?

In [37], Shah found natural algebraic obstructions to equidistribution, and asked if those are the only obstructions. In this article, we give an affirmative answer to Shah's question. This generalizes previous results on $G = \mathrm{SO}(n,1)$ [36, 43], $G = \mathrm{SO}(n,1)^k$ [41], as well as $G = \mathrm{SL}_n(\mathbb{R})$ and $a(t)$ being singular [35, 39]. We also apply the equidistribution result to show that for almost every point on a "non-degenerate" analytic curve in the space of $m \times n$ real matrices, the Dirichlet's theorem cannot be improved. This sharpens a result of Shah and Yang [39].

We remark that our method also applies to analytic submanifolds. For convenience, we restrict our discussions to curves.

2.1.1 Non-escape of mass to infinity

Let $G = \mathbf{G}(\mathbb{R})$ be a semisimple connected real algebraic group of non-compact type, and let L be a Lie group containing G. Let Λ be a lattice in L. Let $\{a(t)\}_{t\in\mathbb{R}^\times}$ be a multiplicative one-parameter subgroup of G with non-trivial projection on each simple factor of G. There is a parabolic subgroup $P = P(a)$ of G associated with $a(t)$:

$$P := \{g \in G\colon \lim_{t\to\infty} a(t)ga(t)^{-1} \text{ exists in } G\}. \tag{2.1.1}$$

Suppose we have a bounded piece of an analytic curve on G given by $\phi : I = [a,b] \to G$, and we fix a point x_0 on L/Λ such that the orbit Gx_0 is dense in L/Λ. Let λ_ϕ denote the parametric measure on L/Λ. If we expect the translated measures to get equidistributed, it is necessary that there is no escape of mass to infinity.

Let us first consider the special case $G = L = \mathrm{SL}_{m+n}(\mathbb{R})$, $\Lambda = \mathrm{SL}_{m+n}(\mathbb{Z})$ and $a(t) = \mathrm{diag}(t^n, \cdots, t^n, t^{-m}, \cdots, t^{-m})$. In [1], Aka, Breuillard, Rosenzweig and de Saxcé defined a family of algebraic sets called *constraining pencils* (see [1, Definition 1.1]), and used it to describe the obstruction to quantitative non-divergence. They remarked that constraining pencils give rise to certain Schubert varieties in Grassmannians.

Inspired by their work, we define the notion of *unstable Schubert varieties*[1] (see Definition 2.2.1) with respect to $a(t)$ for general partial flag variety G/P, which naturally generalizes the notion of constraining pencils. This enables us to describe obstructions to non-divergence in general case.

[1]The name comes from the notion of stability in geometric invariant theory, and should not be confused with unstable manifolds for a diffeomorphism.

Now we project our curve ϕ onto G/P. Consider

$$\begin{aligned} \widetilde{\phi}\colon [a,b] &\longrightarrow G/P \\ s &\longmapsto \phi(s)^{-1}P. \end{aligned} \tag{2.1.2}$$

We are taking inverse here simply because we would like to quotient P on the right, which is the case in most of the literatures.

We are ready to state our first main theorem on non-escape of mass.

Theorem 2.1.1 (Non-escape of mass)**.** *Let $\phi : I = [a,b] \to G$ be an analytic curve such that the image of $\widetilde{\phi}$ is not contained in any unstable Schubert variety of G/P with respect to $a(t)$. Then for any $\epsilon > 0$, there exists a compact subset K of L/Λ such that for any $t > 1$, we have*

$$\frac{1}{b-a}\left|\{s \in [a,b]\colon a(t)\phi(s)x_0 \in K\}\right| > 1 - \epsilon. \tag{2.1.3}$$

To prove Theorem 2.1.1, we consider a certain finite dimensional representation V of G (see Definition 2.3.2), and show that the corresponding curve in V cannot be uniformly contracted to the origin. The key ingredient is the following theorem, which is the main technical contribution of this article.

Theorem 2.1.2 (Linear stability)**.** *Let $\rho\colon G \to \mathrm{GL}(V)$ be any finite-dimensional linear representation of G, with a norm $\|\cdot\|$ on V. Suppose that the image of $\widetilde{\phi}$ is not contained in any unstable Schubert variety of G/P with respect to $a(t)$. Then there exists a constant $C > 0$ such that for any $t > 1$ and any $v \in V$, one has*

$$\sup_{s\in[a,b]} \|a(t)\phi(s)v\| \geq C\|v\|. \tag{2.1.4}$$

Theorem 2.1.2 is of independent interest, as it is also applicable to obtain quantitative non-divergence results (see e.g. [40]). Compared to the previous works on special cases of the theorem, the novel part of our proof is that we use a result in geometric invariant theory, which is Kempf's numerical criterion [17, Theorem 4.2].

Geometric invariant theory was first developed by Mumford to construct quotient varieties in algebraic geometry; its connections to dynamics have been found in recent years. Kapovich, Leeb and Porti [16, Section 7.4] explored the relation with geometric invariant theory for groups of type A_1^n. In a recent preprint [18], Khayutin utilized geometric invariant theory to study the double quotient of a reductive group by a torus. In [30, Section 6], Richard and Shah applied [17, Lemma 1.1(b)] to deal with focusing, which also came from the study of geometric invariant theory.

Theorem 2.1.2 is proved in Section 2.2, and Theorem 2.1.1 is proved in Section 2.3.

2.1.2 Equidistribution of translated measures

Let the notations be as in Section 2.1.1, and suppose that the image of $\widetilde{\phi}\colon s \mapsto \phi(s)^{-1}P$ is not contained in any unstable Schubert variety of G/P with respect to $a(t)$ (see Definition 2.2.1). Due to Theorem 2.1.1, for any sequence $t_i \to \infty$, the sequence of translated measures $a(t)\lambda_\phi$ is tight, i.e. any weak-* limit is a probability measure on L/Λ. If one can further show that any limit measure is the Haar measure on L/Λ, then the translated measure $a(t)\lambda_\phi$ gets equidistributed as $t \to \infty$. In order to achieve this, one needs to exclude a larger family of obstructions.

In a sequence of papers [36, 34, 35], Shah initiated the study of the curve equidistribution problem with several important special cases. For example, when $G = \mathrm{SL}_{n+1}(\mathbb{R})$ and $a(t) = \mathrm{diag}(t^n, t^{-1}, \cdots, t^{-1})$, the obstructions to equidistribution come from linear subspaces of $\mathbb{RP}^n$, which are exactly the unstable Schubert varieties with respect to $a(t)$.

Another interesting case is when $G = \mathrm{SO}(n,1)$ and $\{a(t)\}$ being the geodesic flow on the unit tangent bundle $T^1(\mathbb{H}^n)$ of the hyperbolic space $\mathbb{H}^n \cong \mathrm{SO}(n,1)/\mathrm{SO}(n)$. The visual boundary of $\mathbb{H}^n$ has the identification

$$\partial\mathbb{H}^n \cong \mathbb{S}^{n-1} \cong G/P. \tag{2.1.5}$$

Shah found that the obstructions to equidistribution comes from proper subspheres $\mathbb{S}^{m-1}$ of $\mathbb{S}^{n-1}$ $(m < n)$. However, since the real rank of G is one, the proper Schubert varieties of G/P are just single points. Therefore, these obstructions are not given by Schubert varieties. Nonetheless, the subspheres are still natural geometric objects, as they are closed orbits of the subgroups $\mathrm{SO}(m,1) \subset \mathrm{SO}(n,1)$, which correspond to totally geodesic submanifolds $\mathbb{H}^m \subset \mathbb{H}^n$.

Motivated by these results, Shah [37] found the following algebraic obstruction to equidistribution in the general setting. Suppose that F is a proper subgroup of L containing $\{a(t)\}$, and $g \in G$ is an element such that the orbit Fgx_0 is closed and carries a finite F-invariant measure. Suppose that $\phi(I) \subset P(F \cap G)g$. Then for any sequence $t_i \to \infty$, it follows that any weak-* limit of probability measures $a(t_i)\lambda_\phi$ is a direct integral of measures which are supported on closed sets of the form $bFgx_0$, where $b \in P$. Such limiting measures are concentrated on strictly lower dimensional submanifolds of L/Λ. Shah also asked if these are the only obstructions.

We now state our main theorem on equidistribution, which answers Shah's question affirmatively. Recall that x_0 is an element in L/Λ such that Gx_0 is dense in L/Λ. Let $\widetilde{\phi}$ be as in (2.1.2). For the definition of *unstable Schubert variety*, see Definition 2.2.1.

Theorem 2.1.3. *Let $\phi\colon I = [a, b] \to G$ be an analytic curve such that the following two conditions hold:*

(a) The image of $\widetilde{\phi}$ is not contained in any unstable Schubert variety of G/P with respect to $a(t)$;

(b) For any $g \in G$ and any proper algebraic subgroup F of L containing $\{a(t)\}$ such that Fgx_0 is closed and admits a finite F-invariant measure, the image of ϕ is not contained in $P(F \cap G)g$.

Then for any $f \in C_c(L/\Lambda)$, we have

$$\lim_{t\to\infty} \frac{1}{b-a} \int_a^b f(a(t)\phi(s)x_0)\,\mathrm{d}s = \int_{L/\Lambda} f\,\mathrm{d}\mu_{L/\Lambda}, \tag{2.1.6}$$

where $\mu_{L/\Lambda}$ is the L-invariant probability measure on L/Λ.

Remark 2.1.4. In Theorem 2.1.3, if we assume (a) holds, then by the above discussion we know that (2.1.6) holds if and only if (b) holds. In this sense, our result is sharp.

One can even require $F \cap G$ to be reductive if we replace the family of unstable Schubert varieties with the slightly larger family of *weakly unstable Schubert varieties* (see Definition 2.2.1).

Theorem 2.1.5. *Let $\phi\colon I = [a, b] \to G$ be an analytic curve such that the following two conditions hold:*

(A) The image of $\widetilde{\phi}$ is not contained in any weakly unstable Schubert variety of G/P with respect to $a(t)$;

(B) For any $g \in G$ and any proper algebraic subgroup F of L containing $\{a(t)\}$ such that Fgx_0 is closed and admits a finite F-invariant measure and that $F \cap G$ is reductive, the image of ϕ is not contained in $P(F \cap G)g$.

Then for any $f \in C_c(L/\Lambda)$, we have

$$\lim_{t\to\infty} \frac{1}{b-a} \int_a^b f(a(t)\phi(s)x_0)\,\mathrm{d}s = \int_{L/\Lambda} f\,\mathrm{d}\mu_{L/\Lambda}, \tag{2.1.7}$$

where $\mu_{L/\Lambda}$ is the L-invariant probability measure on L/Λ.

If a reductive subgroup H contains $\{a(t)\}$, then $P_H = P \cap H$ is a parabolic subgroup of H associated with $a(t)$, and HP/P is homeomorphic to H/P_H. Hence we give the following definition.

Definition 2.1.6 (Partial flag subvariety)**.** A *partial flag subvariety of G/P with respect to $a(t)$* is a subvariety of the form gHP/P, where g is an element in G, and H is a reductive subgroup of G containing $\{a(t)\}$.

In view of Definition 2.1.6, Theorem 2.1.5 shows that the obstructions consist of two families of geometric objects: weakly unstable Schubert varieties and partial flag subvarieties.

Theorem 2.1.3 and Theorem 2.1.5 are proved in Section 2.5.

2.1.3 Grassmannians and Dirichlet's approximation theorem on matrices

In this section, we give an application of our equidistribution result to simultaneous Diophantine approximation.

In 1842, Dirichlet proved a theorem on simultaneous approximation of a matrix of real numbers (DT): Given any two positive integers m and n, a matrix $\Psi \in M_{m\times n}(\mathbb{R})$, and $N > 0$, there exist integral vectors $\mathbf{p} \in \mathbb{Z}^n \backslash \{\mathbf{0}\}$ and $\mathbf{q} \in \mathbb{Z}^m$ such that

$$\|\mathbf{p}\| \le N^m \quad \text{and} \quad \|\Psi\mathbf{p} - \mathbf{q}\| \le N^{-n}, \tag{2.1.8}$$

where $\|\cdot\|$ denotes the supremum norm, that is, $\|x\| = \max_{1\leq i\leq k}|x_i|$ for any $\mathbf{x} = (x_1, x_2, \cdots, x_k) \in \mathbb{R}^k$.

Given $0 < \mu < 1$. After Davenport and Schmidt [7], we say that $\Psi \in \mathrm{M}_{m\times n}(\mathbb{R})$ is DT_μ-improvable if for all sufficiently large $N > 0$, there exists nonzero integer vectors $\mathbf{p} \in \mathbb{Z}^n$ and $\mathbf{q} \in \mathbb{Z}^m$ such that

$$\|\mathbf{p}\| \leq \mu N^m \quad \text{and} \quad \|\Psi\mathbf{p} - \mathbf{q}\| \leq \mu N^{-n}. \tag{2.1.9}$$

We say that Ψ is not DT-improvable, if for any $0 < \mu < 1$, Ψ is not DT_μ-improvable.

In [7], it was proved that Dirichlet's theorem cannot be improved for Lebesgue almost every $m \times n$ real matrix. In [8], they also proved that Dirichlet's theorem cannot be (1/4)-improved for almost every point on the curve $\phi(s) = (s, s^2)$ in $\mathbb{R}^2$. This result was generalized by Baker [2] for almost all points on smooth curves in $\mathbb{R}^2$, and by Bugeaud [4] for almost every point on the curve $\phi(s) = (s, s^2, \cdots, s^k)$ in $\mathbb{R}^k$; in each case the result holds for some small value $0 < \mu \leq \epsilon$, where ϵ depends on the curve.

Kleinbock and Weiss [21] recast the problem in the language of homogeneous dynamics, and obtained ϵ-improvable results for general measures. Later Shah [35] studied the case $m = 1$, and showed that if an analytic curve in $\mathbb{R}^n$ is not contained in any proper affine subspace, then almost every point on the curve is not DT-improvable. Lei Yang [42] studied the case $m = n$, and proved an analogous result for square matrices. These results have been generalized to *supergeneric* curves in $M_{m\times n}(\mathbb{R})$ in the recent preprint [39], where an inductive algorithm was introduced to define *generic* and *supergeneric* curves.

In the meantime, Aka, Breuillard, Rosenzweig and de Saxcé [1] worked on extremality of an analytic submanifold of $M_{m\times n}(\mathbb{R})$, and found a sharp condition for extremality in terms of a certain family of algebraic sets called *constraining pencils* (see [1, Definition 1.1]).

Based on [39], and combined with ideas from [36, 1], we replace supergeneric condition by a natural geometric condition, and obtain a sharper result.

We first make some preparations. Let $\mathrm{Gr}(m, m+n)$ denote the real Grassmannian variety of m-dimensional linear subspaces of $\mathbb{R}^{m+n}$.

Definition 2.1.7 (pencil; c.f. [1] Definition 1.1)**.** Given a real vector space $W \subsetneq \mathbb{R}^{m+n}$, and an integer $r \leq m$, we define the pencil $\mathfrak{P}_{W,r}$ to be the set

$$\{V \in \mathrm{Gr}(m, m+n) \colon \dim(V \cap W) \geq r\}. \tag{2.1.10}$$

We call $\mathfrak{P}_{W,r}$ a *constraining pencil* if

$$\frac{\dim W}{r} < \frac{m+n}{m}; \tag{2.1.11}$$

we call $\mathfrak{P}_{W,r}$ a *weakly constraining pencil* if

$$\frac{\dim W}{r} \leq \frac{m+n}{m}. \tag{2.1.12}$$

We say that the pencil $\mathfrak{P}_{W,r}$ is *rational* if W is rational, i.e. W admits a basis in $\mathbb{Q}^{m+n}$.

Remark 2.1.8. (1) If m and n are coprime, then $\frac{m+n}{m}$ is an irreducible fraction, and it follows that (2.1.12) and (2.1.11) are equivalent. Therefore weakly constraining pencils coincide with constraining pencils in this case.

(2) If $m = 1$, then (weakly) constraining pencils are proper linear subspaces of $\mathbb{RP}^n$.

To avoid confusions, we explain the relationship between our pencils and the pencils in [1]. Given $W \subsetneq \mathbb{R}^{m+n}$ and $0 < r < m$, in [1] a pencil $\mathcal{P}_{W,r}$ is defined to be an algebraic subset of $M_{m\times(m+n)}(\mathbb{R})$. More precisely,

$$\mathcal{P}_{W,r} = \left\{x \in M_{m\times(m+n)}(\mathbb{R})\colon \dim(xW) \le r\right\}. \tag{2.1.13}$$

And a pencil $\mathcal{P}_{W,r}$ is called constraining if

$$\frac{\dim W}{r} > \frac{m+n}{m}. \tag{2.1.14}$$

Let x be a full rank $m \times (m+n)$ real matrix. For any subspace $E \subset \mathbb{R}^{m+n}$, let $E^\vee \subset (\mathbb{R}^{m+n})^*$ denote the set of linear functionals on $\mathbb{R}^{m+n}$ which vanish on E. Then $\dim(xW) \le r$ if and only if $\dim\left((\ker x)^\vee \cap W^\vee\right) \ge m - r$. Hence

$$x \in \mathcal{P}_{W,r} \quad \Longleftrightarrow \quad (\ker x)^\vee \in \mathfrak{P}_{W^\vee, m-r}. \tag{2.1.15}$$

Moreover, since $\dim W^\vee = m + n - \dim W$, we have

$$\frac{\dim W}{r} > \frac{m+n}{m} \quad \Longleftrightarrow \quad \frac{\dim W^\vee}{m-r} < \frac{m+n}{m}. \tag{2.1.16}$$

As explained in [1, Section 4], we don't lose any essential information when passing to kernels. Therefore, our constraining pencils are dual to the constraining pencils in [1]. We modified the definition to fit into our framework of Schubert varieties. See Definition 2.2.1 and Theorem 2.6.6 for more details.

To any $\Psi \in M_{m\times n}(\mathbb{R})$, we attach an m-dimensional subspace $V_\Psi \subset \mathbb{R}^{m+n}$ which is spanned by the row vectors of the full rank $m \times (m+n)$ matrix

$$\left[I_{m\times m} | \Psi\right]. \tag{2.1.17}$$

Let $\varphi\colon [a,b] \to M_{m\times n}(\mathbb{R})$ be an analytic curve. It induces a curve on $\mathrm{Gr}(m, m+n)$ by

$$\Phi\colon [a,b] \longrightarrow \mathrm{Gr}(m, m+n)$$

$$s \longmapsto V_{\varphi(s)}.$$

We identify $\mathrm{Gr}(m, m+n)$ with G/P, where $G = \mathrm{SL}_{m+n}(\mathbb{R})$ and $P = P(a)$ is the parabolic subgroup associated with $a(t) = \mathrm{diag}(t^n, \cdots, t^n, t^{-m}, \cdots, t^{-m})$. Hence it makes sense to talk about partial flag subvarieties of $\mathrm{Gr}(m, m+n)$. (See Definition 2.1.6.)

Now we are ready for our main theorem on DT-improvability.

Theorem 2.1.9 (DT-improvability)**.** *Let $\varphi\colon [a,b] \to M_{m\times n}(\mathbb{R})$ be an analytic curve. Suppose that both of the following hold:*

(A) The image of Φ is not contained in any weakly constraining pencil;

(B) The image of Φ is not contained in any proper partial flag subvariety of the Grassmannian variety $\mathrm{Gr}(m, m+n)$ with respect to $a(t)$.

Then for Lebesgue almost every $s \in [a,b]$, $\varphi(s)$ is not DT-improvable.

Theorem 2.1.9 follows from Theorem 2.1.5 and Theorem 2.6.6 via Dani's correspondence, as we explain below. Let $G = \mathrm{SL}_{m+n}(\mathbb{R})$, and let $\Gamma = \mathrm{SL}_{m+n}(\mathbb{Z})$. The homogeneous space can be identified with the space of unimodular lattices of $\mathbb{R}^{m+n}$. Every point $g\Gamma$ corresponds to the unimodular lattice $g\mathbb{Z}^{m+n}$. For $r > 0$, let B_r denote the ball in $\mathbb{R}^{m+n}$ centered at the origin and of radius r. For any $0 < \mu < 1$, the subset

$$K_\mu := \{\Lambda \in G/\Gamma \colon \Lambda \cap B_\mu = \{0\}\}$$

contains an open neighborhood of $\mathbb{Z}^{m+n}$ in G/Γ. Now for any $\Phi \in M_{m\times n}(\mathbb{R})$, set

$$u(\Phi) := \begin{bmatrix} I_m & \Phi \\ & I_n \end{bmatrix}.$$

Suppose for some $0 < \mu < 1$, and any $N > 0$ large enough, there exist nonzero integer vector $\mathbf{p} \in \mathbb{Z}^n$ and integer vector $\mathbf{q} \in \mathbb{Z}^m$ such that $\|\mathbf{p}\| \leq \mu N^m$ and $\|\Phi\mathbf{p} -$

$\mathbf{q}\| \leq \mu N^{-n}$. Then the lattice $a(N)u(\Phi)\mathbb{Z}^{m+n}$ has a vector $a(N)u(\Phi)(-\mathbf{q}, \mathbf{p})$ whose norm is less than μ, i.e. $a(N)u(\Phi)\mathbb{Z}^{m+n} \notin K_\mu$ for all $N > 0$ large enough. Thus, to show that Φ is not DT_μ, it suffices to show that the trajectory $\{a(N)u(\Phi)[e] \colon t > 1\}$ meets K_μ infinitely many times. This will follow from the equidistribution result (see [35]).

The same arguments could be used to prove that for Lebesgue almost every $s \in [a, b]$, $\varphi(s)$ is not DT-improvable along $\mathcal{N}$ (see [38]), where $\mathcal{N}$ is any infinite set of positive integers.

