Moduli spaces of Bridgeland semistable complexes

Dissertation

Presented in Partial Fulfillment of the Requirements for the Degree Doctor of Philosophy in the Graduate School of The Ohio State University

By

Bingyu Xia, B.S.

Graduate Program in Mathematics

The Ohio State University

2017

Dissertation Committee:

Emanuele Macrì, Advisor

David Anderson

Herbert Clemens

Hsian-Hua Tseng

Abstract

This thesis studies moduli spaces of semistable complexes in two aspects: the first one is an interesting example of a moduli space in higher dimension, namely the Hilbert scheme of twisted cubics in the three-dimensional projective space; the second one considers a general conjecture by Bridgeland on the existence of a coarse moduli space for the moduli problem of semistable complexes.

Acknowledgments

I would like to thank my advisor Emanuele Macrì, who introduced me to this topic and gave me advice with great patience. I would also like to thank my co-advisor David Anderson for giving me many useful advice. I am grateful to Yinbang Lin, Giulia Saccà, Benjamin Schmidt, Jun Wang, Yi Xie, Dingxin Zhang and Xiaolei Zhao for helpful conversations.

I express my gratitude to the mathematics department of Ohio State University where most of the work is done. I would also like to thank Northeastern University at which many details were worked out for their hospitality. The research is partially supported by NSF grants DMS-1302730 and DMS-1523496 (PI Emanuele Macrì).

Vita

2011-2017 .... Graduate Teaching Assistant,
Ohio State University

2011 .... B.S. in Mathematics,
Zhejiang University

Publications

Research Publications

Hilbert scheme of twisted cubics as simple wall-crossing. to appear in *Trans. Amer. Soc. Math.*

Fields of Study

Major Field: Mathematics

Specialization: Algebraic Geometry

Table of Contents

Chapter 1: Introduction

In algebraic geometry, we frequently consider a family of objects parametrized by some base variety. Actually, one of the most important success of modern algebraic geometry is the study of families of objects. Among all the families, we always hope there is a universal one that determines all the other families via the pullback of a morphism between the bases. This univeral family, together with its base, is the starting point of the theory of moduli space. In [Bri07], Bridgeland was inspired by the work of Douglas in physics. He introduced the notion of stability condition on a triangulated category. By taking the triangulated category to be the bounded derived category of coherent sheaves on a smooth projective variety, we are led to the study of moduli space of semistable complexes, which is similar to the classical moduli space of Gieseker semistable sheaves. It has many applications in both mathematics and physics. In this thesis, we will first study an interesting example of moduli space of Bridgeland semistable complexes, namely, twisted cubics in three-dimensional projective space. Then we go to the general situation and develop some partial results and conjectures on the existence of a good moduli space of Bridgeland semistable complexes.

1.1 Hilbert Scheme of Twisted Cubics

There have been many successful works studying the moduli space of Bridgeland semistable complexes on surfaces. For example, [ABCH13, CHW14, LZ16] gave complete results on the projective plane $\mathbb{P}^2$; [BM14b] gave a partial result on $K3$ surfaces. It is natural to ask if we can do similar things on threefolds. The first problem here is to construct a Bridgeland stability on threefolds. In the case of the projective space $\mathbb{P}^3$, this problem is solved in [Mac14] by proving a Bogomolov-Gieseker type inequality for an auxiliary tilt stability. Once the existence of a Bridgeland stability is established, we can study the moduli spaces of semistable complexes and see what geometric information we can get by varying stability. This is the motivation of the paper [Xia16]. Its main result is the following:

Theorem 1.1.1. *There is a path γ in the space of stability conditions on $\mathbb{P}^3$ that crosses three walls and four chambers for a fixed Chern character $v = \mathrm{ch}(\mathcal{I}_C)$, where $\mathcal{I}_C$ is the ideal sheaf of a twisted cubic C. If we list the moduli space of semistable objects in each chamber with respect to the path γ, we have:*

(1) *The empty space $\emptyset$;*

(2) *A smooth projective integral variety $\mathbf{M}_1$ of dimension 12 containing the ideal sheaves of twisted cubics as a dense subset;*

(3) *A projective variety $\mathbf{M}_2$ with two irreducible components $\mathbf{B}$ and $\mathbf{P}$, where $\mathbf{P}$ is a $\mathbb{P}^9$-bundle over $\mathbb{P}^3 \times (\mathbb{P}^3)^*$ and $\mathbf{B}$ is the blow-up of $\mathbf{M}_1$ along a 5-dimensional smooth center. The two components of $\mathbf{M}_2$ intersect transversally along the exceptional divisor of $\mathbf{B}$;*

(4) *The Hilbert scheme of twisted cubics* $\mathbf{M}_3$*.* $\mathbf{M}_3$ *is a blow-up of* $\mathbf{M}_2$ *along a* 5*-dimensional smooth center contained in* $\mathbf{P} \setminus \mathbf{B}$*.*

The Hilbert scheme of twisted cubics has been studied intensively in literature. For example, in Piene and Schlessinger's paper [PS85], they proved that the Hilbert scheme has two smooth, projective and rational components: one is twelve-dimensional, containing all twisted cubics as a dense open subset; the other is of fifteen dimension, containing the configurations of a plane cubic and an arbitrary point and their degenerations. They proved further that the two components intersect transversely along a smooth, projective and rational variety of dimension eleven. The above result reproves the classical result and has two interesting features on analyzing the singularities of the moduli space: it is the first time that we introduce the computation of the main obstruction map on the first order deformations to general complexes, and it is purely cohomological and does not use any explicit ideals of twisted cubics, thus allowing for generalizations to other situations.

1.2 More on Moduli Space of Bridgeland Semistable Complexes

In general, the existence of a coarse moduli space or a good moduli space in the sense of [Alp12] for the moduli stack of Bridgeland semistable complexes is a conjecture of Bridgeland:

Conjecture 1.2.1. *([Bri08], Conjecture 16.1) Given a smooth projective variety* X*, a stability condition* $\sigma \in \mathrm{Stab}(X)$ *and a numerical class* $v \in \mathcal{N}(X)$*, there exists a coarse moduli space* $\mathcal{M}_\sigma(v)$ *for complexes in* $\mathrm{D^b}(X)$ *of class* v *which are semistable with respect to* σ*.*

In the classical case of moduli space of Gieseker semistable sheaves, the existence of a good moduli space is essentially due to geometric invariant theory on Quot scheme. But for the moduli stack of Bridgeland semistable complexes, it is unclear that this stack is associated with a GIT problem. In order to attack this problem, we attempt to combine some ideas from two recent works: [AHR15] on the existence of a Luna étale slice for Artin stacks under some mild techinical conditions; and [AS16] on the local structure of moduli space of Gieseker semistable sheaves on a $K3$ surface.

The main result of [AHR15] proves that an Artin stack is étale locally a GIT quotient at closed points with linearly reductive stabilizer. There is also a criterion for the existence of a good moduli space if one can further check this étale local neighborhood satisfies some additional properties. The ideas of [AS16] come later when we are checking those properties for the moduli stack of semistable complexes. At the end, we will provide some partial results and conjectures.

1.3 Notations

X	A smooth projective variety over the complex number $\mathbb{C}$,
$\mathrm{Coh}(X)$	abelian category of coherent sheaves on X,
$\mathrm{D}^{\mathrm{b}}(X)$	bounded derived category of $\mathrm{Coh}(X)$,
$\mathcal{T}_X$	tangent bundle of a smooth projective variety X
$T_{X,x}$	tangent space of X at a point x,
$T_{f,x}$	tangent map $T_{X,x} \longrightarrow T_{Z,f(x)}$ of a morphism $f : X \longrightarrow Z$,
$\mathcal{N}_{Y/X}$	normal bundle of a smooth subvariety Y in X,
$N_{Y/X,y}$	normal space of Y in X at a point y,
$\mathscr{E}xt^1_f(\mathcal{F}, \mathcal{G})$	relative Ext^1 sheaf of $\mathcal{F}$ and $\mathcal{G}$ with respect to a morphism f,
$\mathscr{T}or^1(\mathcal{F}, \mathcal{G})$	Tor^1 sheaf of $\mathcal{F}$ and $\mathcal{G}$.
$\mathrm{ch}(E)$	Chern character of a complex $E \in \mathrm{D}^{\mathrm{b}}(\mathbb{P}^3)$
$c_i(E)$	i-th Chern class of a complex $E \in \mathrm{D}^{\mathrm{b}}(\mathbb{P}^3)$

Chapter 2: Preliminaries

2.1 Stability Conditions

The idea of stability conditions in algebraic geometry goes back to the theory of stable vector bundles and geometric invariant theory where one can single out a nice algebraic family or construct a nice algebraic group quotient. Stability conditions on a triangulated category pushes this idea further by allowing the objects in consideration to be complexes. In this paper, the objects in our moduli problem will always be complexes in the bounded derived category of coherent sheaves on a smooth projective variety. The concept of derived category was first introduced by Grothendieck and Verdier as a correct way to state certain duality theorems. Later on, this category itself becomes more and more interesting and it has many applications. For the definition and basic properties of derived category, one can look at [Chapter 1, Har66].

In this section, we will briefly review the definition of a stability condition on a triangulated category and show how we can construct explicit stability conditions on $\mathbb{P}^3$. We will also define a technical notion called a simple wall-crossing in the stability manifold, which is useful later in the case of twisted cubics.

Definition 2.1.1. *A stability condition $(Z, \mathcal{P})$ on a triangulated category $\mathcal{D}$ consists of a group homomorphism $Z : K(\mathcal{D}) \longrightarrow \mathbb{C}$ called central charge, and full additive subcategories $\mathcal{P}(\phi) \subset \mathcal{D}$ for each $\phi \in \mathbb{R}$, satisfying the following axioms:*

(1) *if $E \in \mathcal{P}(\phi)$ then $Z(E) = m(E)\exp(i\pi\phi)$ for some $m(E) \in \mathbb{R}_{>0}$,*

(2) *for all $\phi \in \mathbb{R}$, $\mathcal{P}(\phi + 1) = \mathcal{P}(\phi)[1]$,*

(3) *if $\phi_1 > \phi_2$ and $A_j \in \mathcal{P}(\phi_j)$, then $\mathrm{Hom}(A_1, A_2) = 0$,*

(4) *for each nonzero object $E \in \mathcal{D}$ there are a finite sequence of real numbers*

$$\phi_1 > \phi_2 > \cdots > \phi_n$$

and a collection of triangles

$$\begin{tikzcd}
0 = E_0 \arrow[r] & E_1 \arrow[r] \arrow[d] & E_2 \arrow[r] \arrow[d] & \cdots \arrow[r] & E_{n-1} \arrow[r] & E_n = E, \arrow[d] \\
& A_1 \arrow[ul, dotted] & A_2 \arrow[ul, dotted] & & & A_n \arrow[ull, dotted]
\end{tikzcd}$$

with $A_j \in \mathcal{P}(\phi_j)$ for all j.

There is a more handy way to define a stability condition, namely defining a stability function on the heart of a bounded t-structure satisfying the Harder-Narasimhan property. This is [Bri07. Proposition 5.3]. One can think of a stability function as parallel to the classical slope stability on the category of coherent sheaves. Let $\mathcal{A}$ be an abelian category and $K(\mathcal{A})$ be its Grothendieck group.

Definition 2.1.2. *A stability function on $\mathcal{A}$ is a group homomorphism $Z : K(\mathcal{A}) \longrightarrow \mathbb{C}$ such that for all $0 \neq E \in \mathcal{A}$, the complex number $Z(E)$ lies in the strict upper half-plane*

$$\{r\exp(i\pi\varphi) | r > 0, 0 < \varphi \leqslant 1\} \subset \mathbb{C}$$

Theorem 2.1.3. *To give a stability function on a triangulated category $\mathcal{D}$ is equivalent to giving a stability function with Harder-Narasimhan property on the heart of a given bounded t-structure on $\mathcal{D}$.*

An easy example of a stability condition is when we take $\mathcal{D}$ to be the bounded derived category of a smooth projective curve C, where our stability coincides with the classical notion of slope stability.

Example 2.1.4. *Take $\mathcal{D} = \mathrm{D^b}(\mathcal{A})$ and $\mathcal{A} = \mathrm{Coh}(C)$ is the heart of the standard t-structure. We define $Z(E) = -\mathrm{deg}(E) + i\mathrm{rank}(E)$, then Theorem 2.1.3 applies and this will define a stability condition on $\mathcal{D}$.*

If we view $\mathrm{rank}(E)$ and $\mathrm{deg}(E)$ as the 0th and 1st Chern character of E, then in fact this idea generalizes to higher dimensional case: we will always assume that the central charge Z is a $\mathbb{C}$-linear combination of the Chern characters pairing with a fixed polarization.

One more technical condition we would like to introduce here is local finiteness of a stability condition. Originally, Bridgeland introduced this property in order to exclude some stability condition with bad behaviors, namely, strictly semistable objects may have infinite Jordan-Hölder filtrations. Later on, as is mentioned in various papers [BMS14] [BMT14] [MS16], the local finiteness property is equivalent to the following so-called support property. First we notice that the image of our central charge Z factors through the numerical Grouthendieck group $K_{\mathrm{num}}(X)$, which is a finite rank lattice which we denote by Λ. We fix a norm $||\cdot||$ on $\Lambda \otimes \mathbb{R}$.

Definition 2.1.5. *A stability condition $(\mathcal{P}, Z)$ satisfies support property if*

$$\inf \left\{ \frac{|Z(\mathrm{ch}(E))|}{||\mathrm{ch}(E)||} : 0 \neq E \in \mathcal{P}(\varphi), \varphi \in \mathbb{R} \right\} > 0$$

Now if we denote the set of stability conditions satisfying support property by $\mathrm{Stab}(\mathcal{D})$, then the main result of [Bri07] tells us that there is a natural topology on $\mathrm{Stab}(\mathcal{D})$ making it a complex manifold.

Next we are going to discuss stability conditions on the three-dimensional projective space $\mathbb{P}^3$ and review some explicit constructions. The main difficulty to construct an explicit stability condition lies in the requirement that the image of the stability function Z has to be inside the upper-half plane. In [Tod09], Toda shows this is not possible on the standard heart $\mathrm{Coh}(\mathbb{P}^3)$. In [BMT14], stability conditions are constructed on a so-called double tilt $\mathscr{A}^{\alpha,\beta}$ of the standard heart. We recall this construction here: First, we identify the cohomology $\mathrm{H}^*(\mathbb{P}^3, \mathbb{Q})$ with $\mathbb{Q}^4$ with respect to the obvious choice of basis. Let $(\alpha, \beta) \in \mathbb{R}_{>0} \times \mathbb{R}$. We define the twisted slope function for $E \in \mathrm{Coh}(\mathbb{P}^3)$ to be

$$\mu_\beta(E) = \frac{c_1(E) - \beta c_0(E)}{c_0(E)}$$

if $c_0(E) \neq 0$, and otherwise we let $\mu_\beta = +\infty$. Then we set

$$\mathcal{T}_\beta = \{E \in \mathrm{Coh}(\mathbb{P}^3) : \text{any quotient sheaf } G \text{ of } E \text{ satisfies } \mu_\beta(G) > 0\}$$

$$\mathcal{F}_\beta = \{E \in \mathrm{Coh}(\mathbb{P}^3) : \text{any subsheaf } F \text{ of } E \text{ satisfies } \mu_\beta(F) \leqslant 0\}.$$

$(\mathcal{F}_\beta, \mathcal{T}_\beta)$ forms a torsion pair in the bounded derived category of $\mathbb{P}^3$, because Harder-Narasimhan filtrations exist for the twisted slope μ_β.

Definition 2.1.6. *Let* $\mathrm{Coh}^\beta(\mathbb{P}^3) \subset \mathrm{D}^\mathrm{b}(\mathbb{P}^3)$ *be the extension-closure* $\langle \mathcal{T}_\beta, \mathcal{F}_\beta[1] \rangle$*. We define the following two functions on* $\mathrm{Coh}^\beta(\mathbb{P}^3)$*:*

$$Z_{\alpha,\beta} = -\left(\mathrm{ch}_2 - \beta \mathrm{ch}_1 + \left(\frac{\beta^2}{2} - \frac{\alpha^2}{2}\right)\mathrm{ch}_0\right) + i\left(\mathrm{ch}_1 - \beta \mathrm{ch}_0\right),$$

$$\nu_{\alpha,\beta} = -\frac{\mathrm{Re}\left(Z_{\alpha,\beta}\right)}{\mathrm{Im}\left(Z_{\alpha,\beta}\right)}$$

if $\mathrm{Im}(Z_{\alpha,\beta}) \neq 0$, *and we let* $\nu_{\alpha,\beta} = +\infty$ *otherwise. An object* $E \in \mathrm{Coh}^{\beta}(\mathbb{P}^3)$ *is called* $\nu_{\alpha,\beta}$*-(semi)stable if for all nontrivial subobjects* F *of* E, *we have* $\nu_{\alpha,\beta}(F) < (\leqslant)\nu_{\alpha,\beta}(E/F)$

An important inequality introduced in [BMT14] and proved in [Mac14] for $\nu_{\alpha,\beta}$-semistable objects is the following. It guarentees the image of the stability function introduced later on the double tilt $\mathscr{A}^{\alpha,\beta}$ lies on the upper-half plane.

Theorem 2.1.7. *(Generalized Bogomolov-Gieseker inequality) For any* $\nu_{\alpha,\beta}$*-semistable object* $E \in \mathrm{Coh}^{\beta}(\mathbb{P}^3)$ *satisfying* $\nu_{\alpha,\beta}(E) = 0$, *we have the following inequality*

$$\mathrm{ch}_3(E) - \beta\mathrm{ch}_2(E) + \frac{\beta^2}{2}\mathrm{ch}_1(E) - \frac{\beta^3}{6}\mathrm{ch}_0(E) \leqslant \frac{\alpha^2}{6}\left(\mathrm{ch}_1(E) - \beta\mathrm{ch}_0(E)\right).$$

On the other hand, for the new slope function $\nu_{\alpha,\beta}$, Harder-Narasimhan filtrations also exist. If we repeat the above construction by defining

$$\mathcal{T}'_{\alpha,\beta} = \{E \in \mathrm{Coh}(\mathbb{P}^3) : \text{any quotient object } G \text{ of } E \text{ satisfies } \nu_{\alpha,\beta}(G) > 0\}$$
$$\mathcal{F}'_{\alpha,\beta} = \{E \in \mathrm{Coh}(\mathbb{P}^3) : \text{any subobject } F \text{ of } E \text{ satisfies } \nu_{\alpha,\beta}(F) \leqslant 0\},$$

then $(\mathcal{F}'_{\alpha,\beta}, \mathcal{T}'_{\alpha,\beta})$ forms a torsion pair of $\mathrm{Coh}^{\beta}(\mathbb{P}^3)$ too.

