Extended Tropicalization of Spherical Varieties

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Abstract

The first steps in defining a notion of spherical tropicalization were recently taken by Tassos Vogiannou in his thesis and by Kiumars Kaveh and Christopher Manon in a related paper. Broadly speaking, the classical notion of tropicalization concerns itself with valuations on the function field of a toric variety that are invariant under the action of the torus. Spherical tropicalization is similar, but considers instead spherical G-varieties and G-invariant valuations.

The core idea of my dissertation is the construction of the extended tropicalization of a spherical embedding. Vogiannou, Kaveh, and Manon only concern themselves with subvarieties of a spherical homogeneous space G/H. My thesis describes how to tropicalize a spherical embedding by tropicalizing the additional G-orbits of X and adding them to the tropicalization of G/H as limit points. This generalizes work done by Kajiwara and Payne for toric varieties and affords a means for understanding how to tropicalize the compactification of a subvariety of G/H in X.

The extended tropicalization construction can be described from three different perspectives. The first uses the polyhedral geometry of the colored fan and the second extends the Gröbner theory definition given by Kaveh and Manon. The third method works by embedding the spherical variety in a specially-constructed toric variety, tropicalizing there with the standard theory, and then applying a particular piecewise-projection map. This final perspective introduces a novel means for tropicalizing a

homogeneous space that allows us to prove several statements about the structure of a spherical tropicalization by transferring results from the toric world where more is known.

We also suggest a definition for the tropicalization of subvarieties of a homogeneous space whose defining equations have coefficients with non-trivial valuation. All the previous theory has been done in the constant coefficient case, i.e. when the coefficients of the defining equations all have trivial valuation.

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Vita

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Publications

The maximum likelihood degree of toric varieties, with C. Améndola, N. Bliss, I. Burke, C. Gibbons, M. Helmer, S. Hoşten, J. Rodriguez, and D. Smolkin, *Journal of Symbolic Computation* (2018)

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Fields of Study

Major Field: Mathematics

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Chapter 1: Introduction

This thesis extends some of the theory of tropical geometry from toric varieties to the more general spherical varieties. In this introduction we give background on the relevant theories: tropical geometry, toric varieties, and spherical varieties. Finally, we describe previous work in spherical tropicalization and outline the content of the remainder of the thesis.

1.1 Tropical Geometry

The aim of this section is to give a brief overview of the first principles of tropical geometry, developing enough theory to state the Fundamental Theorem. We will also say a few words regarding the Structure Theorem, which describes what objects can be obtained as tropicalizations. A more detailed breakdown of the general theory discussed here can be found in [21]. We note that in this thesis we work over $\mathbb{Q}$ as opposed to $\mathbb{R}$, which is much more standard for tropical geometry. We use $\mathbb{Q}$ because this is the more widely-used convention in spherical geometry and spherical tropicalization, our primary focus. The basic theory is not affected by this choice; $\mathbb{Q}$ and $\mathbb{R}$ can be readily interchanged in the following.

Let k be an algebraically closed field and write $\overline{\mathbb{Q}} = \mathbb{Q} \cup \{\infty\}$ with the convention that $x < \infty$ for all $x \in \mathbb{Q}$. Then let $v : k \to \overline{\mathbb{Q}}$ be a valuation, which by definition satisfies:

1. $v(x) = \infty$ if and only if $x = 0$;
2. $v(xy) = v(x) + v(y)$; and
3. $v(x + y) \geq \min\{v(x), v(y)\}$.

With these conditions, it is straightforward to check that $v(x+y) = \min\{v(x), v(y)\}$ if $v(x) \neq v(y)$. For the remainder of this first section, we will only concern ourselves with the restriction of v to $k^* = k \setminus \{0\}$. We denote by $\mathbb{T}$ the algebraic torus $(k^*)^n$.

Now we define the tropicalization of a polynomial. For $\mathbf{u} = (u_1, \ldots, u_n) \in \mathbb{Z}^n$, denote by $\mathbf{x}^{\mathbf{u}}$ the monomial $x_1^{u_1} \cdots x_n^{u_n} \in k[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$. Then if $f = \sum_{\mathbf{u} \in \mathbb{Z}^n} a_{\mathbf{u}} \mathbf{x}^{\mathbf{u}} \in k[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ is a Laurent polynomial, define a function $\operatorname{trop} f : \mathbb{Q}^n \to \mathbb{Q}$, called the *tropicalization of* f, by:

$$\operatorname{trop}_{\mathbb{T}}(f)(\mathbf{w}) = \min_{a_{\mathbf{u}} \neq 0} \{v(a_{\mathbf{u}}) + \mathbf{u} \cdot \mathbf{w}\}.$$

We call the set of $\mathbf{w} = (w_1, \ldots, w_n) \in \mathbb{Q}^n$ where the minimum in $\operatorname{trop} f$ is achieved more than once the *tropical hypersurface* associated to f and denote it by $\operatorname{trop}_{\mathbb{T}}(V(f))$. The subscript $\mathbb{T}$ in these definitions is not standard notation; we explain our use of it here at the beginning of §1.4. This definition extends to an arbitrary subvariety of $(k^*)^n$ by intersecting these tropical hypersurfaces. If $I \subseteq k[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ is an ideal and $V(I) = \{\mathbf{y} \in (k^*)^n \mid f(\mathbf{y}) = 0 \text{ for all } f \in I\}$ is its variety in the algebraic torus, then we call

$$\operatorname{trop}_{\mathbb{T}}(V(I)) := \bigcap_{f \in I} \operatorname{trop}_{\mathbb{T}}(V(f)).$$

the *(toric) tropicalization* of $V(I)$.

It is worth noting that we have taken the intersection over every element of I; in general just considering elements of a generating set won't do. A finite generating set $B \in I$ is called a *tropical basis* if $\operatorname{trop}_{\mathbb{T}}(V(I)) = \bigcap_{f\in B} \operatorname{trop}_{\mathbb{T}}(V(f))$, and every ideal in the ring of Laurent polynomials has a finite tropical basis. If $I = (f)$ is a principal ideal, then $\{f\}$ is a tropical basis for I, i.e. $\operatorname{trop}_{\mathbb{T}}(V(I)) = \operatorname{trop}_{\mathbb{T}}(V(f))$.

Before moving on, we'll compute some examples.

Example 1.1.1. Let $f = 3 + 2x_1 + x_1x_2^{-1} - 7x_2^2 \in \mathbb{C}[x_1^{\pm 1}, x_2^{\pm 1}]$, $I = (f)$, and let $v : \mathbb{C}^* \to \mathbb{Q}$ be the trivial valuation with sends every nonzero element of $\mathbb{C}$ to 0. Then $\operatorname{trop}_{\mathbb{T}} f(w_1, w_2) = \min\{0, w_1, w_1 - w_2, 2w_2\}$ and $\operatorname{trop}_{\mathbb{T}} V(I)$ is the collection of rays shown in Figure 1.1. The 2-dimensional regions are labeled with the expression minimized in that region. The ray separating the "$w_1 - w_2$" and the "0" region has slope 1 and the ray separating the "w_1" and the "$2w_2$" region has slope 2.

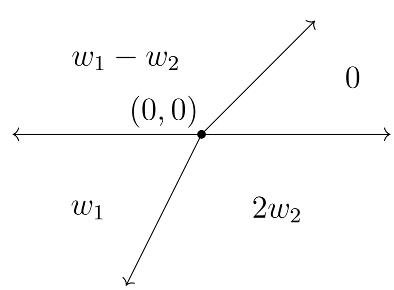


Figure 1.1: The tropicalization of $V(3 + 2x_1 + x_1x_2^{-1} - 7x_2^2)$

Example 1.1.2. We work over the field $\mathbb{C}\{\{t\}\} := \bigcup_{n\in\mathbb{N}} \mathbb{C}(\!(t^{1/n})\!)$, the field of Puiseux series over $\mathbb{C}$. Define a valuation $v : \mathbb{C}\{\{t\}\}^* \to \mathbb{Q}$ by sending a polynomial to the lowest power of t with a nonzero coefficient in its expansion. Thus, if

$$f = 2t + (t^{-1} + 3t^3)x_1 + (7 - t^{1000})x_2 - 6x_1^2 + 4t^{-2}x_1x_2,$$

we have that

$$\operatorname{trop}_{\mathbb{T}} f(w_1, w_2) = \min\{1, w_1 - 1, w_2, 2w_1, w_1 + w_2 - 2\},$$

so the tropical hypersurface $\operatorname{trop}_{\mathbb{T}} V(f)$ is as pictured in Figure 1.2.

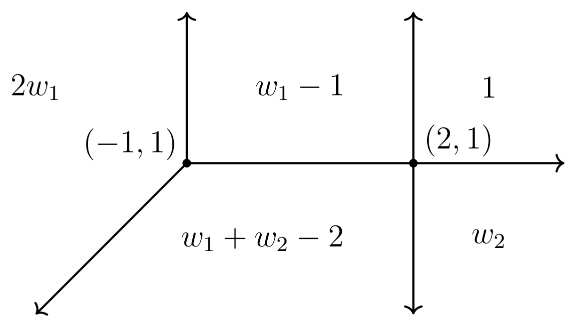


Figure 1.2: The tropicalization of $V(2t+(t^{-1}+3t^3)x_1+(7-t^{1000})x_2-6x_1^2+4t^{-2}x_1x_2)$

The final concept we need to define before we can state the Fundamental Theorem is that of initial forms. Let $v : k^* \to \mathbb{Q}$ be a fixed valuation. Additionally assume that there exists a splitting $\varphi : v(k^*) \to k^*$ such that $(v \circ \varphi)(w) = w$. We will denote by $t^w := \varphi(w)$ the image of $w \in v(k^*)$ under φ. Then we have an associated valuation ring $R := \{x \in k \mid v(x) \geq 0\}$, which contains a maximal ideal $\mathfrak{m} := \{x \in k \mid v(x) > 0\}$. We also have the residue field $\Bbbk := R/\mathfrak{m}$. Denote by $\overline{x}$ the image of $x \in R$ under the projection map $R \to \Bbbk$. Let $f = \sum_{\mathbf{u} \in \mathbb{Z}^n} a_{\mathbf{u}} \mathbf{x}^{\mathbf{u}} \in k[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ be a polynomial and $\mathbf{w} = (w_1, \ldots, w_n) \in \mathbb{Q}^n$ an arbitrary vector. We call $\mathbf{w}$ the weight vector. Write

$$W := \operatorname{trop}_{\mathbb{T}} f(\mathbf{w}) = \min_{a_{\mathbf{u}} \neq 0} \{v(a_{\mathbf{u}}) + \mathbf{u} \cdot \mathbf{w}\}.$$

Then we define the *initial form* $\mathrm{in}_{\mathbf{w}}(f) \in \Bbbk[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ of f with respect to the weight vector $\mathbf{w}$ as follows:

$$\mathrm{in}_{\mathbf{w}}(f) = \sum_{\substack{\mathbf{u}\in\mathbb{Z}^n \\ v(a_{\mathbf{u}})+\mathbf{u}\cdot\mathbf{w}=W}} \overline{t^{-v(a_{\mathbf{u}})}a_{\mathbf{u}}}\mathbf{x}^{\mathbf{u}} = \overline{t^{-W}f(t^{w_1}x_1, \ldots, t^{w_n}x_n)}.$$

The second characterization here is only valid when $\mathbf{w} \in (v(k^*))^n$ since otherwise the t^{w_i} are not defined. This can be a difficult definition to absorb, so we present a brief example.

Example 1.1.3. Let $2t+(t^{-1}+3t^3)x_1+(7-t^{1000})x_2-6x_1^2+4t^{-2}x_1x_2$ in $\mathbb{C}\{\{t\}\}[x_1^{\pm 1}, x_2^{\pm 1}]$ as in Example 1.1.2 so that $\mathrm{trop}\, f(\mathbf{w}) = \min\{1, w_1 - 1, w_2, 2w_1, w_1 + w_2 - 2\}$. If $\mathbf{w}_1 = (-2, 0)$, then $W_1 := \mathrm{trop}\, f(\mathbf{w}_1) = -4$ and this value is met at the monomials $-6x_1^2$ and $4t^{-2}x_1x_2$, so the initial form is as follows:

$$\mathrm{in}_{\mathbf{w}_1}(f) = \overline{t^0(-6)}x_1^2 + \overline{t^2(4t^{-2})}x_1x_2 = -6x_1^2 + 4x_1x_2.$$

Here we identify $-6 = \overline{-6}$ and $4 = \overline{4}$ with their images under the projection. If instead we consider $\mathbf{w}_2 = (0, 2)$, then $W_1 := \mathrm{trop}\, f(\mathbf{w}_2) = -1$ is met solely at the monomial $(t^{-1} + 3t^3)x_1$. Thus, the initial form in this case is

$$\mathrm{in}_{\mathbf{w}_2}(f) = \overline{t(t^{-1} + 3t^3)}x_1 = \overline{1 + 3t^4}x_1 = x_1$$

Having defined initial forms of polynomials, we define the *initial ideal* of an ideal $I \subseteq k[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ with respect to $\mathbf{w} \in \mathbb{Q}^n$ to be

$$\mathrm{in}_{\mathbf{w}}(I) := \langle \mathrm{in}_{\mathbf{w}}(f) \mid f \in I \rangle.$$

The construction of initial ideals in similar in spirit to the theory of Gröbner bases in a polynomial ring, but when we consider ideals of Laurent polynomials, any monomial is a unit in the ring. Thus, if $\text{in}_{\mathbf{w}}(f)$ is a unit for any $f \in I$, then $\text{in}_{\mathbf{w}}(I) = k[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$. It is this consideration that concerns tropical geometry: when is this initial ideal not the entire ring?

We can relate this question to tropicalization through the Fundamental Theorem of Tropical Geometry, which gives multiple characterizations of the tropicalization of an ideal. This theorem will also help us naturally generalize the concept of tropicalization to other settings. For example, in §2.2 we will define the tropicalization of a subvariety of a spherical homogeneous space as the image of a valuation map rather than thinking of it as an intersection of tropical hyperplanes.

Theorem 1.1.4. The Fundamental Theorem of Tropical Geometry. *Let k be an algebraically closed field with a nontrivial valuation v and let $I \subseteq k[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ be an ideal with associated variety $V(I) = \{\boldsymbol{x} \in (k^*)^n \mid f(\boldsymbol{x}) = 0 \text{ for all } f \in I\}$ in $\mathbb{T} = (k^*)^n$. Then the following sets coincide:*

1. $\text{trop}_{\mathbb{T}} V(I) := \bigcap_{f \in I} \text{trop}_{\mathbb{T}} V(f)$;
2. $\left\{\boldsymbol{w} \in \mathbb{Q}^n \mid in_{\boldsymbol{w}}(I) \neq k[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]\right\}$;
3. *The closure in $\mathbb{Q}^n$ of the set*

$$v(V(I)) := \{(v(x_1), \ldots, v(x_n)) \mid (x_1, \ldots, x_n) \in V(I)\}.$$

We have in this theorem the hypothesis that v is nontrivial, but this is not particularly restrictive. In fact, if I is a subvariety of a torus $(k^*)^n$ and K is a valued extension of k (which means $v_K|_k = v_k$), then the tropicalization of I in $(k^*)^n$ is equal to the tropicalization of I in $(K^*)^n$. Indeed, we've already seen this in Examples 1.1.1 and 1.1.2. In Example 1.1.1, we considered the trivial valuation on $\mathbb{C}$, but we just as easily could have considered our ideal being defined over the field $\mathbb{C}\{\{t\}\}$ instead, and the valuation we defined there in Example 1.1.2 restricts to the trivial valuation on $\mathbb{C}$. It is also worth mentioning that the work we did in Example 1.1.3 agrees with this theorem: the vector $\mathbf{w}_1 = (-2, 0)$ lies on the tropical hyperplane and its initial form is a binomial, while the vector $\mathbf{w}_2 = (0, 2)$ does not and its initial form is a monomial, which is a unit in $k[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$.

The next major result in tropical geometry is the Structure Theorem. We describe this in detail in §5.1, but the main idea is that the tropicalization of a toric variety is a polyhedral complex that exhibits a balancing condition at each codimension one polyhedron. In essence, this balancing condition can be thought of as an evenly-matched "tug of war" between the higher dimensional polyhedra.

We can illustrate this in the case of Example 1.1.2, shown in Figure 1.2. This polyhedral complex is of dimension two with two dimension one polyhedra: the points $(-1, 1)$ and $(2, 1)$. At each point, the sum of the vectors arrayed out from the point is zero. That is, at $(2, 1)$, the tropicalization locally consists of rays in the directions $(1, 0)$, $(0, 1)$, $(-1, 0)$, and $(0, -1)$, which sum to zero. Similarly, at $(-1, 1)$ are the rays $(1, 0)$, $(0, 1)$, and $(-1, -1)$, which also cancel each other out. In general, weights may be needed on certain rays to achieve balancing; this is a case where the weights are all one. Example 1.1.1 and Figure 1.1 show a situation where balancing is still possible, but differing weights are needed on the rays to ensure their linear combination is zero.

1.2 Toric Varieties

A toric variety X is a variety carrying the action of a torus $\mathbb{T}$ such that X has an open $\mathbb{T}$-orbit. Toric varieties are widespread in algebraic geometry and provide a wealth of examples to build intuition and generate and test conjectures. Part of the appeal of toric varieties is that they can be modeled by polyhedral objects called fans, and this combinatorial structure has meaningful interplay with tropical geometry. We give here a brief overview of toric geometry, with a bias toward the combinatorial side of the theory. Refer to [9] or [11] for more background on toric varieties.

Let N be a finite-dimensional lattice and $M := \mathrm{Hom}(N, \mathbb{Z})$ its dual. Write $N_{\mathbb{Q}} := N \otimes_{\mathbb{Z}} \mathbb{Q}$ and $M_{\mathbb{Q}} := M \otimes_{\mathbb{Z}} \mathbb{Q}$ to denote the $\mathbb{Q}$-vector spaces associated to these lattices. The fan associated to a toric variety will lie in $N_{\mathbb{Q}}$.

We begin with a string of definitions from polyhedral geometry. Given a finite subset $S \subseteq N_{\mathbb{Q}}$, the *cone* in $N_{\mathbb{Q}}$ spanned by S is the set:

$$\mathrm{cone}(S) := \left\{ \sum_{u \in S} \lambda_u u : \lambda_u \in \mathbb{Q}_{\geq 0} \right\}.$$

A cone is necessarily convex, and it is additionally called *strictly convex* if it contains no lines. The *dual* of a cone $\sigma \in N_{\mathbb{Q}}$ is the set

$$\sigma^{\vee} := \{ u \in M_{\mathbb{Q}} : \langle u, v \rangle \geq 0 \text{ for all } v \in \sigma \},$$

where $\langle \cdot, \cdot \rangle$ is the dot product. If $\sigma \in M_{\mathbb{Q}}$, define $\sigma^{\vee} \in N_{\mathbb{Q}}$ similarly. Given a non-zero vector $u \in M_{\mathbb{Q}}$, note $H_u := u^{\vee}$ is a half-space. If σ is a cone, we say that a cone τ is a *face* of σ if there exists $u \in \sigma^{\vee}$ such that $\tau = H_u \cap \sigma$. This is denoted by $\tau \preceq \sigma$ (or $\tau \prec \sigma$ if $\tau \neq \sigma$). A *fan* Σ is a finite collection of strictly convex cones satisfying two properties:

(i) If $\sigma \in \Sigma$, then every face of σ is in Σ.

(ii) If $\sigma, \tau \in \Sigma$, then $\sigma \cap \tau$ is a face of both σ and τ.

The *support* $\mathrm{supp}(\Sigma)$ of Σ is $\bigcup_{\sigma\in\Sigma}\sigma \subseteq N_{\mathbb{Q}}$, i.e. the subset of the vector space covered by cones in Σ.

The fundamental fact of toric geometry is that each fan is uniquely associated to a normal, separated toric variety and each normal, separated toric variety induces a fan. Each cone σ of the fan Σ_X of a toric variety X corresponds to a torus-orbit $O(\sigma)$ of X. This correspondence is bijective and the polyhedral structure of the fan reflects the geometry of the corresponding orbits in X.

Given a fan Σ, the associated toric variety is obtained as follows. A cone $\sigma \in \Sigma$ corresponds to the affine variety $\mathrm{Spec}\left(k\left[\sigma^\vee \cap M\right]\right)$. If $\sigma, \sigma' \in \Sigma$ share a face $\tau \in \Sigma$, then $\mathrm{Spec}\left(k\left[\sigma^\vee \cap M\right]\right)$ and $\mathrm{Spec}\left(k\left[(\sigma')^\vee \cap M\right]\right)$ are glued together along their shared copy of $\mathrm{Spec}\left(k\left[\tau^\vee \cap M\right]\right)$. Then the variety X_Σ associated to Σ is the gluing of each of these affine varieties along shared torus orbits. The association of cones to orbits in this way is described precisely in the following theorem:

Theorem 1.2.1. [9, Theorem 3.2.6] *Let X_Σ be the toric variety of the fan Σ in $N_{\mathbb{Q}}$. Then:*

(i) There is a bijective correspondence

$$\{\textit{cones } \sigma \textit{ in } \Sigma\} \leftrightarrow \{\mathbb{T}\textit{-orbits in } X_\Sigma\}$$
$$\sigma \leftrightarrow O(\sigma).$$

(ii) Let $n = \dim N_{\mathbb{Q}}$. For each cone $\sigma \in \Sigma$, $\dim O(\sigma) = n - \dim\sigma$.

(iii) The affine open subset $\mathrm{Spec}\left(k\left[\sigma^\vee \cap M\right]\right)$ of X_Σ is the union of torus orbits:

$$\mathrm{Spec}\left(k\left[\sigma^\vee \cap M\right]\right) = \bigcup_{\tau\preceq\sigma} O(\tau).$$

(iv) $\tau \preceq \sigma$ *if and only if* $O(\sigma) \subseteq \overline{O(\tau)}$, *and*

$$\overline{O(\tau)} = \bigcup_{\tau \preceq \sigma} O(\sigma),$$

where $\overline{O(\tau)}$ *denotes the closure in both the classical and Zariski topologies.*

Perhaps the most fundamental connection between toric varieties and tropical geometry is given by Tevelev's lemma:

Proposition 1.2.2. (Tevelev's Lemma) [30, Lemma 2.2] *Suppose* $Y \subseteq \mathbb{T}$ *is a closed subvariety,* X_Σ *is a toric variety for the torus* $\mathbb{T}$*, and* $\overline{Y}$ *is the closure of* Y *in* X_Σ*. Then* $\overline{Y} \cap O(\sigma)$ *for* $\sigma \in \Sigma$ *if and only if* $\mathrm{trop}_{\mathbb{T}}(Y)$ *intersects the interior of* σ*.*

This fact leads to the question of whether there exists a reasonable definition of extended tropicalization that will record the points of the variety added on in torus orbits. Such a theory was developed separately by Kajiwara in 2008 [16] and Payne in 2009 [26].

This theory is described in detail in §2.3, where we develop a similar construction for spherical varieties, but we give here a rough idea of how extended tropicalization works for toric varieties. By definition, a subvariety of the torus is tropicalized by taking the closure of the image of the piecewise valuation map $(k^*)^n \to \mathbb{Q}^n$. The extended tropicalization of an affine variety is the same definition except that the piecewise valuation map is extended to $k^n \to (\mathbb{Q} \cup \{\infty\})^n$, where 0 is mapped to ∞. Tropicalization for general toric varieties is obtained by gluing together these affine tropicalizations along intersections.

1.3 Spherical Varieties

Spherical varieties are generalizations of toric varieties that encompass a wider class of objects, including flag varieties and symmetric varieties. In the same way that toric varieties can be modeled by fans, spherical varieties can be modeled by polyhedral objects called colored fans. This classification was introduced in 1983 by Luna and Vust [20]. Similarly to Theorem 1.2.1, the colored fans record group orbits of the spherical variety. A survey of spherical geometry would be incomplete without this theory, but we delay describing the combinatorial data of a spherical variety to §2.1 as it is somewhat technical. In this introduction, we will limit ourselves to defining spherical varieties and giving some background about their characteristics. There are a number of good references for spherical varieties, including [18], [25], [27], and [32].

Let G be a connected reductive algebraic group and $H \leq G$ a closed subgroup. We call G/H a *spherical homogeneous space* if a Borel subgroup B has an open (dense) orbit under the left group action. We assume through that G/H is a normal variety. A *spherical variety* or *G/H-embedding* is a normal G-variety X that is an equivariant open embedding of G/H. Toric varieties are obtained when $G = B = (k^*)^n$ and H is the trivial subgroup. Much of the difficulty in understanding spherical varieties is in dealing with the gap between G and its Borel subgroup B, an issue that isn't present in the toric case. It is worth observing that a fixed variety X can potentially be realized as a spherical variety under the action of multiple groups. For example, the affine plane $\mathbb{A}^2$ is a toric variety under the action of the two-dimensional torus, but it is also a spherical variety under the action of Gl_2 or Sl_2 (see Example 2.1.6).

A spherical variety can be classified in a number of equivalent ways. For example, any normal G-variety with finitely many B-orbits is spherical. Also, a homogeneous space G/H is spherical if for any irreducible G-module A and any character χ of H, we have

$$\dim \{a \in A : ha = \chi(h)a \text{ for all } h \in H\} \leq 1.$$

This last condition is called the *multiplicity free property*.

The first characterization of spherical varieties regarding the existence of an open B-orbit can be further generalized. The *complexity* of a G-variety X is the minimal codimension of a B-orbit in X. A spherical variety is therefore a normal G-variety of complexity zero. Higher complexity G-varieties are less well understood. Timashev in [31] (or [32], §3.16) develops a combinatorial description of G-varieties of complexity one, but remarks that a similar classification will not hold for higher complexity. To a normal G-variety of complexity one, he associates something he calls a *colored hyperfan*, which is beyond the scope of this thesis to define.

An important example of a spherical variety is the Grassmannian $\mathrm{Gr}(k, n)$ of k-dimensional subspaces of an n-dimensional vector space. The Grassmannian carries a natural action of Gl_n, and this action makes $\mathrm{Gr}(k, n)$ a spherical homogeneous space for $G := \mathrm{Gl}_n$. Explicitly, let $H \leq G$ be the closed subgroup of upper block triangular $n \times n$ matrices with blocks of size k and $n - k$; then G/H is isomorphic to $\mathrm{Gr}(k, n)$.

More generally, write $F(d_1, d_2, \ldots, d_m)$ for the flag variety parametrizing nested subspaces of dimension $0 < d_1 < d_2 < \cdots < d_m = n$ in an n-dimensional vector space. Then $F(d_1, d_2, \ldots, d_m)$ again carries an action of Gl_n and may be realized as a spherical homogeneous space using the subgroup $H \leq G$ of upper block triangular $n \times n$ matrices with blocks of size $d_1, d_2 - d_1, d_3 - d_2, \ldots, d_m - d_{m-1}$. In particular, if

$d_{i+1} - d_i = 1$ for $1 \leq i \leq n-1$, then we obtain the complete flag variety and $H = B$ is the subgroup of upper triangular matrices.

The Bruhat decomposition can be used to verify that that these homogeneous spaces are in fact spherical. If $T \leq B$ is a maximal torus, then the associated *Weyl group* is $W := N_G(T)/T$. This is a finite group, and G can be decomposed into finitely many pieces indexed by the elements of W:

Proposition 1.3.1 (Bruhat decomposition)**.** *$G = \bigsqcup_{w \in W} BwB$, where we abuse notation by identifying $w \in W$ with a representative in G.*

If $P \leq G$ contains B, then we call P *parabolic*, and from the Bruhat decomposition we can see that G/P has finitely many B-orbits and is therefore spherical.

1.4 Spherical Tropicalization

Broadly speaking, classical tropical geometry is concerned with torus-invariant valuations on the field of rational functions of a variety. The tropicalization procedure eloquently records this information in polyhedral structures, the combinatorics of which reflect the geometric structure of the variety. Spherical tropicalization is similar in spirit. It extends the notion of tropicalization from toric varieties to spherical varieties, which carry the action of a connected reductive group G as opposed to an algebraic torus, and tropicalization for spherical varieties records valuations that are G-invariant as opposed to torus-invariant. We notate this distinction by writing the group G as a subscript in the tropicalization. Thus, if $Y \subseteq G/H$ is a closed subvariety, its spherical tropicalization is written $\mathrm{trop}_G(Y)$. This explains our use of the notation $\mathrm{trop}_{\mathbb{T}}$ in the toric case discussed §1.1.

The theory of spherical tropicalization was first developed by Tassos Vogiannou in his 2015 thesis [33]. We delay giving his definition until §2.2 as it requires the theory of §2.1 to fully appreciate. There are some as-yet undefined terms appearing in the following; they will also be explained in §2.1.

The bulk of Vogiannou's work is concerned with tropical compactifications, which were introduced for toric varieties in [30]. The toric definition extends naturally to the spherical setting:

Definition 1.4.1. If $Y \subseteq G/H$ is a closed subvariety and $\overline{Y} \subseteq X$ is its closure in a G/H-embedding X, then $\overline{Y}$ is called a *tropical compactification* if $\overline{Y}$ is complete and the map

$$\mu_{\overline{Y}} : G \times \overline{Y} \to X, \quad (g, x) \mapsto gx$$

is faithfully flat.

The main result of Vogiannou's thesis concerns these tropical compactifications:

Theorem 1.4.2. [33, Theorem 1.2] *Let Y be a closed subvariety of a spherical homogeneous space G/H. Then:*

(i) Tropical compactifications of Y in toroidal spherical varieties exist.

(ii) If $\overline{Y} \subseteq X$ is a tropical compactification, where X is a spherical variety associated to a colored fan Σ, then $\operatorname{supp}(\Sigma) = \operatorname{trop}_G(Y)$.

Along with this, Vogiannou proves an extended version of Tevelev's Lemma:

Proposition 1.4.3. [33, Proposition 4.5] *Let X be a simple toroidal spherical variety with closed G-orbit $\mathcal{O}$, and let $(\sigma, \mathcal{F})$ be the associated colored cone in $\mathcal{N}_{\mathbb{Q}}$. Then*

$\mathrm{trop}_G(Y)$ *intersects the relative interior of* σ *if and only if* $\overline{Y} \subseteq X$ *intersects the closed orbit* $\mathcal{O}$.

We remark here that Tevelev's Lemma and its generalization are both done in the constant coefficient case, that is when the valuation is trivial on the coefficients of the defining polynomials. We discuss how some of these ideas can be extended to the non-constant coefficient case in §5.4.

The only other work in this area is due to Kaveh and Manon [17]. They develop a Gröbner theory for spherical varieties and define the tropicalization of a subvariety of a homogeneous space using this machinery. This construction agrees with Vogiannou's definition, which Kaveh and Manon prove in an extension of the fundamental theorem of tropical geometry.

A particular section of their paper worth highlighting is their discussion of connections between the spherical tropicalization and the Berkovich analytification of a variety. The relation between tropical geometry and Berkovich theory is significant. In [26], Payne proves that the Berkovich analytification of a toric variety is the inverse limit its tropicalizations with respect to different toric embeddings. More recently, Baker, Payne, and Rabinoff in [3] prove a number of theorems relating Berkovich spaces and tropicalizations of curves, including that any finite subgraph embedded in the Berkovich analytification of a curve maps isometrically onto its image in some tropicalization of a toric variety in which the curve is embedded. Kaveh and Manon add to this theory by demonstrating that there is a continuous map from the Berkovich analytification of an affine spherical variety to its spherical tropicalization and that this map commutes with the tropicalization map and the natural inclusion of the variety into its Berkovich analytification.

The principal contribution of this thesis is a description of the topological spaces in which tropicalizations of tropical compactifications lie. We define and construct an extended tropicalization for a spherical embedding X that adds on points to $\mathrm{trop}_G(G/H)$ in a natural way that respects the geometry of the embedding $G/H \hookrightarrow X$. To be precise, the work that has come before has been restricted to considering subvarieties of a homogeneous space G/H. We extend the theory to allow for subvarieties of embeddings as well.

This generalizes the tropicalization of toric varieties independently developed by Kajiwara [16] and Payne [26] in 2008 and 2009, respectively. A diagrammatic representation of how these theories follow from each other is shown in Figure 1.3.

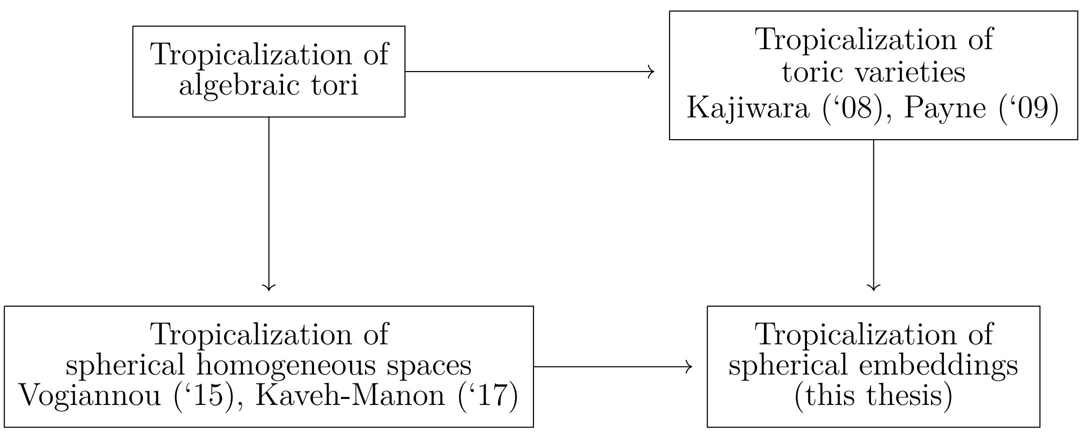


Figure 1.3: Relations between spherical and toric tropicalization

Chapter 2 introduces the theory of spherical varieties and their tropicalizations. We further describe the extended tropicalization construction in detail using the polyhedral geometry of the colored fan. In addition, we describe a means for tropicalizing equivariant morphisms between spherical embeddings. This theory is used later in Chapter 5. In Chapter 3 we explain Kaveh and Manon's Gröbner theory for spherical varieties and how tropicalization works from this perspective. Then we extend their definition to spherical embeddings and prove an extended fundamental theorem that extended Gröbner tropicalization from Chapter 3 and extended tropicalization from Chapter 2 coincide. These two chapters together cover the content of [23].

