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Algebraic K-theory of number fields and rings of integers and the Stickelberger ideal

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ALGEBRAIC K -THEORY OF NUMBER FIELDS AND RINGS OF INTEGERS AND THE STICKELBERGER IDEAL

DISSERTATION

Presented in Partial Fulfillment of the Requirements for
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By

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To My Parents and My Brother

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3) Algebraic Geometry with Prof. A. Ash, Prof. R. Joshua

4) Abelian Varieties with Prof. A. Silverberg

5) Hyperbolic Geometry with Prof. M. Davis

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INTRODUCTION

The classical Stickelberger theorem says, that the class group of an abelian extension F of Q is annihilated by the ideal in $Z[G(F/Q)]$ generated by elements of the form:

$$\Theta_0 = \Theta_0(b) = (b - (b, F)) \sum_{(a;f)=1;1\leq a<f} \zeta_f(a, 0)(a, F)^{-1}, \tag{0.1}$$

where f is the conductor of F, (a, F) denotes the restriction to F of the automorphism of $Q(\xi_f)$, which sends ξ_f to ξ_f^a, $\zeta_f(a, s)$ is the partial zeta function and $(b; 2f) = 1$. Coates and Sinnott [9], defined analogous Stickelberger elements with the values of partial zeta function at negative integers:

$$\Theta_n = \Theta_n(b) = (b^{n+1} - (b, F)) \sum_{(a;f)=1;1\leq a<f} \zeta_f(a, -n)(a, F)^{-1} \tag{0.2}$$

They proved, that $\Theta_n \in Z[G(F/Q)]$ and $\Theta_1(b)$ annihilates the odd part of $K_2(O_F)$ for b relatively prime to the conductor and the order of $K_2(O_F)$. Inspired by the Lichtenbaum conjectures they conjectured, that $\Theta_n(b)$ annihilates the odd part of $K_{2n}(O_F)$ for n odd and b relatively prime to the conductor and $w_{n+1}(F)$. The purpose of this presentation is to give some evidence for their conjecture. This presentation is organized in the following way. Chapter I gives basic definitions and results concerning the Stickelberger elements. Chapter II on the other hand carries some basic definitions and results from algebraic K - theory. Chapter III gives a construction (under some additional conditions) of a map:

$$s : K_{2n-1}(k_w)_l \rightarrow K_{2n-1}(O_E)_l,$$

which splits the natural map if l does not divide n and has some " Galois property", where E is a number field, l is a prime number, w is a place of E relatively prime to l and k_w is the residue field. If l divides n, then the map s is defined on the subgroup of index $|n|_l^{-1}$ in $K_{2n-1}(k_w)_l$. I would like to thank to prof. G. Mislin who has shown to me how to construct such a map using the K- theory with coefficients. The map s is used in chapter IV to define a map Λ. By theorem 1 in chapter IV, the map Λ splits the boundary map in the Quillen localization sequence (more precisely its l torsion part), up to the action of $\Theta_n(b)$ ($n\Theta_n(b)$ resp.) if l does not divide n ($l|n$. resp.). This gives the result, that the Stickelberger elements $\Theta_n(b)$ ($n\Theta_n(b)$ resp.) annihilate the subgroup of divisible elements in $K_{2n}(F)_l$ with b relatively prime to l, f and the order of $K_{2n}(O_F)$. The existence of such a map Λ with the above properties, was suggested to me by my thesis advisor prof. W. Sinnott. Chapter V gives an application of theorem 1 to get some splitting examples in the Quillen localization sequence. Chapter VI describes the construction of the Chern class map of Soulé. Result of Soulé [27] theorem 6 iii and application of some spectral sequences in étale cohomology, as in the paper of Lichtenbaum [20], leads to a construction in chapter VII (for F totally real abelian and n odd) of a surjective map:

$$K_{2n}(F)_l \rightarrow H^2_{cts}(F, Z_l(n+1))_l.$$

This map, results of Schneider [26], theorem of Mazur and Wiles (main conjecture in Iwasawa theory) and results of Soulé [27], give in chapter VIII (theorem 2), the following lower bound of the number of elements of the l torsion part of the group of divisible elements in $K_{2n}(F)_l$. Namely for $l > n$:

$$\#\Big(\bigcap_{r\geq 1} K_{2n}(F)^{l^r}\Big)_l \;\geq\; \Big|\frac{w_{n+1}(F)\zeta_F(-n)}{\prod_{v|l} w_n(F_v)}\Big|_l^{-1} \tag{0.3}$$

This result and corollaries of chapter V, make it possible to determine for each odd prime number l, except irregular prime numbers such that $l \leq n$ and $l \mid nw_{n+1}(Q)\zeta(-n)$, whether or not the short exact sequence:

$$0 \to K_{2n}(Z) \to K_{2n}(Q) \to \bigoplus_v K_{2n-1}(k_v) \to 0$$

splits for the l torsion part. Also the theorem 2 and corollary 1 in chapter IV, give the evidence for the conjecture of Coates and Sinnott. Take $F = Q$ and $l > n$. Then theorem 1 shows that the numerator in the formula of theorem 2 is divisible by the exponent of the l torsion part of the group of divisible elements in $K_{2n}(Q)$. On the other hand $|w_n(Q_l)|_l^{-1} = 1$ (n odd). Hence in this case the lower bound for the group of divisible elements in $K_{2n}(Q)$ is given also by the numerator of the formula in the theorem 2.

I would like to thank to my thesis advisor prof. W. Sinnott for introducing me to the problems relating the Stickelberger ideal to algebraic K- theory and for all his help and support he gave me during my work at Ohio State University.