Definition 2.1.8. *Let* $\mathscr{A}^{\alpha,\beta} \subset \mathrm{D^b}(\mathbb{P}^3)$ *be the extension-closure* $\langle \mathcal{T}'_{\alpha,\beta}, \mathcal{F}_{\alpha,\beta}[1]\rangle$. *We define the following two functions on* $\mathscr{A}^{\alpha,\beta}$, *for* $s > 0$*:*

$$\begin{aligned} Z_{\alpha,\beta,s} = &-\left(\mathrm{ch}_3 - \beta\mathrm{ch}_2 - \left(\left(s+\frac{1}{6}\right)\alpha^2 - \frac{\beta^2}{2}\right)\mathrm{ch}_1 - \left(\frac{\beta^3}{6} - \left(s+\frac{1}{6}\right)\alpha^2\beta\right)\mathrm{ch}_0\right) \\ &+ i\left(\mathrm{ch}_2 - \beta\mathrm{ch}_1 + \left(\frac{\beta^2}{2} - \frac{\alpha^2}{2}\right)\mathrm{ch}_0\right) \\ \lambda_{\alpha,\beta,s} = &-\frac{\mathrm{Re}\left(Z_{\alpha,\beta,s}\right)}{\mathrm{Im}\left(Z_{\alpha,\beta,s}\right)} \end{aligned}$$

if $\mathrm{Im}(Z_{\alpha,\beta,s}) \neq 0$*, and we let* $\lambda_{\alpha,\beta,s} = +\infty$ *otherwise. An object* $E \in \mathscr{A}^{\alpha,\beta}$ *is called* $\lambda_{\alpha,\beta,s}$*-(semi)stable if for all nontrivial subobjects* F *of* E*, we have* $\lambda_{\alpha,\beta,s}(F) < (\leqslant)\lambda_{\alpha,\beta,s}(E/F)$.

Finally by [BMT14, Corollary 5.2.4] and [BMS14, Lemma 8.8], Theorem 2.1.7 implies the following.

Proposition 2.1.9. *The pair* $(\mathscr{A}^{\alpha,\beta}, Z_{\alpha,\beta,s})$ *is a stability condition on* $\mathrm{D^b}(\mathbb{P}^3)$ *for all* $(\alpha, \beta, s) \in \mathbb{R}_{>0} \times \mathbb{R} \times \mathbb{R}_{>0}$*. The function* $(\alpha, \beta, s) \mapsto (\mathscr{A}^{\alpha,\beta}, Z_{\alpha,\beta,s})$ *is continuous.*

2.2 Moduli of complexes

In this section, we are going to introduce some definitions on the moduli problem of semistable complexes. The objects we parametrize are complexes in the bounded derived category of coherent sheaves on a smooth projective variety X, so the first concept we want to introduce is a family of complexes. Let S be a scheme over $\mathbb{C}$ of finite type, then we have the unbounded derived category of quasi-coherent sheaves $\mathrm{D}(\mathrm{Qcoh}(X \times S))$. A complex $E \in \mathrm{D}(\mathrm{Qcoh}(X \times S))$ is called S-perfect if it is locally isomorphic to a bounded complex of flat sheaves over S of finite presentation. We denote the subcategory of S-perfect complexes by $\mathrm{D_{S-perf}}(X \times S)$.

Definition 2.2.1. *A family of complexes over a scheme* S *is an* S*-perfect complex* E *in* $\mathrm{D_{S-perf}}(X \times S)$ *such that* $\mathrm{Ext}^i(E_s, E_s) = 0$ *for all* $s \in S$ *and* $i < 0$*, where* E_s *is the derived restriction of* E *to* $X \times \{s\}$*.*

If we define the moduli functor $\mathfrak{M} : \mathrm{Sch}_{\mathbb{C}} \longrightarrow \mathrm{Grp}$ sending a scheme S to the groupoid of families of complexes over S, then we have the following theorem from [Lie06]. For definitions and properties related to Artin stacks, one can look at [Sta17].

Theorem 2.2.2. *The moduli functor $\mathfrak{M}$ is an Artin stack, locally of finite type, locally quasi-seperated and with seperated diagonal.*

We are mostly interested in the substack of semistable complexes in $\mathfrak{M}$, the precise definition is the following. Suppose $(\mathcal{P}, Z)$ is a stability condition, we fix a numerical class v in $K_{\mathrm{num}}(X)$ and a phase $\sigma \in \mathbb{R}$.

Definition 2.2.3. *The moduli stack of semistable complexes with a primitive numverical class v and phase σ is a substack $\mathfrak{M}^{\mathrm{ss}}_{v,\sigma}$ of $\mathfrak{M}$ sending a scheme S to the groupoid of all families of $(\mathcal{P}, Z)$-semistable complexes with class v and phase σ. Similarly we denote $\mathfrak{M}^{\mathrm{s}}_{v,\sigma} \subset \mathfrak{M}^{\mathrm{ss}}_{v,\sigma}$ to be the substack of stable complexes.*

If $\mathfrak{M}^{\mathrm{s}}_{v,\sigma} = \mathfrak{M}^{\mathrm{ss}}_{v,\sigma}$, meaning that there is no strictly semistable complex or equivalently all semistable complexes are stable, we usually say the stability condition $(\mathcal{P}, Z)$ is not on a wall. We will say $(\mathcal{P}, Z)$ is on a wall if $\mathfrak{M}^{\mathrm{s}}_{v,\sigma} \neq \mathfrak{M}^{\mathrm{ss}}_{v,\sigma}$. The set of stability conditions on a wall is actually a union of locally finite codimension one submanifold of $\mathrm{Stab}(\mathrm{D}^{\mathrm{b}}(X))$, and we can decompose its complement as a disjoint union of connected components. Each connected component is called a chamber and its has the property that all stability conditions inside a same chamber define a same moduli functor of semistable complexes.

One more oberservation here is that stable complexes are always simple, meaning that their endomorphism groups are just scalar multiplications. The next theorem from [Ina02] guarentees the existence of a fine moduli space for $\mathfrak{M}^{\mathrm{s}}_{v,\sigma}$. In particular if a stability condition is not on a wall, we know that $\mathfrak{M}^{\mathrm{s}}_{v,\sigma} = \mathfrak{M}^{\mathrm{ss}}_{v,\sigma}$, hence $\mathfrak{M}^{\mathrm{ss}}_{v,\sigma}$ also has a fine moduli space. We define $\mathfrak{M}_{\mathrm{Spl}}$ to be the functor sending a scheme S to the groupoid of families of simple complexes, and define M_{Spl} to be the functor sending a scheme S to just the set of families of simple complexes.

Theorem 2.2.4. *The moduli functor M_{Spl} is representable by an algebraic space locally of finite type over $\mathbb{C}$, and $\mathfrak{M}_{\mathrm{Spl}}$ is a $\mathbb{C}^*$-gerbe over M_{Spl} via the forgetful morphism.*

Studying how the moduli space of semistable complexes changes when we take stability conditions from different chambers is a very interesting topic, and this is closely related to the geometry of the moduli space. We will study an example in the case of moduli space of twisted cubics in the next chapter. Here we want to introduce a technical definition of simple wall-crossings, which is very useful in Chapter 3. We take two adjacent chambers C_1, C_2 in $\mathrm{Stab}(X)$ and denote the wall between them by W, and we use λ_1, λ_2 for stability conditions in C_1 and C_2 respectively.

Definition 2.2.5. *A wall-crossing (C_1, C_2, W) is simple if there exists two families $\mathcal{U}_A$ and $\mathcal{U}_B$ of semistable complexes with Chern characters v_A and v_B respectively, and with phase σ, for stability conditions in a neighborhood of a point on W meeting C_1 and C_2. We denote the bases of the two families by $\mathbf{M}_A$ and $\mathbf{M}_B$ respectively, and they satisfy the following conditions:*

(1) *$v_A + v_B = v$;*

(2) *if E is λ_1-stable but not λ_2-stable, then there exists a unique pair (A, B) in $\mathbf{M}_A \times \mathbf{M}_B$ such that $0 \longrightarrow B \longrightarrow E \longrightarrow A \longrightarrow 0$ is a nontrivial extension. Conversely, all nontrivial extensions of A by B are λ_1-stable but not λ_2-stable;*

(3) *if F is λ_1-stable but not λ_2-stable, then there exists a unique pair (A, B) in $\mathbf{M}_A \times \mathbf{M}_B$ such that $0 \longrightarrow A \longrightarrow F \longrightarrow B \longrightarrow 0$ is a nontrivial extension. Conversely, all nontrivial extensions of B by A are λ_1-stable but not λ_2-stable.*

In some sense, we can say this is the easiest nontrivial wall-crossing because on the level of sets, the difference of the two moduli space of complexes in adjacent chambers is controlled by the two families $\mathcal{U}_A$ and $\mathcal{U}_B$: if we go from C_1 to C_2, the wall-crossing destablizes a locus of all extensions $0 \longrightarrow B \longrightarrow E \longrightarrow A \longrightarrow 0$ and replaces it by a locus of all reverse extensions $0 \longrightarrow A \longrightarrow F \longrightarrow B \longrightarrow 0$. In the case of moduli space of twisted cubics, Schmidt first studied a series of wall-crossings in the stability manifold in his paper [SchB15], his main result is the following.

Theorem 2.2.6. *The Hilbert scheme of twisted cubics in $\mathbb{P}^3$ can be obtained via four chambers and three simple wall-crossings in* $\mathrm{Stab}(\mathbb{P}^3)$*. The families controlling the three wall-crossings are:*

(1) $(\mathcal{O}_{\mathbb{P}^3}(-2)^3, \mathcal{O}_{\mathbb{P}^3}(-3)[1]^2)$ with its base scheme being a point;

(2) $(\mathcal{I}_p(-1), \mathcal{O}_V(-3))$, where p is a point and V is a hyperplane, with its base scheme being $\mathbb{P}^3 \times (\mathbb{P}^3)^*$;

(3) $(\mathcal{O}_{\mathbb{P}^3}(-1), \mathcal{I}_{q/V}(-3))$, where q is a point lying on a hyperplane V, with its base scheme being the universal hyperplane $H := \{(q, V) \in \mathbb{P}^3 \times (\mathbb{P}^3)^* : q \in V\}$.

In the next chapter, we will show that we can get even more informations based on those families: they can determine the geometry of the moduli spaces, and we are able to describe the singularities in the moduli space after each wall-crossing.

Chapter 3: Hilbert Scheme of Twisted Cubics

In this chapter, we will fix the character v to be $\mathrm{ch}(\mathcal{I}_C)$, where C is a twisted cubic in $\mathbb{P}^3$ and $\mathcal{I}_C$ is its sheaf of ideals. There are three wall-crossings in total to be analyzed by Theorem 2.2.6, so we will treat them seperately in three sections.

3.1 The First Wall-crossing

In this section, we first observe that $\mathrm{Ext}^1(\mathcal{O}_{\mathbb{P}^3}(-2)^3, \mathcal{O}_{\mathbb{P}^3}(-3)[1]^2) = 0$ so on one side of the wall, there is no semistable complexes hence the moduli space is the empty set. We denote the moduli space on the other side of the wall by $\mathbf{M}_1$. We will construct the moduli space $\mathbf{M}_1$ from quiver representations and prove that it is a smooth, projective and integral variety. This part first appears in Theorem 7.1 of [SchB15], we will give more details here.

We start with a quiver $Q = (V, A) : V = \{v_1, v_2\}, A = \{e_i | i = 1, 2, 3, 4\}$, where $s(e_i) = v_1$ and $t(e_i) = v_2$. Explicitly, Q is $\bullet \substack{\longrightarrow \\ \longrightarrow \\ \longrightarrow \\ \longrightarrow} \bullet$ with no relation. We set a dimension vector to be (2,3) and define $\theta : \mathbb{Z} \oplus \mathbb{Z} \longrightarrow \mathbb{Z}$ to be $\theta(m, n) = -3m + 2n$. A representation V with dimension vector $(2, 3)$ is θ-(semi)stable if for any proper nontrivial subrepresentation W we have $\theta(\underline{\dim} W) > (\geqslant) 0$, where $\underline{\dim} W$ is the dimension vector of W. If S is a scheme, we define a family of θ-semistable representations of Q over S with dimension vector $(2, 3)$ to be four homomorphisms

$f_0, f_1, f_2, f_3 : V \longrightarrow W$, where V and W are locally free on S with $\mathrm{rk}(V) = 2$ and $\mathrm{rk}(W) = 3$, such that the representation $f_{0s}, f_{1s}, f_{2s}, f_{3s} : V_s \longrightarrow W_s$ is θ-semistable for any closed point $s \in S$. We define $\mathcal{K}_\theta : \mathbf{Sch}_{\mathbb{C}} \longrightarrow \mathbf{Sets}$ to be the moduli functor sending a scheme S to the set of isomorphism classes of families of θ-semistable representations with dimension vector $(2, 3)$ over S.

Proposition 3.1.1. *The functor $\mathcal{K}_\theta$ is represented by a smooth projective integral variety K_θ.*

Proof. By [Kin94], since the dimension vector $(2, 3)$ is indivisible, $\mathcal{K}_\theta$ is represented by a projective variety K_θ and there is no strictly θ-semistable representation. The path algebra of Q is hereditary since there is no relation between arrows, this means K_θ is smooth and irreducible. □

Theorem 3.1.2. *The two moduli spaces K_θ and $\mathbf{M}_1$ are isomorphic.*

Proof. Fix $(\alpha_0, \beta_0) = (\frac{1}{2} + \varepsilon, -\frac{5}{2})$, where $\varepsilon > 0$ is small. By [SchB15, Theorem 5.3; Theorem 6.1], $\mathbf{M}_1$ is isomorphic to the moduli space $\mathbf{M}^{\mathrm{tilt}}_{\alpha_0,\beta_0}(v)$ of ν_{α_0,β_0}-semistable objects in $\mathrm{Coh}^{\beta_0}(\mathbb{P}^3)$. Since (α_0, β_0) is in the interior of a chamber, there is no strictly semistable objects. Notice that $-3 < \beta_0 < -2$, so by definition $\mathcal{O}(-2)$ and $\mathcal{O}(-3)[1]$ are in $\mathrm{Coh}^{\beta_0}(\mathbb{P}^3)$, and we have

$$Z_{\alpha_0,\beta_0}(\mathcal{O}(-2)) = -\frac{1}{8} + \frac{\alpha_0^2}{2} + \frac{1}{2}i,$$
$$Z_{\alpha_0,\beta_0}(\mathcal{O}(-3)[1]) = \frac{1}{8} - \frac{\alpha_0^2}{2} + \frac{1}{2}i.$$

On the other hand, We denote $\mathrm{Rep}(Q)$ to be the abelian category of quiver representations of Q, and denote $\mathscr{B}$ to be the extension closure of $\mathcal{O}(-2)$ and $\mathcal{O}(-3)[1]$ in $\mathrm{Coh}^{\beta_0}(\mathbb{P}^3)$. By [SchB15, Theorem 5.1], all ν_{α_0,β_0}-semistable objects are in $\mathscr{B}$. By

[Bon89, Theorem 6.2], there is an equivalence $F : \mathrm{D^b}(\mathscr{B}) \longrightarrow \mathrm{D^b}(\mathrm{Rep}(Q))$. This functor F sends $\mathcal{O}(-3)[1]$ and $\mathcal{O}(-2)$ to the two simple representations $\mathbb{C} \longrightarrow 0$ and $0 \longrightarrow \mathbb{C}$. On $\mathscr{B}$, we can define a central charge Z and a slope function η by

$$\begin{aligned} Z(E) &= \theta\left(F^{-1}(E)\right) + i\dim\left(F^{-1}(E)\right), \\ \eta(E) &= -\frac{\mathrm{Re}\left(Z(E)\right)}{\mathrm{Im}\left(Z(E)\right)} = -\frac{\theta\left(F^{-1}(E)\right)}{\dim\left(F^{-1}(E)\right)}, \end{aligned}$$

where dim is the sum of the two components of a dimension vector. This will make $\sigma := (Z, \mathscr{B})$ a stability condition on $\mathrm{D^b}(\mathscr{B})$ by [Bri07, Example 5.5], and F sends σ-semistable objects with Chern character v to θ-semistable representations with dimension vector $(2,3)$. If we denote $\mathbf{M}_\sigma$ to be the moduli of σ-semistable objects in $\mathscr{B}$ with Chern character v, then actually F defines a bijection map of sets between $\mathbf{M}_\sigma$ and K_θ. We will globalize this construction later and get a bijective morphism by using the existence of a universal family. Now we compute that

$$Z\left(\mathcal{O}(-2)\right) = 2 + i,$$

$$Z\left(\mathcal{O}(-3)[1]\right) = -3 + i.$$

If we view Z and $Z_{\alpha_0,\beta_0}|_{\mathrm{D^b}(\mathscr{B})}$ as linear maps from $\mathbb{Z}^2$ to $\mathbb{R}^2$, then an easy computation shows they differ from each other by composing a linear map in $\mathrm{GL}^+(2;\mathbb{R})$. This means they define the same stability condition and hence have the same moduli of semistable objects with Chern character v, so $\mathbf{M}_\sigma = \mathbf{M}^{\mathrm{tilt}}_{\alpha_0,\beta_0}(v)$.

It only remains to show that K_θ is isomorphic to $\mathbf{M}_\sigma$. For any σ-semistable object $E \in \mathrm{D^b}(\mathscr{B})$ with Chern character v, $F(E)$ is a θ-semistable representation $f_1, f_2, f_3, f_4 : \mathbb{C}^3 \longrightarrow \mathbb{C}^2$. We have an obvious exact sequence

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & \mathbb{C}^3 \arrow[r] \arrow[d, "f_i"'] & \mathbb{C}^3 \arrow[d] \\ \mathbb{C}^2 \arrow[r] & \mathbb{C}^2 \arrow[r] & 0 \end{tikzcd}$$

in $\mathrm{Rep}(Q)$ which corresponds to an exact sequence $\mathcal{O}(-2)^3 \longrightarrow E \longrightarrow \mathcal{O}(-3)[1]^2$ in $\mathscr{B}$. By applying the long exact sequence for Hom functor to it, we can see that $\mathrm{Ext}^2(E, E) = 0$. But $\mathrm{Ext}^2(E, E)$ computes the obstruction space of $\mathbf{M}_\sigma$ at E by [Ina02] and [Lie06], so $\mathbf{M}_\sigma$ is smooth and hence a complex manifold. Since there is no strictly σ-semistable object, a universal family $\mathcal{U}$ of σ-semistable objects with Chern character v exists on $\mathbf{M}_\sigma \times \mathbb{P}^3$, and $\mathcal{U}$ is an extension of $p^*\mathcal{O}(-3)^{\oplus 2}[1]$ by $p^*\mathcal{O}(-2)^{\oplus 3}$. If we denote $\mathscr{B}'$ to be the extension closure of $p^*\mathcal{O}(-3)^{\oplus 2}[1]$ and $p^*\mathcal{O}(-2)^{\oplus 3}$ in $\mathrm{D^b}(\mathbf{M}_\sigma \times \mathbb{P}^3)$, and denote $\mathrm{Rep}_{K_\theta}(Q)$ to be the category of families of quiver representations over K_θ. Then there exists an equivalence $F_{K_\theta} : \mathscr{B}' \longrightarrow \mathrm{D^b}(\mathrm{Rep}_{K_\theta}(Q))$ such that when restricted to a fiber $x \times \mathbb{P}^3$, F_{K_θ} is the same as F. Because $F_{K_\theta}(\mathcal{U})|_{x\times\mathbb{P}^3} = F(\mathcal{U}|_{x\times\mathbb{P}^3})$ and $\mathcal{U}|_{x\times\mathbb{P}^3}$ is a σ-semistable object with Chern character v, $F_{K_\theta}(\mathcal{U})|_{x\times\mathbb{P}^3}$ is θ-semistable with dimension vector $(2, 3)$. This means $F_{K_\theta}(\mathcal{U})$ is a family of θ-semistable objects with dimension vector $(2, 3)$, so it induces a morphism $\varphi : \mathbf{M}_\sigma \longrightarrow K_\theta$. As $\mathcal{U}$ is a universal family of σ-semistable objects with Chern character v, and F is a bijection between σ-semistable objects with Chern character v in $\mathscr{B}$ and θ-semistable representations with dimension vector $(2, 3)$, φ is a bijective morphism. We proved that K_θ is smooth in Proposition 3.1, and any bijective morphism between complex manifolds is an isomorphism, so φ is an isomorphism. Therefore K_θ is isomorphic to $\mathbf{M}_1$. $\square$

3.2 The Second Wall-crossing

In this section, we study the wall-crossing controlled by the second family of pairs in Theorem 2.2.6 and prove (3) in Theorem 1.1.1. Throughout this section, we will fix

$$(A, B) = (\mathcal{I}_p(-1), \mathcal{O}_V(-3)),$$

and denote the stability conditions in the chamber of $\mathbf{M}_1$ (resp. $\mathbf{M}_2$) by λ_1 (resp. λ_2). Whenever we take an extension of A and B, we always mean a nontrivial extension class modulo scalar multiplications. The following Hom and Ext group computations are straightforward.