Chapter 4 describes a means for globally tropicalizing G/H-embeddings. That is, the constructions described in Chapters 2 and 3 work by breaking a spherical variety into simple spherical varieties, tropicalizing separately, and then gluing together. This description only uses the tropicalization operation once. It works by embedding a spherical variety in a toric variety, tropicalizing there, and then applying a piecewise linear map that respects the interplay of the torus action and the action of G. There is a means for globally tropicalizing a toric variety, so if the embedding has the A_2-property, we obtain global tropicalization for the spherical variety. In particular, this is a new means for tropicalizing subvarieties of spherical homogeneous spaces.

There is a technical consideration that the embedding in a toric variety will not work unless the spherical variety has the A_2-property (Definition 4.4.1). This is not a particularly restrictive condition. Moreover, a spherical embedding X without the A_2-property can be realized as a collection of spherical embeddings with the A_2-property that are glued together along shared orbits. It follows that even though $\mathrm{trop}_G(X)$ cannot be computed globally with this theory, it can still be recovered by tropicalizing the subvarieties with the A_2-property and gluing together.

To finish the chapter, we use this theory to conclude that the spherical tropicalization procedure commutes with taking closures. This chapter covers the content of [24].

In Chapter 5, we discuss the structure of spherical tropicalizations. The theory from Chapter 4 gives some partial results since we can use the toric structure theorem and then apply the piecewise linear map. However, a structure theorem for spherical tropicalization similar to that for toric tropicalization seems more difficult. The balancing results in the toric case model the intersection theory of the toric variety, but the intersection theory on the spherical variety is too subtle for the combinatorics of the spherical tropicalization to pick up.

Finally, we suggest a definition for tropicalization with respect to non-constant coefficients, which has not yet been considered by any of the previous research. To justify the definition, we prove a result relating the non-constant tropicalization with tropical compactifications, generalizing part (ii) of Theorem 1.4.2.

In an effort to aid readability, necessary portions of the background in this introduction are repeated at the beginning of each chapter.

Chapter 2: Tropicalizing Spherical Embeddings

Marrying algebraic geometric ideas and combinatorics is an active area of research. In recent years, the notion of tropicalization has proven to be a fruitful such tool to answer questions in algebraic geometry. Particularly, toric varieties have benefited from tropical geometric methods meshing with their inherent combinatorial structure. See for example Chapter 6 of [21] for an introduction to some examples of the utility of tropical ideas in the toric world. Toric varieties are examples of spherical varieties, which encompass a wider class of algebraic objects, among them flag varieties and symmetric varieties. Spherical varieties also have combinatorial structure in the form of colored fans, which directly generalize the well-known polyhedral fans of toric geometry. This theory was developed by Luna and Vust [20] in 1983. It is a natural idea to take advantage of the similar combinatorial structure and extend the theory of tropicalization from toric varieties to the more general case.

The first steps in this direction were taken by Tassos Vogiannou in his thesis [33]. Among other results, Vogiannou developed a definition for the tropicalization of subvarieties of a spherical homogeneous space, which is the analogue of the dense torus orbit present in a toric variety. In a forthcoming paper, Kiumars Kaveh and Christopher Manon extend Vogiannou's work by defining a theory of Gröbner bases on spherical varieties and showing that their definition agrees with Vogiannou's via a spherical fundamental theorem. They further consider a notion of spherical amoebas and show, with an additional assumption, that this amoeba approaches the tropicalization.

The purpose of this chapter is to define the tropicalization of a general spherical embedding. Our blueprint for this construction appeared separately in [16] and [26]. These papers define the tropicalization of a toric variety by extending the tropicalization of its dense torus. We will mimic their ideas using the shared polyhedral fan structure of toric and spherical varieties.

The layout of this chapter is as follows. In §2.1, we review the basic theory of spherical varieties, and §2.2 describes Vogiannou's definition of tropicalizing spherical homogeneous spaces. Our construction of the tropicalization of a spherical embedding is explained in §2.3, and §2.4 contains examples. Finally, §2.5 describes a means for tropicalizing morphisms between spherical embeddings.

2.1 The Combinatorics of Spherical Embeddings

There are a number of surveys on spherical varieties and their combinatorial structure. Refer for example to [20], [18], [25], or [27] for more details on the theory discussed in this section. There is occasionally a clash between symbols usually used for toric varieties and their analogs in spherical varieties; whenever possible we have favored the toric conventions as these are more widely known. We work throughout over an algebraically closed field k. Let G be a connected reductive group with a Borel subgroup B. Let $H \leq G$ be a closed subgroup such that the action of B on G/H via the left action of G has an open orbit. In this case, we call G/H a *spherical* *homogeneous space*. A normal G-variety X that contains G/H as an open orbit of the action of G is called a *spherical embedding*.

Let $\mathcal{X}$ denote the group of characters $B \to k^*$. We consider the B semi-invariant rational functions on G/H:

$$k(G/H)^{(B)} := \{f \in k(G/H)^* : \text{there exists } \chi_f \in \mathcal{X} \text{ such that } gf = \chi_f(g)f \text{ for all } g \in B\}.$$

Here, the action of the Borel subgroup on $k(G/H)$ is given by $gf(x) = f(g^{-1}x)$, so gf is only defined on those x such that $g^{-1}x$ is in the domain of f. This affords us a homomorphism $k(G/H)^{(B)} \to \mathcal{X}$ defined by $f \mapsto \chi_f$. Further, the kernel of this map is the set of nonzero constant functions, which we write as k^*. Then denote by $\mathcal{M}$ or $\mathcal{M}(G/H)$ the image of $k(G/H)^{(B)}/k^*$ in $\mathcal{X}$. The lattice $\mathcal{M}$ is finitely generated and free, so we obtain a vector space $\mathcal{N}_{\mathbb{Q}}(G/H) := \operatorname{Hom}(\mathcal{M}, \mathbb{Q}) \cong \mathbb{Q}^m$, simply denoted $\mathcal{N}_{\mathbb{Q}}$ when the underlying homogeneous space is clear. The integer m is called the *rank* of the G/H-embedding.

We must further define the valuation cone, which will lie inside $\mathcal{N}_{\mathbb{Q}}$. We consider G-invariant $\mathbb{Q}$-valuations $k(G/H) \to \mathbb{Q}$ that are trivial on k^*. By restricting such a valuation to $k(G/H)^{(B)}$, we obtain an induced map $k(G/H)^{(B)}/k^* \to \mathbb{Q}$, so we can identify it with a point in $\mathcal{N}_{\mathbb{Q}}$. This identification yields a bijection between the set of such G-invariant valuations and a rational convex cone in $\mathcal{N}_{\mathbb{Q}}$ which we call the *valuation cone*, denoted by $\mathcal{V}(G/H)$ or $\mathcal{V}$. We write $\mathcal{D}(G/H)$ or $\mathcal{D}$ for the (finite) set of B-stable prime divisors in G/H. We refer to $\mathcal{D}(G/H)$ as the *palette* of G/H and call its elements *colors*. Every color D induces a valuation ν_D on $k(G/H)$ given by a function's order of vanishing along the divisor. We write ρ for the map defined by $D \mapsto \nu_D$ and note that ρ need not be injective.

To recap, given a spherical homogeneous space G/H, we associate a vector space $\mathcal{N}_\mathbb{Q}$ containing the valuation cone $\mathcal{V}$, the (finite) palette $\mathcal{D}$, and a map $\rho : \mathcal{D} \to \mathcal{N}_\mathbb{Q}$.

Definition 2.1.1. Let $\sigma \subseteq \mathcal{N}_\mathbb{Q}$ be a cone and $\mathcal{F} \subseteq \mathcal{D}$. We call the pair $(\sigma, \mathcal{F})$ a *colored cone* if the following properties are satisfied:

1. σ is generated by $\rho(\mathcal{F})$ and finitely many elements of $\mathcal{V}$;

2. $\operatorname{int}(\sigma) \cap \mathcal{V} \neq \emptyset$.

We say that $(\sigma, \mathcal{F})$ is *strictly convex* if in addition σ is a strictly convex cone and $0 \notin \rho(\mathcal{F})$.

Definition 2.1.2. A colored cone $(\tau, \mathcal{F}')$ is a *(colored) face* of a colored cone $(\sigma, \mathcal{F})$ if τ is a face of σ satisfying $\operatorname{int}(\tau) \cap \mathcal{V} \neq \emptyset$ such that $\mathcal{F}' = \mathcal{F} \cap \rho^{-1}(\tau)$. In this case we write $(\tau, \mathcal{F}') \preceq (\sigma, \mathcal{F})$ or $\tau \preceq \sigma$ if the colors are understood.

Definition 2.1.3. A *colored fan* is a finite collection Σ of colored cones such that the following hold:

1. If $(\sigma, \mathcal{F}) \in \Sigma$ is a colored cone and $(\tau, \mathcal{F}')$ is a face of $(\sigma, \mathcal{F})$, then $(\tau, \mathcal{F}') \in \Sigma$.

2. Every $v \in \mathcal{V}$ is in the interior of at most one colored cone in Σ.

We say in addition that Σ is *strictly convex* if each of its colored cones is strictly convex.

We also define here the support of a colored fan, though it will not come up again until Chapter 5.

Definition 2.1.4. The *support* of a colored fan Σ is the following subset of $\mathcal{V}$:

$$\operatorname{supp}(\Sigma) := \left(\bigcup_{(\sigma,\mathcal{F}) \in \Sigma} \sigma \right) \cap \mathcal{V}.$$

We emphasize that in this definition the second condition only applies to points of $\mathcal{V}$, so that colored cones of the colored fan are allowed to overlap nontrivially outside the valuation cone. With these definitions in hand, we can describe the colored fan associated to a G/H-embedding X. Note that if D is a B-stable prime divisor on X that is not G-stable, then the intersection $D \cap G/H$ is a color of G/H. Conversely, the closure in X of a color is a B-stable prime divisor D on X that is not G-stable. Thus we can identify the palette with the set of all such divisors. For a closed G-orbit $\mathcal{O}$ of X, let $\mathcal{F} \subseteq \mathcal{D}$ consist of the B-stable prime divisors containing $\mathcal{O}$ that are not G-stable. The colored cone $(\sigma, \mathcal{F})$ associated to $\mathcal{O}$ is spanned by $\rho(\mathcal{F})$ and the set of G-stable prime divisors containing $\mathcal{O}$. Taking these colored cones over every G-orbit of X, we obtain a colored fan. If a G/H-embedding X has a single closed G-orbit and hence a single maximal colored cone, we call X *simple*. A G/H-embedding consists of some finite number of simple embeddings glued together along G-orbits; this structure is reflected in the polyhedral geometry of the colored fan.

Theorem 2.1.5. [20, Prop. 8.10], [18, Thm. 3.3] *There is a bijection between simple G/H-embeddings and strictly convex colored cones in $\mathcal{N}_{\mathbb{Q}}$, and there is a bijection between G/H-embeddings and strictly convex colored fans in $\mathcal{N}_{\mathbb{Q}}$.*

Example 2.1.6. Let $G = \mathrm{Sl}_2$ with B the subgroup of upper triangular matrices and

$$H = \{M \in G \mid M \text{ is upper triangular with 1's on the diagonal}\}.$$

Then $G/H = \mathbb{A}^2 \setminus \{0\}$ where the action of G is given by matrix multiplication of a column vector $[x\ y]^T$. Every character $B \to k^*$ is of the form

$$\chi_n : \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \mapsto a^n$$

for some $n \in \mathbb{Z}$, so $\mathcal{X} \cong \mathbb{Z}$. Under the prescribed action of G, we can see that $k(G/H)^{(B)}/k^* = \{y^n \mid n \in \mathbb{Z}\}$ and that the character associated to y^n is χ_n. It follows that $\mathcal{M} \cong \mathbb{Z}$ and hence $\mathcal{N}_{\mathbb{Q}} \cong \mathbb{Q}$.

We now turn to the valuation cone $\mathcal{V}$. Consider the following two valuations of $k(G/H)$, which are G-invariant:

$$\frac{f}{g} \mapsto \operatorname{mindeg} f - \operatorname{mindeg} g \qquad\qquad \frac{f}{g} \mapsto \deg g - \deg f$$

Here, mindeg denotes the minimum degree of a monomial in a polynomial in $k[x, y]$. After restricting to $k(G/H)^{(B)}$, we see that the valuation on the left corresponds to sending $y^m \mapsto m$ (i.e. χ_1^*) and the one on the right to $y^m \mapsto -m$ (i.e. χ_{-1}^*). Thus, positive multiples of these valuations induce every possible element of $\mathcal{N}_{\mathbb{Q}} = \operatorname{Hom}(\mathcal{M}, \mathbb{Q})$ and so $\mathcal{V} = \mathcal{N}_{\mathbb{Q}}$.

The only closed B-orbit contained in G/H is the divisor $D := V(y)$, so the palette $\mathcal{D}$ consists solely of D. This means that an embedding of G/H can have at most one color, corresponding to the divisor where y vanishes. This divisor gives the ray spanned by χ_1^*.

We'll finish by explicitly computing the fan associated to $\mathbb{P}^2$ with homogeneous coordinates W, X, and Y. We can realize $\mathbb{P}^2$ as an embedding of $\mathbb{A}^2 \setminus \{0\}$ via $[x\ y]^T \mapsto [1 : x : y]$. There are three G-orbits in $\mathbb{P}^2$:

$$\begin{aligned} \mathbb{A}^2 \setminus \{0\} &:= \{[1 : x : y] \mid x, y \in k \text{ not both zero}\} \\ V(W) &:= \{[0 : x : y] \mid x, y \in k \text{ not both zero}\} \\ O &:= \{[1 : 0 : 0]\}. \end{aligned}$$

The latter two orbits are closed, so our fan will have two maximal cones. The orbit $V(W)$ is itself a B-stable prime divisor. This divisor is G-stable, so we will have a cone without color. The function y in $k(\mathbb{A}^2 \setminus \{0\})$ can be written as Y/W on $\mathbb{P}^2$. On $V(W)$, Y/W has a pole of order 1, so the cone associated to this orbit is the ray spanned by χ^*_{-1}. The other closed orbit O is contained in one B-stable prime divisor as well: $V(Y)$. This divisor is not G-stable, so the corresponding ray will have color. Clearly y vanishes with order 1 on $V(Y)$, so this will give the cone spanned by χ^*_1. This example is drawn in Table 2.1 along with the other colored fans of $\mathbb{A}^2 \setminus \{0\}$. This table also appears in [33] except for the final column, which will be explained in Example 2.4.3. Note how the color is indicated by a bullseye.

Variety	Closed G-orbits	Colored Fan	Tropicalization
$\mathbb{A}^2 \setminus \{0\}$	$\mathbb{A}^2 \setminus \{0\}$		
$\mathbb{A}^2$	$\{0\}$		
$\mathrm{Bl}_0(\mathbb{A}^2)$	E		
$\mathbb{P}^2 \setminus \{0\}$	$V(W)$		
$\mathbb{P}^2$	$V(W)$, $\{0\}$		
$\mathrm{Bl}_0(\mathbb{P}^2)$	$V(W)$, E		

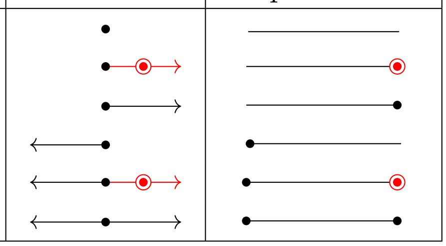


Table 2.1: Colored fans and colored tropicalizations associated to $\mathbb{A}^2 \setminus \{0\}$. Here, E denotes the exceptional divisor of the blowup.

2.2 Tropicalizing Homogeneous Spaces

In his thesis [33], Tassos Vogiannou defines the tropicalization of a subvariety of the spherical homogeneous space G/H, extending the well-known theory of subvarieties of a torus. We outline his construction here; more details and examples can be found in his thesis. Suppose G/H is a spherical homogeneous space over k, let $K := k(\!(t)\!)$ denote the field of Laurent series, and let $\overline{K} := \bigcup_{n\in\mathbb{N}} k(\!(t^{1/n})\!)$ denote the field of Puiseux series. We use the valuation $\nu : \overline{K}^* \to \mathbb{Q}$ that gives the lowest power of t

appearing with nonzero coefficient. Note that this restricts naturally to K^* and is trivial on k^*.

We will define a map $G/H(K) \to \mathcal{N}_{\mathbb{Q}}$. Let $\gamma : \operatorname{Spec} K \to G/H$ be a K-point of G/H. We will define a G-invariant discrete valuation ν_γ on $k(G/H)^*$ associated to γ. To do this, we need to describe how ν_γ acts on rational functions, so let $f \in k(G/H)^*$ be arbitrary. The domain of f may not contain the image of γ, but we may find $g \in G$ such that the image of γ is in the domain of gf. There is a pullback map $\gamma^* : k(G/H) \to K$ given by evaluation at γ, so we consider $\gamma^*(gf) \in K$. Then we write $\nu_\gamma(f) = \nu(\gamma^*(gf))$. This is not a priori well-defined since it may depend on g. To overcome this, we take g so that $\nu(\gamma^*(gf))$ is minimized; this minimum is achieved on an open set of G and we call such g *sufficiently general*.

Thus we have a map $G/H(K) \to \{G\text{-invariant discrete valuations on } k(G/H)^*\}$ given by $\gamma \mapsto \nu_\gamma$. As discussed, G-invariant discrete valuations on $k(G/H)^*$ determine elements of $\mathcal{V}$, so this is really a map $G/H(K) \to \mathcal{V}$. Further, we can extend this map so it is defined over $G/H(\overline{K})$. Indeed, suppose $\gamma : \operatorname{Spec} \overline{K} \to G/H$ is a $\overline{K}$-point. This induces a homomorphism of k-algebras $\gamma^* : A \to \overline{K}$ since the image of γ must lie in some open affine $\operatorname{Spec} A \subseteq G/H$. Since G/H is of finite type, A is finitely-generated as a k-algebra, and so it follows that γ^* factors through $k(\!(t^{1/n})\!)$ for some sufficiently large n. Thus, γ factors as $\operatorname{Spec} \overline{K} \to \operatorname{Spec} k(\!(t^{1/n})\!) \to G/H$. We can think of $\operatorname{Spec} k(\!(t^{1/n})\!)$ as the spectrum of Laurent polynomials in an indeterminate variable $t^{1/n}$. This morphism induces a valuation by the work above; dividing this valuation by n gives a valuation ν_γ, which we associate to γ. This extension in fact gives a surjection $\operatorname{val} : G/H(\overline{K}) \twoheadrightarrow \mathcal{V}$, which allows us to finally define the tropicalization of a subvariety of a homogeneous space.

Definition 2.2.1. If $Y \subseteq G/H$ is a subvariety, the *tropicalization* of Y is $\mathrm{trop}_G(Y) := \mathrm{val}\left(Y\left(\overline{K}\right)\right)$.

In particular, note that $\mathrm{trop}_G(G/H) = \mathcal{V}(G/H)$, a fact we will use in §2.3.

2.3 The Construction

Again let G/H be a spherical homogeneous space and let $\mathcal{N}_{\mathbb{Q}}$ and $\mathcal{V} \subseteq \mathcal{N}_{\mathbb{Q}}$ be the associated vector space and valuation cone. Let X be a simple G/H-embedding with maximal colored cone $(\sigma, \mathcal{F})$. We will show how to tropicalize X and then see how the tropicalization of a general embedding can be obtained by tropicalizing simple embeddings and gluing. Then we will describe how to tropicalize a subvariety.

Each colored face τ of σ corresponds to a G-orbit $\mathcal{O}_\tau$ of X, and [18, Corollary 2.2] says that each orbit is a spherical homogeneous G-variety with an open orbit of the same Borel subgroup B. As a spherical homogeneous space, an orbit $\mathcal{O}_\tau$ associated to τ has a valuation cone $\mathcal{V}_\tau := \mathcal{V}(\mathcal{O}_\tau)$ that lies in a $\mathbb{Q}$-vector space $\mathcal{N}_{\mathbb{Q}}(\mathcal{O}_\tau)$. Then as a set, we define $\mathrm{trop}_G(X) := \bigsqcup_{\tau \preceq \sigma} \mathcal{V}_\tau$. This is similar in spirit to the construction of [16] and [26]; we break X up into orbits and tropicalize each of them separately. It only remains to define a topology on this set.

Let $\overline{\mathbb{Q}} := \mathbb{Q} \cup \{\infty\}$. We write

$$\overline{\mathcal{N}}(\sigma) := \mathrm{Hom}\left(\sigma^\vee \cap \mathcal{M}, \overline{\mathbb{Q}}\right),$$

where the homomorphisms in the set on the right are semigroup homomorphisms. We will show that as sets $\bigsqcup_{\tau \preceq \sigma} \mathcal{N}_{\mathbb{Q}}(\mathcal{O}_\tau) \subseteq \overline{\mathcal{N}}_{\mathbb{Q}}$.

We fix our attention on a colored face $\tau \preceq \sigma$. There is a copy of $\mathcal{N}_{\mathbb{Q}}(\mathcal{O}_\tau)$ in $\overline{\mathcal{N}}(\sigma)$ given by considering those semigroup homomorphisms $\varphi : \sigma^\vee \cap \mathcal{M} \to \overline{\mathbb{Q}}$ for which

$\varphi^{-1}(\mathbb{Q}) = \tau^{\perp} \cap (\sigma^{\vee} \cap \mathcal{M})$. More explicitly,

$$\operatorname{Hom}\left(\tau^{\perp} \cap (\sigma^{\vee} \cap \mathcal{M}), \mathbb{Q}\right) \cong \operatorname{Hom}\left(\tau^{\perp} \cap \mathcal{M}, \mathbb{Q}\right) \cong \mathcal{N}_{\mathbb{Q}}(\mathcal{O}_{\tau}),$$

which we see as follows. We have that $\tau^{\perp} \cap \mathcal{M}$ consists of those functions in $\mathcal{M} = k(G/H)^{(B)}/k^*$ that do not have zeroes or poles along the orbit $\mathcal{O}_{\tau}$. These are precisely the functions in $\mathcal{M}$ that can be restricted to B semi-invariant rational functions on $\mathcal{O}_{\tau}$. Restriction thus gives a map $\tau^{\perp} \cap (\sigma^{\vee} \cap \mathcal{M}) \to \mathcal{M}(\mathcal{O}_{\tau})$. Theorem 6.3 of [18] shows that this map is an isomorphism, so after dualizing we have $\operatorname{Hom}\left(\tau^{\perp} \cap (\sigma^{\vee} \cap \mathcal{M}), \mathbb{Q}\right) \cong \mathcal{N}_{\mathbb{Q}}(\mathcal{O}_{\tau})$. Further, homomorphisms in $\operatorname{Hom}\left(\tau^{\perp} \cap (\sigma^{\vee} \cap \mathcal{M}), \mathbb{Q}\right)$ extend uniquely to homomorphisms in $\overline{\mathcal{N}}(\sigma)$ by sending every character outside $\tau^{\perp} \cap \mathcal{M}$ to ∞.

In the toric case, it now follows that $\bigsqcup_{\tau \preceq \sigma} \mathcal{N}_{\mathbb{Q}}(\mathcal{O}_{\tau})$ is in bijective correspondence with $\overline{\mathcal{N}}(\sigma)$. For a general spherical variety, this is not necessarily true. This is because a colored cone may contain a subcone which is a face in the sense of polyhedral geometry but which lies outside the valuation cone and therefore does not correspond to an orbit in the spherical variety. To address this, we write $\overline{\mathcal{N}}_{\mathcal{V}}(\sigma)$ to denote the set of homomorphisms $\varphi \in \overline{\mathcal{N}}(\sigma)$ such that $\varphi^{-1}(\mathbb{Q}) = \tau^{\perp} \cap (\sigma^{\vee} \cap \mathcal{M})$ for some colored face $\tau \preceq \sigma$.

Now every homomorphism $\varphi \in \overline{\mathcal{N}}_{\mathcal{V}}(\sigma)$ is realized as an extension of a homomorphism $\tau^{\perp} \cap (\sigma^{\vee} \cap \mathcal{M}) \to \mathbb{Q}$ for some unique $\tau \preceq \sigma$ where $\varphi^{-1}(\mathbb{Q}) = \tau^{\perp} \cap (\sigma^{\vee} \cap \mathcal{M})$. Thus we have a bijective correspondence:

$$\overline{\mathcal{N}}_{\mathcal{V}}(\sigma) \leftrightarrow \bigsqcup_{\tau \preceq \sigma} \mathcal{N}_{\mathbb{Q}}(\mathcal{O}_{\tau}).$$

Placing the topology on $\overline{\mathcal{N}}_{\mathcal{V}}(\sigma)$ inherited from $\overline{\mathbb{Q}}^{\sigma^{\vee} \cap \mathcal{M}}$, we obtain a topology on $\bigsqcup_{\tau \preceq \sigma} \mathcal{N}_{\mathbb{Q}}(\mathcal{O}_{\tau})$. Under this topology, $\bigsqcup_{\tau \preceq \sigma} \mathcal{N}_{\mathbb{Q}}(\mathcal{O}_{\tau})$ is isomorphic to a subspace of $\overline{\mathbb{Q}}^{m}$, where m is the rank of $\mathcal{N}$.

We still must see how $\bigsqcup_{\tau \preceq \sigma} \mathcal{V}_\tau$ lies in $\overline{\mathcal{N}}_{\mathcal{V}}(\sigma) = \bigsqcup_{\tau \preceq \sigma} \mathcal{N}_{\mathbb{Q}}(\mathcal{O}_\tau)$ under this topology. Any homomorphism in $\operatorname{Hom}(\tau^\perp \cap \mathcal{M}, \mathbb{Q})$ that is induced by a G-invariant valuation is the restriction of a G-variant valuation in $\mathcal{V}(G/H) \subseteq \operatorname{Hom}(\mathcal{M}, \mathbb{Q})$. Indeed, by Corollary 1.5 in [18], any G-invariant valuation on $k(\mathcal{O}_\tau)$ can be lifted to a G-invariant valuation on $k(G)$, which can then be restricted to $k(G/H)$. Thus, the valuation cone $\mathcal{V}_\tau$ is the image of the valuation cone $\mathcal{V}$ under the map $\operatorname{Hom}(\mathcal{M}, \mathbb{Q}) \to \operatorname{Hom}\left(\tau^\perp \cap \mathcal{M}, \mathbb{Q}\right)$ induced by the inclusion $\tau^\perp \cap \mathcal{M} \hookrightarrow \mathcal{M}$. Thus $\operatorname{trop}_G(X) = \bigsqcup_{\tau \preceq \sigma} \mathcal{V}_\tau$ inherits the subspace topology from $\overline{\mathcal{N}}_{\mathcal{V}}(\sigma)$.

To tropicalize a non-simple spherical embedding, we tropicalize the simple embeddings corresponding to each of its maximal cones and then glue these together along the tropicalizations of their shared orbits. If $Y \subseteq G/H$ is a closed subvariety and $\overline{Y} \subseteq X$ its closure in X, then we define

$$\operatorname{trop}_G\left(\overline{Y}\right) := \bigsqcup_{\tau \preceq \sigma} \operatorname{trop}_G\left(\overline{Y} \cap \mathcal{O}_\tau\right) \subseteq \bigsqcup_{\tau \preceq \sigma} \mathcal{V}_\tau = \operatorname{trop}_G(X).$$

Here, $\operatorname{trop}_G\left(\overline{Y} \cap \mathcal{O}_\tau\right)$ denotes the tropicalization as a subvariety of a homogeneous space and we give $\operatorname{trop}_G\left(\overline{Y}\right)$ the subspace topology inherited from $\operatorname{trop}_G(X)$.

Our construction as discussed thus far only recognizes the polyhedral structure of the colored fan but ignores whether or not that fan has colors. We need this information to completely classify all spherical embeddings, so it seems useful to remember the presence of colors when we tropicalize. We address this as follows. In a colored fan, we can think of our palette of colors as a collection of points in $\mathcal{N}_{\mathbb{Q}}$ corresponding to B-stable prime divisors. If no colors appear in our fan, we call the spherical variety *toroidal*. If a color appears, it will lie in some number of colored cones, which is to say the associated prime divisor contains the orbits corresponding to those cones. Each such orbit gives a cone in the stratification of the tropicalization; we simply record whether the color appears in the colored fan by labeling its associated valuation cone with that color. We show in Example 2.4.3 two different spherical embeddings of the same homogeneous space that have different colored tropicalizations; if color is ignored the tropicalizations become the same.

Before proceeding to examples, we prove a result showing that the extended tropicalization reflects whether or not the variety is complete. The proof requires the following result:

Theorem 2.3.1. (cf. [20, Prop. 8.10]) *A G/H-embedding X with colored fan Σ is complete if and only if* $\operatorname{supp}(\Sigma) = \mathcal{V}(G/H)$.

To state our result cleanly, we also must work over $\mathbb{R}$ rather than $\mathbb{Q}$. That is, we proceed as before with the construction, but start with $\operatorname{Hom}(\sigma^\vee \cap \mathcal{M}, \mathbb{R} \cup \{\infty\})$ rather than $\operatorname{Hom}\left(\sigma^\vee \cap \mathcal{M}, \overline{\mathbb{Q}}\right)$. In other words, $\operatorname{trop}_G(X)$ as we have defined it lies in a stratification of $\mathbb{Q}$-vector spaces, and for the following result we tensor each of those vector spaces with $\mathbb{R}$. The combinatorial structure of the tropicalization is unchanged; this technical point is only necessary because $\mathbb{R}$ is a complete metric space with respect to the Euclidean metric and $\mathbb{Q}$ is not.

Theorem 2.3.2. *A G/H-embedding X is complete if and only if* $\operatorname{trop}_G(X)$ *(taken with respect to $\mathbb{R}$, not $\mathbb{Q}$) is a compact topological space.*

Proof. Let X be a complete G/H-embedding with colored fan Σ and let $\mathcal{U}$ be a covering of $\operatorname{trop}_G(X)$ by open sets. The tropicalization $\operatorname{trop}_G(X)$ is a stratification of (finitely-many) valuation cones lying in $\mathbb{R}$-vector spaces. Each full-dimensional colored cone $(\sigma, \mathcal{F}) \in \Sigma$ tropicalizes to a 0-dimensional vector space $\mathcal{V}_\sigma$ in $\operatorname{trop}_G(X)$. For each such σ, choose $U(\sigma) \in \mathcal{U}$ that contains the point $\mathcal{V}_\sigma$.

Now let τ be a codimension one colored cone in Σ. Because X is complete, Σ covers $\mathcal{V}(G/H)$, and so τ either lies between two full-dimensional colored cones or lies on the boundary of $\mathcal{V}(G/H)$ and is the face of a full-dimensional colored cone. In either case, $\mathcal{V}_\tau \setminus \left(\bigcup_{\sigma \succeq \tau} U(\sigma)\right)$ is a closed bounded subset of $\mathcal{V}_\tau$ under the topology on $\mathrm{trop}_G(X)$. This subset is compact, so it can it be covered by finitely-many elements $\{U_i(\tau)\} \subseteq \mathcal{U}$. As τ was arbitrary, we may do this for every codimension one cone in Σ.

For each codimension two cone, the associated valuation cone is similarly closed and bounded upon removal of the $U(\sigma)$'s and $U_i(\tau)$'s, so it can be covered by finitely-many elements of $\mathcal{U}$. Proceeding in this way through cones of higher and higher codimension generates a finite sub-covering of $\mathrm{trop}_G(X)$, as needed.

Conversely, this process will fail precisely when $\mathrm{supp}(\Sigma) \neq \mathcal{V}(G/H)$ as it will run into unbounded valuation cones that cannot necessarily be covered by finitely many open sets. $\square$

2.4 Examples

Example 2.4.1. In the toric case, $G = T^n$ is a torus of dimension n, H is trivial, and $B = G$, so there are no colors. The B semi-invariant rational functions are precisely the monomials in the variables $\{x_1, \ldots, x_n\}$, so $\mathcal{N}_\mathbb{Q} \cong \mathbb{Q}^n$, spanned by cocharacters χ_i^* defined as follows:

$$\chi_i^*(x_j) = \begin{cases} 1 & i = j \\ 0 & \text{otherwise.} \end{cases}$$

We can also see that the valuation cone $\mathcal{V}$ is all of $\mathcal{N}_\mathbb{Q}$. Indeed, consider the valuations

$$\frac{f}{g} \mapsto \mathrm{mindeg}_i(f) - \mathrm{mindeg}_i(g) \qquad\qquad \frac{f}{g} \mapsto \deg_i(g) - \deg_i(f),$$

where $\deg_i$ and mindeg_i respectively denote the degree and minimum degree in x_i. These valuations are G-invariant and induce the cocharacters χ_i^* and $-\chi_i^*$, so $\mathcal{V} = \mathcal{N}_{\mathbb{Q}}$.

Having realized the torus as a spherical homogeneous space, the colored fan associated to a toric variety when viewed as a spherical embedding is the same as the fan coming from the theory of toric geometry. Our definition of the tropicalization only relies on the polyhedral structure of this fan and is identical to the process described in [16] and [26].

Example 2.4.2. A flag variety is a spherical homogeneous space G/P where P is a parabolic subgroup, one which contains a Borel subgroup. Such a homogenous space has a trivial valuation cone, so the tropicalization of a flag variety is a point under this theory.

Example 2.4.3. We return now to the example and notation of $\mathbb{A}^2 \setminus \{0\}$ discussed in Example 2.1.6. In $\mathrm{Bl}_0(\mathbb{P}^2)$, there are three G-orbits: G/H, $V(W)$, and the exceptional divisor E. We have already seen that the valuation cone of G/H is a copy of $\mathbb{Q}$, so we move on to $V(W)$ and E. Both of these are copies of $\mathbb{P}^1$ and the action of $G = \mathrm{Sl}_2$ on both of them is given by matrix multiplication. In this case $G/B \cong \mathbb{P}^1$, so the closed orbits are flag varieties and our discussion in Example 2.4.2 tells us the tropicalizations are trivial. Let us show this explicitly. The rational functions $k(G/B) = k(\mathbb{P}^1)$ are quotients of two homogeneous polynomials of the same degree in the variables X and Y. The action of the Borel subgroup B on these functions is the same as it is on $k(G/H)$, so the only B semi-invariant rational functions on $k(G/B)$ are powers of Y. The only power of Y in $k(\mathbb{P}^1)$ is the constant function, so $k(\mathbb{P}^1)^{(B)}$ is trivial and hence so are the associated $\mathcal{M}$, $\mathcal{N}_{\mathbb{Q}}$, and $\mathcal{V}$.