2.2 Linear stability and Kempf's one-parameter subgroups

Let $G = \mathbf{G}(\mathbb{R})$ be a semisimple connected real algebraic group. If $\delta \colon \mathbb{G}_m \to \mathbf{G}$ is a homomorphism of real algebraic groups, we call δ a *multiplicative one-parameter subgroup* of G. We associate a parabolic subgroup with δ as:

$$P(\delta) := \{g \in G \colon \lim_{t \to \infty} \delta(t) g \delta(t)^{-1} \text{ exists in } G\}, \tag{2.2.1}$$

Let $\Gamma(G)$ be the set of all multiplicative one-parameter subgroups of G. Following Kempf [17], we define the *Killing length* of a multiplicative one-parameter subgroup δ by the equation

$$2\|\delta\|^2 = \mathrm{Trace}[(\mathrm{ad}(\delta_* \mathrm{d}/\mathrm{d}t))^2], \tag{2.2.2}$$

and it follows from the invariance of the Killing form that the Killing length is G-invariant.

Now fix a multiplicative one-parameter subgroup a of G. We choose and fix a maximal $\mathbb{R}$-split torus T of G containing $\{a(t)\}$. Let $\Gamma(T)$ be the set of the multiplicative one-parameter subgroups of T, and $X(T)$ be the set of characters of T. We define a pairing as following: if $\chi \in X(T)$ and $\delta \in \Gamma(T)$, $\langle \chi, \delta \rangle$ is the integer which occurs in the formula $\chi(\delta(t)) = t^{\langle \chi, \delta \rangle}$. Let $(\cdot, \cdot)$ denote the positive definite bilinear form on $\Gamma(T)$ such that $(\delta, \delta) = \|\delta\|^2$.

By a suitable choice of positive roots R^+, we may assume that a is a dominant cocharacter of in T. Recall that the set $\Gamma^+(T)$ of dominant cocharacters of T is defined by:

$$\Gamma^+(T) = \{\delta \in \Gamma(T) \colon \langle \delta, \alpha \rangle \geq 0, \, \forall \alpha \in R^+\}. \tag{2.2.3}$$

Let B be the corresponding minimal parabolic subgroup of G whose Lie algebra consists of all the non-positive root spaces.

Let $P = P(a)$ be the parabolic subgroup associated with a. Let W^P denote set of minimal length coset representatives of the quotient W_G/W_P, where $W_G = N_G(T)/Z_G(T)$ and $W_P = N_P(T)/Z_P(T)$ are Weyl groups of G and P. Then W_G acts on $\Gamma(T)$ by conjugation: $w \cdot \delta = w \delta w^{-1}$. Denote $\delta^w = w \cdot \delta$. We take the Bruhat order on W^P such that $w' \leq w$ if and only if the closure of the Schubert cell BwP contains $Bw'P$. We note that the Bruhat order coincides with the folding order defined in [16] (See [16, Remark 3.8]).

Definition 2.2.1 (Schubert variety)**.** Given an element $w \in W^P$, the standard Schubert variety X_w is the Zariski closure of the Schubert cell BwP. A *Schubert variety* is a subvariety of G/P of the form gX_w, where $g \in G$ and $w \in W^P$.

We say that a Schubert variety gX_w is **unstable** with respect to $a(t)$ if there exists $\delta \in \Gamma^+(T)$ such that $(\delta, a^w) > 0$. We say that gX_w is **weakly unstable** with respect to $a(t)$ if there exists non-trivial $\delta \in \Gamma^+(T)$ such that $(\delta, a^w) \geq 0$.

For short, we will just say unstable or weakly unstable Schubert variety if $a(t)$ is clear in the context.

Remark 2.2.2. In this article, when we project from G to G/P, we always take the following map

$$\begin{aligned} \pi_P \colon G &\longrightarrow G/P \\ g &\longmapsto g^{-1}P. \end{aligned} \tag{2.2.4}$$

When we write BwP, we treat it as a subvariety of G/P; while $Pw^{-1}B$ is treated as a subset of G.

For $\delta \in \Gamma^+(T)$, define the subset $W^+(\delta, a)$ of W^P as

$$W^+(\delta, a) = \{w \in W^P \colon (\delta, a^w) > 0\}, \tag{2.2.5}$$

and we define $W^-(\delta, a), W^{0+}(\delta, a)$ and $W^{0-}(\delta, a)$ similarly, with $<$, $\geq$ and $\leq$ in place of $>$ in (2.2.5) respectively. We note that $W^+(\delta, a)$ is a "metric thickening" as defined in [16, Section 3.4].

Lemma 2.2.3. *(a) Let $w' \leq w$ be elements in W^P, and $\delta \in \Gamma^+(T)$. Then one has $(\delta, a^{w'}) \geq (\delta, a^w)$.*

(b) $\bigsqcup_{w \in W^+(\delta,a)} BwP$ is a finite union of unstable Schubert subvarieties of G/P.

(c) $\bigsqcup_{w \in W^{0+}(\delta,a)} BwP$ is a finite union of weakly unstable Schubert subvarieties of G/P.

Proof. Both (b) and (c) follow from (a). For a proof of (a), see e.g. [16, Lemma 3.4]. □

Let $\rho : G \to \mathrm{GL}(V)$ be any finite dimensional linear representation of G. Let us recall some notions from geometric invariant theory (see e.g. [24] for more details). A nonzero vector v is called *unstable* if the closure of the G-orbit Gv contains the origin. v is called *semistable* if it is not unstable. For any $v \in V\backslash\{0\}$ and $\delta \in \Gamma(G)$, by [17, Lemma 1.2] we can write $v = \sum v_i$ where $\delta(t)v_i = t^i v_i$. Define the numerical function $m(v, \delta)$ to be the maximal[1] i such that $v_i \neq 0$.

By a theorem of Kempf (see [17, Theorem 4.2]), the function $m(v, \delta)/\|\delta\|$ has a negative minimum value B_v on the set of non-trivial multiplicative one-parameter subgroups δ. Let $\Lambda(v)$ denote the set of primitive multiplicative one-parameter subgroup δ such that $m(v, \delta) = B_v \cdot \|\delta\|$. Kempf [17, Theorem 4.2] shows that the parabolic subgroup $P(\delta)$ does not depend on the choice of $\delta \in \Lambda(v)$, which is denoted by $P(v)$. Moreover, $\Lambda(v)$ is a principal homogeneous space under conjugation by the unipotent radical of $P(v)$. In particular, for any δ in $\Lambda(v)$ and b in $P(v)$, we know that $b\delta b^{-1}$ is also contained in $\Lambda(v)$.

For $v \in V\backslash\{0\}$, define

$$G(v, V^-(a)) = \{g \in G \colon gv \in V^-(a)\}, \tag{2.2.6}$$

where

$$V^-(a) = \{v \in V \colon \lim_{t\to\infty} a(t)v = 0\}. \tag{2.2.7}$$

As noted in [18, Section 3.3], though the limits in [17] are defined algebraically, they coincide with limits in the Hausdorff topology induced from the usual topology on $\mathbb{R}$, by [17, Lemma 1.2].

Now we proceed to the main result of this section.

[1] It is "minimal" in Kempf's original definition. Since we are taking limit as t tends to ∞ instead of 0, our numerical function is actually opposite to Kempf's.

Proposition 2.2.4. *For any $v \in V\backslash\{0\}$, there exits $\delta_0 \in \Gamma^+(T)$ and $g_0 \in G$ such that*

$$G(v, V^-(a)) \subset \bigsqcup_{w \in W^+(\delta_0, a)} Pw^{-1}Bg_0^{-1}. \tag{2.2.8}$$

Proof. By definition we have the following identities because of G-equivariance:

$$G(gv, V^-(a)) = G(v, V^-(a))g^{-1}, \quad \forall g \in G; \tag{2.2.9}$$

$$\Lambda(gv) = g\Lambda(v)g^{-1}, \quad \forall g \in G. \tag{2.2.10}$$

If v is semistable, then $G(v, V^-(a))$ is empty, and the conclusion trivially holds. From now on we assume that v is unstable, and thus $\Lambda(v)$ is non-empty. Take $\delta_1 \in \Lambda(v)$, then there exists $g_0 \in G$ and $\delta_0 \in \Gamma^+(T)$ such that $g_0^{-1}\delta_1 g_0 = \delta_0$. It follows from (2.2.10) that $\delta_0 \in \Lambda(g_0^{-1}v)$.

We prove by contradiction. Suppose that (2.2.8) does not hold. Considering the Bruhat decomposition

$$G = \bigsqcup_{w \in W^P} Pw^{-1}B, \tag{2.2.11}$$

we can take $g \in G(g_0^{-1}v, V^-(a))$ such that it can be written as

$$g = pw^{-1}b, \text{ where } p \in P,\ w \in W^{0-}(\delta_0, a),\ b \in B. \tag{2.2.12}$$

Write $v' = bg_0^{-1}v$. In view of (2.2.10), by [17, Theorem 4.2(3)] we have $\Lambda(g_0^{-1}v) = \Lambda(v')$. Hence δ_0 is an element in $\Lambda(v')$.

We also have $v' \in V^-(a^w)$. Indeed, $gg_0^{-1}v \in V^-(a)$ implies that $pw^{-1}v' \in V^-(a)$. Since $V^-(a)$ is P-invariant, we know that $w^{-1}v' \in V^-(a)$. Hence $v' \in V^-(a^w)$.

Take a large integer N, we define $\delta_N = N\delta_0 + a^w$. We claim that for a sufficiently large N, one has

$$\frac{m(v', \delta_N)}{\|\delta_N\|} < \frac{m(v', \delta_0)}{\|\delta_0\|}, \tag{2.2.13}$$

and this will contradict the fact that $\delta_0 \in \Lambda(v')$.

To prove the claim, consider the weight space decomposition $V = \bigoplus V_\chi$, where T acts on V_χ by multiplication via the character χ of T. It suffices to prove that for any χ such that the projection of v' on V_χ is nonzero, one has

$$\frac{\langle \chi, \delta_N \rangle}{\|\delta_N\|} < \frac{\langle \chi, \delta_0 \rangle}{\|\delta_0\|}. \tag{2.2.14}$$

To prove (2.2.14), we define an auxiliary function:

$$\begin{aligned} f(s) &= \frac{\langle \chi, \delta_0 + s \cdot a^w \rangle^2}{\|\delta_0 + s \cdot a^w\|^2} \\ &= \frac{\langle \chi, \delta_0 \rangle^2 + 2s\langle \chi, \delta_0 \rangle \langle \chi, a^w \rangle + s^2 \langle \chi, a^w \rangle^2}{(\delta_0, \delta_0) + 2s(\delta_0, a^w) + s^2(a^w, a^w)} \end{aligned} \tag{2.2.15}$$

Compute its derivative at 0:

$$f'(0) = \frac{2\langle \chi, \delta_0 \rangle \langle \chi, a^w \rangle (\delta_0, \delta_0) - 2(\delta_0, a^w)\langle \chi, \delta_0 \rangle^2}{(\delta_0, \delta_0)^2} \tag{2.2.16}$$

Since $v' \in V^-(a^w)$, we know that $\langle \chi, a^w \rangle < 0$. Since $\delta_0 \in \Lambda(v')$, we know that $\langle \chi, \delta_0 \rangle < 0$. Also by the choice of w we know that $(\delta_0, a^w) \leq 0$. Combining the above one gets $f'(0) > 0$. Hence for N large we have

$$f(1/N) > f(0), \tag{2.2.17}$$

and (2.2.14) follows because each side of (2.2.17) is the square of each side of (2.2.14). Therefore (2.2.13) holds, contradicting the fact that $\delta_0 \in \Lambda(v')$. □

Now we are ready to prove Theorem 2.1.2.

Proof of Theorem 2.1.2. We prove by contradiction. Suppose that for all $C > 0$, there exist t and v such that (2.1.4) does not hold. We take a sequence $C_i \to 0$. Then after passing to a subsequence we can find $t_i \to \infty$ and a sequence $(v_i)_{i\in\mathbb{N}}$ in V such that

$$\sup_{s\in[a,b]} \|a(t_i)\phi(s)v_i\| < C_i\|v_i\|. \tag{2.2.18}$$

Without loss of generality we may assume that $\|v_i\| = 1$. Then after passing to a subsequence, we may assume that $v_i \to v_0$. Hence we have

$$\sup_{s\in[a,b]} \|a(t_i)\phi(s)v_0\| \overset{t_i\to\infty}{\longrightarrow} 0. \tag{2.2.19}$$

Therefore $\phi(s)v_0$ is contained in $V^-(a)$ for all $s \in [a,b]$, and it follows that the image of ϕ is contained in $G(v_0, V^-(a))$. (See (2.2.6).) By Lemma 2.2.3(b) and Proposition 2.2.4, the image of $G(v_0, V^-(a))$ under π_P in G/P is a finite union of unstable Schubert varieties. But ϕ is analytic, which implies that the image of $\widetilde{\phi}$ is contained in one single unstable Schubert variety. This contradict our assumption on ϕ. □

Proposition 2.2.4 and Theorem 2.1.2 will play a central role in proving the non-divergence of translated measures. To handle non-focusing, one needs a slightly generalized version, motivated by the work of Richard and Shah [30, Section 6]. We need the following result due to Kempf.

Lemma 2.2.5 ([17] Lemma 1.1(b))**.** *Let G be a connected reductive algebraic group over a field k, and X be any affine G-scheme. If S is a closed G-subscheme of X, then there is a G-equivariant morphism $f : X \to W$, where W is a representation of G, such that S is the scheme-theoretic inverse image $f^{-1}(0)$ of the reduced closed subscheme of W supported by zero.*

In view of Kempf's Lemma 2.2.5, the following is a corollary of Proposition 2.2.4.

Corollary 2.2.6. *Let the notations be as in the beginning of this section. Let S be the real points of any G-subscheme of V. For any $v \in V$, define the following subset of G:*

$$G(v, S, a) = \{g \in G\colon \lim_{t \to \infty} a(t) g v \in S\}. \tag{2.2.20}$$

Then for any $v \in V \backslash S$, there exists $\delta_0 \in \Gamma^+(T)$ and $g_0 \in G$ such that

$$G(v, S, a) \subset \bigsqcup_{w \in W^+(\delta_0, a)} P w^{-1} B g_0^{-1}. \tag{2.2.21}$$

Proof. By Lemma 2.2.5, there exist a G-equivariant morphism $f : V \to W$ where $f^{-1}(0) = S$. Hence it follows from the definition that

$$G(v, S, a) \subset G(f(v), W^-(a)). \tag{2.2.22}$$

Now it remains to apply Proposition 2.2.4 for W and $f(v)$. □

Now we present the following variant of Proposition 2.2.4.

Proposition 2.2.7. *Let $v \in V$ such that the G-orbit Gv is not closed. Define*

$$G(v, V^{0-}(a)) = \{g \in G : gv \in V^{0-}(a)\}, \tag{2.2.23}$$

where

$$V^{0-}(a) = \{v \in V : \lim_{t \to \infty} a(t) v \text{ exists}\}. \tag{2.2.24}$$

Then there exists $\delta_0 \in \Gamma^+(T)$ and $g_0 \in G$ such that

$$G(v, V^{0-}(a)) \subset \bigsqcup_{w \in W^{0+}(\delta_0, a)} P w^{-1} B g_0^{-1}. \tag{2.2.25}$$

Proof. Let $S = \partial(Gv)$. Since any G-orbit is open in its closure, we know that S is closed and G-invariant. By Lemma 2.2.5, there exists a G-equivariant morphism

$f : V \to W$ where $f^{-1}(0) = S$. Notice that $f(v)$ is unstable in W. We claim that

$$G(f(v), W^{0-}(a)) \subset \bigsqcup_{w \in W^{0+}(\delta_0, a)} Pw^{-1}Bg_0^{-1}. \tag{2.2.26}$$

To prove the claim, we argue with W and $f(v)$ in exactly the same way as in the proof of Proposition 2.2.4. The only difference is the following. When showing $f'(0) > 0$, one needs $\langle \chi, a^w \rangle < 0$ and $(\delta_0, a^w) \leq 0$ there; but here one has $\langle \chi, a^w \rangle \leq 0$ and $(\delta_0, a^w) < 0$, which also implies that $f'(0) > 0$. Hence (2.2.26) holds.

Finally, since f is G-equivariant, we have $f(V^{0-}) \subset W^{0-}$. Hence

$$G(v, V^{0-}(a)) \subset G(f(v), W^{0-}(a)). \tag{2.2.27}$$

Therefore (2.2.25) holds. $\square$

2.3 Non-divergence of the limiting distribution

Let $G = \mathbf{G}(\mathbb{R})$ be a connected semisimple real algebraic group, and L be a real Lie group containing G. Let $\{a(t)\}_{t\in\mathbb{R}^\times}$ be a multiplicative one-parameter subgroup of G with non-trivial projection on each simple factor of G. Let $P = P(a)$ be the parabolic subgroup of G whose real points consists of the elements $g \in G$ such that the limit $\lim_{t\to\infty} a(t)ga(t)^{-1}$ exists. Let $\phi\colon I = [a, b] \to G$ be an analytic map, and let $\pi_P\colon G \to G/P$ be the projection which maps g to $g^{-1}P$. Then $\widetilde{\phi} = \pi_P \circ \phi$ is an analytic curve on G/P. In this section we assume that the image of $\widetilde{\phi}$ is not contained in any unstable Schubert variety of G/P with respect to $a(t)$.

Let $x_0 = l\Lambda \in L/\Lambda$. We will assume that the orbit of x_0 under G is dense in L/Λ; that is $\overline{Gx_0} = L/\Lambda$. Let $t_i \to \infty$ be any sequence in $\mathbb{R}_{>0}$. Let μ_i be the parametric measure supported on $a(t_i)\phi(I)x_0$, that is, for any compactly supported function $f \in C_c(L/\Lambda)$ one has

$$\int_{L/\Lambda} f \, d\mu_i = \frac{1}{|I|} \int_I f(a(t_i)\phi(s)x_0) \, \mathrm{d}s. \tag{2.3.1}$$

Theorem 2.3.1. *Given $\epsilon > 0$ there exists a compact set $\mathcal{F} \subset L/\Lambda$ such that $\mu_i(\mathcal{F}) \geq 1 - \epsilon$ for all large $i \in \mathbb{N}$.*

This theorem will be proved via linearization technique combined with Theorem 2.1.2. We follow [35, Section 3] closely, as most of the arguments there work not only for $G = \mathrm{SL}_n(\mathbb{R})$ but also for general G.

Definition 2.3.2. Let $\mathfrak{l}$ denote the Lie algebra of L, and denote $d = \dim L$. We define

$$V = \bigoplus_{i=1}^{d} \bigwedge^i \mathfrak{l},$$

and let L act on V via $\bigoplus_{i=1}^{d} \bigwedge^i \mathrm{Ad}(L)$. This defines a linear representation of L (and of G by restriction):

$$L \to \mathrm{GL}(V).$$

The following theorem due to Kleinbock and Margulis is the basic tool to prove that there is no escape of mass to infinity:

Theorem 2.3.3 (see [5], [19] and [36])**.** *Fix a norm $\|\cdot\|$ on V. There exist finitely many vectors $v_1, v_2, \cdots, v_r \in V$ such that for each $i = 1, 2, \cdots, r$, the orbit Λv_i is discrete, and moreover, the following holds: for any $\epsilon > 0$ and $R > 0$, there exists a compact set $K \subset L/\Lambda$ such that for any $t > 0$ and any subinterval $J \subset I$, one of the following holds:*

(I) There exist $\gamma \in \Lambda$ and $j \in \{1, \cdots, r\}$ such that

$$\sup_{s \in J} \|a(t)\phi(s) l \gamma v_j\| < R;$$

(II)

$$|\{s \in J \colon a(t)\phi(s)x_0 \in K\}| \geq (1-\epsilon)|J|.$$

The key ingredient of the proof, as explained in [35, Section 3.2] and [36, Section 2.1], is the following growth property called the (C,α)-good property, which is due to [19, Proposition 3.4]. Following Kleinbock and Margulis, we say that a function $f \colon I \to \mathbb{R}$ is (C,α)-good if for any subinterval $J \subset I$ and any $\epsilon > 0$, the following holds:

$$|\{s \in J \colon |f(s)| < \epsilon\}| \leq C \left(\frac{\epsilon}{\sup_{s\in J}|f(s)|}\right)^{\alpha} |J|.$$

Now we are ready to prove the main result of this section.

Proof of Theorem 2.3.1. Take any $\epsilon > 0$. Take a sequence $R_k \to 0$ as $k \to \infty$. For each $k \in \mathbb{N}$, let $\mathcal{F}_k \subset L/\Lambda$ be a compact set as determined by Theorem 2.3.3 for these ϵ and R_k. If the theorem fails to hold, then for each $k \in \mathbb{N}$ we have $\mu_i(\mathcal{F}_k) > 1-\epsilon$ for infinitely many $i \in \mathbb{N}$. Therefore after passing to a subsequence of $\{\mu_i\}$, we may assume that $\mu_i(\mathcal{F}_i) < 1-\epsilon$ for all i. Then by Theorem 2.3.3, after passing to a subsequence, we may assume that there exists v_0 and $\gamma_i \in \Lambda$ such that

$$\sup_{s\in I}\|a(t_i)\phi(s)l\gamma_i v_0\| \leq R_i \xrightarrow{i\to\infty} 0.$$

Since $\Lambda \cdot v_0$ is discrete, there exists $r_0 > 0$ such that $\|l\gamma_i v_0\| \geq r_0$ for each i. We put $v_i = l\gamma_i v_0/\|l\gamma_i v_0\|$. Then $v_i \to v \in V$ and $\|v\| = 1$. Therefore

$$\sup_{s\in I}\|a(t_i)\phi(s)v_i\| \leq R_i/r_0 \xrightarrow{i\to\infty} 0. \tag{2.3.2}$$

Then it follows that

$$\sup_{s\in I}\|a(t_i)\phi(s)v\| \xrightarrow{i\to\infty} 0. \tag{2.3.3}$$

This contradict Theorem 2.1.2. ☐

As a consequence of Theorem 2.3.1, we deduce the following:

Corollary 2.3.4. *After passing to a subsequence, $\mu_i \to \mu$ in the space of probability measures on L/Λ with respect to the weak-* topology.*

We note that Theorem 2.1.1 follows from Theorem 2.3.1.