Lemma 3.2.1. $\mathrm{Hom}(A, B) = \mathrm{Hom}(B, A) = 0$, $\mathrm{Hom}(A, A) = \mathrm{Hom}(B, B) = \mathbb{C}$*;*

$\mathrm{Ext}^1(A, B) = \mathbb{C}$ *if* $p \in V$*, and* 0 *otherwise,*

$\mathrm{Ext}^1(A, A) = \mathrm{Ext}^1(B, B) = \mathbb{C}^3$*,* $\mathrm{Ext}^1(B, A) = \mathbb{C}^{10}$*;*

$\mathrm{Ext}^2(A, B) = \mathbb{C}$*,* $\mathrm{Ext}^2(B, B) = 0$*,* $\mathrm{Ext}^2(A, A) = \mathbb{C}^3$*,* $\mathrm{Ext}^2(B, A) = 0$*;*

$\mathrm{Ext}^3(A, B) = \mathrm{Ext}^3(A, A) = \mathrm{Ext}^3(B, B) = \mathrm{Ext}^3(B, A) = 0.$

Moduli space of nontrivial extensions. In this subsection, we construct two moduli spaces H and $\mathbf{P}$, where H parametrizes nontrivial extensions of A by B and $\mathbf{P}$ parametrizes the reverse nontrivial extensions. We show that with the universal extensions on those moduli spaces, H is embedded into $\mathbf{M}_1$ and $\mathbf{P}$ is embedded into $\mathbf{M}_2$. Then we do some detailed computations on Ext groups for later uses.

Let us first introduce some more notations here: we will denote the family of A on $\mathbf{M}_A \times \mathbb{P}^3 = \mathbb{P}^3 \times \mathbb{P}^3$ by $\mathcal{U}_A$, and the family of B on $\mathbf{M}_B \times \mathbb{P}^3 = (\mathbb{P}^3)^* \times \mathbb{P}^3$ by $\mathcal{U}_B$. Denote two projections by

$$\mathbf{M}_A \times \mathbb{P}^3 \xleftarrow{\pi_A} \mathbf{M}_A \times \mathbf{M}_B \times \mathbb{P}^3 \xrightarrow{\pi_B} \mathbf{M}_B \times \mathbb{P}^3.$$

We also denote the projection onto the first two factors by $\mathbf{M}_A \times \mathbf{M}_B \times \mathbb{P}^3 \xrightarrow{\pi} \mathbf{M}_A \times \mathbf{M}_B$. Let H be the incidence hyperplane $\{(p, V) \in \mathbb{P}^3 \times (\mathbb{P}^3)^* | p \in V\}$, and denote the restriction of the above three projections to $H \times \mathbb{P}^3$ by π_A^H, π_B^H and π_H. Define $\mathcal{F}$ to be $\pi_A^*\mathcal{U}_A$ and $\mathcal{G}$ to be $\pi_B^*\mathcal{U}_B$, and define $\mathcal{F}_H$ to be $\left(\pi_A^H\right)^*\mathcal{U}_A$ and $\mathcal{G}_H$ to be $\left(\pi_B^H\right)^*\mathcal{U}_B$. Let $S \longrightarrow \mathbf{M}_A \times \mathbf{M}_B$ and $S_H \longrightarrow H$ be any morphisms of schemes, and denote the pullbacks of these two morphisms with respect to π and π_H by q^S and q_H^S.

Proposition 3.2.2. *There exists an extension on $H \times \mathbb{P}^3$*

$$0 \longrightarrow \mathcal{G}_H \otimes \pi_H^*\mathcal{L} \longrightarrow \mathcal{U}_E \longrightarrow \mathcal{F}_H \longrightarrow 0, \tag{3.1}$$

$\mathcal{L} = \mathscr{E}xt^1_{\pi_H}(\mathcal{F}_H, \mathcal{G}_H)^*$ is a line bundle, which is universal on the category of noetherian H-schemes for the classes of nontrivial extensions of $\left(q_H^S\right)^*\mathcal{F}_H$ by $\left(q_H^S\right)^*\mathcal{G}_H$ on $(H \times \mathbb{P}^3) \times_H S_H$, modulo the scalar mutiplication of $H^0(S_H, \mathcal{O}^*_{S_H})$.

Proof. We apply [Lan85, Proposition 4.2; Corollary 4.5] to $\mathcal{F}_H$, $\mathcal{G}_H$ and π_H. We only need to check that $\mathscr{E}xt^0_{\pi_H}(\mathcal{F}_H, \mathcal{G}_H) = 0$ and $\mathscr{E}xt^1_{\pi_H}(\mathcal{F}_H, \mathcal{G}_H)$ commutes with base change in the sense that over any point $(p_0, V_0) \in H$, $\mathscr{E}xt^1_{\pi_H}(\mathcal{F}_H, \mathcal{G}_H)$ restricts to $\mathrm{Ext}^1(A_0, B_0)$. First notice that $\mathscr{E}xt^3_{\pi_H}(\mathcal{F}_H, \mathcal{G}_H)$ restricts to $\mathrm{Ext}^3(A_0, B_0)$ over (p_0, V_0), where the latter is 0 by Lemma 4.1. Then [Lan85, Theorem 1.4] tells us $\mathscr{E}xt^2_{\pi_H}(\mathcal{F}_H, \mathcal{G}_H)$ restricts to $\mathrm{Ext}^2(A_0, B_0)$ over (p_0, V_0), where the latter is $\mathbb{C}$ for all points in H. Hence $\mathscr{E}xt^2_{\pi_H}(\mathcal{F}_H, \mathcal{G}_H)$ is a line bundle. Again [Lan85, Theorem 1.4] tells us $\mathscr{E}xt^1_{\pi_H}(\mathcal{F}_H, \mathcal{G}_H)$ restricts to $\mathrm{Ext}^1(A_0, B_0)$ over (p_0, V_0). By Lemma 4.1 we have $\mathrm{Ext}^1(A_0, B_0) = \mathbb{C}$ for all points in H, so $\mathscr{E}xt^1_{\pi_H}(\mathcal{F}_H, \mathcal{G}_H)$ is a line bundle. Applying [Lan85, Theorem 1.4] a third time, $\mathscr{E}xt^0_{\pi_H}(\mathcal{F}_H, \mathcal{G}_H)$ will restrict to $\mathrm{Hom}(A_0, B_0)$, where the latter is 0 by Lemma 4.1. Hence $\mathscr{E}xt^0_{\pi_H}(\mathcal{F}_H, \mathcal{G}_H) = 0$. $\square$

Proposition 3.2.3. *The relative Ext sheaf $\mathscr{E}xt^1_\pi(\mathcal{G}, \mathcal{F})$ is locally free of rank 10 on $\mathbf{M}_A \times \mathbf{M}_B$. If we denote its projectivization $\mathbb{P}(\mathscr{E}xt^1_\pi(\mathcal{G}, \mathcal{F})^*)$ by $\mathbf{P}$, then there exists* *an extension on* $\mathbf{P} \times \mathbb{P}^3$

$$0 \longrightarrow h^*\mathcal{F} \otimes \pi_{\mathbf{P}}^*\mathcal{O}_{\mathbf{P}}(1) \longrightarrow \mathcal{U}_F \longrightarrow h^*\mathcal{G} \longrightarrow 0, \tag{3.2}$$

h is the projection $\mathbf{P} \times \mathbb{P}^3 \longrightarrow \mathbf{M}_A \times \mathbf{M}_B \times \mathbb{P}^3$, $\pi_{\mathbf{P}}$ is the projection $\mathbf{P} \times \mathbb{P}^3 \longrightarrow \mathbf{P}$ and $\mathcal{O}_{\mathbf{P}}(1)$ is the relative $\mathcal{O}(1)$ on $\mathbf{P}$, which is universal on the category of noetherian $\mathbf{M}_A \times \mathbf{M}_B$-schemes for the classes of nontrivial extensions of $\left(q^S\right)^* \mathcal{F}$ by $\left(q^S\right)^* \mathcal{G}$ on $(\mathbf{M}_A \times \mathbf{M}_B \times \mathbb{P}^3) \times_{\mathbf{M}_A \times \mathbf{M}_B} S$, modulo the scalar multiplication of $H^0(S, \mathcal{O}_S^*)$.

Proof. The proof is completely analogous to the proof of Proposition 3.2.2. □

The existence the above extension $\mathcal{U}_E$ (resp. $\mathcal{U}_{\mathcal{F}}$) gives a flat family of λ_1-stable (resp. λ_2-stable) sheaves on H (resp. $\mathbf{P}$), hence it induces a morphism $\varphi_E : H \longrightarrow \mathbf{M}_1$ (resp. $\varphi_F : \mathbf{P} \longrightarrow \mathbf{M}_2$).

Proposition 3.2.4. *(1) The induced morphism φ_E is a closed embedding;*

(2) The induced morphism φ_F is injective on the level of sets and Zariski tangent spaces.

Proof. On the level of sets, φ_E maps an extension $0 \longrightarrow B \longrightarrow E \longrightarrow A \longrightarrow 0$ to E. If we have two extensions $0 \longrightarrow B \longrightarrow E \longrightarrow A \longrightarrow 0$ and $0 \longrightarrow B' \longrightarrow E' \longrightarrow A' \longrightarrow 0$ such that $E \cong E'$ as stable sheaves, then $E' = E$ and this isomorphism is just a scalar multiplication by some $c \in \mathbb{C}^*$. By the definition of a simple wall-crossing with a pair of destabilizing object, we must have $A' = A$ and $B' = B$. This implies that φ_E is injective on the level of sets.

On the level of Zariski tangent spaces, a tangent vector v of H at a point (p, V) can be represented by a morphism $\mathrm{Spec}\mathbb{C}[\varepsilon]/(\varepsilon^2) \longrightarrow H$. By pulling back the universal extension (1) to $(H \times \mathbb{P}^3) \times_H \mathrm{Spec}\mathbb{C}[\varepsilon]/(\varepsilon^2) = \mathrm{Spec}\mathbb{C}[\varepsilon]/(\varepsilon^2) \times \mathbb{P}^3$, we get an exact sequence of flat families

$$0 \longrightarrow \mathcal{G}_\varepsilon \longrightarrow \mathcal{E}_\varepsilon \longrightarrow \mathcal{F}_\varepsilon \longrightarrow 0$$

and $\mathcal{G}_\varepsilon$, $\mathcal{E}_\varepsilon$ and $\mathcal{F}_\varepsilon$ restrict to B, E and A on the closed fiber respectively. In particular, $\mathcal{E}_\varepsilon$ is a flat family of λ_1-stable objects. It gives rise to a morphism $\mathrm{Spec}\mathbb{C}[\varepsilon]/(\varepsilon^2) \longrightarrow \mathbf{M}_1$ corresponding to $T_{\varphi_E,(p,V)}(v)$. Suppose we have two tangent vectors v, v' represented by morphisms $\xi, \xi' : \mathrm{Spec}\mathbb{C}[\varepsilon]/(\varepsilon^2) \longrightarrow H$ and $T_{\varphi_E,(p,V)}(v) = T_{\varphi_E,(p,V)}(v')$. Then there exists an isomorphism $\eta : \mathcal{E}_\varepsilon \longrightarrow \mathcal{E}'_\varepsilon$ between the resulting flat families of λ_1-stable objects such that η restricts to identity on the closed fiber. By [Ina02] and [Lie06], η corresponds to the following diagram in the derived category:

$$\begin{array}{ccc} E & = & E \\ {\scriptstyle\zeta}\downarrow & & {\scriptstyle\zeta'}\downarrow \\ E[1] & \xrightarrow{c} & E[1], \end{array}$$

where c is a multiplication by some nonzero constant c. By composing ξ and ξ' with the natural projections

$$\mathbf{M}_A = \mathbb{P}^3 \longleftarrow H \longrightarrow (\mathbb{P}^3)^* = \mathbf{M}_B,$$

we can complete ζ and ζ' to commutative diagrams

$$\begin{array}{ccccc} B & \longrightarrow & E & \longrightarrow & A \\ \downarrow & & {\scriptstyle\zeta}\downarrow & & \downarrow \\ B[1] & \longrightarrow & E[1] & \longrightarrow & A[1] \end{array} \qquad \begin{array}{ccccc} B & \longrightarrow & E & \longrightarrow & A \\ \downarrow & & {\scriptstyle\zeta'}\downarrow & & \downarrow \\ B[1] & \longrightarrow & E[1] & \longrightarrow & A[1], \end{array}$$

Via the two diagrams, the above diagram of η will induce two diagrams

$$\begin{array}{ccc} B & = & B \\ {\scriptstyle\zeta_B}\downarrow & & {\scriptstyle\zeta'_B}\downarrow \\ B[1] & \xrightarrow{c} & B[1] \end{array} \qquad \begin{array}{ccc} A & = & A \\ {\scriptstyle\zeta_A}\downarrow & & {\scriptstyle\zeta'_A}\downarrow \\ A[1] & \xrightarrow{c} & A[1] \end{array}$$

corresponding to isomorphisms $\eta_B : \mathcal{G}_\varepsilon \longrightarrow \mathcal{G}'_\varepsilon$ and $\eta_A : \mathcal{F}_\varepsilon \longrightarrow \mathcal{F}'_\varepsilon$ such that they restrict to identities on closed fiber and they make the following diagram commutative:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{G}_\varepsilon & \longrightarrow & \mathcal{E}_\varepsilon & \longrightarrow & \mathcal{F}_\varepsilon & \longrightarrow & 0 \\
 & & \eta_B \big\downarrow & & \eta \big\downarrow & & \eta_A \big\downarrow & & \\
0 & \longrightarrow & \mathcal{G}'_\varepsilon & \longrightarrow & \mathcal{E}'_\varepsilon & \longrightarrow & \mathcal{F}'_\varepsilon & \longrightarrow & 0,
\end{array}$$

which implies the two morphisms ξ and ξ' are the same. Therefore $v = v'$ and $T_{\varphi_E, E}$ is injective. This proves that φ_E is a closed embedding. The proof of (2) is completely analogous to the above argument. $\square$

Now we study the normal sequence of the embedding $\varphi_E : H \longrightarrow \mathbf{M}_1$. Fix a nontrivial extension $0 \longrightarrow B \longrightarrow E \longrightarrow A \longrightarrow 0$, then we have the following lemma.

Lemma 3.2.5. *The following diagram is coming from taking the long exact sequences for* Hom *functor in two directions, it is commutative with exact rows and columns and all boundary homomorphisms are* 0.

$$\begin{array}{ccccccc}
\mathrm{Ext}^1(A,B) = \mathbb{C} & \xrightarrow{0} & \mathrm{Ext}^1(A,E) = \mathbb{C}^2 & \longrightarrow & \mathrm{Ext}^1(A,A) = \mathbb{C}^3 & \longrightarrow & \mathbb{C} \\
{\scriptstyle 0}\big\downarrow & & \big\downarrow & & \big\downarrow & & \big\downarrow \\
\mathrm{Ext}^1(E,B) = \mathbb{C}^2 & \longrightarrow & \mathrm{Ext}^1(E,E) = \mathbb{C}^{12} & \longrightarrow & \mathrm{Ext}^1(E,A) = \mathbb{C}^{10} & \longrightarrow & 0 \\
\big\downarrow & & \big\downarrow & & \big\downarrow & & \big\downarrow \\
\mathrm{Ext}^1(B,B) = \mathbb{C}^3 & \longrightarrow & \mathrm{Ext}^1(B,E) = \mathbb{C}^{13} & \longrightarrow & \mathrm{Ext}^1(B,A) = \mathbb{C}^{10} & \longrightarrow & 0 \\
\big\downarrow & & \big\downarrow & & \big\downarrow & & \big\downarrow \\
\mathrm{Ext}^2(A,B) = \mathbb{C} & \xrightarrow{0} & \mathrm{Ext}^2(A,E) = \mathbb{C}^3 & \longrightarrow & \mathrm{Ext}^2(A,A) = \mathbb{C}^3 & \longrightarrow & 0
\end{array}$$

Proof. This diagram is a straightforward computation by using that $(A, B) = (\mathcal{I}_p(-1), \mathcal{O}_V(-3))$ and that E satisfies a triangle $\mathcal{O}(-2)^3 \longrightarrow E \longrightarrow \mathcal{O}(-3)[1]^2$. $\square$

The Kodaira-Spencer map $\mathrm{KS} : T_{\mathbf{M}_1, E} \longrightarrow \mathrm{Ext}^1(E,E)$ is known to be an isomorphism by deformation theory of complexes in [Ina02] and [Lie06]. If we let θ_E to be the composition $\mathrm{Ext}^1(E,E) \longrightarrow \mathrm{Ext}^1(E,A) \longrightarrow \mathrm{Ext}^1(B,A)$ (or $\mathrm{Ext}^1(E,E) \longrightarrow \mathrm{Ext}^1(B,E) \longrightarrow \mathrm{Ext}^1(B,A)$) in the diagram of Lemma 3.2.5, and let the kernel of θ_E to be K_E, then we have

Proposition 3.2.6. *The Kodaira-Spencer map* KS *restricts to an isomorphism between* $T_{H,E}$ *and* K_E*, and we have the following commutative diagram:*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & T_{H,E} & \longrightarrow & T_{\mathbf{M}_1,E} & \longrightarrow & N_{H/\mathbf{M}_1,E} & \longrightarrow & 0 \\
& & \downarrow{\scriptstyle \mathrm{KS}} & & \downarrow{\scriptstyle \mathrm{KS}} & & \downarrow & & \\
0 & \longrightarrow & K_E & \longrightarrow & \mathrm{Ext}^1(E,E) & \xrightarrow{\theta_E} & \mathrm{Ext}^1(B,A) & &
\end{array}$$