Thus, $\bigsqcup_\tau \mathcal{V}(\mathcal{O}_\tau)$ in this case consists of a copy of $\mathbb{Q}$ and two points. The two points attach to $\mathbb{Q}$ by thinking of them as ∞ and $-\infty$. We can think of $\mathrm{Bl}_0(\mathbb{P}^2)$ as the two simple spherical varieties $\mathrm{Bl}_0(\mathbb{A}^2)$ and $\mathbb{P}^2 \setminus \{0\}$ glued together along $G/H = \mathbb{A}^2 \setminus \{0\}$. In $\mathrm{Bl}_0(\mathbb{A}^2)$, we add in limit points over 0, which correspond to extended valuations taking $y \in k(G/H)^{(B)}$ to ∞, giving a copy of $\overline{\mathbb{Q}}$. In $\mathbb{P}^2 \setminus \{0\}$, we add in limit points at infinity, which similarly correspond to extended valuations and we again get $\overline{\mathbb{Q}}$. We finally glue these copies of $\overline{\mathbb{Q}}$ along their shared copy of $\mathbb{Q}$ by identifying a number in one copy of $\mathbb{Q}$ with its reciprocal in the other copy. This is illustrated in Figure 2.1. The gluing is reminiscent of the tropicalization of $\mathbb{P}^1$ viewed as a toric variety, as described in [21, §6.2].

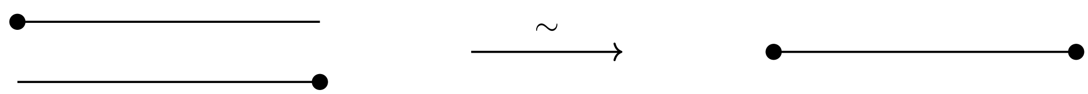


Figure 2.1: The colored tropicalization of $\mathrm{Bl}_0(\mathbb{P}^2)$

If instead we consider the G/H-embedding $\mathbb{P}^2$, the simple spherical varieties are $\mathbb{P}^2 \setminus \{0\}$ and $\mathbb{A}^2$. The former can be tropicalized as before, but $\mathbb{A}^2$ has a G-fixed point $[1:0:0]$ whose associated cone has color. The point has a trivial valuation cone, just like the effective divisor in $\mathrm{Bl}_0(\mathbb{P}^2)$. The gluing operation works the same as with $\mathrm{Bl}_0(\mathbb{P}^2)$, so topologically we again obtain a line segment. This is shown in Figure 2.2; the colored point associated to $[1:0:0]$ is on the right. The other

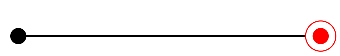


Figure 2.2: The colored tropicalization of $\mathbb{P}^2$

embeddings of $\mathbb{A}^2 \setminus \{0\}$ are obtained from $\mathrm{Bl}_0(\mathbb{P}^2)$ or $\mathbb{P}^2$ by omitting certain orbits. Thus, their tropicalizations omit the corresponding pieces, and we obtain the other results depicted in Table 2.1.

Example 2.4.4. This example extends [33, §5.3]. Our group is $G = \mathrm{Gl}_2 \times \mathrm{Gl}_2$ and the subgroup H is the diagonal, so that $G/H \cong \mathrm{Gl}_2$. the action of G on G/H is given by $(g, h) \cdot X = gXh^{-1}$ and the Borel subgroup is

$$B = \{(U, L) \mid U \text{ is upper triangular and } L \text{ is lower triangular}\}.$$

The Borel subgroup has an open orbit $\{(x_{ij}) \in \mathrm{Gl}_2 \mid x_{22} \neq 0\}$, so this is a spherical homogeneous space. We will embed G/H into $\mathrm{Bl}_0(\mathbb{A}^4)$ by sending $X = (x_{ij})$ to its image in the principal open subset $D(x_{11}x_{22} - x_{12}x_{21}) \subset \mathbb{A}^4 \setminus \{0\}$. Viewing $\mathrm{Bl}_0(\mathbb{A}^4)$ as a subvariety of $\mathbb{A}^4 \times \mathbb{P}^3$, we give $\mathbb{P}^3$ the coordinates y_{ij} and denote elements by $((x_{ij}), [y_{ij}])$, so that the blowup is cut out by the equations $x_{ij}y_{k\ell} = x_{k\ell}y_{ij}$. The action of G on $\mathrm{Bl}_0(\mathbb{A}^4)$ is then given by matrix multiplication in both components:

$$(g, h) \cdot ((x_{ij}), [y_{ij}]) = (g(x_{ij})h^{-1}, g[y_{ij}]h^{-1}), \quad (g, h) \in G, ((x_{ij}), [y_{ij}]) \in \mathrm{Bl}_0(\mathbb{A}^4).$$

In this case, the lattice of B semi-invariant rational functions $\mathcal{M} = k(G/H)^{(B)}/k^*$ on G/H is spanned by $f_1 := (x_{11}x_{22} - x_{12}x_{21})/x_{22}$ and $f_2 := x_{22}$. We choose these particular generators because the associated cocharacters are cleaner for computations. The palette $\mathcal{D}$ in this case consists of one B-stable prime divisor: $V(x_{22})$. The vector space $\mathcal{N}_{\mathbb{Q}}$ of cocharacters is two-dimensional, spanned by χ_1^* and χ_2^* defined as follows:

$$\chi_i^*(f_j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}.$$

The valuation cone $\mathcal{V}$ associated to G/H is $\{\alpha_1\chi_1^* + \alpha_2\chi_2^* \mid \alpha_1 \geq \alpha_2\}$. The valuation cone and palette of G/H are shown in Figure 2.3.

Let us now determine the colored fan associated to $\mathrm{Bl}_0(\mathbb{A}^4)$ as a Gl_2-embedding. Under the prescribed action of G, there are four G-orbits:

$$\begin{aligned}
\mathrm{Gl}_2 &:= \{((x_{ij}), [y_{ij}]) \mid (x_{ij}) \in \mathrm{Gl}_2\} \\
R_1 &:= \{((x_{ij}), [y_{ij}]) \mid (x_{ij}) \text{ has rank } 1\} \\
\mathbb{P}(\mathrm{Gl}_2) &:= \{(0, [y_{ij}]) \mid (y_{ij}) \in \mathrm{Gl}_2\} \\
\mathbb{P}(R_1) &:= \{(0, [y_{ij}]) \mid (y_{ij}) \text{ has rank } 1\}
\end{aligned}$$

Only one of these orbits is closed: $\mathbb{P}(R_1)$, so we will have one maximal colored cone.

There are three B-stable prime divisors of $\mathrm{Bl}_0(\mathbb{A}^4)$: the exceptional divisor E, $V(y_{11}y_{22} - y_{12}y_{21}) = V(\det(y_{ij}))$, and the color $V(x_{22})$. The only B-stable prime divisors containing $\mathbb{P}(R_1)$ are E and $V(\det(y_{ij}))$. Both of these are also G-stable, so our fan will have no colors. Along E, f_2 clearly vanishes with order 1 and f_1 can be written in the form $f_1 = f_2 \cdot (y_{11}y_{22} - y_{12}y_{21})/y_{22}^2$, so it also vanishes with order 1 along E. Thus we obtain a ray $\sigma_{1,1}$ in the direction $(1,1)$. Along $V(\det(y_{ij}))$, f_1 vanishes with order 1 and f_2 doesn't vanish, so we obtain a ray $\sigma_{1,0}$ in the direction $(1,0)$. Together $\sigma_{1,0}$ and $\sigma_{1,1}$ span a single two-dimensional cone σ. Figure 2.4 exhibits the colored cone associated to $\mathrm{Bl}_0(\mathbb{A}^4)$.

Now we apply our construction. We start with $\bigsqcup \mathcal{N}_{\mathbb{Q}}(\mathcal{O}_i)$, where the disjoint union is over four separate colored cones corresponding to the four G-orbits in $\mathrm{Bl}_0(\mathbb{A}^4)$. There is one colored cone of dimension zero (Gl_2), two colored cones of dimension one (R_1 and $\mathbb{P}(\mathrm{Gl}_2)$), and one colored cone of dimension two ($\mathbb{P}(R_1)$), so $\bigsqcup \mathcal{N}_{\mathbb{Q}}(\mathcal{O}_i)$ consists of one copy of $\mathbb{Q}^2$, two copies of $\mathbb{Q}^1$, and one zero-dimensional vector space. They are shown in Figure 2.5. The vertical line corresponds to the orbit R_1 and the slanted line to $\mathbb{P}(\mathrm{Gl}_2)$.

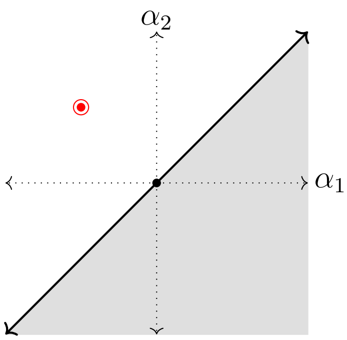


Figure 2.3: The valuation cone and palette of Gl_2

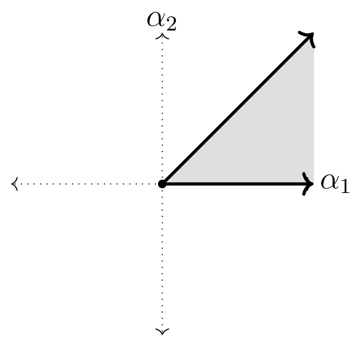


Figure 2.4: The colored cone of $\mathrm{Bl}_0(\mathbb{A}^4)$.

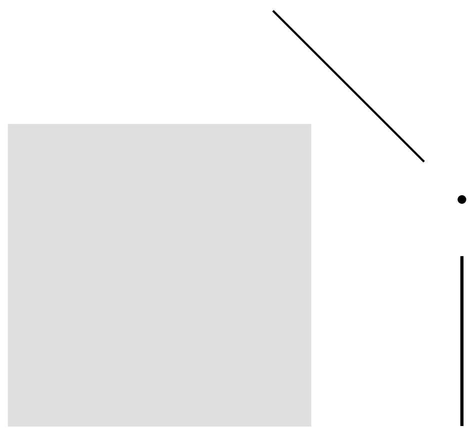


Figure 2.5: $\bigsqcup \mathcal{N}_{\mathbb{Q}}(\mathcal{O}_i)$

Now we consider the valuation cone $\mathcal{V}$ of each orbit. We have already seen that the valuation cone of Gl_2 is given by $\{(\alpha_1, \alpha_2) \mid \alpha_1 \geq \alpha_2\}$ and the valuation cone of $\mathbb{P}(R_1)$ is necessarily trivial. The orbit R_1 is a spherical homogeneous space isomorphic to $V(x_{11}x_{22} - x_{12}x_{21}) \subset \mathbb{A}^4 \setminus \{0\}$. The B semi-invariant rational functions $k(R_1)^{(B)}$ are just the powers of f_2 since f_1 vanishes along R_1. We can define valuations that consider the degree of a rational function similarly to Example 2.1.6, so the valuation cone $\mathcal{V}(R_1)$ is a copy of $\mathbb{Q}$. Alternatively, $\mathcal{V}(R_1)$ is the image of $\mathcal{V}(\mathrm{Gl}_2)$ under the projection map $\mathcal{N}_{\mathbb{Q}}(\mathrm{Gl}_2) \to \mathcal{N}_{\mathbb{Q}}(R_1)$. This map is defined by taking a product

$f_1^{\alpha_1} f_2^{\alpha_2} \in k(\mathrm{Gl}_2)^{(B)}/k^*$ to $f_2^{\alpha_2} \in k(R_1)^{(B)}/k^*$. The image of $\mathcal{V}(\mathrm{Gl}_2)$ is all of $\mathcal{N}_{\mathbb{Q}}(R_1)$, so $\mathcal{V}(R_1) = \mathcal{N}_{\mathbb{Q}}(R_1)$.

Finally, the orbit $\mathbb{P}(\mathrm{Gl}_2)$ is $D(y_{11}y_{22} - y_{12}y_{21}) \subset \mathbb{P}^3$. The B semi-invariant rational functions $k(\mathbb{P}(\mathrm{Gl}_2))^{(B)}$ are spanned by $(y_{11}y_{22} - y_{12}y_{21})/y_{22}^2$ since they must have the same degree in the numerator and denominator. Since $\nu(f_1) \geq \nu(f_2)$ for any G-invariant valuation ν on $\mathcal{M}(\mathrm{Gl}_2)$, we have $\nu((y_{11}y_{22} - y_{12}y_{21})/y_{22}^2) \geq 0$ for any G-invariant valuation ν on $\mathcal{M}(\mathbb{P}(\mathrm{Gl}_2))$. Thus, $\mathcal{V}(\mathbb{P}(\mathrm{Gl}_2))$ is a ray in $\mathcal{N}_{\mathbb{Q}}(\mathbb{P}(\mathrm{Gl}_2))$. The union of the valuation cones and their gluing is illustrated in Figure 2.6. In Table 2.2 we exhibit several other embeddings of Gl_2 along with their associated colored cones and colored tropicalizations.

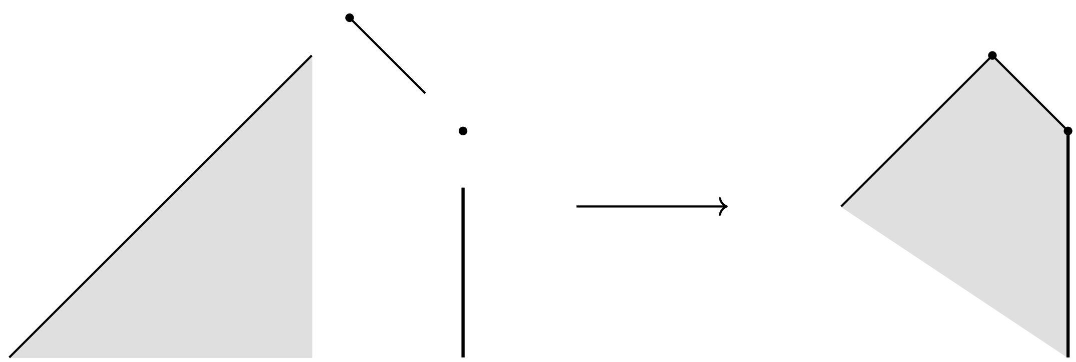


Figure 2.6: $\bigsqcup \mathcal{V}_i$ and the tropicalization of $\mathrm{Bl}_0(\mathbb{A}^4)$

2.5 Tropicalizing Morphisms

In this section, we describe how a morphism between spherical varieties induces a morphism between their tropicalizations. This theory is somewhat independent of the rest of the chapter, but it will be an important part of Chapter 4. We include it here rather than Chapter 4 because there we work over $\mathbb{C}$ and this section holds

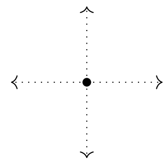
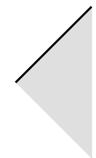
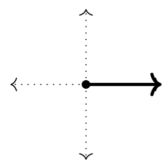
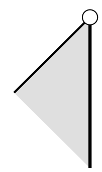
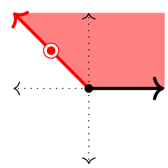
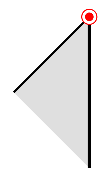
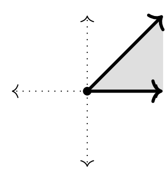
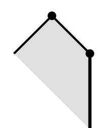
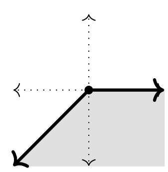
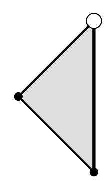
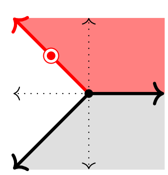
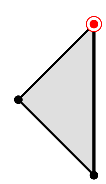
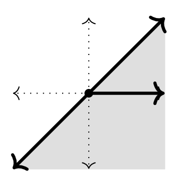
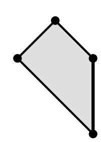
Variety	Colored Fan	Colored Tropicalization
Gl_2		
$\mathbb{A}^4 \setminus \{0\}$		
$\mathbb{A}^4$		
$\mathrm{Bl}_0(\mathbb{A}^4)$		
$\mathbb{P}^4 \setminus \{0\}$		
$\mathbb{P}^4$		
$\mathrm{Bl}_0(\mathbb{P}^4)$		

Table 2.2: Colored fans and colored tropicalizations of Gl_2-embeddings

in the generality of k an arbitrary algebraically-closed field. Our main result is that these tropicalized morphisms commute with tropicalization (Proposition 2.5.2), which extends [21, Corollary 6.2.17].

Let X and X' respectively be a G/H-embedding and a G/H'-embedding. If $\Phi : X \to X'$ is a G-morphism induced by a surjective G-equivariant morphism $G/H \to G/H'$, then there is an induced continuous map $\mathrm{trop}(\Phi) : \mathrm{trop}(X) \to \mathrm{trop}(X')$.

Before describing the map, we collect some facts about morphisms between spherical varieties as stated in [18, §4]. First, a G-equivariant dominant morphism $\Phi : G/H \to G/H'$ between spherical homogeneous spaces induces an injection $\Phi^* : \mathcal{M}(G/H') \hookrightarrow \mathcal{M}(G/H)$. This in turn induces a surjection $\Phi_* : \mathcal{N}_{\mathbb{Q}}(G/H) \to \mathcal{N}_{\mathbb{Q}}(G/H')$, which restricts to $\Phi_* : \mathcal{V}(G/H) \to \mathcal{V}(G/H')$. Note that Φ^* is defined by $f \mapsto f \circ \Phi$ and Φ_* is defined by $\mu \mapsto \mu \circ \Phi^*$.

Let $\mathcal{O}_i \subseteq X$ be a G-orbit with valuation cone $\mathcal{V}_i$. Because Φ is G-equivariant, it takes orbits of X to orbits of X', so $\Phi(\mathcal{O}_i) \subseteq \mathcal{O}'_i$ for some orbit $\mathcal{O}'_i$ of X'. Let $\mathcal{V}'_i$ denote the valuation cone of $\mathcal{O}'_i$. By restriction, we have a map $\mathcal{O}_i \to \mathcal{O}'_i$ and hence an induced map $\mathcal{V}(\mathcal{O}_i) \to \mathcal{V}(\mathcal{O}'_i)$. Since this holds for each G-orbit of X, we can define a map $\mathrm{trop}(\Phi) : \bigsqcup_i \mathcal{V}_i \to \bigsqcup_j \mathcal{V}'_j$ by taking the disjoint union of the pushforwards.

Suppose that we have a morphism $\Phi : G/H \to G'/H'$ of spherical homogeneous spaces that is equivariant with respect to some surjective homomorphism of algebraic groups $\varphi : G \to G'$. We would like to define the tropicalization of Φ in this setting where G is not necessarily equal to G'. The homomorphism φ gives an action of G on G'/H'; we show that this action makes G'/H' a spherical homogeneous space with respect to G.

By choosing an appropriate basepoint, we may assume that $\Phi(H) = H'$. Equivariance then implies that for any $g \in G$, $\Phi(gH) = \varphi(g)H'$. Note that we also have $\phi(H) \leq H'$. Indeed, if $h \in H$, we have $H' = \Phi(H) = \Phi(hH) = \varphi(h)H'$, so $\varphi(h) \in H'$. Then Φ can be factored as the natural projection $G/H \to G/\varphi^{-1}(H')$ followed by $\overline{\varphi} : G/\varphi^{-1}(H') \to G'/H'$ given by $g\varphi^{-1}(H') \mapsto \varphi(g)H'$. By Zariski's Main Theorem (see for example [22, §III.9]), $G/\varphi^{-1}(H') \cong G'/H'$ as varieties as G'/H' is normal, so $\Phi : G/H \to G'/H'$ may be realized as the projection $G/H \to G/\varphi^{-1}(H')$, and we may define $\operatorname{trop}(\Phi) : \operatorname{trop}_G(G/H) \to \operatorname{trop}_G(G/\varphi^{-1}(H'))$ as we did when $G' = G$ and φ was the identity. This makes G'/H' a spherical homogeneous space with respect to the action of G.

Proposition 2.5.1. *Let X and X' be a G/H-embedding and a G'/H'-embedding, respectively. Suppose $\Phi : X \to X'$ is a dominant morphism equivariant with respect to a surjective homomorphism $\varphi : G \to G'$. Then* $\operatorname{trop}(\Phi)$ *is continuous.*

Proof. By the argument preceding the proposition, we may assume that $G = G'$ and φ is the identity map.

Let $\mu \in \operatorname{trop}_G(\mathcal{O})$ where $\mathcal{O}$ is a G-orbit in X and let the G-orbit $\Phi(\mathcal{O})$ correspond to a colored cone $(\sigma', \mathcal{F}')$. Then there is a sequence $\{\mu_\ell\}_{\ell=1}^{\infty} \subset \operatorname{trop}_G(G/H)$ of valuations such that $\lim_{\ell\to\infty} \mu_\ell = \mu$ in the topology on $\operatorname{trop}_G(X)$. Let $f \in (\sigma')^\vee \cap \mathcal{M}'$ be an arbitrary B semi-invariant rational function on the orbit $\Phi(\mathcal{O})$. Then:

$$\begin{aligned}\lim_{\ell\to\infty} (\operatorname{trop}(\Phi)(\mu_\ell))(f) &= \lim_{\ell\to\infty} (\mu_\ell \circ \Phi^*)(f) \\ &= \lim_{\ell\to\infty} \mu_\ell \circ (f \circ \Phi) \\ &= \mu \circ (f \circ \Phi) \\ &= (\operatorname{trop}(\Phi)(\mu))(f).\end{aligned}$$

It follows that $\operatorname{trop}(\Phi)(\mu_\ell) \to \operatorname{trop}(\Phi)(\mu)$ as $\ell \to \infty$, and the claim follows. $\square$

Proposition 2.5.2. *Let X and X' be spherical embeddings with respect to groups G and G', respectively. Let $\Phi : X \to X'$ be a dominant morphism equivariant with respect to a surjective homomorphism $\varphi : G \to G'$ such that $G/\varphi^{-1}(H) \cong G'/H'$ as varieties. Then if $Y \subseteq X$ is a subvariety,* $\operatorname{trop}_{G'}(\Phi(Y)) = \operatorname{trop}(\Phi)(\operatorname{trop}_G(Y))$.

Proof. As before, we may assume that $G = G'$ and φ is the identity map.

Let $\mathcal{O} \subseteq X$ be an arbitrary G-orbit of X and consider the restriction of $\operatorname{trop}(\Phi)$ to the valuation cone $\mathcal{V}(\mathcal{O})$ associated to $\mathcal{O}$. By definition of $\operatorname{trop}(\Phi)$, the image of $\mathcal{V}(\mathcal{O})$ lies in the valuation cone $\mathcal{V}(\mathcal{O}')$ of some orbit $\mathcal{O}' \subseteq X'$. We will prove the statement in the case that $Y \subseteq \mathcal{O}$ is a subvariety of the spherical homogeneous space $\mathcal{O}$. The full statement follows directly by considering the individual intersections of a subvariety with each orbit.

Let $\gamma : \operatorname{Spec} \overline{K} \to Y$ be a $\overline{K}$-point of Y. Then $\Phi \circ \gamma$ is a $\overline{K}$-point of $\Phi(Y)$ and all $\overline{K}$-points of $\Phi(Y)$ arise in this way. This is because $\Phi : \mathcal{O} \to \mathcal{O}'$ is surjective as it is G-equivariant with respect to the surjective map φ. Thus, $\operatorname{trop}_G(\Phi(Y))$ is defined as follows:

$$\operatorname{trop}_G(\Phi(Y)) := \left\{ \begin{array}{c} k(\Phi(Y))^{(B)} \to \mathbb{Q} \\ f \mapsto \nu((\Phi \circ \gamma)^*(gf)) \end{array} : \gamma \in Y\left(\overline{K}\right) \right\},$$

where g is chosen to be sufficiently general for a given f. The tropicalization of Y is

$$\operatorname{trop}_G(Y) := \left\{ \begin{array}{c} k(Y)^{(B)} \to \mathbb{Q} \\ f \mapsto \nu(\gamma^*(gf)) \end{array} : \gamma \in Y\left(\overline{K}\right) \right\},$$

again for sufficiently general g. Applying $\operatorname{trop}(\Phi)$ means taking $\nu_\gamma \mapsto \nu_\gamma \circ \Phi^*$, so this gives us $\operatorname{trop}(\Phi)(\operatorname{trop}_G(Y))$:

$$\operatorname{trop}(\Phi)(\operatorname{trop}_G(Y)) := \left\{ \begin{array}{c} k(\Phi(Y))^{(B)} \to \mathbb{Q} \\ f \mapsto \nu(\gamma^*(g\Phi^*(f))) \end{array} : \gamma \in Y\left(\overline{K}\right) \right\}.$$

To prove the proposition, we need to show that $\nu((\Phi \circ \gamma)^*(gf)) = \nu(\gamma^*(g\Phi^*(f)))$ for all $f \in k(\Phi(Y))^{(B)}$ where g is chosen to be sufficiently general for both f and $\Phi^*(f)$. Note that $(\Phi \circ \gamma)^* = \gamma^* \circ \Phi^*$, so it will be sufficient to show that $\Phi^*(gf) = g\Phi^*(f)$. For $x \in Y$ we have the following, using the fact that Φ is G-equivariant:

$$\Phi^*(gf)(x) = (gf \circ \Phi)(x) = f(g^{-1}\Phi(x)) = (f \circ \Phi)(g^{-1}x) = g(f \circ \Phi)(x) = g\Phi^*(f)(x).$$

Thus, $\Phi^*(gf) = g\Phi^*(f)$ and so the proposition holds. $\square$

Chapter 3: An Extended Fundamental Theorem

Along with Vogiannou's work, Kaveh and Manon in [17] give an alternate means for tropicalizing subvarieties of spherical homogeneous spaces. Their approach uses a Gröbner theory for spherical varieties that they develop. Ultimately, they prove via a fundamental theorem that the two constructions coincide. In this chapter, we describe how their construction can be extended to spherical embeddings and show that this notion of extended spherical tropicalization coincides with that described in §2.3. We retain the conventions and notation used previously.

In §3.1, we recall the Fundamental Theorem of Tropical Geometry from the toric case and its extension to toric varieties. Then §3.2 discusses Gröbner tropicalization of spherical homogeneous spaces and §3.3 shows how this can be extended to spherical embeddings.

3.1 The Fundamental Theorem

We begin by outlining the Fundamental Theorem of Tropical Geometry in the toric case (cf. [28, Theorem 2.1] or [21, Theorem 3.2.3]). As in Chapter 1, we work over $\mathbb{Q}$ as this is the standard convention in spherical geometry; our results can be modified by tensoring with $\mathbb{R}$ to no ill effect.

Let $v : k \to \overline{\mathbb{Q}}$ be a valuation with a splitting $\varphi : v(k^*) \to k^*$ such that $(v \circ \varphi)(w) = w$; we write $t^w := \varphi(w)$. The valuation v has an associated valuation ring $R := \{x \in k \mid v(x) \geq 0\}$ with maximal ideal $\mathfrak{m} := \{x \in k \mid v(x) > 0\}$. Denote by $\overline{x}$ the image of $x \in R$ under the projection map $R \to \Bbbk := R/\mathfrak{m}$. Let $f = \sum_{\mathbf{u} \in \mathbb{Z}^n} a_{\mathbf{u}} \mathbf{x}^{\mathbf{u}} \in k[x_1^{\pm 1}, \ldots, x_m^{\pm 1}]$ and let $\mathbf{w} = (w_1, \ldots, w_m) \in \mathbb{Q}^m$ be an arbitrary vector. We call $\mathbf{w}$ the weight vector. Write

$$W := \operatorname{trop} f(\mathbf{w}) = \min_{a_{\mathbf{u}} \neq 0} \{v(a_{\mathbf{u}}) + \mathbf{u} \cdot \mathbf{w}\}.$$

Definition 3.1.1. The *initial form* $\operatorname{in}_{\mathbf{w}}(f) \in \Bbbk[x_1^{\pm 1}, \ldots, x_m^{\pm 1}]$ of $f := \sum_{\mathbf{u} \in \mathbb{Z}^m} a_{\mathbf{u}} \mathbf{x}^{\mathbf{u}}$ with respect to the weight vector $\mathbf{w}$ is:

$$\operatorname{in}_{\mathbf{w}}(f) := \sum_{\substack{\mathbf{u} \in \mathbb{Z}^m \\ v(a_{\mathbf{u}}) + \mathbf{u} \cdot \mathbf{w} = W}} \overline{t^{-v(a_{\mathbf{u}})} a_{\mathbf{u}}} \mathbf{x}^{\mathbf{u}} = \overline{t^{-W} f(t^{w_1} x_1, \ldots, t^{w_m} x_m)}.$$

The second characterization here is only valid when $\mathbf{w} \in (v(k^*))^m$ since otherwise the t^{w_i} are not defined.

Definition 3.1.2. The *initial ideal* of an ideal $I \subseteq k[x_1^{\pm 1}, \ldots, x_m^{\pm 1}]$ with respect to $\mathbf{w} \in \mathbb{Q}^m$ is

$$\operatorname{in}_{\mathbf{w}}(I) := \langle \operatorname{in}_{\mathbf{w}}(f) \mid f \in I \rangle.$$

We can now recall the Fundamental Theorem from Chapter 1:

Theorem 3.1.3. Fundamental Theorem of Tropical Geometry. *Let k be an algebraically closed field with a nontrivial valuation v and let $I \subseteq k[x_1^{\pm 1}, \ldots, x_m^{\pm 1}]$ be an ideal with associated variety $V(I) = \{\boldsymbol{x} \in (k^*)^m \mid f(\boldsymbol{x}) = 0$ for all $f \in I\}$ in $(k^*)^m$. Then the following subsets of $\mathbb{Q}^m$ coincide:*

1. $\operatorname{trop} V(I) := \bigcap_{f \in I} \operatorname{trop} V(f)$;

2. $\left\{\boldsymbol{w} \in \mathbb{Q}^m \mid in_{\boldsymbol{w}}(I) \neq k[x_1^{\pm 1}, \ldots, x_m^{\pm 1}]\right\}$;

3. *The closure in* $\mathbb{Q}^m$ *of the set*

$$v(V(I)) := \{(v(x_1), \ldots, v(x_m)) \mid (x_1, \ldots, x_m) \in V(I)\}.$$

We now describe how Theorem 3.1.3 can be extended to the tropicalization of toric varieties in the spirit of [16] and [26].

If $f = \sum c_{\mathbf{u}} x^{\mathbf{u}} \in k[x_1, \ldots, x_m]$ and $\mathbf{w} \in \overline{\mathbb{Q}}^m$, we define

$$\operatorname{trop}(f)(\mathbf{w}) := \min \{v(c_{\mathbf{u}}) + \mathbf{w} \cdot \mathbf{u} : \mathbf{u} \in (\mathbb{N} \cup \{0\})^m\}$$

with the convention that if $w_i = \infty$, then $w_i \cdot u_i = \infty$ when $u_i \neq 0$ and $w_i \cdot u_i = 0$ when $u_i = 0$. When $\operatorname{trop}(f)(\mathbf{w}) < \infty$, then we define the initial form $\operatorname{in}_{\mathbf{w}}(f)$ exactly as in Definition 3.1.1. If $\operatorname{trop}(f)(\mathbf{w}) = \infty$, then we define $\operatorname{in}_{\mathbf{w}}(f) = 0$. The ideal $\operatorname{in}_{\mathbf{w}}(I) = \langle \operatorname{in}_{\mathbf{w}}(f) : f \in I \rangle \in \Bbbk[x_1, \ldots, x_m]$ is defined as before.

We establish one final piece of notation before stating the theorem. If $\sigma \subseteq \{1, 2, \ldots, n\}$ and $\mathbf{w} \in \mathbb{Q}^{m-|\sigma|}$ is indexed by $\{i : i \notin \sigma\}$, then we write $\mathbf{w} \times \infty^{\sigma} \in \mathbb{Q}^m$ to be the vector that is w_i when the coordinate $i \notin \sigma$ and ∞ otherwise. If $\Sigma \subseteq \mathbb{Q}^{m-|\sigma|}$, then $\Sigma \times \infty^{\sigma} := \{\mathbf{w} \times \infty^{\sigma} : \mathbf{w} \in \Sigma\}$. Now we can state the theorem:

Theorem 3.1.4. [21, Theorem 6.2.15] *Let* Y *be a subvariety of* $\mathbb{A}^m$*, and let* $I \subseteq k[x_1, \ldots, x_m]$ *be its ideal. Then the following subsets of* $\overline{\mathbb{Q}}^m = \operatorname{trop}(\mathbb{A}^m)$ *coincide:*

1. $\bigcap_{f \in I} \operatorname{trop}(V(f))$;

2. *the set of vectors* $\boldsymbol{w} \in \mathbb{Q}^m$ *for which* $\operatorname{in}_{\boldsymbol{w}}(I) \subseteq \Bbbk[x_1, \ldots, x_m]$ *does not contain a monomial; and*

3. the set

$$\bigcup_{\sigma\subseteq\{1,\dots,m\}} \operatorname{trop}(Y\cap O_\sigma)\times\infty^\sigma,$$

where $O_\sigma = \{\boldsymbol{x}\in\mathbb{A}^m : x_i = 0 \textit{ for } i\in\sigma, \textit{ and } x_j\neq 0 \textit{ for } j\notin\sigma\}$.

This statement is restricted to subvarieties of affine space, but it can be extended to arbitrary toric varieties by tropicalizing a quotient of a subvariety of the Cox ring of the variety. This is described explicitly in [21, Corollary 6.2.16].

3.2 Spherical Gröbner tropicalization

We now describe the Kaveh-Manon notion of the tropicalization of a spherical homogeneous space. We will give an overview of the relevant points of [17, §4], which is relatively self-contained. There is a great deal of general theory on this subject from [17] that will be omitted.