2.4 Invariance under a unipotent flow

Let $G = \mathbf{G}(\mathbb{R})$ be a connected semisimple real algebraic group, and $\{a(t)\}_{t\in\mathbb{R}^\times}$ be a multiplicative one-parameter subgroup of G with non-trivial projection on each simple factor of G. Define

$$P = \{g \in G\colon \lim_{t\to\infty} a(t)ga(t)^{-1} \text{ exists}\}. \tag{2.4.1}$$

Let $\mathfrak{X}$ be a locally compact second countable Hausdorff topological space, with a continuous G-action. Let $\phi\colon I = [a,b] \to G$ be an analytic curve, whose projection under $g \mapsto g^{-1}P$ on G/P is non-constant. Let $\mathfrak{g}$ denote the Lie algebra of G.

Since the exponential map $\exp\colon \mathfrak{g} \to G$ is a local homeomorphism, we can take a sufficiently small $\eta > 0$ such that for any $s \in I$ and $0 < \xi < \eta$, there exists $\Psi(s,\xi)$ in $\mathfrak{g}$ such that

$$\phi(s+\xi)\phi(s)^{-1} = \exp\Psi(s,\xi). \tag{2.4.2}$$

Moreover, Ψ is an analytic map in both s and ξ. Since $s \mapsto \phi(s)^{-1}P$ is not constant, $\Psi(s,\xi)$ does not belong to the Lie algebra of P.

Lemma 2.4.1. *There exists $m > 0$ and a nilpotent element Y_s in $\mathfrak{g}$ such that for all $s \in I$,*

$$\operatorname{Ad} a(t)\, \Psi(s, t^{-m}) \to Y_s, \quad t \to \infty. \tag{2.4.3}$$

Moreover, one can assume that the map $s \to Y_s$ is non-zero and analytic, and the convergence is uniform in s.

Proof. Since Ψ is an analytic map in both s and ξ, we can write

$$\Psi(s, \xi) = \sum_{i=1}^{\infty} \xi^i \psi_i(s), \tag{2.4.4}$$

where $\psi_i \colon I \to \mathfrak{g}$ is analytic for each i.

Notice that $\operatorname{Ad} a(t)$ is semisimple and acts on the finite dimensional vector space $\mathfrak{g}$, then for each i there exist $m_i \in \mathbb{Z}$ such that

$$\operatorname{Ad} a(t)\psi_i(s) = \sum_{j \le m_i} t^j \psi_{i,j}(s), \tag{2.4.5}$$

where $\psi_{i,j}(s)$ is analytic in s, and $\psi_{i,m_i}(s) \neq 0$ for all but finitely many $s \in I$. Since the projection of ϕ on G/P is non-trivial, there exists i such that $m_i > 0$.

Combining (2.4.4)(2.4.5), we get

$$\operatorname{Ad} a(t)\, \Psi(s, \xi) = \sum_{i=1}^{\infty} \sum_{j \le m_i} t^j \xi^i \psi_{i,j}(s). \tag{2.4.6}$$

Now set $m = \max_{i \ge 1}\{m_i/i\}$. Since m_i are all eigenvalues of $\operatorname{Ad} a(t)$, they are uniformly bounded from above. Hence we know that m exists and $m > 0$. Denote $\mathbb{I} = \{i \ge 1 \colon m_i/i = m\}$, and we see that $\mathbb{I}$ is a finite set. We set

$$Y_s = \sum_{i \in \mathbb{I}} \psi_{i,m_i}(s). \tag{2.4.7}$$

Since $\operatorname{Ad} a(t)^{-1} Y_s \to 0$ as $t \to \infty$, Y_s is nilpotent.

In view of (2.4.6),

$$\operatorname{Ad} a(t)\, \Psi(s, t^{-m}) = Y_s + \sum_{j-im<0} t^{j-im} \psi_{i,j}(s), \tag{2.4.8}$$

and (2.4.3) follows. □

We could then twist Y_s into a single direction due to the following lemma.

Lemma 2.4.2. *There are only finitely many G-conjugacy classes of the nilpotent elements in the Lie algebra $\mathfrak{g}$ of G.*

Proof. This result has been proved for groups over the complex numbers $\mathbb{C}$ (see [31]). Let X be any non-zero nilpotent element in $\mathfrak{g}$. Now it remains to show that there are only finitely many $\mathbf{G}(\mathbb{R})$-orbits in the real points of $\mathbf{G}(\mathbb{C})\cdot X$. Let $\mathbf{H}$ be the stabilizer of X in $\mathbf{G}$. Then $\mathbf{H}$ is an algebraic group defined over $\mathbb{R}$. It is well known that the $\mathbf{G}(\mathbb{R})$-orbits in $(\mathbf{G}/\mathbf{H})(\mathbb{R})$ are parametrized by the Galois cohomology $H^1(\mathrm{Gal}(\mathbb{C}/\mathbb{R}), \mathbf{H}(\mathbb{C}))$. Then the statement of the lemma follows from the finiteness of $H^1(\mathrm{Gal}(\mathbb{C}/\mathbb{R}), \mathbf{H}(\mathbb{C}))$, which is guaranteed by [25, Theorem 6.14]. □

Since there are only finitely many conjugacy classes of nilpotent elements in $\mathfrak{g}$, up to at most finitely many points we may assume that all the Y_s are in the same conjugacy class. Hence there exists w_0 in $\mathfrak{g}$, and $\delta(s)$ in G which is also analytic in s, such that for all but finitely many $s \in I$ one has

$$\operatorname{Ad}(\delta(s)) \cdot Y_s = w_0 \neq 0. \tag{2.4.9}$$

Define a non-trivial unipotent one-parameter subgroup of G as

$$W = \{\exp(t w_0) \colon t \in \mathbb{R}\}. \tag{2.4.10}$$

Let $(t_i)_{i\in\mathbb{N}}$ be a sequence in $\mathbb{R}$ such that $t_i \to \infty$ as $i \to \infty$. Let $x_i \to x$ a convergent sequence in $\mathfrak{X}$. For each $i \in \mathbb{N}$, let λ_i be the probability measure on $\mathfrak{X}$ such that

$$\int_{\mathfrak{X}} f \,\mathrm{d}\lambda_i = \frac{1}{|I|}\int_{s\in I} f(\delta(s)a(t_i)\phi(s)x_i)\,\mathrm{d}s, \quad \forall f \in \mathrm{C}_c(\mathfrak{X}). \tag{2.4.11}$$

The following theorem is the main result of this section. The new idea here due to Nimish Shah is that we can actually twist the curve after translating by $a(t)$.

Theorem 2.4.3. *Suppose that $\lambda_i \to \lambda$ in the space of finite measures on $\mathfrak{X}$ with respect to the weak-* topology, then λ is invariant under W.*

Proof. Given $f \in \mathrm{C}_c(\mathfrak{X})$ and $\epsilon > 0$. Since f is uniformly continuous, there exists a neighborhood Ω of the neutral element in G such that

$$|f(\omega y) - f(y)| < \epsilon, \quad \forall \omega \in \Omega, \forall y \in \mathfrak{X}. \tag{2.4.12}$$

Define

$$\Omega' = \bigcap_{s\in I} \delta(s)^{-1}\Omega\delta(s), \tag{2.4.13}$$

and Ω' is non-empty and open because $\{\delta(s)\}_{s\in I}$ is compact.

By Lemma 2.4.1, there exists $T > 0$ such that for all $t > T$ and for all but finitely many $s \in I$, there exists $\omega_{t,s} \in \Omega'$ such that

$$a(t)\exp\Psi(s, t^{-m})a(t)^{-1} = \omega_{t,s}\exp Y_s. \tag{2.4.14}$$

Take $\xi_i = t_i^{-m}$. In view of (2.4.2), for i large enough we have

$$\phi(s + \xi_i) = \exp\Psi(s, \xi_i)\phi(s). \tag{2.4.15}$$

Hence there exists $i_0 \in \mathbb{N}$ such that for all $i > i_0$,

$$\begin{aligned}
\delta(s)a(t_i)\phi(s+\xi_i) &= \delta(s)a(t_i)\exp\Psi(s,\xi_i)\phi(s) \\
&= \delta(s)\omega_{t_i,s}\exp Y_s a(t_i)\phi(s) \\
&= \left(\delta(s)\omega_{t_i,s}\delta(s)^{-1}\right)\delta(s)\exp Y_s a(t_i)\phi(s) \\
&= \left(\delta(s)\omega_{t_i,s}\delta(s)^{-1}\right)(\exp w_0)\delta(s)a(t_i)\phi(s) \\
&\in \Omega(\exp w_0)\delta(s)a(t_i)\phi(s).
\end{aligned} \tag{2.4.16}$$

By (2.4.12) we know that for all but finitely many $s \in I$,

$$|f((\exp w_0)\delta(s)a(t_i)\phi(s)x_i) - f(\delta(s)a(t_i)\phi(s+\xi_i)x_i)| < \epsilon. \tag{2.4.17}$$

It follows that for all $i > i_0$,

$$\left|\frac{1}{|I|}\int_I f((\exp w_0)\delta(s)a(t_i)\phi(s)x_i)\,\mathrm{d}s - \frac{1}{|I|}\int_I f(\delta(s)a(t_i)\phi(s+\xi_i)x_i)\,\mathrm{d}s\right| < \epsilon. \tag{2.4.18}$$

On the other hand, since f is bounded on $\mathfrak{X}$, there exists $i_1 \in \mathbb{N}$ such that for all $i > i_1$,

$$\left|\frac{1}{|I|}\int_I f(\delta(s)a(t_i)\phi(s+\xi_i)x_i)\,\mathrm{d}s - \frac{1}{|I|}\int_I f(\delta(s)a(t_i)\phi(s)x_i)\,\mathrm{d}s\right| < \epsilon. \tag{2.4.19}$$

Combining the above two equations we get

$$\left|\frac{1}{|I|}\int_I f((\exp w_0)\delta(s)a(t_i)\phi(s)x_i)\,\mathrm{d}s - \frac{1}{|I|}\int_I f(\delta(s)a(t_i)\phi(s)x_i)\,\mathrm{d}s\right| < 2\epsilon. \tag{2.4.20}$$

Therefore, for i large enough we have

$$\left|\int_{\mathfrak{X}} f((\exp w_0)\cdot x)\,\mathrm{d}\lambda_i - \int_{\mathfrak{X}} f(x)\,\mathrm{d}\lambda_i\right| < 2\epsilon. \tag{2.4.21}$$

Taking $i \to \infty$,

$$\left|\int_{\mathfrak{X}} f((\exp w_0)\cdot x)\,\mathrm{d}\lambda - \int_{\mathfrak{X}} f(x)\,\mathrm{d}\lambda\right| \le 2\epsilon. \tag{2.4.22}$$

Since ϵ is arbitrary, we conclude that λ is $\exp w_0$-invariant.

If we replace w_0 with any scalar multiple of w_0, the above arguments still work. Hence λ is invariant under $W = \{\exp(tw_0) \colon t \in \mathbb{R}\}$. $\square$

2.5 Dynamical behavior of translated trajectories near singular sets

Let notations be as in Section 2.3. Recall that the image of $\widetilde{\phi}$ is not contained in any unstable Schubert varieties of G/P with respect to $a(t)$. Let $\{\lambda_i : i \in \mathbb{N}\}$ be the sequence of probability measures on L/Λ as define in (2.4.11), where we take $\mathfrak{X} = L/\Lambda$ and $x_i = x_0$. Due to Theorem 2.3.1, by passing to a subsequence we assume that $\lambda_i \to \lambda$ as $i \to \infty$, where λ is a probability measure on L/Λ. By Theorem 2.4.3, λ is invariant under a unipotent subgroup W. We would like to describe the limit measure λ using the description of ergodic invariant measures for unipotent flows on homogeneous spaces due to Ratner [27]. We follow the treatment in [36, Section 4].

2.5.1 Ratner's theorem and linearization technique

Let $\pi \colon L \to L/\Lambda$ denote the natural quotient map. Let $\mathcal{H}$ denote the collection of closed connected subgroups H of L such that $H \cap \Lambda$ is a lattice in H, and that a unique unipotent one-parameter subgroup of H acts ergodically with respect to the H-invariant probability measure on $H/H \cap \Lambda$. Then $\mathcal{H}$ is a countable collection (see [27]).

For a closed connected subgroup H of L, define

$$N(H, W) = \{g \in L \colon g^{-1}Wg \subset H\}. \tag{2.5.1}$$

Now, suppose that $H \in \mathcal{H}$. We define the associated singular set

$$S(H,W) = \bigcup_{\substack{F \in \mathcal{H} \\ F \subsetneq H}} N(F,W). \tag{2.5.2}$$

Note that $N(H,W)N_L(H) = N(H,W)$. By [23, Proposition 2.1, Lemma 2.4],

$$N(H,W) \cap N(H,W)\gamma \subset S(H,W),\ \forall \gamma \in \Lambda \backslash N_L(H). \tag{2.5.3}$$

By Ratner's theorem [27, Theorem 1], as explained in [23, Theorem 2.2], we have the following.

Theorem 2.5.1 (Ratner)**.** *Given the W-invariant probability measure λ on L/Λ, there exists $H \in \mathcal{H}$ such that*

$$\lambda(\pi(N(H,W))) > 0 \quad \text{and} \quad \lambda(\pi(S(H,W))) = 0. \tag{2.5.4}$$

Moreover, almost every W-ergodic component of λ on $\pi(N(H,W))$ is a measure of the form $g\mu_H$, where $g \in N(H,W) \backslash S(H,W)$ and μ_H is a finite H-invariant measure on $\pi(H) \cong H/H \cap \Lambda$. In particular if H is a normal subgroup of L then λ is H-invariant.

Let V be as in Section 2.3. Let $d = \dim H$, and fix $p_H \in \bigwedge^d \mathfrak{h} \backslash \{0\}$. Due to [6, Theorem 3.4], the orbit Λp_H is a discrete subset of V. We note that for any $g \in N_L(H)$, $gp_H = \det(\operatorname{Ad} g|_{\mathfrak{h}})p_H$. Hence the stabilizer of p_H in L equals

$$N_L^1(H) := \{g \in N_L(H) \colon \det(\operatorname{Ad} g|_{\mathfrak{h}}) = 1\}. \tag{2.5.5}$$

Recall that $\operatorname{Lie}(W) = \mathbb{R}w_0$. Let

$$\mathcal{A} = \{v \in V : v \wedge w_0 = 0\}, \tag{2.5.6}$$

where V is defined in Definition 2.3.2. Then $\mathcal{A}$ is a linear subspace of V. We observe that

$$N(H,W) = \{g \in L \colon g \cdot p_H \in \mathcal{A}\}. \tag{2.5.7}$$

Recall that $x_0 = l\Lambda \in L/\Lambda$. Using the fact that ϕ is analytic, we obtain the following consequence of the linearization technique and (C, α)-good property (see [36][35][38]).

Proposition 2.5.2. *Let C be a compact subset of $N(H, W) \setminus S(H, W)$. Given $\epsilon > 0$, there exists a compact set $\mathcal{D} \subset \mathcal{A}$ such that, given a relatively compact neighborhood Φ of $\mathcal{D}$ in V, there exists a neighborhood $\mathcal{O}$ of $\pi(C)$ in L/Λ such that for any $t \in \mathbb{R}$ and subinterval $J \subset I$, one of the following statements holds:*

(I) $|\{s \in J \colon \delta(s)a(t)\phi(s)x_0 \in \mathcal{O}\}| \leq \epsilon |J|$.

(II) There exists $\gamma \in \Lambda$ such that $\delta(s)a(t)\phi(s)l\gamma p_H \in \Phi$ for all $s \in J$.

2.5.2 Algebraic consequences of positive limit measure on singular sets

Recall the definition of λ_i in (2.4.11), where we take $\mathfrak{X} = L/\Lambda$ and $x_i = x_0$. After passing to a subsequence, $\lambda_i \to \lambda$ in the space of probability measures on L/Λ, and by Theorem 2.3.1 and Theorem 2.4.3, we know that there exists $H \in \mathcal{H}$ such that

$$\lambda(\pi(N(H, W) \setminus S(H, W)) > 0. \tag{2.5.8}$$

In this section, we use Proposition 2.5.2 and Theorem 2.1.2 to obtain the following algebraic consequence, which is an analogue of [36, Proposition 4.8].

Proposition 2.5.3. *Let $l \in L$ such that $x_0 = l\Lambda$. Suppose $\lambda_i \to \lambda$, then there exists $\gamma \in \Lambda$ such that*

$$\phi(s)l\gamma p_H \in V^{0-}(a), \quad \forall s \in I. \tag{2.5.9}$$

Proof. By (2.5.4) there exists a compact subset $C \subset N(H, W) \setminus S(H, W)$ and a constant $c_0 > 0$ such that $\lambda(\pi(C)) > c_0$. We fix $0 < \epsilon < c_0$, and apply Proposition 2.5.2

to obtain $\mathcal{D}$. We choose any relatively compact neighborhood Φ of $\mathcal{D}$, and obtain an $\mathcal{O}$ such that either (I) or (II) holds.

Since $\lambda_i \to \lambda$, there exists $i_0 \in \mathbb{N}$ such that for all $i > i_0$, (I) does not hold. Therefore (II) holds for all $i > i_0$. In other words, there exists a sequence $\{\gamma_i\}$ in Λ and a subinterval $J \subset I$ such that

$$\delta(s)a(t_i)\phi(s)l\gamma_i p_H \in \Phi, \quad \forall i > i_0, \forall s \in J. \tag{2.5.10}$$

By Theorem 2.1.2, we know that $\{\gamma_i p_H\}$ is bounded. Indeed, if $\gamma_i p_H$ is unbounded, the left hand side of (2.5.10) will also be unbounded and cannot stay in Φ. Hence after passing to a subsequence, we may assume that there exists $\gamma \in \Lambda$ such that $\gamma_i p_H = \gamma p_H$ holds for all i. It follows that $a(t_i)\phi(s)l\gamma p_H$ remains bounded in V. This concludes the proof. $\square$

Next we are able to obtain more algebraic information from Proposition 2.5.3. First we show that the limiting process actually happens inside the G-orbit $G \cdot l\gamma p_H$.

Proposition 2.5.4. *Let the notations be as in Proposition 2.5.3. Then for all but finitely many $s \in I = [a, b]$, there exists $\xi(s) \in P$ such that*

$$\lim_{t\to\infty} a(t)\phi(s)l\gamma p_H = \xi(s)\phi(s)l\gamma p_H. \tag{2.5.11}$$

Proof. Denote $v = l\gamma p_H$. According to Proposition 2.5.3, the limit on the left-hand side of (2.5.11) exists. We claim that the limit actually lies in the G-orbit Gv for all but finitely many $s \in I$.

Consider the boundary $S = \partial(Gv) = \overline{Gv} \backslash Gv$. If S is empty then the claim holds automatically. Now suppose that S is non-empty, and that there exist infinitely many $s \in I$ such that $\lim_{t\to\infty} a(t)\phi(s)v$ is contained in S. Since ϕ is analytic, we have that for any $s \in I$, $\lim_{t\to\infty} a(t)\phi(s)v$ is contained in S. Hence in view of (2.2.20),

$$\phi(s) \in G(v, S, a), \quad \forall s \in J. \tag{2.5.12}$$

Moreover, by Corollary 2.2.6 there exists $\delta_0 \in \Gamma^+(T)$ and $g_0 \in G$ such that

$$G(v, S, a) \subset \bigsqcup_{w \in W^+(\delta_0, a)} P w^{-1} B g_0^{-1}. \tag{2.5.13}$$

By (2.5.12),(2.5.13) and Lemma 2.2.3(b), the image of $\widetilde{\phi}$ is contained in an unstable Schubert variety with respect to $a(t)$, which contradicts our assumption.

Hence for all but finitely many $s \in I$, there exists $\eta(s) \in G$ such that

$$\lim_{t\to\infty} a(t)\phi(s)v = \eta(s)\phi(s)v. \tag{2.5.14}$$

Now fix any s such that (2.5.14) holds. Take $t_0 > 0$, and set $w = a(t_0)\phi(s)v$. Then

$$\lim_{t\to\infty} a(t)w = \eta(s)a(t_0)^{-1}w. \tag{2.5.15}$$

By taking t_0 large enough, we may assume that $\eta(s)a(t_0)^{-1}$ is contained in a small neighborhood of the neutral element in G. Let F denote the stabilizer of $\eta(s)a(t_0)^{-1}w$ $= \eta(s)\phi(s)v$ in G, and let $\mathfrak{f}$ be the Lie algebra of F. It is easy to see that F contains $\{a(t)\}$.

Now the Lie algebra $\mathfrak{f}$ of F is $\operatorname{Ad} a(t)$-invariant, and thus we have the following decomposition as a consequence of $a(t)$ being semisimple:

$$\mathfrak{g} = \mathfrak{f}^\perp \oplus \mathfrak{f}, \tag{2.5.16}$$

where $\mathfrak{f}^\perp$ is an $\operatorname{Ad} a(t)$-invariant subspace of $\mathfrak{g}$.