CHAPTER I
STICKELBERGER IDEAL

§ 1. BASIC PROPERTIES OF STICKELBERGER ELEMENTS

Let F/Q be an abelian extension of the field of the rational numbers Q. $G = G(F/Q)$ will denote its Galois group. Let f be the smallest natural number such that F is contained in $Q(\xi_f)$. Such an f is called the conductor of F. ξ_f denotes the f - power root of unity. Let σ_a denote the element of $G(Q(\xi_f)/Q)$ such that $\sigma_a(\xi_f) = \xi_f^a$ for $(a, f) = 1$. Let us define (a, F) to be the restriction of σ_a to F. Let $f > 1$. The Stickelberger element $\Theta_n(b) = \Theta_n \in Z[G]$ is defined [9] in the following way:

$$\Theta_n = (b^{n+1} - (b; F)) \sum_{(a,f)=1; 1 \le a < f} \zeta_f(a; -n)(a; F)^{-1}. \tag{1.1}$$

where $\zeta_f(a; s) = \sum_{k \ge 1; k \equiv a \bmod f} k^{-s}$ is the partial zeta function, s is a complex number. When $n = 0$ then Θ_0 is essentially the classical Stickelberger element. We can write the Stickelberger element Θ_n in the following way:

$$\Theta_n = \sum_{(a,f)=1; 1 \le a < f} \Delta_{n+1}(a,b,f)\sigma_a^{-1} \tag{1.2}$$

Where $\Delta_{n+1}(a,b,f) = b^{n+1}\zeta_f(a; -n) - \zeta_f(ab; -n)$. By Coates and Sinnott [9] theorems 1.2 and 1.3 we have:

$$\Delta_{n+1}(a,b,f) \text{ are integer numbers if } (b, w_{n+1}(Q(\xi_f))) = 1 \text{ and} \tag{1.3}$$

$$\Delta_{n+1}(a,b,f) \equiv (ab)^n \Delta_1(a,b,f) \bmod f_n \text{ where } f_n = f \prod_{p \mid f} p^{v_p(n)} \tag{1.4}$$

v_p is the valuation corresponding to the prime number p.

§ 2. STICKELBERGER ELEMENTS FOR Q.

Let $f = 1$. Then $F = Q$. In this case the Stickelberger element is defined in the following way.

DEFINITION 1. *$\Theta_n = (b^{n+1} - 1)\zeta_Q(-n)$ denotes the Stickelberger element for Q.*

NOTATION. From now on let us write ζ instead of ζ_Q. If E and F are finite Galois extensions of Q with $E \supset F$, then Res denotes the restriction map $Res : G(E/Q) \rightarrow G(F/Q)$.

LEMMA 1. *Let $f = 1$. Let ξ be the l^k power root of unity. Let Θ_n^* be the Stickelberger element for $Q(\xi)$. Then $Res\ \Theta_n^* = (1 - l^n)\Theta_n$.*

PROOF: Put $g = l^k$. $Res\Theta_n^* \doteq (b^{n+1} - 1) \sum_{(a;\, g)=1;1\leq a<g} \zeta_g(a\,; -n)$. Hence

$$\sum_{(a;\, g)=1;1\leq a<g} \zeta_g(a; s) = \sum_{(m;\, g)=1} m^{-s} = \prod_{p \neq l} (1 - p^{-s})^{-1} = (1 - l^{-s})\zeta(s). \text{ QED.} \tag{1.5}$$

CHAPTER II
ALGEBRAIC K-THEORY

§ 1. BASIC DEFINITIONS AND RESULTS

Let A be a commutative ring with identity. Algebraic K-groups were defined by Quillen [24] in the following way:

$$K_n(A) = \pi_{n+1}(BQP(A)) \text{ for } n \geq 0. \tag{2.1}$$

In the above formula the notation is as follows:

$P(A)$ denotes the category of finitely generated, projective A - modules,

$QP(A)$ denotes the Quillen Q construction for $P(A)$,

$BQP(A)$ is the classifying space of the category $QP(A)$,

π_{n+1} denotes the n+1 homotopy group.

If $n \geq 1$ then by [12] p. 228.

$$K_n(A) = \pi_n(BGL(A)^+) \tag{2.2}$$

Where $BGL(A)^+$ is the plus construction described in [21]. Browder [3] has defined algebraic K- theory with coefficients in the following way. Let m be a natural number. Then:

$$K_n(A; Z/m) = [Y^n_m; BGL(A)^+] \text{ if } n > 0, \tag{2.3}$$

where $[Y^n_m; BGL(A)^+]$ denotes the group of homotopy classes of maps (with a base point) from Y^n_m to $BGL(A)^+$. The space Y^n_m is obtained by attaching the boundary of the n - dimensional disk to an (n - 1) -dimensional sphere by the map:

$$S^{n-1} \xrightarrow{\times m} S^{n-1}$$

For $n > 1$:

$$K_n(A;\, Z/m) = [Y_m^{n+1};\, BQP(A)] \text{ [12] p. 228.} \tag{2.4}$$

For $n = 1$ as in [27] p.259 we put (this differs from $K_1(A;\, Z/m)$ of Browder):

$$K_1(A;\, Z/m) \stackrel{def}{=} [Y_m^2;\, BQP(A)], \tag{2.5}$$

$$K_0(A;\, Z/m) \stackrel{def}{=} K_0(A)/mK_0(A). \tag{2.6}$$

We have also the Bockstein exact sequence [27] p. 259:

$$\to K_n(A) \to K_n(A) \to K_n(A;\, Z/m) \xrightarrow{b} K_{n-1}(A) \to$$

obtained from the cofibration sequence

$$S^{n-1} \to S^{n-1} \to Y_m^n\ .$$

The map b is called the boundary map or the Bockstein map. For simplicity we introduce the following notation;

$$K_n(A;\, Z/m) = K_n(A;\, m). \tag{2.7}$$

In this presentation A will usually denote a number field, the ring of integers of a number field, or a finite field. Let F be a number field, $O_F = O$ be the ring of integers in F, k_v be the residue field for a finite place v. We have the localization sequence [24] corollary of theorem 5:

$$\to K_n(O_F) \to K_n(F) \xrightarrow{\delta_F} \bigoplus_v K_{n-1}(k_v) \to K_{n-1}(O_F) \to K_{n-1}(F\,) \to$$