Let G/H be a spherical homogeneous space and let $B\leq G$ be a Borel subgroup with associated palette $\mathcal{D}$. We denote the open orbit of B in G/H by $(G/H)_B$, and it is given as follows:

$$(G/H)_B = (G/H)\setminus\bigcup_{D\in\mathcal{D}} D.$$

For fixed $v\in\mathcal{V}$ and $a\in\mathbb{Q}$, define $k\left[(G/H)_B\right]_{v\geq a} := \{f\in k\left[(G/H)_B\right] : v(f)\geq a\}$ and similarly $k\left[(G/H)_B\right]_{v>a}$. We define a graded algebra $\operatorname{gr}_v(k\left[(G/H)_B\right])$ as follows:

$$\operatorname{gr}_v\left(k\left[(G/H)_B\right]\right) = \bigoplus_{a\in\mathbb{Q}} k\left[(G/H)_B\right]_{v\geq a}/k\left[(G/H)_B\right]_{v>a}. \tag{3.1}$$

(We note that there is an oversight in [17] as to this definition: the direct sum is taken over $\mathbb{Q}_{\geq 0}$ rather than $\mathbb{Q}$; $\mathbb{Q}$ is the correct notion.) Let $f\in k\left[(G/H)_B\right]$ and $v\in\mathcal{V}$ and write $v(f)=a$. Then we define $\operatorname{in}_v(f)$ to be the quotient of f in

$k\left[(G/H)_B\right]_{v\geq a}/k\left[(G/H)_B\right]_{v>a}$. If $J_B \subseteq k\left[(G/H)_B\right]$ is an ideal, we define

$$\mathrm{in}_v(J_B) := \langle \mathrm{in}_v(f) : f \in J_B \rangle \subseteq \mathrm{gr}_v\left(k\left[(G/H)_B\right]\right).$$

We write $\mathcal{V}_B(J_B)$ to denote the set of $v \in \mathcal{V}$ such that $\mathrm{in}_v(J_B) \neq \mathrm{gr}_v\left(k\left[(G/H)_B\right]\right)$.

Definition 3.2.1. Let $Y \subseteq G/H$ be a closed subvariety defined by an ideal $J \subseteq k[G/H]$. For each Borel subgroup $B \leq G$, let $J_B \subseteq k\left[(G/H)_B\right]$ denote the ideal that defines $Y \cap (G/H)_B$ as a subvariety of $(G/H)_B$. Then we define the *Gröbner tropicalization* of Y to be

$$\mathrm{trop}_G^{\mathrm{gr}}(Y) := \bigcup_B \mathcal{V}_B\left(J_B\right),$$

where the union is indexed over all Borel subgroups B of G.

Kaveh and Manon show ([17, Proposition 4.10]) that in fact we need only take the union over a finite number of Borel subgroups of G. The following theorem and example illustrate the necessity of taking a union of multiple Borel subgroups. Ultimately, we want to show that Gröbner tropicalization agrees with Vogiannou's tropicalization, and valuations lying in Vogiannou's tropicalization can be missed if not enough Borel subgroups are used.

Theorem 3.2.2. [17, Theorem 4.6] *Let $Y \subseteq (G/H)_B$ be a subvariety defined by an ideal $J \subseteq k\left[(G/H)_B\right]$. Let $v \in \mathcal{V}$ be a valuation, let X be the G/H-embedding whose colored fan consists of the ray spanned by v, and let $\mathcal{O}$ be the closed G-orbit in X. Then v lies in $\mathcal{V}_B(J)$ if and only if the closure of Y in X intersects the open B-orbit of $\mathcal{O}$.*

Example 3.2.3. (cf. [14, Example 4.3]) Let $G = \mathrm{Sl}_2$, H be the diagonal torus, and B be the upper triangular matrices. Then $G/H \cong (\mathbb{P}^1 \times \mathbb{P}^1) \setminus \mathcal{O}$, where $\mathcal{O}$ denotes the diagonal and G acts naturally on each copy of $\mathbb{P}^1$. The map $G \to (\mathbb{P}^1 \times \mathbb{P}^1) \setminus \mathcal{O}$ is given as follows; its kernel is H:

$$\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \mapsto \left(\begin{bmatrix} x_{11} \\ x_{21} \end{bmatrix}, \begin{bmatrix} x_{12} \\ x_{22} \end{bmatrix} \right).$$

Then the palette $\mathcal{D}$ consists of two colors: $V(x_{21})$ and $V(x_{22})$. The B semi-invariant rational functions are integer powers of $g := (x_{11}x_{22} - x_{12}x_{21})/x_{21}x_{22}$ and the lattice $\mathcal{N}$ is isomorphic to $\mathbb{Z}$. We claim $\mathcal{V}$ is the ray generated by the G-invariant valuation

$$f \mapsto \text{order of vanishing of } f \text{ along } \mathcal{O}.$$

Note that $g \mapsto 1$ under this valuation. We show that no valuation sending g to a negative number can be G-invariant. Note first that

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \cdot \frac{x_{22}}{x_{12}} = -\frac{x_{12}}{x_{22}} \quad \text{and} \quad \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \cdot \frac{x_{21}}{x_{11}} = -\frac{x_{11}}{x_{21}}$$

This implies any G-invariant valuation must take both $\frac{x_{12}}{x_{22}}$ and $\frac{x_{11}}{x_{21}}$ to 0. Thus, if v is a G-invariant valuation, we have that

$$v(g) = v\left(\frac{x_{11}x_{22} - x_{12}x_{21}}{x_{21}x_{22}}\right) = v\left(\frac{x_{11}}{x_{21}} - \frac{x_{12}}{x_{22}}\right) \geq \min\left\{ v\left(\frac{x_{11}}{x_{21}}\right), v\left(\frac{x_{12}}{x_{22}}\right)\right\} = 0.$$

Therefore, any G-invariant valuation must take g to a non-negative number.

Both colors map to the same point outside the valuation cone and the only G/H-embedding is $X \cong \mathbb{P}^1 \times \mathbb{P}^1$, whose colored fan is the ray $\mathcal{V}$. The data from the homogeneous space is shown in Figure 3.1.

Figure 3.1: Spherical data for $(\mathbb{P}^1 \times \mathbb{P}^1) \setminus \mathcal{O}$; $\mathcal{V}$ is the ray to the right and both colors lie on the left

We have that

$$\begin{aligned}(G/H)_B &= (G/H) \setminus \{V(x_{21}), V(x_{22})\} \\ &= \{([x_{11} : x_{21}], [x_{12} : x_{22}]) : x_{21}, x_{22}, x_{11}x_{22} - x_{12}x_{21} \neq 0\} \\ &= \{([x_{11} : 1], [x_{12} : 1]) : x_{11} \neq x_{12}\} \\ &\cong \mathbb{A}^2 \setminus \Delta.\end{aligned}$$

Here, Δ represents the diagonal of $\mathbb{A}^2$. This tells us that:

$$k\left[(G/H)_B\right] \cong k[x_{11}, x_{12}, (x_{11} - x_{12})^{\pm}].$$

Let us consider $J = (x_{21})$ and the corresponding subvariety $Y := V(x_{21}) \subset G/H$. The closure of Y in X intersects $\mathcal{O}$ in the point $([1 : 0], [1 : 0])$, which does not lie in the open B-orbit of $\mathcal{O}$. Thus, Theorem 3.2.2 tells us $\mathrm{gr}_B(J_B)$ consists solely of the trivial valuation since it can contain no other valuation along the ray $\mathcal{V}$.

If instead we had used the Borel subgroup B' of lower triangular matrices, then the resulting palette $\mathcal{D}'$ would consist of $V(x_{11})$ and $V(x_{12})$ and we would still have $(G/H)_{B'} \cong \mathbb{A}^2 \setminus \Delta$. The important difference is that in this case we have:

$$(G/H)_{B'} = \{([1 : x_{21}], [1 : x_{22}]) : x_{21} \neq x_{22}\}.$$

The subvariety Y still intersects the closed G-orbit of X in the point $([1 : 0], [1 : 0])$, but this point lies in the open B'-orbit. Thus, $\mathrm{gr}_{B'}(J_{B'})$ consists of the valuation cone $\mathcal{V}$, which is the tropicalization of $V(x_{21})$. In general, if $Y \subseteq G/H$ is cut out by an ideal J, we have $\mathrm{trop}_G^{\mathrm{gr}}(Y) = \mathcal{V}_B(J_B) \cup \mathcal{V}_{B'}(J_{B'})$.

This notion of spherical tropicalization via Gröbner theory agrees with Vogiannou's tropicalization:

Theorem 3.2.4. [17, Theorem 5.15] *Let $Y \subseteq G/H$ be a closed subvariety. Then Gröbner tropicalization coincides with spherical tropicalization:*

$$\mathrm{trop}_G(Y) = \mathrm{trop}_G^{\mathrm{gr}}(Y).$$

3.3 Extended Gröbner tropicalization

In this section we extend Gröbner tropicalization to encompass spherical embeddings. Let X be a simple spherical embedding defined by a colored cone $(\sigma, \mathcal{F})$. Write $\mathcal{O}$ for the unique closed G-orbit of X, write $\mathcal{D}(X)$ for the set of B-stable prime divisors of X, and write $\mathcal{D}_{\mathcal{O}}(X)$ for the B-stable prime divisors of X that contain $\mathcal{O}$. There is a B-stable subset X_B of X defined by:

$$X_B := X \setminus \bigcup_{D \in \mathcal{D}(X) \setminus \mathcal{D}_{\mathcal{O}}(X)} D,$$

in other words by throwing out the B-stable prime divisors not containing $\mathcal{O}$. Note that this is consistent with the earlier notation $(G/H)_B$ and that $(G/H)_B = X_B \cap (G/H)$. This theory appears with more detail in [18, §2], where the notation X_0 is used rather than X_B. The following is a straightforward extension of [17, Theorem 4.1].

Theorem 3.3.1. *Let X be a simple G/H-embedding with closed orbit $\mathcal{O}$. The subset X_B is open and affine, and $\mathcal{O} \cap X_B$ is a B-orbit. Further, the regular functions on X_B can be described as follows:*

$$k\left[X_B\right] = \left\{ f \in k\left[(G/H)_B\right] : \nu_D(f) \geq 0 \text{ for all } D \in \mathcal{D}_{\mathcal{O}}(X) \right\}.$$

Proof. The first claims are from [18, Theorem 2.1]. Turning to the regular functions, note that because $(G/H)_B$ is open in X_B, we can identify a function in $k\left[X_B\right]$ with its restriction to $(G/H)_B$. The functions in $k[(G/H)_B]$ that are restrictions in this way are precisely those that do not have poles when evaluated at points in $X_B \setminus (G/H)_B$. In other words, they are the functions f such that $\nu_D(f) \geq 0$ for all $D \in \mathcal{D}_{\mathcal{O}}(X)$. Conversely, if we have a function in $k[(G/H)_B]$, it extends (uniquely) to $k[X_B]$ precisely when it does not have poles along any $D \in \mathcal{D}_{\mathcal{O}}(X)$. $\square$

The orbit $\mathcal{O}$ is a spherical homogeneous space under the action of G with open B-orbit $\mathcal{O} \cap X_B$. It follows that $\mathcal{D}(\mathcal{O}) = \{D \cap \mathcal{O} : D \in \mathcal{D}(X) \setminus \mathcal{D}_{\mathcal{O}}(X)\}$.

Now denote by $\mathcal{V}(X_B)$ the extended G-invariant $\overline{\mathbb{Q}}$-valuations on $k[(G/H)_B]$ that are finite on $k[X_B]$:

$$\mathcal{V}(X_B) := \left\{ v : k[(G/H)_B] \to \overline{\mathbb{Q}} : k[X_B] \subseteq v^{-1}(\mathbb{Q}) \right\}.$$

Observe that $\mathcal{V}(X_B)$ can be identified with a subset of $\overline{\mathcal{N}}_{\mathcal{V}}(\sigma)$. Indeed, a G-invariant $\overline{\mathbb{Q}}$-valuation on $k[(G/H)_B]$ induces a semigroup homomorphism $\sigma^{\vee} \cap \mathcal{M} \to \overline{\mathbb{Q}}$ in the same way a G-invariant $\overline{\mathbb{Q}}$-valuation on $k[G/H]$ does. The only difference is that valuations on $k[(G/H)_B]$ cannot take infinite values on functions cutting out the B-stable divisors because these functions are invertible in $k[(G/H)_B]$.

For any $v \in \mathcal{V}(X_B)$, we define $\mathrm{gr}_v(k[X_B])$ as follows (cf. Equation (3.1)).

$$\mathrm{gr}_v(k[X_B]) = k[(G/H)_B]_{v=\infty} \oplus \bigoplus_{a \in \mathbb{Q}} k[(G/H)_B]_{v \geq a} / k[(G/H)_B]_{v > a}, \tag{3.2}$$

where $k[(G/H)_B]_{v=\infty} := \{f \in k[(G/H)_B] : v(f) = \infty\}$. If we further assert that $k[(G/H)_B]_{v>\infty} := \{0\}$, then we may write

$$\mathrm{gr}_v(k[X_B]) = \bigoplus_{a \in \overline{\mathbb{Q}}} k[(G/H)_B]_{v \geq a} / k[(G/H)_B]_{v > a}.$$

Note that if $v \in \mathcal{V}$, the first summand of Equation (3.2) vanishes and $\mathrm{gr}_v(k[X_B]) = \mathrm{gr}_v(k[G/H_B])$. For $f \in k[X_B]$, $J \subseteq k[X_B]$, and $v \in \mathcal{V}(X_B)$, we define $\mathrm{in}_v(f)$

and $\mathrm{in}_v(J)$ as before. Similarly, $\mathcal{V}_B(J)$ is the set of $v \in \mathcal{V}(X_B)$ such that $\mathrm{in}_v(J) \neq \mathrm{gr}_v\left(k\left[X_B\right]\right)$. Now we can define the extended Gröbner tropicalization of a subvariety of a spherical embedding:

Definition 3.3.2. Let X be a simple G/H-embedding associated to a colored cone $(\sigma, \mathcal{F})$ and $Y \subseteq G/H$ a closed subvariety defined by an ideal $J \subseteq k[G/H]$. Let $\overline{Y} \subseteq X$ denote the closure of Y in X. For each Borel subgroup $B \leq G$, let $\overline{J}_B \subseteq k\left[X_B\right]$ denote the ideal that defines $\overline{Y} \cap X_B$ as a subvariety of X_B. The *extended Gröbner tropicalization* of $\overline{Y}$ in $\overline{\mathcal{N}}_{\mathcal{V}}(\sigma)$ is:

$$\mathrm{trop}_G^{\mathrm{gr}}\left(\overline{Y}\right) := \bigcup_B \mathcal{V}_B\left(\overline{J}_B\right).$$

In particular, if $Y = G/H$, then

$$\mathrm{trop}_G^{\mathrm{gr}}(X) = \bigcup_B \mathcal{V}_B(0).$$

If X is a non-simple G/H-embedding, we define $\mathrm{trop}_G^{\mathrm{gr}}\left(\overline{Y}\right)$ as follows. For each simple G/H-embedding $X' \subseteq X$, compute $\mathrm{trop}_G^{\mathrm{gr}}\left(\overline{Y} \cap X'\right) \subseteq \mathrm{trop}_G^{\mathrm{gr}}(X')$ and glue together along shared valuations.

We finally prove an extended fundamental theorem equating our two notions of extended spherical tropicalization. We first prove the statement for simple embeddings and then deduce the full result as a corollary.

Theorem 3.3.3. *Let X be a simple G/H-embedding and $Y \subseteq G/H$ be a closed subvariety. Then*

$$\mathrm{trop}_G(\overline{Y}) = \mathrm{trop}_G^{\mathrm{gr}}(\overline{Y})$$

as subspaces of $\overline{\mathcal{N}}_{\mathcal{V}}(\sigma)$, where the closure $\overline{Y}$ is taken in X.

Proof. We first show that $\operatorname{trop}_G(X) = \operatorname{trop}_G^{\mathrm{gr}}(X)$. Let $(\sigma, \mathcal{F})$ be the colored cone associated to X and let $\mathcal{O}$ be the G-orbit associated to a colored face $\tau \preceq \sigma$. If $B \leq G$ is a Borel subgroup, denote by $\mathcal{D}_{\mathcal{O}}(X) \subseteq \mathcal{D}(X)$ the set of B-stable divisors of X that contain $\mathcal{O}$. Write

$$X_B(\mathcal{O}) := X \setminus \bigcup_{D \in \mathcal{D}(X) \setminus \mathcal{D}_{\mathcal{O}}(X)} D \subseteq X_B$$

and note that $X_B(\mathcal{O}) = X_B$ when $\mathcal{O}$ is the unique closed orbit of X. We can characterize the regular functions on $X_B(\mathcal{O})$ as we did in Theorem 3.3.1:

$$k\left[X_B(\mathcal{O})\right] = \left\{f \in k\left[(G/H)_B\right] : \nu_D(f) \geq 0 \text{ for all } D \in \mathcal{D}_{\mathcal{O}}(X)\right\}.$$

Now consider the following subset:

$$\left\{v : k\left[(G/H)_B\right] \to \overline{\mathbb{Q}} : k\left[X_B(\mathcal{O})\right] = v^{-1}(\mathbb{Q})\right\} \subseteq \mathcal{V}\left(X_B\right) \subseteq \overline{\mathcal{N}}_{\mathcal{V}}(\sigma).$$

The condition that $k\left[X_B(\mathcal{O})\right] = v^{-1}(\mathbb{Q})$ is the same as the condition that v satisfies $v^{-1}(\mathbb{Q}) = \tau^{\perp} \cap (\sigma^{\vee} \cap \mathcal{M})$, with the additional requirement that functions vanishing along the B-stable prime divisors aren't sent to ∞. Taking the union over all Borel subgroups thus gives a copy of $\mathcal{V}(\mathcal{O}) \subseteq \operatorname{Hom}\left(\tau^{\perp} \cap (\sigma^{\vee} \cap \mathcal{M}), \overline{\mathbb{Q}}\right)$ in $\operatorname{trop}_G^{\mathrm{gr}}(X)$. This holds for all G-orbits $\mathcal{O}$, so $\operatorname{trop}_G^{\mathrm{gr}}(X) = \operatorname{trop}_G(X)$.

Now if $Y \subseteq G/H$ is a closed variety, suppose $\overline{Y} \cap \mathcal{O} \neq \emptyset$ for some G-orbit $\mathcal{O}$. Then Theorem 3.2.4 ensures that $\operatorname{trop}_G\left(\overline{Y} \cap \mathcal{O}\right) = \operatorname{trop}_G^{\mathrm{gr}}\left(\overline{Y} \cap \mathcal{O}\right)$ in $\mathcal{V}(\mathcal{O}) \subseteq \overline{\mathcal{N}}_{\mathcal{V}}(\sigma)$. As this holds for every orbit $\mathcal{O}$, the statement follows. $\square$

Corollary 3.3.4. *Let X be a G/H-embedding and $Y \subseteq G/H$ a closed subvariety. Then*

$$\operatorname{trop}_G(\overline{Y}) \cong \operatorname{trop}_G^{\mathrm{gr}}(\overline{Y}),$$

where the closure $\overline{Y}$ is taken in X.

Proof. Theorem 3.3.3 proves this for simple embeddings. The gluing operation between tropicalizations of simple embeddings is identical for both constructions, so the result follows. □

Chapter 4: Global Spherical Tropicalization via Toric Embeddings

In his thesis [33], Tassos Vogiannou introduces a notion of spherical tropicalization for a spherical homogeneous space G/H. Broadly speaking, this operation records the G-invariant divisors on the field of rational functions while the standard toric tropicalization instead records torus-invariant divisors. Kaveh and Manon in [17] recover this construction using a Gröbner theory for spherical varieties that they develop. In Chapters 2 and 3, we proposed a means for tropicalizing embeddings of spherical homogeneous spaces. There we define two constructions—one that mimics the theory for toric varieties developed by Kajiwara [16] and Payne [26] and one that extends the Gröbner theory definition of Kaveh and Manon—and show that they coincide.

In this chapter, we give a third method for obtaining the tropicalization of a spherical embedding. This method is global, by which we mean the tropicalization operation need only be applied once to a single object. It is possible to globally tropicalize a toric variety by tropicalizing the quotient construction of a toric variety via the Cox ring. See [26, Remark 3.5] or [21, §6.2] for a description of this. For details on the quotient construction, refer to [7] and [8] and for a comprehensive overview of Cox rings in general, see [2]. Our aim is to mimic this construction for spherical varieties. This is noteworthy because in the constructions of spherical tropicalization from Chapters 2 and 3, the spherical variety is divided into simple G/H-embeddings, these are tropicalized separately, and then the tropicalizations are glued together.

The construction follows Gagliardi's work in [14], where he describes the Cox ring of a spherical embedding using its combinatorial data (see also [5]). We begin by embedding the spherical variety in a toric variety, which is an example of the embedding of a Mori dream space into a toric variety described by Hu and Keel [15]. Once embedded in the toric variety, we can tropicalize there using the standard toric theory. Then applying a particular piecewise projection map will deliver the spherical tropicalization.

This process does not work for all spherical varieties. When the spherical variety does not have the A_2-property (Definition 4.4.1), embedding into a toric variety is not possible. The tropicalization of a spherical variety without the A_2-property can still be described using this theory, but it cannot be done globally.

The layout of this chapter is as follows. In §4.1, we review some of the theory of spherical varieties and their tropicalization. We explain a novel means of tropicalizing subvarieties of a spherical homogeneous spaces using toric embeddings in §4.2. Theorems 4.2.2 and 4.2.5 prove that this new method coincides with the original theory of [33]. In §4.3 we show how to embed toroidal spherical varieties in toric varieties and in §4.4 we extend this to spherical embeddings with color. Finally, §4.5 shows how to recover the extended tropicalization of an embedding by using the toric tropicalization. We also show in this section that taking the closure commutes with the tropicalization operation.

In this chapter we work over the complex numbers $\mathbb{C}$ rather than an arbitrary algebraically closed field k as in Chapters 2 and 3. Some of the results we cite in this chapter are given over $\mathbb{C}$, so we restrict ourselves to this setting to ensure accuracy.

4.1 Background

We review some of the theory of spherical varieties. Throughout this chapter, we work over $\mathbb{C}$. For additional background, refer to [20], [18], [25], or [27]. If G is a connected reductive group and $H \leq G$ is a closed subgroup, then the quotient G/H is a *(spherical) homogeneous space* if it is a normal variety containing a dense orbit of a Borel subgroup B. A *spherical embedding* X is a normal G-variety with an open equivariant embedding $G/H \hookrightarrow X$. We will also call X a *spherical variety* or a *G/H-embedding* if we wish to highlight the underlying homogeneous space. A spherical variety X is modeled by a combinatorial object called a colored fan. We recall the outline of the theory here; refer to the cited papers for further details.

Denote by $\mathcal{X}$ the group of characters $B \to \mathbb{C}^*$ on B. The set $\mathbb{C}(G/H)^{(B)}$ of B semi-invariant rational functions on G/H is defined by

$$\mathbb{C}(G/H)^{(B)} := \{f \in \mathbb{C}(G/H)^* : \text{there exists } \chi_f \in \mathcal{X} \text{ such that } gf = \chi_f(g)f \text{ for all } g \in B\},$$

where the action of g on f is given by $gf(x) := f\left(g^{-1}x\right)$ for $x \in G/H$. The subset $\mathcal{M} \subseteq \mathcal{X}$ of characters χ_f associated to $f \in \mathbb{C}(G/H)^{(B)}/\mathbb{C}^*$ is a lattice. We write $\mathcal{N} := \operatorname{Hom}(\mathcal{M}, \mathbb{Z})$ to denote its dual. Further, $\mathcal{M}_{\mathbb{Q}} := \mathcal{M} \otimes_{\mathbb{Z}} \mathbb{Q}$ and $\mathcal{N}_{\mathbb{Q}} := \mathcal{N} \otimes_{\mathbb{Z}} \mathbb{Q}$ are the associated $\mathbb{Q}$-vector spaces. We write $\mathcal{V}(G/H)$ to denote the set of G-invariant valuations on $\mathbb{C}(G/H)^*$ that are trivial when restricted to $\mathbb{C}^*$. A given $v \in \mathcal{V}(G/H)$ induces a homomorphism $\mathbb{C}(G/H)^{(B)}/\mathbb{C}^* \to \mathbb{Q}$, namely $f \mapsto v(f)$. Because f can be associated to a character χ_f, this homomorphism induces an element of $\mathcal{N}_{\mathbb{Q}}$. Thus, we obtain a map $\mathcal{V}(G/H) \to \mathcal{N}_{\mathbb{Q}}$, which is an inclusion ([20, Proposition 7.4]). We will identify $\mathcal{V}(G/H)$ with its image in $\mathcal{N}_{\mathbb{Q}}$, which is a convex polyhedral cone. We call $\mathcal{V}(G/H)$ the *valuation cone* of G/H and write it as $\mathcal{V}$ when G/H is understood.

We also obtain a *palette* $\mathcal{D}$ of colors associated to G/H. These are the prime divisors of G/H that are B-stable but not G-stable. Each $D \in \mathcal{D}$ induces a valuation v_D given by vanishing along D. Thus, we obtain a map $\rho : \mathcal{D} \to \mathcal{N}$ given by $D \mapsto v_D$; it need not be injective. We now have enough background to define colored cones and colored fans:

Definition 4.1.1. A *colored cone* $(\sigma, \mathcal{F})$ consists of a cone $\sigma \subseteq \mathcal{N}_{\mathbb{Q}}$ and a subset $\mathcal{F} \subseteq \mathcal{D}$ such that:

(i) σ is generated by $\rho(\mathcal{F})$ and finitely many elements of $\mathcal{V}$;

(ii) $\operatorname{int}(\sigma) \cap \mathcal{V} \neq \emptyset$,

where $\operatorname{int}(\sigma)$ denotes the relative interior of σ. We call $(\sigma, \mathcal{F})$ *strictly convex* if in addition σ is strictly convex and $0 \notin \rho(\mathcal{F})$.

Definition 4.1.2. A colored cone $(\tau, \mathcal{F}')$ is a *face* of a colored cone $(\sigma, \mathcal{F})$ if τ is a face of σ and $\mathcal{F}' = \mathcal{F} \cap \rho^{-1}(\tau)$. We write $\tau \preceq \sigma$ in this case.

Definition 4.1.3. A *colored fan* Σ is a collection of colored cones such that:

(i) If $(\sigma, \mathcal{F}) \in \Sigma$, the every face of $(\sigma, \mathcal{F})$ is in Σ;

(ii) For every $v \in \mathcal{V}$, there is at most one $(\sigma, \mathcal{F}) \in \Sigma$ with $v \in \operatorname{int}(\sigma)$.

We say Σ is *strictly convex* if in addition every colored cone in Σ is strictly convex.

Note that the relative interiors of two colored cones in a colored fan may intersect nontrivially outside $\mathcal{V}$; we present an instance of this in Example 4.4.5. There is a bijective correspondence between G/H-embeddings and strictly convex colored fans. The colored cones in a strictly convex colored fan represent the G-orbits of an embedding X. Strictly convex colored cones correspond to *simple* G/H-embeddings, which have one closed G-orbit. If the colored fan associated to a spherical variety has no colors, we say that the embedding is *toroidal*.

Throughout this paper, we denote by $\nu : \mathbb{C}\{\{t\}\} \to \mathbb{Q}$ the valuation that takes a Puiseux series to the lowest power of t appearing with nonzero coefficient. Given a closed subvariety $Y \subseteq G/H$, we define its spherical tropicalization as follows. For each $\mathbb{C}\{\{t\}\}$-point $\gamma \in Y(\mathbb{C}\{\{t\}\})$, we obtain the following associated valuation ν_γ : $\mathbb{C}(G/H)^* \to \mathbb{Q}$ in $\mathcal{V}$:

$$f \mapsto \nu\left(\gamma^*(gf)\right) \text{ for } f \in \mathbb{C}(G/H)^* \text{ and } g \in G \text{ sufficiently general.}$$

We must explain what we mean by sufficiently general g. For g in an open subset U of G, the image of γ is in the domain of gf. Furthermore, the minimum of $\nu\left(\gamma^*(gf)\right)$ over all $g \in G$ is met on an open subset W. A sufficiently general g lies in $U \cap W$. The tropicalization $\operatorname{trop}_G(Y) \subseteq \mathcal{V}$ is the collection of ν_γ for all $\gamma \in Y(\mathbb{C}\{\{t\}\})$. Note that the subscript G records which group action we are tropicalizing with respect to and observe that $\operatorname{trop}_G(G/H) = \mathcal{V}(G/H)$.

We briefly describe the extended tropicalization construction for spherical embeddings from [23]. If X is a G/H-embedding and $\mathcal{O} \subseteq X$ is a G-orbit, then $\mathcal{O}$ is itself a homogeneous space with respect to the action of G. Thus, $\mathcal{O}$ has a valuation cone

$\mathcal{V}(\mathcal{O})$. As a set, the spherical tropicalization is

$$\operatorname{trop}_G(X) := \bigsqcup_{\mathcal{O} \subseteq X} \mathcal{V}(\mathcal{O}).$$

Note that the subscript G is used to record which group action we are considering. The subset $\mathcal{V}(\mathcal{O}) \in \operatorname{trop}_G(X)$ can be thought of as extended valuations on $\mathbb{C}[G/H]$, i.e. semigroup homomorphisms $\mathbb{C}[G/H] \to \mathbb{Q} \cup \{\infty\}$. Which elements of $\mathbb{C}[G/H]$ that are allowed to be sent to ∞ is determined by which functions in $\mathbb{C}[G/H]$ vanish along $\mathcal{O}$. The topology on $\operatorname{trop}_G(X)$ in essence adds on these extended valuations by considering them as limit points of finitely-valued $\mathbb{Q}$-valuations. If $Y \subseteq X$ is a closed subvariety, we obtain its tropicalization $\operatorname{trop}_G(Y)$ by separately tropicalizing its intersection $Y \cap \mathcal{O}$ with each G-orbit $\mathcal{O}$ of X. We define

$$\operatorname{trop}_G(Y) := \bigsqcup_{\mathcal{O} \subseteq X} \operatorname{trop}_G(Y \cap \mathcal{O}) \subseteq \operatorname{trop}_G(X)$$

and apply the subspace topology inherited from $\operatorname{trop}_G(X)$.

We can identify the tropicalization of a simple G/H-embedding consisting of one maximal colored cone $(\sigma, \mathcal{F})$ with a set of semigroup homomorphisms $\operatorname{Hom}^{\mathcal{V}}(\sigma^\vee \cap \mathcal{M}, \overline{\mathbb{Q}}) \subseteq \operatorname{Hom}(\sigma^\vee \cap \mathcal{M}, \overline{\mathbb{Q}})$. This set consists of all semigroup homomorphisms $\mu \in \operatorname{Hom}(\sigma^\vee \cap \mathcal{M}, \overline{\mathbb{Q}})$ such that $\mu^{-1}(\mathbb{Q}) = \tau^\perp \cap \sigma^\vee \cap \mathcal{M}$ where $(\tau, \mathcal{F}')$ is a face of $(\sigma, \mathcal{F})$.

We establish some notational conventions. Throughout, X denotes a spherical embedding, $Y \subseteq G/H$ denotes a closed subvariety of a homogeneous space, and Z denotes a toric variety. The combinatorial data associated to spherical varieties is written in non-calligraphic font when referring to toric varieties. That is, a toric variety has the lattices M and N rather than $\mathcal{M}$ and $\mathcal{N}$. Finally, we write $\overline{\mathbb{Q}} := \mathbb{Q} \cup \{\infty\}$.

4.2 Tropicalizing Homogeneous Spaces via Toric Embeddings

In [14], Gagliardi proves a theorem relating the valuation cone of G/H to tropicalization. Inspired by his result, we find an alternate means for tropicalizing a subvariety of a spherical homogeneous space. The purpose of this section is to set up Gagliardi's theory and show how to recover from it Vogiannou's spherical tropicalization (Theorem 4.2.2).

Throughout, we will always assume that G is of simply connected type, which means that $G = G^{ss} \times C$ where G^{ss} is semi-simple simply connected and C is a torus. Every connected reductive group G has a finite covering $p : G' \to G$ by a group G' of simply connected type and G/H is isomorphic to the spherical homogeneous space $G'/p^{-1}(H)$, so this is not a restrictive assumption.

We work initially over a spherical homogeneous space G/H with trivial divisor class group, which implies that G/H is quasi-affine and $\mathbb{C}[G/H]$ is a unique factorization domain. The case of non-trivial divisor class group will be handled later. We will describe how to associate a toric variety Z_0 to G/H. Each color in the palette of G/H is a prime divisor $D_i = V(f_i)$. The orbit of f_i in $\mathbb{C}[G/H]$ under the action of G spans a G-module of some rank s_i. We choose a basis $\{f_{i1} := f_i, f_{i2}, \ldots, f_{is_i}\} \subseteq G \cdot f_i$ for this G-module. Further, $\Gamma\left(G/H, \mathcal{O}^*_{G/H}\right)/\mathbb{C}^*$ is a finitely generated free abelian group with basis $\{g_k\}_{k=1}^m$. The characters associated to f_i and g_k span the lattice $\mathcal{M}$ of characters of B semi-invariant rational functions $k(G/H)^{(B)}$. Denote these characters by v_i^* and w_k^*, respectively. Define a toric variety

$$Z_0 := (\mathbb{C}^{s_1} \setminus \{0\}) \times \cdots \times (\mathbb{C}^{s_r} \setminus \{0\}) \times (\mathbb{C}^*)^m.$$

with coordinates f_{ij} and g_k, matching the basis of $\mathcal{M}$. Denote the lattice of cocharacters of Z_0 by $N \cong \mathbb{Z}^{s_1+\cdots+s_r+m}$ with basis $\{v_{11}, v_{12}, \ldots, v_{rs_r}, w_1, \ldots, w_m\}$. Write $\mathbb{T} \cong (\mathbb{C}^*)^{s_1+\cdots+s_r+m}$ for the dense torus in Z_0. The inclusion $\iota : G/H \hookrightarrow Z_0$ defined by $x \mapsto (f_{ij}(x))_{i,j} \times (g_k(x))_k$ is a closed embedding. This induces a natural action of G on Z_0 commuting with ι. The coordinate ring of Z_0 has coordinates S_{ij} for $1 \leq i \leq r$ and $1 \leq j \leq s_i$ and T_k for $1 \leq k \leq m$. The inclusion ι is dual to the map $\Psi : \mathbb{C}[Z_0] \to \mathbb{C}[G/H]$ defined by $S_{ij} \mapsto f_{ij}$ and $T_k \mapsto g_k$. Let $\mathfrak{p}$ denote the kernel of Ψ so that $\mathbb{C}[Z_0]/\mathfrak{p} \cong \mathbb{C}[G/H]$. Finally, the lattice $\mathcal{N}$ dual to $\mathcal{M}$ has basis $\{v_1, \ldots, v_r, w_1, \ldots, w_m\}$ and we define an inclusion $\mathcal{N} \hookrightarrow N$ by $v_i \mapsto v_{i1} + \cdots + v_{is_i}$ and $w_k \mapsto w_k$.