On the other hand, according to the eigenvalues of $\operatorname{Ad} a(t)$, we can decompose $\mathfrak{g}$ into

$$\mathfrak{g} = \mathfrak{g}^- \oplus \mathfrak{g}^0 \oplus \mathfrak{g}^+. \tag{2.5.17}$$

Combining the above two decompositions (2.5.16)(2.5.17), we get

$$\mathfrak{g} = \mathfrak{g}^- \oplus \mathfrak{g}^0 \oplus (\mathfrak{g}^+ \cap \mathfrak{f}^\perp) \oplus (\mathfrak{g}^+ \cap \mathfrak{f}). \tag{2.5.18}$$

Hence there exist $X_s^{0-} \in \mathfrak{g}^0 \oplus \mathfrak{g}^-$ and $X_s^+ \in \mathfrak{g}^+ \cap \mathfrak{f}^\perp$ such that

$$a(t_0)\eta(s)^{-1} \in \exp X_s^{0-} \exp X_s^+ F. \tag{2.5.19}$$

By (2.5.15), we have that $X_s^+ = 0$. Hence

$$a(t_0)\eta(s)^{-1} \in \exp X_s^{0-} F. \tag{2.5.20}$$

Set $\xi(s) = \exp(-X_s^{0-})a(t_0)$, and one can verify that (2.5.11) holds. ☐

If we consider the slightly larger family of *weakly* unstable Schubert varieties, and further assume that the image of $\widetilde{\phi}$ is not contained in any weakly unstable Schubert variety, then we could obtain the following refinement of Proposition 2.5.4.

Proposition 2.5.5. *In the situation of Proposition 2.5.3, further assume that the image of $\widetilde{\phi}$ is not contained in any weakly unstable Schubert variety of G/P with respect to $a(t)$. Then the orbit $G \cdot l\gamma p_H$ is closed, and the stabilizer of $l\gamma p_H$ in G is reductive.*

Proof. Write $v = l\gamma p_H$. Suppose that Gv is not closed, then the boundary $S = \partial(Gv)$ is non-empty. By Proposition 2.2.7 there exists $\delta_0 \in \Gamma^+(T)$ and $g_0 \in G$ such that

$$G(v, V^{0-}(a)) \subset \bigsqcup_{w \in W^{0+}(\delta_0, a)} Pw^{-1}Bg_0^{-1}. \tag{2.5.21}$$

Also by (2.5.9) we know

$$\phi(s) \in G(v, V^{0-}(a)), \quad \forall s \in I. \tag{2.5.22}$$

By (2.5.21), (2.5.22) and Lemma 2.2.3(c), the image of $\widetilde{\phi}$ is contained in a weakly unstable Schubert variety, which contradicts our assumption on ϕ.

Therefore Gv is closed, i.e. $G \cdot l\gamma p_H$ is closed. By Matsushima's criterion, the stabilizer of $l\gamma p_H$ in G is reductive. □

The following proposition describes the obstructions to equidistribution. (C.f. [39, Theorem 6.1].)

Proposition 2.5.6. *Suppose that the image of $\widetilde{\phi}$ is not contained in any unstable Schubert variety of G/P with respect of $a(t)$, and that $\lambda_i \to \lambda$. Then there exists $g \in G$ and an algebraic subgroup F of L containing $\{a(t)\}$ such that $Fgl\Lambda$ is closed and admits a finite F-invariant measure, and that*

$$\phi(s) \in P(F \cap G)g, \quad \forall s \in I. \tag{2.5.23}$$

Furthermore, if the image of $\widetilde{\phi}$ is not contained in any weakly unstable Schubert variety, then we can choose F such that $F \cap G$ is reductive.

Proof. Let $\xi(s)$ be defined as in Proposition 2.5.4. Since the right hand side of (2.5.23) is left $a(t)$-invariant, without loss of generality we may replace $\phi(s)$ with $a(t_0)\phi(s)$ for some large $t_0 > 0$, and assume that $\xi(s)$ lies in a small neighborhood of e in G, for all $s \in I$. Hence we may take $\xi(s) \in P$.

Fix any $s_0 \in I$. Let $g = \xi(s_0)\phi(s_0)$ and $v = l\gamma p_H$. We set $F = \mathrm{Stab}_L(gv) = gl\gamma N_L^1(H)\gamma^{-1}l^{-1}g^{-1}$. By Proposition 2.5.4 we have $\{a(t)\} \subset F$. Since $\Lambda \cdot p_H$ discrete, $N_L^1(H) \cdot \Lambda$ is closed. Hence $Fgl\Lambda$ is also closed.

Now the Lie algebra $\mathfrak{f}$ of F is $\mathrm{Ad}\, a(t)$-invariant, and thus we have the following decomposition as a consequence of $a(t)$ being semisimple:

$$\mathfrak{g} = \mathfrak{f}^\perp \oplus \mathfrak{f}, \tag{2.5.24}$$

where $\mathfrak{f}^{\perp}$ is an $\operatorname{Ad} a(t)$-invariant subspace of $\mathfrak{g}$.

On the other hand, according to the eigenvalues of $\operatorname{Ad} a(t)$, we can decompose $\mathfrak{g}$ into

$$\mathfrak{g} = \mathfrak{g}^{-} \oplus \mathfrak{g}^{0} \oplus \mathfrak{g}^{+}. \tag{2.5.25}$$

Combining the above two decompositions (2.5.24)(2.5.25), we get

$$\mathfrak{g} = \mathfrak{g}^{-} \oplus \mathfrak{g}^{0} \oplus (\mathfrak{g}^{+} \cap \mathfrak{f}^{\perp}) \oplus (\mathfrak{g}^{+} \cap \mathfrak{f}). \tag{2.5.26}$$

Hence for all s near s_0, there exist $X_s^{0-} \in \mathfrak{g}^{0} \oplus \mathfrak{g}^{-}$ and $X_s^{+} \in \mathfrak{g}^{+} \cap \mathfrak{f}^{\perp}$ such that

$$\xi(s_0)\phi(s)g^{-1} \in \exp X_s^{0-} \exp X_s^{+} F. \tag{2.5.27}$$

Since $a(t_i)\phi(s)v$ converge in V as $i \to \infty$, by Proposition 2.5.4 we know that $a(t_i)\phi(s)g^{-1}F$ converge in G/F as $i \to \infty$. It follows that

$$X_s^{+} = 0, \quad \forall s \in I. \tag{2.5.28}$$

Since $X_s^{0-} \in \mathfrak{g}^{0} \oplus \mathfrak{g}^{-}$, we have

$$\exp X_s^{0-} \in P. \tag{2.5.29}$$

Combining (2.5.27)(2.5.28)(2.5.29) we get

$$\phi(s) \in PFg, \tag{2.5.30}$$

for all $s \in I$. This implies (2.5.23). Moreover, by [32, Theorem 2.3], there exists a subgroup F_1 of F containing all Ad-unipotent one-parameter subgroups of L contained in F such that $F_1 gl\Lambda$ admits a finite F_1-invariant measure. Since F contains $\{a(t)\}$, P contains the central torus of F. Hence $PFg = PF_1g$, and we may replace F by F_1.

If we further assume that the image of $\widetilde{\phi}$ is not contained in any weakly unstable Schubert variety, then by Proposition 2.5.5 we know that the stabilizer of $l\gamma p_H$ in G is reductive, i.e. $g^{-1}Fg \cap G$ is reductive. Hence $F \cap G$ is also reductive. $\square$

2.5.3 Lifting of obstructions and proof of equidistribution results

In this section, we show that the conditions in Theorem 2.1.5 are preserved under projections. This enables us to use induction to prove the equidistribution results.

Lemma 2.5.7. *Let G be a connected semisimple real algebraic group, and $p\colon G \to \overline{G}$ be a surjective homomorphism. Let $a(t)$ be a multiplicative one-parameter subgroup of G, and $\overline{a(t)}$ be its image in $\overline{G}$. Suppose that $\overline{a(t)}$ is non-trivial. Define (weakly) unstable Schubert varieties and partial flag subvarieties of $\overline{G}/\overline{P}$ with respect to $\overline{a(t)}$, $\overline{T}$ and $\overline{B}$. Then the preimage of any unstable (resp. weakly unstable) Schubert subvariety of $\overline{G}/\overline{P}$ with respect to $\overline{a(t)}$ is an unstable (resp. weakly unstable) Schubert subvariety of G/P with respect to $a(t)$.*

Proof. Let $X_{\overline{w}}$ be an unstable Schubert subvariety of $\overline{G}/\overline{P}$, where $\overline{w} \in W^{\overline{P}}$ such that $(\overline{\delta}, \overline{a}^{\overline{w}}) \geq 0$ for some $\overline{\delta} \in \Gamma^+(\overline{T})$. Let G_1 denote the kernel of p, and we have $W_G = W_{G_1} \times W_{\overline{G}}$. Let w_0 denote the unique maximal element in W^{P_1}. Then the preimage of $X_{\overline{w}}$ is $X_{(w_0,\overline{w})}$. Now it remains to check instability. We note that the Killing form on $\mathfrak{g}$ is the sum of the Killing forms on $\mathfrak{g}_1$ and $\overline{\mathfrak{g}}$. Hence we consider the lifted multiplicative one-parameter subgroup $(e, \overline{\delta}) \in \Gamma^+(T)$, and use it to check that $X_{(w_0,\overline{w})}$ is unstable.

The same proof also works for weakly unstable Schubert varieties. □

We now proceed to the equidistribution results. Recall that $l \in L$ such that $x_0 = l\Lambda$, and λ_i are probability measures on L/Λ as defined in (2.4.11).

Proposition 2.5.8. *Let ϕ be an analytic curve on G such that the following two conditions hold:*

(a) the image of $\widetilde{\phi}$ is not contained in any unstable Schubert variety of G/P with respect to $a(t)$;

(b) For any $g \in G$ and any proper algebraic subgroup F of L containing $\{a(t)\}$ such that Fgx_0 is closed and admits a finite F-invariant measure, the image of ϕ is not contained in $P(F \cap G)g$.

Suppose that $\lambda_i \to \lambda$ in the weak- topology, then λ is the unique L-invariant probability measure on L/Λ.*

Proof. By Proposition 2.5.6, there exists an algebraic subgroup F of L such that (2.5.23) holds. Then condition (b) implies that $F = G$, and thus G fixes $l\gamma p_H$. Arguing as in the proof of [35, Theorem 5.6], we know that $L = N_L^1(H)$, i.e. H is normal in L.

Now we can prove the theorem by induction on the number of simple factors in L. If L is simple, then we have $H = L$, and λ is $H = L$-invariant. For the inductive step, we consider the natural quotient map $p \colon L \to L/H$. For any subset $E \subset L$, let $\overline{E}$ denote its image under the quotient map. By Lemma 2.5.7, $\overline{\phi(I)}$ is not contained in any unstable Schubert variety with respect to $\overline{a(t)}$. Hence $\overline{\phi}$ still satisfies condition (a). One can also verify that $\overline{\phi}$ still satisfies condition (b). Indeed, if the image of $\overline{\phi}$ is contained in $\overline{P}(F_0 \cap \overline{G})\overline{g}$ for some $F_0 \subsetneq \overline{L}$ such that $F_0\overline{gx_0}$ is closed, then the image of ϕ is contained in $P(p^{-1}(F_0) \cap G)g$ and $p^{-1}(F_0)gx_0$ is also closed.

Now both conditions still hold for the projected curve $\overline{\phi}$. By inductive hypothesis we know that the projected measure $\overline{\lambda}$ is the L/H-invariant measure on $L/H\Lambda$. In addition, we already know that λ is H-invariant. Therefore λ is L-invariant. $\square$

Corollary 2.5.9. *Let ϕ be an analytic curve satisfying (a) and (b) in Proposition 2.5.8. Let μ_i be the probability measure on L/Λ as defined in (2.3.1). Suppose that $\mu_i \to \mu$ with respect to the weak-* topology, then μ is the unique L-invariant probability measure on L/Λ.*

Proof. The deduction of Corollary 2.5.9 from Proposition 2.5.8 is analogous to the proof of [35, Corollary 5.7]. $\square$

Parallel to Proposition 2.5.8 and Corollary 2.5.9, the following results could be proved with the same arguments.

Proposition 2.5.10. *Let ϕ be an analytic curve on G such that the following two conditions hold:*

(A) the image of $\widetilde{\phi}$ is not contained in any weakly unstable Schubert variety of G/P with respect to $a(t)$;

(B) For any $g \in G$ and any proper algebraic subgroup F of L containing $\{a(t)\}$ such that Fgx_0 is closed and admits a finite F-invariant measure and that $F \cap G$ is reductive, the image of ϕ is not contained in $P(F \cap G)g$.

Suppose that $\lambda_i \to \lambda$ in the weak- topology, then λ is the unique L-invariant probability measure on L/Λ.*

Corollary 2.5.11. *Let ϕ be an analytic curve satisfying (A) and (B) in Proposition 2.5.10. Let μ_i be the probability measure on L/Λ as defined in (2.3.1). Suppose that $\mu_i \to \mu$ with respect to the weak-* topology, then μ is the unique L-invariant probability measure on L/Λ.*

Now we are ready to prove the main theorems in Section 2.1.2.

Proof of Theorem 2.1.3. If (2.1.6) fails to hold, then there exist $\epsilon > 0$ and a sequence $t_i \to \infty$ such that for each i,

$$\left| \frac{1}{b-a} \int_a^b f(a(t_i)\phi(s)x_0)\, \mathrm{d}s - \int_{L/\Lambda} f \, \mathrm{d}\mu_{L/\Lambda} \right| \geq \epsilon. \tag{2.5.31}$$

In view of (2.3.1) and Corollary 2.3.4, this statement contradicts Corollary 2.5.9. □

Proof of Theorem 2.1.5. If (2.1.7) fails to hold, then there exist $\epsilon > 0$ and a sequence $t_i \to \infty$ such that for each i,

$$\left| \frac{1}{b-a} \int_a^b f(a(t_i)\phi(s)x_0)\, \mathrm{d}s - \int_{L/\Lambda} f \, \mathrm{d}\mu_{L/\Lambda} \right| \geq \epsilon. \tag{2.5.32}$$

In view of (2.3.1) and Corollary 2.3.4, this statement contradicts Corollary 2.5.11. □

2.6 Grassmannians and Schubert varieties

In this section we consider the special case where $G = L = \mathrm{SL}_{m+n}(\mathbb{R})$, and $\Lambda = \mathrm{SL}_{m+n}(\mathbb{Z})$. Define

$$a(t) = \begin{bmatrix} t^n I_m & \\ & t^{-m} I_n \end{bmatrix}.$$

Then $\{a(t)\}$ is a multiplicative one-parameter subgroup of G. In this section, all the unstable and weakly unstable Schubert varieties are with respect to this $a(t)$. Let P be the parabolic subgroup associated with $\{a(t)\}$. We have

$$P = \left\{ \begin{bmatrix} A & \mathbf{0} \\ C & D \end{bmatrix} \in \mathrm{SL}_{m+n}(\mathbb{R}) \colon A \in M_{m\times m}(\mathbb{R}),\, C \in M_{n\times m}(\mathbb{R}),\, D \in M_{n\times n}(\mathbb{R}) \right\}. \tag{2.6.1}$$

Hence the partial flag variety G/P coincide with $\mathrm{Gr}(m, m+n)$, the Grassmannian of m-dimensional subspaces of $\mathbb{R}^{m+n}$. It is an irreducible projective variety of dimension mn.

2.6.1 Schubert cells and Schubert varieties

Let B be the Borel subgroup of lower triangular matrices in G, and T the group of diagonal matrices in G. The Weyl group $W = N_G(T)/Z_G(T)$ is isomorphic to S_{m+n}, the permutation group on $m+n$ elements. The Weyl group W_P of P is isomorphic to $S_m \times S_n$, and the set W^P of minimal length coset representatives of W/W_P consists of the permutations $w = (w_1, \cdots, w_{m+n})$ such that $w_1 < \cdots < w_m$ and $w_{m+1} < \cdots < w_{m+n}$. We identify w in W^P with the subset $I_w = \{w_1, \cdots, w_m\}$ of $\{1, 2, \cdots, m+n\}$. The cosets wP are exactly the T-fixed points of G/P. The Schubert cell C_w is by definition BwP, and the Schubert variety X_w is defined to be $\overline{BwP}$, the closure of C_w in G/P. For $w, w' \in W^P$, $w' \in X_w$ if and only if $w' \leq w$ in the Bruhat order. We note that the Bruhat order here is the order on the tuples $(w_1, \cdots, w_m)$ given by

$$(w_i) \leq (v_i) \iff w_i \leq v_i, \forall 1 \leq i \leq m.$$

The dimension of X_w is given by $l(w)$, which equals $\sum_{k=1}^{m}(w_k - k)$.

The definitions above coincide with the classical definitions. For $1 \leq k \leq m+n$, let F_k be the standard k-dimensional subspace of $\mathbb{R}^{m+n}$ spanned by $\{e_1, \cdots, e_k\}$. We have the complete flag of subspaces

$$0 = F_0 \subset F_1 \subset F_2 \cdots \subset F_{m+n-1} \subset F_{m+n} = \mathbb{R}^{m+n}. \tag{2.6.2}$$

For an m-dimensional subspace $V \in \mathrm{Gr}(m, m+n)$ of $\mathbb{R}^{m+n}$, consider the intersections of the subspace with the flag:

$$0 \subset (F_1 \cap V) \subset (F_2 \cap V) \cdots \subset (F_{m+n-1} \cap V) \subset W. \tag{2.6.3}$$

For $w \in W^P$, we have a tuple $(w_1, \cdots, w_m)$, and the Schubert cell C_w has the following description:

$$C_w = \{V \in \mathrm{Gr}(m, m+n) \colon \dim(V \cap F_{w_k}) = k;\ \dim(V \cap F_l) < k,\ \forall l < w_k\}\,. \tag{2.6.4}$$

In other words, the tuple $(w_1, \cdots, w_m)$ gives the indices where the dimension jumps.

Similarly, the Schubert variety X_w has the following description:

$$X_w = \{V \in \mathrm{Gr}(m, m+n) \colon \dim(V \cap F_{w_k}) \geq k,\ 1 \leq k \leq m\}\,. \tag{2.6.5}$$

Now it is easy to see that

$$X_w = \bigsqcup_{w' \leq w} C_{w'}. \tag{2.6.6}$$

Hence the Schubert cells give a stratification of the Grassmannian variety.

Example 2.6.1. (1) For $m = 1$, the Grassmannian $\mathrm{Gr}(1, n)$ is just the projective space $\mathbb{RP}^n$, and the Schubert varieties form a flag of linear subspaces $X_0 \subset X_1 \subset \cdots \subset X_n$, where $X_j \cong \mathbb{RP}^j$.

(2) For $m = n = 2$ one gets the following poset of Schubert varieties in $\mathrm{Gr}(2, 4)$:

$$\begin{tikzcd}
 & X_{34} \arrow[d, dash] & \\
 & X_{24} \arrow[dl, dash] \arrow[dr, dash] & \\
X_{14} \arrow[dr, dash] & & X_{23} \arrow[dl, dash] \\
 & X_{13} \arrow[d, dash] & \\
 & X_{12} &
\end{tikzcd} \tag{2.6.7}$$

where X_{12} is one single point, and X_{34} is $\mathrm{Gr}(2, 4)$.

2.6.2 Pencils

The main goal of this section is to show that maximal (weakly) constraining pencils coincide with maximal (weakly) unstable Schubert varieties in the Grassmannian case, and hence the latter is a natural generalization to all partial flag varieties.

Given a real vector space $W \subsetneq \mathbb{R}^{m+n}$, and an integer $r \leq m$, we recall from Definition 2.1.7 that the pencil $\mathfrak{P}_{W,r}$ is the set

$$\{V \in \mathrm{Gr}(m, m+n) \colon \dim(V \cap W) \geq r\}.$$

Denote $d = \dim W$. Let $w \in W^P$ be the element such that $(w_1, \cdots, w_m)$ is the tuple

$$(d - r + 1, \cdots, d, r + 1, \cdots, m).$$

One can verify that the pencil $\mathfrak{P}_{W,r}$ is the Schubert variety gX_w, where g is an element in $\mathrm{SL}_{m+n}(\mathbb{R})$ such that $W = g \cdot F_d$. The pencil is called constraining (resp. weakly constraining) if the inequality (2.1.11) (resp. (2.1.12)) holds.

On the other hand, we recall that the Schubert variety X_w is unstable (resp. weakly unstable) if there exists a non-trivial multiplicative one-parameter subgroup δ in $\Gamma^+(T)$ such that $(\delta, a^w) > 0$ (resp. ≥ 0). Let Δ be the element in the Lie algebra $\mathfrak{t}$ of T such that $\delta(t) = \exp(\log t \cdot \Delta)$. Then Δ could be written as $\mathrm{diag}(t_1, t_2, \cdots, t_{m+n})$, where $t_1 \geq t_2 \geq \cdots \geq t_{m+n}$ and $\sum t_i = 0$. Hence in the case of Grassmannian we have the following criterion of stability.

Lemma 2.6.2. *Let w be an element in W^P, then the corresponding Schubert variety X_w is unstable (resp. weakly unstable) if and only if the following system is soluble:*

$$t_1 \geq \cdots \geq t_k > 0 \geq t_{k+1} \geq \cdots \geq t_{m+n} \tag{2.6.8}$$

$$\sum_{i=1}^{m+n} t_i = 0 \tag{2.6.9}$$

$$\sum_{j=1}^{m} t_{w_j} > 0 \; (\textit{resp.} \; \sum_{j=1}^{m} t_{w_j} \geq 0) \tag{2.6.10}$$

Example 2.6.3 ($m = n = 2$). We continue with Example 2.6.1(2). If $w = (14)$, then we can take $t_1 = 3, t_2 = t_3 = t_4 = -1$, which gives $t_1 + t_4 > 0$. Hence by Lemma 2.6.2 we have X_{14} is unstable. Similarly we can show that X_{23} is unstable by taking $t_1 = t_2 = t_3 = 1, t_4 = -3$.