Proof. θ_E is the composition of $\mathrm{Ext}^1(E,E) \longrightarrow \mathrm{Ext}^1(E,A) \longrightarrow \mathrm{Ext}^1(B,A)$, where the first map is surjective with a two-dimensional kernel $\mathrm{Ext}^1(E,B)$ and the second map has a 3-dimensional kernel $\mathrm{Ext}^1(A,A)$ by Lemma 3.2.5. This implies K_E is 5-dimensional since K_E is an extension of $\mathrm{Ext}^1(A,A)$ by $\mathrm{Ext}^1(E,B)$, so $\mathrm{dim}K_E = \mathrm{dim}T_{H,E}$. On the other hand, as shown in the proof of Proposition 3.2.4, a vector v in $T_{H,E}$ is represented by a commutative diagram:

$$\begin{array}{ccccc}
B & \longrightarrow & E & \longrightarrow & A \\
\downarrow & & {\scriptstyle \mathrm{KS}(v)}\downarrow & & \downarrow \\
B[1] & \longrightarrow & E[1] & \longrightarrow & A[1]
\end{array}.$$

$\theta_E(\mathrm{KS}(v))$ is equal to the composition $B \longrightarrow E \xrightarrow{\mathrm{KS}(v)} E[1] \longrightarrow A[1]$, which is zero since by using the commutativity of the diagram. Hence $T_{H,E}$ is mapped into K_E under KS. Since we have proved $\mathrm{dim}K_E = \mathrm{dim}T_{H,E}$, KS canonically induces an isomorphism between them. $\square$

We can also define $\theta_F : \mathrm{Ext}^1(F,F) \longrightarrow \mathrm{Ext}^1(A,B)$ for any nontrivial extension $0 \longrightarrow A \longrightarrow F \longrightarrow B \longrightarrow 0$ in a similar way. Denote its kernel by K_F, then we have :

Corollary 3.2.7. *The tangent space* $T_{\mathbf{P},F}$ *is canonically identified with* K_F *under the Kodaira-Spencer map.*

Proof. The reason that $T_{\mathbf{P},F}$ is mapped into K_F under the Kodaira-Spencer map is the same as in the case of Proposition 3.2.6. Conversely, take any $\zeta \in K_F$, we have that the composition $A \longrightarrow F \xrightarrow{\zeta} F[1] \longrightarrow B[1]$ is 0. By using the universal property of a triangle in the derived category, there exists morphisms $A \longrightarrow A[1]$ and $B \longrightarrow B[1]$ such that the following diagram is commutative:

$$\begin{array}{ccccc} A & \longrightarrow & F & \longrightarrow & B \\ \downarrow & & \zeta\downarrow & & \downarrow \\ A[1] & \longrightarrow & F[1] & \longrightarrow & B[1] \end{array}.$$

This diagram will correspond to an exact sequence of flat families on $\mathrm{Spec}\mathbb{C}[\varepsilon]/(\varepsilon^2) \times \mathbb{P}^3$

$$0 \longrightarrow \mathcal{F}_\varepsilon \longrightarrow \mathcal{F}'_\varepsilon \longrightarrow \mathcal{G}_\varepsilon \longrightarrow 0$$

where $\mathcal{F}_\varepsilon$, $\mathcal{F}'_\varepsilon$ and $\mathcal{G}_\varepsilon$ will restrict to A, F and B on the closed fiber. By the universal property of $\mathbf{P}$ proved in Proposition 3.2.3, this sequence induces a morphism from $\mathrm{Spec}\mathbb{C}[\varepsilon]/(\varepsilon^2)$ to $\mathbf{P}$ corresponding to a tangent vector v of $\mathbf{P}$ at F. It is not hard to check $\mathrm{KS}(v) = \zeta$, so KS is also surjective between $T_{\mathbf{P},F}$ and K_F. $\square$

We can use the exact sequence (3.1) to write down the following globalization of the diagram in Proposition 3.2.6.

Proposition 3.2.8. *The following diagram has exact rows. Among the three vertical morphisms, the left one and middle one are isomorphisms, and the right one is an injection.*

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{T}_H \arrow[r] \arrow[d] & \mathcal{T}_{\mathbf{M}_1}|_H \arrow[r] \arrow[d, "\mathrm{KS}"'] & \mathcal{N}_{H/\mathbf{M}_1} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \mathcal{K}_E \arrow[r] & \mathscr{E}xt^1_{\pi_H}(\mathcal{U}_E, \mathcal{U}_E) \arrow[r] & \mathscr{E}xt^1_{\pi_H}(\mathcal{G}_H \otimes \pi_H^*\mathcal{L}, \mathcal{F}_H) & \end{tikzcd}$$

From this proposition we see that the normal bundle $\mathcal{N}_{H/\mathbf{M}_1}$ embeds into $\mathscr{E}xt^1_{\pi_H}(\mathcal{G}_H \otimes \pi_H^*\mathcal{L}, \mathcal{F}_H)$, hence its projectivization $\mathbb{P}(\mathcal{N}^*_{H/\mathbf{M}_1})$ is embedded in

$$\mathbb{P}(\mathscr{E}xt^1_{\pi_H}(\mathcal{G}_H \otimes \pi_H^*\mathcal{L}, \mathcal{F}_H)^*) = \mathbb{P}(\mathscr{E}xt^1_{\pi_H}(\mathcal{G}_H, \mathcal{F}_H)^*),$$

where the latter is the preimage of H under the projection $\mathbb{P}(\mathscr{E}xt^1_\pi(\mathcal{G}, \mathcal{F})^*) = \mathbf{P} \longrightarrow \mathbb{P}^3 \times (\mathbb{P}^3)^*$.

Next we are going to compute the dimension of the Zariski tangent space $T_{\mathbf{M}_2,F} \cong \mathrm{Ext}^1(F,F)$ for a nontrivial extension $0 \longrightarrow A \longrightarrow F \longrightarrow B \longrightarrow 0$. First let us introduce some notations: we denote $e : A \longrightarrow B[1]$ the nontrivial extension of A by B and name the arrows $B \xrightarrow{h} E \xrightarrow{j} A$. Similarly let $f : B \longrightarrow A[1]$ be the extension we fix and name the arrows $A \xrightarrow{k} F \xrightarrow{l} B$. There are three cases and they are taken care of by the following three propositions.

Proposition 3.2.9. *If $F \in \mathbb{P}(\mathcal{N}^*_{H/\mathbf{M}_1})$, then we have the following commutative diagram with exact rows and columns. All boundary homomorphisms are 0 except at $\mathrm{Ext}^1(B,A)$, where the two homomorphisms $\mathrm{Ext}^1(F,A) \longleftarrow \mathrm{Ext}^1(B,A) \longrightarrow \mathrm{Ext}^1(B,F)$ have a same 1-dimensional kernel $\mathbb{C}f$.*

$$\begin{tikzcd}
\mathrm{Ext}^1(B,A)=\mathbb{C}^{10} \arrow[r] \arrow[d] & \mathrm{Ext}^1(F,A)=\mathbb{C}^{12} \arrow[r] \arrow[d] & \mathrm{Ext}^1(A,A)=\mathbb{C}^{3} \arrow[d] \\
\mathrm{Ext}^1(B,F)=\mathbb{C}^{12} \arrow[r] \arrow[d] & \mathrm{Ext}^1(F,F)=\mathbb{C}^{16} \arrow[r] \arrow[d] & \mathrm{Ext}^1(A,F)=\mathbb{C}^{4} \arrow[d] \\
\mathrm{Ext}^1(B,B)=\mathbb{C}^{3} \arrow[r] \arrow[d] & \mathrm{Ext}^1(F,B)=\mathbb{C}^{4} \arrow[r] \arrow[d, "0"'] & \mathrm{Ext}^1(A,B)=\mathbb{C} \arrow[d, "0"'] \\
0 \arrow[r] \arrow[d] & \mathrm{Ext}^2(F,A)=\mathbb{C}^{3} \arrow[r] \arrow[d] & \mathrm{Ext}^2(A,A)=\mathbb{C}^{3} \arrow[d] \\
0 \arrow[r] \arrow[d] & \mathrm{Ext}^2(F,F)=\mathbb{C}^{4} \arrow[r] \arrow[d] & \mathrm{Ext}^2(A,F)=\mathbb{C}^{4} \arrow[d] \\
0 \arrow[r] & \mathrm{Ext}^2(F,B)=\mathbb{C} \arrow[r] & \mathrm{Ext}^2(A,B)=\mathbb{C}
\end{tikzcd}$$

Proof. We show that the diagram holds if and only if $F \in \mathbb{P}(\mathcal{N}^*_{H/\mathbf{M}_1})$. If the diagram holds, then $\theta_F \neq 0$. We can find $\zeta \in \mathrm{Ext}^1(F,F)$ such that $e = l[1]\circ\zeta\circ k$. Now we have $f\circ e[-1] = f\circ l\circ\zeta[-1]\circ k[-1] = 0$ because $f\circ l = 0$. This means $f : B \longrightarrow A[1]$ factors through $h : B \longrightarrow E$, i.e. $f = x\circ h$ for some $x : E \longrightarrow A[1]$. On the other hand, from the diagram in Lemma 3.2.5 we see that $\mathrm{Ext}^1(E,E) \xrightarrow{j_*} \mathrm{Ext}^1(E,A)$ is surjective, hence $x : E \longrightarrow A[1]$ lifts to some $\xi : E \longrightarrow E[1]$. So we have $f = j[1]\circ\xi\circ h$ and f is in the image of θ_E. By Proposition 3.2.6, this means f is in $\mathbb{P}(\mathcal{N}^*_{H/\mathbf{M}_1})$. Conversely, if f is in $\mathbb{P}(\mathcal{N}^*_{H/\mathbf{M}_1})$, then we can write $f = j[1]\circ\xi\circ h$ for some nontrivial $\xi : E \longrightarrow E[1]$. Then $f[1]\circ e = j[2]\circ\xi[1]\circ h[1]\circ e = 0$ because $h[1]\circ e = 0$. This means $e : A \longrightarrow B[1]$ factors through $l[1] : F[1] \longrightarrow B[1]$, i.e. $e = l[1]\circ z$ for some $z : A \longrightarrow F[1]$. On the other hand, $\mathrm{Ext}^1(F,F) \xrightarrow{k^*} \mathrm{Ext}^1(A,F)$ is surjective because its cokernel $\mathrm{Ext}^2(B,F) = 0$. This implies that $z = \zeta\circ k$ for some $\zeta : E \longrightarrow E[1]$. So we have $e = l[1]\circ\zeta\circ k$ and e is in the image of θ_F. Therefore $\theta_F \neq 0$. By Corollary 4.7, the kernel of θ_F is $T_{\mathbf{P},F}$, which is 15-dimensional since $\mathbf{P}$ is a $\mathbb{P}^9$-bundle over $\mathbb{P}^3\times(\mathbb{P}^3)^*$.

Hence $\mathrm{Ext}^1(F,F) = \mathbb{C}^{16}$. The rest of the diagram will follow automatically due to exactness. $\square$

Proposition 3.2.10. *If $F \in \mathbb{P}(\mathscr{E}xt^1_{\pi_H}(\mathcal{G}_H, \mathcal{F}_H)^*) \setminus \mathbb{P}(\mathcal{N}^*_{H/\mathbf{M}_1})$, then we have the following commutative diagram with exact rows and columns. All boundary homomorphisms are* 0 *except at* $\mathrm{Ext}^1(B,A)$*, where the two homomorphisms* $\mathrm{Ext}^1(F,A) \longleftarrow \mathrm{Ext}^1(B,A) \longrightarrow \mathrm{Ext}^1(B,F)$ *have a same* 1*-dimensional kernel* $\mathbb{C}f$*.*

$$\begin{array}{ccccc}
\mathrm{Ext}^1(B,A)=\mathbb{C}^{10} & \longrightarrow & \mathrm{Ext}^1(F,A)=\mathbb{C}^{12} & \longrightarrow & \mathrm{Ext}^1(A,A)=\mathbb{C}^{3} \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{Ext}^1(B,F)=\mathbb{C}^{12} & \longrightarrow & \mathrm{Ext}^1(F,F)=\mathbb{C}^{15} & \longrightarrow & \mathrm{Ext}^1(A,F)=\mathbb{C}^{3} \\
\downarrow & & \downarrow & & {\scriptstyle 0}\downarrow \\
\mathrm{Ext}^1(B,B)=\mathbb{C}^{3} & \longrightarrow & \mathrm{Ext}^1(F,B)=\mathbb{C}^{4} & \longrightarrow & \mathrm{Ext}^1(A,B)=\mathbb{C} \\
\downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \mathrm{Ext}^2(F,A)=\mathbb{C}^{3} & \longrightarrow & \mathrm{Ext}^2(A,A)=\mathbb{C}^{3} \\
\downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \mathrm{Ext}^2(F,F)=\mathbb{C}^{3} & \longrightarrow & \mathrm{Ext}^2(A,F)=\mathbb{C}^{3} \\
\downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \mathrm{Ext}^2(F,B)=\mathbb{C} & \longrightarrow & \mathrm{Ext}^2(A,B)=\mathbb{C}
\end{array}$$

Proof. By the proof of previous proposition, we know that $\theta_F = 0$ since F is not in $\mathbb{P}(\mathcal{N}^*_{H/\mathbf{M}_1})$. Therefore $\mathrm{Ext}^1(F,F) = \mathbb{C}^{15}$. By Lemma 3.2.1, we know $\mathrm{Ext}^1(A,B) = \mathbb{C}$, since F is mapped into H under the bundle projection $\mathbf{P} \longrightarrow \mathbb{P}^3 \times (\mathbb{P}^3)^*$. The rest of the diagram then follows automatically due to exactness. $\square$

Proposition 3.2.11. *If $F \in \mathbf{P} \setminus \mathbb{P}(\mathscr{E}xt^1_{\pi_H}(\mathcal{G}_H, \mathcal{F}_H)^*)$, then we have the following commutative diagram with exact rows and columns. All boundary homomorphisms are* 0 *except at* $\mathrm{Ext}^1(B,A)$*, where the two homomorphisms* $\mathrm{Ext}^1(F,A) \longleftarrow \mathrm{Ext}^1(B,A) \longrightarrow \mathrm{Ext}^1(B,F)$ *have a same* 1*-dimensional kernel* $\mathbb{C}f$*.*

$$
\begin{array}{ccccc}
\mathrm{Ext}^1(B,A)=\mathbb{C}^{10} & \longrightarrow & \mathrm{Ext}^1(F,A)=\mathbb{C}^{12} & \longrightarrow & \mathrm{Ext}^1(A,A)=\mathbb{C}^{3} \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{Ext}^1(B,F)=\mathbb{C}^{12} & \longrightarrow & \mathrm{Ext}^1(F,F)=\mathbb{C}^{15} & \longrightarrow & \mathrm{Ext}^1(A,F)=\mathbb{C}^{3} \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{Ext}^1(B,B)=\mathbb{C}^{3} & \longrightarrow & \mathrm{Ext}^1(F,B)=\mathbb{C}^{3} & \longrightarrow & \mathrm{Ext}^1(A,B)=0 \\
\downarrow & & {\scriptstyle 0}\downarrow & & \downarrow \\
0 & \longrightarrow & \mathrm{Ext}^2(F,A)=\mathbb{C}^{3} & \longrightarrow & \mathrm{Ext}^2(A,A)=\mathbb{C}^{3} \\
\downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \mathrm{Ext}^2(F,F)=\mathbb{C}^{4} & \longrightarrow & \mathrm{Ext}^2(A,F)=\mathbb{C}^{4} \\
\downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \mathrm{Ext}^2(F,B)=\mathbb{C} & \longrightarrow & \mathrm{Ext}^2(A,B)=\mathbb{C}
\end{array}
$$

Proof. Since F is not in $\mathbb{P}(\mathcal{N}^*_{H/\mathbf{M}_1})$, we have $\theta_F = 0$ and $\mathrm{Ext}^1(F,F) = \mathbb{C}^{15}$. By Lemma 3.2.1, we know $\mathrm{Ext}^1(A,B) = 0$ since F is mapped outside H under the bundle projection $\mathbf{P} \longrightarrow \mathbb{P}^3 \times (\mathbb{P}^3)^*$. The rest of the diagram then follows automatically due to exactness. □

Remark 3.2.12. *From the above propositions, we can see that for $F \in \mathbf{P} \backslash \mathbb{P}(\mathcal{N}^*_{H/\mathbf{M}_1})$, $\mathbf{P}$ is smooth at F and $\dim T_{\mathbf{P},F} = \dim T_{\mathbf{M}_2,F} = 15$. By Proposition 3.2.4 (2), $T_{\varphi_F,F}$ is injective. This implies φ_F is an isomorphism at F and $\mathbf{M}_2$ is smooth at F.*

Elementary modification. In this subsection, we construct a flat family of λ_2-stable objects on the blow-up of $\mathbf{M}_1$ along H. The key is to perform a so-called elementary modification on the pullback of universal family of λ_1-stable objects along the exceptional divisor with respect to the extension (3.1) in Proposition 3.2.2.