Theorem 4.2.1 shows how the embedding $G/H \hookrightarrow Z_0$ can be related to the valuation cone of G/H:

Theorem 4.2.1. [14, Theorem 1.7] $\mathcal{V}(G/H) = \mathrm{trop}_{\mathbb{T}}(G/H \cap \mathbb{T}) \cap \mathcal{N}_{\mathbb{Q}}$.

We can also write this as $\mathrm{trop}_G(G/H) = \mathrm{trop}_{\mathbb{T}}(G/H \cap \mathbb{T}) \cap \mathcal{N}_{\mathbb{Q}}$. In fact, because $\mathcal{N}_{\mathbb{Q}} \subseteq N_{\mathbb{Q}} = \mathrm{trop}_{\mathbb{T}}(\mathbb{T})$, we may write $\mathrm{trop}_G(G/H) = \mathrm{trop}_{\mathbb{T}}(G/H) \cap \mathcal{N}_{\mathbb{Q}}$. Note that $\mathrm{trop}_{\mathbb{T}}(G/H)$ is potentially ill-defined if one does not have a notion of extended tropicalization since G/H is not necessarily contained in $\mathbb{T}$.

Theorem 4.2.1 does not extend to subvarieties $Y \subseteq G/H$. That is, $\mathrm{trop}_G(Y) \neq \mathrm{trop}_{\mathbb{T}}(Y) \cap \mathcal{N}_{\mathbb{Q}}$ in general. Ultimately, accounting for subvarieties of G/H requires a new result independent of Theorem 4.2.1, which we describe now. Note that

$$\mathrm{trop}_{\mathbb{T}}(Z_0) = \left(\overline{\mathbb{Q}}^{s_1} \setminus \{\infty\}\right) \times \cdots \times \left(\overline{\mathbb{Q}}^{s_r} \setminus \{\infty\}\right) \times \mathbb{Q}^m$$

and define a map $\psi : \mathrm{trop}_{\mathbb{T}}(Z_0) \to \mathcal{N}_{\mathbb{Q}}$ as follows:

$$\psi : \mathrm{trop}_{\mathbb{T}}(Z_0) \to \mathcal{N}_{\mathbb{Q}}$$
$$(a_{11}, \dots, a_{1s_1}, a_{21}, \dots, a_{rs_r}, b_1, \dots, b_m) \mapsto \left(\min_{1 \le j \le s_1} \{a_{1j}\}, \dots, \min_{1 \le j \le s_r} \{a_{rj}\}, b_1, \dots, b_m \right).$$

The following theorem is the core idea of this chapter. The further work ultimately simply expands on this result.

Theorem 4.2.2. *If $Y \subseteq G/H$ is a closed subvariety and G/H has trivial divisor class group, then* $\mathrm{trop}_G(Y) = \psi\left(\mathrm{trop}_{\mathbb{T}}(Y)\right)$.

Proof. We first consider the left-to-right inclusion. Let γ be a $\mathbb{C}\{\{t\}\}$-point of $Y \subseteq G/H$. Vogiannou's definition gives us a G-invariant valuation $\nu_\gamma \in \mathrm{trop}_G(Y) \subseteq \mathcal{N}_{\mathbb{Q}}$ defined by $f \mapsto \nu(\gamma^*(gf))$ for sufficiently general $g \in G$. The dual lattice $\mathcal{M}$ to $\mathcal{N}$ is spanned by characters associated to the f_i's and g_k's, so we only need to know how ν_γ behaves on these functions to completely determine it as an element of $\mathcal{N}_{\mathbb{Q}}$. Consider $\nu_\gamma(f_i)$ for some i, suppose that $g \in G$ is sufficiently general so that $\nu_\gamma(f_i) = \nu(\gamma^*(gf_i))$, and write $gf_i = \sum_{j=1}^{s_i} a_j f_{ij}$ where $a_j \in \mathbb{C}$. Generically, the minimum $\min_j \{\nu(a_j f_{ij}(\gamma))\}$ is met at only one monomial, so we may write $\nu_\gamma(f_i) = \nu(a_j f_{ij}(\gamma))$ for some j. Note that $a_j f_{ij}(\gamma) \neq 0$ as otherwise f_i would be zero on all of G/H. Further, observe that by this argument $\nu_\gamma(f_{ij}) = \nu_\gamma(f_{ik})$ for all j and k as ν_γ is G-invariant and f_{ij} and f_{ik} lie in the same G-orbit. Thus, $\mathrm{trop}_G(Y) \subseteq \mathcal{N}_{\mathbb{Q}} \subseteq N_{\mathbb{Q}}$. Applying ψ to the $\mathbb{T}$-invariant valuation induced by γ gives ν_γ, defined by $f_i \mapsto \nu(a_j f_{ij}(\gamma))$.

By [19, Proposition 1.3], the g_k are all G eigenvectors, so it follows that the $\mathbb{T}$-invariant and G-invariant valuations induced by γ act identically on the g_k. This implies the first inclusion.

Conversely, suppose that γ is a $\mathbb{C}\{\{t\}\}$-point of $Y \subseteq Z_0$ and consider $\psi(\widetilde{\nu}_\gamma)$, where $\widetilde{\nu}_\gamma$ is the $\mathbb{T}$-invariant valuation induced by γ. For fixed i, choose some k such that $\min_j \{\nu(f_{ij}(\gamma))\} = \nu(f_{ik}(\gamma))$. Note that by definition of Z_0 we cannot have $f_{ij}(\gamma) = 0$ for all j, so the minimum is finite and thus the image of Y under ψ is contained in $\mathcal{N}_\mathbb{Q}$. For generic $g \in G$, gf_i has a nonzero coefficient a_{ik} on f_{ik}, so it follows that $\nu_\gamma(f_i) = \nu(a_{ik}f_{ik}(\gamma)) = \psi(\widetilde{\nu}_\gamma)(f_i)$, where ν_γ is the G-invariant valuation induced by γ. Because the g_k are G-eigenvectors, $\widetilde{\nu}_\gamma(g_k) = \nu_\gamma(g_k)$ for all k and so $\psi(\widetilde{\nu}_\gamma) = \nu_\gamma \in \mathrm{trop}_G(Y)$. $\square$

Example 4.2.3. Let $G = \mathrm{Gl}_2$ and H be the subgroup of upper triangular matrices with ones on the diagonal. Then $G/H \cong \mathbb{C}^2 \setminus \{0\}$ where the action of G is given by matrix multiplication on a vector $(x\ y)^T$. The Borel subgroup B of upper triangular matrices has an open orbit $D(y)$, the principal open set where y doesn't vanish. There is one color $V(y) \subset G/H$ and $\mathcal{V} = \mathcal{N}_\mathbb{Q} \cong \mathbb{Q}$.

In this setting, $m = 0$, $r = 1$, $s_1 = 2$, and we may write $f_{11} := y$ and $f_{12} := x$. The lattice $\mathcal{N}$ is spanned by a single ray v_1. The left-hand side of Figure 4.1 shows the vectors v_1, v_{11}, and v_{12}. Then $Z_0 \cong \mathbb{C}^2 \setminus \{0\}$ and the inclusion $G/H \hookrightarrow Z_0$ is an isomorphism given by switching coordinates: $(a, b) \mapsto (b, a)$. The tropicalization $\mathrm{trop}_\mathbb{T}(G/H)$ is $\overline{\mathbb{Q}}^2 \setminus \{\infty\}$ and the inclusion $\mathcal{N}_\mathbb{Q} \hookrightarrow N_\mathbb{Q}$ is the diagonal map $\mathbb{Q} \hookrightarrow \mathbb{Q}^2$. Figure 4.1 illustrates Theorem 4.2.2 in the case where $Y = G/H$. In the figure, $\mathrm{trop}_G(G/H) = \mathcal{V}$ appears as a dotted line and we conclude that $\mathcal{V} = \mathcal{N}_\mathbb{Q}$ since $\mathrm{trop}_\mathbb{T}(\mathbb{C}^2 \setminus \{0\})$ surjects onto $\mathcal{N}_\mathbb{Q}$ via ψ.

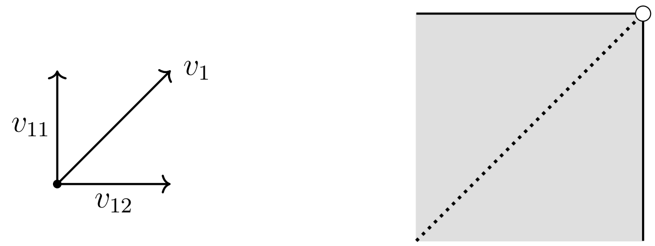


Figure 4.1: The vectors v_1, v_{11}, and v_{12} of Example 4.2.3 (left) and the tropicalization of the punctured plane with $\mathcal{N}_{\mathbb{Q}}$ embedded as a dotted line (right)

This result allows us to recover a description due to Vogiannou of the tropicalization of subvarieties of G/H in this setting ([33, Example 3.10]). Vogiannou considers a curve C in $G/H = \mathbb{C}^2 \setminus \{0\}$ given by a polynomial $f(x, y)$. He shows that $\mathrm{trop}_G(C)$ is a ray oriented to the left in $\mathcal{V} \cong \mathbb{Q}$ when the constant term of f is nonzero and is all of $\mathcal{V}$ when the constant term is zero. The condition that the constant term of $f(x, y)$ is zero is equivalent to saying that the zero vector is contained in the closure of C in $\mathbb{C}^2$. By [21, Proposition 6.3.5], this occurs if and only if $\mathrm{trop}_{\mathbb{T}}(C)$ intersects the interior of the first quadrant. In this case, $\mathrm{trop}_{\mathbb{T}}(C)$ projects onto the right-hand ray of $\mathcal{V}$ under the map ψ, so we conclude that $\mathrm{trop}_G(C) = \mathcal{V}$.

Example 4.2.4. Let $G := \mathrm{Sl}_2 \times \mathrm{Sl}_2$, H be the diagonal subgroup, and B consist of ordered pairs of upper and lower triangular matrices. Then $G/H \cong \mathrm{Sl}_2$ with coordinates x_{ij} for $i, j = 1, 2$ and the action of G is $(g, h) \cdot (x_{ij}) = g(x_{ij})h^{-1}$. The only B semi-invariant rational function is $f_1 := x_{22}$ with associated character $\chi : B \to \mathbb{C}^*$ defined by $((a_{ij}), (b_{ij})) \mapsto a_{22}^{-1} b_{22}$. The orbit of x_{22} under the action of G is

$$G \cdot x_{22} = \{-g_{21}h_{12}x_{11} + g_{21}h_{11}x_{12} - g_{22}h_{12}x_{21} + g_{22}h_{11}x_{22} : (g_{ij}), (h_{ij}) \in \mathrm{Sl}_2\}$$

It follows that $G \cdot f_1$ has rank four and we may choose as our basis $f_{11} := f_1 = x_{22}$, $f_{12} := x_{21}$, $f_{13} := x_{12}$, and $f_{14} := x_{11}$. Finally, $\Gamma\left(G/H, \mathcal{O}^*_{G/H}\right)/\mathbb{C}^*$ is trivial, so $Z_0 \cong \mathbb{C}^4 \setminus \{0\}$. The inclusion $G/H \hookrightarrow Z_0$ is then given by $(x_{ij}) \mapsto (x_{22}, x_{21}, x_{12}, x_{11})$.

The image of G/H in Z_0 is $V(f_{11}f_{14} - f_{12}f_{13} - 1)$. The tropicalization of this variety is the set of extended valuations μ such that the minimum

$$\min\left\{\mu(f_{11}) + \mu(f_{14}), \mu(f_{12}) + \mu(f_{13}), 0\right\}$$

is met more than once. This forces $\mu(f_{1j}) \leq 0$ for some j. Applying ψ therefore takes us to the ray in $\mathcal{N}_{\mathbb{Q}} \cong \mathbb{Q}$ spanned by the valuation $f_1 \mapsto -1$, so $\mathcal{V}$ is a half-space.

We must still consider the case that the spherical homogeneous space has non-trivial divisor class group. Following [14], we will use bold-faced characters $\boldsymbol{G}$ and $\boldsymbol{H}$ when the spherical homogeneous space $\boldsymbol{G}/\boldsymbol{H}$ does not have trivial divisor class group. In general, the spherical data associated to $\boldsymbol{G}/\boldsymbol{H}$ is notated as with G/H but bold-faced. We still assume that $\boldsymbol{G} = \boldsymbol{G}^{ss} \times \boldsymbol{C}$ is of semisimple type. For each $\boldsymbol{D}_i$ in the palette $\boldsymbol{\mathfrak{D}} := \{\boldsymbol{D}_1, \ldots, \boldsymbol{D}_r\}$, we consider the pullback of $\boldsymbol{D}_i$ under the projection map $\boldsymbol{G} \to \boldsymbol{G}/\boldsymbol{H}$. By results in [5], there is a unique $\boldsymbol{f}_i \in \mathbb{C}[\boldsymbol{G}]$ such that $V(\boldsymbol{f}_i)$ cuts out the pre-image of $\boldsymbol{D}_i$, $\boldsymbol{f}_i$ is $\boldsymbol{C}$-invariant, and $\boldsymbol{f}_i(1) = 1$. Then $\boldsymbol{H}$ acts from the right on $\boldsymbol{f}_i$ with character $\boldsymbol{\chi}_i \in \mathcal{X}(\boldsymbol{H})$. We will come back to these $\boldsymbol{f}_i$ in §4.4.

Gagliardi defines $G := \boldsymbol{G} \times (\mathbb{C}^*)^{\boldsymbol{\mathfrak{D}}}$ and

$$H := \left\{(h, \boldsymbol{\chi}_1(h), \ldots, \boldsymbol{\chi}_r(h)) : h \in \boldsymbol{H}\right\}.$$

The Borel subgroup is $B := \boldsymbol{B} \times (\mathbb{C}^*)^{\boldsymbol{\mathfrak{D}}}$. There is a natural isomorphism $H \cong \boldsymbol{H}$ given by projection to the first coordinate and a natural morphism $\boldsymbol{\pi} : G/H \to \boldsymbol{G}/\boldsymbol{H}$ with the induced surjective map $\boldsymbol{\pi}_* : \mathcal{N}_{\mathbb{Q}} \to \boldsymbol{\mathcal{N}}_{\mathbb{Q}}$. The B-stable prime divisors in the palette $\mathcal{D}$ of G/H are precisely the pullbacks under $\boldsymbol{\pi}$ of the colors in $\boldsymbol{\mathcal{D}}$. The character group of $(\mathbb{C}^*)^{\boldsymbol{\mathcal{D}}}$ is isomorphic to $\mathbb{Z}^{\boldsymbol{\mathcal{D}}}$, and we denote its basis by $\{\eta_1, \ldots, \eta_r\}$. Gagliardi shows that G/H has trivial divisor class group ([14, Corollary 3.2]) and that $\mathcal{V} = \boldsymbol{\pi}_*^{-1}(\boldsymbol{\mathcal{V}})$ ([14, Proposition 3.3]).

Theorem 4.2.2 can now be easily generalized using the theory of §2.5:

Theorem 4.2.5. *If $\boldsymbol{Y} \subseteq \boldsymbol{G}/\boldsymbol{H}$ is a closed subvariety, then*

$$\mathrm{trop}_{\boldsymbol{G}}(\boldsymbol{Y}) = (\mathrm{trop}(\boldsymbol{\pi}) \circ \psi) \left(\mathrm{trop}_{\mathbb{T}} \left(\boldsymbol{\pi}^{-1} (\boldsymbol{Y})\right)\right).$$

Proof. By Theorem 4.2.2, this is equivalent to proving

$$\mathrm{trop}_{\boldsymbol{G}}(\boldsymbol{Y}) = \mathrm{trop}(\boldsymbol{\pi}) \left(\mathrm{trop}_{G} \left(\boldsymbol{\pi}^{-1}(\boldsymbol{Y})\right)\right),$$

which follows directly from Proposition 2.5.2. □

Example 4.2.6. Suppose $\boldsymbol{G} = \mathrm{Sl}_2$, $\boldsymbol{H}$ is the diagonal torus, and $\boldsymbol{B}$ is the upper triangular matrices. Then $\boldsymbol{G}/\boldsymbol{H} \cong \mathbb{P}^1 \times \mathbb{P}^1 \setminus \Delta$, where Δ is the diagonal and G acts on each component on the left by matrix multiplication. The homogeneous space $\mathbb{P}^1 \times \mathbb{P}^1 \setminus \Delta$ has divisor class group $\mathbb{Z}^2$. The palette $\boldsymbol{\mathcal{D}}$ consists of two colors: $V(x_{21})$ and $V(x_{22})$. The associated characters are both defined by $(h_{ij}) \mapsto h_{22}$; call this character $\boldsymbol{\chi}$. The vector space $\boldsymbol{\mathcal{N}}_{\mathbb{Q}}$ associated to $\boldsymbol{G}/\boldsymbol{H}$ is one-dimensional, $\boldsymbol{\mathcal{V}}$ is ray, and both colors lie outside of the valuation cone.

Now $G \cong \boldsymbol{G} \times (\mathbb{C}^*)^{\boldsymbol{\mathcal{D}}}$ and $H := \{((h_{ij}), h_{22}, h_{22}) : (h_{ij}) \in \boldsymbol{H}\}$. Then G acts on $\mathbb{C}^2 \times \mathbb{C}^2 \times \mathbb{C}^*$. In this action, $\boldsymbol{G}$ acts by left matrix multiplication on the copies of $\mathbb{C}^2$ and trivially on $\mathbb{C}^*$ and $(\mathbb{C}^*)^{\boldsymbol{\mathcal{D}}}$ respectively acts with weights $-\eta_1$ and $-\eta_2$ on the first and second copy of $\mathbb{C}^2$ and with weight $-\eta_1 - \eta_2$ on $\mathbb{C}^*$. If the coordinates of the copies of $\mathbb{C}^2$ are respectively given by S_{11}, S_{21} and S_{12}, S_{22} and the coordinate of $\mathbb{C}^*$

is T, then

$$G/H = V(S_{11}S_{22} - S_{12}S_{21} - T) \subset \mathbb{C}^2 \times \mathbb{C}^2 \times \mathbb{C}^*.$$

The map $\boldsymbol{\pi} : G/H \to \boldsymbol{G}/\boldsymbol{H}$ is then defined by

$$((S_{11}, S_{21}), (S_{12}, S_{22}), T) \mapsto [S_{11} : S_{21}] \times [S_{12} : S_{22}] \subset \mathbb{P}^1 \times \mathbb{P}^1 \setminus \Delta.$$

Consider $\boldsymbol{Y} = \boldsymbol{G}/\boldsymbol{H}$; we will recover $\boldsymbol{\mathcal{V}}$ as $\mathrm{trop}_{\boldsymbol{G}}(\boldsymbol{G}/\boldsymbol{H})$ using Theorem 4.2.5. The colors of G/H are given by $V(S_{21})$ and $V(S_{22})$ and we may set $f_1 = f_{11} := S_{21}$ and $f_2 = f_{21} := S_{22}$. Further, $s_1 = s_2 = 2$ and $f_{12} = S_{11}$ and $f_{22} = S_{12}$, so the toric variety Z_0 associated to G/H is $(\mathbb{C}^2 \setminus \{0\}) \times (\mathbb{C}^2 \setminus \{0\}) \times \mathbb{C}^*$ with these coordinates. The pre-image $\boldsymbol{\pi}^{-1}(\boldsymbol{G}/\boldsymbol{H})$ is $G/H = V(S_{11}S_{22} - S_{12}S_{21} - T)$. The tropicalization $\mathrm{trop}_{\mathbb{T}}(G/H)$ is the set of points

$$(a_{11}, a_{12}, a_{21}, a_{22}, b_1) \in \mathrm{trop}_{\mathbb{T}}(Z_0) \cong \left(\overline{\mathbb{Q}}^2 \setminus \{\infty\}\right) \times \left(\overline{\mathbb{Q}}^2 \setminus \{\infty\}\right) \times \mathbb{Q}$$

such that the minimum $\min\{a_{11} + a_{22}, a_{12} + a_{21}, b_1\}$ is met at least twice. It follows that the image of $\mathrm{trop}_{\mathbb{T}}(G/H)$ under the map ψ is the set of ordered triples (a_1, a_2, b_1) such that $a_1 + a_2 \leq b_1$.

The $\boldsymbol{B}$ semi-invariant rational functions on $\boldsymbol{G}/\boldsymbol{H}$ are one-dimensional, given by integer powers of $\boldsymbol{f} := \frac{S_{11}S_{22} - S_{12}S_{21}}{S_{12}S_{22}}$. The lattice $\boldsymbol{\mathcal{N}}$ is spanned by the $\boldsymbol{G}$-invariant valuation $\boldsymbol{f} \mapsto 1$. Under $\mathrm{trop}(\boldsymbol{\pi})$, an element $(a_1, a_2, b_1) \in \mathrm{trop}_G(G/H)$ is mapped to the valuation $(\boldsymbol{f} \mapsto b_1 - a_1 - a_2) \in \mathrm{trop}_{\boldsymbol{G}}(\boldsymbol{G}/\boldsymbol{H})$. Because $a_1 + a_2 \leq b_1$ on $\psi\left(\mathrm{trop}_G(G/H)\right)$, it follows that this valuation is always non-negative on $\boldsymbol{f}$ and so we recover the fact that $\boldsymbol{\mathcal{V}} = \mathrm{trop}_{\boldsymbol{G}}(\boldsymbol{G}/\boldsymbol{H})$ is a ray in the positive direction.

4.3 Embedding Toroidal Varieties in Toric Varieties

In [14], Gagliardi considers spherical embeddings X of G/H whose associated colored fans only include rays without color and exclude all higher-dimensional cones. He does this because he is interested in computing the Cox ring and this is the only information needed to do that. Given such an embedding, he finds an explicit toric variety Z in which G/H embeds such that $\overline{G/H} = X$ in Z. He restricts himself to such embeddings because to compute the Cox ring of a spherical embedding, only information about the G-stable prime divisors is needed, i.e. rays without color.

In this section we extend his construction to arbitrary toroidal spherical embeddings, whose colored fans consist of rays with colors and potentially higher dimensional cones spanned by them. We maintain the notation and conventions established in §4.2. In particular, we start in the case where G/H has trivial divisor class group. Let X be a G/H-embedding given by a fan Σ whose one-dimensional cones are $\{u_1, \ldots, u_n\} \subseteq \mathcal{V}$. For each colored cone $(\sigma, \emptyset) \in \Sigma$, write $\sigma(1) \subseteq \{u_1, \ldots, u_n\}$ to denote the set of one-dimensional (non-colored) faces of a σ. Define

$$\mathfrak{A} := \{\mathfrak{a} \subset \{v_{ij}\} : \text{ for each } i \text{ there is at least one } j \text{ with } v_{ij} \notin \mathfrak{a}\}.$$

We note that $\mathfrak{A}$ is defined slightly differently in [14], where the phrase "at least one" is replaced by "exactly one". For each σ and each $\mathfrak{a} \in \mathfrak{A}$, we define

$$\sigma_{\mathfrak{a}} := \operatorname{cone}(\mathfrak{a} \cup \sigma(1)) \subset N_{\mathbb{Q}}.$$

The dimension of the cone $\sigma_{\mathfrak{a}}$ is $\dim \sigma_{\mathfrak{a}} = |\mathfrak{a}| + \dim \sigma$. This follows because for every i at least one v_{ij} is absent from $\mathfrak{a}$, which means each ray of σ is linearly independent of the rays in $\mathfrak{a}$. Then we define the fan Σ_Z to be the set of cones $\sigma_{\mathfrak{a}}$ for all $(\sigma, \emptyset) \in \Sigma$

and $\mathfrak{a} \in \mathfrak{A}$:

$$\Sigma_Z := \{\sigma_{\mathfrak{a}} : (\sigma, \emptyset) \in \Sigma, \mathfrak{a} \in \mathfrak{A}\}.$$

That Σ_Z is well-defined is a corollary of Proposition 4.4.6, which we state and prove later. Let Z be the toric variety associated to the fan Σ_Z. Note that $Z_0 \subseteq Z$ since the fan for Z_0 can be obtained from the definition by taking σ to be the trivial cone and letting $\mathfrak{a}$ range over $\mathfrak{A}$. We claim that the closure of G/H in Z is isomorphic to X. The argument mimics the procedure in [14]; many of the proofs proceed essentially identically. This claim is proved in Proposition 4.4.7; we delay the proof until §4.4, where we simultaneously consider non-toroidal varieties.

We now show that the action of G on Z_0 extends to Z. Define $\widehat{N} := N \oplus \mathbb{Z}^n$ where $\mathbb{Z}^n$ is given the basis $\{e_1, \ldots, e_n\}$. For each $\sigma_{\mathfrak{a}} \in \Sigma_Z$, define the cone

$$\widehat{\sigma}_{\mathfrak{a}} := \operatorname{cone}\left(\mathfrak{a} \cup \bigcup_{u_k \in \sigma(1)} \{e_k\}\right) \subseteq \widehat{N}_{\mathbb{Q}}$$

and let

$$\Sigma_{\widehat{Z}} := \operatorname{fan}\left(\widehat{\sigma}_{\mathfrak{a}} : \sigma_{\mathfrak{a}} \in \Sigma_Z\right).$$

In essence, the difference between $\sigma_{\mathfrak{a}}$ and $\widehat{\sigma}_{\mathfrak{a}}$ is that the rays of $\sigma_{\mathfrak{a}}$ in $\sigma(1)$ are not necessarily orthogonal. In $\widehat{\sigma}_{\mathfrak{a}}$, each u_k is replaced with a vector e_k that is orthogonal to every vector in $\mathfrak{a}$ and every other e_ℓ.

If $\widehat{Z}$ is the associated toric variety with torus $\widehat{\mathbb{T}} := \mathbb{T} \times (\mathbb{C}^*)^n$, then

$$\widehat{Z} \cong \mathbb{C}^{s_1 + \cdots + s_r} \times (\mathbb{C}^*)^m \times \mathbb{C}^n \setminus \widehat{S},$$

where $\widehat{S}$ is a closed set of codimension at least two. We revisit this theory in more detail in §4.5. There is a natural toric morphism $p : \widehat{Z} \to Z$ induced by the map $\widehat{N} \to N$ defined by $v_{ij} \mapsto v_{ij}$, $w_k \mapsto w_k$, and $e_\ell \mapsto u_\ell$. By [29, Theorem 4.1], p is a good quotient. In fact, it is a geometric quotient (cf. [1, Proposition 3.2]). The quotient is with respect to a subtorus $\Gamma \cong (\mathbb{C}^*)^n$ of $\mathbb{T} \times \mathbb{C}^n$, which is parametrized as follows:

$$\Gamma := \left\{ \left(\prod_{\ell=1}^n t_\ell^{-\langle u_\ell, v_{11}^* \rangle}, \dots, \prod_{\ell=1}^n t_\ell^{-\langle u_\ell, v_{rs_r}^* \rangle}, \prod_{\ell=1}^n t_\ell^{-\langle u_\ell, w_1^* \rangle}, \dots, \prod_{\ell=1}^n t_\ell^{-\langle u_\ell, w_k^* \rangle}, t_1, \dots, t_n \right) : t_\ell \in \mathbb{C}^* \right\}.$$

We may extend the natural action of G on $\mathbb{C}^{s_1+\cdots+s_r} \times (\mathbb{C}^*)^m$ to $\widehat{Z}$ by having G act trivially on the additional $\mathbb{C}^n$ summand. Then the action of G on $\widehat{Z}$ commutes with the action of Γ. As a result, we obtain an action of G on Z. More explicitly, because p is a good quotient, [9, Proposition 5.0.7] says that for any point $x \in X$, the preimage $p^{-1}(x)$ contains a unique closed Γ-orbit. If $g \in G$ and $z \in Z$, we define $g \cdot z := p(g \cdot \widehat{z})$, where $\widehat{z}$ is an element of the unique closed Γ-orbit in $p^{-1}(z)$. Because the actions of G and Γ commute, g takes the closed orbit of $p^{-1}(z)$ to another Γ-orbit. Since p is constant on orbits, $p(g \cdot \widehat{z})$ does not depend on the choice of $\widehat{z}$.

Further, the inclusion $N \hookrightarrow \widehat{N}$ induces a morphism $Z_0 \to \widehat{Z}$ that commutes with $p : \widehat{Z} \to Z$ to give the inclusion $Z_0 \hookrightarrow Z$, so the described action of G on Z extends that of G on Z_0.

Example 4.3.1. We again consider Example 4.2.3 and in particular the embedding of $\mathbb{C}^2 \setminus \{0\}$ in $\mathbb{P}^2 \setminus \{0\}$. The corresponding colored cone has one ray u_1 spanned by $-v_1$ in $\mathcal{N}_{\mathbb{Q}}$ and $Z = \mathbb{P}^2 \setminus \{0\}$. The variety $\widehat{Z} = (\mathbb{C}^2 \times \mathbb{C}) \setminus (\{0\} \times \mathbb{C}) = (\mathbb{C}^2 \setminus \{0\}) \times \mathbb{C}$ is given by adding one $\mathbb{C}$ summand for the cone $-v_1$ and removing the subvariety where both of the first two coordinates vanish. The morphism $p : \widehat{Z} \to Z = \mathbb{P}^2 \setminus \{0\}$ is then given by $(x, y, z) \mapsto [x : y : z]$. In Figure 4.2, we illustrate the compatible map of fans between $\Sigma_{\widehat{Z}}$ and Σ_Z that induces p. The preimage of a point $[a : b : c] \in \mathbb{P}^2 \setminus \{0\}$

is $\{(at, bt, ct) : a, b, c \in \mathbb{C}, t \in \mathbb{C}^*\}$, so p is a geometric quotient by the action of the torus $\Gamma := \{(t, t, t) : t \in \mathbb{C}^*\}$.

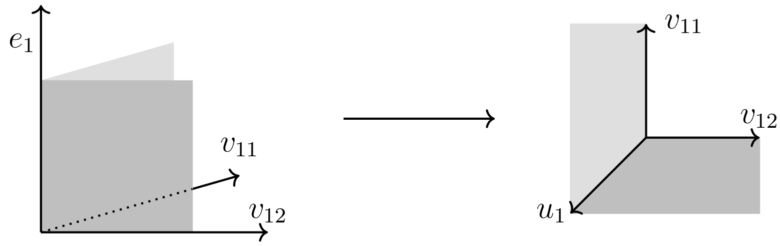


Figure 4.2: The compatible map of fans induced by $\widehat{N}_{\mathbb{Q}} \to N_{\mathbb{Q}}$ for the embedding $\mathbb{P}^2 \setminus \{0\}$.

If instead we consider $\mathrm{Bl}_0(\mathbb{C}^2) \subset \mathbb{C}^2 \times \mathbb{P}^1$ with one ray spanned by v_1, we get the same $\widehat{Z}$ and the map $\widehat{Z} \to Z = \mathrm{Bl}_0(\mathbb{C}^2)$ is given by $(x, y, z) \mapsto (xz, yz) \times [x : y]$. The preimage of a point $(ac, bc) \times [a : b] \in \mathrm{Bl}_0(\mathbb{C}^2)$ is $\{(at^{-1}, bt^{-1}, ct) : t \in \mathbb{C}^*\}$, so it is a geometric quotient by the torus $\Gamma := \{(t^{-1}, t^{-1}, t) : t \in \mathbb{C}^*\}$.

We have not yet discussed the case of a $\boldsymbol{G}/\boldsymbol{H}$-embedding $\boldsymbol{X}$ where the homogeneous space $\boldsymbol{G}/\boldsymbol{H}$ has nontrivial divisor class group. Given such a homogeneous space, we earlier defined an associated homogeneous space G/H with trivial divisor class group along with a map $\boldsymbol{\pi} : G/H \to \boldsymbol{G}/\boldsymbol{H}$. Gagliardi shows ([14, Proposition 3.4]) that the pushforward $\boldsymbol{\pi}_* : \mathcal{N}_{\mathbb{Q}} \to \boldsymbol{\mathcal{N}}_{\mathbb{Q}}$ is an isomorphism when restricted to the subspace $(\mathcal{N}_T)_{\mathbb{Q}} := \operatorname{span}\{w_1, \ldots, w_m\}$.

If $\boldsymbol{\Sigma}$ is the colored fan associated to a toroidal embedding $\boldsymbol{X}$, we define a G/H-embedding X by considering the fan Σ, which is the preimage of $\boldsymbol{\Sigma}$ under $\boldsymbol{\pi}_*|_{(\mathcal{N}_T)_{\mathbb{Q}}}$. Let Z be the toric variety associated to X as constructed earlier in this section.

There is a good geometric quotient of Z by the torus $(\mathbb{C}^*)^{\mathfrak{D}}$, so we may extend $\boldsymbol{\pi}$ to $\boldsymbol{\pi} : Z \to \boldsymbol{Z}$, where $\boldsymbol{Z}$ is a toric variety. Refer again to the theory in [29], as well as [1, Proposition 3.2]. By Proposition 4.4.7, the spherical variety X associated to the fan Σ is the closure of G/H in Z. Then $\boldsymbol{\pi}$ takes X to the closure of $\boldsymbol{G}/\boldsymbol{H}$ in $\boldsymbol{Z}$ (cf. [9, Definition 5.0.5] and [9, Theorem 5.0.6]). By the construction of the colored fan, the image of X under $\boldsymbol{\pi}$ is $\boldsymbol{X}$ and hence $\boldsymbol{X}$ is the closure of $\boldsymbol{G}/\boldsymbol{H}$ in $\boldsymbol{Z}$. We collect these facts later in Proposition 4.4.8 where we also consider the possibility of colors.

