Analogies to Integer Knowledge Facilitate Fraction Learning

Thesis

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Abstract

Fractions are important in mathematics but notoriously difficult to learn. One obstacle for learners is the "whole number bias," in which the whole numbers in fractions (i.e., in numerators and denominators) overshadows fractional magnitudes in numeric reasoning. How to overcome the "whole number bias" and make use of prior whole number knowledge? Here we investigated whether comparison to whole numbers on the number line can facilitate children's learning of fraction magnitudes.

In Experiment 1, 30 children identified a number (integer or fraction) from a mark on a number line or placed a mark on a number line to indicate a target number. Estimates of integers were as much as 2x more accurate than estimates of fractions. Therefore, we hypothesized that comparing fraction to integer scales might improve representations of fractions as much as 2x.

Using a pretest-training-posttest design (Experiment 2), 64 children were trained to estimate and compare positions of fraction and integers on number lines (e.g., 3/8:1::3:8). At post-test, children solved number-line problems either with (Cue group) or without (No Cue group) visually-aligned cues to facilitate source retrieval. Children in the Cue groups did improve their estimates from pretest to post-test as much as 2x, suggesting that children's understanding of fractional magnitudes came from the

comparison to integers. Further, visually-aligned cues activated inert knowledge at post-test.

Together, the two experiments suggest that comparisons of integers and fractions on number lines can reduce the whole number bias and teach fractional magnitudes.

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Chapter 1. Introduction

Mathematical knowledge is important for educational and financial success. Knowledge of algebra and high school mathematics strongly predicts college entrance, college grades, college graduation, early earnings and earnings growth (Murnane, Willett, & Levy, 1995; National Mathematics Advisory Panel, 2008). Knowledge of fraction magnitudes is positively related to overall mathematics achievements both concurrently (Siegler, Thompson, & Schneider, 2011; Siegler & Pyke, 2013) and longitudinally (Siegler et al., 2012; Bailey, Hoard, Nugent, & Geary, 2012; Booth, Newton, & Twiss-Garrity, 2014). In high school mathematics, knowledge of fractions seems particularly important. In both the USA and the UK, fifth graders' knowledge of fractions uniquely predicts their knowledge of algebra and overall mathematics achievement in high school, controlling for integer knowledge, general intellectual ability, working memory, and family income and education (Siegler et al., 2012). Together, these findings suggest that knowledge of fraction magnitudes plays an essential role in mathematics achievement, and thus is of vital importance in education.

Despite its importance, the development of fraction proficiency has been controversial over the past decades. In some theories, development of fraction magnitudes understanding is fundamentally different from – and thus impaired by – development of integer magnitude understanding. One reason this might be the case is that there are many properties of integers that cannot generalize to fractions. Among these properties, integers have unique representations of single Arabic numerals, have unique successors, are countable, never decrease with multiplication, never increase with division, and so on. Thus, knowledge of the properties of integers might distort understanding of fractions, sometimes termed as “the whole number bias” (Ni & Zhou, 2005). More specifically, an early developed and solid understanding of integers can interfere with children’s later developing understanding of fractions.

A somewhat similar claim is made by the privileged domain theories which posit that an innate cognitive system of representing numbers discretely is incommensurate with ordered and continuous fractions (Gallistel & Gelman, 1992; Hartentt & Gelman, 1998). In this view, too, knowledge of integers may serve to distort understanding of fractions (Gelman & Williams, 1998). Similarly, from an evolutionary perspective, Geary (2006) posits that integer representation is biologically primary whereas fraction representation is biologically secondary, and the biological constrains that facilitate integer knowledge can impede fraction understanding. To sum up, both the privileged domain theories and the evolutionary theory suggest that there are qualitative differences between integer and fraction learning, and better understanding of integer magnitude is independent from (and might lead to worse) representations of fraction magnitudes.

In contrast, an alternative theory, the integrated theory of numerical development, emphasizes the commonality between learning integers and fractions (Siegler et al., 2011). This theory posits that learning the magnitude of numbers is a basic process uniting the development of understanding all real numbers, and the development of numerical magnitude can be captured as a progressively broadening set of numbers whose magnitudes can be accurately represented. In their view, although there are many differences between the development of fractions and that of integers, fractions and integers share the developmental continuity that knowledge of magnitude is essential in overall understanding. Fractions, like integers, have magnitudes, and fraction magnitudes, like integer magnitudes, can be placed on a number line as a linear function of their actual value. Siegler and Lotie-Forgues (2014) further propose that the development of knowledge of symbolic numerical magnitude can be depicted as extending mental number line from small whole numbers to larger whole numbers to rational numbers, including fractions, decimals, and integers.

The integrated theory of numerical development proposed a unified perspective of development of rational number magnitudes understanding, but it also raises new question: How do children extend their existing mental number line from integers to fractions? More broadly, how can children develop representations of a potentially infinite number of numbers on a continuous number line when they can only have limited experiences with particular numbers? What are the mechanisms of extending limited knowledge of small whole numbers to magnitudes children rarely (if ever) meet?

Analogies are one mechanism that can bootstrap limited experience far beyond the training space (Case & Okamoto, 1996; Gentner 1983; Gick & Holyoak, 1983). The domain-general ability of analogy – not privileged domains -- is what makes humans so smart relative to other species (Gentner, 2003). Analogical ability enables people to better understand a novel concept based on their related prior knowledge, to extract the structure of relational similarity, to make inferences in novel situations, and to transfer learning across contexts. Analogy also serves as a mechanism of representational change in childhood, making it possible for people to re-organize existing knowledge in light of new information (Opfer & Siegler, 2007). Therefore, analogies from poorly- to well-understood magnitudes might be a general mechanism in numerical development, including development of fraction understanding.

In the current study, we aim to investigate whether analogies to integer magnitude would improve children's understanding of fraction magnitudes, and to determine different cognitive supports that would facilitate this analogical transfer. This issue is important theoretically because the privileged domains approach posits that integer and fraction understanding must develop in parallel, whereas the integrated approach posits a necessary connection between the two. This issue is also important practically because fractions are central to math achievement and because fractions are always taught after integers, thereby requiring an approach that could reduce the whole number bias. In the next sections, we will detail (a) analogy as a mechanism of developmental change across numeric representations, (b) cognitive supports for analogical transfer, and (c) more specific empirical questions that were examined in the present studies.

1. Analogy as a Mechanism of Representational Change

1.1 Development of Representations of Integer Magnitudes

Typically, children's representation of integers undergoes a transition from a logarithmic to a linear function (Siegler & Opfer, 2003; Siegler & Booth, 2004). Children have an early erroneous logarithmic representation of linearly increasing numbers, i.e., estimated magnitudes at the low end of the scale are further apart than estimated magnitudes at the high end, which later turn into a more accurate linear representation. This logarithmic to linear shift happens from kindergarteners to second graders on the scale of 0 to 100, and from second graders to sixth graders on the scale of 0 to 1000. More specifically, most kindergarteners have logarithmic distributions of estimates with numbers from 0 to 100, with which scale most second graders have linear distributions (Siegler & Booth, 2004); most second graders have logarithmic distributions of estimates with numbers from 0 to 1000, with which scale most sixth graders have linear distributions (Siegler & Opfer, 2003). These findings are consistent with the integrated theory of numerical development, showing that the accurate representations of integers on a mental number line gradually extends from small whole numbers to large whole numbers.

Studies have shown that analogy plays an important role in extending children's numerical knowledge of small integers to large integers (Opfer & Siegler, 2007; Thompson & Opfer, 2008; Thompson & Opfer, 2010). Cognitive supports of analogy can dramatically promote the logarithmic to linear shift over a very short period of time (Opfer & Siegler, 2007; Thompson & Opfer, 2010). In a microgenetic study of numerical estimation, researchers found that often after a single trial of feedback of 150 on a 0-1000 number line, second graders abruptly improved their estimates with the entire numerical range from 0 to1000 (Opfer & Siegler, 2007). The underlying mechanism of this abrupt and broad representational change is hypothesized to be analogical mapping from smaller to larger numerical contexts, i.e., 150 is to the 0-1000 range as 15 is to the 0-100 range.

To directly test the hypothesis that analogies to small whole numbers will prompt more accurate linear representations of large whole numbers, Thompson and Opfer (2010) provided second graders opportunities to align number line problems of small, familiar magnitudes (0-100) to number line problems of larger, less familiar numerical scales (0-1,000, 0-10,000, 0-100,000), for instance, the alignment between 15 pears on a 0 to 100 pears line and 150 on a 0 to 1,000 number line. Through this alignment invention, second graders greatly improved their estimates with large numerical scales even far beyond the training space. This research line of studies indicates that analogical mapping from small numerical contexts to larger numerical contexts facilitate extraction of the abstract structure of decimal system, and thus the learning can be generalized to even very large number that children seldom experienced.