Let us first introduce some notations: denote the blow-up of $\mathbf{M}_1$ along H by $\mathbf{B}$, the blow-up morphism $\mathbf{B} \times \mathbb{P}^3 \longrightarrow \mathbf{M}_1 \times \mathbb{P}^3$ by b and its restriction to the exceptional divisor $\mathbb{P}(\mathcal{N}^*_{H/\mathbf{M}_1}) \times \mathbb{P}^3 \longrightarrow H \times \mathbb{P}^3$ by b_H. Denote the universal family of λ_1-stable objects on $\mathbf{M}_1 \times \mathbb{P}^3$ by $\mathcal{U}_1$, then $\mathcal{U}_1|_{H\times\mathbb{P}^3}$ and $\mathcal{U}_E$ both induce the embedding $\varphi_E : H \longrightarrow \mathbf{M}_1$, so they differ from each other by tensoring a pullback of a line bundle from H via projection. Assume $\mathcal{U}_1|_{H\times\mathbb{P}^3} = \mathcal{U}_E \otimes \pi_H^*\mathcal{L}'$ for some line bundle $\mathcal{L}'$ on H. Consider the composition of the restriction map and the pullback of surjection in (3.1) by b_H:

$$b^*\mathcal{U}_1 \twoheadrightarrow b^*\mathcal{U}_1|_{\mathbb{P}(\mathcal{N}^*_{H/\mathbf{M}_1})\times\mathbb{P}^3} = b_H^*\mathcal{U}_E \otimes b_H^*\pi_H^*\mathcal{L}' \twoheadrightarrow b_H^*\mathcal{F}_H \otimes b_H^*\pi_H^*\mathcal{L}'$$

Denote the kernel of this composition by $\mathcal{K}$ then we have:

Proposition 3.2.13. *The sheaf $\mathcal{K}$ is a flat family of λ_2-stable objects.*

Proof. $\mathcal{K}$ is a flat family of λ_2-stable objects outside the exceptional divisor because it is identical to $\mathcal{U}_1$. If we restrict the exact sequence $0 \longrightarrow \mathcal{K} \longrightarrow b^*\mathcal{U}_1 \longrightarrow b_H^*\mathcal{F}_H \otimes b_H^*\pi_H^*\mathcal{L}' \longrightarrow 0$ to the exceptional divisor $\mathbb{P}(\mathcal{N}^*_{H/\mathbf{M}_1}) \times \mathbb{P}^3$, we will get

$$\begin{aligned} 0 \longrightarrow \mathscr{T}or^1(b_H^*\mathcal{F}_H \otimes b_H^*\pi_H^*\mathcal{L}', \mathcal{O}_{\mathbb{P}(\mathcal{N}^*_{H/\mathbf{M}_1})\times\mathbb{P}^3}) \longrightarrow \mathcal{K}|_{\mathbb{P}(\mathcal{N}^*_{H/\mathbf{M}_1})\times\mathbb{P}^3} \longrightarrow \\ b_H^*\mathcal{U}_E \otimes b_H^*\pi_H^*\mathcal{L}' \longrightarrow b_H^*\mathcal{F}_H \otimes b_H^*\pi_H^*\mathcal{L}' \longrightarrow 0 \end{aligned}$$

On the other hand, tensoring $b_H^*\mathcal{F}_H \otimes b_H^*\pi_H^*\mathcal{L}'$ to the exact sequence

$$0 \longrightarrow \mathcal{I}_{\mathbb{P}(\mathcal{N}^*_{H/\mathbf{M}_1})\times\mathbb{P}^3} \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}_{\mathbb{P}(\mathcal{N}^*_{H/\mathbf{M}_1})\times\mathbb{P}^3} \longrightarrow 0,$$

we have

$$\begin{aligned} 0 \longrightarrow \mathscr{T}or^1(b_H^*\mathcal{F}_H \otimes b_H^*\pi_H^*\mathcal{L}', \mathcal{O}_{\mathbb{P}(\mathcal{N}^*_{H/\mathbf{M}_1})\times\mathbb{P}^3}) \xrightarrow{=} b_H^*\mathcal{F}_H \otimes b_H^*\pi_H^*\mathcal{L}' \otimes \mathcal{I}_{\mathbb{P}(\mathcal{N}^*_{H/\mathbf{M}_1})\times\mathbb{P}^3} \\ \xrightarrow{0} b_H^*\mathcal{F}_H \otimes b_H^*\pi_H^*\mathcal{L}' \xrightarrow{=} b_H^*\mathcal{F}_H \otimes b_H^*\pi_H^*\mathcal{L}' \longrightarrow 0. \end{aligned}$$

Hence

$$\begin{aligned} \mathscr{T}or^1(b_H^*\mathcal{F}_H \otimes b_H^*\pi_H^*\mathcal{L}', \mathcal{O}_{\mathbb{P}(\mathcal{N}^*_{H/\mathbf{M}_1})\times\mathbb{P}^3}) &= b_H^*\mathcal{F}_H \otimes b_H^*\pi_H^*\mathcal{L}' \otimes \mathcal{I}_{\mathbb{P}(\mathcal{N}^*_{H/\mathbf{M}_1})\times\mathbb{P}^3} \\ &= b_H^*\mathcal{F}_H \otimes b_H^*\pi_H^*\mathcal{L}' \otimes \mathcal{N}^*_{\mathbb{P}(\mathcal{N}^*_{H/\mathbf{M}_1})\times\mathbb{P}^3}. \end{aligned}$$

Also notice that the kernel of

$$b_H^*\mathcal{U}_E \otimes b_H^*\pi_H^*\mathcal{L}' \longrightarrow b_H^*\mathcal{F}_H \otimes b_H^*\pi_H^*\mathcal{L}'$$

is $b_H^*\mathcal{G}_H \otimes b_H^*\pi_H^*\mathcal{L} \otimes b_H^*\pi_H^*\mathcal{L}'$, so $\mathcal{K}|_{\mathbb{P}(\mathcal{N}^*_{H/\mathbf{M}_1})\times\mathbb{P}^3}$ satisfies

$$0 \longrightarrow b_H^*\mathcal{F}_H \otimes b_H^*\pi_H^*\mathcal{L}' \otimes \mathcal{N}^*_{\mathbb{P}(\mathcal{N}^*_{H/\mathbf{M}_1})\times\mathbb{P}^3} \longrightarrow \mathcal{K}|_{\mathbb{P}(\mathcal{N}^*_{H/\mathbf{M}_1})\times\mathbb{P}^3} \longrightarrow \\ b_H^*\mathcal{G}_H \otimes b_H^*\pi_H^*\mathcal{L} \otimes b_H^*\pi_H^*\mathcal{L}' \longrightarrow 0. \qquad (3.3)$$

This means on each fiber $x \times \mathbb{P}^3$, the restriction $\mathcal{K}_x$ is an extension of B by A. In particular $\mathcal{K}_x$ has the same Chern character as other fibers, therefore $\mathcal{K}$ is flat since $\mathbf{B}$ is smooth. To prove it is a family of λ_2-stable objects, we need to show $\mathcal{K}_x$ is a nontrivial extension of B by A. Actually since $x \in \mathbb{P}(\mathcal{N}^*_{H/\mathbf{M}_1})$ represents a nonzero normal direction of H in $\mathbf{M}_1$, we expect $\mathcal{K}_x$ to be $\theta_E(\mathrm{KS}(x))$ in $\mathrm{Ext}^1(B, A)$. This is indeed the case because $\mathcal{K}|_{\mathbb{P}(\mathcal{N}^*_{H/\mathbf{M}_1})\times\mathbb{P}^3}$ can be interpreted in the following way: First we use the injection

$$b_H^*\mathcal{G}_H \otimes b_H^*\pi_H^*\mathcal{L} \otimes b_H^*\pi_H^*\mathcal{L}' \longrightarrow b_H^*\mathcal{U}_E \otimes b_H^*\pi_H^*\mathcal{L}'$$

to pull back the exact sequence

$$0 \longrightarrow b^*\mathcal{U}_1 \otimes \mathcal{I}_{\mathbb{P}(\mathcal{N}^*_{H/\mathbf{M}_1})\times\mathbb{P}^3} \longrightarrow b^*\mathcal{U}_1 \longrightarrow b_H^*\mathcal{U}_E \otimes b_H^*\pi_H^*\mathcal{L}' \longrightarrow 0,$$

we get

$$0 \longrightarrow b^*\mathcal{U}_1 \otimes \mathcal{I}_{\mathbb{P}(\mathcal{N}^*_{H/\mathbf{M}_1})\times\mathbb{P}^3} \longrightarrow \mathcal{K} \longrightarrow b_H^*\mathcal{G}_H \otimes b_H^*\pi_H^*\mathcal{L} \otimes b_H^*\pi_H^*\mathcal{L}' \longrightarrow 0.$$

Then we push out the resulting exact sequence using the surjection

$$b^*\mathcal{U}_1 \otimes \mathcal{I}_{\mathbb{P}(\mathcal{N}^*_{H/\mathbf{M}_1})\times\mathbb{P}^3} \longrightarrow b_H^*\mathcal{F}_H \otimes b_H^*\pi_H^*\mathcal{L}' \otimes \mathcal{I}_{\mathbb{P}(\mathcal{N}^*_{H/\mathbf{M}_1})\times\mathbb{P}^3} = b_H^*\mathcal{F}_H \otimes b_H^*\pi_H^*\mathcal{L}' \otimes \mathcal{N}^*_{\mathbb{P}(\mathcal{N}^*_{H/\mathbf{M}_1})\times\mathbb{P}^3},$$

we will get (3.3). On a fiber $x \times \mathbb{P}^3$, this means first we take an extension

$$0 \longrightarrow E \longrightarrow G \longrightarrow E \longrightarrow 0$$

representing $x \in \mathrm{Ext}^1(E,E)$, then do a pullback using $B \longrightarrow E$ followed by a pushout using $E \longrightarrow A$. The resulting extension

$$0 \longrightarrow A \longrightarrow \mathcal{K}_x \longrightarrow B \longrightarrow 0$$

is exactly $\theta_E(\mathrm{KS}(x))$. This shows that $\mathcal{K}$ is a flat family of λ_2-stable objects. $\square$

If we denote the induced morphism of $\mathcal{K}$ by $\delta : \mathbf{B} \longrightarrow \mathbf{M}_2$, then

Proposition 3.2.14. *(1) The induced morphism δ is an isomorphism outside $\mathbb{P}(\mathcal{N}^*_{H/\mathbf{M}_1})$, and the restriction $\delta|_{\mathbb{P}(\mathcal{N}^*_{H/\mathbf{M}_1})}$ coincides with $\varphi_F|_{\mathbb{P}(\mathcal{N}^*_{H/\mathbf{M}_1})}$;*

(2) The induced morphism δ is injective on the level of sets and Zariski tangent spaces.

Proof. δ is an isomorphism outside $\mathbb{P}(\mathcal{N}^*_{H/\mathbf{M}_1})$ because $\mathcal{K}$ is the same with $\mathcal{U}_1$. On the other hand, under the identification

$$\begin{aligned}
&\mathrm{Ext}^1\left(b_H^*\mathcal{G}_H \otimes b_H^*\pi_H^*\mathcal{L} \otimes b_H^*\pi_H^*\mathcal{L}', b_H^*\mathcal{F}_H \otimes b_H^*\pi_H^*\mathcal{L}' \otimes \mathcal{N}^*_{\mathbb{P}(\mathcal{N}^*_{H/\mathbf{M}_1})\times\mathbb{P}^3}\right)\\
&= \mathrm{Ext}^1\left(b_H^*\mathcal{G}_H \otimes b_H^*\pi_H^*\mathcal{L}, b_H^*\mathcal{F}_H \otimes \mathcal{N}^*_{\mathbb{P}(\mathcal{N}^*_{H/\mathbf{M}_1})\times\mathbb{P}^3}\right)\\
&= \mathrm{H}^0\left(\mathbb{P}(\mathcal{N}^*_{H/\mathbf{M}_1}), \mathscr{E}xt^1_{\pi_{\mathbb{P}(\mathcal{N}^*_{H/\mathbf{M}_1})}}\left(b_H^*\mathcal{G}_H \otimes b_H^*\pi_H^*\mathcal{L}, b_H^*\mathcal{F}_H \otimes \pi^*_{\mathbb{P}(\mathcal{N}^*_{H/\mathbf{M}_1})}\mathcal{O}_{\mathbb{P}(\mathcal{N}^*_{H/\mathbf{M}_1})}(1)\right)\right)\\
&= \mathrm{H}^0\left(H, \mathscr{E}xt^1_{\pi_H}\left(\mathcal{G}_H \otimes \pi_H^*\mathcal{L}, \mathcal{F}_H\right) \otimes \mathcal{N}^*_{H/\mathbf{M}_1}\right)\\
&= \mathrm{Hom}\left(\mathcal{N}_{H/\mathbf{M}_1}, \mathscr{E}xt^1_{\pi_H}\left(\mathcal{G}_H \otimes \pi_H^*\mathcal{L}, \mathcal{F}_H\right)\right),
\end{aligned}$$

the extension (3.3) corresponds to the injection i from $\mathcal{N}_{H/\mathbf{M}_1}$ to $\mathscr{E}xt^1_{\pi_H}(\mathcal{G}_H \otimes \pi_H^*\mathcal{L}, \mathcal{F}_H)$ constructed in Proposition 3.2.8 via the Kodaira-Spencer map. Similarly in Proposition 3.2.3, the extension (3.2) corresponds to the identity id in $\mathrm{Hom}(\mathscr{E}xt^1_\pi(\mathcal{G},\mathcal{F}), \mathscr{E}xt^1_\pi(\mathcal{G},\mathcal{F})) =$

$\mathrm{Ext}^1(h^*\mathcal{G}, h^*\mathcal{F}\otimes\pi_{\mathbf{P}}\mathcal{O}_{\mathbf{P}}(1))$. Notice that i is the restriction of id to $\mathcal{N}_{H/\mathbf{M}_1}$, this means (3.3) is a restriction of (3.2) to $\mathbb{P}(\mathcal{N}^*_{H/\mathbf{M}_1})\times\mathbb{P}^3$ up to tensoring a pullback of some line bundle on $\mathbb{P}(\mathcal{N}^*_{H/\mathbf{M}_1})$. Therefore $\delta|_{\mathbb{P}(\mathcal{N}^*_{H/\mathbf{M}_1})\times\mathbb{P}^3} = \varphi_F|_{\mathbb{P}(\mathcal{N}^*_{H/\mathbf{M}_1})\times\mathbb{P}^3}$. In particular, $\delta|_{\mathbb{P}(\mathcal{N}^*_{H/\mathbf{M}_1})\times\mathbb{P}^3}$ is injective on the level of Zariski tangent spaces since φ_F is. To show δ is injective on the level of Zariski tangent spaces, it only remains to show that the normal direction v_x of $\mathbb{P}(\mathcal{N}^*_{H/\mathbf{M}_1})$ in $\mathbf{B}$ at a point $x\in\mathbb{P}(\mathcal{N}^*_{H/\mathbf{M}_1})$ is not sent to the image of $T_{\mathbb{P}(\mathcal{N}^*_{H/\mathbf{M}_1}),x}$ under $T_{\delta,x}$. If it were so, we suppose $\xi : \mathrm{Spec}\mathbb{C}[\varepsilon]/(\varepsilon^2)\longrightarrow\mathbf{B}$ represents v_x. Notice that we have a pullback diagram

$$\begin{array}{ccc} \mathbb{P}(\mathcal{N}^*_{H/\mathbf{M}_1}) & \longrightarrow & \mathbf{P} \\ \Big\downarrow & & \Big\downarrow{\scriptstyle \varphi_F} \\ \mathbf{B} & \xrightarrow{\delta} & \mathbf{M}_2 \end{array}$$

since $\delta(\mathbf{B})\cap\varphi_F(\mathbf{P}) = \delta(\mathbb{P}(\mathcal{N}^*_{H/\mathbf{M}_1}))$. Because $T_{\delta,x}(T_{\mathbb{P}(\mathcal{N}^*_{H/\mathbf{M}_1}),x})$ is contained in $T_{\varphi_F,x}$, we can lift $\delta\circ\xi$ to $\xi' : \mathrm{Spec}\mathbb{C}[\varepsilon]/(\varepsilon^2)\longrightarrow\mathbf{P}$ that makes the pullback diagram above commutative, hence ξ factors through $\mathbb{P}(\mathcal{N}^*_{H/\mathbf{M}_1})$. This implies v_x is in $T_{\mathbb{P}(\mathcal{N}^*_{H/\mathbf{M}_1}),x}$, which is a contradiction. □

Remark 3.2.15. *(1) The last argument also shows that the normal direction v_x is not mapped to the image of $T_{\mathbf{P},\mathcal{K}_x}$ under $T_{\varphi_F,F}$. By Corollary 4.7, $T_{\varphi_F,F}(T_{\mathbf{P},\mathcal{K}_x})$ is the kernel of θ_F, so we must have $\theta_F(v_x)\neq 0$;*

(2) Since $T_{\varphi_F,F}(T_{\mathbf{P},F}) = \mathbb{C}^{15}$ and $T_{\delta,F}(T_{\mathbf{B},F}) = \mathbb{C}^{12}$, the pullback diagram in the above proof also implies $T_{\varphi_F,F}(T_{\mathbf{P},F})\cap T_{\delta,F}(T_{\mathbf{B},F}) = T_{\delta,F}(T_{\mathbb{P}(\mathcal{N}^*_{H/\mathbf{M}_1}),F}) = \mathbb{C}^{11}$.

Obstruction computation. In this subsection, we study the deformation theory of complexes on the intersection of the two irreducible components of $\mathbf{M}_2$. We give explicit local equations defining $\mathbf{M}_2$ at a point in the intersection. In particular, this will imply the two irreducible components of $\mathbf{M}_2$ intersect transversely.

Recall that we have constructed two morphisms $\delta : \mathbf{B} \longrightarrow \mathbf{M}_2$ and $\varphi_F : \mathbf{P} \longrightarrow \mathbf{M}_2$, both of them are injective on the level of sets and Zariski tangent spaces. By the definition of a simple wall-crossing, any λ_2-stable object has to lie in the image of one of the two morphisms. Thus $\mathbf{M}_2$ has two irreducible components corresponding to the image of δ and φ_F. The intersection of the two components is the image of the exceptional divisor $\mathbb{P}(\mathcal{N}^*_{H/\mathbf{M}_1})$ by Proposition 3.2.14. Outside the intersection of the two components, $\mathbf{M}_2$ is smooth by Remark 3.2.12 and Remark 3.2.15 (1). To study the singularity of $\mathbf{M}_2$, we fix an λ_2-semistable object F in $\mathbb{P}(\mathcal{N}^*_{H/\mathbf{M}_1})$, then we have

Proposition 3.2.16. *The tangent vectors of $\mathbf{M}_2$ at F in the subspaces $T_{\varphi_F,F}(T_{\mathbf{P},F})$ and $T_{\delta,F}(T_{\mathbf{B},F})$ correspond to miniversal deformations of F.*

Proof. Suppose a Zariski tangent vector of $\mathbf{M}_2$ at F in $T_{\varphi_F,F}(T_{\mathbf{P},F})$ is represented by a morphism $\eta : \mathrm{Spec}\mathbb{C}[\varepsilon]/(\varepsilon^2) \longrightarrow \mathbf{M}_2$, then η factors through $\varphi_F : \mathbf{P} \longrightarrow \mathbf{M}_2$:

$$\begin{tikzcd} \mathrm{Spec}\mathbb{C}[\varepsilon]/(\varepsilon^2) \arrow[r] \arrow[d, "\eta'"] \arrow[dr, "\eta"] & \mathrm{Spec}S \arrow[dl, "\xi"'] \\ \mathbf{P} \arrow[r, "\varphi_F"] & \mathbf{M}_2 \end{tikzcd}$$

If S is a finite dimensional local Artin $\mathbb{C}$-algebra with a local surjection $S \longrightarrow \mathbb{C}[\varepsilon]/(\varepsilon^2)$, then we can lift η' to $\xi : \mathrm{Spec}S \longrightarrow \mathbf{P}$ since $\mathbf{P}$ is smooth. By composing ξ with φ_F, we get a lift of η. Hence η corresponds to a miniversal deformation. A similar argument works for tangent vectors in $T_{\delta,F}(T_{\mathbf{B},F})$. $\square$

In order to show $T_{\varphi_F,F}(T_{\mathbf{P},F})$ and $T_{\delta,F}(T_{\mathbf{B},F})$ are all the miniversal deformations of F, we study the quadratic part of the Kuranishi map $\kappa_2 : T_{\mathbf{M}_2,F} \cong \mathrm{Ext}^1(F,F) \longrightarrow \mathrm{Ext}^2(F,F)$. First we give a decomposition of $T_{\mathbf{M}_2,F} \cong \mathrm{Ext}^1(F,F)$ with respect to some geometric structures. In the blow-up $\mathbf{B}$, we have $T_{\mathbf{B},F} = N_{\mathbb{P}(\mathcal{N}^*_{H/\mathbf{M}_1})/\mathbf{B},F} \oplus T_{\mathbb{P}(\mathcal{N}^*_{H/\mathbf{M}_1}),F}$

and $N_{\mathbb{P}(\mathcal{N}^*_{H/\mathbf{M}_1})/\mathbf{B},F}$ is 1-dimensional. Suppose it is generated by a vector v_F, then we have

Proposition 3.2.17. *The Zariski tangent space $T_{\mathbf{M}_2,F} \cong \mathrm{Ext}^1(F,F)$ has the following decomposition*

$$T_{\mathbf{M}_2,F} = \mathbb{C}v_F \oplus T_{\mathbb{P}(N^*_{H/\mathbf{M}_1,E}),F} \oplus N_{\mathbb{P}(N^*_{H/\mathbf{M}_1,E})/\mathbb{P}(\mathrm{Ext}^1(B,A)^*),F} \oplus T_{H,E} \oplus N_{H/\mathbb{P}^3\times(\mathbb{P}^3)^*,E}. \quad (3.4)$$