4.4 The Problem of Colors

The methods of §4.3 are illustrations of [15, Proposition 2.11], which says that any Mori dream space can be embedded in a projective toric variety. Projective spherical varieties are Mori dream spaces, but not every spherical embedding is a Mori dream space. This issue is addressed in [13], another paper of Gagliardi.

Definition 4.4.1. A normal variety X is said to have the *A_2-property* if any two points in X lie in some affine open subset of X.

Definition 4.4.2. A colored fan Σ is *polyhedral* if the relative interiors of any two colored cones in Σ have empty intersection.

Theorem 4.4.3. [13, Theorem 1.5] *Let Σ be a colored fan with associated spherical embedding $G/H \hookrightarrow X$. Then X has the A_2-property if and only if Σ is polyhedral.*

Finally, we make use of the following theorem of Włodarczyk, which Gagliardi cites in [13]:

Theorem 4.4.4 ([34])**.** *A normal variety X has the A_2-property if and only if it admits a closed embedding $X \hookrightarrow Z$ into a toric variety Z.*

All of this serves to tell us that the spherical varieties that cannot be embedded in a toric variety are those whose cones are not polyhedral. Intuitively, this is reflected by the fact that a spherical embedding with a non-polyhedral colored fan will be associated by the process described in §4.3 to a toric variety with a non-polyhedral fan, which is an impossibility. A colored fan can only exhibit non-polyhedral behavior outside of the valuation cone. We illustrate this in the following extended example.

Example 4.4.5. Let $G = \mathrm{Sl}_3$ and $H = \mathrm{Sl}_2$ embedded in G as the lower right entries. Then G has an action on $\mathbb{C}^3 \times \mathbb{C}^3$ given by

$$g \cdot (x, y) = \left(gx, \left(g^{-1}\right)^* y\right),$$

where $*$ indicates taking the conjugate transpose. Under this action, the point $((1,0,0),(1,0,0))$ has isotropy group H. If the coordinates of $\mathbb{C}^3 \times \mathbb{C}^3$ are given as $((x_1, x_2, x_3),(y_1, y_2, y_3))$, then the orbit of this point is $V(x_1y_1 + x_2y_2 + x_3y_3 - 1) = G/H$. Taking the Borel group B consisting of the upper triangular matrices, G/H is a spherical homogeneous space.

There are two colors: $V(x_3)$ and $V(y_1)$, and $\Gamma(G/H, \mathcal{O}^*_{G/H})$ is trivial. Thus, $\mathcal{N}$ is two-dimensional, spanned by valuations v_1 and v_2 respectively associated to the colors $V(x_3)$ and $V(y_1)$. The G-modules $G \cdot x_3$ and $G \cdot y_1$ both have rank three and we may define an embedding as follows:

$$V(x_1y_1 + x_2y_2 + x_3y_3 - 1) = G/H \hookrightarrow Z_0 := \left(\mathbb{C}^3 \setminus \{0\}\right) \times \left(\mathbb{C}^3 \setminus \{0\}\right)$$
$$((x_1, x_2, x_3),(y_1, y_2, y_3)) \mapsto (x_3, x_2, x_1, y_1, y_2, y_3)$$

The tropicalization $\mathrm{trop}_{\mathbb{T}}(G/H)$ of G/H in Z_0 is the set of extended valuations μ where the minimum

$$\min \{\mu(x_1) + \mu(y_1), \mu(x_2) + \mu(y_2), \mu(x_3) + \mu(y_3), 0\}$$

is met at least twice. By Theorem 4.2.1, $\mathcal{V} = \mathrm{trop}_{\mathbb{T}}(G/H) \cap \mathcal{N}_{\mathbb{Q}}$. On $\mathcal{N}_{\mathbb{Q}}$, $\mu(x_1) = \mu(x_2) = \mu(x_3)$ and $\mu(y_1) = \mu(y_2) = \mu(y_3)$, and so for the above minimum to be met twice, we must have $\mu(x_3) + \mu(y_1) \leq 0$. Thus, $\mathcal{V} = \{v_1 + v_2 \leq 0\}$. Figure 4.3 shows the valuation cone and palette of this homogeneous space.

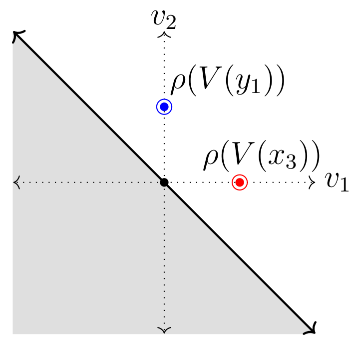


Figure 4.3: The valuation cone and palette of $V(x_1y_1 + x_2y_2 + x_3y_3 - 1)$.

The interest here is that there are two colors that lie outside the valuation cone, so there is potential for valid colored cones that are not polyhedral. Indeed, we may have a simple G/H-embedding corresponding to the colored cone spanned by the color $\rho(V(x_3))$ and $-2v_1 + v_2$ since the interior of this cone intersects the valuation cone. This is illustrated by the red cone in Figure 4.4. Similarly, there exists an embedding whose colored cone is spanned by $\rho(V(y_1))$ and $v_1 - 2v_2$, the blue cone in Figure 4.4. The union of these cones is a valid colored fan and thus corresponds to a G/H-embedding. Even though this fan is not polyhedral, it is a colored fan because the intersection is outside the valuation cone.

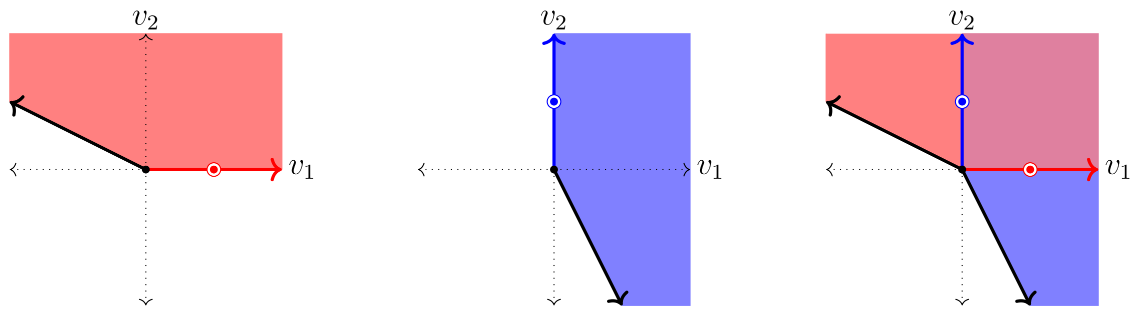


Figure 4.4: Colored cones corresponding to three spherical embeddings. The embedding on the far right is the gluing of the other two along G/H; it has two maximal colored cones that overlap in the first quadrant.

Throughout this section we therefore only consider colored fans that are polyhedral. As before, we assume for the moment that G/H has trivial divisor class group. Let X be a G/H-embedding with colored fan Σ and let Z_0 be the toric variety associated to G/H defined in §4.3. Let $(\sigma, \mathcal{F}) \in \Sigma$ be a colored cone. Then write $\sigma(1) := \{u_1, \ldots, u_n\}$ for the set of one-dimensional non-colored faces of σ and define

$$\mathfrak{A}(\mathcal{F}) := \{\mathfrak{a} \subseteq \{v_{ij}\} : \text{for each } i \text{ such that } D_i \notin \mathcal{F}, \text{ there is at least one } j \text{ with } v_{ij} \notin \mathfrak{a}\}.$$

Note that $\mathfrak{A}(\emptyset)$ is the same as $\mathfrak{A}$, as defined at the beginning of §4.3. The set $\mathfrak{A}(\mathcal{F})$ simply extends $\mathfrak{A}$ by allowing the entire set $\{v_{ij}\}_j$ to be present when the color D_i lies in $\mathcal{F}$. For a given $\mathfrak{a} \in \mathfrak{A}(\mathcal{F})$, we define a cone in $N_{\mathbb{Q}}$ as follows:

$$\sigma_{\mathfrak{a}} := \operatorname{cone}(\mathfrak{a} \cup \sigma(1)) \subset N_{\mathbb{Q}}.$$

The fan Σ_Z is defined similarly to before:

$$\Sigma_Z := \operatorname{fan}(\{\sigma_{\mathfrak{a}} : (\sigma, \mathcal{F}) \in \Sigma, \mathfrak{a} \in \mathfrak{A}(\mathcal{F})\}).$$

If Σ has no colors, then Σ_Z is consistent with the object we described in §4.3. By a similar argument to the toroidal case, we can see that the action of G on Z_0 also extends to Z in this setting.

Proposition 4.4.6. *If X is a G/H-embedding whose associated colored fan Σ is polyhedral, then the fan Σ_Z is well-defined.*

Proof. We must check that Σ_Z is closed under taking faces and that the intersection of any two cones in Σ_Z is a face of each. If $\sigma_{\mathfrak{a}}$ is a cone in Σ_Z, then its faces are precisely those $\sigma'_{\mathfrak{a}'}$ such that $\mathfrak{a}' \subseteq \mathfrak{a}$ and $\sigma' \preceq \sigma$ in Σ. Such a cone $\sigma'_{\mathfrak{a}'}$ is in Σ_Z, as needed. Suppose $\sigma_{\mathfrak{a}}, \sigma'_{\mathfrak{a}'} \in \Sigma_Z$ are two cones. Then because Σ is a polyhedral fan, the intersection of σ and σ' is a colored cone $(\sigma'', \mathcal{F}'') \in \Sigma$ that is a colored face of each. The intersection $\sigma_{\mathfrak{a}} \cap \sigma'_{\mathfrak{a}'}$ then equals $\sigma''_{\mathfrak{a}\cap\mathfrak{a}'}$, which is in Σ_Z. $\square$

The following proposition is what this section has been working towards. It is an extension of [14, Lemma 2.13] and [14, Proposition 2.14], and the proof owes much of its structure to those two results.

Proposition 4.4.7. (cf. [14, Proposition 2.14]) *Suppose G/H has trivial divisor class group and X is a G/H-embedding corresponding to a colored fan Σ. Then X is isomorphic to the closure $\overline{G/H}$ of G/H in the associated toric variety Z.*

Proof. The claim is proved in [14, Proposition 2.14] when Σ consists solely of non-colored rays. If Σ consists of a single ray σ spanned by the color D_i, then on every additional G-orbit of Z outside of Z_0, f_i vanishes and f_j and g_k do not for every $j \neq i$ and k. Thus, this embedding corresponds to a ray in the direction of σ. Moreover, this ray must have color since the f_{ij} may vanish with different multiplicities along the orbits added. After gluing together along shared orbits, we conclude that the statement of the proposition holds when Σ consists solely of rays, both colored and not.

We now turn to proving the full claim. It will be sufficient to prove it for simple embeddings since they can then be glued together along shared orbits. Let Σ consist of a single colored cone $(\sigma, \mathcal{F})$ with non-colored rays $u_1, \ldots, u_n \in \sigma(1)$.

Let the maximal cone in Σ_Z spanned by the sets $\sigma(1)$ and $\bigcup_{D_i \in \mathcal{F}} \{v_{ij}\}_{j=1}^{s_i}$ be denoted τ. The cone τ corresponds to an affine variety U_τ. For each ray $u_\ell \in \sigma(1)$, there is an open affine subset U_ℓ of U_τ with two torus orbits. Similarly, for each $D_i \in \mathcal{F}$ and each $1 \leq j \leq s_i$, there is an open affine subset U_{ij} corresponding to the ray v_{ij}. Let $\pi_\ell \in \mathbb{C}[U_\tau]$ for $1 \leq \ell \leq n$ denote prime elements that cut out the closures of the U_ℓ in U_τ. Similarly choose prime elements $\pi_{ij} \in \mathbb{C}[U_\tau]$ that cut out the U_{ij}. Then we have the following commutative diagram:

$$\begin{tikzcd}
\mathbb{C}[U_\tau] \arrow[rrr, hook] \arrow[d] & & & (\mathbb{C}[S_{11}, \ldots, S_{rs_r}, T_1, \ldots, T_m])_{\mathfrak{p}} \arrow[d] \\
\mathbb{C}[U_\tau]/(\mathfrak{p} \cap \mathbb{C}[U_\tau]) \arrow[r, hook, "\mathfrak{N}"] & R_1 \arrow[r, hook, "\mathfrak{L}"] & R_2 \arrow[r] & (\mathbb{C}[S_{11}, \ldots, S_{rs_r}, T_1, \ldots, T_m])_{\mathfrak{p}}/\mathfrak{p} \cong \mathbb{C}(G/H)
\end{tikzcd}$$

In the diagram, $\mathfrak{N}$ is normalization and $\mathfrak{L}$ is localization. For each π_ℓ and each π_{ij}, there are respectively prime elements $\widetilde{\pi}_\ell, \widetilde{\pi}_{ij} \in R_2$ such that $V(\pi_\ell) = V(\widetilde{\pi}_\ell)$ and $V(\pi_{ij}) = V(\widetilde{\pi}_{ij})$ in $\operatorname{Spec} R_2$. Now $L \in \{S_{ij}, T_k\}$ can be written in the form

$$L = c \cdot \prod_{\ell=1}^{n} \left(\pi_\ell^{d_{\ell,1}} / \pi_\ell^{d_{\ell,2}}\right) \cdot \prod_{D_i \in \mathcal{F}, 1 \leq j \leq s_i} \left(\pi_{ij}^{d_{ij,1}} / \pi_{ij}^{d_{ij,2}}\right)$$

for $d_\ell, d_{i,j} \in \mathbb{Z}_{\geq 0}$ and $c \in \mathbb{C}[U_\tau]^*$. We may further write

$$L = \widetilde{c} \cdot \prod_{\ell=1}^{n} \left(\widetilde{\pi}_\ell^{e_\ell d_{\ell,1}} / \widetilde{\pi}_\ell^{e_\ell d_{\ell,2}}\right) \cdot \prod_{D_i \in \mathcal{F}, 1 \leq j \leq s_i} \left(\widetilde{\pi}_{ij}^{e_{ij} d_{ij,1}} / \widetilde{\pi}_{ij}^{e_{ij} d_{ij,2}}\right)$$

where $e_\ell, e_{ij} \in \mathbb{Z}_{\geq 0}$ and $\widetilde{c} \in R_2^*$.

The valuations in the colored cone $(\sigma, \mathcal{F})$ are positive $\mathbb{Q}$-linear combinations of the u_ℓ and the valuations induced by the colors $D_i \in \mathcal{F}$. It follows from the above argument that every such valuation is induced by a torus invariant one in Z. This proves the claim for simple spherical embeddings. Gluing together along shared orbits gives the full result. $\square$

It still remains to consider when the homogeneous space $\boldsymbol{G}/\boldsymbol{H}$ has nontrivial divisor class group. Recall that the colors $D_1, \ldots, D_r$ of the associated homogeneous space G/H are precisely the pullbacks of the colors $\boldsymbol{D}_1, \ldots, \boldsymbol{D}_r$ of $\boldsymbol{G}/\boldsymbol{H}$. For each D_i, Gagliardi defines $f_i \in \Gamma(G/H, \mathcal{O}_{G/H})$ such that $V(f_i) = D_i$. There is an inclusion from the character lattice of $(\mathbb{C}^*)^{\boldsymbol{\mathfrak{D}}}$ to $\mathbb{C}[G]^*$ that takes a character χ to a monomial $\epsilon^{\chi} \in \mathbb{C}\left[(\mathbb{C}^*)^{\boldsymbol{\mathfrak{D}}}\right] \subset \mathbb{C}[G]^*$. Then we may write $f_i := \boldsymbol{f}_i \epsilon^{-\eta_i}$, where η_i is the character that is trivial on every coordinate not equal to i. The function f_i is invariant under the action of H from the right and $V(f_i) = D_i$. Note that as a result $(\mathbb{C}^*)^{\boldsymbol{\mathfrak{D}}}$ has a nontrivial action on f_i for all i.

Then let $\boldsymbol{X}$ be a $\boldsymbol{G}/\boldsymbol{H}$-embedding with polyhedral colored fan $\boldsymbol{\Sigma}$. For each color $\boldsymbol{\mathfrak{D}_i}$ appearing in $\boldsymbol{\Sigma}$, include in Σ a colored ray corresponding to D_i. For each ray without color, take the preimage of this ray under $\boldsymbol{\pi}_*|_{(\mathcal{N}_T)_{\mathbb{Q}}}$ as we did for toroidal embeddings. Then add in higher-dimensional cones between these rays in $\mathcal{N}_{\mathbb{Q}}$ if they exist in $\boldsymbol{\Sigma}$. As in the toroidal case, there is a good geometric quotient $\boldsymbol{\pi} : Z \to \boldsymbol{Z}$ that extends $\boldsymbol{\pi} : G/H \to \boldsymbol{G}/\boldsymbol{H}$. We obtain the following result, similarly to the toroidal case:

Proposition 4.4.8. *Let $\boldsymbol{G}/\boldsymbol{H} \hookrightarrow \boldsymbol{X}$ be a spherical embedding with associated spherical embedding $G/H \hookrightarrow X$ where G/H has trivial divisor class group. Then $\overline{\boldsymbol{G}/\boldsymbol{H}} \cong \boldsymbol{X}$, where the closure is taken in $\boldsymbol{Z}$.*

4.5 Extended (Global) Tropicalization

This section finally gives a global construction for the tropicalization of a spherical embedding. Because of the issues raised in §4.4, this construction can only work for a certain class of spherical embeddings, namely those that have the A_2-property. For many homogeneous spaces, this consideration will not raise any issues. For example, if G/H is horospherical or if it has fewer than two colors lying outside the valuation cone, the global tropicalization will always go through without issue.

As before, we begin discussing the case when G/H has trivial divisor class group and we maintain the notation introduced previously. Let X be a G/H-embedding with the associated toric varieties Z and $\widehat{Z}$ and morphism $p : \widehat{Z} \to Z$. Our methodology will be to work in the toric world with $\widehat{Z}$ and Z where results are known. Then we will apply an extension of the map ψ. The general theory in the following discussion can be found in more generality in [21, §6.1]; also refer to [9, §5.1].

There is a short exact sequence

$$0 \to N \hookrightarrow \widehat{N} \to A_{n-1}\left(\widehat{Z}\right) \to 0,$$

where $A_{n-1}\left(\widehat{Z}\right)$ is the cokernel of the natural inclusion $N \hookrightarrow \widehat{N}$. Applying the functor $\mathrm{Hom}(-, \mathbb{C}^*)$, we obtain the following exact sequence:

$$\mathrm{Hom}(N, \mathbb{C}^*) \leftarrow \mathrm{Hom}\left(\widehat{N}, \mathbb{C}^*\right) \hookleftarrow Q \leftarrow 0,$$

where $Q := \mathrm{Hom}\left(A_{n-1}\left(\widehat{Z}\right), \mathbb{C}^*\right)$. Let E_ℓ denote the coordinate in $\mathbb{C}\left[\widehat{Z}\right]$ corresponding to the ray e_ℓ so that

$$\mathbb{C}\left[\widehat{Z}\right] = \mathbb{C}[S_{11}, \ldots, S_{rs_r}, T_1, \ldots, T_m, E_1, \ldots, E_n].$$

Then the *irrelevant ideal* in $\mathbb{C}\left[\widehat{Z}\right]$ is

$$F := \left\langle \prod_{v_{ij}\notin\sigma} S_{ij} \cdot \prod_{w_k\notin\sigma} T_k \cdot \prod_{e_\ell\notin\sigma} E_\ell : \sigma \in \Sigma_{\widehat{Z}} \right\rangle$$

and

$$\widehat{Z} \cong \mathbb{C}^{s_1+\cdots+s_r+m+n} \setminus V(F) \cong \mathbb{C}^{s_1+\cdots+s_r} \times (\mathbb{C}^*)^m \times \mathbb{C}^n \setminus \widehat{S}$$

for some $\widehat{S}$ of codimension at least two. Finally, we have that

$$Z \cong (\mathbb{C}^{s_1+\cdots+s_r+m+n} \setminus V(F))/Q.$$

This quotient construction of Z tropicalizes in the following sense (see [21, Proposition 6.2.6] and [21, Corollary 6.2.16]):

Proposition 4.5.1. *Suppose G/H has trivial divisor class group and suppose $Y \subseteq G/H$ is a closed subvariety. Let X be a G/H-embedding with the A_2-property with associated toric varieties Z and $\widehat{Z}$ and let $\overline{Y}$ be the closure of Y in Z. Then*

$$\operatorname{trop}_{\mathbb{T}}\left(\overline{Y}\right) \cong \left(\overline{\mathbb{Q}}^{s_1+\cdots+s_r+m+n} \setminus \operatorname{trop}_{\widehat{\mathbb{T}}}(V(F))\right) / \operatorname{trop}_{\widehat{\mathbb{T}}}(Q).$$

This result provides a means for globally tropicalizing a toric variety. That is to say, we do not need to consider any separate pieces and then glue together, we may simply tropicalize a single toric variety and take an appropriate quotient. To obtain a universal tropicalization for spherical embeddings, it will be sufficient to describe how to recover $\operatorname{trop}_G\left(\overline{Y}\right)$ from $\operatorname{trop}_{\mathbb{T}}\left(\overline{Y}\right)$. When $\overline{Y}$ is replaced by Y, we described this in Theorem 4.2.2 using the piecewise projection map ψ:

$$\psi : \operatorname{trop}_{\mathbb{T}}(Z_0) \to \mathcal{N}_{\mathbb{Q}}$$
$$(a_{11}, \ldots, a_{1s_1}, a_{21}, \ldots, a_{rs_r}, b_1, \ldots, b_m) \mapsto \left(\min_{1\le j\le s_1} \{a_{1j}\}, \ldots, \min_{1\le j\le s_r} \{a_{rj}\}, b_1, \ldots, b_m\right).$$

We now define an extension $\overline{\psi} : \mathrm{trop}_{\mathbb{T}}(Z) \to \mathrm{trop}_G(X)$ of ψ that takes extended $\mathbb{T}$-invariant valuations to extended G-invariant valuations. Applying this map to the global toric tropicalization above will afford a global spherical tropicalization. Let $(\sigma, \mathcal{F})$ be a colored cone in the colored fan of the G/H-embedding X and let $\mathfrak{a} \in \mathfrak{A}(\mathcal{F})$. In the extended tropicalization of the spherical variety, tropicalizing the orbit corresponding to the colored cone $(\sigma, \mathcal{F})$ corresponds to adding in semigroup homomorphisms in $\mathrm{Hom}^{\mathcal{V}}\left(\sigma^\vee \cap \mathcal{M}, \overline{\mathbb{Q}}\right)$ and viewing them as limit points of $\mathrm{trop}_G(G/H) := \mathcal{V} \subseteq \mathrm{Hom}(\mathcal{M}, \mathbb{Q})$.

Before proceeding, we deal with a slight clash of notation that comes up here. Thought of as a valuation, $\mu \in \mathrm{Hom}\,(\sigma_{\mathfrak{a}}^\vee \cap \mathcal{M}, \overline{\mathbb{Q}})$ acts on a lattice of torus semi-invariant rational functions, where the group action is multiplicative. As an element of $\mathrm{Hom}\,(\sigma_{\mathfrak{a}}^\vee \cap \mathcal{M}, \overline{\mathbb{Q}})$, however, μ acts on the additive semigroup $\sigma_{\mathfrak{a}}^\vee \cap M$. This can be addressed by identifying an element $(a_{11}, \ldots, a_{rs_r}, b_1, \ldots, b_m) \in \sigma_{\mathfrak{a}}^\vee \cap M$ with the function $f_{11}^{a_{11}} \cdots f_{rs_r}^{a_{rs_r}} g_1^{b_1} \cdots g_m^{b_m}$. Similarly, $(a_1, \ldots, a_r, b_1, \ldots, b_m) \in \sigma^\vee \cap \mathcal{M}$ is identified with $f_1^{a_1} \cdots f_r^{s_r} g_1^{b_1} \cdots g_m^{b_m}$.

The extension $\overline{\psi}$ will take a valuation $\mu \in \mathrm{Hom}\,(\sigma_{\mathfrak{a}}^\vee \cap M, \overline{\mathbb{Q}})$ to $\mathrm{Hom}^{\mathcal{V}}\,(\sigma^\vee \cap \mathcal{M}, \overline{\mathbb{Q}})$. Suppose $\mu \in \mathrm{trop}_{\mathbb{T}}(O)$ where O corresponds to the cone $\sigma_{\mathfrak{a}}$. Further let $(\sigma, \mathcal{F})$ be the associated colored cone corresponding to a G-orbit $\mathcal{O}$. Define a set $\Omega \subset \mathbb{Z}^r$ as follows:

$$\Omega := \{\omega \in \mathbb{Z}^r : 1 \leq \omega(i) \leq s_i \text{ for all } 1 \leq i \leq r\}.$$

Then for any $\mu \in \mathrm{Hom}\left(\sigma_{\mathfrak{a}}^\vee \cap M, \overline{\mathbb{Q}}\right)$, we define $\overline{\psi}(\mu) \in \mathrm{Hom}^{\mathcal{V}}\left(\sigma^\vee \cap \mathcal{M}, \overline{\mathbb{Q}}\right)$ to be infinite on $(\sigma^\vee \setminus \sigma^\perp) \cap \mathcal{M}$ and to act on $f_1^{a_1} \cdots f_r^{a_r} g_1^{b_1} \cdots g_m^{b_m} \in \sigma^\perp \cap \mathcal{M}$ as follows:

$$\overline{\psi}(\mu)\left(f_1^{a_1} \cdots f_r^{a_r} g_1^{b_1} \cdots g_m^{b_m}\right) = \min\left\{\mu\left(f_{1\omega(1)}^{a_1} \cdots f_{r\omega(r)}^{a_r} g_1^{b_1} \cdots g_m^{b_m}\right) : \omega \in \Omega\right\}.$$

Remark 4.5.2. There is a tacit assumption here that we only consider $\omega \in \Omega$ such that

$$f_{1\omega(1)}^{a_1} \cdots f_{r\omega(r)}^{a_r} g_1^{b_1} \cdots g_m^{b_m} \in \sigma_{\mathfrak{a}}^{\vee} \cap M$$

to ensure that μ is well-defined. When $\mu = \nu_\gamma$ is induced by a $\mathbb{C}\{\{t\}\}$-point γ, then the minimum may be taken over the entirety of Ω. An explanation of this appears in the proof of Theorem 4.5.5.

Example 4.5.3. We will extend Example 4.2.3. Recall that in this setting $r = 1$, $f_{11} = y$, and $f_{12} = x$. Consider the embedding $\mathrm{Bl}_0(\mathbb{C}^2)$ of $G/H = \mathbb{C}^2 \setminus \{0\}$ given by a single non-colored ray in the direction of v_1, which we call σ. The associated toric variety Z is also $\mathrm{Bl}_0(\mathbb{C}^2)$, given by the fan shown in Figure 4.5.

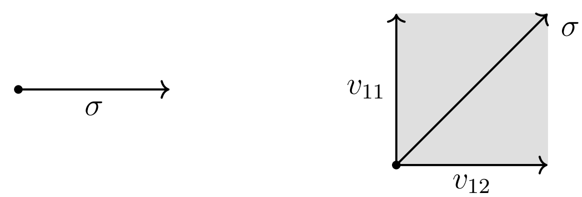


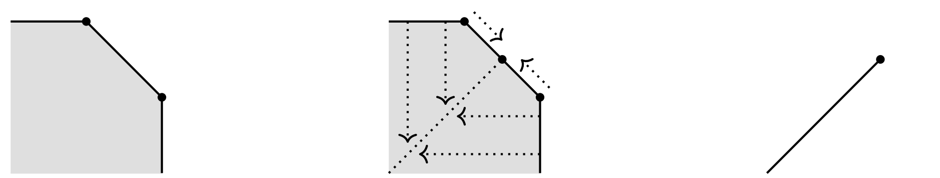
Figure 4.5: Fans for $\mathrm{Bl}_0(\mathbb{C}^2)$ as a spherical embedding (left) and a toric embedding (right)

In the vector space $N_{\mathbb{Q}}$ corresponding to the toric variety, σ is the ray in the fan of $\mathrm{Bl}_0(\mathbb{C}^2)$ corresponding to the exceptional divisor. There are three possibilities for $\mathfrak{a}$: $\mathfrak{a} = \emptyset$, $\{v_{11}\}$, or $\{v_{12}\}$. These respectively correspond to three cones $\sigma_{\mathfrak{a}}$: the diagonal ray σ, the two-dimensional cone spanned by σ and v_{11}, and the two-dimensional cone spanned by σ and v_{12}.

The spherical tropicalization of $\mathrm{Bl}_0(\mathbb{C}^2)$ is isomorphic to $\overline{\mathbb{Q}}$, where the tropicalization of the exceptional divisor is the point ∞. In other words, $\sigma^\perp \cap \mathcal{M}$ is the origin in

$\mathcal{N}$ and corresponds to the map $\mu \in \operatorname{Hom}\left(\sigma^\vee \cap \mathcal{M}, \overline{\mathbb{Q}}\right)$ sending every non-zero element of $\sigma^\vee \cap \mathcal{M}$ to ∞, i.e. the extended G-invariant map $\mathbb{C}[x, y] \to \overline{\mathbb{Q}}$ that is ∞ on all non-constant functions.

Therefore, if $\mu \in \sigma_{\mathfrak{a}}^\vee \cap M$ for some choice of $\mathfrak{a}$, then it must by definition of $\overline{\psi}$ be sent to ∞ in $\overline{\mathbb{Q}}$. This can be viewed as collapsing the diagonally-oriented one-dimensional vector space and the two zero-dimensional vector spaces in $\operatorname{trop}_{\mathbb{T}}(\operatorname{Bl}_0(\mathbb{C}^2))$ to a point. See Figure 4.6 for reference.



$\operatorname{trop}_{\mathbb{T}}(\operatorname{Bl}_0(\mathbb{C}^2))$ Action of $\overline{\psi}$ $\operatorname{trop}_{G}(\operatorname{Bl}_0(\mathbb{C}^2))$

Figure 4.6: $\overline{\psi}(\operatorname{trop}_{\mathbb{T}}(\operatorname{Bl}_0(\mathbb{C}^2))) = \operatorname{trop}_{G}(\operatorname{Bl}_0(\mathbb{C}^2))$

In the previous example we saw that $\overline{\psi}$ takes the extended toric tropicalization to the extended spherical tropicalization. Our goal now is to prove that this holds in general. To accomplish this, we need an additional theorem:

Theorem 4.5.4. [21, Theorem 6.2.18] *Let $Y \subseteq \mathbb{T}$, and let $\overline{Y}$ be the closure of Y in a toric variety Z. Then*

$$\operatorname{trop}_{\mathbb{T}}\left(\overline{Y}\right) = \overline{\operatorname{trop}_{\mathbb{T}}(Y)}.$$

Theorem 4.5.5. *If G/H has trivial divisor class group, $Y \subseteq G/H$ is a subvariety, and X is a G/H-embedding with the A_2-property, then*

$$\overline{\psi}\left(\operatorname{trop}_{\mathbb{T}}\left(\overline{Y}\right)\right) = \operatorname{trop}_{G}\left(\overline{Y}\right),$$

where the closure on the left is taken in Z and the closure on the right is taken in X.

Proof. Let $\gamma : \operatorname{Spec} \mathbb{C}\{\{t\}\} \to \overline{Y}$ be a $\mathbb{C}\{\{t\}\}$-point of $\overline{Y}$ and suppose $\widetilde{\nu}_\gamma$ and ν_γ are respectively the $\mathbb{T}$-invariant and G-invariant extended valuations induced by γ. Then γ lies in a G-orbit corresponding to a colored cone $(\sigma, \mathcal{F})$ and a $\mathbb{T}$-orbit corresponding to a cone $\sigma_{\mathfrak{a}}$. We will show that $\overline{\psi}\,(\widetilde{\nu}_\gamma) = \nu_\gamma$.

Let $f_1^{a_1} \cdots f_r^{a_r} g_1^{b_1} \cdots g_m^{b_m} \in \sigma^\perp \cap \mathcal{M}$. Then a sufficiently general element of G takes this B semi-invariant rational function to a function of the form

$$(c_{11} f_{11} + \cdots + c_{1s_1} f_{1s_1})^{a_1} \cdots (c_{r1} f_{r1} + \cdots + c_{rs_r} f_{rs_r})^{a_r} \cdot c g_1^{b_1} \cdots g_m^{b_m}$$

for $c, c_{ij} \in \mathbb{C}^*$. Generically, this is a non-zero rational function on the G-orbit corresponding to $(\sigma, \mathcal{F})$ that is defined at γ, so the valuation $\widetilde{\nu}_\gamma$ takes it to a rational number. By Theorem 4.5.4, $\widetilde{\nu}_\gamma$ lies in the closure of $\operatorname{trop}_{\mathbb{T}}(Y)$, so there exists a sequence $\{\nu_\ell\}_{\ell=1}^\infty$ of $\mathbb{T}$-invariant valuations associated to $\mathbb{C}\{\{t\}\}$-points γ_ℓ of Y such that $\lim_{\ell\to\infty} \nu_\ell = \widetilde{\nu}_\gamma$ in the topology on $\operatorname{trop}_{\mathbb{T}}(Z)$. Then we have the following, which verifies the claim:

$$\begin{aligned}
&\nu_\gamma \left(f_1^{a_1} \cdots f_r^{a_r} g_1^{b_1} \cdots g_m^{b_m} \right) \\
&= \widetilde{\nu}_\gamma \left((c_{11} f_{11} + \cdots + c_{1s_1} f_{1s_1})^{a_1} \cdots (c_{r1} f_{r1} + \cdots + c_{rs_r} f_{rs_r})^{a_r} \cdot c g_1^{b_1} \cdots g_m^{b_m} \right) \\
&= \lim_{\ell\to\infty} \nu_\ell \left((c_{11} f_{11} + \cdots + c_{1s_1} f_{1s_1})^{a_1} \cdots (c_{r1} f_{r1} + \cdots + c_{rs_r} f_{rs_r})^{a_r} \cdot c g_1^{b_1} \cdots g_m^{b_m} \right) \\
&= \lim_{\ell\to\infty} \sum_{i=1}^r a_i \min_j \{\nu_\ell(f_{ij})\} + \sum_{k=1}^m b_k \nu_\ell(g_k) \\
&= \lim_{\ell\to\infty} \min \left\{ \nu_\ell \left(f_{1\omega(1)}^{a_1} \cdots f_{r\omega(r)}^{a_r} g_1^{b_1} \cdots g_m^{b_m} \right) : \omega \in \Omega \right\} \\
&= \min \left\{ \widetilde{\nu}_\gamma \left(f_{1\omega(1)}^{a_1} \cdots f_{r\omega(r)}^{a_r} g_1^{b_1} \cdots g_m^{b_m} \right) : \omega \in \Omega \right\} = \overline{\psi}\,(\widetilde{\nu}_\gamma) \left(f_1^{a_1} \cdots f_r^{a_r} g_1^{b_1} \cdots g_m^{b_m} \right)
\end{aligned}$$

Pursuant to Remark 4.5.2, note that the minimum here is indexed over the entirety of the set Ω since every c_{ij} is nonzero. $\square$

Having established the result when the homogeneous space has trivial divisor class group, we turn to the case of non-trivial divisor class group. As before, we write $\boldsymbol{G}/\boldsymbol{H}$ to denote a homogeneous space with non-trivial divisor class group and write G/H for the homogeneous space with trivial divisor class group and dominant map $\boldsymbol{\pi} : G/H \to \boldsymbol{G}/\boldsymbol{H}$. If X is the G/H-embedding associated to a $\boldsymbol{G}/\boldsymbol{H}$-embedding $\boldsymbol{X}$, then the extended map $\boldsymbol{\pi} : X \to \boldsymbol{X}$ is equivariant with respect to the natural surjection $G \to \boldsymbol{G}$. We can therefore extend Theorem 4.2.5 to spherical embeddings of homogeneous spaces with non-trivial divisor class group; the proof proceeds almost identically.