1.2 Development of Representations of Fractional Magnitudes

In terms of fraction learning, however, analogies to integers have been shown to distort, instead of facilitating, understanding of fraction magnitudes (DeWolf & Vosniadou, 2015; Opfer & DeVries, 2008; Thompson & Opfer, 2008). In fact, a major difficulty of learning fraction is the whole number bias – the tendency to focus on the whole number components of fractions instead of processing the fractional magnitudes holistically, and the tendency to erroneously generalize properties of integers to fractions (Ni & Zhou, 2005).

Studies have shown that both adults and children are influenced by the whole number bias (Behr, Wachsmuth, Post, & Lesh, 1984; Bonato, Fabbri, Umilta, & Zorzi, 2007; Harnett & Gelman, 1998; Vamvakoussi & Vosniadou, 2010). A distance effect of denominators' values, instead of fraction magnitudes, was observed when adults were asked to compare fractions with same numerators (Bonato et al., 2007). Accuracy and response time for fraction magnitude comparison depend on the comparison's consistency with whole number ordering, i.e., whether the larger fractions are with larger whole number parts. (DeWolf & Vosniadou, 2015). Similarly, children often only compare whole number parts when they are asked to compare two fractions (Behr et al., 1984; Harnett & Gelman, 1998). Even years after learning fractions from school curriculum, some children still fail to understand that some properties of integers do not generalize to fractions, such as, fractions do not have unique successors, and there are infinite number of numbers between two fractions (Vamvakoussi & Vosniadou, 2010).

More strikingly, research has shown that better representations of integers might impede representations of fractions. Opfer & DeVries (2008) presented adults and second graders with money lines indicating different values of salaries (e.g., from $1/minute to $1/1440 minutes) and asked participants to estimate a target salary on this money line (e.g., $1/60 minutes). People tended to use only the denominator's value, instead of the fraction magnitude, as a guide for placing their estimates. Interestingly, as children typically have logarithmic representations of integer ranging from 0 to 1000 (Siegler & Opfer, 2003), whereas adults have more accurate linear representations of large integer numbers, children's less accurate representations of the denominator's value favor this fractional scale where fraction magnitudes increase as a power function of denominator's value. Thus, second graders provided more accurate estimates with fraction magnitudes of common numerators than adults. Consistent with this result, when second graders were trained to form more linear representations of integers, their performance on the following fraction number line tasks became worse, indicating that children's retrieval of wrong analogical sources caused a negative transfer for estimates of fractions (Thompson & Opfer, 2008). Thus, consistent with privileged domains and evolutionary perspectives, a better understanding of integer magnitudes has been shown to lead to worse understanding of fraction magnitudes.

On the other hand, there is also evidence suggesting that analogies to integer magnitudes might have a positive transfer to fraction understanding (Kalchman, Moss, & Case, 2001; Moss & Case, 1999; Siegler et al., 2011). Siegler et al. (2011) argue that analogies to integers do not necessarily interfere with fraction learning – it is the incorrect analogies to integers that interfere with fraction learning. Correct analogies to integers, such as thinking of fractions as magnitudes that can be placed on a number line just like integers, may help children have better representations of fractions.

Supporting this claim indirectly, evidence from neuroscience (Ischebeck, Schocke, & Delazer, 2009), correlational studies (Bailey, Siegler, & Geary, 2014) and empirical studies (Kalchman et al., 2001; Moss & Case, 1999; Siegler et al., 2011) indicates that knowledge of integer might have a positive transfer to fraction understanding. Biologically, neuroscience evidence shows that brain areas associated with fraction magnitude representations overlap with those associated with integer magnitude representations (Ischebeck et al., 2009), providing potential biological mechanisms of analogical transfer from integers to fractions. Longitudinally, integer magnitude knowledge in the first grade predicts knowledge of fraction magnitude in middle school, controlling for integer arithmetic proficiency, domain general cognitive skills, family income, parental education, race, and gender (Bailey et al., 2014).

Empirically, rational number curriculum that emphasizes the relation between fraction and integer magnitudes has shown to be more effective than traditional curriculum (Moss & Case, 1999; Kalchman et al., 2001). To give fourth graders a familiar analogical source for understanding fractions, researchers began by teaching percentages, which shared more similarities to children's prior knowledge of integers than fractions. Throughout twenty forty-minute lessons in a five-month period, students were exposed to different learning activities emphasizing the equivalence of percentages, decimals and fraction magnitudes. Among these activities, students were encouraged to estimate percentages of various quantities, map decimals with time magnitudes using stopwatches, place decimals on a number line, and translate among representations of percentages, decimals and fractions. When tested after this curriculum, fraction knowledge of fourth graders was better that of eighth grader who received a traditional curriculum, and was as good as a group of preservice teachers.

Moss and Case (1999)'s rational number curriculum has shown to be promising in improving children's fraction understanding. However, the independent role of providing a familiar analogical source could not be determined due to the complexity of their training, which involved board games with decimals, lessons in translating among percentages, decimals, and fractions, lessons in fraction arithmetic, and so on. A more rigorously designed study might be helpful to further confirm the effect of familiar analogical sources of integers for fraction learning.

2. Cognitive Supports to Facilitate Analogical Transfer

The mixed results of both harmful and beneficial effects from analogies to integer on fraction understanding might be due to the difficulties in correct analogical inference. Although analogical transfer can lead to considerable insights when it occurs, in many cases it also fails to occur among both children and adults (Catrambone & Holyoak, 1989; Gick & Holyoak, 1980; Novick & Holyoak, 1991; Novick, 1988; Opfer & DeVries, 2008; Reed, Dempster, & Ettinger, 1985; Richland, Morrison, & Holyoak, 2006; Thompson & Opfer, 2008). Fortunately, there are also ways to facilitate analogical inference, which might also encourage positive transfer from integer knowledge to fraction magnitudes understanding.

One way to boost analogies is to compare analogical sources and targets. According to structure-mapping theory (Gentner, 1983; Gentner & Markman, 1997), the process of comparison can help people extract the maximal structural similarity between two representations (Falkenhainer, Forbus, & Gentner, 1989; Wolff & Gentner, 2000). Consistent with the structural-mapping theory, cross-cultural studies indicate that mathematics teachers in East Asian classes, where students outperform their U.S. peers in international mathematics tests, provide more cognitive supports for analogical inferences than teachers in U.S classrooms (Richland, Zur, & Holyoak, 2007). Teachers in East Asian schools are more likely to visually present familiar source analog along with the target being taught, so that students are encouraged to compare the source and the target.

To determine the effect of structural alignment in understanding fractions by analogies, Opfer et al. (2017) designed a simplified version of Moss and Case (1999)'s curriculum. Using a pretest-training-posttest design, experimenters provided fourth and fifth graders alignment between fractions and one of the three sources of analogical bases: integers (e.g., 3:8::3/8:1), percentages (e.g., 37.5%:100%::3/8:1), fractions (e.g., 3/8:1:3/8:1), or no analogical bases as a control group. Students estimated the magnitudes of the analogical bases and fractions on vertically aligned number lines, so that they could have a chance to draw analogies between the base problem and the fraction problem. Results showed that aligning target fraction problems with analogical sources greatly increased accuracy of estimates. However, effect of alignment was short-lived for number line problems. The performance of children in the analogical sources group plummeted to the same level of that of children in the control group at post-test, indicating that additional support might be necessary to for children to spontaneously remember the analogical sources in the future.

Another way to facilitate analogical transfer is to provide source retrieval cues (Gick & Holyoak, 1980; Gick & Holyoak, 1983; Novick & Holyoak, 1991). In the primary study of Gick and Holyoak (1980), experimenters first presented people with a source story disguised as a story recall task. After completing this story recall task, participants were then asked to solve a structurally similar target story. Without any cues, only about 30% of participants successfully solved the target story. However, after a simple nonspecific cue that the second story can be solved with the help of the previous story, the percentage of people who successfully solved the target story strikingly increased to 75%, suggesting that successful mapping between source and target itself might be insufficient for successful analogy transfer – reviving inert knowledge and retrieving the correct prior knowledge is also essential to analogy inference.

In this thesis, we presented a further investigation of different cognitive supports towards analogical inference from integer to fraction magnitudes. To determine the independent role of analogies to integers for fraction learning, we applied a simplified version of Moss and Case (1999)'s real number curriculum, same as the training program with analogical sources of integers in Opfer et al. (2017). During training, children were presented aligning structure of integers and fraction magnitudes on different scales. From the perspective of structure mapping theory, this cognitive support might be beneficial to improve fourth and fifth graders mapping of fraction magnitudes to better understood integers.

Our study is unique from Opfer et al. (2017) in two important ways, (a) to determine cognitive supports of retrieval cues to the analogy base, at post-test we provided a group of children with cues to remind them the magnitudes of integers. If children's failure of analogical transfer is due to failure of remembering and retrieving the correct analogical base, then they might revive and transfer their inert knowledge of integer magnitude with retrieval cues; (b) we reduced the interval between pretest and posttest from about one semester to about one week, with an aim of minimizing the effect of school curriculum.