Soulé [27] theorem 3, p. 274 proved that the above sequence splits into short exact sequences as follows

$$0 \to K_{2n}\,(O_F) \to K_{2n}(F) \xrightarrow{\delta_F} \bigoplus_v K_{2n-1}(k_v) \to 0$$

From now on l will denote an odd prime number. Because all groups in the above exact sequence are torsion we can consider only the l torsion part of the above exact sequence. In future all the calculation will be done for each l separately. Algebraic K-theory of finite fields was determined by Quillen [25] theorem 8, p. 583:

$$K_0(k_v) = Z, \tag{2.8}$$

$$K_{2n-1}(k_v) = Z/(q_v^n - 1), \tag{2.9}$$

$$K_{2n}(k_v) = 0 \text{ for } n > 0, \tag{2.10}$$

where $q_v = \#k_v$.

It was determined by Quillen that $K_n(O_F)$ has structure of a finitely generated abelian group. Borel [2] determined the rank of $K_n(O_F)$ as follows:

$$K_n(O_F) \otimes Q = Q \text{ if } n = 0, \tag{2.11}$$

$$K_n(O_F) \otimes Q = Q^{r_1+r_2-1} \text{ if } n = 1, \tag{2.12}$$

$$K_n(O_F) \otimes Q = Q^{r_2} \text{if } n \equiv 3 mod\ 4, \tag{2.13}$$

$$K_n(O_F) \otimes Q = Q^{r_1+r_2} \text{ if } n \equiv 1 mod\ 4,\ n \neq 1, \tag{2.14}$$

$$K_n(O_F) \otimes Q = 0 \text{ if } n \text{ is even and } n > 0. \tag{2.15}$$

Other results from K - theory will be quoted later.

CHAPTER III
GALOIS SECTION

§ 1. INTRODUCTORY COMPUTATIONS.

Let E be a number field, O be its ring of integers. Let $k \geq 1$ be an integer. In this chapter we assume that $\xi \in E$, where ξ is the l^k power root of unity. For any finite place w or the associated prime ideal $\beta_w = \beta$ in O, the residue field $k_w = O/\beta$ has $N_{E/Q}(\beta) = p^f = q$ elements, where p is a prime number and $f = [k_w;Z/p]$. Throughout this chapter we assume that $l^k \| (q - 1)$. We have the following commutative diagram:

$$\begin{tikzcd}
{} \arrow[r] & K_n(O) \arrow[r, "l^k"] \arrow[d, "\pi"] & K_n(O) \arrow[r] \arrow[d, "\pi"] & K_n(O\,;l^k) \arrow[r, "b"] \arrow[d, "\pi"] & K_{n-1}(O) \arrow[r] \arrow[d, "\pi"] & {} \\
{} \arrow[r] & K_n(k_w) \arrow[r, "l^k"] & K_n(k_w) \arrow[r] & K_n(k_w\,;l^k) \arrow[r, "b"] & K_{n-1}(k_w) \arrow[r] & {}
\end{tikzcd}$$

Diagram 3.1.

The horizontal sequences are the Bockstein sequences, π is the natural map induced by the map $\pi : O \to O/\beta$ and is denoted in the same way. Take $n = 2$. Then the above diagram and the assumptions about O and k_w give the following diagram:

$$\begin{array}{ccccccc} & \longrightarrow & K_2(O;l^k) & \xrightarrow{\ \ b\ \ } & K_1(O)_l & \xrightarrow{\ l^k\ } & 0 \\ & & \big\downarrow{\scriptstyle \pi} & & \big\downarrow{\scriptstyle \pi} & & \\ 0 & \longrightarrow & K_2(k_w;l^k) & \xrightarrow{\ \ b\ \ } & K_1(k_w)_l & \xrightarrow{\ l^k\ } & 0 \end{array}$$

Diagram 3.2.

The element $\xi \in K_1(O) = O^*$ maps by π onto the generator of $K_1(k_w)_l = (k_w^*)_l$ that will be denoted by ξ too. If G is an abelian group then G_l denotes the torsion part of G. According to [3] theorem 1.7, $K_2(O;\ l^k)$ has exponent l^k. So we can find an element $x \in K_2(O;\ l^k)$ of order l^k such that $b(x) = \xi$. It is possible by diagram 3.2 above. Let us define $y = \pi(x) \in K_2(k_w;\ l^k)$. Observe that $b\pi(x) = \xi$. We can define a homomorphism on generator y in the following way:

$$s_2 : K_2(k_w;\ l^k) \to K_2(O;\ l^k),$$
$$s_2(y) = x. \tag{3.1}$$

This is well defined section of the left vertical arrow π in the diagram 3.2. Obviously the natural map $\pi : K_1(O)_l \to K_1(k_w)_l$ is an isomorphism because of the root of unity. Let

$$K_1(k_w)_l \to K_1(O)_l \to K_1(O)_l \to K_1(k_w)_l$$

ıaps:

(where the first map from left is s_1, second is σ_a, third is π) we get:

$$\pi\ \sigma_a s_1(\xi) = \xi^a.$$

get:

Let us consider the commutative diagram:

$$\pi\ \sigma_a s_1(\xi) = \xi^a. \tag{3.2}$$

Let us consider the commutative diagram:

$$\begin{array}{ccccccc}
K_2(k_w;l^k) & \xrightarrow{s_2} & K_2(O;l^k) & \xrightarrow{\sigma_a} & K_2(O;l^k) & \xrightarrow{\pi} & K_2(k_w;l^k) \\
\downarrow b & & \downarrow b & & \downarrow b & & \downarrow b \\
K_1(k_w)_l & \xrightarrow{s_1} & K_1(O)_l & \xrightarrow{\sigma_a} & K_1(O)_l & \xrightarrow{\pi} & K_1(k_w)_l
\end{array}$$

Diagram 3.3.