When $w = (24)$, $t_2 + t_4 \geq 0$ is soluble as we can take $t_1 = t_2 = 1, t_3 = t_4 = -1$. However, $t_2 + t_4 > 0$ is insoluble. Indeed, suppose $t_2 + t_4 > 0$, then $t_1 + t_3 \geq t_2 + t_4 > 0$, and it follows that $t_1 + t_2 + t_3 + t_4 > 0$, which contradicts (2.6.9). Therefore we conclude that X_{24} is weakly unstable but not unstable.

Now we are ready for the main results of this section.

Proposition 2.6.4. *Every constraining (resp. weakly constraining) pencil is an unstable (resp. weakly unstable) Schubert variety of* $\mathrm{Gr}(m, m+n)$.

Proof. Let $\mathfrak{P}_{W,r}$ be a constraining pencil, and thus by definition we have

$$\frac{d}{r} < \frac{m+n}{m}, \tag{2.6.11}$$

where $d = \dim W$. Then $\mathfrak{P}_{W,r} = gX_w$, where $g \in G$ and $w \in W^P$ such that

$$(w_1, \cdots, w_m) = (d - r + 1, \cdots, d, n + r + 1, \cdots, m + n). \tag{2.6.12}$$

Now set $t_1 = \cdots = t_d = m+n-d$ and $t_{d+1} = \cdots = t_{m+n} = -d$. It is clear that (2.6.8) and (2.6.9) are satisfied. Moreover,

$$\begin{aligned} \sum_{j=1}^{m} t_{w_j} &= r(m+n-d) - (m-r)d \\ &= r(m+n) - md \\ &= mr\left(\frac{m+n}{m} - \frac{d}{r}\right) \\ &> 0. \end{aligned} \tag{2.6.13}$$

Hence (2.6.10) also holds. Therefore, by Lemma 2.6.2 we conclude that $\mathfrak{P}_{W,r}$ is an unstable Schubert variety. The same proof also works for weakly constraining pencils. □

Proposition 2.6.5. *Every unstable (resp. weakly unstable) Schubert variety of* $\mathrm{Gr}(m, m+n)$ *is contained in a constraining (resp. weakly constraining) pencil.*

Proof. Let X_w be an unstable Schubert variety and consider the set $I_w = \{w_1, \cdots, w_m\}$. Let J_w be the subset of I_w consisting of the elements with jump, that is, w_k is contained in J_w if and only if $w_{k+1} - w_k > 1$. Here we set $w_{m+1} = 0$. Notice that for any $w_k \in J_w$, if we set $W = F_{w_k}$ and $r = k$, then X_w is contained in the pencil $\mathfrak{P}_{W,r}$. Now it suffices to show that there exists $w_k \in J_w$ such that

$$\frac{w_k}{k} < \frac{m+n}{m}. \tag{2.6.14}$$

Actually, the function $k \mapsto w_k/k$ achieves its minimum at some k such that $w_k \in J_w$. Hence it suffices to prove the following claim.

Claim. There exists $1 \leq k \leq m$ such that (2.6.14) holds.

We prove the claim by contradiction. Suppose that for any $1 \leq k \leq m$ we have

$$\frac{w_k}{k} \geq \frac{m+n}{m}. \tag{2.6.15}$$

For $1 \le i \le m+n$, consider the auxiliary function

$$g(i) = \begin{cases} -m & i \notin I_w; \\ n & i \in I_w. \end{cases} \tag{2.6.16}$$

For any $1 \le i < m+n$, let w_k be the largest element in I_w such that $w_k \le i$ (and set $w_k = 0$ if $i < w_1$). As a consequence of (2.6.15), we have

$$\begin{aligned} \sum_{j=1}^{i} g(i) &\le \sum_{j=1}^{w_k} g(i) \\ &= -m(w_k - k) + nk \\ &= (m+n)k - mw_k \\ &\le 0. \quad \text{By (2.6.15)} \end{aligned} \tag{2.6.17}$$

It is also clear that

$$\sum_{j=1}^{m+n} g(i) = 0. \tag{2.6.18}$$

Since X_w is unstable, we may find $t_1, \cdots, t_{m+n}$ satisfying (2.6.8)(2.6.9)(2.6.10). Denote

$$A = \sum_{i \in I_w} t_i; \tag{2.6.19}$$

$$B = \sum_{i \notin I_w} t_i. \tag{2.6.20}$$

Then $A > 0$ and $A + B = 0$ by (2.6.9)(2.6.10). Hence $B < 0$, and $nA - mB > 0$.

On the other hand, summation by parts leads to

$$\begin{aligned} nA - mB &= n \sum_{i \in I_w} t_i - m \sum_{i \notin I_w} t_i \\ &= \sum_{i=1}^{m+n} g(i) t_i \\ &= \sum_{i=1}^{m+n-1} \left[(t_i - t_{i+1}) \sum_{j=1}^{i} g(j) \right] + t_{m+n} \sum_{j=1}^{m+n} g(j) \\ &= \sum_{i=1}^{m+n-1} \left[(t_i - t_{i+1}) \sum_{j=1}^{i} g(j) \right] \\ &\le 0. \end{aligned} \tag{2.6.21}$$

This is a contradiction.

Therefore we have proved the claim, and thus $\mathfrak{P}_{W,r}$ is a constraining pencil containing the Schubert variety X_w. The same proof works for weakly unstable Schubert varieties. □

Combining Proposition 2.6.4 and Proposition 2.6.5, we conclude the following.

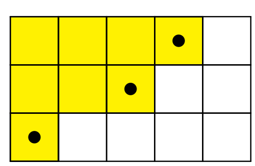
Theorem 2.6.6. *Let E be any subset of $\mathrm{Gr}(m, m+n) \cong G/P$. Then E is contained in an unstable (resp. weakly unstable) Schubert variety with respect to $a(t)$ if and only if E is contained in a constraining (resp. weakly constraining) pencil.*

2.6.3 Young diagrams

In this section, we will give a combinatorial description of pencils and (weakly) constraining pencils, using Young diagrams. This will enable us to quickly see whether a Schubert variety is a pencil, and whether a pencil is (weakly) constraining. The readers are referred to Fulton's book [12] for more details.

A *partition* is a sequence of integers $\lambda = (\lambda_1, \cdots, \lambda_m)$ such that $\lambda_1 \geq \cdots \geq \lambda_m \geq 0$. Let $\Pi_{m,n}$ denote the set of partitions such that $\lambda_1 \leq n$. A *Young diagram* is a set of boxes arranged in a left justified array, such that the row lengths weakly decrease from top to bottom. To any partition λ we associate the Young diagram D_λ whose i-th row contains λ_i boxes. An *outside corner* of the Young diagram D_λ is a box in D_λ such that removing the box we still get a Young diagram.

Example 2.6.7. Let $m = 3$, $n = 5$, and $\lambda = (4, 3, 1) \in \Pi_{m,n}$. The Young diagram D_λ fits inside an $m \times n$ rectangle.



There are three outside corners, which are marked with a dot in the diagram.

Given $\lambda \in \Pi_{m,n}$, the associated Schubert variety $X_\lambda \subset \mathrm{Gr}(m, m+n)$ is defined by the conditions

$$\dim(V \cap F_{n+i-\lambda_i}) \geq i, \quad 1 \leq i \leq m. \tag{2.6.22}$$

Actually we only need outside corners to define X_λ; the pairs (i, λ_i) which are not outside corners are redundant. (See [12, Exercise 9.4.18].) Therefore, we have the following lemma.

Lemma 2.6.8. *Given $\lambda \in \Pi_{m,n}$, the Schubert variety X_λ is a pencil if and only if the Young diagram D_λ has only one outside corner.*

The Schubert variety given by Example 2.6.7 is not a pencil, as the Young diagram has three outside corners. However, every Schubert variety can be written as an intersection of pencils.

One can also recognize constraining and weakly constraining pencils with the help of Young diagrams.

For an $m \times n$ rectangle, we draw the diagonal connecting the northeast and the southwest of the rectangle. A *node* is a vertex of a box. We call a node *unstable* if it is lying below the diagonal, and *weakly unstable* if it is lying on or below the diagonal. See Figure 2.1 for an example.

Now we can reformulate the definition of constraining and weakly constraining pencils.

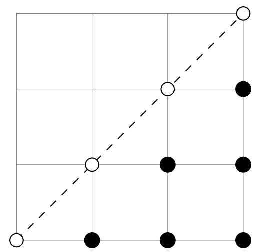
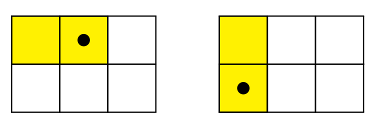


Figure 2.1: Unstable and weakly unstable nodes in a 3×3 rectangle. The black nodes are unstable, while the white nodes are weakly unstable but not unstable.

Lemma 2.6.9. *A pencil X_λ is constraining (resp. weakly constraining) if and only if the bottom-right vertex of the outside corner of D_λ is an unstable (resp. weakly unstable) node.*

Example 2.6.10. Let $m = 2$ and $n = 3$. By Lemma 2.6.9 there are 5 constraining pencils: $X_{12}, X_{15}, X_{23}, X_{25}$ and X_{34}. Among those X_{25} and X_{34} are the maximal ones, and they give the obstruction to non-divergence.



As noted in Remark 2.1.8, the weakly constraining pencils coincide with the constraining pencils in the case that m and n are coprime. This also follows from the simple observation that there are no nodes lying on the diagonal of D_λ.

Chapter 3: Expanding translates of shrinking submanifolds and Diophantine approximation

3.1 Background and main results

After Davenport and Schmidt [7], given $0 < \lambda \leq 1$, we say that $\boldsymbol{z} = (z_1, \ldots, z_n) \in \mathbb{R}^n$ is $\mathrm{DT}(\lambda)$ if for each sufficiently large $N \in \mathbb{N}$, there exist integers $q_1, \ldots, q_n$ and p such that

$$|(q_1 z_1 + \ldots + q_n z_n) - p| \leq \lambda/N^n \text{ and } 0 < \max_{1 \leq i \leq n} |q_i| \leq \lambda N. \tag{3.1.1}$$

In a dual manner, we say that $\boldsymbol{z} \in \mathbb{R}^n$ is $\mathrm{DT}'(\lambda)$ if for each sufficiently large $N \in \mathbb{N}$ there exist integers q and $p_1, \ldots, p_n$ such that

$$\max_{1 \leq i \leq n} |q z_i - p_i| \leq \lambda/N \text{ and } 0 < |q| \leq \lambda N^n. \tag{3.1.2}$$

Dirichlet's simultaneous approximation theorem states that every $\boldsymbol{z} \in \mathbb{R}^n$ is $\mathrm{DT}(1)$ and $\mathrm{DT}'(1)$. Davenport and Schmidt [7] proved that for any $\lambda < 1$, almost every $\boldsymbol{z} \in \mathbb{R}^n$ is not $\mathrm{DT}(\lambda)$ and not $\mathrm{DT}'(\lambda)$. In other words, Dirichlet's theorem cannot be improved for almost all $\boldsymbol{z} \in \mathbb{R}^n$. In [8] they showed that for almost every $z \in \mathbb{R}$, the vector $\boldsymbol{z} = (z, z^2) \in \mathbb{R}^2$ is not $\mathrm{DT}(1/4)$, opening an investigation of whether almost all points on a sufficiently curved submanifold in $\mathbb{R}^n$ are not $\mathrm{DT}(\lambda)$ for any

$\lambda < 1$. The question was taken up in [2, 4], where several non-improvability results were obtained for small $\lambda > 0$. Later Kleinbock and Weiss [21] reformulated this question in terms of dynamics on homogeneous spaces using an observation due to Dani [5] relating simultaneous Diophantine approximation to asymptotic properties of individual orbits of diagonal subgroups. Using the non-divergence techniques from [19], they [21] proved that almost all points on the image of a l-nondegenerate differentiable map from an open set in $\mathbb{R}^d$ to $\mathbb{R}^n$ are not $\mathrm{DT}(\lambda)$ for some very small $\lambda > 0$, where l-nondegenerate means that at almost every point all partial derivatives of the map up to order l span $\mathbb{R}^n$.

In [35] by proving an equidistribution result for expanding translates of analytic curve segments on the space of unimodular lattices in $\mathbb{R}^{n+1}$, it was shown that if an analytic curve in $\mathbb{R}^n$ is not contained in a proper affine subspace then almost all points on this curve are not $\mathrm{DT}(\lambda)$ and not $\mathrm{DT}'(\lambda)$ for every $\lambda \in (0, 1)$. The analyticity is a technical assumption because of a fundamental limitation of the method of proof; namely the (C, α)-good property [19] of differentiable maps do not survive under composition by non-linear polynomial maps. To overcome this limitation we would require a quantitative local avoidance result, which was conjectured in [37, Section 5]. In this article, we resolve this conjecture and prove a stronger equidistribution result for expanding translates of sufficiently slowly shrinking curves (cf. [34] for $G = \mathrm{SO}(n, 1)$). The new equidistribution result leads to non-improvability of Dirichlet's approximation theorem for nondegenerate manifolds as defined by Pyartli [26].

Definition 3.1.1 (cf. [26, §2])**.** We say that a curve $\zeta : (c, d) \to \mathbb{R}^k$ is *nondegenerate at* $s \in (c, d)$, if $\zeta^{(k-1)}(s)$ exists and the vectors $\zeta^{(0)}(s) := \zeta(s), \zeta^{(1)}(s), \ldots, \zeta^{(k-1)}(s)$ span $\mathbb{R}^k$.

Let Ω be an open subset of $\mathbb{R}^d$ and $\phi : \Omega \to \mathbb{R}^{n+1}$ be a C^n-map. We say that ϕ is *nondegenerate* at $s \in \Omega$ if the following conditions are satisfied:

1. The derivative $D\phi(s) : \mathbb{R}^d \to \mathbb{R}^{n+1}$ is injective. Let $\mathcal{T} := D\phi(s)(\mathbb{R}^d)$.

2. There exists a subspace $\mathcal{L}$ of $\mathbb{R}^{n+1}$ containing $\phi(s)$ such that $\mathcal{T} \oplus \mathcal{L} = \mathbb{R}^{n+1}$, and there exists $0 \neq v \in \mathcal{T}$ such that the map $\rho_v : (-r_0, r_0) \to \mathbb{R}v + \mathcal{L}$ defined by $\rho_v(r) = \phi(\Omega_1) \cap (rv + \mathcal{L})$ for all $|r| < r_0$, for a neighborhood Ω_1 of s and some $r_0 > 0$, is nondegenerate at 0.

We say that ϕ is *nondegenerate* if it is nondegenerate at all $s \in \Omega$.

Theorem 3.1.2. *Let $\psi : \Omega \to \mathbb{R}^n$ be a $(n+1)$-times differentiable map, where Ω is open in $\mathbb{R}^d$. Suppose that $\tilde{\psi} : \Omega \to \mathbb{R}^{n+1}$ given by, $\tilde{\psi}(s) = (1, \psi(s))$ for all $s \in \Omega$, is nondegenerate. Then given an infinite set $\mathcal{N} \subset \mathbb{N}$, for almost every $s \in \Omega$ and any $\lambda \in (0,1)$, there are no integral solutions to (3.1.1) and (3.1.2) for $\boldsymbol{z} = \psi(s)$ and infinitely many $N \in \mathcal{N}$.*

In particular, $\psi(s)$ is not $DT(\lambda)$ or $DT'(\lambda)$ for almost any $s \in \Omega$ and any $\lambda \in (0,1)$.

Remark 3.1.3. (1) The manifold $(\Omega, \psi, \mathbb{R}^n)$ is *nondegenerate* at $\psi(s)$ as per Pyartli [26] if and only if the corresponding map $\tilde{\psi}$ is nondegenerate at s.

(2) It will be interesting to know whether the following holds: If a manifold is l-nondegenerate in the sense of Kleinbock and Margulis [19] then almost every point of the manifold is nondegenerate in the sense of Pyartli.

(3) If ψ is analytic and $\psi(\Omega)$ is not contained in a proper affine subspace of $\mathbb{R}^n$ then $\tilde{\psi}$ is nondegenerate on $\Omega \setminus Z$, where Z is a proper analytic subvariety of Ω with strictly lower dimension and with zero Lebesgue measure.

As shown by Kleinbock and Weiss [21] and Shah [35, Section 2], due to the Dani's correspondence the Theorem 3.1.2 can be derived as a consequence of Theorem 3.1.4, which is the main goal of this article.

Notation

Let $1 \leq d \leq n$, and $G = \mathrm{SL}(n+1, \mathbb{R})$. For $t > 0$, let $a(t) := \mathrm{diag}(t^n, t^{-1}, \ldots, t^{-1}) \in G$. Let $\Omega \subset \mathbb{R}^d$ be open and $\Phi : \Omega \to G$ be a C^1-map. Then for any $s \in \Omega$,

$$a(t)\Phi(s) = t^n I_0 \Phi(s) + t^{-1} I_n \Phi(s), \tag{3.1.3}$$

where $I_0 = \mathrm{diag}(1, 0, \ldots, 0)$, $I_n = \mathrm{diag}(0, 1, \ldots, 1) \in \mathrm{M}(n+1, \mathbb{R})$. For any $g \in \mathrm{M}(n+1, \mathbb{R})$, we identify $I_0 g$ with the top row of g which is realized as an element of $\mathbb{R}^{n+1}$. We define $\phi : \Omega \to \mathbb{R}^{n+1}$ by $\phi(s) = I_0 \Phi(s) \in \mathbb{R}^{n+1}$ for all $s \in \Omega$.

Theorem 3.1.4. *Suppose that ϕ is a nondegenerate $(n+1)$-times differentiable map. Let L be a Lie group containing G, Λ a lattice in L, and let $x \in L/\Lambda$. Then there exists $E_x \subset \Omega$ of zero Lebesgue measure such that for every $s \in \Omega \setminus E_x$, and any bounded open convex neighborhood C of 0 in $\mathbb{R}^d$,*

$$\lim_{t \to \infty} \frac{1}{\mathrm{vol}(C)} \int_C f(a(t)\Phi(s + t^{-1}\eta)x)\, d\eta = \int_{\overline{Gx}} f\, d\mu_x, \tag{3.1.4}$$

where $\mathrm{vol}(\cdot)$ denotes the Lebesgue measure, and μ_x is the G-invariant probability measure on the homogeneous space $\overline{Gx}$.

In particular, for any probability measure ν on Ω which is absolutely continuous with respect to the Lebesgue measure,

$$\lim_{t \to \infty} \int_\Omega f(a(t)\Phi(\eta)x)\, d\nu(\eta) = \int_{\overline{Gx}} f\, d\mu_x. \tag{3.1.5}$$

To derive Theorem 3.1.2, we need (3.1.5) for $\Phi(s) = \left(\begin{smallmatrix} 1 & \psi(s) \\ 0 & I_n \end{smallmatrix}\right)$ and a suitably chosen embedding of G into $L = G \times G$ [35, §1.0.1]. But to justify (3.1.5) for differentiable maps, we need to prove the equidistribution of local expansion given by (3.1.4), which is new even for the horospherical case of $L = G$, $d = n$ and $\psi(s) = s$; cf. [13, Lemma 16] and [14, Theorem 20].

Our proof of (3.1.4) is quite different from the arguments of [35] for proving (3.1.5) for analytic maps. A new identity observed in this chapter allows us to describe the limiting distribution of expansion of shrinking pieces in the curve ($d = 1$) case using equidistribution of long polynomial trajectories on homogeneous spaces [33].

This chapter is organized as follows. In §3.2 we obtain the key identity as mentioned above. In §3.3, we combine the result on limiting distributions of polynomial trajectories with the key identity to obtain the algebraic description of the limiting distribution of the stretching translates of the shrinking segments of the curve ($d = 1$) around any given point $\Phi(s)x$ in $\Phi(\Omega)x$ (Theorem 3.3.4). In §3.4 we will derive the analogous result for shrinking balls around any given point in the submanifold (Theorem 3.4.1). For this purpose we will fiber the shrinking balls into shrinking nondegenerate curves segments using a twisting trick due to Pyartli [26]. A point $s \in \Omega$ is called exceptional if the limiting distribution of expanding translates of the shrinking balls in $\Phi(\Omega)x$ about the point $\Phi(s)x$ is not G-invariant. In §3.5, we will obtain a geometric description of the set of exceptional points (Proposition 3.5.3) and prove that it is Lebesgue null (Proposition 3.5.1). We will show that in many standard examples the exceptional points are dense in Ω (Proposition 3.5.4).

3.2 Basic identity

The main new ingredient in the proof of Theorem 3.1.4 is the following:

Lemma 3.2.1 (Basic Identity)**.** *Let $d = 1$, $\Omega \subset \mathbb{R}$ open, $\Phi : \Omega \to G$ a C^1-map, and $s \in \Omega$ be such that the map $\phi = I_0\Phi : \Omega \to \mathbb{R}^{n+1}$ is $(n+1)$-times differentiable and nondegenerate at s. Then there exists a is a nilpotent matrix $B_s \in M(n+1, \mathbb{R})$ of rank n such that for any $t \neq 0$ with $s + t^{-1} \in \Omega$, we have*

$$a(|t|)\Phi(s + t^{-1}) = (I + o(t^{-1})t)\xi_s(\sigma)(I - tB_s)^{-1}, \tag{3.2.1}$$

where $\sigma = t/|t| = \pm 1$, $\xi_s(\pm 1) \in G$, and $o(t^{-1}) \in \mathrm{M}_{n+1}(\mathbb{R})$ is such that $o(t^{-1})t \to 0$ as $t \to \infty$.