3. The Current Study

In the current study, the central purpose was to examine and compare children's estimates with integers and fractions (Experiment 1), and to determine whether aligning structure of integer and fraction scales, as well as retrieval cues to remember the analogical sources, will prompt children to have a better understanding of fractional magnitudes (Experiment 2).

The purpose of Experiment 1 was to examine an important premise of improving children's fraction understanding by analogical sources of integers: whether children's representation of integer is more accurate than fractional magnitudes. One strategy of estimating fractions is to transform the problem to an integer estimation problem on a different scale. For example, to estimate 3/8 on a 0-1 number line, one can enlarge the scale of the number line by 8 times and then estimate 3 on a 0-8 number line. If children's estimates with fraction scales are worse than their estimates with integer scales, then analogies to integers could potentially facilitate children's estimates with fractions. Otherwise, there is not much room left for children to benefit from analogical sources of integers.

Previous studies have shown that estimates of fractional magnitudes overall are less accurate than integer magnitudes (Iuculano & Butterworth, 2011; Fazio, Bailey, Thompson, & Siegler, 2014). Our task was designed slightly differently from previous studies to directly examine whether children's fraction magnitudes representation would benefit from analogies to integer representation. In previous studies, researchers investigated integer representations on fixed scales (e.g., 0-100 in Iuculano & Butterworth, 2011; 0-1000 in Fazio et al., 2014), whereas in the current one, the right ending point for integer estimation tasks varied to correspond to the value of the denominator of the fraction magnitude. Therefore, we directly examined whether a representational change from fraction to integer scales (e.g., 3/8 stands in relation to 1 as the same ratio as the relation between 3 and 8) might lead to potential benefits for children's understanding of fraction magnitudes.

In Experiment 2, we attempted to (a) determine whether structural alignment of integer and fractions would facilitate children's understanding of fractions; (b) determine whether source retrieval cues can revive children's inert knowledge and foster their analogical transfer in the context of estimating fraction magnitudes on number lines. To meet our goals for Experiment 2, sixty-four fourth and fifth graders were presented opportunities to compare and draw analogies between fraction and integer magnitudes. To test the effect of source retrieval cues, at post-test children either received source retrieval cues of integer magnitude (Cue group) or did not receive the cues (No Cue group). We hypothesized that (a) comparisons of integers and fractions on number lines can bootstrap fractional magnitudes understanding beyond the training space, and (b) cues of familiar sources would support analogical transfer in fraction learning.

Chapter 2. Experiment 1

The premise of teaching fractions by highlighting structural similarities of integers and fractions (e.g., 3/8:1::3:8) is that children are better at estimates with integers than fractions. To examine this premise, in Experiment 1, we investigated children's estimates with fractions (e.g., 3/8 on a 0 to 1 number line) and integers (e.g., 3 on a 0 to 8 number line) on equivalent number lines.

1. Method

1.1 Participants

Thirty 3rd, 4th, and 5th graders participated in the study (3rd graders: 3 boys and 4 girls, M = 9.50 years, SD = .49; 4th graders: 8 boys and 5 girls, M = 10.08 years, SD = .55; 5th graders: 5 boys and 5 girls, M = 10.99 years, SD = .54).

1.2 Materials and Procedure

Participants completed 4 different number line estimation tasks in a 2 (format: integers, fractions) * 2 (task: Position-to-Number, Number-to-Position) fully-crossed design (Figure 1). The order of tasks was determined by a balanced Latin Square. The tasks were presented on a laptop and programmed by MATLAB.

In the Position-to-Number (PN) task, in each trial children were presented a number line with two endpoints. There was also a vertical hatch mark somewhere on the line, and children were asked to estimate what number goes with the mark. Children would type their answer in boxes and click ‘Next’ to continue. In the Number-to-Position (NP) task, children were presented a number line and were asked to estimate a given number by dragging a vertical hatch mark on the line in each trial. They would click ‘Next’ to see the next trial after making their decisions.

For fraction estimation tasks, the number lines ranged from 0 to 1. The numbers to be estimated were 1/11, 1/7, 1/4, 3/8, 2/5, 4/7, 2/3, 7/9, 5/6, and 9/10, consisting of a total of 10 trials in PN and NP tasks respectively. Two of these fractions were drawn from each fifth of the number line. For integer estimation tasks, the numbers to be estimated were designed based on the magnitudes in the fraction tasks, in order to examine children’s estimates of fraction and integers on equivalent number lines. More specifically, for each magnitude tested in fraction estimation tasks (e.g., 3/8), the number line would range from 0 to the denominator of the fraction (e.g., 8) in the corresponding integer estimation trial and the to be estimated number would be the numerator of the fraction (e.g., 3).

There was a total of 40 trials in all 4 tasks. The order of items within each task was randomized. It took approximately 10-15 minutes for children to complete all the tasks.

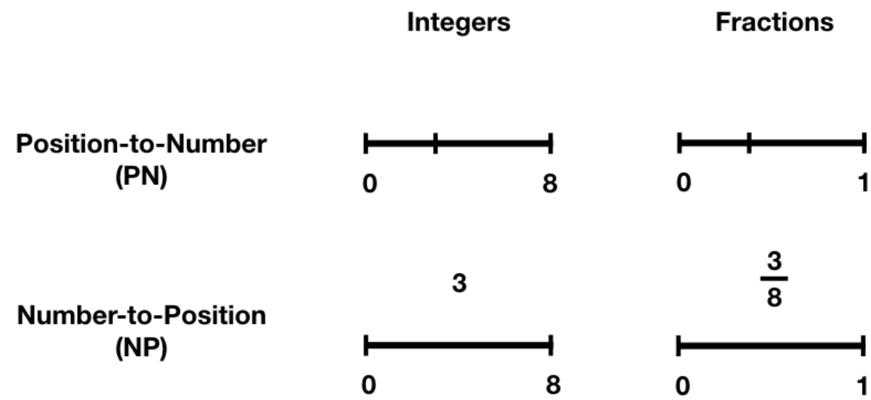

Figure 1. Illustrations of number line tasks in Experiment 1.

2. Results

To assess accuracy, we first calculated percent absolute error (PAE), which equals the difference between the child's estimate and the correct answer, divided by the total numerical range. For example, if a child answers 2 for estimating 3 on a 0 to 8 number line, then the child's PAE for this item is $| 2 - 3 | / 8 * 100\% = 12.5\%$. Thus, lower PAEs correspond to more accurate estimates. In some cases, it is impossible to estimate the answer exactly correct, for example, in NP tasks where children need to estimate a position on a continuous line. Therefore, in our analysis, an estimate was considered correct (i.e., accuracy equals 1) if the estimate was within 2.5% of the correct answer (i.e., PAE is equal to or smaller than 2.5%). One child missed one trial on Fraction PN tasks. Responses with 0 as value of denominator were excluded from the analysis (0.8% of trials).

Effects of gender, grade (3^{rd}, 4^{th}, 5^{th}), task (NP, PN), and format (integers, fractions) on PAE were accessed with mixed-linear models using the lme4 package in the R environment (Bates, Maechler, Bolker, & Walker, 2015; Bates & Sarkar, 2006).

Participants were denoted as a random effect to control for their associated intraclass correlations (Pinheiro & Bates, 2000). Similarly, children's accuracy was assessed by a generalized mixed linear model with a logit link, with effects of gender, grade (3^{rd}, 4^{th}, 5^{th}), task (NP, PN), and format (integers, fractions) as fixed effects and participants as random effects.

Children's PAE did not differ across gender, $\beta = .01$, $SE = .05$, $p = .850$. PAE improved (decreased) from third grade (46%) to fourth grade (14%), $\beta = .32$, $SE = .13$, $p < .05$, and third grade to fifth grade (8%), $\beta = .39$, $SE = .14$, $p < .01$, but was not different from fourth grade to fifth grade, $\beta = .07$, $SE = .12$, $p = .576$. As expected, PAE was lower for integers (8%) than fractions (32%) in general, $\beta = .12$, $SE = .02$, $p < .001$. PAE was also observed to be lower for NP tasks (10%) than PN tasks (30%), $\beta = .10$, $SE = .02$, $p < .001$. However, the differences between estimates with integers and fractions were larger in PN tasks than NP tasks, $\beta = .10$, $SE = .02$, $p < .001$. On PN tasks, estimates with integers (8%) were more accurate than estimates with fractions (61%), $\beta = .22$, $SE = .03$, $p < .001$; whereas on NP tasks, PAE for estimates with integers (9%) were not significantly different from estimates with fractions (12%), $\beta = .01$, $SE = .02$, $p = .555$ (Figure 2A).