Observe that we get:

$$\pi\ \sigma_a\ s_2(y) = b^{-1}\ \pi\ \sigma_a\ s_1 b\ (y) = b^{-1}\ \pi\ \sigma_a\ s_1(\xi) =$$

$$b^{-1}(\xi^a) = (b^{-1}(\xi))^a = y^a. \qquad (3.3)$$

Let us put $2n$ instead of n in the diagram 3.1. We need to consider two cases:

(i) l does not divide n ,

(ii) l divides n.

We will continue to consider only case (i). At the end of this chapter we will explain how to deal with the second case. Hence considering (i) we get the following diagram

$$\begin{array}{ccccccccc}
 & \longrightarrow & K_{2n}(O;\ l^k) & \xrightarrow{b} & K_{2n-1}(O)_l & \xrightarrow{l^k} & \\
 & & \downarrow \pi & & \downarrow \pi & & \\
0 & \longrightarrow & K_{2n}(\,k_w;\ l^k) & \xrightarrow{b} & K_{2n-1}(\,k_w)_l & \xrightarrow{l^k} & 0
\end{array}$$

Diagram 3.4.

There is a product structure on the K-theory:

$$*: K_n(A) \otimes K_m(A) \to K_{n+m}(A),$$

for n and m non negative. For reference see [11] and [29]. The product structure is associative and commutative in graded sense. It makes $K_*(A)$ into a graded ring. The same holds for K-theory with coefficients $K_*(A;\ l^k)$. Any homomorphism $A \to B$

induces naturally a ring homomorphism on both K-theory with coefficients and without. Observe that l torsion part of $K_{2n-1}(k_w)$ is cyclic of order l^k by [25] (l does not divide n). Let us define:

$$z = y * y * \ldots * y, \tag{3.4}$$

where on the right we have product of y n-times by itself. As in [3] corollary 2.5 z generates $K_{2n}(k_w; l^k)$.

Let us define a homomorphism:

$$s_{2n} : K_{2n}(k_w; l^k) \to K_{2n}(O; l^k),$$

$$s_{2n}(z) = s_2(y) * s_2(y) * \ldots * s_2(y). \tag{3.5}$$

Observe that by product properties we get:

$$\pi(s_{2n}(z)) = \pi(s_2(y)) * \ldots * \pi(s_2(y)) = y * \ldots * y = z. \tag{3.6}$$

So s_{2n} is a section of π. Using the diagram 3.4, let us define the homomorphism:

$$s : K_{2n-1}(k_w)_l \to K_{2n-1}(O)_l,$$

$$s(\xi_w) = b(s_{2n}(b^{-1}(\xi_w))), \tag{3.7}$$

where ξ_w is the generator of $K_{2n-1}(k_w)_l$ such that $b^{-1}(\xi_w) = z$.

§ 2. PROPERTIES OF THE MAP s.

PROPOSITION 1. $\pi\ \sigma_a\ s\ (\xi_w) = \xi_w^{a^n}$. (3.8)

PROOF. First we want to prove that $\pi\ \sigma_a\ s_{2n}(z) = z^{a^n}$. But we have:

$$\pi\ \sigma_a\ s_{2n}\ (z) = \pi\ (\sigma_a\ \{s_2(y) * \dots * s_2(y)\}) =$$

$$\pi\ \sigma_a\ s_2(y) * \dots * \pi\ \sigma_a\ s_2(y) = y^a * \dots * y^a = (y * \dots * y)^{a^n} = z^{a^n}. \quad (3.9)$$

Let us consider the commutative diagram:

$$\begin{array}{ccccccc}
K_{2n}(k_w\ ;l^k) & \xrightarrow{s_{2n}} & K_{2n}(O\ ;l^k) & \xrightarrow{\sigma_a} & K_{2n}(O;\ l^k) & \xrightarrow{\pi} & K_{2n}(k_w\ ;l^k) \\
\downarrow b & & \downarrow b & & \downarrow b & & \downarrow b \\
K_{2n-1}(k_w)_l & \xrightarrow{s} & K_{2n-1}(O)_l & \xrightarrow{\sigma_a} & K_{2n-1}(O)_l & \xrightarrow{\pi} & K_{2n-1}(k_w)_l
\end{array}$$

Diagram 3.5.

From this we can get the following relations:

$$\pi\ \sigma_a\ s\ (\xi_w) = b\ \pi\ \sigma_a\ s_{2n}b^{-1}(\xi_w) = b\ (\ (b^{-1}(\xi_w))^{a^n}\ ) = \xi_w^{a^n}.\ \text{QED.} \quad (3.10)$$

Let us now consider the case (ii). Observe that $l^{k+r} \parallel (q^n - 1)$ where $l^k \parallel (q - 1)$ and $l^r \parallel n$. Then we do the same calculation as for the case (i). The difference is (as far as the above method is concerned) that we must use the coefficients in Z/l^k. We cannot use the Z/l^{k+r} coefficients because there are only l^k power roots of unity available in O. As a result in this case we get the map s :

$$s : K_{2n-1}(k_w)[l^k] \rightarrow K_{2n-1}(O)[l^k],$$

which splits π :

$$\pi\ : K_{2n-1}(O)[l^k] \rightarrow K_{2n-1}(k_w)[l^k],$$

where $G[m]$ denotes the elements of exponent m in an abelian group G.