Theorem 4.5.6. *Let $\boldsymbol{G}/\boldsymbol{H} \hookrightarrow \boldsymbol{X}$ be a spherical embedding with associated spherical embedding $G/H \hookrightarrow X$ where G/H has trivial divisor class group. If $\boldsymbol{Y} \subseteq \boldsymbol{G}/\boldsymbol{H}$ is a closed subvariety and $\boldsymbol{X}$ is a $\boldsymbol{G}/\boldsymbol{H}$-embedding, then*

$$\operatorname{trop}_{\boldsymbol{G}}\left(\overline{\boldsymbol{Y}}\right) = \left(\operatorname{trop}(\boldsymbol{\pi}) \circ \overline{\psi}\right)\left(\operatorname{trop}_{\mathbb{T}}\left(\boldsymbol{\pi}^{-1}\left(\overline{\boldsymbol{Y}}\right)\right)\right),$$

where the closures are taken in $\boldsymbol{X}$.

Proof. By Theorem 4.5.5, this is equivalent to showing

$$\operatorname{trop}_{\boldsymbol{G}}\left(\overline{\boldsymbol{Y}}\right) = \operatorname{trop}(\boldsymbol{\pi})\left(\operatorname{trop}_{G}\left(\boldsymbol{\pi}^{-1}\left(\overline{\boldsymbol{Y}}\right)\right)\right).$$

Because $\boldsymbol{\pi}$ is equivariant with respect to the surjective group homomorphism $G \to \boldsymbol{G}$, the statement follows from Proposition 2.5.2. □

Developing the interplay between spherical tropicalization and toric tropicalization has the potential to give insight into the structure of spherical tropicalizations by translating known results from the toric world. As an example, we can generalize Theorem 4.5.4 to spherical varieties:

Theorem 4.5.7. *If $Y \subseteq G/H$ is a closed subvariety and X is a G/H-embedding, then*

$$\operatorname{trop}_G\left(\overline{Y}\right) = \overline{\operatorname{trop}_G(Y)},$$

where the closure on the left is taken in X and the closure on the right is taken in $\operatorname{trop}_G(X)$.

Proof. We prove the theorem when X is a simple G/H-embedding because X always has the A_2-property in this setting. If X is not simple, we may break it up into simple G/H-embeddings, where the result holds, and then glue together along shared orbits. Note that this will work even if X does not have the A_2-property.

We suppose first that G/H has trivial divisor class group. Let $\nu_\gamma \in \operatorname{trop}_G\left(\overline{Y}\right) \setminus \operatorname{trop}_G(Y)$ be an extended G-invariant valuation corresponding to a $\mathbb{C}\{\{t\}\}$-point γ. Then γ induces a $\mathbb{T}$-invariant valuation $\widetilde{\nu}_\gamma$ in $\operatorname{trop}_{\mathbb{T}}\left(\overline{Y}\right)$. By Theorem 4.5.4, there exists a sequence of $\mathbb{C}\{\{t\}\}$-points $\gamma_\ell \in Y\left(\mathbb{C}\{\{t\}\}\right)$ with associated $\mathbb{T}$-invariant valuations $\widetilde{\nu}_\ell$ such that $\lim_{\ell\to\infty} \widetilde{\nu}_\ell = \widetilde{\nu}_\gamma$ in the topology on $\operatorname{trop}_{\mathbb{T}}(Z)$. We claim that $\lim_{\ell\to\infty} \overline{\psi}\left(\widetilde{\nu}_\ell\right) = \nu_\gamma$. Suppose γ lies in a G-orbit corresponding to a colored cone $(\sigma, \mathcal{F})$ and a $\mathbb{T}$-orbit corresponding to $\sigma_{\mathfrak{a}}$ and let $f_1^{a_1} \cdots f_r^{a_r} g_1^{b_1} \cdots g_m^{b_m} \in \sigma^\perp \cap \mathcal{M}$ be arbitrary. Then we have the following, recalling from Remark 4.5.2 that the minimums are indexed over all of Ω:

$$\begin{aligned}
\lim_{\ell\to\infty} \overline{\psi}(\nu_\ell)\left(f_1^{a_1} \cdots f_r^{a_r} g_1^{b_1} \cdots g_m^{b_m}\right) &= \lim_{\ell\to\infty} \min\left\{\widetilde{\nu}_\ell\left(f_{1\omega(1)}^{a_1} \cdots f_{r\omega(r)}^{a_r} g_1^{b_1} \cdots g_m^{b_m}\right) : \omega \in \Omega\right\} \\
&= \min\left\{\widetilde{\nu}_\gamma\left(f_{1\omega(1)}^{a_1} \cdots f_{r\omega(r)}^{a_r} g_1^{b_1} \cdots g_m^{b_m}\right) : \omega \in \Omega\right\} \\
&= \overline{\psi}\left(\widetilde{\nu}_\gamma\right)\left(f_1^{a_1} \cdots f_r^{a_r} g_1^{b_1} \cdots g_m^{b_m}\right) \\
&= \nu_\gamma\left(f_1^{a_1} \cdots f_r^{a_r} g_1^{b_1} \cdots g_m^{b_m}\right).
\end{aligned}$$

The last equality here follows from the proof of Theorem 4.5.5. It follows that ν_γ is the limit of the sequence $\left\{\overline{\psi}\left(\widetilde{\nu}_\ell\right)\right\}_{\ell=1}^{\infty}$. Hence, $\nu_\gamma \in \overline{\operatorname{trop}_G(Y)}$ and the first inclusion follows.

Conversely, suppose that $\mu \in \overline{\operatorname{trop}_G(Y)}$ and let $\{\gamma_\ell\}_{\ell=1}^{\infty}$ be a sequence of $\mathbb{C}\{\{t\}\}$-points of Y such that the G-invariant valuations $\mu_\ell := \nu_{\gamma_\ell}$ satisfy $\lim_{\ell\to\infty}\mu_\ell = \mu$ in the topology on $\operatorname{trop}_G(X)$. Each γ_ℓ induces a $\mathbb{T}$-invariant valuation $\widetilde{\mu}_\ell$. By possibly replacing $\{\mu_\ell\}_{\ell=1}^{\infty}$ with a subsequence, there is $\widetilde{\mu} \in \overline{\operatorname{trop}_{\mathbb{T}}(Y)}$ with $\widetilde{\mu} = \lim_{\ell\to\infty}\widetilde{\mu}_\ell$. By Theorem 4.5.4, $\widetilde{\mu}$ is induced by a $\mathbb{C}\{\{t\}\}$-point γ of $\overline{Y}$. We claim that $\mu = \overline{\psi}\left(\widetilde{\mu}\right) \in \operatorname{trop}_G\left(\overline{Y}\right)$. Let $f_1^{a_1}\cdots f_r^{a_r} g_1^{b_1}\cdots g_m^{b_m} \in \sigma^\perp \cap \mathcal{M}$ be arbitrary, where σ corresponds to the G-orbit in whose tropicalization μ lies. Then the equality

$$\mu\left(f_1^{a_1}\cdots f_r^{a_r} g_1^{b_1}\cdots g_m^{b_m}\right) = \overline{\psi}\left(\widetilde{\mu}\right)\left(f_1^{a_1}\cdots f_r^{a_r} g_1^{b_1}\cdots g_m^{b_m}\right)$$

follows from an argument similar to the previous inclusion. Hence, μ is induced by γ and so $\mu \in \operatorname{trop}_G\left(\overline{Y}\right)$.

Now suppose $\boldsymbol{G}/\boldsymbol{H}$ has nontrivial divisor class group and let G/H be the associated spherical homogeneous space with trivial divisor class group. Then we have the following string of inclusions and equalities:

$$\begin{aligned}
\overline{\operatorname{trop}_{\boldsymbol{G}}\left(\boldsymbol{Y}\right)} &= \overline{\operatorname{trop}(\boldsymbol{\pi})\left(\operatorname{trop}_G\left(\boldsymbol{\pi}^{-1}\left(\boldsymbol{Y}\right)\right)\right)} \\
&\supseteq \operatorname{trop}(\boldsymbol{\pi})\left(\overline{\operatorname{trop}_G\left(\boldsymbol{\pi}^{-1}\left(\boldsymbol{Y}\right)\right)}\right) \\
&= \operatorname{trop}(\boldsymbol{\pi})\left(\operatorname{trop}_G\left(\overline{\boldsymbol{\pi}^{-1}\left(\boldsymbol{Y}\right)}\right)\right) \\
&\subseteq \operatorname{trop}(\boldsymbol{\pi})\left(\operatorname{trop}_G\left(\boldsymbol{\pi}^{-1}\left(\overline{\boldsymbol{Y}}\right)\right)\right) \\
&= \operatorname{trop}_{\boldsymbol{G}}\left(\overline{\boldsymbol{Y}}\right)
\end{aligned}$$

From top to bottom, these containments and equalities respectively follow from Theorem 4.2.5, Proposition 2.5.1, the first half of this proof, the continuity of $\boldsymbol{\pi}$, and

Theorem 4.5.6. Completing the proof now comes down to showing that the two containments are equalities.

For the first containment, suppose that $\mu \in \overline{\operatorname{trop}(\boldsymbol{\pi})(W)}$, where we write $W := \operatorname{trop}_G\left(\boldsymbol{\pi}^{-1}\left(\boldsymbol{Y}\right)\right)$. Then there exists a sequence $\{\mu_\ell\}_{\ell=1}^{\infty}$ of valuations in $\operatorname{trop}(\boldsymbol{\pi})\left(W\right)$ such that $\lim_{\ell\to\infty}\mu_\ell = \mu$. For each ℓ, choose an element $\boldsymbol{\mu}_\ell \in \operatorname{trop}(\boldsymbol{\pi})^{-1}(\mu_\ell)$. The set $\{\boldsymbol{\mu}_\ell\}_{\ell=1}^{\infty}$ contains a convergent subsequence in W. This subsequence converges to an extended valuation $\widetilde{\mu} \in \overline{W}$, which necessarily maps to μ under $\operatorname{trop}(\boldsymbol{\pi})$. Thus, the first inclusion is equality.

Now we consider the second inclusion. We will show that $\operatorname{trop}_{\mathbb{T}}\left(\overline{\boldsymbol{\pi}^{-1}\left(\boldsymbol{Y}\right)}\right) \supseteq \operatorname{trop}_{\mathbb{T}}\left(\boldsymbol{\pi}^{-1}\left(\overline{\boldsymbol{Y}}\right)\right)$ since applying the map $\operatorname{trop}(\boldsymbol{\pi})\circ\overline{\psi}$ will deliver the needed inclusion by Theorem 4.5.6. Theorem 4.5.4 says that $\overline{\operatorname{trop}_{\mathbb{T}}\left(\boldsymbol{\pi}^{-1}\left(\boldsymbol{Y}\right)\right)} = \operatorname{trop}_{\mathbb{T}}\left(\overline{\boldsymbol{\pi}^{-1}\left(\boldsymbol{Y}\right)}\right)$, and the proof in [21] only uses the fact that $\overline{\boldsymbol{\pi}^{-1}\left(\boldsymbol{Y}\right)}$ is a closed set in Z whose intersection with $\mathbb{T}$ is the variety $\boldsymbol{\pi}^{-1}\left(\boldsymbol{Y}\right)\cap\mathbb{T}$. Because $\boldsymbol{\pi}^{-1}\left(\overline{\boldsymbol{Y}}\right)$ satisfies these properties, we can conclude that $\overline{\operatorname{trop}_{\mathbb{T}}\left(\boldsymbol{\pi}^{-1}\left(\boldsymbol{Y}\right)\right)} = \operatorname{trop}_{\mathbb{T}}\left(\boldsymbol{\pi}^{-1}\left(\overline{\boldsymbol{Y}}\right)\right)$ and hence the equality $\operatorname{trop}_{\mathbb{T}}\left(\overline{\boldsymbol{\pi}^{-1}\left(\boldsymbol{Y}\right)}\right) = \operatorname{trop}_{\mathbb{T}}\left(\boldsymbol{\pi}^{-1}\left(\overline{\boldsymbol{Y}}\right)\right)$. This completes the second inclusion and the proof. $\square$

Chapter 5: Toward A Structure Theorem

Beyond the fundamental theorem, the next significant result in classical tropical geometry is the structure theorem. This describes what types of objects we obtain upon tropicalizing a subvariety of an algebraic torus. In the toric setting, tropicalizations take the form of balanced polyhedral complexes. In the spherical case, they do not exhibit such consistent behavior, so it is unclear whether a similarly concise classification is possible. This section is devoted to the first partial steps in that direction, with some commentary on where the theory collapses when moving from the toric case to the spherical case.

This chapter is organized as follows. In §5.1, we describe the structure theorem in the toric case. In §5.2, we use the theory of Chapter 4 to obtain some partial results on the structure of tropical spherical varieties. Then §5.3 gives some background on the intersection theory of spherical varieties and how this theory informs the balancing condition when restricted to toric varieties. Finally, §5.4 proposes a means for spherical tropicalization when the ideals are defined over the field of Puiseux series rather than the base field and §5.5 has examples. We work throughout over $\mathbb{C}$ with torus $\mathbb{T} = (\mathbb{C}^*)^n$.

5.1 The Structure Theorem for Tropical Toric Varieties

Theorem 5.1.1 is the structure theorem for toric varieties. The remainder of this section defines and explains the terminology used in the theorem. All of this theory can be found in more detail in [21].

Theorem 5.1.1. [21, Theorem 3.3.5] *Let Y be an irreducible d-dimensional subvariety of $(\mathbb{C}\{\{t\}\}^*)^n$. Then $\text{trop}_{\mathbb{T}}(Y)$ is the support of a balanced weighted rational polyhedral complex pure of dimension d. Moreover, that polyhedral complex is connected through codimension one.*

A *rational polyhedron* P is a subset of $\mathbb{Q}^n$ of the form

$$P = \{\mathbf{x} \in \mathbb{Q}^n : A\mathbf{x} \leq \mathbf{b}\}$$

where A is a $d \times n$ matrix and $\mathbf{b}$ is a vector, both with entries in $\mathbb{Q}$. A *polyhedral complex* is a collection Σ of polyhedra such that $P, Q \in \Sigma$ implies $P \cap Q \subseteq P, Q$ is a face of each and $P \cap Q \in \Sigma$. Further, a polyhedral complex Σ is *pure of dimension d* if the maximal polyhedra in Σ all have dimension d.

We now turn to the balancing condition. Let Σ be a fan in $\mathbb{Q}^n$ that is pure of dimension d. For each cone $\sigma \in \Sigma$ of dimension d, fix a weight $w(\sigma)$. Let $\tau \in \Sigma$ be a cone of dimension $d-1$ and denote by L the subspace spanned by τ. Then $(\sigma + L)/L$ is a ray in $\mathbb{Q}^n/L$; let $\mathbf{v}_\sigma$ be the first lattice point on this ray. We say that Σ is *balanced at τ* if:

$$\sum_{\tau \preceq \sigma} w(\sigma)\mathbf{v}_\sigma = 0.$$

The fan Σ is called *balanced* if it is balanced at each $(d-1)$-dimensional cone $\tau \in \Sigma$.

If a polyhedral complex Σ is pure of dimension d, we can place weights on polyhedra of maximal dimension and define a notion of balancing by considering the local structure of the polyhedral complex, which we make precise below.

If σ is a polyhedron in Σ, then $\mathrm{star}_\Sigma(\sigma)$ is a fan with a cone $\overline{\tau}$ for each $\tau \in \Sigma$ containing σ as a face:

$$\overline{\tau} := \{\lambda(\mathbf{x} - \mathbf{y}) : \lambda \geq 0, \mathbf{x} \in \tau, \mathbf{y} \in \sigma\}.$$

If Σ is a polyhedral complex regular of dimension d, we say it is *balanced* if $\mathrm{star}_\Sigma(\sigma)$ is balanced for all $\sigma \in \Sigma$ of dimension $d-1$.

Finally, if Σ is a polyhedral complex that is pure of dimension d, then it is *connected through codimension one* if for any two d-dimensional cells $P, P' \in \Sigma$, there is a chain $P = P_1, P_2, \ldots, P_s = P'$ for which P_i and P_{i+1} share a common face of dimension $d-1$.

5.2 Partial Balancing Results

In this section we state some simple balancing results on spherical tropicalizations. Before proving these, we remark that in [17], the authors give a first basic structure result:

Proposition 5.2.1. [17, Corollary 4.19] *The tropicalization* $\mathrm{trop}_G(Y)$ *is the support of a fan with convex rational polyhedral cones.*

We will expand on this by giving a slightly finer description of what fans are possible under spherical tropicalization.

Our methodology will be to consider the tropicalization in a toric variety as we did in Chapter 4. We recall our notation from that chapter. We write N for the lattice of the torus $\mathbb{T}$ whose basis consists of the vectors v_{ij}'s and w_k's and $\mathcal{N}$ for the lattice of G/H whose basis consists of the vectors v_i's and w_k's. The tropicalization $\operatorname{trop}_{\mathbb{T}}(Z_0)$ is isomorphic to

$$\left(\overline{\mathbb{Q}}^{s_1} \setminus \{\infty\}\right) \times \cdots \times \left(\overline{\mathbb{Q}}^{s_r} \setminus \{\infty\}\right) \times \mathbb{Q}^m$$

and contains $N_{\mathbb{Q}}$ as a copy of $\mathbb{Q}^{s_1+\cdots+s_r+m}$. The map $\psi : \operatorname{trop}_{\mathbb{T}}(Z_0) \to \mathcal{N}_{\mathbb{Q}}$ is then defined by

$$(a_{11}, \ldots, a_{1s_1}, a_{21}, \ldots, a_{rs_r}, b_1, \ldots, b_m) \mapsto \left(\min_{1\leq j\leq s_1} \{a_{1j}\}, \ldots, \min_{1\leq j\leq s_r} \{a_{rj}\}, b_1, \ldots, b_m\right).$$

We now describe how polyhedral complexes behave under projections. Let $p : N \to \mathcal{N}$ be a homomorphism of lattices given by a matrix A; by an abuse of notation, we will use the same notation for the extension to $p : N_{\mathbb{Q}} \to \mathcal{N}_{\mathbb{Q}}$. Let Σ be a polyhedral complex in $N_{\mathbb{Q}}$. After refining Σ if necessary, we may assume that $p(\Sigma) := \{p(\sigma) : \sigma \in \Sigma\}$ forms a polyhedral complex in $\mathcal{N}_{\mathbb{Q}}$. We note here that a refinement of a pure balanced polyhedral complex in $\mathbb{Q}^n$ is also balanced ([21, Lemma 3.6.2]). The polyhedral complex $p(\Sigma)$ may not be pure, however, so a meaningful notion of balancing cannot be applied directly. Given Σ and p, denote by Σ^p the subset of $p(\Sigma)$ containing only cones of maximal dimension and their faces. Then Σ^p inherits a balancing condition from Σ:

Lemma 5.2.2. [21, Lemma 3.6.3] *If Σ is a balanced fan, then Σ^p is also balanced.*

Furthermore, the balancing is given by placing the following weight on a maximal cell $\sigma' \in \Sigma^p$:

$$w(\sigma') := \sum_{\sigma\in\Sigma,\ p(\sigma)=\sigma'} w(\sigma) \cdot \left[\mathcal{N}_{\sigma'} : p\left(N_{\sigma}\right)\right].$$

Here, $w(\sigma)$ for $\sigma \in \Sigma$ is a weight on σ in a balancing of Σ and $[\mathcal{N}_{\sigma'} : p\,(N_\sigma)]$ is the greatest common divisor of the maximal minors of the matrix AV, where the columns of V form a basis for N_σ, the sublattice spanned by σ.

We will concern ourselves with the following homomorphisms of lattices. Let $p_{\mathbb{T}} : N \to N$ and $p_G : \mathcal{N} \to \mathcal{N}$ be the coordinate projections to span $\{w_1, \ldots, w_m\}$ in their respective lattices. Let $p^i_{\mathbb{T}} : N \to N$ similarly be projection to span $\{v_{i1}, \ldots, v_{is_i}\}$ and let $p^i_G : \mathcal{N} \to \mathcal{N}$ be projection to span $\{v_i\}$.

Remark 5.2.3. We also note that for an arbitrary closed subvariety $Y \subseteq G/H$, $\psi\,(\mathrm{trop}_{\mathbb{T}}(Y)) = \psi\,(\mathrm{trop}_{\mathbb{T}}(Y) \cap N_{\mathbb{Q}})$. This is because points in $\mathrm{trop}_{\mathbb{T}}(Y) \setminus N_{\mathbb{Q}}$ are limits of points in $\mathrm{trop}_{\mathbb{T}}(Y) \cap N_{\mathbb{Q}}$. Thus, their images under ψ are limits of points in $\mathrm{trop}_G(Y)$ and hence appear in $\mathrm{trop}_G(Y)$ as it is closed in $\mathcal{N}_{\mathbb{Q}}$.

Then we have the following results:

Theorem 5.2.4. *If $Y \subseteq G/H$ is a subvariety of a spherical homogeneous space G/H with trivial divisor class group, then the projection of* $\mathrm{trop}_G(Y)$ *to* $\mathrm{span}(w_1, \ldots, w_m)$ *is the support of a balanced fan upon removal of maximal cones not of maximum dimension.*

Proof. Note that $\psi \circ p_{\mathbb{T}} = p_G \circ \psi : N_{\mathbb{Q}} \to \mathcal{N}_{\mathbb{Q}}$. Then using Theorem 4.2.2 and Remark 5.2.3, this gives:

$$\begin{aligned} p_G\,(\mathrm{trop}_G(Y)) &= (p_G \circ \psi)\,(\mathrm{trop}_{\mathbb{T}}(Y)) \\ &= (p_G \circ \psi)\,(\mathrm{trop}_{\mathbb{T}}(Y) \cap N_{\mathbb{Q}}) \\ &= (\psi \circ p_{\mathbb{T}})\,(\mathrm{trop}_{\mathbb{T}}(Y) \cap N_{\mathbb{Q}}) \\ &= p_{\mathbb{T}}\,(\mathrm{trop}_{\mathbb{T}}(Y) \cap N_{\mathbb{Q}})\,. \end{aligned}$$

The last equality follows because ψ is the identity when restricted to $\operatorname{span}(w_1, \ldots, w_m)$. By Lemma 5.2.2, $p_{\mathbb{T}}(\operatorname{trop}_{\mathbb{T}}(Y) \cap N_{\mathbb{Q}})$ is the support of a balanced fan upon removal of maximal cones not of maximum dimension. $\square$

Theorem 5.2.5. *Suppose G/H is a spherical homogeneous space with trivial divisor class group, $Y \subseteq G/H$, and $D_i \in \mathcal{D}$ is a color of G/H. Then the image of the projection of* $\operatorname{trop}_G(Y)$ *to the subspace spanned by v_i cannot be a single ray in the direction of v_i.*

Proof. Similarly to the proof of Theorem 5.2.4, we have that $\psi \circ p_{\mathbb{T}}^i = p_G^i \circ \psi : N_{\mathbb{Q}} \to \mathcal{N}_{\mathbb{Q}}$, which gives:

$$p_G^i(\operatorname{trop}_G(Y)) = (p_G^i \circ \psi)(\operatorname{trop}_{\mathbb{T}}(Y) \cap N_{\mathbb{Q}}) = (\psi \circ p_{\mathbb{T}}^i)(\operatorname{trop}_{\mathbb{T}}(Y) \cap N_{\mathbb{Q}}).$$

Thus, for $p_G^i(\operatorname{trop}_G(Y))$ to consist of a single ray in the direction of v_i, the image $p_{\mathbb{T}}^i(\operatorname{trop}_{\mathbb{T}}(Y) \cap N_{\mathbb{Q}})$ must be contained in the first quadrant of $\operatorname{span}(v_{i1}, \ldots, v_{is_i})$. That is, the coefficient on v_{ij} must be positive for all j. This follows because ψ acts on $\operatorname{span}(v_{i1}, \ldots, v_{is_i})$ by taking the minimum of these coefficients. By Lemma 5.2.2, $p_{\mathbb{T}}^i(\operatorname{trop}_{\mathbb{T}}(Y) \cap N_{\mathbb{Q}})$ is the support of a balanced fan upon removal of lower-dimensional cones. Therefore, if $p_{\mathbb{T}}^i(\operatorname{trop}_{\mathbb{T}}(Y) \cap N_{\mathbb{Q}})$ is contained in the first quadrant of $\operatorname{span}(v_{i1}, \ldots, v_{is_i})$, then to be balanced it must consist solely of the origin, which will not map to a ray. $\square$

It is worth noting that any other set of rays is possible after projection to $\operatorname{span}(v_i)$: there may be a ray in the direction of $-v_i$, there may be rays in both directions, and the result may just be the origin.

5.3 Intersection Theory

In the toric setting, the weights that balance a tropical variety can be recovered from intersection theory. In this section we discuss how this theory translates and breaks down when we attempt to extend from the toric case to the spherical case. Ultimately, the problem is the gap between the action of the group G and the action of a Borel subgroup B. Intersection theory on toric varieties is discussed in [12] and the theory is extended to spherical varieties in [10].

Theorem 5.3.1. [10, Theorem 3] *If a connected solvable linear algebraic group acts on a complete scheme X with only finitely many orbits, then the Kronecker duality map $A^k(X) \to \mathrm{Hom}(A_k X, \mathbb{Z})$ is an isomorphism.*

In a spherical variety X, the Borel subgroup satisfies all the hypotheses of this theorem. Denote by Σ the set of B-orbits and $\Sigma^{(k)}$ the set of k-dimensional orbits. Write V_σ for the orbit closure of $\sigma \in \Sigma$ and $\tau \prec \sigma$ whenever $V_\tau \subset V_\sigma$. Then Theorem 5.3.1 says a cohomology class in $A^k(X)$ can be identified with a map $w : \Sigma^{(k)} \to \mathbb{Z}$. The map w is balanced in the following sense. Given $\tau \in \Sigma^{(k+1)}$ and a B-eigenfunction f on V_τ,

$$\Sigma_{\sigma \prec \tau} \operatorname{ord}_\sigma(f) w(\sigma) = 0,$$

where $\operatorname{ord}_\sigma(f)$ denotes the order of vanishing of f along V_σ.

The balancing condition on the tropicalization of toric varieties reflects a deeper balancing condition on cohomology classes of a tropical compactification of the variety. Explicitly:

Theorem 5.3.2. [21, Theorem 6.7.7] *Let Y be a subvariety of the torus, and let $\overline{Y}$ be any tropical compactification in a toric variety X associated to a fan Σ. The* *balanced fan associated to $\overline{Y}$ after taking a completion of Σ, has support* $\mathrm{trop}(Y)$*, and its weights agree with the multiplicities on* $\mathrm{trop}(Y)$*. If X is smooth and the compactification $\overline{Y}$ is tropical, then the weight on $V(\sigma)$ in the balancing is given by $\overline{Y} \cdot V(\sigma)$ for any maximal cone $\sigma \in \Sigma$. Thus the tropicalization of Y determines the class $\left[\overline{Y}\right] \in A^*(X)$.*

The operative sentence of this theorem is the last one: the tropicalization models the cohomology class. In the toric case, Σ can be modeled by the fan of the toric variety and the intersection theory can be read of from the polyhedral geometry of the fan. In the spherical case, the colored fan does not necessarily exhibit this balancing behavior. There are essentially two reasons for this. First, the colored fan records data at the level of G-orbits and therefore does not always see the granularity of the B-orbits, which is where the balancing of cohomology classes occurs. Second, in the toric case the balancing can be modeled by the associated fan in part because the dimension of a cone is directly related to the dimension of the corresponding orbit. That is, if a toric variety has a d-dimensional torus orbit, this orbit corresponds to an $(m-d)$-dimensional cone, where m is the dimension of the vector space $\mathcal{N}_{\mathbb{Q}}$.

For toric varieties, m is both the dimension of $\mathcal{N}_{\mathbb{Q}}$ and the dimension of the associated toric variety. This is not the case in the spherical world, where we need two separate notions: the dimension of G/H and the dimension of $\mathcal{N}_{\mathbb{Q}}$. To differentiate the two, we call the latter the *rank* of G/H and denote it by $\mathrm{rk}(G/H)$. The dimension and rank are related by work of Brion [4], which Knop explains in [18].

For each $D \in \mathcal{D}$, write $P_D \leq G$ for the stabilizer of D. Note that P_D is a parabolic subgroup of G, as our notation suggests. If $\mathcal{F} \subseteq \mathcal{D}$ is a subset of the palette, then we write $P_{\mathcal{F}} := \bigcap_{D \in \mathcal{F}} P_D$.

Theorem 5.3.3. [18, Theorem 6.6] *Let X be a simple G/H-embedding defined by a colored cone $(\sigma, \mathcal{F})$ with unique closed orbit $\mathcal{O}$. Then:*

$$\dim \mathcal{O} = \operatorname{rk}(G/H) - \dim \sigma + \dim G/P_{\mathcal{D}\setminus\mathcal{F}},$$

where dim *indicates either the dimension as a variety or dimension as a polyhedral cone.*

As a result of this theorem, colored cones of the same dimension may correspond to orbits of differing dimensions. The combinatorial data of the colored fan alone is therefore not sufficient to give us information about the relationship of the dimension and rank. As such, it is not clear how to identify which colored cones correspond to orbits of a particular dimension.

The holdup here is the colors, specifically the term $\dim G/P_{\mathcal{D}\setminus\mathcal{F}}$ in Theorem 5.3.3. Proposition 5.3.4 illustrates a case where this term is equal for two separate G-orbits. Under the given hypotheses, the relationship between the dimensions of two G-orbits is modeled by the dimensions of the associated colored cones in the same way as it is in the toric case:

Proposition 5.3.4. [18, Lemma 6.4] *Let $(\sigma, \mathcal{F})$ be a colored cone and $(\sigma', \mathcal{F}')$ one of its colored faces with $\mathcal{F} = \mathcal{F}'$. Let $\mathcal{O}$ and $\mathcal{O}'$ respectively denote the unique closed G-orbit in the spherical embeddings corresponding to σ and σ'. Then:*

$$\dim \mathcal{O}' - \dim \mathcal{O} = \dim \sigma - \dim \sigma'.$$

Essentially, the colored cones and their faces can give the difference in dimension of the associated G-orbits, but only when every color in the colored cone is present in its face. In general, these dimensions will not be so well-behaved and so the colored fan does not obviously model the balancing of cohomology classes. In short, even if the issue of the gap between B-orbits and G-orbits can be overcome, the balancing of cohomology classes concerns the dimension of the orbits and any balancing condition on the colored fan must be done with regard to the dimension of their associated cones. Because these dimensions can disagree, the tropicalization seems inadequate to model the balancing induced by cohomology classes.

5.4 Non-Constant Coefficients

Write $\overline{K} := \mathbb{C}\{\{t\}\}$. Throughout this thesis, we have only concerned ourselves with tropicalizing varieties defined over $\mathbb{C}[G/H]$, but it is natural to consider ideals defined over $\overline{K}[G/H]$. We distinguish these situations by calling them respectively the constant coefficient and non-constant coefficient case. This convention is justified by the fact that $\nu : \overline{K} \to \mathbb{Q}$ is trivial when restricted to $\mathbb{C}$, so the coefficients add no extra information when tropicalizing. In both [33] and [17], the authors only consider the case of constant coefficients.

Despite this, Vogiannou's original definition for the constant coefficient case can just as easily be applied to the non-constant coefficient case. That is, if $I \in \overline{K}[G/H]$ is an ideal, then we can define the tropicalization of $Y := V(I)$ as before:

$$\operatorname{trop}_G(Y) := \left\{\nu_\gamma : \gamma \in Y\left(\overline{K}\right)\right\},$$

where $\nu_\gamma(f) := \nu(\gamma^*(gf))$ for sufficiently general $g \in G$. We note here that to properly extend this definition, we must work with the original group G with coefficients in $\mathbb{C}$ rather than K. For example, if $G = \mathrm{Gl}_2(\mathbb{C})$, we should use this group rather than $G = \mathrm{Gl}_2\left(\overline{K}\right)$ even in the non-constant coefficient case.

Vogiannou justifies his definition by showing that it has an appropriate theory with respect to compactifications. Explicitly, in Theorem 1.4.2 he shows that in a tropical compactification, the tropicalization equals the support of the colored fan of the G/H-embedding. The purpose of this section is to show that tropicalization with respect to non-constant coefficients also has nice behavior with respect to compactifications.