The pattern of accuracy was similar to that of PAE. Children's accuracy was not significantly different across gender, $\beta = .17$, $SE = .17$, $p = .308$. Accuracy improved from third grade (26%) to fourth grade (43%), $\beta = .95$, $SE = .44$, $p < .05$, and from third grade to fifth grade (47%), $\beta = 1.22$, $SE = .46$, $p < .01$, but not from fourth grade to fifth grade, $\beta = .27$, $SE = .38$, $p = .483$. As expected, accuracy was higher for integers (52%)

than fractions (28%), $\beta = .59$, $SE = .07$, $p < .001$. Accuracy was higher for PN tasks (49%) than NP tasks (32%), $\beta = .41$, $SE = .07$, $p < .001$. However, the differences between estimates with integers and fractions were larger in PN tasks than NP tasks, $\beta = .51$, $SE = .07$, $p < .001$. On PN tasks, estimates with integers (71%) were more accurate than estimates with fractions (26%), $\beta = 1.11$, $SE = .10$, $p < .001$; whereas in NP tasks, estimates with integers (33%) were not significantly different from estimates with fractions (30%), $\beta = .08$, $SE = .09$, $p = .400$ (Figure 2B).

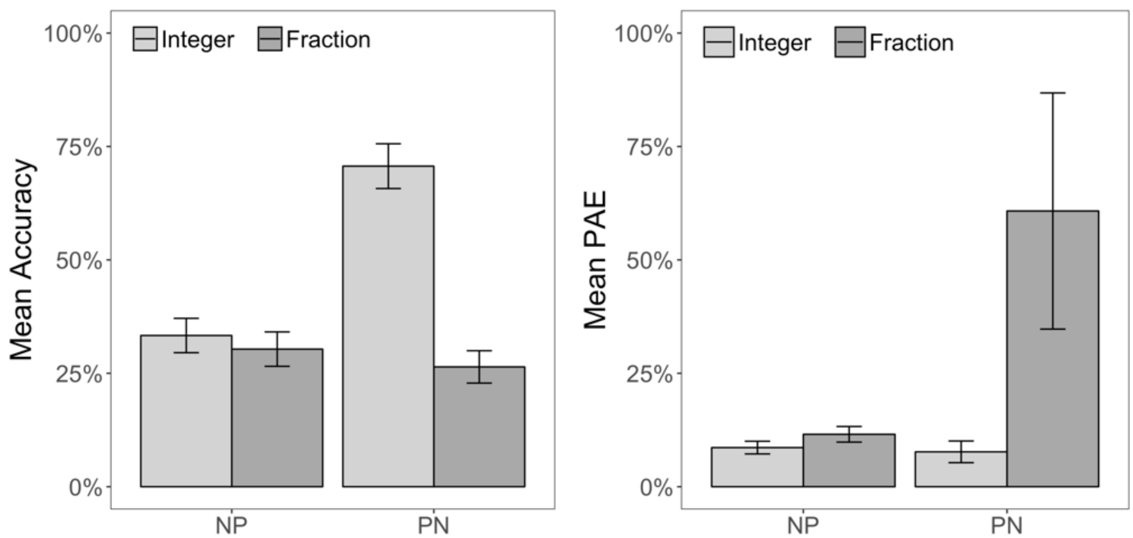

Figure 2. Average PAE (Panel A) and accuracy (Panel B) for estimates across different tasks (PN, NP) and formats (integers, fractions) in Experiment 1. Error bars indicate standard errors.

3. Discussion

In Experiment 1, we investigated children's estimates with integers and fractions when they identified (PN) or placed (NP) a number on a number line. Results revealed a general trend of more accurate estimates with integers than fractions, though this trend seems to be task dependent.

Not surprisingly, there was a huge advantage of integers (PAE: 8%, accuracy: 71%) over fractions (PAE: 61%, accuracy: 26%) in identifying numbers, which is consistent with previous studies (Iuculano & Butterworth, 2011); surprisingly, children's estimates with fractions (PAE: 12%, accuracy: 30%) were as good as their estimates with integers (PAE: 9%, accuracy: 33%) when they were asked to place a number on a number line, which is inconsistent with previous studies (Iuculano & Butterworth, 2011; Fazio et al., 2014). The inconsistent results from previous studies in our NP tasks might be accounted for by relatively accurate overall representations of numbers of our participants. Compared to Fazio et al. (2014) where average PAEs equals 12% for integer estimates, and 20% for fraction estimates, in our study, PAEs in NP were lower for both formats (integers: PAE = 9%; fractions: PAE = 12%).

The discrepancy between PN and NP has been observed in previous studies (Iuculano & Butterworth, 2011). Iuculano & Butterworth (2011) provided evidence that, contrary to representation of integers, representation of fraction magnitudes seems to be "task dependent", namely, both adults and children had a linear representation of fractions on NP tasks, but not PN tasks. It is worth noting that although both tasks target on participants' representations of numbers, there are also essential differences between two tasks. For PN tasks, since children were asked to generate a number according to a position on a number line, children can generate any number including numbers that are beyond the range of two ending points. Therefore, errors are unbounded for PN tasks. What is more, PN tasks also require children's knowledge of writing a fraction, e.g., denominators are above the line and numerators are below the line. For NP tasks, however, children only needed to mark somewhere on the line to estimate a target number. In this case, children's responses were bounded between the two ending points of the line. The larger discrepancy between estimates with integers and fractions in PN tasks might denote the more possibilities of making mistakes in generating a fraction than placing a fraction. Consistent with this claim, children had larger average PAEs with fractions on PN (61%) than on NP (12%) tasks.

According to the results from Experiment 1, we hypothesized that children would benefit from analogical resources of integers in estimating fractions on PN. Among the 4 different tasks, performance on PN task with integers was the best on average, with lowest PAE (8%) and highest accuracy (71%). Thus, integer PN task might be a good source analog for teaching fractional magnitudes. In Experiment 2, we designed a progressive alignment training program to help children align fraction magnitudes to integers on equivalent number lines, starting with integer PN task.

Chapter 3. Experiment 2

In Experiment 2, we intended to explore (a) effects of aligning structures between integers and fraction magnitudes, and (b) effects of retrieval cues on children's analogical transfer in the context of estimating fractions on a number line, using a pretest-training-posttest design. First, we tested children's estimates with fractions in both PN and NP tasks at pretest. Then we trained children to map fraction magnitude to familiar analogs, i.e., integers, in a progressive alignment way. The training program was adapted from Opfer et al. (2017). Immediately after training, we again examined children's estimates with fractions either with (Cue group) or without (No Cue group) retrieval cues of magnitudes of integers.

1. Method

1.1 Participants

Sixty-four 4^{th} graders and 5^{th} graders participated in the study. (4^{th} graders: 15 boys and 25 girls, *M* = 10.02 years, *SD* = .50; 5^{th} graders: 11 boys and 13 girls, *M* = 10.93 years, *SD* = .52). They were recruited from elementary schools in the Midwestern United States.

1.2 Materials

Pretest and post-test measures of fractional representations.

Fraction Position-to-Number (FPN) Task: The FPN task is identical to the FPN task we used in Experiment 1. In each trial, children were presented a number line from 0 to 1 on a computer screen. There was also a vertical hatch mark somewhere on the line, and children were asked to estimate what number went with the mark. Children would type their answer in a box and click 'Next' to continue. The numbers to be estimated were 1/11, 1/7, 1/4, 3/8, 2/5, 4/7, 2/3, 7/9, 5/6, and 9/10, consisting of a total of 10 trials. Two of these fractions were drawn from each fifth of the number line 0 – 1.

Fraction Number-to-Position (FNP) Task: The FNP task is almost identical to what we used in Experiment 1, except that we extended the sets of fractions to be estimated. In each trial, children were presented a 0-1 number line and were asked to estimate a given number by dragging a vertical hatch mark on the line. They would click 'Next' to see the next trial after making their decisions. The numbers to be estimated were 1/11, 1/7, 1/4, 3/8, 2/5, 4/7, 2/3, 7/9, 5/6, 9/10, 2/22, 2/14, 6/16, 4/10, 8/14, 4/6, 14/18, 10/12 and 18/20, consisting of a total of 20 trials. Ten of those fractions that were overlapped with PN tasks (i.e., 1/11, 1/7, 1/4, 3/8, 2/5, 4/7, 2/3, 7/9, 5/6, 9/10) were also covered in training, i.e., trained items. The other ten magnitudes (i.e., 2/22, 2/14, 6/16, 4/10, 8/14, 4/6, 14/18, 10/12) are improper fractions of the same magnitudes and they were not covered in training. Thus, we refer them in the following analysis as untrained items.

Training Program.

During the training program, children were provided analogical sources of integers in alignment with fraction problems (3:8::3/8:1). First, children received conceptual instructions of integer number lines (Figure 3). Children were shown a tall rectangle partly filled with water and asked, ‘How much of the glass is filled in? Some? A little bit? Less than half? We can be more Exact. To be exact, we can use numbers on a measuring glass.’ Then children were presented two vertical number lines aligned with the rectangle and they were told, ‘Numbers on a measuring glass are like numbers on a number line. Numbers on a measuring glass tell us how many milliliters there are. 0 is none. 100 is all. 1 is almost nothing. 99 is almost everything. 50 is half. With numbers on the measuring glass, we can be more exact than with ordinary words, like ‘some’, ‘a little’, or ‘less than half’, so we can say the glass is 25 milliliters full”.