CHAPTER IV
STICKELBERGER SPLITTING

§ 1. THE MAP Λ.

Similarly as in Chapter III we will consider the case when l does not divide n. We will state the results for both cases but the detailed proofs will be done only for l which does not divide n. At the end of this chapter we will describe what happens when l divides n. Let us consider the exact sequence :

$$0 \to K_{2n}(O_F) \to K_{2n}(F) \to \bigoplus_{v} K_{2n-1}(k_v) \to 0$$

The direct sum in the sequence is over all finite places of O_F. We always work with the l part of the sequence even though it is not explicitly mentioned. Let ξ_v be the element of $K_{2n-1}(k_v)$ of order l^k where $l^k \| (q^n - 1)$ and $q = \# k_v$. Hence $l^k = \# K_{2n-1}(k_v)_l$.

DEFINITION 2. *Let us define a homomorphism Λ_v as follows:*

(i) *if l does not divide n:*

$$\Lambda_v: K_{2n-1}(k_v)_l \to K_{2n}(F)_l,$$

$$\Lambda_v(\xi_v) = Tr_{E/F}(\lambda_b(\beta) * s(\xi_w^{b^n})) \text{ if } l \text{ divides } f, \tag{4.1}$$

$$\Lambda_v(\xi_v) = Tr_{E/F}(\{\lambda_b(\beta) * s(\xi_w^{b^n})\}^{\gamma_l}) \text{ if } l \text{ does not divide } f. \tag{4.2}$$

(ii) *if l divides n:*

$$\Lambda_v: K_{2n-1}(k_v)_l \to K_{2n}(F)_l,$$

$$\Lambda_v(\xi_v) = Tr_{E/F}(\lambda_b(\beta) * s(\xi_w^{nb^n})) \text{ if } l \text{ divides } f, \tag{4.3}$$

$$\Lambda_v(\xi_v) = Tr_{E/F}(\{\lambda_b(\beta) * s(\xi_w^{nb^n})\}^{\gamma_l}) \text{ if } l \text{ does not divide } f. \tag{4.4}$$

We will now explain the notation used in the above formulas.

a) $E = F(\xi)$ where ξ is the l^k power root of unity,

b) $Tr: K_n(E) \to K_n(F)$ is the trace homomorphism,

c) w is a finite place of E over v in F,

d) β is the prime ideal corresponding to w,

e) ξ_w is any element of $K_{2n-1}(k_w)$ such that $N(\xi_w) = \xi_v$ where N denotes the trace homomorphism $N: K_{2n-1}(k_w) \to K_{2n-1}(k_v)$

f) s is the map from the previous chapter,

g) $\lambda_b(\beta)$ denotes any generator of the principal ideal $\beta^{\Theta_0^*}$, where Θ_0^* is the Stickelberger element for the field E when $n = 0$. The ideal $\beta^{\Theta_0^*}$ is principal by the classical Stickelberger theorem.

h) If l does not divide f then $G(E/Q) = G(F/Q) \oplus G(Q(\xi)/Q)$. Hence we can consider the unique element (which we call σ_l for simplicity) such that its restriction to $G(Q(\xi)/Q)$ is the trivial automorphism and its restriction to $G(F/Q)$ is the automorphism (l, F). Now we are ready to define γ_l:

$$\gamma_l = 1 + l^n \sigma_l^{-1} + l^{2n} \sigma_l^{-2} + l^{3n} \sigma_l^{-3} + \ldots \tag{4.5}$$

LEMMA 2. *The map Λ_v is well defined.*

PROOF. It is enough to prove that ξ_v has the same order as ξ_w. We will observe that N: $K_{2n-1}(k_w)_l \to K_{2n-1}(k_v)_l$ is an isomorphism. Let us notice that $\# K_{2n-1}(k_w) = q_w^n - 1$ where $q_w = \# k_w$. Observe that $E_w = F_v(\xi)$ so $O_w = O_v[\xi]$. Hence $k_w = O_w/(\pi_w) = O_w/(\pi_v) = O_v[\xi]/(\pi_v) = k_v(\xi)$. But $l^k \| (q_v^n - 1)$ so k_w is contained in the finite extension of k_v of degree n. Put $q_v = q$. Hence $[k_w : k_v] = r$ divides n and $\# K_{2n-1}(k_w) = (q^r)^n - 1$. Let log_l denote the l-adic logarithm. We get that $log_l(q^{nr}) = n\ log_l(1 + (q^r - 1)) = n\ (q^r - 1)u_1$ where u_1 is an l-adic unit. But $log_l(q^{rn}) = log_l(1 + (q^{rn} - 1)) = (q^{rn} - 1)u_2$ where u_2 is an l-adic unit too. It follows that $l^k \| (q_w - 1)$ and $l^k \| (q_w^n - 1)$. Hence we see that N is an isomorphism on l torsion. QED.

REMARKS. E_w and F_v are the completions of E and F with respect to w and v respectively, O_w and O_v are their rings of integers and π_w and π_v are their uniformizers respectively. Observe that we can take $\pi_w = \pi_v$ because E_w/F_v is unramified. Observe also that we can take ξ_w to be the element constructed in the chapter III and take $\xi_v = N(\xi_w)$ to start with. It does not affect the definition of Λ_v at all.

DEFINITION . *Let us denote by Λ the following map* :

$$\Lambda : \bigoplus_v K_{2n-1}(k_v)_l \to K_{2n}(F)_l,$$
$$\Lambda = \prod_v \Lambda_v. \tag{4.6}$$

REMARK : In the theorem below ξ_v will denote (by abuse of notation) the element $(\ldots 1 \ldots 1, \xi_v, 1 \ldots 1 ..)$ of $\bigoplus_v K_{2n-1}(k_v)$.