In this decomposition,

$$T_{\delta,F}(T_{\mathbf{B},F}) = \mathbb{C}v_F \oplus T_{\mathbb{P}(N^*_{H/\mathbf{M}_1,E}),F} \oplus T_{H,E}$$

$$T_{\varphi_F,F}(T_{\mathbf{P},F}) = T_{\mathbb{P}(N^*_{H/\mathbf{M}_1,E}),F} \oplus N_{\mathbb{P}(N^*_{H/\mathbf{M}_1,E})/\mathbb{P}(\mathrm{Ext}^1(B,A)^*),F} \oplus T_{H,E} \oplus N_{H/\mathbb{P}^3\times(\mathbb{P}^3)^*,E}$$

Proof. By Remark 3.2.15 (1), $\theta_F(v_F) \neq 0$, hence we can decompose $\mathrm{Ext}^1(F,F) = \mathbb{C}v_F \oplus T_{\mathbf{P},F}$ because the kernel of θ_F is $T_{\mathbf{P},F}$. On the other hand, $\mathbf{P} = \mathbb{P}(\mathscr{E}xt^1_\pi(\mathcal{G},\mathcal{F})^*)$ is a projective bundle over $\mathbb{P}^3\times(\mathbb{P}^3)^*$, so we have $T_{\mathbf{P},F} = T_{\mathbb{P}(\mathrm{Ext}^1(B,A)^*),F} \oplus T_{\mathbb{P}^3\times(\mathbb{P}^3)^*,(A,B)}$. To give further decomposition, denote E the nontrivial extension of A by B, we have that $\mathbb{P}(N^*_{H/\mathbf{M}_1,E})$ is embedded in $\mathbb{P}(\mathrm{Ext}^1(B,A)^*)$ via the Kodaira-Spencer map by Proposition 3.2.6, so $T_{\mathbb{P}(\mathrm{Ext}^1(B,A)^*),F} = T_{\mathbb{P}(N^*_{H/\mathbf{M}_1,E}),F} \oplus N_{\mathbb{P}(N^*_{H/\mathbf{M}_1,E})/\mathbb{P}(\mathrm{Ext}^1(B,A)^*),F}$. Also notice that the incidence hyperplane H is embedded in $\mathbb{P}^3\times(\mathbb{P}^3)^*$, so $T_{\mathbb{P}^3\times(\mathbb{P}^3)^*,(A,B)} = T_{H,E} \oplus N_{H/\mathbb{P}^3\times(\mathbb{P}^3)^*,E}$. By composing all the decompositions above, we have the proposition. $\square$

The importance of this decomposition is that some of the summands have direct relations with the Ext^2 groups in Lemma 3.2.5, Proposition 3.2.6 and Proposition 3.2.9, which becomes crucial later when we compute κ_2. Fix a nontrivial $\zeta \in \mathrm{Ext}^1(F,F)$. Let $e : A \longrightarrow B[1]$ correspond to the nontrivial extension E and $f : B \longrightarrow A[1]$

correspond to F, name the arrows $A \xrightarrow{k} F \xrightarrow{l} B$. Then we have the following two lemmas:

Lemma 3.2.18. *The normal space $N_{\mathbb{P}(N^*_{H/\mathbf{M}_1,E})/\mathbb{P}(\mathrm{Ext}^1(B,A)^*),F}$ can be identified with $\mathrm{Ext}^2(A,A)$ under a canonical isomorphism. If ζ belongs to $N_{\mathbb{P}(N^*_{H/\mathbf{M}_1,E})/\mathbb{P}(\mathrm{Ext}^1(B,A)^*),F}$ in (3.4), then $\zeta = k[1] \circ t \circ l$ for some $t \in \mathrm{Ext}^1(B,A)$ such that $t[1] \circ e$ is nonzero in $\mathrm{Ext}^2(A,A)$.*

Proof. By Lemma 3.2.5, we know that the cokernel of $\theta_E : \mathrm{Ext}^1(E,E) \longrightarrow \mathrm{Ext}^1(B,A)$ is $\mathrm{Ext}^2(A,A)$. By Proposition 3.2.6, we know that the Kodaira-Spencer map KS induces an isomorphism between the image of θ_E and $N_{H/\mathbf{M}_1,E}$. On the other hand, $N_{\mathbb{P}(N^*_{H/\mathbf{M}_1,E})/\mathbb{P}(\mathrm{Ext}^1(B,A)^*),F}$ is equal to the quotient $\mathrm{Ext}^1(B,A)/N_{H/\mathbf{M}_1,E}$, so we have

$$N_{\mathbb{P}(N^*_{H/\mathbf{M}_1,E})/\mathbb{P}(\mathrm{Ext}^1(B,A)^*),F} \cong \mathrm{Ext}^2(A,A).$$

To prove the second statement, we look at the square

$$\begin{array}{ccc}
\mathrm{Ext}^1(B,A) & \xrightarrow{l^*} & \mathrm{Ext}^1(F,A) \\
{\scriptstyle k[1]_*}\downarrow & & {\scriptstyle k[1]_*}\downarrow \\
\mathrm{Ext}^1(B,F) & \xrightarrow{l^*} & \mathrm{Ext}^1(F,F)
\end{array}$$

in Proposition 3.2.9. There is an injection $\mathrm{Ext}^1(B,A)/\mathbb{C}f \longrightarrow \mathrm{Ext}^1(F,F)$, which is the same as $T_{\mathbb{P}(\mathrm{Ext}^1(B,A)^*),F} \longrightarrow \mathrm{Ext}^1(F,F)$. Notice the fact that $N_{\mathbb{P}(N^*_{H/\mathbf{M}_1,E})/\mathbb{P}(\mathrm{Ext}^1(B,A)^*),F}$ is contained in $T_{\mathbb{P}(\mathrm{Ext}^1(B,A)^*),F}$, ζ has to be in $T_{\mathbb{P}(\mathrm{Ext}^1(B,A)^*),F}$, this means $\zeta = k[1] \circ t \circ l$ for some $t \in \mathrm{Ext}^1(B,A)$. For ζ to be nontrivial and lying in $\mathrm{Ext}^2(A,A)$, t has to be nonzero under the cokernel map $(-)[1] \circ e : \mathrm{Ext}^1(B,A) \longrightarrow \mathrm{Ext}^2(A,A)$, so $t[1] \circ e \neq 0$ □

Lemma 3.2.19. *The normal space $N_{H/\mathbb{P}^3\times(\mathbb{P}^3)^*,E}$ can be identified with $\mathrm{Ext}^2(A,B)$ under a canonical isomorphism. If ζ belongs to $N_{H/\mathbb{P}^3\times(\mathbb{P}^3)^*,E}$ in (3.4), then ζ can* *be completed to the following commutative diagram with* $e[1] \circ t + r[1] \circ e \neq 0$ *in* $\mathrm{Ext}^2(A, B)$*:*

$$\begin{array}{ccccc} A & \xrightarrow{k} & F & \xrightarrow{l} & B \\ {\scriptstyle t}\downarrow & & {\scriptstyle \zeta}\downarrow & & {\scriptstyle r}\downarrow \\ A[1] & \xrightarrow{k[1]} & F[1] & \xrightarrow{l[1]} & B[1] \end{array}$$

Proof. Recall that K_E is the kernel of θ_E, and by Proposition 3.2.6 it can be identified with $T_{H,E}$ via the Kodaira-Spencer map. From the diagram in Lemma 3.2.5, we have an exact sequence

$$0 \longrightarrow K_E \longrightarrow \mathrm{Ext}^1(A, A) \oplus \mathrm{Ext}^1(B, B) \xrightarrow{(e[1]\circ -)+(-[1]\circ e)} \mathrm{Ext}^2(A, B) \longrightarrow 0\cdot$$

On the other hand, we have the canonical normal sequence of H embedded in $\mathbb{P}^3 \times (\mathbb{P}^3)^*$

$$0 \longrightarrow T_{H,E} \longrightarrow T_{\mathbb{P}^3 \times (\mathbb{P}^3)^*, (A,B)} \longrightarrow N_{H/\mathbb{P}^3 \times (\mathbb{P}^3)^*, E} \longrightarrow 0,$$

since $\mathrm{Ext}^1(A, A) \oplus \mathrm{Ext}^1(B, B)$ can also be identified with $T_{\mathbb{P}^3 \times (\mathbb{P}^3)^*, (A,B)}$ via the Kodaira-Spencer map, this induces a canonical isomorphism between $N_{H/\mathbb{P}^3 \times (\mathbb{P}^3)^*, E}$ and $\mathrm{Ext}^2(A, B)$.

Notice that $N_{H/\mathbb{P}^3 \times (\mathbb{P}^3)^*, E}$ is contained in $T_{\mathbf{P},F}$ and the latter is kernel of θ_F. We have $\theta_F(\zeta) = 0$. By using the universal property of triangles, ζ can be completed to a commutative diagram:

$$\begin{array}{ccccc} A & \xrightarrow{k} & F & \xrightarrow{l} & B \\ {\scriptstyle t}\downarrow & & {\scriptstyle \zeta}\downarrow & & {\scriptstyle r}\downarrow \\ A[1] & \xrightarrow{k[1]} & F[1] & \xrightarrow{l[1]} & B[1] \end{array} .$$

Since ζ is nontrivial, (t, r) has to be sent to a nonzero element in $\mathrm{Ext}^2(A, B)$ under the last map of the exact sequence above, therefore $e[1] \circ t + r[1] \circ e \neq 0$. □

With respect to the decomposition (3.4), we let

$$\zeta = u_1 v_F + w_1 + u_2 s_1 + u_3 s_2 + u_4 s_3 + w_2 + u_5 s_4, \tag{3.5}$$

where $w_1 \in T_{\mathbb{P}(N^*_{H/\mathbf{M}_1,E}),F}$, $\{s_1, s_2, s_3\}$ forms a basis of $N_{\mathbb{P}(N^*_{H/\mathbf{M}_1,E})/\mathbb{P}(\mathrm{Ext}^1(B,A)^*),F}$, $w_2 \in T_{H,E}$, $\{s_4\}$ is a basis of $N_{H/\mathbb{P}^3\times(\mathbb{P}^3)^*,E}$ and $u_i \in \mathbb{C}$ are coefficients. (3.5) is inspired by the explicit basis chosen in the proof of [PS85, Lemma 6]. In the next theorem, we will see that the equations cutting out miniversal deformations by using (3.5) is the same as using Piene and Schlessinger's basis in the case of deformations of ideals.

Proposition 3.2.20. *The quadratic part of Kuranishi map takes the following form with respect to (3.5)*

$$\kappa_2(\zeta) = \zeta \cup \zeta = \sum_{i=1}^{4} u_1 u_{i+1}(v_F + s_i) \cup (v_F + s_i),$$

where $\cup$ *is the Yoneda pairing of extensions.* $\{(v_F + s_i) \cup (v_F + s_i) | i = 1, 2, 3, 4\}$ *forms a basis of the obstruction space* $\mathrm{Ext}^2(F, F)$*.*

Proof. The equality $\kappa_2(\zeta) = \zeta \cup \zeta$ is known for complexes in [Ina02], [Lie06] and [KLS06]. The second equality is a straightforward computation. It only uses the fact that for any v in $T_{\mathbf{B},F}$ or $T_{\mathbf{P},F}$, we have $v \cup v = 0$ since v is a miniversal deformation by Proposition 3.2.16.

To prove the last statement, we first show that $\{(v_F + s_i) \cup (v_F + s_i) | i = 1, 2, 3\}$ is linearly independent. If not, then a certain nontrivial linear combination $\sum_{i=1}^{3} a_i(v_F + s_i) \cup (v_F + s_i) = 0$. We can rewrite it as $v_F[1] \circ s + s[1] \circ v_F = 0$, where $s = \sum_{i=1}^{3} a_i s_i$ is a nontrivial first deformation of F in $N_{\mathbb{P}(N^*_{H,E})/\mathbb{P}(\mathrm{Ext}^1(B,A)^*),F}$. By Lemma 3.2.18, we can write $s = k[1] \circ t \circ l$ for some $t \in \mathrm{Ext}^1(B, A)$ such that $t[1] \circ e$ is nonzero in $\mathrm{Ext}^2(A, A)$. Now

$$\begin{aligned} 0 &= (v_F[1] \circ s + s[1] \circ v_F) \circ k \\ &= v_F[1] \circ k[1] \circ t \circ l \circ k + k[2] \circ t[1] \circ l[1] \circ v_F \circ k. \end{aligned}$$

Since $l \circ k = 0$ and $l[1] \circ v_F \circ k = \theta_F(v_F) = e$, we have $k[2] \circ t[1] \circ e = 0$. From the diagram in Proposition 3.2.9, we know that $\mathrm{Ext}^2(A, A) \xrightarrow{k[2]_*} \mathrm{Ext}^2(A, F)$ is an injection, hence $t[1] \circ e = 0$, which is a contradiction.

It only remains to show that $(v_F + s_4) \cup (v_F + s_4)$ is not a linear combination of $\{(v_F + s_i) \cup (v_F + s_i) | i = 1, 2, 3\}$. For this we will show for $i = 1, 2, 3$

$$l[2] \circ ((v_F + s_i) \cup (v_F + s_i)) = 0,$$

$$l[2] \circ ((v_F + s_4) \cup (v_F + s_4)) \neq 0.$$

By Lemma 3.2.18, we can assume $s_i = k[1] \circ t_i \circ l$ for some $t_i \in \mathrm{Ext}^1(B, A)$ satisfying $t_i[1] \circ e \neq 0$. Then

$$\begin{aligned}&l[2] \circ ((v_F + s_i) \cup (v_F + s_i)) \\ =&l[2] \circ v_F[1] \circ k[1] \circ t_i \circ l + l[2] \circ k[2] \circ t_i[1] \circ l[1] \circ v_F.\end{aligned}$$

Since $l[2] \circ v_F[1] \circ k[1] = e[1]$ and $l[2] \circ k[2] = 0$, we have $l[2] \circ ((v_F + s_i) \cup (v_F + s_i)) = e[1] \circ t_i \circ l$. Notice that $e[1] \circ t_i \in \mathrm{Ext}^2(B, B) = 0$, so $l[2] \circ ((v_F + s_i) \cup (v_F + s_i)) = 0$. On the other hand, s_4 is a nontrivial element in $N_{H/\mathbb{P}^3 \times (\mathbb{P}^3)^*, E}$. By Lemma 3.2.19, s_4 can be completed to the following commutative diagram with $e[1] \circ t_4 + r_4[1] \circ e \neq 0$ in $\mathrm{Ext}^2(A, B)$:

$$\begin{array}{ccccc} A & \xrightarrow{k} & F & \xrightarrow{l} & B \\ {\scriptstyle t_4}\downarrow & & {\scriptstyle s_4}\downarrow & & {\scriptstyle r_4}\downarrow \\ A[1] & \xrightarrow{k[1]} & F[1] & \xrightarrow{l[1]} & B[1] \end{array}$$

Now

$$
\begin{aligned}
& l[2] \circ ((v_F + s_4) \cup (v_F + s_4)) \circ k \\
=& l[2] \circ v_F[1] \circ s_4 \circ k + l[2] \circ s_4[1] \circ v_F \circ k \\
=& l[2] \circ v_F[1] \circ k[1] \circ t_4 + r_4[1] \circ l[1] \circ v_F \circ k \\
=& e[1] \circ t_4 + r_4[1] \circ e \neq 0.
\end{aligned}
$$

By the diagram in Proposition 3.2.9, $k^* : \mathrm{Ext}^2(F, B) \longrightarrow \mathrm{Ext}^2(A, B)$ is an isomorphism, hence $l[2] \circ ((v_F + s_4) \cup (v_F + s_4)) \neq 0$. $\square$

Corollary 3.2.21. *The two irreducible components of $\mathbf{M}_2$ intersect transversely.*

Proof. Proposition 3.2.20 shows that $\kappa_2^{-1}(0)$ is cut out by equations $u_1u_2, u_1u_3, u_1u_4, u_1u_5$ in $\mathrm{Ext}^1(F, F)$, so all first order deformations that can be lifted to the second order form a $\mathbb{C}^{15} \cup \mathbb{C}^{12}$ satisfying $\mathbb{C}^{15} \cap \mathbb{C}^{12} = \mathbb{C}^{11}$ in $\mathrm{Ext}^1(F, F)$. But $T_{\varphi_F, F}(T_{\mathbf{P}, F}) \cup T_{\delta, F}(T_{\mathbf{B}, F}) = \mathbb{C}^{15} \cup \mathbb{C}^{12}$ and $T_{\varphi_F, F}(T_{\mathbf{P}, F}) \cap T_{\delta, F}(T_{\mathbf{B}, F}) = T_{\varphi_F, F}(T_{\mathbb{P}(\mathcal{N}^*_{H/\mathbf{M}_1}), F}) = \mathbb{C}^{11}$ by Remark 3.2.15 (2), so indeed we have exhibited all miniversal deformations of F and the two components of $\mathbf{M}_2$ intersect transversely. $\square$

We end this section by proving $\mathbf{M}_2$ is a projective variety.

Theorem 3.2.22. *The moduli space $\mathbf{M}_2$ is a projective variety.*

Proof. $\mathbf{M}_2$ is smooth outside the intersection of its two components by Remark 3.2.12 and Remark 3.2.15 (1) . For any $F \in \mathbb{P}(\mathcal{N}^*_{H/\mathbf{M}_1})$, since no first order deformation other than a versal one can be lifted to the second order, $\mathbf{M}_2$ is reduced at F. This proves $\mathbf{M}_2$ is reduced. Now we can view $\mathbf{M}_2$ as the pushout of the closed embeddings $\mathbf{B} \longleftarrow \mathbb{P}(\mathcal{N}^*_{H/\mathbf{M}_1}) \longrightarrow \mathbf{P}$. In general a pushout diagram does not exist in the category of schemes, but when the two morphisms are closed embeddings it exists [SchK05, Lemma 3.9]. This proves that $\mathbf{M}_2$ is a scheme. The fact that $\mathbf{M}_2$ is projective and of finite type comes after the analysis of the third wall-crossing in the next section, where we prove that $\mathbf{M}_3$ is a blow-up of $\mathbf{M}_2$ along a smooth center contained in $\varphi_F(\mathbf{P}) \setminus \delta(\mathbf{B})$. Since $\mathbf{M}_3$ is the Hilbert scheme, it is automatically projective and of finite type, so $\mathbf{M}_2$ is a projective variety. □

3.3 The Third Wall-crossing

In this section, we study the wall-crossing controlled by the third family of pairs in Theorem 2.2.6 and prove (4) in Theorem 1.1.1. In this section, we will fix

$$(A, B) = \left(\mathcal{O}(-1), \mathcal{I}_{q/V}(-3)\right).$$

The methods are almost the same as the ones in the previous section, but the situation here is easier since we expect no extra components or singularities occur after the wall-crossing, and $\mathbf{M}_3$ is a blow-up of $\mathbf{M}_2$ along a smooth center.

The following Hom and Ext group computations are straightforward.