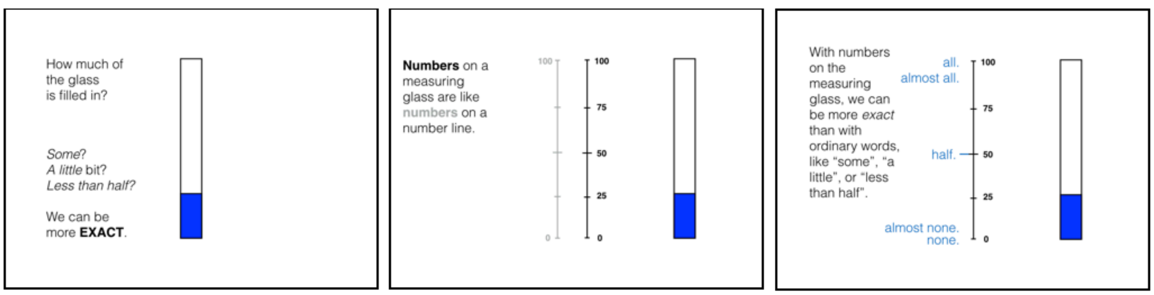

Figure 3. An illustration of conceptual instructions of integer number lines.

The numbers trained were: 1/11, 1/7, 1/4, 3/8, 2/5, 4/7, 2/3, 5/6, 7/9 and 9/10, which were all included in the testing battery at pretest and post-test. For each magnitude, children learned to map integer PN (IPN) to integer NP (INP), integer NP (INP) to fraction NP (FNP), fraction NP (FNP) to fraction PN (FPN), and fraction PN (FPN) to integer PN (IPN), consisting of 8 number line estimation trials in total (see Figure 4 for an illustration of the training program). For each mapping, children were first presented a base problem on the top part of the screen (i. e., problems without alignment). They were given feedbacks and the correct answer after they solved the base problem. Their answer was considered correct if it was within +/- 2.5% of the locations of correct answer, the same criterion as in Experiment 1. And then, participants were presented a target problem at the bottom part of the screen (i. e., problems with alignment) while they could still see the base problem and the correct answer for it. They also received feedbacks after they finished the target problem so that they were able to see the alignment between two problems. On some trials, participants were also asked to categorize the answer. The sequence of mapping was designed to start from relatively easy questions (i.e., integer problems) and gradually become more difficult (i.e., fraction problems).

For example, for magnitude 3/8, first children solved an integer PN problem at the top part of the screen and they were given corrective feedbacks. A green check mark would appear if their answer was within +/-2.5% of the correct answer (i.e., PAE was lower than or equal to 2.5%). After finishing the first problem, children would click next and then the second problem, an integer NP showed up at the bottom part of the screen while the first problem and its correct answer remained on the screen. Children also received feedbacks and correct answer for this second problem. These two problems were vertically aligned so that children can compare and see the alignment between them.

The screen cleared and then a third problem, the same integer NP problem as the second problem appeared on the top part of the screen to examine children's performance on this question without alignment, and whether they could retrieve the magnitude right after solving it and receiving feedbacks. After children received feedbacks on this problem, they were asked to categorize the answer as "Almost Nothing", "Some", "Around Half", "Most", or "Almost Everything" out of the whole range (e.g., "How much is this number if 8 is everything?"). The choices were horizontally arranged bellow the number line. Children clicked a choice to response and did not receive feedback on categorization questions. The categorization tasks were only presented when children needed to map between integers and fractions, in order to examine children's belief of relations between the target number and the ending points across equivalent number lines, e.g., 3/8 to 1 is the same as 3 to 8.

After that, a fraction NP problem (i.e., the fourth problem) appeared at the bottom part of the screen, vertically aligned with the integer NP problem, so that children had an opportunity to draw an analogy that a fraction stands in relation to 1 as being the same ratio as the relation between its numerator and denominator. After receiving feedbacks, children were asked to categorize the answer (e.g., "How much is this number if 1 is everything?").

The screen cleared and the fifth problem, the same fraction NP problem as the fourth problem, was presented. The sixth problem was a fraction PN problem, vertically aligned with the fifth problem. Again, the screen cleared and children answered another fraction PN problem and were asked to categorize the number. Finally, the eighth problem, an integer PN problem, was aligned with the seventh problem, and children also had to categorize the number.

Then children were asked to estimate the next largest number across 8 problems. The magnitude of trained numbers gradually increased from smallest to largest to give children a sense of the relative magnitude among trained numbers. There was a total of

80 number line estimation problems. The training program was self-paced and it took approximately 30 minutes for children to complete the program.

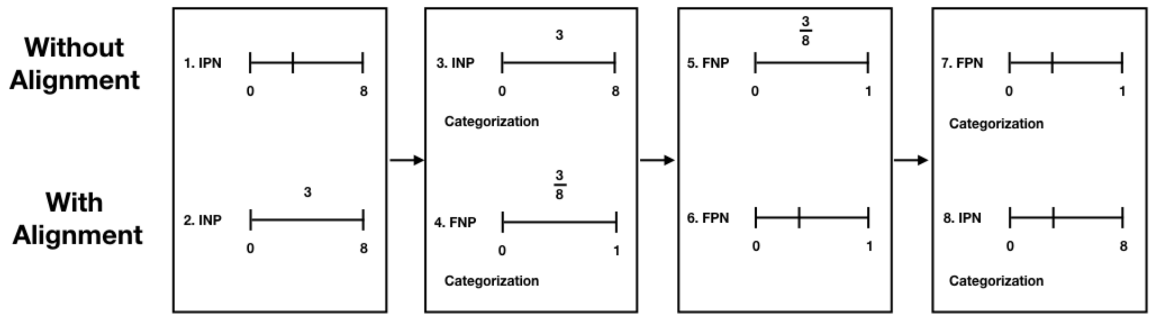

Figure 4. An illustration of stimuli in the training procedure. For each magnitude, children solved 8 problems on 4 screens. On each screen, the problems showed up one by one. Children could see the top problem (the problem without alignment) when they solved the bottom problem (the problem with alignment).

Table 1. Fraction problems across different test phases and conditions in Experiment 2.

<table>
<tr><th></th><th>Pretest</th><th colspan="2">Training</th><th>Post-test</th></tr>
<tr><td></td><td></td><td>With alignment</td><td>Without alignment</td><td></td></tr>
<tr><td>Cue</td><td rowspan="2">NP (20 items),
PN (10 items)</td><td rowspan="2">NP (10 items),
PN (10 items)</td><td rowspan="2">NP (10 items),
PN (10 items)</td><td>NP with cues (20 items),
PN with cues (10 items)</td></tr>
<tr><td>No Cue</td><td>NP (20 items),
PN (10 items)</td></tr>
</table>

1.3 Procedure

Two sessions were conducted on separate days, with an interval of 7.23 days (*SD* = 4.23). In Session 1, children completed pretest battery, including 10 trials of PN tasks and 20 trials of NP tasks. The order of two tasks was counterbalanced. In Session 2, children completed training and post-test. The post-test took place immediately after training. To test the effect of retrieval cues on transfer from training to post-test problems, at post-test, children were randomly assigned to one of two groups: a Cue group ($N = 33$) or a No Cue group (the control group, $N = 31$). Both groups of children received the same pretest and training, and the only difference between two groups was whether there were retrieval cues for analogical sources at post-test (Table 1).

At post-test, children in the Cue group were asked to solve problems in the post-test testing battery (10 PN problems and 20 NP problems) in a randomized order. The problems were the same as pretest battery. Children were also instructed to think about how they solved the problem in training before solving the problems (e.g., 'Now you will practice some problems like you did before, but you won't get any feedbacks about whether your answer was right or wrong. Just do your best. Also, before you solve each problem, think about how you solved the problem in the last game.'). In each trial, they would see two problems vertically aligned on the screen (Figure 5). The bottom problem was a fraction number line estimation problem from testing batteries, and the top problem was the corresponding integer estimation problem serving as a cue. Children were asked to only solve the bottom problem, but to think about how they solved the top problem before making their decision.

In the No Cue group, the procedure was the same with that of the Cue group, except that children were not presented any cues nor instructed to think about how they solved the problems in their last game.