§ 2. STICKELBERGER SPLITTING PROPERTY OF Λ. AUXILIARY COMPUTATIONS.

THEOREM 1. (i) $\delta_F \circ \Lambda(\xi_v) = \xi_v^{\Theta_n}$ *if l does not divide n,* (4.7)

(ii) $\delta_F \circ \Lambda(\xi_v) = \xi_v^{n\Theta_n}$ *if l divides n.* (4.8)

Before giving proof of this theorem let us prove some lemmas.

LEMMA 3. *The following diagram commutes*

$$\begin{tikzcd} K_{2n}(E) \arrow[r, "\delta_E"] \arrow[d, "Tr_{E/F}"'] & \bigoplus_{v} \bigoplus_{w|v} K_{2n-1}(k_w) \arrow[d, "N"] \\ K_{2n}(F) \arrow[r, "\delta_F"] & \bigoplus_{v} K_{2n-1}(k_v) \end{tikzcd}$$

Diagram 4.1.

where N is defined in the usual way. In the above diagram v ranges through all finite primes which do not divide l.

PROOF. It is done in [27] p. 276.

LEMMA 4. *The following diagram commutes*

$$\begin{tikzcd} K_{2n}(E) \arrow[r, "\delta_E"] \arrow[d, "\sigma"] & \bigoplus_w K_{2n-1}(k_w) \arrow[d, "\sigma"] \\ K_{2n}(E) \arrow[r, "\delta_E"] & \bigoplus_w K_{2n-1}(k_w) \end{tikzcd}$$

Diagram 4.2.

where $\sigma \in G(E/Q)$ and induces naturally the maps on K- theory, denoted in the diagram 4.2 *in the same way.*

PROOF. Let $M(O_E)$ be the category of finitely generated O_E -modules, let $M(O_E)_{tor}$ be the subcategory of torsion O_E - modules, let $M(E)$ denote the category of finite dimensional E - vector spaces and let $M(k_w)$ denote the category of finite dimensional k_w - vector spaces for a valuation w. The lemma follows by the diagrams 4.3 and 4.4 below. Observe that the horizontal arrows in the diagram 4.4 are homotopy equivalences by the devissage theorem [24] theorem 4. QED

$$\begin{tikzcd} M(O_E)_{tor} \arrow[r] \arrow[d, "\sigma"] & M(O_E) \arrow[r] \arrow[d, "\sigma"] & M(E) \arrow[d, "\sigma"] \\ M(O_E)_{tor} \arrow[r] & M(O_E) \arrow[r] & M(E) \end{tikzcd}$$

Diagram 4.3.

$$\begin{tikzcd}
\bigoplus_w M(k_w) \arrow[r] \arrow[d, "\sigma"] & M(O_E)_{tor} \arrow[d, "\sigma"] \\
\bigoplus_w M(k_w) \arrow[r] & M(O_E)_{tor}
\end{tikzcd}$$

Diagram 4.4.

Assume first that l divides f. Then by diagram 4.1:

$$\delta_F \circ \Lambda(\xi_v) = N \circ \delta_E \{\lambda_b(\beta) * s(\xi_w^{b^n})\}. \tag{4.9}$$

To calculate $\delta_E \{\lambda_b(\beta) * s(\xi_w^{b^n}\}$ we use the following result of Gillet [11] p. 268. There is the following commutative diagram:

$$\begin{tikzcd}
K_1(E) \times K_{2n-1}(O_E) \arrow[r, "*"] \arrow[d, "\delta_E \times id"] & K_{2n}(E) \arrow[d, "\delta_E"] \\
\bigoplus_w K_0(k_w) \times K_{2n-1}(O_E) \arrow[r, "*"] & \bigoplus_w K_{2n-1}(k_w)
\end{tikzcd}$$

Diagram 4.5.

The boundary map $\delta_E: K_1(E) \longrightarrow \bigoplus_w K_0(k_w)$ can be seen as:

$$\delta_E(x) = \bigoplus_w w(x)[k_w], \tag{4.10}$$

where $[k_w]$ denotes the element of $K_0(k_w)$, generated by the one dimensional vector space k_w over k_w. Let f^* denote the conductor of E and let Θ_n^* denote the Stickelberger element for the field E. In addition let w_a^{-1} denote the place w shifted by $(a, E)^{-1} = (a^{-1}, E)$. To make notation easier to read we will write σ_a instead of (a, E). It does not lead to any confusion, because we consider only the Galois action on E. Then we can define the section $s_{a^{-1}} = \sigma_{a^{-1}} s \sigma_a$ which gives the following diagram:

$$\begin{tikzcd} K_{2n-1}(k_w) \arrow[r, "s"] \arrow[d, "\sigma_{a^{-1}}"] & K_{2n-1}(O) \arrow[d, "\sigma_{a^{-1}}"] \\ K_{2n-1}(k_{w_a^{-1}}) \arrow[r, "s_{a^{-1}}"] & K_{2n-1}(O) \end{tikzcd}$$

Diagram 4.6.

Hence $s = \sigma_a s_{a^{-1}} \sigma_{a^{-1}}$. Moreover we have the following commutative diagram:

$$\begin{tikzcd} K_{2n-1}(O) \arrow[r, "\pi"] \arrow[d, "\sigma_{a^{-1}}"] & K_{2n-1}(k_w) \arrow[d, "\sigma_{a^{-1}}"] \\ K_{2n-1}(O) \arrow[r, "\pi_{a^{-1}}"] & K_{2n-1}(k_{w_a^{-1}}) \end{tikzcd}$$

Diagram 4.7.

Let $\pi_{a^{-1}} : O \to k_{w_a^{-1}}$ be the residue map. We will denote also by $\pi_{a^{-1}}$ the natural map induced on K-theory.