Further, we will show that spherical tropicalizations with non-constant coefficients are polyhedral complexes in $\mathcal{V}$. We first note that Theorem 4.2.2 extends to the non-constant coefficient definition; the proof goes through identically. We recall the statement here:

Theorem 4.2.2. If $Y \subseteq G/H$ is a closed subvariety and G/H has trivial divisor class group, then $\mathrm{trop}_G(Y) = \psi\left(\mathrm{trop}_{\mathbb{T}}(Y)\right)$.

Using this, we can then use the toric structure theorem to deduce the structure of the spherical tropicalization. Before proceeding to results, we note some properties of the map $\boldsymbol{\pi} : G/H \to \boldsymbol{G}/\boldsymbol{H}$ we discussed in Chapter 4. We will consider the induced map $\mathrm{trop}(\boldsymbol{\pi}) := \boldsymbol{\pi}_* : \mathcal{N}_{\mathbb{Q}} \to \boldsymbol{\mathcal{N}}_{\mathbb{Q}}$. Recall that $\mathcal{N}_T \subseteq \mathcal{N}$ is the sublattice $\mathrm{span}(w_1, \ldots, w_m)$ and

$$\boldsymbol{\pi}_*|_{(\mathcal{N}_T)_{\mathbb{Q}}} : (\mathcal{N}_T)_{\mathbb{Q}} \to \boldsymbol{\mathcal{N}}_{\mathbb{Q}}$$

is an isomorphism by [14, Proposition 3.4]. Further, the pullbacks of the colors in $\boldsymbol{\mathcal{D}}$ are precisely the colors in $\mathcal{D}$. Thus, the map $\boldsymbol{\pi}_*$ takes $v_i \in \mathcal{N}_{\mathbb{Q}}$ corresponding to $D_i \in \mathcal{D}$ to the vector in $\boldsymbol{\mathcal{N}}_{\mathbb{Q}}$ that measures vanishing along the color $\boldsymbol{D}_i \in \boldsymbol{\mathcal{D}}$. In particular, the map $\boldsymbol{\pi}_* : \mathcal{N}_{\mathbb{Q}} \to \boldsymbol{\mathcal{N}}_{\mathbb{Q}}$ is a linear transformation.

Theorem 5.4.1. *If $Y \subseteq G/H$ is a closed subvariety, then* $\mathrm{trop}_G(Y)$ *is the support of a polyhedral complex.*

Proof. Suppose first that G/H has trivial divisor class group. By Theorem 5.1.1, $\operatorname{trop}_{\mathbb{T}}\left(Y \cap \left(\overline{K}^{*}\right)^{n}\right)$ is the support of a polyhedral complex in $N_{\mathbb{Q}}$. The map ψ restricted to $N_{\mathbb{Q}}$ is a piecewise projection over finitely many regions $R_{\ell} \subseteq N_{\mathbb{Q}}$. Then the image of $\operatorname{trop}_{\mathbb{T}}(Y) \cap R_{\ell}$ for each ℓ is the projection of the support of a polyhedral complex and hence is the support of a polyhedral complex after potentially refining. The image of $\operatorname{trop}_{\mathbb{T}}\left(Y \cap \left(\overline{K}^{*}\right)^{n}\right)$ under ψ is thus a finite union of polyhedral complexes. Repeating this argument on $\operatorname{trop}_{\mathbb{T}}(Y \cap O)$ for each $\mathbb{T}$-orbit O, we conclude that $\psi\left(\operatorname{trop}_{\mathbb{T}}(Y)\right)$ is a finite union of polyhedral complexes. After possibly refining this finite union, we obtain the statement of the theorem.

Now we show that $\operatorname{trop}_{\boldsymbol{G}}(Y)$ is the support of a polyhedral complex when $Y \subseteq \boldsymbol{G}/\boldsymbol{H}$ and $\boldsymbol{G}/\boldsymbol{H}$ does not have trivial divisor class group. Let $\boldsymbol{\pi} : G/H \to \boldsymbol{G}/\boldsymbol{H}$ be the associated projection from a homogeneous space G/H with trivial divisor class group. By the first paragraph of the proof, $\operatorname{trop}_{G}\left(\boldsymbol{\pi}^{-1}(Y)\right)$ is the support of a polyhedral complex. By Theorem 4.2.5, $\operatorname{trop}_{\boldsymbol{G}}(Y) = \boldsymbol{\pi}_{*}\left(\operatorname{trop}_{G}\left(\boldsymbol{\pi}^{-1}(Y)\right)\right)$ and so because $\operatorname{trop}_{G}\left(\boldsymbol{\pi}^{-1}(Y)\right)$ is the support of a polyhedral complex and $\boldsymbol{\pi}_{*}$ is a linear transformation, $\operatorname{trop}_{\boldsymbol{G}}(Y)$ is also the support of a polyhedral complex. $\square$

We finally describe how the structure of this polyhedral complex is related to the fan associated to the constant coefficient case. Note that a polyhedron $P \subseteq \mathbb{Q}^{n}$ can be written as

$$P = \left\{\mathbf{x} \in \mathbb{Q}^{n} : A\mathbf{x} \leq \mathbf{b}\right\},$$

where $A \in \mathbb{Q}^{m \times n}$ is a matrix and $\mathbf{b} \in \mathbb{Q}^{n}$ is a vector. The $\leq$ symbol signifies that $(A\mathbf{x})_{i} \leq \mathbf{b}_{i}$ for each component i.

Definition 5.4.2. The *recession fan* of a polyhedron $P = \{\mathbf{x} \in \mathbb{Q}^n : A\mathbf{x} \leq \mathbf{b}\} \subseteq \mathbb{Q}^m$ is

$$\mathrm{rec}(P) := \{\mathbf{x} \in \mathbb{Q}^n : A\mathbf{x} \leq \mathbf{0}\}.$$

If Σ is a polyhedral complex in $\mathbb{Q}^n$, then its *recession fan* is

$$\mathrm{rec}(\Sigma) := \bigcup_{P \in \Sigma} \mathrm{rec}(P).$$

Despite its name, the recession fan does not have a canonical fan structure, but it is the support of a fan. As such, we understand $\mathrm{rec}(\Sigma)$ to signify the union of the $\mathrm{rec}(P)$ as a set, independent of fan structure.

An alternative way to define $\mathrm{rec}(P)$ is

$$\mathrm{rec}(P) := \{\mathbf{x} \in \mathbb{Q}^n : P + \mathbf{x} \subseteq P\}.$$

See [6] and [21] for further details on recession fans and polyhedral geometry.

Throughout the remainder, we let $Y \subseteq G/H$ be a closed subvariety whose defining equations have coefficients in $\overline{K}$. We denote by $\mathrm{trop}_G^{\mathrm{triv}}(Y)$ the tropicalization of Y where we pass to the field $\mathbb{K} := \overline{K}\{\{s\}\}$ of Puiseux series over $\overline{K}$ and use the valuation $\overline{\nu}$ that takes the smallest power of s appearing with nonzero coefficient. In other words, we take the trivial valuation on $\overline{K}$ and extend to the field of Puiseux series over it, just as we did when we passed from $\mathbb{C}$ to $\overline{K}$ in Chapter 4. Given a $\mathbb{K}$-point γ of Y, we similarly obtain a G-invariant valuation $\overline{\nu}_\gamma$ as follows:

$$\overline{\nu}_\gamma(f) := \overline{\nu}\left(\gamma^*\left(gf\right)\right),$$

where f is a rational function and g is sufficiently general for f. We define

$$\mathrm{trop}_G^{\mathrm{triv}}(Y) := \{\overline{\nu}_\gamma : \gamma \in Y\left(\mathbb{K}\right)\}.$$

Note that if Y is given by equations with coefficients in $\mathbb{C}$, then $\mathrm{trop}_G^{\mathrm{triv}}(Y) = \mathrm{trop}_G(Y)$. Note also that Theorems 4.2.2 and 4.2.5 also apply to tropicalization with respect to the trivial valuation. That is,

$$\psi\left(\mathrm{trop}_{\mathbb{T}}^{\mathrm{triv}}(Y)\right) = \mathrm{trop}_G^{\mathrm{triv}}(Y);$$
$$(\mathrm{trop}(\boldsymbol{\pi}) \circ \psi)\left(\mathrm{trop}_{\mathbb{T}}^{\mathrm{triv}}\left(\boldsymbol{\pi}^{-1}(Y)\right)\right) = \mathrm{trop}_{\boldsymbol{G}}^{\mathrm{triv}}(Y).$$

In the toric case, the polyhedral complex $\mathrm{trop}_G(Y)$ is related to $\mathrm{trop}_G^{\mathrm{triv}}(Y)$ in the following way:

Theorem 5.4.3. [21, Theorem 3.5.6] *Let $Y \subseteq \mathbb{T}^n$ be a closed subvariety. Then as subsets of $N_{\mathbb{Q}}$:*

$$\mathrm{trop}_{\mathbb{T}}^{\mathrm{triv}}(Y) = \mathrm{rec}\left(\mathrm{trop}_{\mathbb{T}}(Y)\right).$$

We can directly extend this theorem to the spherical case, which requires us to first prove a lemma.

Lemma 5.4.4. *If $P \subseteq \mathbb{Q}^n$ is a polyhedron and $\pi : \mathbb{Q}^n \to \mathbb{Q}^m$ is a linear transformation, then*

$$\mathrm{rec}(\pi(P)) = \pi(\mathrm{rec}(P)).$$

Proof. Because π is linear, if $P + \mathbf{x} \subseteq P$ for some $\mathbf{x} \in \mathbb{Q}^n$, then $\pi(P) + \pi(\mathbf{x}) \subseteq \pi(P)$. It follows that $\mathrm{rec}(\pi(P)) \supseteq \pi(\mathrm{rec}(P))$.

Conversely, suppose $\mathbf{y} \in \mathrm{rec}(\pi(P))$. Note that $\pi(\mathbb{Q}^n)$ is a subspace of $\mathbb{Q}^m$, so because $\pi(P) + \mathbf{y} \subseteq \pi(P)$, $\mathbf{y}$ must lie in this subspace. Thus, $\mathbf{y} \in \pi(\mathbb{Q}^n)$ and so $\pi^{-1}(\mathbf{y})$ is an affine linear subspace of $\mathbb{Q}^n$. We may write $\pi^{-1}(\mathbf{y}) = \mathbf{x} + \ker(\pi)$ for some $\mathbf{x} \in \mathbb{Q}^n$. Then we have $\pi(P+\mathbf{x}) = \pi(P)+\mathbf{y} \subseteq \pi(P)$. Because $\pi^{-1}(\pi(P)) = P+\ker(\pi)$, this implies that $P + \mathbf{x} \subseteq P + \ker(\pi)$. Thus, there exists some $\mathbf{z} \in \ker(\pi)$ such that $P + \mathbf{x} - \mathbf{z} \subseteq P$. Because $\pi(\mathbf{x} - \mathbf{z}) = \mathbf{y}$, it follows that $\mathrm{rec}(\pi(P)) \subseteq \pi(\mathrm{rec}(P))$. □

Theorem 5.4.5. *Let $Y \subseteq G/H$ be a closed subvariety. Then:*

$$\operatorname{trop}_G^{\mathrm{triv}}(Y) = \operatorname{rec}(\operatorname{trop}_G(Y)).$$

Proof. Suppose first that G/H has trivial divisor class group. Then after taking a refinement of $\operatorname{trop}_{\mathbb{T}}(Y)$, we may assume that ψ acts as a projection $\psi|_{P_\ell} : P_\ell \to \mathcal{N}_{\mathbb{Q}}$ when restricted to each polyhedron $P_\ell \in \operatorname{trop}_{\mathbb{T}}(Y)$. Chaining the equalities from Lemma 5.4.4, Theorem 5.4.3, and Theorem 4.2.2, we obtain:

$$\operatorname{rec}(\operatorname{trop}_G(Y)) = \bigcup_\ell \operatorname{rec}\left(\psi|_{P_\ell}\left(P_\ell\right)\right) = \bigcup_\ell \psi|_{P_\ell}\left(\operatorname{rec}\left(P_\ell\right)\right) = \psi\left(\operatorname{trop}_{\mathbb{T}}^{\mathrm{triv}}(Y)\right) = \operatorname{trop}_G^{\mathrm{triv}}(Y).$$

Now suppose that $\boldsymbol{G}/\boldsymbol{H}$ has nontrivial divisor class group. Then if G/H is the associated homogeneous space with trivial divisor class group, the first paragraph of this proof applies to $\operatorname{trop}_G\left(\boldsymbol{\pi}^{-1}(Y)\right)$. Hence, we have the following:

$$\begin{aligned}
\operatorname{rec}\left(\operatorname{trop}_{\boldsymbol{G}}(Y)\right) &= \operatorname{rec}\left(\operatorname{trop}(\boldsymbol{\pi})\left(\operatorname{trop}_G\left(\boldsymbol{\pi}^{-1}\left(Y\right)\right)\right)\right) && \text{Theorem 4.2.5}\\
&= \operatorname{trop}(\boldsymbol{\pi})\left(\operatorname{rec}\left(\operatorname{trop}_G\left(\boldsymbol{\pi}^{-1}\left(Y\right)\right)\right)\right) && \text{Lemma 5.4.4}\\
&= \operatorname{trop}(\boldsymbol{\pi})\left(\operatorname{trop}_G^{\mathrm{triv}}\left(\boldsymbol{\pi}^{-1}\left(Y\right)\right)\right) && \\
&= \operatorname{trop}_{\boldsymbol{G}}^{\mathrm{triv}}\left(Y\right) && \text{Theorem 4.2.5.}
\end{aligned}$$

□

Finally, we can generalize part (*ii*) of Theorem 1.4.2, which says that if a closed subvariety $Y \subseteq G/H$ has a tropical compactification in a G/H-embedding X, then the support of the colored fan of X equals $\operatorname{trop}_G^{\mathrm{triv}}(Y)$. Recall the definition of support from §2.1 (Definition 2.1.4).

Corollary 5.4.6. *Let $Y \subseteq G/H$ be a closed subvariety. If $\overline{Y} \subseteq X$ is a tropical compactification in a G/H-embedding X with colored fan Σ_X, then:*

$$\operatorname{supp}(\Sigma_X) = \operatorname{rec}(\operatorname{trop}_G(Y)).$$

Proof. By part (ii) of Theorem 1.4.2, $\mathrm{supp}(\Sigma_X) = \mathrm{trop}_G^{\mathrm{triv}}(Y)$ under the hypotheses given. Theorem 5.4.5 then tells us that $\mathrm{supp}(\Sigma_X) = \mathrm{rec}(\mathrm{trop}_G(Y))$, as claimed. □

5.5 Examples

Example 5.5.1. We first consider $G/H = \mathbb{A}^2 \setminus \{0\}$ as in Example 2.1.6. Let $f \in \overline{K}[G/H]$ be an arbitrary polynomial. Then we claim $\mathrm{trop}_G(V(f))$ is either the entirety of $\mathcal{N}_{\mathbb{Q}}$ or is a ray oriented to the left with vertex at some point $a \in \mathbb{Q}$.

Suppose first that f has a non-zero constant term c with $\nu(c) = a \in \mathbb{Q}$ and write $f_0 := f - c$. We assume $f_0 \neq 0$ since the variety $V(f)$ is trivial otherwise. Then for any $\overline{K}$-point γ and sufficiently general $g \in G$,

$$\nu(\gamma^*(gf)) = \nu(\gamma^*(gf_0) + c) = \min\{\nu(\gamma^*(gf_0)), a\} \leq a.$$

Thus, $\mathrm{trop}_G(V(f))$ is a ray to the left with vertex at a.

If $f \in \overline{K}[G/H]$ has constant term zero, then there is no such restriction on $\nu(\gamma^*(gf))$ and hence the tropicalization is all of $\mathcal{N}_{\mathbb{Q}}$.

For more interesting higher-dimensional examples, we consider $G = \mathrm{Gl}_2 \times \mathrm{Gl}_2$ and $G/H \cong \mathrm{Gl}_2$ as in Example 2.4.4. In his thesis, Vogiannou describes a way to compute the tropicalization of a subvariety of G/H in this case:

Theorem 5.5.2. [33, Theorem 1.3] *Let Y be a closed subvariety of Gl_n, defined by some ideal $I \subseteq \mathbb{C}[\mathrm{Gl}_n]$. Then $\mathrm{trop}_G(Y)$ consists of the n-tuples $(\alpha_1, \ldots, \alpha_n)$ of invariant factors (in decreasing order) of invertible matrices with entries in $\overline{K}$, that satisfy the equations of I.*

Vogiannou proves this for the constant coefficient case, but his proof also applies to the non-constant coefficient case as we have described it since the definitions are

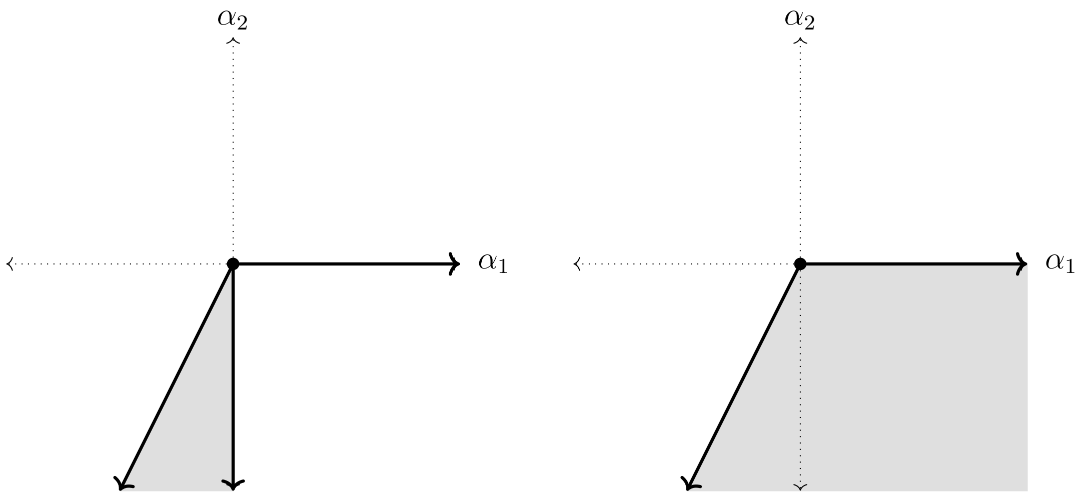


Figure 5.1: The subset of $\mathrm{trop}_G(V(x_{11}-1, x_{21}-x_{12}^2))$ with $\nu(z(t)) \neq \nu(y(t)^3)$ (left) and $\mathrm{trop}_G(V(x_{11}-1, x_{21}-x_{12}^2))$ (right)

the same. We will consider the particular case of Gl_2. Suppose we have a matrix $(x_{ij}) \in \mathrm{Gl}_2$ with coefficients in K that satisfies the equations of some ideal I. Then the invariant factors of (x_{ij}) are $\alpha_2 = \min_{i,j}\{\nu(x_{ij})\}$ and $\alpha_1 = \nu(x_{11}x_{22}-x_{12}x_{21})-\alpha_2$. We illustrate this by building on [33, Example 5.4], which has an oversight we explain below.

Example 5.5.3. Consider the ideal $I := (x_{11}-1, x_{21}-x_{12}^2)$. This ideal can be parametrized as follows:

$$\begin{pmatrix} 1 & y(t) \\ y(t)^2 & z(t) \end{pmatrix}, \quad y(t), z(t) \in \overline{K}.$$

If one assumes that $\nu(z(t)) \neq \nu(y(t)^3)$, then the possible ordered pairs (α_1, α_2) of invariant factors are shown in Figure 5.1 on the left; the ray pointing down and to the left has primitive vector $(-1,-2)$. Vogiannou claims this is $\mathrm{trop}_G\left(V\left(x_{11}-1, x_{21}-x_{12}^2\right)\right)$ ([33, Example 5.4]); we show that this is incorrect.

We claim that $\mathrm{trop}_G\left(V\left(x_{11}-1, x_{21}-x_{12}^2\right)\right)$ also contains the entirety of the fourth quadrant. That is, any point $(\alpha_1, \alpha_2) = (a, b)$ with $b \leq 0 \leq a$ lies in the tropicalization as well. These points occur when $\nu(z(t)) = \nu(y(t)^3)$, i.e. when the valuation of the determinant may be arbitrarily large. Consider the following $\overline{K}$-point that lies in $V(I)$:

$$\begin{pmatrix} 1 & t^{b/3} \\ t^{2b/3} & t^b + t^{a+b} \end{pmatrix}, \quad b \leq 0 \leq a.$$

Note that the determinant of this matrix is t^{a+b}. Then $\alpha_2 = \min\{0, b/3, 2b/3, b\} = b$ and $\alpha_1 = \nu(t^{a+b}) - \alpha_2 = (a+b) - b = a$. Thus, $(\alpha_1, \alpha_2) = (a, b)$ lies in the tropicalization and so we get the entirety of the fourth quadrant as well. This is shown on the right in Figure 5.1.

Example 5.5.4. In this example, we give a twist on Example 5.5.3 by including nonconstant coefficients. Consider the ideal $I := (x_{11}-1, t^2x_{21}-x_{12}^2)$. This ideal can be parametrized as follows:

$$\begin{pmatrix} 1 & ty(t) \\ y(t)^2 & z(t) \end{pmatrix}, \quad y(t), z(t) \in \overline{K}.$$

For simplicity of notation, we write $y_\nu := \nu(y(t))$ and $z_\nu := \nu(z(t))$. Then Theorem 5.5.2 tells us that $\mathrm{trop}_G(V(I))$ consists of tuples (α_1, α_2) such that $\alpha_2 = \min\{0, 1+y_\nu, 2y_\nu, z_\nu\}$ and $\alpha_1 = \nu(z(t) - ty(t)^3) - \alpha_2$. Let us assume first that $z_\nu \neq 3y_\nu + 1$ so that $\alpha_1 = \min\{z_\nu, 3y_\nu + 1\} - \alpha_2$. Then there are eight cases to consider because there are four possibilities for $\min\{0, 1+y_\nu, 2y_\nu, z_\nu\}$ and two possibilities for $\min\{z_\nu, 3y_\nu+1\}$. The cases where the first minimum is met at $y_\nu + 1$ cannot occur because $y_\nu + 1 \leq 2y_\nu$ only when $y_\nu \geq 1$, so 0 would be the minimum in this case. The six other cases can occur and we consider them in turn below; the resulting polyhedral object is shown in Figure 5.2.

1. Minimums met at $0, z_\nu$.

In this case, $\alpha_2 = 0$ and $\alpha_1 = z_\nu - 0$ for $z_\nu \geq 0$, so we obtain a ray on the α_1-axis starting at the origin oriented in the positive direction.

2. Minimums met at $0, 3y_\nu + 1$.

Now $\alpha_2 = 0$ and $\alpha_1 = 3y_\nu + 1$, the only restriction being that $y_\nu \geq 0$. Thus, we obtain the ray starting at the point $(\alpha_1, \alpha_2) = (1, 0)$ oriented in the direction $(1, 0)$.

3. Minimums met at z_ν, z_ν.

Here $\alpha_2 = z_\nu$ and $\alpha_1 = 0$ for $z_\nu \leq 0$, so we have the ray starting at the origin in the direction $(0, -1)$.

4. Minimums met at $z_\nu, 3y_\nu + 1$.

Again $\alpha_2 = z_\nu$ but now $\alpha_1 = 3y_\nu + 1 - z_\nu$. The set where the given minimums are met is cut out by the inequalities $3y_\nu + 1 \leq z_\nu \leq 0, y_\nu + 1, 2y_\nu$, which gives a two-dimensional cone in the $y_\nu z_\nu$-plane with vertex $(-1, -2)$ spanned by rays in the directions $(-1, -2)$ and $(-1, -3)$. The image of this cone under the linear transformation from the $y_\nu z_\nu$-plane to the $\alpha_1\alpha_2$-plane defined by $(y_\nu, z_\nu) \mapsto (3y_\nu + 1 - z_\nu, z_\nu)$ is the cone with vertex $(0, -2)$ spanned by rays in the directions $(0, -1)$ and $(-1, -2)$.

5. Minimums met at $2y_\nu, z_\nu$.

In this case, $\alpha_2 = 2y_\nu$ and $\alpha_1 = z - 2y_\nu$. The minimums are met when the inequalities $2y_\nu \leq 0, z_\nu$ and $z_\nu \leq 3y_\nu + 1$ are met. These cut out the triangle in the $y_\nu z_\nu$-plane with vertices $(0, 0)$, $(-1, -2)$, and $(0, 1)$. Under $(y_\nu, z_\nu) \mapsto$

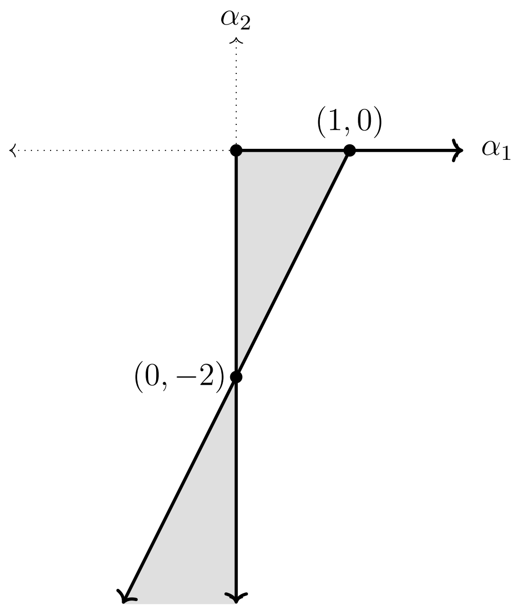


Figure 5.2: Valuations in $\mathrm{trop}_G(V(x_{11}-1, t^2x_{21}-x_{12}^2))$ with $z_\nu \neq 3y_\nu + 1$

$(z_\nu - 2y_\nu, 2y_\nu)$, this set is mapped to the triangle in the $\alpha_1\alpha_2$-plane with vertices $(0,0)$, $(0,-2)$, and $(1,0)$.

6. Minimums met at $2y_\nu, 3y_\nu + 1$.

Now $\alpha_2 = 2y_\nu$ and $\alpha_1 = y_\nu + 1$. Thus, we obtain points in the $\alpha_1\alpha_2$-plane of the form $(y_\nu + 1, 2y_\nu)$ where $y_\nu \leq 0$. This is a ray with vertex $(1,0)$ in the direction of $(-1,-2)$.

Now we consider when $z_\nu = 3y_\nu + 1$. In this setting, $\alpha_1 \geq z_\nu - \alpha_2 = 3y_\nu + 1 - \alpha_2$. There are three subcases: when $\alpha_2 = \min\{0, 1+y_\nu, 2y_\nu, z_\nu\}$ is 0, $2y_\nu$, or z_ν. The resulting tropicalization is shown in Figure 5.3. The added regions not present in Figure 5.2 are labeled by where the minimum $\min\{0, y_\nu + 1, 2y_\nu, z_\nu\}$ is met.

(i) If $\alpha_2 = 0$, then $\alpha_1 \geq 3y_\nu + 1 \geq 1$ and we get the ray starting at $(1,0)$ in the direction $(1,0)$.

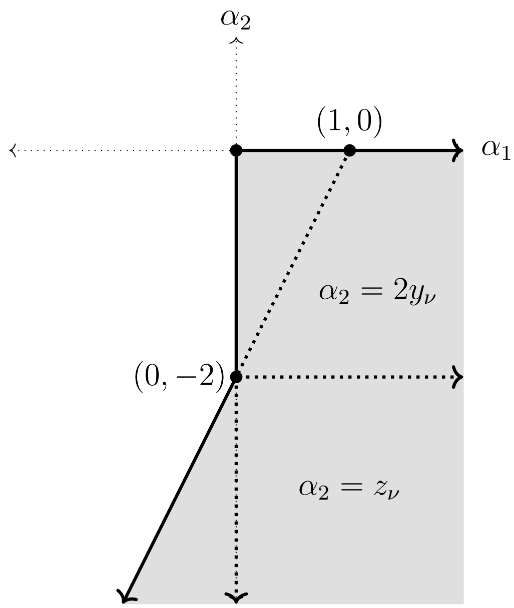


Figure 5.3: $\mathrm{trop}_G(V(x_{11}-1, t^2x_{21}-x_{12}^2))$

(ii) If $\alpha_2 = 2y_\nu$, then $2y_\nu \leq 0$ and $2y_\nu \leq z_\nu = 3y_\nu + 1$. These inequalities force $-1 \leq y_\nu \leq 0$. We also have that $\alpha_1 \geq (3y_\nu + 1) - 2y_\nu = y_\nu + 1$. The ordered pairs (α_1, α_2) that satisfy these conditions are those such that $-2 \leq \alpha_2 \leq 0$ and $\alpha_1 \geq \frac{1}{2}\alpha_2 + 1$.

(iii) If $\alpha_2 = z_\nu$, then $z_\nu \leq 2y_\nu$ and so $3y_\nu + 1 \leq 2y_\nu$, which implies $y_\nu \leq -1$ and hence $\alpha_2 \leq -2$. Further, $\alpha_1 \geq z_\nu - z_\nu = 0$. This gives the points in the fourth quadrant such that $\alpha_2 \leq -2$.

Example 5.5.5. Consider the ideal $I := (x_{22} - t)$ in Gl_2. We will recover $\mathrm{trop}_G(V(I))$ using the three methods discussed in this thesis. This tropicalization appears in Figure 5.4.

1. Vogiannou tropicalization.

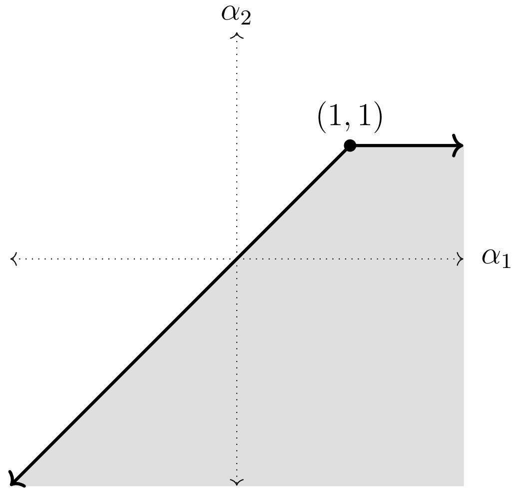


Figure 5.4: $\mathrm{trop}_G(V(x_{22}-t))$

Using Theorem 5.5.2, $\mathrm{trop}_G(V(I))$ consists of the ordered pairs (α_1, α_2) such that $\alpha_2 = \min\{1, \nu(x_{11}), \nu(x_{12}), \nu(x_{21})\}$ and $\alpha_1 = \nu(tx_{11} - x_{12}x_{21}) - \alpha_2$ for all $x_{11}, x_{12}, x_{21} \in K$ such that $tx_{11} \neq x_{12}x_{21}$. Thus, $\alpha_2 \leq 1$ and α_1 can vary arbitrarily as long as (α_1, α_2) remains in the valuation cone. This recovers the tropicalization in Figure 5.4.

2. Gröbner tropicalization.

Let v be a G-invariant valuation. There are three cases to consider: $v(x_{22}) > 1$, $v(x_{22}) = 1$, and $v(x_{22}) < 1$. If $v(x_{22}) > 1$, then $\mathrm{in}_v(x_{22}-t) = t$, which is a unit in $\mathrm{gr}_v\left(K\left[(G/H)_B\right]\right)$, and so v doesn't lie in the tropicalization.

If $v(x_{22}) = 1$, then $\mathrm{in}_v(x_{22}-t) = x_{22}-t$, which is not a unit, so these valuations lie in the tropicalization.

Finally, suppose $v(x_{22}) < 1$ so that $\mathrm{in}_v(x_{22}-t) = x_{22}$. The Borel subgroup B we've been working with is the group of ordered pairs (U, L) where U is upper triangular and L is lower triangular. In this setting, the divisor $V(x_{22})$ is B-stable and so x_{22} is a unit in $\mathrm{gr}_v(K[(G/H)_B])$. If we instead work with the group B' of ordered pairs (L, U), then the only B'-stable divisor is $V(x_{11})$. Thus, $\mathrm{in}_v(x_{22} - t) = x_{22}$ is not a unit in $\mathrm{gr}_v(K[(G/H)_{B'}])$, and so these valuations are included in the tropicalization.

There is no restriction on the valuation of the determinant, so the tropicalization must consist of every element in the valuation cone such that $\alpha_2 \leq 1$, which matches Figure 5.4.

3. Toric Embedding Tropicalization.

   This is similar to Example 4.2.4. We have that $V(x_{22})$ is the only B-stable prime divisor and the G-module spanned by $G \cdot x_{22}$ has as a basis x_{22}, x_{21}, x_{12}, and x_{11}. In keeping with the notation of Chapter 4.5, we write $f_{11} := x_{22}$, $f_{12} := x_{21}$, $f_{13} := x_{12}$, and $f_{14} := x_{11}$. The only rational functions on Gl_2 are constant multiples of powers of the determinant $g_1 := x_{11}x_{22} - x_{12}x_{21}$.

   Then the toric variety Z_0 associated to Gl_2 is the subvariety of $\left(\overline{K}^4 \setminus \{0\}\right) \times \overline{K}^*$ cut out by the equations $f_{11} - t$ and $f_{11}f_{14} - f_{12}f_{13} - g_1$.

   The image of $\mathrm{trop}_{\mathbb{T}}(V(f_{11} - t))$ under the map $\psi : \mathrm{trop}_{\mathbb{T}}(Z_0) \to \mathcal{N}_{\mathbb{Q}}$ consists of those valuations that are less than or equal to 1 on x_{22}. This is because for $v \in \mathrm{trop}_{\mathbb{T}}(V(f_{11} - t))$, we have:

$$\begin{aligned}\psi(v)(x_{22}) &= \min\{v(f_{11}), v(f_{12}), v(f_{13}), v(f_{14})\} = \min\{v(t), v(f_{12}), v(f_{13}), v(f_{14})\} \\ &= \min\{1, v(f_{12}), v(f_{13}), v(f_{14})\} \leq 1.\end{aligned}$$

Under ψ, $\mathrm{trop}_{\mathbb{T}}(V(f_{11}f_{14} - f_{12}f_{13} - g_1))$ is simply the valuation cone, so we again conclude that $\mathrm{trop}_G(V(I))$ is as shown in Figure 5.4.

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