We note that $B_s^n \neq 0$ and $B_s^{n+1} = 0$, so

$$P_s(t) := (I - tB_s)^{-1} = I + \sum_{k=1}^{n} t^k B_s^k \in \mathrm{SL}(n+1, \mathbb{R}) = G. \tag{3.2.2}$$

Proof. We want to find a nilpotent matrix $B_s \in \mathrm{M}_{n+1}(\mathbb{R})$ such that

$$\lim_{t\to\infty} a(|t|)\Phi(s + t^{-1})(I - tB_s) \in G.$$

Let $t \neq 0$ such that $s + t^{-1} \in \Omega$. In view of (3.1.3), by Taylor's expansion,

$$I_0\Phi(s + t^{-1}) = \phi(s + t^{-1}) = \sum_{k=0}^{n+1} \frac{\phi^{(k)}(s)}{k!} t^{-k} + o(t^{-(n+1)}).$$

For any $B_s \in \mathrm{M}(n+1, \mathbb{R})$ and $\sigma = t/|t| = \pm 1$, we have

$$\begin{aligned}
&a(|t|)I_0\Phi(s + t^{-1})(I - tB_s) = |t|^n \phi(s + t^{-1})(I - tB_s) \\
&= \sigma^n\Big(\big(\sum_{k=0}^{n+1} \frac{\phi^{(k)}(s)}{k!} t^{n-k}\big) + o(t^{-1}) \Big)(I - tB_s) \\
&= \sigma^n\Big(-\phi(s)B_s t^{n+1} + \sum_{k=1}^{n} \big(\frac{\phi^{(k-1)}(s)}{(k-1)!} - \frac{\phi^{(k)}(s)}{k!} B_s \big) t^{n-k+1} \Big) \\
&\quad + \sigma^n \xi_{s,1} + o(t^{-1})t,
\end{aligned} \tag{3.2.3}$$

where

$$\xi_{s,1} = \frac{\phi^{(n)}(s)}{n!} - \frac{\phi^{(n+1)}(s)}{(n+1)!} B_s. \tag{3.2.4}$$

We want to choose B_s such that all the coefficients of positive powers of t vanish in (3.2.3); in other words, we want

$$\phi(s)B_s = 0 \text{ and } \frac{\phi^{(k)}(s)}{k!}B_s = \frac{\phi^{(k-1)}(s)}{(k-1)!} \text{ for } 1 \leq k \leq n. \tag{3.2.5}$$

By our assumption, $\{\phi^{(k)}(s)/k! : 0 \leq k \leq n\}$ is a basis of $\mathbb{R}^{n+1}$. Therefore there exists a unique matrix B_s such that (3.2.5) holds. Moreover, B_s is nilpotent matrix of rank n. In particular, $\det(I - tB_s) = 1$ for all $t \in \mathbb{R}$.

Now by (3.2.3) and (3.2.5), we have the following key identity:

$$a(|t|)I_0\Phi(s + t^{-1})(I - tB_s) = \sigma^n\xi_{s,1} + o(t^{-1})t. \tag{3.2.6}$$

Also, since Φ is differentiable at s,

$$\begin{aligned} a(|t|)I_n\Phi(s + t^{-1})(I - tB_s) &= |t|^{-1}(I_n\Phi(s) + O(t^{-1}))(I - tB_s) \\ &= \sigma\xi_{s,2} + O(t^{-1}), \end{aligned} \tag{3.2.7}$$

where

$$\xi_{s,2} = -I_n\Phi(s)B_s. \tag{3.2.8}$$

In view of (3.1.3), combining (3.2.6) and (3.2.7):

$$a(|t|)\Phi(s + t^{-1})(I - tB_s) = \xi_s(\sigma) + o(t^{-1})t, \tag{3.2.9}$$

where in view of (3.2.4) and (3.2.8), $\sigma = t/|t| = \pm 1$ and

$$\xi_s(\sigma) = \sigma^n\xi_{s,1} + \sigma\xi_{s,2}. \tag{3.2.10}$$

Now (3.2.1) follows from (3.2.9). Since the left hand side of (3.2.9) belongs to G for all t, by taking $t \to \pm\infty$, we get $\xi_s(\pm 1) \in G$. $\square$

Though it is straightforward to verify the basic identity, the path that lead us to conceive the identity involved an intricate study of interactions of linear dynamics of intertwining copies of $\mathrm{SL}(2, \mathbb{R})$ in G using Weyl group actions using [39, Lemma 4.1].

3.3 Limiting distribution of polynomial trajectories and stretching translates of shrinking curves

Our proof of Theorem 3.1.4 for $d = 1$ is based on Lemma 3.2.1, and the following result on limiting distribution of polynomial trajectories on homogeneous spaces which was proved using Ratner's description [27] of ergodic invariant measures for unipotent flows.

Notation

Let L be a Lie group containing G and Λ be a lattice in G. Let $x \in L/\Lambda$. Let $\mathcal{H}_x$ denote the collection of all connected Lie subgroups H of L such that Hx is closed and admits an H-invariant probability measure, say μ_H, which is ergodic with respect to an Ad_L-unipotent one-parameter subgroup of L. Then $\mathcal{H}_x$ is countable [27, 6].

If $H_1, H_2 \in \mathcal{H}_x$ then there exists $H \in \mathcal{H}_x$ such that $H \subset H_1 \cap H_2$ and H contains all Ad_L-unipotent one-parameter subgroups of $H_1 \cap H_2$ [32, §2].

Theorem 3.3.1 (Shah [33]). *Let $Q : \mathbb{R} \to G = \mathrm{SL}(n+1, \mathbb{R})$ be a map whose each coordinate is a polynomial and the identity element $I \in Q(\mathbb{R})$. Let H be the smallest Lie subgroup of L containing $Q(\mathbb{R})$ such that Hx is closed. Then $H \in \mathcal{H}_x$, and for any $f \in C_c(L/\Lambda)$,*

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T f(Q(t)x)\, dt = \int_{Hx} f\, d\mu_H.$$

The following is its straightforward reformulation via change of variable.

Corollary 3.3.2. *Let the notation be as in Theorem 3.3.1. Then for any $f \in C_c(L/\Lambda)$ and $c < d$,*

$$\lim_{T\to\infty} \frac{1}{d-c} \int_c^d f(Q(Ts)x)\, ds = \int_{Hx} f\, d\mu_H.$$

From this result we can deduce its following variation.

Corollary 3.3.3. *Let the notation be as in Theorem 3.3.1. Let $\rho : \mathbb{R} \to G$ be a measurable map and ν be a finite measure on $\mathbb{R}$ which is absolutely continuous with respect to the Lebesgue measure. Then for any $f \in C_c(L/\Lambda)$,*

$$\int_{\mathbb{R}} f(\rho(\eta)Q(T\eta)x)\, d\nu(\eta) \overset{T\to\infty}{\longrightarrow} \int_{\mathbb{R}} \Big[\int_{Hx} f(\rho(\eta)y)\, \mu_H(y)\Big]\, d\nu(\eta). \tag{3.3.1}$$

Proof. We can assume that $|f| \leq 1$. And since ν is finite, due to Lusin's theorem, we can replace ρ and $d\nu(\eta)/d\eta$ by continuous functions with compact support. Let $s \in \mathbb{R}$. Given $\epsilon > 0$, there exists $\delta_s > 0$ such that for all $\eta \in (s - \delta_s/2, s + \delta_s/2)$ and $y \in L/\Lambda$,

$$|(d\nu/d\eta)(\eta) - (d\nu/d\eta)(s)| \leq \epsilon \quad \text{and} \quad |f(\rho(\eta)y) - f(\rho(s)y)| \leq \epsilon.$$

Using these approximations, by Corollary 3.3.2 for any $0 < \delta < \delta_s$ there exists $T_{s,\delta} \geq 1$ such that

$$\Big|\int_{s-\delta/2}^{s+\delta/2} f(\rho(\eta)Q(T\eta)x)\, d\nu(\eta) - \delta \cdot (d\nu/d\eta)(s) \cdot \int_{Hx} f(\rho(s)y)\, d\mu_H(y)\Big| \leq 2\epsilon\delta,$$

for all $T \geq T_{s,\delta}$. We use convergence in measure to complete the proof. □

Theorem 3.3.4. *Let $d = 1$ and the notation be as in Theorem 3.1.4 and Notation 3.3. Let $s \in \Omega$. Then there exists $H_s \in \mathcal{H}_x$ such that the following holds: Let ν be a finite measure on $\mathbb{R}$ which is absolutely continuous with respect to the Lebesgue measure. Then for any $f \in C_c(L/\Lambda)$,*

$$\begin{aligned} &\lim_{t\to\infty} \int_{\mathbb{R}} f(a(t)\Phi(s + \eta t^{-1})x)\, d\nu(\eta) \\ &= \int_{\mathbb{R}} \Big[\int_{H_s x} f(a(|\eta|)\xi_s(\operatorname{sign}(\eta))y)\, d\mu_{H_s}(y)\Big] d\nu(\eta), \end{aligned} \tag{3.3.2}$$

where $\operatorname{sign}(\eta) = \eta/|\eta| = \pm 1$ *and* $\xi_s(\pm 1) \in G$ *are given by* (3.2.10).

Moreover if $H_s \supset G$*, then* $\overline{Gx} = H_s x$*,* $\mu_x = \mu_{H_s}$*, and*

$$\lim_{t\to\infty} \int_{\mathbb{R}} f(a(t)\Phi(s+\eta t^{-1})x)\, d\nu(\eta) = |\nu| \cdot \int_{\overline{Gx}} f\, d\mu_x.$$

Proof. Let $\eta \neq 0$. For $t \gg 1$, writing $h = \eta^{-1}t$, by (3.2.1) and (3.2.2),

$$\begin{aligned} a(t)\Phi(s+\eta t^{-1})x &= a(|\eta|)a(|h|)\Phi(s+h^{-1})x \\ &= a(|\eta|)(I + o(h^{-1})h)\xi_s(\operatorname{sign}(\eta))P_s(h)x \\ &= (I + |\eta|^{-(n+1)}o(h^{-1})h))a(|\eta|)\xi_s(\operatorname{sign}(\eta))P_s(h)x \\ &= (I + |\eta|^{-(n+1)}o(t^{-1})t)a(|\eta|)\xi_s(\operatorname{sign}(\eta))P_s(t\eta^{-1})x. \end{aligned}$$

Since f is bounded, we can ignore the integration over a small neighborhood of 0, outside which $|\eta|^{-(n+1)}o(t^{-1})t$ is close to 0 uniformly for all large t. So by uniform continuity of f we can ignore the factor $(I + |\eta|^{-(n+1)}o(t^{-1})t)$, and hence

$$\begin{aligned} &\lim_{t\to\infty} \int_{\mathbb{R}} f(a(t)\Phi(s+\eta t^{-1})x)\, d\nu(\eta) \\ &= \lim_{t\to\infty} \int_{\mathbb{R}} f(a(|\eta|)\xi_s(\operatorname{sign}(\eta))P_s(t\eta^{-1})x)\, d\nu(\eta). \end{aligned} \tag{3.3.3}$$

By (3.2.2), $P_s(0) = I$. Let $H_s \in \mathcal{H}_x$ be the smallest subgroup containing $P_s(\mathbb{R})$. Applying Corollary 3.3.3 to the image of ν on $\mathbb{R}$ under the map $\eta \mapsto \eta^{-1}$, from (3.3.3) we obtain (3.3.2).

∎

3.4 Stretching translates of shrinking submanifolds

In this section we will obtain the analogue of Theorem 3.3.4 for $d \geq 2$.

Notation

Let $d \geq 2$ and $n \geq 2$. Let $\Phi : \Omega \subset \mathbb{R}^d \to G = \mathrm{SL}(n+1, \mathbb{R})$ be a C^1-map. Fix $s \in \Omega$, and suppose that $\phi = I_0\Phi : \Omega \to \mathbb{R}^{n+1}$ is $(n+1)$-differentiable and nondegenerate at s. So the derivative $D\phi(s) : \mathbb{R}^d \to \mathbb{R}^{n+1}$ of ϕ as s is injective. Let $\mathrm{SO}(d)$ be the special orthogonal group acting on $\mathcal{T} := D\phi(s)(\mathbb{R}^d)$. Since $\phi(s) \neq 0$, by Definition 3.1.1(2), $d \leq n$.

Theorem 3.4.1. *There exists a rational function $\xi_s : \mathrm{SO}(d) \to G$ such that the following holds. Let L be a Lie group containing G, Λ be a lattice in L, and $x \in L/\Lambda$. Then there exists a closed subgroup H_s of L such that $H_s x$ is closed and admits an H_s-invariant probability measure, say μ_{H_s}, and for any open bounded convex neighbourhood C of 0 in $\mathbb{R}^d$ and any $f \in C_c(L/\Lambda)$,*

$$\begin{aligned} &\lim_{t\to\infty} \frac{1}{\mathrm{vol}(C)} \int_C f(a(t)\Phi(s+t^{-1}\eta)x)\, d\eta \\ &= \int_{g\in \mathrm{SO}(d)} \int_0^{r_g} \left[\int_{H_s x} f(a(r)\xi_s(g)y) d\,\mu_{H_s}(y) \right] r^{d-1}\, dr\, dg. \end{aligned} \tag{3.4.1}$$

Here we fix a unit vector $e_1 \in \mathcal{T}$, and let $r_g = \sup\{r \geq 0 : rge_1 \in D\phi(s)(C)\}$, and $\int \cdot\, dg$ denotes a suitably normalized Haar integral on $\mathrm{SO}(d)$.

Moreover if $H_s \supset G$, then $\overline{Gx} = H_s x$, $\mu_x = \mu_{H_s}$, and

$$\lim_{t\to\infty} \frac{1}{\mathrm{vol}(C)} \int_C f(a(t)\Phi(s+t^{-1}\eta)x)\, d\eta = \int_{\overline{Gx}} f\, d\mu_x. \tag{3.4.2}$$

Realizing the manifold as a graph over a tangent

Let $\mathcal{L}$ be a subspace of $\mathbb{R}^{n+1}$ containing $\phi(s)$ as in Definition 3.1.1. Since $D\phi(s)$ is an injection, and $\mathcal{T} \oplus \mathcal{L} = \mathbb{R}^{n+1}$, by the implicit function theorem, there exist open neighborhoods Δ of 0 in $\mathcal{T}$ and Ω_1 of s in $\mathbb{R}^d$, and a C^{n+1}-diffeomorphism $\Psi : \Delta \to \Omega_1$

and a C^{n+1} map $F : \Delta \to \mathcal{L}$ such that $\Psi(0) = s$ and

$$\phi(\Psi(\eta)) = \phi(s) + \eta + F(\eta), \quad \forall \eta \in \Delta. \tag{3.4.3}$$

In particular, $DF(0) = 0$ and $D\Psi(0) = D\phi(s)^{-1}$.

Fix an open bounded convex neighborhood C of 0 in $\mathbb{R}^d$. Let $C_1 = D\phi(0)(C) \subset \mathcal{T}$. Then for any $f \in C_c(L/\Lambda)$,

$$\lim_{t\to\infty} \frac{1}{\mathrm{vol}(C)} \int_C f(a(t)\Phi(s + t^{-1}\kappa)x)\, d\kappa,$$

changing the variable κ to η such that $s + t^{-1}\kappa = \Psi(t^{-1}\eta)$,

$$\begin{aligned} &= \lim_{t\to\infty} \frac{1}{\mathrm{vol}(C)} \int_{t\Psi^{-1}(s+t^{-1}C)} f(a(t)\Phi(\Psi(t^{-1}\eta)x) \cdot |\det(D\Psi(t^{-1}\eta))|\, d\eta \\ &= \lim_{t\to\infty} \frac{1}{\mathrm{vol}(C_1)} \int_{C_1} f(a(t)\Phi(\Psi(t^{-1}\eta)x)\, d\eta, \end{aligned} \tag{3.4.4}$$

if any of the limits exists. Because since

$$\eta = t\Psi^{-1}(s + t^{-1}\kappa) = D\Psi(0)^{-1}(\kappa) + O(t^{-2})t = D\phi(s)(\kappa) + O(t^{-1}),$$

$\lim_{t\to\infty} \mathrm{vol}(t\Psi^{-1}(s + t^{-1}C)\Delta C_1) = 0$, and $\mathrm{vol}(C) = |\det(D\Psi(0))|\,\mathrm{vol}(C_1)$.

Nondegenerate curves on the manifold via Pyartli's twisting

Let $r_0 > 0$ be such that for any $0 \neq w \in \mathcal{T}$, the curve $\rho_w : (-r_0, r_0) \to \mathbb{R}w + \mathcal{L} \cong \mathbb{R}^{1+(n+1)-d}$ given by

$$\rho_w(r) = \phi(\Psi(rw))) = \phi(s) + rw + F(rw),$$

parametrizes the one-dimensional submanifold $\phi(\Omega_1) \cap (\mathbb{R}w + \mathcal{L})$. By Definition 3.1.1, we pick $0 \neq v \in \mathcal{T}$ such that ρ_v is nondegenerate at 0.

Remark 3.4.2. Fix a basis of $\mathcal{T}$. Let $w = (w_1, \ldots, w_d) \in \mathcal{T}$. Then $\rho_w(0) = \phi(s) \in \mathcal{L}$, $\rho_w^{(1)}(0) = w$, and for $2 \leq i \leq \dim \mathcal{L}$,

$$\rho_w^{(i)}(0) = \sum_{i_j \geq 0,\ i_1 + \ldots + i_d = i} \frac{i!}{i_1! \cdots i_d!} \cdot \partial_1^{i_1} \cdots \partial_d^{i_d} F(0) \cdot w_1^{i_1} \cdots w_d^{i_d} \in \mathcal{L}. \tag{3.4.5}$$

Therefore ρ_w is nondegenerate at 0, if and only if the determinant of the matrix whose 1-st row is $\phi(s)$ and the i-th row is $\rho_w^{(i)}(0)$ for $2 \le i \le \dim \mathcal{L}$ with respect to a fixed basis in $\mathcal{L}$ is nonzero. Since ρ_v is nondegenerate, ρ_w is nondegenerate for all $w \in \mathcal{T}$ outside an $\mathbb{R}$-invariant algebraic subvariety of strictly lower dimension.

Choose an orthonormal basis $\{e_i : 1 \le i \le d\}$ for $\mathcal{T}$. Let $\gamma : \mathbb{R} \to \mathcal{T}$ be the curve given by

$$\gamma(r) = re_1 + \sum_{i=2}^{d} r^{n-d+i} e_i \in \mathcal{T},\ \forall r \in \mathbb{R}.$$

For $g \in \mathrm{SO}(d)$, let $\zeta_{g\gamma} : (-r_0, r_0) \to \phi(\Omega_1)$ be the curve given by

$$\zeta_{g\gamma}(r) = \phi(\Psi(g\gamma(r))) = \phi(s) + g\gamma(r) + F(g\gamma(r)). \tag{3.4.6}$$

Lemma 3.4.3 ([26, Lemma 5])**.** *Let $g \in \mathrm{SO}(d)$ be such that the curve ρ_{ge_1} is nondegenerate at 0 in $\mathcal{L} + \mathbb{R}ge_1$. Then $\zeta_{g\gamma}$ is nondegenerate at 0 in $\mathbb{R}^{n+1}$.*

Proof. We observe that

$$\begin{aligned} \zeta_{g\gamma}^{(k)}(0) &= \rho_{ge_1}^{(k)}(0) \text{ for } 0 \le k \le (n-d+1) = \dim(\mathcal{L} + \mathbb{R}ge_1) - 1, \\ \zeta_{g\gamma}^{(n-d+i)}(0) &= ge_i \text{ modulo } \mathcal{L} + \mathbb{R}ge_1, \quad \text{for } 2 \le i \le d. \end{aligned} \tag{3.4.7}$$

So $\{\zeta_{g\gamma}^{(k)}(0) : 0 \le k \le n\}$ spans $\mathbb{R}^{n+1}$. □

Polar fibering

For $t \ge 1$, let $T_t : \mathrm{SO}(d) \times [0, \infty) \to \mathbb{R}^d$ be given by

$$T_t(g, r) = tg\gamma(t^{-1}r) = g \cdot t\gamma(t^{-1}r) = g \cdot (re_1 + \sum_{i=2}^{d} t^{-(n-d+i-1)} r^{n-d+i} e_i). \tag{3.4.8}$$

We recall that $2 \le d \le n$. Let dg denote a Haar integral on $\mathrm{SO}(d)$. For a fixed $r > 0$, under $T_t(\cdot, r)$, the Haar measure on $\mathrm{SO}(d)$ projects to a rotation invariant measure on the sphere of radius $|t\gamma(t^{-1}r)|$ in $\mathbb{R}^d$ centered at 0. Then the image of the integral $\mathrm{d}g \times |t\gamma(t^{-1}r)|^{d-1}\mathrm{d}|t\gamma(t^{-1}r)|$ under the map T_t corresponds to a multiple of the Lebesgue integral on $\mathbb{R}^d$.