Lemma 3.3.1. *$\mathrm{Hom}(A, B) = \mathrm{Hom}(B, A) = 0$, $\mathrm{Hom}(A, A) = \mathrm{Hom}(B, B) = \mathbb{C}$;*

$\mathrm{Ext}^1(A, B) = \mathbb{C}$, $\mathrm{Ext}^1(A, A) = 0$, $\mathrm{Ext}^1(B, B) = \mathbb{C}^5$, $\mathrm{Ext}^1(B, A) = \mathbb{C}^{10}$;

$\mathrm{Ext}^2(A, B) = 0$, $\mathrm{Ext}^2(B, B) = \mathbb{C}^2$, $\mathrm{Ext}^2(A, A) = 0$, $\mathrm{Ext}^2(B, A) = \mathbb{C}$;

$\mathrm{Ext}^3(A, B) = \mathrm{Ext}^3(A, A) = \mathrm{Ext}^3(B, B) = \mathrm{Ext}^3(B, A) = 0$.

Similar to Proposition 3.2.2, the incidence hyperplane H is the moduli space of nontrivial extensions of A by B. Similar to Proposition 3.2.4, we can construct an embedding $\varphi'_E : H \longrightarrow \mathbf{M}_2$. Since $\mathbf{M}_2$ has two irreducible components $\mathbf{B}$ and $\mathbf{P}$, we want to know which component H lies in.

Proposition 3.3.2. *Under the induced morphism φ'_E, H is embedded into $\mathbf{P} \setminus \mathbf{B}$.*

Proof. Take any $E \in H$, we have a nontrivial extension $0 \longrightarrow B \longrightarrow E \longrightarrow A \longrightarrow 0$. By using long exact sequences for Hom functor, we get the following commutative diagram with exact rows and columns, and all boundary homomorphisms are 0.

$$
\begin{array}{ccccc}
\mathrm{Ext}^1(A,B)=\mathbb{C} & \xrightarrow{0} & \mathrm{Ext}^1(E,B)=\mathbb{C}^5 & \longrightarrow & \mathrm{Ext}^1(B,B)=\mathbb{C}^5 \\
{\scriptstyle 0}\big\downarrow & & \big\downarrow & & \big\downarrow \\
\mathrm{Ext}^1(A,E)=0 & \longrightarrow & \mathrm{Ext}^1(E,E) & \longrightarrow & \mathrm{Ext}^1(E,B) \\
\big\downarrow & & \big\downarrow & & \big\downarrow \\
\mathrm{Ext}^1(A,A)=0 & \longrightarrow & \mathrm{Ext}^1(E,A)=\mathbb{C}^{10} & \longrightarrow & \mathrm{Ext}^1(B,A)=\mathbb{C}^{10} \\
\big\downarrow & & \big\downarrow & & \big\downarrow \\
\mathrm{Ext}^2(A,B)=0 & \longrightarrow & \mathrm{Ext}^2(E,B)=\mathbb{C}^2 & \longrightarrow & \mathrm{Ext}^2(B,B)=\mathbb{C}^2 \\
\big\downarrow & & \big\downarrow & & \big\downarrow \\
0 & \longrightarrow & \mathrm{Ext}^2(E,E) & \longrightarrow & \mathrm{Ext}^2(B,E) \\
\big\downarrow & & \big\downarrow & & \big\downarrow \\
0 & \longrightarrow & \mathrm{Ext}^2(E,A)=\mathbb{C} & \longrightarrow & \mathrm{Ext}^2(B,A)=\mathbb{C}
\end{array}
$$

If $E \in \mathbf{B} \setminus \mathbf{P}$, then $\mathrm{Ext}^1(E,E) = \mathbb{C}^{12}$, but this violates the exactness of the central column of the above diagram. If $E \in \mathbf{P} \cap \mathbf{B}$, then by Proposition 4.9 we have $\mathrm{Ext}^1(E,E) = \mathbb{C}^{16}$ and $\mathrm{Ext}^2(E,E) = \mathbb{C}^4$, which also does not fit into the above diagram. Hence $E \in \mathbf{P} \setminus \mathbf{B}$. $\square$

Remark 3.3.3. *This proposition means that the third wall-crossing only modifies one irreducible component of $\mathbf{M}_2$, namely $\mathbf{P}$. It does not touch the other component $\mathbf{B}$.*

On the other hand, we can construct a morphism $\varphi'_F : \mathbf{P}' \longrightarrow \mathbf{M}_3$ that is injective on the level of sets and Zariski tangent spaces, where $\mathbf{P}'$ is a $\mathbb{P}^9$-bundle over H parametrizing all nontrivial extensions of B by A. This implies that for any F in the image of φ'_F, $\mathrm{Ext}^1(F,F)$ is at least 14-dimensional since $\dim\mathbf{P'} = 14$ and $\mathbf{P'}$ is smooth.

If we denote the blow-up of $\mathbf{M}_2$ along H by $\mathbf{B}'$, then we can perform the elementary modification on the pullback of the universal family over $\mathbf{M}_2$ along the exceptional divisor of $\mathbf{B}'$ to get a flat family $\mathcal{K}'$. Similar to Proposition 3.2.14, $\mathcal{K}'$ induces a morphism $\delta' : \mathbf{B}' \longrightarrow \mathbf{M}_3$ which is injective on the level of sets and Zariski tangent spaces.

Theorem 3.3.4. *The induced morphism δ' is an isomorphism.*

Proof. $\mathcal{K}'$ is the same as the universal family over $\mathbf{M}_2$ outside the exceptional divisor, so δ' is an isomorphism outside the exceptional divisor. For any F lying in the exceptional divisor, δ' induces an injection $T_{\mathbf{B}',F} \longrightarrow \mathrm{Ext}^1(F,F) = T_{\mathbf{M}_3,F}$. To prove δ' is an isomorphism at F, we only need to show $\mathrm{Ext}^1(F,F) = \mathbb{C}^{15} = T_{\mathbf{B}',F}$. Since we have an exact sequence $0 \longrightarrow A \longrightarrow F \longrightarrow B \longrightarrow 0$, this can be done by writing down the long exact sequences for Hom functor again.

$$\begin{array}{ccccc}
\mathrm{Ext}^1(B,A)=\mathbb{C}^{10} & \longrightarrow & \mathrm{Ext}^1(F,A)=\mathbb{C}^{9} & \longrightarrow & \mathrm{Ext}^1(A,A)=0 \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{Ext}^1(B,F)=\mathbb{C}^{14} & \longrightarrow & \mathrm{Ext}^1(F,F)=\mathbb{C}^{15} & \longrightarrow & \mathrm{Ext}^1(A,F)=\mathbb{C} \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{Ext}^1(B,B)=\mathbb{C}^{5} & \longrightarrow & \mathrm{Ext}^1(F,B)=\mathbb{C}^{6} & \longrightarrow & \mathrm{Ext}^1(A,B)=\mathbb{C} \\
{\scriptstyle 0}\downarrow & & {\scriptstyle 0}\downarrow & & \downarrow \\
\mathrm{Ext}^2(B,A)=\mathbb{C} & \longrightarrow & \mathrm{Ext}^2(F,A)=\mathbb{C} & \longrightarrow & \mathrm{Ext}^2(A,A)=0 \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{Ext}^2(B,F)=\mathbb{C}^{3} & \longrightarrow & \mathrm{Ext}^2(F,F)=\mathbb{C}^{3} & \longrightarrow & \mathrm{Ext}^2(A,F)=0 \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{Ext}(B,B)=\mathbb{C}^{2} & \longrightarrow & \mathrm{Ext}^2(F,B)=\mathbb{C}^{2} & \longrightarrow & \mathrm{Ext}^2(A,B)=0
\end{array}$$

$\square$

Chapter 4: More on Moduli Spaces of Bridgeland Semistable Complexes

In this chapter, we give some general discussions on the moduli stack of Bridgeland semistable complexes. Our goal is to prove Conjecture 1.2.1, at least in the case of $K3$ surfaces. As we have already mentioned in the introduction, the difficulty of this conjecture is that we do not know whether the moduli stack of Bridgeland semistable complex associates with a GIT problem while the classical moduli space of Gieseker semistable sheaves does. The recent development [AHR15] in stack theory has shed some light on how we may proceed for a proof, which we are now going to discuss.

Let $(\mathcal{P}, Z)$ be a stability condition on $\mathrm{D}^{\mathrm{b}}(X)$ satisfying support property, where $\mathcal{P}$ is the slicing and Z is the central charge. We fix a numerical class v in the numerical Grothendieck group $K_{\mathrm{num}}(X)$ and a phase $\sigma \in \mathbb{R}$. For simplicity, we will just denote the moduli functor of semistable complexes with class v and phase σ to be $\mathcal{M}_\sigma(v)$, which was denoted by $\mathfrak{M}^{\mathrm{ss}}_{v,\sigma}$ in Chapter 2. In the rest of this chapter, we will always be with respect to the stability condition $(\mathcal{P}, Z)$ when we say a semistable complex. By Theorem 2.2.2, $\mathcal{M}_\sigma(v)$ is an algebraic stack locally of finite type over $\mathbb{C}$. Our goal is to show that $\mathcal{M}_\sigma(v)$ has a good moduli space in the sense of [Alp12], and the first key ingredient is the following criterion.

Theorem 4.0.1. *([AHR15], Theorem 2.15) Let $\mathcal{X}$ be an algebraic stack, locally of finite type over $\mathbb{C}$, with affine diagonal. Then $\mathcal{X}$ admits a good moduli space if and only if*

1. *For every point $y \in \mathcal{X}(\mathbb{C})$, there exists a unique closed point in the closure $\overline{\{y\}}$.*

2. *For every closed point $x \in \mathcal{X}(\mathbb{C})$, the stabilizer group scheme G_x is linearly reductive and the morphism $\hat{\mathcal{X}}_x \to \mathcal{X}$ from the coherent completion of $\mathcal{X}$ at x satisfies:*

   (a) *The morphism $\hat{\mathcal{X}}_x \to \mathcal{X}$ is stabilizer preserving at every point; that is, $\hat{\mathcal{X}}_x \to \mathcal{X}$ induces an isomorphism of stabilizer groups for every point $\xi \in |\hat{\mathcal{X}}_x|$.*

   (b) *The morphism $\hat{\mathcal{X}}_x \to \mathcal{X}$ maps closed points to closed points.*

   (c) *The map $\hat{\mathcal{X}}_x(\mathbb{C}) \to \mathcal{X}(\mathbb{C})$ is injective.*

The theorem basically tells us that local behaviors of an algebraic stack at the most degenerated points determine whether it has a good moduli space. We want to check these conditions on $\mathcal{M}_\sigma(v)$.

The first thing we check is the assumption that $\mathcal{M}_\sigma(v)$ has affine diagonal. The following proposition mimicks Corollary 5.5 in [AHR15], and it proves $\mathcal{M}_\sigma(v)$ has affine diagonal. Let $\mathcal{F}$ and $\mathcal{G}$ be two families over a scheme S parametrizing semistable complexes with Chern character v and phase σ.

Proposition 4.0.2. *The functor $\underline{\mathrm{Hom}}(\mathcal{F}, \mathcal{G})$ assigning to any $f : S' \to S$ the set $\mathrm{Hom}_{\mathrm{D^b}(X \times S')}(\mathbf{L}f^*\mathcal{F}, \mathbf{L}f^*\mathcal{G})$ is representable by an affine S-scheme.*

Proof. Assume f is affine, then

$$\begin{aligned}\mathrm{Hom}_{\mathrm{D^b}(X\times S')}\left(\mathbf{L}f^*\mathcal{F},\mathbf{L}f^*\mathcal{G}\right) &= \mathrm{Hom}_{\mathrm{D^b}(X\times S)}\left(\mathcal{F},f_*\mathbf{L}f^*\mathcal{G}\right)\\ &= \mathrm{Hom}_{\mathrm{D^b}(X\times S)}\left(\mathcal{F},\mathcal{G}\otimes^{\mathbf{L}}\pi^*(f_*\mathcal{O}_{S'})\right)\end{aligned}$$

The functor $\mathrm{Hom}_{\mathrm{D^b}(X\times S)}\left(\mathcal{F},\mathcal{G}\otimes^{\mathbf{L}}\pi^*(-)\right):\mathrm{QCoh(S)}\to\mathrm{Ab}$ is coherent by a variation of [AHR15] Proposition 5.4 to the case $X\times S\xrightarrow{\pi}S$. By the observation in [Hal14] Example 3.10, this functor is corepresentable by some quasicoherent sheaf Q. Hence

$$\begin{aligned}\mathrm{Hom}_{\mathrm{D^b}(X\times S')}\left(\mathbf{L}f^*\mathcal{F},\mathbf{L}f^*\mathcal{G}\right) &= \mathrm{Hom}_{\mathrm{D^b}(X\times S)}\left(\mathcal{F},\mathcal{G}\otimes^{\mathbf{L}}\pi^*(f_*\mathcal{O}_{S'})\right)\\ &= \mathrm{Hom}_{\mathcal{O}_S}(Q,f_*\mathcal{O}_{S'})\\ &= \mathrm{Hom}_{O_S-\mathrm{Alg}}(\mathrm{Sym}^{\bullet}_{\mathcal{O}_S}Q,f_*\mathcal{O}_{S'})\\ &= \mathrm{Hom}_{\mathrm{Sch}/S}(S',\mathrm{Spec}_{\mathcal{O}_S}\mathrm{Sym}^{\bullet}_{\mathcal{O}_S}Q).\end{aligned}$$

This proves the proposition. □

Remark 4.0.3. *Theorem 1.2 of [AHR15] can be applied to $\mathcal{M}_\sigma(v)$ after this proposition. It provides an equivariant étale local neighborhood around every closed point of $\mathcal{M}_\sigma(v)$ with linearly reductive stabilizer group. Actually we will prove in next section that every closed point of $\mathcal{M}_\sigma(v)$ has linearly reductive stabilizer group.*

4.1 On the Topology of $|\mathcal{M}_\sigma(v)|$

In this section, we will try to understand the topology of the underlying topological space $|\mathcal{M}_\sigma(v)|$ of $\mathcal{M}_\sigma(v)$ and prove partially that the first condition in Theorem 4.0.1 is true for $\mathcal{M}_\sigma(v)$.

First we recall the definition of the underlying topological space $|\mathcal{X}|$ for an algebraic stack. Let $\mathcal{X}$ be an algebraic stack, $|\mathcal{X}|$ is defined to be the set of equivalent classes of morphisms $p : \mathrm{Spec}(K) \to \mathcal{X}$, where K is a field. We say that two points $p : \mathrm{Spec}(K) \to \mathcal{X}$ and $q : \mathrm{Spec}(L) \to \mathcal{X}$ are equivalent if there exists a field Ω and a 2-commutative diagram

$$\begin{tikzcd} \mathrm{Spec}(\Omega) \arrow[r] \arrow[d] & \mathrm{Spec}(L) \arrow[d, "q"] \\ \mathrm{Spec}(K) \arrow[r, "p"] & \mathcal{X}. \end{tikzcd}$$

$|\mathcal{X}|$ is equiped with the following topology: take any $\pi : U \to \mathcal{X}$ which is is surjective, flat, and locally of finite presentation with U an algebraic space, $W \subseteq |\mathcal{X}|$ is open if and only if $\pi^{-1}(W)$ is open in U.

In our case, since our stack is over the complex number $\mathbb{C}$, points in $|\mathcal{M}_\sigma(v)|$ can be represented by $\mathcal{M}_\sigma(v)(\mathbb{C})$, where the latter is just the set of semistable complexes. Sometimes we will use "points" and "semistable complexes" interchangably. We are particularly interested in characterizing $E' \in \overline{\{E\}}$ for two semistable complexes E and E'. We need the following definition: we say that E isotrivially specializes to E' if there is a smooth curve C and a family of semistable complexes $\mathcal{E} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{perf}}(C \times X)$, such that $\mathcal{E}$ is a trivial family of E over an open subset of C and $\mathcal{E}$ has some special fiber E'. By [ASvdW10], we have

Proposition 4.1.1. *The complex E isotrivially specializes to E' if and only if $E' \in \overline{\{E\}}$.*

In particular, if we have three complexes E, F and G such that E isotrivially specializes to F and F isotrivially specializes to G, then E isotrivially specializes to G by topology. Now we are ready to state the main result in this section which partially proves the first condition in Theorem 4.0.1.

Proposition 4.1.2. *For any semistable complex $E \in \mathcal{M}_\sigma(v)(\mathbb{C})$, there is a semistable complex $F \in \overline{\{E\}}$ that is a closed point with respect to the topology of $|\mathcal{M}_\sigma(v)|$. Moreover, $F = \mathrm{gr}(E)$ is equal to the direct sum of the stable Jordan-Hölder factors of E.*

We will need the following two lemmas.

Lemma 4.1.3. *For any nonsplit exact sequence $0 \to B \to E \to A \to 0$ in $\mathcal{P}(\sigma)$, E isotrivially specializes to $A \oplus B$.*

Proof. First, we construct a natural family over $\mathrm{Ext}^1(A, B)$ parametrizing extensions of A by B, which can be viewed as a generalization of Example 2.1.12 in [HL97]. Let $S = \mathbb{P}(\mathrm{Ext}^1(A, B)^*)$ and denote the projections by $S \xleftarrow{p} S \times X \xrightarrow{q} X$, then

$$\begin{aligned}
\mathrm{Ext}^1(q^*A, q^*B \otimes p^*\mathcal{O}_S(1)) &= \mathrm{H}^1(S \times X, \mathrm{Hom}^\cdot(q^*A, q^*B \otimes p^*\mathcal{O}_S(1)) \\
&= \mathrm{H}^1(S \times X, \mathrm{Hom}^\cdot(q^*A, q^*B) \otimes \mathcal{O}_{S\times X}(1)) \\
&= \mathrm{H}^1(S \times X, q^*\mathrm{Hom}^\cdot(A, B) \times \mathcal{O}_{S\times X}(1)) \\
&= \mathrm{H}^1(X, \mathrm{Hom}^\cdot(A, B)) \otimes \mathrm{Ext}^1(A, B)^* \\
&= \mathrm{Ext}^1(A, B) \otimes \mathrm{Ext}^1(A, B)^* \\
&= \mathrm{Hom}(\mathrm{Ext}^1(A, B), \mathrm{Ext}^1(A, B)).
\end{aligned}$$

The extension corresponds to the identity would be the desired family.

Now we denote the given extension $0 \to B \to E \to A \to 0$ by $e \in \mathrm{Ext}^1(A, B)$, then by restricting the family constructed above to the affine line $\mathbb{C}e$, we get an trivial family of E on $\mathbb{C} \setminus 0$, and the fiber over 0 is $0 \cdot e = 0 \in \mathrm{Ext}^1(A, B)$, which is the trivial extension $A \oplus B$. □

Lemma 4.1.4. *If a semistable complex E isotrivially specializes to F, then $E \oplus E'$ isotrivially specializes to $F \oplus E'$.*

Proof. By assumption there exists an isotrivial family of E to F. Then by adding a trivial family of E' to the isotrivial family of E to F, we will get a isotrivial family of $E \oplus E'$ to $F \oplus E'$. □

Proof of Proposition 4.1.2. Recall that the full subcategory $\mathcal{P}(\sigma)$ consists of all semistable complexes of phase σ. It is an excercise to show that $\mathcal{P}(\sigma)$ is an Artinian and Noetherian abelian category, hence every object has Jordan-Hölder filtrations. By repeatedly using Lemma 4.1.3 and 4.1.4 together with a Jordan-Hölder filtration of E, we see that E isotrivially specializes to $F := \mathrm{gr}(E)$, which is the direct sum of the stable Jordan-Hölder factors of E. We need to show that F is a closed point in $\overline{\{E\}}$. Suppose $F' \in \overline{\{F\}}$ is another point in the closure, then there exists a flat family $\mathcal{F}$ over a smooth curve C such that $\mathcal{F}|_0 = F'$ for some point $0 \in C$ and $\mathcal{F}|_{C\setminus 0} = \mathcal{O}_{C\setminus 0} \otimes F$. Let $F = \bigoplus_i F_i^{n_i}$ be the unique decomposition of F into stable factors, where F_i runs through a complete set of representative of isomorphism classes of stable complexes in $\mathcal{P}(\sigma)$ and $n_i = \hom(F_i, F)$. Since $\mathcal{F}$ is flat, the function from C to $\mathbb{Z}$ sending a point x to $\hom(F_i, \mathcal{F}_x)$ is semicontinuous for every i. It equals n_i for every point $x \neq 0$. Thus $n_i' = \mathrm{hom(F_i, F')} \geqslant n_i$. Now if we consider the evaluation map $e_i : F_i \otimes \mathrm{Hom}(F_i, F') \longrightarrow F'$, it has to be injective and its image is isomorphic to $F_i^{n_i'}$. Finally, the sum $\sum_i F_i^{n_i'}$ has to be direct. This can only happen when all n_i' are equal to n_i. Hence $F' = \bigoplus_i F_i^{n_i} = F$.