LEMMA 5. *$s_{a^{-1}}$ is a section of $\pi_{a^{-1}}$* and *$\pi_{a^{-1}}\sigma_c\, s_{a^{-1}}$ = raising to the c^n power which we denote in the proof by c^n.*

PROOF. $\pi_{a^{-1}}\, s_{a^{-1}} = (\sigma_{a^{-1}}\pi\ \sigma_a)(\sigma_{a^{-1}}\, s\ \sigma_a) = \sigma_{a^{-1}}\pi\ s\ \sigma_{a^{-1}}$ = identity. Moreover

$$\pi_{a^{-1}}\sigma_c\, s_{a^{-1}} = (\sigma_{a^{-1}}\ \pi\ \sigma_a)\sigma_c\,(\sigma_{a^{-1}}\, s\ \sigma_a) = \sigma_{a^{-1}}\pi\ \sigma_c\, s\ \sigma_a = \sigma_{a^{-1}}\, c^n\ \sigma_a = c^n. \quad (4.11)$$

QED.

§ 3. PROOF OF THEOREM 1.

PROOF. Using the above information we will calculate the image of $\delta_E\ \{\lambda_b(\beta) * s\ (\xi_w^{b^n})\}$ in each $K_{2n-1}(k_u)$ separately. Starting first with the case when $u = w_a^{-1}$ we get:

$$\delta_E\ \{\lambda_b(\beta) * s\ (\xi_w^{b^n})\} = \delta_E\ \{\lambda_b(\beta) * (\sigma_a s_{a^{-1}} \sigma_{a^{-1}}(\xi_w^{b^n}))\} =$$

$$= \Delta_1(a,b,f^*)[k_{w_a^{-1}}] * (\pi_{a^{-1}} \sigma_a s_{a^{-1}} \sigma_{a^{-1}}(\xi_w^{b^n}) =$$

$$= \Delta_1(a,b,f^*)[k_{\ w_a^{-1}}] * \sigma_{a^{-1}}(\xi_w)^{a^n b^n} = \sigma_{a^{-1}}(\xi_w)^{a^n b^n \Delta_1(a,b,f^*)} \text{ in } K_{2n-1}(k_u), \tag{4.12}$$

where $u = w_a^{-1}$. We obtained the second equality from diagram 4.5. The last equality follows, because $\Delta_1(a,b,f^*)$ is an integer and the product by $[k_{w_a^{-1}}]$ is the identity map, because of the agreement of this product with the product defined in [24] p.103. On the other hand if u is not a $G(Q(\xi_{f^*})/Q)$ conjugate of w then $u\ (\lambda_b(\beta)) = 0$. So $\delta_E\ \{\lambda_b(\beta) * s\ (\xi_w^{b^n})\} = u\ (\lambda_b(\beta))[k_u] * \pi_u\ (s\ (\xi_w^{b^n})) = 1$ by bilinearity of the product. But Coates-Sinnott result [9] theorem 1.3 gives:

$\Delta_{n+1}(a,b,f^*) \equiv a^n b^n \Delta_1(a,b,f^*)\ mod\ f_n^*$ and by definition $l^k \mid f_n^*$ so

$$\delta_E\ \{\lambda_b\ (\beta) * s\ (\xi_w^{b^n})\} = \xi_w{}^{\Theta_n^*} \quad \text{and} \tag{4.13}$$

$$\delta_F\ (Tr_{E/F}\ \{\lambda_b(\beta) * s\ (\xi_w^{b^n})\}) = N\ (\xi_w{}^{\Theta_n^*}). \tag{4.14}$$

Observe that we have the following diagram in which the upper square commutes:

$$\begin{tikzcd}
K_n(k_w) \arrow[r, "\sigma_{a^{-1}}"] \arrow[d, "N"] & K_n(k_{w_a^{-1}}) \arrow[d, "N"] \\
K_n(k_v) \arrow[r, "\sigma_{a^{-1}}"] \arrow[d, "i"] & K_n(k_{v_a^{-1}}) \arrow[d, "i"] \\
K_n(k_w) \arrow[r, "\sigma_{a^{-1}}"] & K_n(k_{w_a^{-1}})
\end{tikzcd}$$

Diagram 4.8.

Indeed the lower square commutes, the lower vertical maps are imbeddings by [25] theorem 8 and the big rectangle commutes, because $i \circ N = \sum \sigma$, where the summation is over respective Galois group. Using the diagram we get:

$N\,(\xi_w{}^{\Theta_n^*}) = N\,(\xi_w)^{Res\,\Theta_n^*} = \xi_v{}^{\Theta_n}$ because $Res\,(\Theta_n^*) = \Theta_n$, [9] p.159. (4.15)

Now let us discuss the case when l does not divide the conductor f. In the same way as above we need to calculate:

$\delta_F \circ Tr_{E/F}\,(\{\lambda_b(\beta) * s(\xi_w^{b^n})\}^{\gamma_l}) = N \circ \delta_E\,(\{\lambda_b(\beta) * s(\xi_w^{b^n})\}^{\gamma_l}) =$

$N((\delta_E\,\{\lambda_b(\beta) * s(\xi_w^{b^n})\})^{\gamma_l}) = N(\xi_w{}^{\Theta_n^*\gamma_l}) = N(\xi_w)^{\Theta_n(1-l^n\sigma_l{}^{-1})\gamma_l} = \xi_v{}^{\Theta_n}.$ (4.16)

The first equality follows by diagram 4.1 and the second equality follows by diagram 4.2. The third equality is computed in the same way as the case $l|f$ and the forth equality follows from the diagram 4.8 and by the following property [9] p.159 of Stickelberger elements:

$$Res\,(\Theta_n^*) = (1 - l^n\sigma_l^{-1})\Theta_n. \text{ QED.} \qquad (4.17)$$