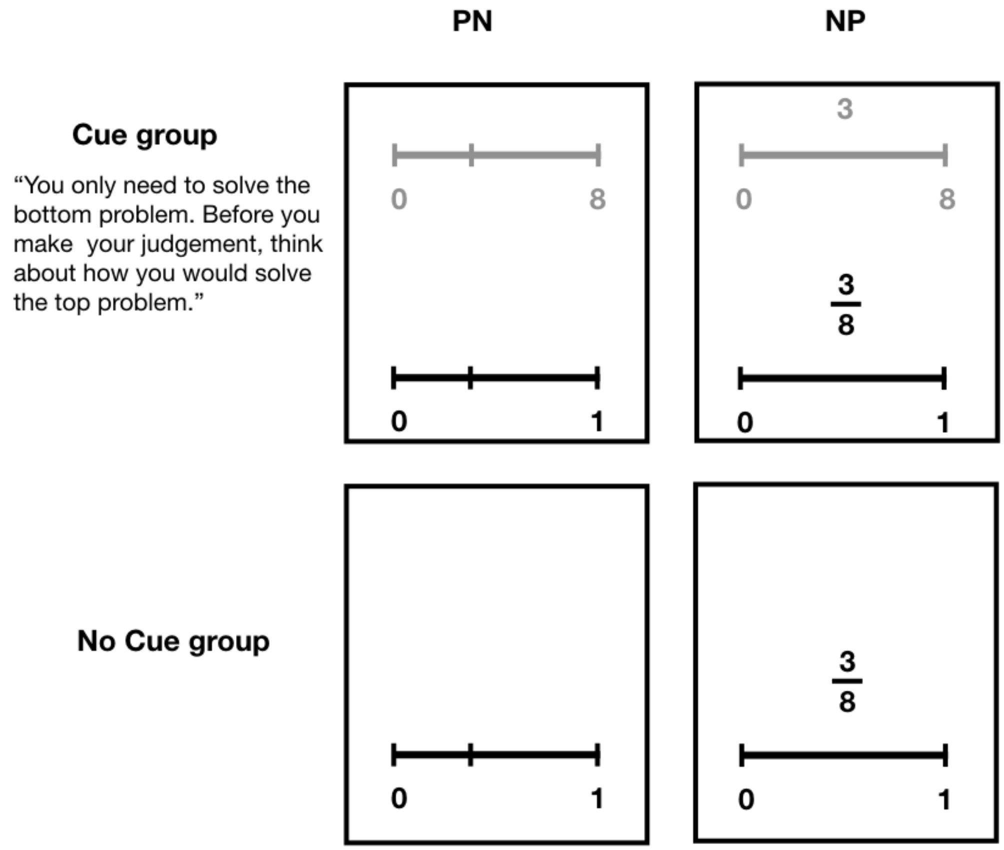

Figure 5. An illustration of two conditions (i.e., Cue and No Cue) at post-test in Experiment 2.

2. Results

Same as Experiment 1, Children's performance was measured as percent absolute error (PAE): | child's estimate – correct answer| / numerical range * 100%. In addition, children's response was considered correct (i.e., accuracy = 1) if PAE is equal to or smaller than 2.5%, which was the same criterion with what we used to give feedbacks in training and what we used in Experiment 1. Responses with 0 as value of denominator were excluded from the analysis (0.02% of trials).

In the following sessions, we will first analyze children's number line estimates with fraction magnitudes that were trained and tested across 4 test phases (pretest, training with alignment, training without alignment, and post-test), in order to investigate a full trajectory of learning. Two questions we aimed to answer are (a) whether children were capable of using alignment to map fractions to integers, and (b) whether children benefited from retrieval cues of familiar analogical sources. Next, we will compare children's estimates with trained and untrained fraction magnitude, to investigate where children were able to generalize analogical inferences to novel problems.

2.1 Trained items

Effects of gender, grade (4^{th}, 5^{th}), test phases (pretest, training with alignment, training without alignment, post-test), and condition (Cue group, No Cue group) on PAE were accessed with mixed-linear models using the lme4 package in the R environment (Bates et al., 2015; Bates & Sarkar, 2006). This model included by-subject random intercept and by-subject random slopes for test phases. Similarly, children's accuracy was assessed by a generalized mixed linear model with a logit link, with effects of gender, grade (4^{th}, 5^{th}), test phases (pretest, training with alignment, training without alignment, post-test), and condition (Cue group, No Cue group) as fixed effects, and by-subject random intercept and by-subject random slopes for test phases as random effects.

In both tasks, children's performance did not differ across gender (PN: PAE, β = .00, SE = .02, p = .840, accuracy, β = .15, SE = .15, p = .308; NP: PAE, β = .00, SE = .00, p = .872, accuracy, β = .04, SE = .07, p = .582), or grade (PN: PAE, β = .01, SE = .02, p =

.491, accuracy, $\beta = .28$, $SE = .17$, $p = .096$; NP: PAE, $\beta = .00$, $SE = .00$, $p = .411$, accuracy, $\beta = .07$, $SE = .08$, $p = .345$).

In the following sections, we will report children's number line estimation during pretest, training with alignment, training without alignment, and at post-test to determine whether children were capable of using an analogy to integers to improve their representations of fraction magnitude, and also whether retrieval cues would facilitate analogical transfer.

Pretest. First, we examined children's number line estimation at pretest to make sure that children in both groups (Cue group and No Cue group) were equally good at fractions estimation before training (Figure 6). As expected, on both tasks, there were no differences in accuracy for estimates between two groups (PN: PAE, $\beta = .07$, $SE = .13$, $p = .556$, accuracy, $\beta = .03$, $SE = .19$, $p = .876$; NP: PAE, $\beta = .01$, $SE = .02$, $p = .443$; accuracy, $\beta = .08$, $SE = .18$, $p = .656$).

Alignment. To test the effect of alignment, we examined the accuracy of the training problems with alignment. On both tasks, there were significant gains from pretest to training problems with alignment (PN: PAE, $\beta = .51$, $SE = .12$, $p < .001$, accuracy, $\beta = 6.00$, $SE = .53$, $p < .001$; NP: PAE, $\beta = .15$, $SE = .02$, $p < .001$, accuracy, $\beta = 5.00$, $SE = .37$, $p < .001$).

No Alignment. To investigate whether the effect of alignment can generalize to the same problem without alignment, we examined the accuracy of the training problems without alignment. On both tasks, there were significant gains from pretest to training without alignment problems (PN: PAE, $\beta = .49$, $SE = .12$, $p < .001$, accuracy, $\beta = 4.90$, *SE* = .42, $p < .001$; NP: PAE, $\beta = .13$, *SE* = .02, $p < .001$, accuracy, $\beta = 1.75$, *SE* = .20, $p < .001$). We then compared estimates of problems without alignment and problems with alignment to determine whether children were able to remember the aligned magnitude immediately after estimations and feedbacks, since each problem without alignment were presented immediately after the same problem with alignment. We observed that generalization of alignment to the same problem immediately after seeing the alignment was task dependent. On PN tasks, estimates of training without alignment problems were not different from training with alignment in both PAE and accuracy (PAE, $\beta = .01$, *SE* = .02, $p = .577$, accuracy, $\beta = .69$, *SE* = .44, $p = .118$), suggesting that, shortly after training, the effect of alignment generalized to problems without alignment. On NP tasks, however, estimates of training without alignment problems were less accurate than with alignment, even if they just solved the same problem with alignment seconds ago (PAE, $\beta = .01$, *SE* = .00, $p < .01$, accuracy, $\beta = 1.62$, *SE* = .19, $p < .001$). The discrepancy between PN and NP tasks also suggested that retrieving visuo-spatial information in working memory is more difficult than retrieving semantic information, such as symbolic numbers.

Post-test. To examine whether children learned to use the analogy to integers at post-test when the analogical bases were not presented, we examined accuracy for estimates of children in the No Cue group. Children's estimates improved from pretest to post-test on most measures in both tasks (PN: Accuracy, $\beta = .81$, *SE* = .23, $p < .001$; NP: PAE, $\beta = .10$, *SE* = .02, $p < .001$; accuracy, $\beta = .07$, *SE* = .02, $p < .01$), except PAE on PN task ($\beta = .25$, *SE* = .19, $p = .188$), presumably due to large standard deviations of PAEs on PN task, for instance, the standard deviation (1.06) for children in the No Cue group at post-test was almost 3 times larger than the mean PAE (36%).

To examine the effect of retrieval cues, we compared the accuracy of children in the Cue group and No Cue group at post-test. Estimates did not differ between two groups on most measures in both tasks (PN: PAE, $\beta = .13$, $SE = .09$, $p = .176$; NP: PAE, $\beta = .00$, $SE = .01$, $p = .914$; accuracy, $\beta = .00$, $SE = .11$, $p = .988$), except on PN accuracy ($\beta = .65$, $SE = .15$, $p < .001$). Accuracy for children in the Cue group (66%) were higher than children in the No Cue group (40%) on PN. The absence of effect of retrieval cues on NP tasks might be due to the ceiling effect of performance without cues (No Cue group at post-test: PAE = 8%; accuracy = 32%).