Remark 4.1.5. *For a complete proof of the first condition in Theorem 4.0.1, we need to show that this closed point F is unique $\overline{\{E\}}$.*

4.2 Étale Local Neighborhood at Closed Points

This section is devoted to the second condition in Theorem 4.0.1, but it is merely a plan with only a few rigorous proofs and details. We will label the statements without a proof by "claims".

The key concept in the second condition of Theorem 4.0.1 is the coherent completion of $\mathcal{M}_\sigma(v)$ at a closed point F. But as far as the author knows, there is no good way in general to construct this stack and to check 2.(a), 2.(b) and 2.(c), unless one can construct an explicit étale local neighborhood of $\mathcal{M}_\sigma(v)$ at F with reasonable properties.

Definition 4.2.1. *Let $\mathcal{X}$ be an algebraic stack of finite type over $\mathbb{C}$ and $x \in \mathcal{X}(\mathbb{C})$. We call $f : ([\mathrm{Spec}A/G_x], v) \longrightarrow \mathcal{X}$ an étale GIT presentation around x if*

- *A is a finite type $\mathbb{C}$-algebra and G_x is linearly reductive.*
- *f is étale, affine and $f(v) = x$.*
- *f is stabilizer preserving at v.*

The second condition in Theorem 4.0.1 is now equivalent to the following statement.

Claim 4.2.2. *Let F be a closed point in $\mathcal{M}_\sigma(v)$ (which means F is a polystable complex), then*

- *There exists an étale local GIT presentation $f : ([\mathrm{Spec}A/\mathrm{Aut}(F)], w) \longrightarrow (\mathcal{M}_\sigma(v), F)$ around F.*
- *f is stabilizer preserving, and it sends closed points to closed points. The induced map $[\mathrm{Spec}A/\mathrm{Aut}(F)](\mathbb{C}) \longrightarrow \mathcal{M}_\sigma(v)(\mathbb{C})$ is injective.*

As mentioned in Remark 4.0.3, there is a first candidate for such an étale local GIT presentation provided by Theorem 1.2 of [AHR15], where it is constructed by some deep theories on algebraic stacks. A disadvantage of this candidate is that it is not explicit enough, and one will not be able to check the the second statement in Claim 4.2.2. However, if we assume the polystable complex F to have one more property, we will be able to construct an explicit candidate for the étale local GIT presentation by using the ideas in [AS16]. The property we want to introduce here is "formality".

Definition 4.2.3. *Let E be a complex in $\mathrm{D}^{\mathrm{b}}(X)$. If the differential graded Lie algebra $\mathrm{RHom}^{\cdot}(E, E)$ is quasi-isormorphic to its cohomology complex, we say that E has the formality property.*

For more on this formality property, one can look at Section 3 of [AS16]. At least in the case of $K3$ surfaces, the expectation is that any Gieseker semistable sheaves have such property. From now on, we will always assume that F has formality property. An important consequence of the formality property is the following proposition on the miniversal deformation space of F.

Proposition 4.2.4. *Let E be a complex that has formality property and*

$$\kappa : \widehat{\mathrm{Ext}^1}(E, E) \longrightarrow \mathrm{Ext}^2(E, E).$$

be the formal Kuranishi map, where $\widehat{\mathrm{Ext}^1}(E, E)$ is the completion of $\mathrm{Ext}^1(E, E)$ at 0. Let

$$\kappa_2 : \mathrm{Ext}^1(E, E) \longrightarrow \mathrm{Ext}^2(E, E)$$

be the main obstruction map, it equals the usual Yoneda pairing of extensions. Then $\kappa^{-1}(0) = \widehat{\kappa_2}^{-1}(0)$ where $\widehat{\kappa_2}^{-1}(0)$ is the completion of $\kappa_2^{-1}(0)$ at 0.

If we start with the miniversal deformation space $(\Sigma, \widehat{\mathcal{F}})$ of F where $\Sigma = \mathrm{Spec}B$ for some complete local $\mathbb{C}$-algebra B and $\widehat{\mathcal{F}}$ is a formal universal family on $\widehat{\mathcal{F}}$, then Σ can be identified with $\kappa^{-1}(0)$, and the above proposition implies that the germ $(\kappa_2^{-1}(0), 0)$ is an algebraization of Σ in the sense that Σ is the completion of $\kappa_2^{-1}(0)$ at 0. Notice that Σ has a natural $\mathrm{Aut}(F)$-action on it, we expect that $(\kappa_2^{-1}(0), 0)$ is an $\mathrm{Aut}(F)$-equivariant algebraization of Σ. To be precise, we need to prove the following statement.

Claim 4.2.5. *There exists an* $\mathrm{Aut}(F)$*-action and an* $\mathrm{Aut}(F)$*-equivariant family* $\mathcal{U}$ *on the germ* $(\kappa_2^{-1}(0), 0)$ *such that* $\mathcal{U}$ *restricts to* F *at* 0 *and pullbacks to* $\widehat{\mathcal{F}}$ *via the completion morphism.*

The germ $(\kappa_2^{-1}(0), 0)$ is expected to be the second candidate for an étale local GIT presentation of $\mathcal{M}_\sigma(v)$ at F. We will explain later why $(\kappa_2^{-1}(0), 0)$ is explicit enough to check the second statement in Claim 4.2.2. At the moment, it is still not clear why $(\kappa_2^{-1}(0), 0)$ is an étale local GIT presentation. We propose to show this by proving the following statement.

Claim 4.2.6. *There exists an* $\mathrm{Aut}(F)$*-equivariant analytic isomorphism between the germ of the étale local GIT presentation* $(\mathrm{Spec}A, w)$ *provided by Theorem 1.2 of [AHR15] and the germ* $(\kappa_2^{-1}(0), 0)$*.*

Since $(\mathrm{Spec}R, w)$ is an étale local GIT presentation by construction, so is $(\kappa_2^{-1}(0), 0)$ if they are $\mathrm{Aut}(F)$-equivariantly isomorphic. The main ingredients of a proof of the above claim would be to answer the following question: If we have a linearly reductive group G acting on two varieties X and Y with fixed points $x \in X$ and $y \in Y$, and if there exists a G-equivariant isomorphism between the completions $\widehat{\mathcal{O}}_{X,x}$ and $\widehat{\mathcal{O}}_{Y,y}$, when can we lift this isomorphism to a G-equivariant analytic isomorphism between the germs (X, x) and (Y, y)?

Now we give some more detailed description of $(\kappa_2^{-1}(0), 0)$ in terms of quiver representations. Let $F = \bigoplus_i^{r=1} F_i^{n_i}$, where the sum runs through all nonisomorphic stable factors F_1, F_2, $\ldots$, F_r of F. We define the following Ext^1 quiver Q of F_1, F_2, $\ldots$, F_r: the vertex set of Q is in 1-to-1 correspondence with F_1, F_2, $\ldots$, F_r; between i-th vertex and j-th vertex, we put $\mathrm{ext}^1(F_i, F_j)$ arrows.

Proposition 4.2.7. *There exists a canonical identification of* $\mathrm{Ext}^1(F, F)$ *and the space of quiver representations* $\mathrm{Rep}(Q, \bar{n})$*, where the dimension vector* $\bar{n} = (n_1, n_2, \ldots, n_r)$*.*

Proof. Let V_i to be an n_i-dimensional vector space and write $F = \bigoplus_i F_i \otimes V_i$. Then

$$\mathrm{Ext}^1(F, F) = \bigoplus_{i,j} \mathrm{Ext}^1(F_i, F_j) \otimes \mathrm{Hom}(V_i, V_j). \tag{4.1}$$

This exactly means to give $\mathrm{ext}^1(F_i, F_j)$ linear maps between V_i and V_j. $\square$

If we further introduce some relations of arrows on the quiver Q, we will be able to characterize the locus $\kappa_2^{-1}(0)$. For any $\xi \in \mathrm{Ext}^1(F, F)$, we can represent ξ as a matrix with respect to the decomposition (4.1): $M_\xi = (\xi_{i,j} \otimes f_{i,j})_{r \times r}$ where $\xi_{i,j} \otimes f_{i,j}$ is in $\mathrm{Ext}^1(F_i, F_j) \otimes \mathrm{Hom}(V_i, V_j)$. Then

$$\kappa_2(\xi) = \xi \cup \xi = M_\xi^2,$$

where M_ξ^2 is just the usual matrix product with the product on elements defined by

$$(\xi_{i,j} \otimes f_{i,j}) \cdot (\xi_{j,k} \otimes f_{j,k}) = (\xi_{i,j} \cup \xi_{j,k}) \otimes (f_{j,k} \circ f_{i,j}).$$

The resulting $(\xi_{i,j} \cup \xi_{i',j'}) \otimes (f_{i',j'} \circ f_{i,j})$ is in $\mathrm{Ext}^2(F_i, F_k) \otimes \mathrm{Hom}(V_i, V_k)$, which can be viewed as a direct summand in $\mathrm{Ext}^2(F, F)$. The relations we want to introduce on the arrows of Q are exactly the same as the above description. We first identify the vector space generated by arrows between i-th vertex and j-th vertex with $\mathrm{Ext}^1(F_i, F_j)$, then we can introduce Yoneda pairing on the space of all arrows $\bigoplus_{i,j} \mathrm{Ext}^1(F_i, F_j)$. The relations of arrows are generated by $R \cup R = 0$ for any $R \in \bigoplus_{i,j} \mathrm{Ext}^1(F_i, F_j)$. Let us denote the Ext^1 quiver of $F_1, F_2, \ldots, F_r$ with the above relations by Q_Y, then we can summarise the above construction to be the following proposition.

Proposition 4.2.8. *There is a canonical identification between $\kappa_2^{-1}(0)$ and the space of quiver representations $\mathrm{Rep}(Q_Y, \bar{n})$, where the dimension vector $\bar{n} = (n_1, n_2, \ldots, n_r)$.*

Interpreting $\kappa_2^{-1}(0)$ as quiver representation will help us understand the étale local GIT presentation if we assume Claim 4.2.5 and Claim 4.2.6. To be precise, we propose the following statement. Let $\mathrm{mod}Q_Y$ be the category of quiver representations over Q_Y.

Claim 4.2.9. *There exists a canonical inclusion $h : \mathrm{mod}Q_Y \hookrightarrow \mathcal{P}(\sigma)$ making $\mathrm{mod}Q_Y$ a full subcategory of the semistable complexes of phase σ. Its image is the full subcategory generated by the stable factors $F_1, F_2, \ldots, F_r$. The étale local GIT presentation*

$$f : \left([\kappa_2^{-1}(0)/\mathrm{Aut}(F), 0]\right) \longrightarrow (\mathcal{M}_\sigma(v), F)$$

will send an element $\xi \in \kappa_2^{-1}(0)$ to $h(\xi)$.

On the level of simple objects and extensions of simple objects, it is clear what h does: it sends a simple object in $\mathrm{mod}Q_Y$ to its corresponding stable factor F_i in $\mathcal{P}(\sigma)$; it sends an extension of two simple objects to the corresponding extensions in $\mathrm{Ext}^1(F_i, F_j)$. But in general if one has a quiver representation with a long Jordan-Hölder filtration, we hope that h will send it to a complex in $\mathrm{P}(\sigma)$ with same filtration.

Assuming Claim 4.2.9, we will be able to partially prove the second statement in Claim 4.2.2.

Corollary 4.2.10. *The étale local GIT presentation* $f : \left([\kappa_2^{-1}(0)/\mathrm{Aut}(F), 0]\right) \longrightarrow (\mathcal{M}_\sigma(v), F)$ *is stabilizer preserving, and it sends closed points to closed points.*

Proof. f is stabilizer preserving, because it sends ξ to $h(\xi)$, where h is an inclusion from a full subcategory, so we indeed have $\mathrm{Aut}(\xi) = \mathrm{Aut}(h(\xi))$.

As a GIT quotient, closed points in the stack $[\kappa_2^{-1}(0)/\mathrm{Aut}(F)]$ means points whose orbit under the $\mathrm{Aut}(F)$-action is closed. If we still use the matrix M_ξ to represent a point in $\kappa_2^{-1}(0)$, it is not hard to see ξ has a closed orbit if and only if M_ξ is diagonal and $f_{i,i} \in \mathrm{Hom}(V_i, V_i)$ is diagonal for any i. As a quiver representation, this means all maps between different vertices are 0 and all endomorphisms are diagonal, hence it is a direct sum of simple representations. Since h preserves direct sum and sends simple representations to stable complexes, $h(\xi)$ will be a polystable complex, which means it is a closed point in $\mathcal{M}_\sigma(v)$. $\square$

Bibliography

[ABCH13] D. Arcara, A. Bertram, I. Coskun and J. Huizenga, The minimal model program for the Hilbert scheme of points on $\mathbb{P}^2$ and Bridgeland stability, *Adv. Math.*, **235**, 580-626, 2013.

[AHR15] J. Alper, J. Hall and D. Rydh, A luna étale slice theorem for algebraic stacks, 2015, arXiv:1504.06467.

[Alp12] J. Alper, Good moduli space for Artin stacks, *Ann. Inst. Fourier*, **63.6** (2013), 2349-2402.

[AP06] D. Abramovich and A. Polishchuk. Sheaves of t-structures and valuative criteria for stable complexes. *J. reine. angew. Math.* **590**, 89–130, 2006.

[ASvdW10] J. Alper, D. Smyth and F. van der Wyck, Weakly proper moduli stacks of curves.

[AS16] E. Arbarello and G. Saccà, Singularities of moduli spaces of sheaves on K3 surfaces and Nakajima quiver varieties, 2016, arXiv:1505.00759.

[BM14a] A. Bayer and E. Macrì, Projectivity and birational geometry of Bridgeland moduli spaces, J. Amer. Math. Soc. **27** (2014), 707-752.

[BM14b] A. Bayer and E. Macrì, MMP for moduli of sheaves on $K3$s via wall-crossing: nef and movable cones, Lagrangian fiberations. *Invent. Math.*, **198**(3), 505-590, 2014.

[Bon89] A. I. Bondal, Representation of associative algebras and coherent sheaves, *Izv. Akad. Nauk SSSR Ser. Mat.* **53**, 25-44, 1989.

[BMS14] A. Bayer, E. Macrì and P. Stellari. The Space of Stability Conditions on Abelian Threefolds, and on some Calabi-Yau Threefolds, arXiv:1410.1585.

[BMT14] A. Bayer, E. Macrì and Y. Toda. Bridgeland Stability on threefolds I: Bogomolov-Gieseker type inequalities. *J. Algebraic Geom.* **23**, 117–163, 2014.

[Bri07] T. Bridgeland, Stability conditions on triangulated categories, *Ann. of Math.* **166**, 317-345, 2007.

[Bri08] T. Bridgeland, Stability conditions on K3 sufaces, *Duke. Math. J.*, 141(2):241-291, 2008.

[CHW14] I. Coskun, J. Huizenga and M. Woolf, The effective cone of moduli of sheaves on the plane, 2014, arXive:1401.1613.

[Hal14] J. Hall, Cohomology and base change for algebraic stacks, *Math. Z.* **278**(2014), no. 1-2, 401-429.

[Har66] R. Hartshorne, Residue and duality, *Springer*, Berlin, 1966.

[HL97] D. Huybrechts and M. Lehn, The geometry of moduli of sheaves. Number E 31 in *Aspects of Mathematics*, Viewveg, 1997.

[Ina02] M. Inaba. Toward a definition of moduli of complexes of coherent sheaves on a projective scheme. *J. Math. Kyoto Univ.* **42**, 317–329, 2002.

[KLS06] D. Kaledin, M. Lehn and C. Sorger. Singular symplectic moduli spaces. *Invent. Math.* **164**, 591–614, 2006.

[Lan83] H. Lange. Universal families of extensions. *Journal of Algebra* **83**, 101–113, 1983.

[Lie06] M. Lieblich. Moduli of complexes on a proper morphism. *J. Algebraic Geom.* **15**, 175–206, 2006.

[LLMS16] M. Lahoz, M. Lehn, E. Macrì and P. Stellari, Generalized twisted cubics on a cubic fourfold as a moduli space of stable objects, 2016, arXiv:1609.04573.

[LZ16] C. Li and X. Zhao, Birational models of moduli spaces of coherent sheaves on the projective plane, 2016, arXive:1603.05035.

[M2] D. R. Grayson and M. E. Stillman, Macaulay2, a software system for research in algebraic geometry. Available at http://www.math.uiuc.edu/Macaulay2.

[Mac14] E. Macrì, A generalized Bogomolov-Gieseker inequality for the three-dimensional projective space. *Algebra Number Theory* **8**, 173–190, 2014.

[MS16] E. Macri and B. Schmidt, Lectures on Bridgeland stability, arXiv:1607.01262, 2016

[Kin94] A.D. King, Moduli of representations of finite-dimensional algebras. *Quart. J. Math.* **45**, 515–530, 1994.

[PS85] R. Piene and M. Schlessinger, On the Hilbert scheme compactification of the space of twisted cubics. *Amer. J. Math* **107**, 761-774, 1985.

[PT16] D. Piyaratne and Y. Toda. Moduli of Bridgeland semistable objects on 3-folds and Donaldson-Thomas invariants, arxiv:1504.01177.

[SchK05] K. Schwede, Gluing schemes and a scheme without closed points, *Contemp. Math.* **386**, 157-172, 2005.

[SchB15] B. Schmidt, Bridgeland stability on threefolds - some wall crossings, 2015, arXiv:1509.04608.

[SM] The SageMath Developers, SageMath (Version 7.0), 2016. http://www.sagemath.org

[Sta17] the Stacks Project authors, Stacks Project, http://stacks.math.columbia.edu, 2017

[Tra16] R. Tramel, New stability conditions on surfaces and new Castelnuovo-type inequalities for curves on complete-intersection surfaces. *Ph.D thesis*. University of Edinburgh. 2015.

[Tod09] Y. Toda. Limit stable objects on Calabi-Yau 3-folds. *Duke Math J.* **149**, 157–208, 2009.

[Xia16] B. Xia, Hilbert scheme of twisted cubics as simple wall-crossing, 2016, arXiv:1608.04609v1