Let $r_{g,t} = \sup\{r \geq 0 : T_t(g,r) \in C_1\}$. Now $T_t(g,r) = rge_1 + O(t^{-1})$ uniformly in g and bounded r. Therefore $r_{g,t} = r_g + O(t^{-1})$, where

$$r_g = \sup\{r \geq 0 : gre_1 \in C_1\}.$$

By (3.4.8), $\frac{|t\gamma(t^{-1}r)|^{d-1}}{r^{d-1}} \cdot \frac{d}{dr}|t\gamma(t^{-1}r)| = 1 + O(t^{-1}r)^2$. Therefore continuing (3.4.4), by the change of variable $\eta = T_t(g,r)$,

$$\begin{aligned}
&\lim_{t\to\infty} \frac{1}{\mathrm{vol}(C_1)} \int_{C_1} f(a(t)\Phi(\Psi(t^{-1}\eta)x))\, d\eta \\
&= \lim_{t\to\infty} \int_{g\in \mathrm{SO}(d)} \left[\int_0^{r_{g,t}} f((a(t)\Phi(\Psi(t^{-1}T_t(g,r))|t\gamma(t^{-1}r)|^{d-1}\, d|t\gamma(t^{-1}r)|\right] dg \\
&= \lim_{t\to\infty} \int_{g\in \mathrm{SO}(d)} \left[\int_0^{r_g} f(a(t)\Phi(\Psi(g\gamma(t^{-1}r)))x)r^{d-1}dr\right] dg,
\end{aligned} \tag{3.4.9}$$

where for each t the Haar integral dg is normalized such that the integral of the expression equals 1 for the constant function $f \equiv 1$.

Proof of Theorem 3.4.1

In view of (3.4.6) and (3.4.9),

$$I_0\Phi(\Psi(g\gamma(r))) = \zeta_{g\gamma}(r) \in \mathbb{R}^{n+1}. \tag{3.4.10}$$

Let $\{\tilde{e}_k : 0 \leq k \leq n\}$ denote the standard basis of $\mathbb{R}^{n+1}$ consisting of row vectors. For $g \in \mathrm{SO}(d)$, let $M(g) \in \mathrm{M}(n+1, \mathbb{R})$ be such that with respect to the right action $\mathbb{R}^{n+1}$,

$$\tilde{e}_k M(g) = \zeta_{g\gamma}^{(k)}(0)/k!,\ \forall\, 0 \leq k \leq n.$$

Now $\zeta_{g\gamma}$ is nondegenerate at 0 if and only if $\det(M(g)) \neq 0$. By (3.4.5) and (3.4.7), $\det(M(g))$ is a polynomial in coordinates of g. Therefore the set $Z_s = \{g \in \mathrm{SO}(d) : \det(M(g)) = 0\}$ is an affine subvariety of $\mathrm{SO}(d)$. Since ϕ is nondegenerate at s, there exists $g \in \mathrm{SO}(d)$ such that $ge_1 = v$ and ρ_v is nondegenerate at 0. Therefore by Lemma 3.4.3, we have that $g \notin Z_s$. Therefore Z_s is a strictly lower dimensional subvariety of $\mathrm{SO}(d)$, where $d \geq 2$. Hence Z_s is null with respect to dg.

Let $g \in \mathrm{SO}(d) \setminus Z_s$. Let $B(g) = M(g)^{-1} B M(g)$, where B is the lower triangular matrix such that $\tilde{e}_0 B = 0$ and $\tilde{e}_k B = \tilde{e}_{k-1}$ for $1 \leq i \leq n$. Then

$$\zeta_{g\gamma}(0) B(g) = 0 \text{ and } (\zeta_{g\gamma}^{(k)}(0)/k!) B(g) = \zeta_{g\gamma}^{(k-1)}(0)/(k-1)!, \ \forall 1 \leq k \leq n,$$

as in (3.2.5). In view of (3.2.10), let

$$\xi_s(g) = I_0(\zeta_{g\gamma}^{(n)}(0)/n! - \zeta_{g\gamma}^{(n+1)}(0)/(n+1)! \cdot B(g)) - I_n \Phi(s) B(g).$$

Then by (3.4.10) and (3.2.9),

$$a(t)\Phi(\Psi(g\gamma(t^{-1}))) = (I + o(t^{-1})t)\xi_s(g)(I - tB(g))^{-1}. \tag{3.4.11}$$

In particular, $\xi_s(g) \in G$. As in (3.2.2),

$$(I - tB(g))^{-1} = \sum_{k=0}^{n} t^k B(g)^k, \ \forall t \in \mathbb{R}. \tag{3.4.12}$$

Let $\mathfrak{f}(g)$ be the $\mathbb{R}$-span of $\{B^k(g) : 0 \leq k \leq n\}$. Then one has

$$\mathfrak{f}(g) = M(g)^{-1} \mathfrak{f} M(g), \tag{3.4.13}$$

where $\mathfrak{f}$ is the $\mathbb{R}$-span of $\{B^k : 0 \leq k \leq n\}$.

We fix $x \in L/\Lambda$. Let $H(g) \in \mathcal{H}_x$ be the smallest Lie subgroup such that its Lie algebra contains $\mathfrak{f}(g)$. By Theorem 3.3.4, in view of (3.4.9) and (3.4.11),

$$\begin{aligned}
&\lim_{t\to\infty}\int_0^{r_g} f(a(t)\Phi(\Psi(g\gamma(t^{-1}r)))r^{d-1}\,dr \\
&=\int_0^{r_g}\Big[\int_{H(g)x} f(a(r)\xi_s(g)y)\,d\mu_{H(g)}(y)\Big]r^{d-1}dr.
\end{aligned}\tag{3.4.14}$$

Claim 1.

There exists $H_s \in \mathcal{H}_x$ such that $H(g) = H_s$, $\forall\, g \in \mathrm{SO}(d) \setminus (Z_s \cup Z_{x,s})$, where $Z_{x,s}$ is a Haar-null subset of $\mathrm{SO}(d)$.

To prove this, let $H \in \mathcal{H}_x$. For $g \in \mathrm{SO}(d) \setminus Z_s$, we have $H(g) \in \mathcal{H}_x$ and

$$H(g) \subset H \iff \mathfrak{f}(g) \subset \mathrm{Lie}(H) \iff M(g)^{-1}\mathfrak{f}M(g) \subset \mathrm{Lie}(H). \tag{3.4.15}$$

So define

$$Z_s(H) = \{g \in \mathrm{SO}(d) : \mathfrak{f}M(g) \subset M(g)\cdot \mathrm{Lie}(H)\}.$$

Then $Z_s(H)$ is an affine subvariety of $\mathrm{SO}(d)$. So $\mathrm{SO}(d) \setminus Z_s$ is locally compact, and hence of Baire second category. For every $g \in \mathrm{SO}(d) \setminus Z_s$,

$$Z_s(H(g)) \supset \{g' \in \mathrm{SO}(d) \setminus Z_s : H(g') \subset H(g)\} \ni g. \tag{3.4.16}$$

Since $\mathcal{H}_x$ is countable, $\mathrm{SO}(d) \setminus Z_s$ is covered by a countable union of closed sets $Z_s(H(g))$, where $g \in \mathrm{SO}(d) \setminus Z_s$. So there exists $g_0 \in \mathrm{SO}(d)$ such that $Z_s(H(g_0))$ contains a non-empty open subset of $\mathrm{SO}(d)$. Since $d \geq 2$, any non-empty open subset of $\mathrm{SO}(d)$ is Zariski dense in $\mathrm{SO}(d)$. Therefore $Z_s(H(g_0)) = \mathrm{SO}(d)$. So $H(g) \subset H(g_0)$ for all $g \in \mathrm{SO}(d) \setminus Z_s$. Define

$$Z_{x,s} = \cup\{g \in \mathrm{SO}(d) \setminus Z_s :\ H_s(g_0) \not\subset H(g)\}.$$

Let $g \in Z_{x,s}$. By (3.4.16), $g_0 \notin Z_s(H(g))$. So $Z_s(H(g))$ is a proper affine subvariety of $\mathrm{SO}(d)$ of strictly lower dimension, and it is Haar-null on $\mathrm{SO}(d)$. Also $g \in Z_s(H(g))$ and $H(g) \in \mathcal{H}_x$. Therefore, since $\mathcal{H}_x$ is countable, $Z_{x,s}$ is Haar-null on $\mathrm{SO}(d)$. Put $H_s = H(g_0)$. Then $H(g) = H_s$ for all $g \in \mathrm{SO}(d) \setminus (Z_s \cup Z_{x,s})$. So the Claim 1 holds.

Continuing (3.4.9) using (3.4.14), by Claim 1, since $Z_s \cup Z_{x,s}$ is Haar-null,

$$\begin{aligned}
&\lim_{t\to\infty} \int_{g\in\mathrm{SO}(d)} \left[\int_0^{r_g} f((a(t)\Phi(\Psi(g\gamma(t^{-1}r))))x) r^{d-1}\, dr\right] dg \\
&= \int_{g\in\mathrm{SO}(d)\setminus(Z_{x,s}\cup Z_s)} \int_0^{r_g} \left[\int_{y\in H_s x} f(a(r)\xi_s(g)y)\, d\mu_{H_s}\right] r^{d-1}\, dr\, dg. \qquad (3.4.17)
\end{aligned}$$

This completes the proof of Theorem 3.4.1. □

3.5 Equidistribution of translates of nondegenerate manifolds

Let the notation be as in the statement of Theorem 3.1.4. In view of (3.4.2) in Theorem 3.4.1, let

$$E_x = \{s \in \Omega : G \not\subset H_s\}. \qquad (3.5.1)$$

To derive Theorem 3.1.4 from Theorem 3.4.1, we will show that E_x is a countable union of sets of the form $\phi^{-1}(W)$, where W is a proper subspace of $\mathbb{R}^{n+1}$ (Proposition 3.5.3), and the following holds:

Proposition 3.5.1. *Let $\Omega \subset \mathbb{R}^d$ be an open set, and $\phi : \Omega \to \mathbb{R}^{n+1}$ be a n-times differentiable nondegenerate map. Then for any nonzero linear functional $\ell : \mathbb{R}^{n+1} \to \mathbb{R}$, the set $\{s \in \Omega : \ell(\phi(s)) = 0\}$ has zero Lebesgue measure. In fact, if $d = 1$, then $\phi^{-1}(\ker \ell)$ is discrete in Ω.*

Proof. Let $d = 1$. Suppose $s \in \Omega \subset \mathbb{R}$ and a sequence $\{s_i\} \subset \Omega \setminus \{s\}$ are such that $\ell(\phi(s_i)) = 0$ and $s_i \to s$ as $i \to \infty$. By Taylor's expansion,

$$0 = \ell(\phi(s_i)) = \sum_{k=0}^{n} \frac{\ell(\phi^{(k)}(s))}{k!}(s_i - s)^k + o(s - s_i)^n$$

for all i. Let $0 \leq m \leq n$ be such that $\ell(\phi^{(k)}(s)) = 0$ for $0 \leq k < m$. Then dividing both sides by $(s_i - s)^m$, and letting $i \to \infty$, $\ell(\phi^{(m)}(s)) = 0$. By induction $\ell(\phi^{(k)}(s)) = 0$ for all $0 \leq k \leq n$. But this contradicts our assumption that $\phi^{(k)}(s)$ for $k = 0, \ldots, n$ are linearly independent in $\mathbb{R}^{n+1}$. Therefore $\phi^{-1}(\ker \ell)$ is discrete in Ω if $d = 1$.

Now we consider the case of $d \geq 2$. Suppose on the contrary that $\phi^{-1}(\ker \ell) \subset \Omega$ has strictly positive Lebesgue measure in $\mathbb{R}^d$. Let $s \in \Omega$ be a Lebesgue density point of $\phi^{-1}(\ker \ell)$. Then for a unit ball C about 0 in $\mathbb{R}^d$, by (3.4.4), (3.4.9) and (3.4.6),

$$\begin{aligned}
1 &= \lim_{t \to \infty} \frac{1}{\mathrm{vol}(C)} \int_{\eta \in C} \chi_{\ker \ell}(\phi(s + t^{-1}\eta))\, d\eta \\
&= \lim_{t \to \infty} \int_{g \in \mathrm{SO}(d)} \int_0^{r_g} \chi_{\ker \ell}(\phi(\Psi(g\gamma(t^{-1}r))r^{d-1}\, dr\, dg \\
&= \lim_{t \to \infty} \int_{g \in \mathrm{SO}(d)} \int_0^{r_g} \chi_{\ker \ell}(\zeta_{g\gamma}(t^{-1}r))r^{d-1}\, dr = 0,
\end{aligned}$$

because for every $g \in \mathrm{SO}(d) \setminus Z_s$, there exists $t_g > 0$, such that $r \mapsto \zeta(g\gamma)(r)$ is nondegenerate for all $|r| < t_g^{-1} r_g$, and hence by the case of $d = 1$, $\{r \in [0, t_g^{-1} r_g) : \ell(\zeta_{g\gamma}(r)) = 0\} = \zeta_{g\gamma}^{-1}(\ker \ell) \cap (0, t_g^{-1} r_g]$ is a countable set. $\square$

In order to describe the exceptional set E_x, we will use a crucial result from [35], which is generalized in [44] for arbitrary G. We begin with some some notation and observations. Let

$$P^- = \{g \in G : \overline{\{a(t)ga(t)^{-1} : t \geq 1\}} \text{ is compact}\}$$

$$U = \{u(\boldsymbol{z}) := \left(\begin{smallmatrix} 1 & \boldsymbol{z} \\ 0 & I_n \end{smallmatrix}\right) : \boldsymbol{z} \in \mathbb{R}^n\}.$$

Let $\{\tilde{e}_k : 0 \leq k \leq n\}$ denote the standard basis of $\mathbb{R}^{n+1}$, which is treated as the space of top rows of matrices in $\mathrm{M}(n+1, \mathbb{R})$. We identify $\mathbb{R}^n$ with $\mathrm{span}\{\tilde{e}_k : 1 \leq k \leq n\}$. Then P^- is the stabilizer of the line $\mathbb{R} \cdot \tilde{e}_0$ for the right action of G on $\mathbb{R}^{n+1}$ and $\tilde{e}_0 u(\boldsymbol{z}) = \tilde{e}_0 + \boldsymbol{z} \in \mathbb{R}^{n+1}$, $\forall \boldsymbol{z} \in \mathbb{R}^n$. Therefore

$$P^- U = \{g \in G : g_{00} := \langle \tilde{e}_0 g, \tilde{e}_0 \rangle \neq 0\} = \{g \in G : I_0 g \notin \{0\} \times \mathbb{R}^n\}, \tag{3.5.2}$$

and it is a Zariski open dense neighborhood of the identity in G.

For a finite dimensional representation V of G, define

$$V^+ = \{v \in V : \lim_{t \to \infty} a(t)^{-1} v = 0\}, \quad V^- = \{v \in V : \lim_{t \to \infty} a(t) v = 0\},$$
$$V^0 = \{v \in V : a(t) v = v, \ \forall t > 0\},$$

and let π_+, π_0, and π_- denote the natural projections onto V^+, V^0 and V^-, respectively, with respect to the decomposition $V = V^+ \oplus V^0 \oplus V^-$.

Proposition 3.5.2 ([35, Corollary 4.4]). *Let $\mathcal{E} \subset P^- U$ be such that $I_0 \mathcal{E}$ is not contained in the union of any n proper subspaces of $\mathbb{R}^{n+1}$. Then for any finite dimensional representation V of G and a nonzero $v \in V$, if*

$$gv \in V^0 + V^-, \quad \forall g \in \mathcal{E}, \tag{3.5.3}$$

then $\pi_0(gv) \neq 0$ for all $g \in \mathcal{E}$ and $Z_G(\{a(t) : t > 0\})$ fixes $\pi_0(gv)$.

Proof. For every $g \in P^- U$, there exists a unique $\bar{g} \in \mathbb{R}^n$ such that $P^- g = P^- u(\bar{g})$, and $I_0 g = g_{00}(\tilde{e}_0 + \bar{g})$. Since P^- stabilizes $V^0 + V^-$, by (3.5.3)

$$u(\bar{g}) v \subset V^0 + V^-, \quad \forall g \in \mathcal{E}. \tag{3.5.4}$$

Claim 1

Let $h \in \mathcal{E}$. Then for any proper subspaces W_k of $\mathbb{R}^n$ for $1 \leq k \leq n$,

$$\{\bar{g} - \bar{h} : g \in \mathcal{E}\} \not\subset \cup_{k=1}^{n} W_k.$$

On the contrary, suppose $\{\bar{g} - \bar{h} : g \in \mathcal{E}\} \subset \cup_{k=1}^{n} W_k$. For every $g \in P^{-}U$, $\bar{g} - \bar{h} = g_{00}^{-1} I_0 g - h_{00}^{-1} I_0 h$. Therefore $I_0 \mathcal{E} \subset \cup_{k=1}^{n}(W_k \oplus \mathbb{R} I_0 h)$. Since $W_k \oplus \mathbb{R} I_0 h$ is a proper subspace of $\mathbb{R}^{n+1}$, this leads to a contradiction, so Claim 1 holds.

According to [35, Corollary 4.4], if (3.5.4) and Claim 1 hold, $\pi_0(u(\bar{h})v) \neq 0$ and it is fixed by $Z_G(\{a(t) : t > 0\})$. For any $b \in P^{-}$ and $w \in V$, $\pi_0(bw) = \lambda \pi_0(w)$ for some $\lambda \neq 0$. Therefore we conclude that $\pi_0(hv) \neq 0$ and it is fixed by $Z_G(\{a(t) : t > 0\})$. □

Proposition 3.5.3. *Let $H \in \mathcal{H}_x$ be such that $G \not\subset H$. Let*

$$E_H = \{s \in \Omega : \Phi(s) \in P^{-}U \text{ and } H_s \subset H\}. \tag{3.5.5}$$

For any $s \in E_H$, there there exists a neighborhood Ω_2 of s such that $\phi(\Omega_2 \cap E_H)$ is contained in the union of at most n proper subspaces of $\mathbb{R}^{n+1}$.

Proof. Let F be the closure of the subgroup of G generated by all unipotent elements of G contained in H. Then $F \neq G$. Since H is a connected Lie group, F is a real algebraic subgroup of G. Since F admits no nontrivial characters, we choose a finite dimensional representation V of G with a vector $p_F \in V$ such that F fixes p_F and V has no nonzero G-fixed vector.

Claim 2

For any $s \in E_H$, $\Phi(s) p_F \in V^0 + V^{-}$.

To see this, for any $g \in \mathrm{SO}(d)$, since $\Psi(0) = s$, we have

$$\lim_{t\to\infty} \pi_+(\Phi(\Psi(g\gamma(t^{-1})))p_F) = \pi_+(\Phi(s)p_F). \tag{3.5.6}$$

Now let $s \in E_H$ and $g \in \mathrm{SO}(d) \setminus (Z_s \cup Z_{x,s})$. By (3.4.11),

$$\begin{aligned} a(t)\Phi(\Psi(g\gamma(t^{-1})))p_F &= (I + o(t^{-1})t)\xi_s(g)(I - tB_s(g))^{-1}p_F \\ &= (I + o(t^{-1})t)\xi_s(g)p_F. \end{aligned} \tag{3.5.7}$$

because by (3.4.12), (3.4.13) and (3.4.15), $(I - tB_s(g))^{-1}$ is a unipotent element of G contained in H, so it fixes p_F. Therefore from (3.5.6) we conclude that $\pi_+(\Phi(s)p_F) = 0$, otherwise (3.5.7) diverges as $t \to \infty$. So Claim 2 holds.

Now suppose $s \in E_H$ is such that for any neighborhood Ω_2 of s, $\phi(\Omega_2 \cap E_H)$ is not contained in the union of any n proper subspaces of $\mathbb{R}^{n+1}$.

Therefore in view of Claim 2, by Proposition 3.5.2 applied to $\mathcal{E} = \Phi(E_H)$,

$$\pi_0(\Phi(s)p_F) \neq 0, \text{ and it is fixed by } Z_G(\{a(t) : t > 0\}). \tag{3.5.8}$$

Claim 3

$\pi_0(\Phi(s)p_F)$ is fixed by $u(\boldsymbol{z}) \in U$ for some $\boldsymbol{z} \in \mathbb{R}^n \setminus \{0\}$.

To prove this, by our assumption we pick a sequence $(s_i) \subset E_H$ such that $s_i \to s$ and $\phi(s_i) \notin \mathbb{R}\phi(s)$, $\forall\, i$. Since $\phi(s_i) = I_0\Phi(s_i)$ and $I_0P^-\Phi(s) \subset \mathbb{R}\phi(s)$, we have $\Phi(s_i) \notin P^-\Phi(s)$ for all i. Therefore $\Phi(s_i)\Phi(s)^{-1} = b_i u(\boldsymbol{z}_i)$, where $b_i \to I$ in P^- and $0 \neq \boldsymbol{z}_i \to 0$ as $i \to \infty$. Let $t_i = |\boldsymbol{z}_i|^{-1/(n+1)}$. After passing to a subsequence, there exist $0 \neq \boldsymbol{z} \in \mathbb{R}^n$ such that as $i \to \infty$,

$$a(t_i)\Phi(s_i)\Phi(s)^{-1}a(t_i)^{-1} = a(t_i)b_i a(t_i^{-1}) \cdot u(\boldsymbol{z}_i/|\boldsymbol{z}_i|) \to u(\boldsymbol{z}). \tag{3.5.9}$$

Now since $\Phi(s_i)p_F \in V^0 + V^-$,

$$\lim_{i\to\infty} a(t_i)\Phi(s_i)p_F = \lim_{i\to\infty} \pi_0(\Phi(s_i)p_F) = \pi_0(\Phi(s)p_F).$$

On the other hand by (3.5.9), as $i \to \infty$,

$$a(t_i)\Phi(s_i)p_F = a(t_i)\Phi(s_i)\Phi(s)^{-1}a(t_i)^{-1} \cdot a(t_i)\Phi(s)p_F \to bu(\boldsymbol{z}) \cdot \pi_0(\Phi(s)p_F).$$

Therefore $\pi_0(\Phi(s)p_F)$ is fixed by $u(\boldsymbol{z})$. This proves Claim 3.