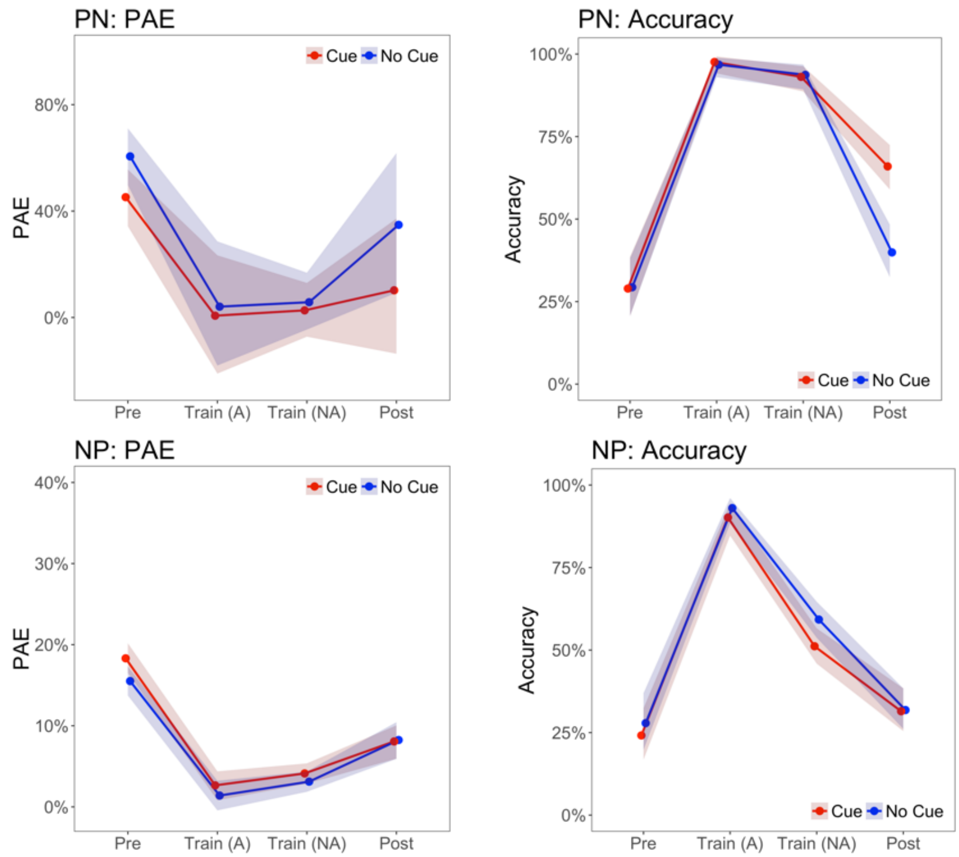

Figure 6. Experiment 2: Mean PAEs, accuracy and 95% confidence intervals for trained items on PN and NP tasks across 4 test phases (pretest, training with alignment, training without alignment, and post-test) from 1000 simulated data using mixed linear models. N.B., y-axes are not identical.

2.2 Untrained items

To investigate effects of training and retrieval cues on untrained items compared to trained items, we further analyzed data with NP tasks at pretest and post-test using a mixed linear model on PAEs. In the model, we included gender, grade (4^{th}, 5^{th}), type (trained, untrained), condition (Cue group, No Cue group), phase (pretest, post-test) as fixed effects, by-subject random intercept, and by-subject random slopes for test phases.

PAEs were lower at post-test (8%) than pretest (16%), $\beta = .04$, $SE = .01$, $p < .001$ (Figure 7). This effect was larger in trained items (pretest: 17%, post-test: 8%) than untrained items (pretest: 15%, post-test: 9%), $\beta = .01$, $SE = .00$, $p < .05$. Not only did PAEs with trained problems significantly decreased from pretest to post-test, $\beta = .03$, $SE = .01$, $p < .001$, PAEs with untrained problems also significantly decreased, $\beta = .03$, $SE = .01$, $p < .001$. More specifically, for untrained problems, estimates at post-test were better than pretest in both Cue group and No Cue group (Cue group: $\beta = .08$, $SE = .02$, $p < .001$; No Cue group: $\beta = .05$, $SE = .02$, $p < .05$), indicating that the alignment effect generalized to untrained problems as well even when retrieval cues were not presented.

A generalized mixed linear model analysis on accuracy with the same model structure as PAE analysis, found that accuracy was higher at post-test (30%) than pretest (22%), $\beta = .30$, $SE = .06$, $p < .001$. Again, accuracy with both trained ($\beta = .33$, $SE = .08$, $p < .001$) and untrained ($\beta = .29$, $SE = .08$, $p < .001$) items significantly increased from pretest to post-test. For untrained problems, estimates at post-test were more accurate than pretest in both Cue group and No Cue group (Cue group: $\beta = .62$, $SE = .21$, $p < .01$; No Cue group: $\beta = .52$, $SE = .21$, $p < .05$).

No other effects or interactions were found. Specifically, we did not observe an effect of retrieval cues on both trained and untrained problems, which was probably due to the ceiling effect of estimates with fractions in NP tasks at post-test.

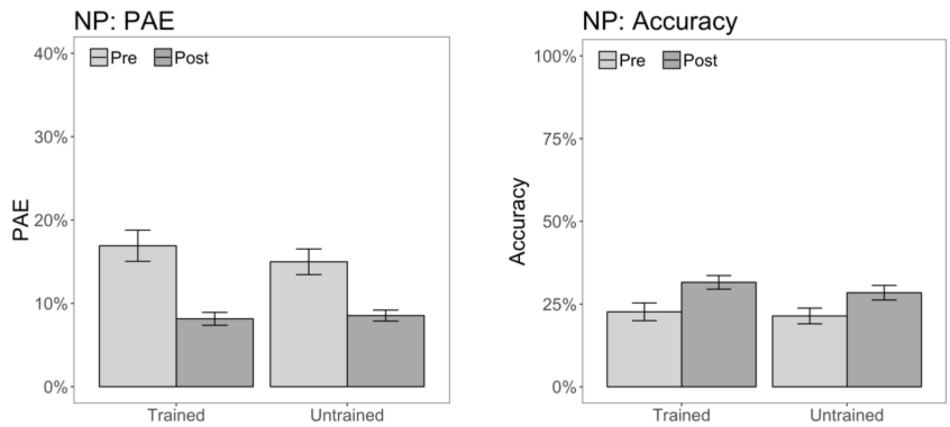

Figure 7. Average PAEs and accuracy for estimates of trained and untrained items at pretest and post-test on NP tasks in Experiment 2. Error bars indicate standard errors.

3. Discussion

In Experiment 2, we designed a training program to provide children with structural alignment to familiar analogical sources of integers. Results revealed that analogies to integers significantly improved children's estimates of fractions over a very short period. In NP tasks, even without any cognitive supports of retrieval cues, children's estimates of fractions at post-test (PAE = 8%, Accuracy = 32%) were as good as estimates with integers (PAE = 9%, Accuracy = 33%) we observed in Experiment 1. This effect of analogies between integer and fraction scales can also facilitate better representations of untrained items (pretest, PAE = 15%, Accuracy = 21%; post-test, PAE = 9%, Accuracy = 28%). Similarly, in PN tasks, simply supports of aligning structures between integers and fractions improved children's estimates from pretest (PAE = 61%, Accuracy = 26%) to post-test (PAE = 36%, Accuracy = 40%), though not as good as estimates with integers (PAE = 8%, Accuracy = 71%) we observed in Experiment 1 (also noted that the difference between PAEs at pretest and post-test were not significant).

To determine whether cognitive supports of retrieval cues would further improve children's fraction estimates, we provided a group of children with cues to remind them of integer magnitudes at post-test. On NP tasks, retrieval cues did not show additional help to improve children's estimates compared to children without cues. Both groups of children's estimates with fraction at post-test (Cue: PAE = 8%, Accuracy = 32%; No Cue: PAE = 8%, Accuracy = 32%) reached accuracy as high as estimates with integers (PAE = 9%, Accuracy = 33%) we observed in Experiment 1, indicating that simply aligning structure of integer and fractional magnitudes on different scales improved children's fraction understanding over a very short period of time, and children were capable of retrieving analogical bases on integers when the alignment was not presented.

On PN tasks, however, retrieval cues did boost a more accurate representation of fractions. Children in the Cue group outperformed those in the No Cue group at post-test. With retrieval cues, children's estimates with fractions at post-test (PAE = 10%, Accuracy = 66%) were almost as good as those with integers (PAE = 8%, Accuracy = 71%) that we observed in Experiment 1.

Retrieval cues of integer magnitude on an equivalent number line are more useful in PN tasks than NP tasks might be due to different features of these two tasks. For NP tasks, even without any cues, the problem itself provides enough information for people to convert it to an integer estimation problem, i.e., multiplying the right ending point of the number line by the denominator of the to be estimated fraction, and estimating the numerator on the new number line. For PN tasks, it is difficult to know the exact denominator of the to be estimated fraction without any cues. Though not explicitly told, children seemed to understand that the alignment between integer and fraction number line, and that the retrieval cues provided values of denominator, and thus the fraction PN problems could be converted to integer estimation problems.

However, in our current design, improvement from pretest to post-test might also be accounted for by practice and feedbacks on number line estimations of fractions. A future follow-up study where children are presented the same aligning structure of integers and fractions, but with feedbacks only to integer problems as a control group might provide a more rigorous picture.

Chapter 4. General Discussion

Analogies to prior knowledge is a key ability to learn mathematics. In the current study, we investigated whether analogies between integer and fractional scales and different cognitive supports for analogical transfer might lead to better understandings of fraction magnitudes.