By Claim 3 and (3.5.8), $\pi_0(\Phi(s)p_F) \neq 0$ and is fixed by the subgroup generated by $u(\boldsymbol{z})$ and $Z_G(\{a(t) : t > 0\})$. Since every nontrivial element of U is conjugated to $u(\boldsymbol{z})$ by an element of $Z_G(\{a(t) : t > 0\})$, we have that $\pi_0(\Phi(s)p_F)$ is fixed by $Z_G(\{a(t) : t > 0\})U$, which is a parabolic subgroup of G. So $\pi_0(\Phi(s)p_F) \neq 0$ is fixed by G, a contradiction to our choice of V. $\square$

Proof of Theorem 3.1.4. By (3.5.1), (3.5.2), and (3.5.5),

$$E_x = \phi^{-1}(\{0\} \times \mathbb{R}^n) \bigcup \cup \{E_H : H \in \mathcal{H}_x \text{ and } H \not\supset G\}.$$

Since $\mathcal{H}_x$ is countable, by Proposition 3.5.3, E_x is a countable union of sets of the form $\phi^{-1}(W)$, where W is a proper subspace of $\mathbb{R}^{n+1}$. Therefore by Proposition 3.5.1, the Lebesgue measure of E_x is zero. Let $s \in \Omega \setminus E_x$. Then $H_s \supset G$. Therefore by Theorem 3.4.1, we get (3.4.2), which is same as (3.1.4). Now (3.1.5) can deduced from (3.1.4) using the Lebesgue points of ν and convergence in measure. $\square$

Next we show that the exceptional set is dense in many examples.

Proposition 3.5.4. *Let $L = G = \mathrm{SL}(n+1, \mathbb{R})$ and $\Lambda = \mathrm{SL}(n+1, \mathbb{Z})$. Let $\phi : \mathbb{R}^d \to \mathbb{R}^{n+1}$ be a polynomial map with coefficients in $\mathbb{Q}$ and that its image is not contained in a proper subspace of $\mathbb{R}^{n+1}$, in particular it is nondegenerate on a Zariski open dense* *set* $\Omega \subset \mathbb{R}^d$. *Let* $\Phi : \Omega \subset \mathbb{R}^d \to G$ *be a map such that* $I_0\Phi(s) = \phi(s)$ *for all* $s \in \Omega$. *Let* $x \in \mathrm{SL}(n+1, \mathbb{Q})/\Lambda \subset L/\Lambda$. *Then* $E_x \supset \Omega \cap \mathbb{Q}^d$.

Proof. Let $s \in \Omega \cap \mathbb{Q}^d$. In the notation of §3.4, $\mathcal{T}$ and $\mathrm{SO}(d)$ are defined over $\mathbb{Q}$. We choose an orthonormal basis $\{e_i : 1 \leq i \leq d\}$ of $\mathcal{T}$ to be defined over $\mathbb{Q}$. Let $g \in \mathrm{SO}(d)(\mathbb{Q})$. Then $M(g) \in \mathrm{GL}(n+1, \mathbb{Q})$, and hence $\mathfrak{f}(g)$ is defined over $\mathbb{Q}$, and it is an abelian subalgebra consisting of nilpotent matrices. Therefore $H(g)$ is an abelian unipotent group defined over $\mathbb{Q}$. Since $\mathrm{SO}(d)(\mathbb{Q})$ is Zariski dense in $\mathrm{SO}(d)$, H_s is a unipotent group defined over $\mathbb{Q}$, so it does not contain G. So $s \in E_x$. $\square$

Appendix A: Linearized non-divergence and the action of Weyl group

In this appendix, we prove a result concerning the dynamical behavior of translates of shrinking curve in a finite dimensional linear representation of $G = \mathrm{SL}(n+1,\mathbb{R})$. This result leads to an altenative proof of the main results in Chapter 3. It can be regarded as an infinisimal analogue of Theorem 2.1.2.

Proposition A.0.1 (Linear stability)**.** *Let $G = SL(n+1,\mathbb{R})$. For $t > 0$ and $s \in \mathbb{R}$, define*

$$a(t) = \begin{bmatrix} e^{nt} & & & & \\ & e^{-t} & & & \\ & & e^{-t} & & \\ & & & \ddots & \\ & & & & e^{-t} \end{bmatrix}, u(s) = \begin{bmatrix} 1 & s & s^2 & \dots & s^n \\ & 1 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{bmatrix}.$$

Let $\rho : G \to GL(V)$ be any finite dimensional linear representation of G, with a norm $||\cdot||$ on V. Then for any $\eta > 0$, there exists $c > 0$ such thats for any $v \in V$ and any $t > 0$, we have

$$\sup_{s\in[0,\eta e^{-t}]} ||a(t)u(s)v|| \geq c||v||. \tag{A.0.1}$$

Before we prove this proposition, let's first make some preparations.

Consider the diagonal subgroup

$$T = \left\{ \begin{bmatrix} a_1 & & & \\ & a_2 & & \\ & & \ddots & \\ & & & a_{n+1} \end{bmatrix} : \prod_{i=1}^{n+1} a_i = 1, a_i > 0 \right\},$$

and let $\mathfrak{t} = \mathrm{Lie}(T)$ be its Lie algebra. Let L_i be the character of $\mathfrak{t}$ which maps a diagonal element to its ith coordinate, for $1 \le i \le n+1$. Then $\alpha_i = L_i - L_{i+1}(1 \le i \le n)$ form a set of simple roots. Define

$$\beta_i = L_1 - L_{i+1} = \sum_{j=1}^{i} \alpha_j, 1 \le i \le n.$$

Let $\langle \cdot, \cdot \rangle$ be the inner product given by the Killing form. Then

$$\langle \beta_i, \beta_j \rangle = \begin{cases} 2 & \text{if } i = j, \\ 1 & \text{if } i \ne j. \end{cases}$$

For $1 \le i < j \le n$, let w_{ij} be the Weyl element which switches L_i and L_j. Then w_{ij} switches β_i and β_j but fixes the other β_k's. Hence for any weight λ in the weight lattice Λ_W, we have

$$\begin{aligned} w_{ij}(\lambda) &= \lambda - 2\frac{\langle \lambda, \beta_i - \beta_j \rangle}{\langle \beta_i - \beta_j, \beta_i - \beta_j \rangle}(\beta_i - \beta_j) \\ &= \lambda - \langle \lambda, \beta_i - \beta_j \rangle (\beta_i - \beta_j). \end{aligned} \tag{A.0.2}$$

For $1 \le i \le n$, let r_i be the Weyl element which is the reflection with respect to β_i. More precisely,

$$r_i(\lambda) = \lambda - 2\frac{\langle \lambda, \beta_i \rangle}{\langle \beta_i, \beta_i \rangle}\beta_i = \lambda - \langle \lambda, \beta_i \rangle \beta_i. \tag{A.0.3}$$

Consider the following element in $\mathfrak{t}$:

$$H = \begin{bmatrix} n & & & & \\ & -1 & & & \\ & & -1 & & \\ & & & \ddots & \\ & & & & -1 \end{bmatrix}.$$

Then for any weight λ, we have

$$\lambda(H) = \langle \lambda, \sum_{i=1}^{n} \beta_i \rangle. \tag{A.0.4}$$

In our study of linear dynamics, the following notions turn out to be convenient.

Definition A.0.2. Suppose λ and μ are weights. We define the *effective value* of μ with respect to λ as

$$\mathrm{eff}_\lambda(\mu) = \begin{cases} \mu(H) - \sum_{i=1}^{n} i m_i & \text{if } \mu = \lambda + \sum_{i=1}^{n} m_i \beta_i \\ -\infty & \text{otherwise} \end{cases} \tag{A.0.5}$$

Now fix a finite dimensional representation $\rho : G \to \mathrm{GL}(V)$ of G, and let Λ_V denote its weight diagram. We say that $\lambda \in \Lambda_V$ is *potentially non-negative* if $\mathrm{eff}_\lambda(\mu) \geq 0$ for some weight $\mu \in \Lambda_V$.

Remark A.0.3. It is clear from the definition that if $\lambda(H) \geq 0$, then λ is potentially non-negative.

Lemma A.0.4. *Suppose that $\lambda \in \Lambda_V$ is not potentially non-negative(see Definition A.0.2), and that there exists $k \in \mathbb{N}$ and $1 \leq i_1 < i_2 < \cdots < i_k \leq n$ such that*

$$\langle \lambda, \beta_{i_k} \rangle < \langle \lambda, \beta_{i_{k-1}} \rangle < \cdots < \langle \lambda, \beta_{i_2} \rangle < \langle \lambda, \beta_{i_1} \rangle < 0. \tag{A.0.6}$$

Then the following inequality holds:

$$\langle \lambda, \sum_{i=1}^{n} \beta_i - \sum_{r=1}^{k} (i_{r+1} - i_r) \beta_{i_r} \rangle < 0, \tag{A.0.7}$$

where we set $i_{k+1} = n + 1$.

Proof. Let w_{ij} and r_i be the Weyl elements as defined in equation (A.0.2) and (A.0.3). Since the Weyl group acts on the weight diagram Λ_V, we know that $r_{i_k}(\lambda) = \lambda -$

$\langle\lambda, \beta_{i_k}\rangle\beta_{i_k}$ is also contained in Λ_V. Furthermore, applying $w_{i_k i_{k-1}}, w_{i_{k-1} i_{k-2}}, \cdots, w_{i_2 i_1}$ in order we obtain the following weight μ which is still in Λ_V:

$$\begin{aligned}
\mu &= w_{i_2 i_1} \circ \cdots \circ w_{i_{k-1} i_{k-2}} \circ w_{i_k i_{k-1}} \circ r_{i_k}(\lambda) \\
&= w_{i_2 i_1} \circ \cdots \circ w_{i_{k-1} i_{k-2}} \circ w_{i_k i_{k-1}}(\lambda - \langle\lambda, \beta_{i_k}\rangle\beta_{i_k}) \\
&= w_{i_2 i_1} \circ \cdots \circ w_{i_{k-1} i_{k-2}}(\lambda - \langle\lambda, \beta_{i_k} - \beta_{i_{k-1}}\rangle\beta_{i_k} - \langle\lambda, \beta_{i_{k-1}}\rangle\beta_{i_{k-1}}) \\
&= \cdots \\
&= \lambda - \sum_{r=1}^{k}\langle\lambda, \beta_{i_r} - \beta_{i_{r-1}}\rangle\beta_{i_r},
\end{aligned} \tag{A.0.8}$$

where we set $\beta_{i_0} = 0$.

Since λ is not potentially non-negative, by Definition A.0.2 we have

$$\begin{aligned}
\mathrm{eff}_\lambda(\mu) &= \mu(H) - \sum_{r=1}^{k} i_k\langle\lambda, \beta_{i_r} - \beta_{i_{r-1}}\rangle \\
&= \langle\lambda, \sum_{i=1}^{n}\beta_i\rangle - (n+1)\langle\lambda, \beta_{i_k}\rangle - \sum_{r=1}^{k} i_k\langle\lambda, \beta_{i_r} - \beta_{i_{r-1}}\rangle \\
&= \langle\lambda, \sum_{i=1}^{n}\beta_i - \sum_{r=1}^{k}(i_{r+1} - i_r)\beta_{i_r}\rangle \\
&< 0.
\end{aligned} \tag{A.0.9}$$

This finishes the proof of the lemma. □

Lemma A.0.5. *Every weight $\lambda \in \Lambda_V$ is potentially non-negative.*

Proof. We prove by contradiction. Suppose λ is not potentially non-negative, then by Remark A.0.3 we have

$$\lambda(H) = \langle\lambda, \sum_{i=1}^{n}\beta_i\rangle < 0.$$

Therefore there exists $1 \leq m \leq n$ such that $\langle\lambda, \beta_m\rangle < 0$. Take m to be the smallest such index, i.e.

$$\langle\lambda, \beta_i\rangle \geq 0, \ \forall i < m. \tag{A.0.10}$$

Now let $(i_1, i_2, \cdots, i_k)$ be the lexicographically smallest tuple satisfying

(i) $m = i_1 < i_2 < \cdots < i_k \leq n$,

(ii) $\langle \lambda, \beta_{i_k} \rangle < \langle \lambda, \beta_{i_{k-1}} \rangle < \cdots < \langle \lambda, \beta_{i_2} \rangle < \langle \lambda, \beta_{i_1} \rangle < 0$.

Then by Lemma A.0.4,

$$\langle \lambda, \sum_{i=1}^{n} \beta_i - \sum_{r=1}^{k} (i_{r+1} - i_r) \beta_{i_r} \rangle < 0, \tag{A.0.11}$$

where we set $i_{k+1} = n + 1$.

Combining equation (A.0.10) and (A.0.11), we have

$$\langle \lambda, \sum_{\substack{i \geq m \\ i \notin \{i_1, \cdots i_k\}}} \beta_i - \sum_{r=1}^{k} (i_{r+1} - i_r - 1) \beta_{i_r} \rangle < 0. \tag{A.0.12}$$

Rewrite the above inequality as

$$\langle \lambda, \sum_{r=1}^{k} \sum_{j=i_r+1}^{i_{r+1}-1} (\beta_j - \beta_{i_r}) \rangle < 0, \tag{A.0.13}$$

it follows that there exists $1 \leq r \leq k$ and $i_r < j < i_{r+1}$ such that

$$\langle \lambda, \beta_j - \beta_{i_r} \rangle < 0. \tag{A.0.14}$$

Therefore, the tuple $(i_1, i_2, \cdots, i_r, j)$ is a tuple which also satisfies (i)(ii), but lexicographically smaller than $(i_1, i_2, \cdots, i_k)$. This contradicts to the choice of $(i_1, i_2, \cdots, i_k)$.

□

In the above lemma we actually prove Proposition A.0.1 in the case that v is an eigenvector. Furthermore, one needs to show that different eigenvectors do not cancel completely at all weight spaces with non-negative effective values. Therefore, we introduce the following notions.

Definition A.0.6. Given any $b \in \mathbb{R}$, define the *competing hyperplane* C_b as

$$C_b = \{\lambda \in \mathfrak{t}^* : \langle \lambda, n_C \rangle = b\}, \tag{A.0.15}$$

where $n_C = \sum_{i=1}^{n}(i - \frac{n}{2})\beta_i$.
We also define the *zero hyperplane* Z_b as

$$Z_b = \{\lambda \in \mathfrak{t}^* : \langle \lambda, n_Z \rangle = b\}, \tag{A.0.16}$$

where $n_Z = \sum_{i=1}^{n}(i - \frac{n}{2} - 1)\beta_i$.
Similarly, denote the corresponding halfspaces by

$$C_b^- = \{\lambda \in \mathfrak{t}^* : \langle \lambda, n_C \rangle \leq b\}; \tag{A.0.17}$$

$$Z_b^- = \{\lambda \in \mathfrak{t}^* : \langle \lambda, n_Z \rangle \leq b\}. \tag{A.0.18}$$

Remark A.0.7. It can be verified that if $\lambda, \mu \in \Lambda_V$, and there exists $b \in \mathbb{R}$ such that $\lambda \in C_b$ and $\mu \in Z_b$, then $\mathrm{eff}_\lambda(\mu) = 0$.

We will also need the following lemma:

Lemma A.0.8 ([39] Lemma 4.1)**.** *Let V be a finite dimensional linear representation of* $\mathrm{SL}(2, \mathbb{R})$*. Let*

$$A = \left\{a(t) := \begin{bmatrix} e^t & \\ & e^{-t} \end{bmatrix} : t \in \mathbb{R}\right\}, U = \left\{u := \begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix} : s \in \mathbb{R}\right\}.$$

Express V as the direct sum of eigenspaces with respect to the action of A:

$$V = \bigoplus_{\lambda \in \mathbb{R}} V^\lambda(A), \textit{where } V^\lambda(A) := \{v \in V : a(t)v = e^{\lambda t}v : \forall t \in \mathbb{R}\}.$$

For any $v \in V \backslash \{0\}$ and $\lambda \in \mathbb{R}$, let $v^\lambda = v^\lambda(A)$ denote the $V^\lambda(A)$-component of v, $\lambda^{max}(v) = \max\{\lambda : v^\lambda \neq 0\}$. Then for any $r \neq 0$,

$$\lambda^{max}(u(r)v) \geq -\lambda^{max}(v).$$

Proof. See [39]. □

Now given any subset S of $\mathfrak{t}^*$, we define

$$V^S := \bigoplus_{\lambda \in S \cap \Lambda_V} V^\lambda; \tag{A.0.19}$$

and for any $v \in V$, let v^S denote the V^S-component of v.

The following lemma is our key observation in the proof of Lemma A.0.10.

Lemma A.0.9. *Suppose we define the following half-spaces inductively:*

$$\begin{aligned} Q_0 &:= C_b^-; \\ Q_i &:= r_i(Q_{i-1}), 1 \leq i \leq n. \end{aligned} \tag{A.0.20}$$

with r_i being the reflection defined in (A.0.3). Then

$$Q_n = Z_b^-. \tag{A.0.21}$$

Proof. By direct computation,

$$Q_i = \{\lambda \in \mathfrak{t}^* : \langle \lambda, n_{Q_i} \rangle \leq b\}, \tag{A.0.22}$$

where

$$n_{Q_i} := \sum_{j=1}^{i} (j - \frac{n}{2} - 1)\beta_j + \sum_{j=i+1}^{n} (j - \frac{n}{2})\beta_j. \tag{A.0.23}$$

Take $i = n$, we see that Q_n coincides with Z_b^-. □

Lemma A.0.10 (injectivity)**.** *Let $u_0 = u(1) = \exp X_{\beta_n} \cdots \exp X_{\beta_1}$, and C_b^-, Z_b^- be defined as in Definition A.0.6. Then for any non-zero $v \in V^{C_b^-}$, we have*

$$(u_0 \cdot v)^{Z_b^-} \neq 0. \tag{A.0.24}$$

Proof. We prove by induction on n. The case $n = 2$ is valid by Lemma A.0.8. Now suppose that the lemma is true for $n-1$, we shall prove that it is also true for n. Take any non-zero $v \in V^{C_b^-}$, denote

$$\Lambda(v) := \{\lambda : v^\lambda \neq 0\} \subset C_b^-. \tag{A.0.25}$$

Since the set $\{\beta_i\}_{1 \leq i \leq n}$ forms a $\mathbb{R}$-basis of the character space, it follows that every weight λ can be written uniquely in the form

$$\lambda = \sum_{i=1}^{n} m_i \beta_i. \tag{A.0.26}$$

Take any $\lambda_0 \in \Lambda(v)$ which maximizes the linear functional $\lambda \mapsto \sum_{i=2}^n m_i$. Applying Lemma A.0.8 to the $\mathrm{SL}_2(\beta_1)$-submodule associated with the weight string $\lambda_0 + \mathbb{Z}\beta_1$, and to the canonical projection of v onto this submodule, we could find a weight $\lambda_1 = \lambda_0 + m_1\beta_1$ contained in $Q_1 = r_1(C_b^-)$ such that $(\exp X_{\beta_1} \cdot v)^{\lambda_1} \neq 0$. Due to the choice of λ_0 we can apply the inductive hypothesis to the $\mathrm{SL}_{n-1}(\beta_2, \cdots, \beta_n)$-submodule associated with λ_1, and to the canonical projection of $\exp X_{\beta_1} \cdot v$ onto this submodule. Repeating the above process, in the i-th step we could find a weight $\lambda_i \in Q_i$ such that $(\exp X_{\beta_i} \cdots \exp X_{\beta_1} \cdot v)^{\lambda_i} \neq 0$. Finally, we get $\lambda_n \in Q_n$, and we finish the proof by applying Lemma A.0.9. $\square$

Now we are ready to prove Proposition A.0.1.

Proof of Proposition A.0.1. The conclusion is trivial for $v = 0$. Now suppose $v \neq 0$, we write $v = \sum_\lambda v^\lambda$. Without loss of generality we may assume that $\eta = 1$. Let

$$u_0 := u(1) = \exp X_{\beta_n} \exp X_{\beta_{n-1}} \cdots \exp X_{\beta_1}. \tag{A.0.27}$$

Then for any $\lambda, \mu \in \Lambda_V$, one has the following:

$$(a(s)u(e^{-t})v^\lambda)^\mu = e^{\mathrm{eff}_\lambda(\mu)t}(u_0 v^\lambda)^\mu, \tag{A.0.28}$$

where $\text{eff}_\lambda(\mu)$ is the effective value of μ (see Definition A.0.2).

Now take any λ such that $v^\lambda \neq 0$. By Lemma A.0.5, λ is potentially non-negative, i.e. there exists $\mu \in \Lambda_V$ such that $\text{eff}_\lambda(\mu) \geq 0$.

Now it remains to show that there exist a constant $a \geq 0$ and $\mu \in \Lambda_V$ such that

$$\sum_{\text{eff}_\lambda(\mu)=a} (u_0 v^\lambda)^\mu \neq 0. \tag{A.0.29}$$

Notice that there exist $b \in \mathbb{R}$ such that the set $\{\lambda : \text{eff}_\lambda(\mu) = a\}$ is actually contained in the competing hyperplane C_b (see Definition A.0.6). Since $v^\lambda \neq 0$, it follows that $v^{C_b} \neq 0$. Hence by Lemma A.0.10, there exists $\mu \in Z_b^-$ such that

$$\sum_{\text{eff}_\lambda(\mu)=a} (u_0 v^\lambda)^\mu = (u_0 v^{C_b})^\mu \neq 0. \tag{A.0.30}$$

Combining equation (A.0.28)(A.0.30), we finish the proof of the propositon. $\square$

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