In Experiment 1, third-to-fifth graders identified (Position-to-Number) or placed (Number-to-Position) fractions and integers on number lines. Our results indicated a general more accurate representation of integers over fractions, confirming the premise of using integer knowledge to help fraction understanding. In Experiment 2, we designed a training program to help children project fractions to familiar analogical sources of integers. Our results indicated that analogies to integers might prompt better understanding of fraction magnitudes given cognitive supports of structural alignment and retrieval cues. Children were capable of making abrupt and broad analogical transfer and estimated fractions as good as integers over a short period of one-session training.

1. Re-examine the Whole Number Bias

Our results indicated that integer knowledge can have a positive transfer in fraction learning. Children were capable of building relatively novel concepts on their prior solid knowledge of integer magnitude, provided correct analogies that the fraction magnitude to 1 is the same ratio as value of its numerator to its denominator. Consistent with the integrated theories of numerical development (Siegler et al., 2011), we provided evidence that the development of fractions and integers share developmental continuities. In contrast with the unintegrated theories of numerical development (e.g., the privileged domain theory), knowledge of integers does not necessarily interfere with development of fraction understanding, instead, integer magnitude understanding can be a good source analog to facilitate fraction learning.

In the current study, we are not denying the importance of the issue of whole number bias, but the interference effect of integer magnitudes on fraction magnitude should not be the end of the story. We are trying to further investigate mechanisms that might facilitate positive whole number bias, and ways to improve fraction understanding based on prior solid knowledge of integers. Several recent studies also targeted on the factors influencing the variability in whole number bias (Alibali & Sidney, 2015; Vamvakoussi, Van Dooren, & Verschaffel, 2012). Alibali and Sidney (2015) conclude that the whole number bias depends on contexts and experiences. In their view, the whole number bias arises when the representations of fraction magnitudes are not activated strongly enough to guide performance in a specific context, and thus the more strongly activated representations of integers might instead guide the performance.

Furthermore, Braithwaite and Siegler (2017) propose that there is a 'componential-to-fraction magnitude shift' in the development of fraction magnitudes understanding. Parallel to the logarithmic-to-linear shift in the development of integer magnitude representations, the development of fraction magnitude understanding undergoes a shift from componential or hybrid representations to fraction magnitudes.

Braithwaite and Siegler examined fourth-to-eighth grader's representations of fraction and demonstrated that the componential-to-fraction shift might occur more slowly than the logarithmic-to-linear shift and might not occur at all for some people.

How to facilitate this componential-to-fraction shift in fraction learning? Our study indicated that correct analogies to integer magnitudes might speed this componential-to-fraction magnitude shift.

2. Analogy as a Mechanism to Speed the Componential-to-Fraction Shift

Our findings revealed that analogies to integer magnitude can lead to better understanding of fraction magnitudes over a short period of time, which is parallel to the findings that analogies to small whole numbers lead to representational changes of large whole numbers (Thompson & Opfer, 2010). In our study, we provided fourth and fifth graders aligning structures of fractional magnitudes on 0-1 number lines and numerators' values on number lines from 0 to the value of denominators. By comparing equivalent integer and fraction magnitude on different scales, children were implicitly shown how numeral symbols of a fraction function. Therefore, analogies that emphasized relative magnitudes between integers and fractions helped children understanding fraction magnitudes better.

Similarly, studies have shown that knowledge of integer arithmetic can also lead to positive transfer to fraction arithmetic through analogies (Sidney & Alibali, 2015, 2017). To facilitate knowledge of fraction division, Sidney and Alibali (2015) provided fifth and sixth graders either surface analogue of other fraction operation or structure analogue of integer division, and found that students gained more conceptual knowledge of fraction division when they were provided analogical bases of integer division. Moreover, this positive transfer can even happen when analogies were not explicit instructed. In a follow-up study, Sidney and Alibali (2017) showed that solving integer division problems immediately before fraction division problems generated implicit analogies between integer and fraction knowledge, and thus improved learners' conceptual structure of fraction division.

How to facilitate positive analogical transfer between integers and fractions? In the current investigation, we provided children with cognitive supports of structural alignment and retrieval cues, and both were shown to be beneficial to positive analogical transfer.

2.1 Cognitive Supports of Alignment for Analogies

In Experiment 2, an important feature of our invention is the structure alignment. According to the structure-mapping theory (Gentner, 1983; Gentner & Markman, 1997), comparisons facilitate maximum structural similarity between two representations. Several features of our training program might provide further support for analogical transfer from integer to fraction magnitudes: visual alignment, multiple examples, and progressive alignment. For instance, we provided children with vertically aligned number lines of integer and fraction scales over 10 different fractional magnitudes. For each magnitude, the alignment started with the easiest problem (i.e., integer PN).

2.2 Cognitive Supports of Retrieval Cues for Analogies

Our second experiment also indicated that source retrieval cues facilitate analogical transfer in fraction learning. At post-test, children with analogical cues produced more accurate estimates when asked to identify fractions on a number line than those who did not receive any cues of analogical bases. Consistent with classic findings of Gick and Holyoak (1980), we provided further evidence in the context of fraction learning that even if target and source has been successfully mapped, failure of retrieving the analogical sources can prevent analogical inference.

The effect of retrieval cues in analogical transfer has also been shown in previous studies (Novick & Holyoak, 1991). For example, college students were asked to solve a target mathematics problem after they learned the solution of a structurally analogous problem. Prior to solving the problem, they were also provided different kinds of retrieval cues. Researchers found that cues mapping numbers between base and target were most beneficial, cues mapping concepts were the second, and non-specific retrieval cues that reminded students of the base problem were least beneficial. In our study, children were provided visually aligned cues. Due to the one-to-one correspondence of alignment, these retrieval cues implicitly mapped both numbers and concepts between analogical bases and target. These features of retrieval cues appeared to provide critical opportunities for children to make analogical inferences from integer to fraction magnitudes.

3. Advantages of Number Line Training

We used number lines in our invention to help children map between integer and fraction magnitudes, and have a better understanding of fraction magnitudes and properties. With number lines, children have opportunities of mapping the relations between a portion of the length of the line and the total length of the line, to the ratio of relations between values of the numerator and denominator of a fraction. What is more, the continuous feature of the line may hint children that there are infinite fractions between two integers.

Previous findings have also indicated that number line training can improve children's understanding of fraction magnitudes (Hamdan & Gunderson, 2017; Fazio, Kennedy, & Siegler, 2016). Using a pretest-training-posttest design, Hamdan and Gunderson (2017) presented second and third graders with one of the three training programs, a number line training program where children learned to mark magnitudes on very thin two-dimensional number lines, an area model training program where children learned to divide circles to indicate fractional magnitudes, and a reading crossword puzzle control program, and found that number line training plays a causal role in children's fractional magnitudes understanding, and is more beneficial than the educational widely-used area model.

Likewise, Fazio et al. (2016) provided further evidence that training to map fractional magnitudes on unidimensional number lines and understanding of unit fractions help deepen children's understanding of fractions. In their experiment, children were trained to map fraction magnitudes on a number line with more and more accurate estimations. Using the technique of successful approximation, children's estimates had to be within 20% of the target fraction at the start of training, and if they were successful in estimating with this criterion, the criterion reduced to 15% the target fraction, and finally to 10%. In our training program, we set a very strict criterion of 2.5% within the correct answer at the beginning of the program, and we provided evidence that this kind of training is also beneficial to children's understanding of fractions.

4. Limitations and Future Directions

4.1 Limitations of the Current Study

Several limitations of the current study should be noted. First, there are also alternative explanations for the improvement from pretest to post-test other than analogies to integers magnitudes. One explanation is that children could benefit from school curriculums during the interval between pretest and post-test (*Mean* = 7.23 days, *SD* = 4.23 days). A second explanation is that simply practice and feedbacks of fraction number line estimation problems facilitated children's understanding of fraction magnitudes, as observed in previous studies (Fazio et al., 2016; Hamdan & Gunderson, 2017). Unfortunately, we could not address this two explanations in the current design. A future follow-up study could add a group of children who only receive feedbacks on the integer estimation problems as a control group, to further separate the effect of analogical bases, the effect of school education, and the effect of feedbacks.

Second, our post-test took place immediately after training. It would be worth investigating whether the effect of analogies can facilitate children's fraction understanding in the long-term.

4.2 Highlights for Future Studies and Educational Implications

Finally, the mechanism of analogy to well understood magnitude to extend mental number line highlights for future studies and educational implications. Our program was limited to fractions 0-1, but analogy can serve as a broad mechanism to expand the mental number line, from small whole numbers to large whole numbers, from whole numbers to fractions 0-1, from fractions 0-1 to fractions 0-N, from positive numbers to negative numbers, and so on. Also, we found positive results of emphasizing the analogy between fraction magnitudes and familiar sources of integers, with cognitive supports of visual alignment and retrieval cues, which might have further educational implications.

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