Let us now discuss the case $l|n$. Observe that the map Λ is well defined. Indeed if $l^{k+r}||(q^n - 1)$ for some $k > 0$ then $l^k||(q^{n_0} - 1)$ where $n = n_0 l^r$, $(n; n_0) = 1$. Again, put $E = F(\xi)$ where ξ is the l^k power root of unity. In the same way

$k_w = k_v(\xi)$ and k_w is contained in the the finite extension of k_v of degree n_0. Hence $[k_w:k_v] = s$ for some s dividing n_0. So using again the l - adic logarithm we can prove that $l^k \| (q^s - 1)$ and $l^{k+r} \| (q^{ns} - 1)$ so the norm homomorphism will be an isomorphism again. Repeating the proof of the case (i) we get the case (ii). Theorem 1 gives:

COROLLARY 1. *Let* $D = \bigcap_{r \geq 1} K_{2n}(F)^r$ *be the group of all divisible elements in* $K_{2n}(F)$ and let b be relatively prime to lf. *Then* $\Theta_n = \Theta_n(b)$ *annihilates* D_l *if* l *does not divide* n *and* $n\Theta_n = n\Theta_n(b)$ *annihilates* D_l *if* l *divides* n.

PROOF. Let $d \in D_l$. Take some big natural number t such that $d = x^{l^t}$ where x is an element of $K_{2n}(F)$. Hence

$$\delta_F(x) = (\ldots\alpha_v\ldots) \in \bigoplus_v K_{2n-1}(k_v). \tag{4.18}$$

Let us denote Λ_v the map from theorem 1 for the place v. If $K_{2n-1}(k_v)_l$ is trivial then we define $\Lambda_v = 1$. Let $\alpha = (\ldots\alpha_v\ldots)$ and $\Lambda = \prod_v \Lambda_v$. Hence $\delta_F\, \Lambda(\alpha) = \alpha^{\Theta_n}$ or $\alpha^{n\Theta_n}$ if l does not divide n or l divides n respectively. But $\delta_F(x^{\Theta_n}) = (\delta_F(x))^{\Theta_n} = \alpha^{\Theta_n}$ which is a consequence of diagram 4.2. In the same way $\delta_F(x^{n\Theta_n}) = \alpha^{n\Theta_n}$. Hence $\Lambda(\alpha)\, x^{-\Theta_n} \in K_{2n}(O)$ or $\Lambda(\alpha)\, x^{-n\Theta_n} \in K_{2n}(O)$ respectively. Assume that we have taken t such that l^t is the exponent of $K_{2n}(O)_l$. Then raising to the power l^t we get $\Lambda(\alpha^{l^t})(x^{l^t})^{-\Theta_n} = 1$ or $\Lambda(\alpha^{l^t})(x^{l^t})^{-n\Theta_n} = 1$ But $\alpha^{l^t} = 1$ and $x^{l^t} = d$. QED.

CHAPTER V
EXAMPLES

We will now give some applications of our theorem 1 in the case when $F = Q$. Observe that in this case theorem 1 says:

COROLLARY 2. $\delta_Q \circ \Lambda(\xi_v) = \xi_v^{(b^{n+1}-1)\zeta(-n)}$ *if l does not divide n,* (5.1)

$$\delta_Q \circ \Lambda(\xi_v) = \xi_v^{n(b^{n+1}-1)\zeta(-n)} \textit{ if } l \textit{ divides } n. \qquad (5.2)$$

REMARK. For any fixed l we can assume that b is relatively prime to l ,$w_{n+1}(Q)$ and $\# K_{2n}(Z)$. But by [6] p. 293:

$$GCD\{b^{n+1} - 1;\ b \text{ prime and } (b, l\, w_{n+1}(Q)\, \# K_{2n}(Z)) = 1\} = w_{n+1}(Q). \qquad (5.3)$$

COROLLARY 3. *If l does not divide $nw_{n+1}(Q)\zeta(-n)$ then the l torsion part of the exact sequence* $0 \to K_{2n}(Z) \to K_{2n}(Q) \to \bigoplus_v K_{2n-1}(k_v) \to 0$ *splits.*

REMARK. $|w_{n+1}(Q)\zeta(-n)|_l^{-1} = \#\{[A \otimes_{Z_l} \tau(n)]^{G_\infty}\}$ is equal to 1 if l is an odd regular prime number. We denote by τ the Tate module and we put $\tau(1) = \tau$, $\tau(0) = Z_l$, $\tau(n) = \tau(n-1) \otimes_{Z_l} \tau$. Also $A = \varprojlim_m A_m$ where A_m is the l torsion part of the ideal class group of $F_m = Q(\mu_{l^m})$. In addition $F_\infty = Q(\mu_{l^\infty})$, $G_\infty = G(F_\infty/Q)$.

PROPOSITION 2. *If a prime number l is regular and $l|n$ then the exact sequence :*

$$0 \to K_{2n}(Z) \to K_{2n}(Q) \to \bigoplus_v K_{2n-1}(k_v) \to 0$$

splits for the l torsion part.

PROOF. Let $q = q_v = \#k_v$. Let $l^{k+r} \| (q^n - 1)$ where $n = n_0 l^r$, $(n_0;l) = 1$ and let $l^k \| (q^{n_0} - 1)$. Let us consider the field $E = Q(\xi)$ where ξ is the l^k power root of unity. Call O its ring of integers and pick k_w a residue field for some place w over v. By lemma 1, $k_w = k_v(\xi)$ and k_w is contained in a finite extension of k_v of degree n_0. By the result of Harris-Segal [17] corollary 3.2, p.27 there is a section:

$$s': K_{2n-1}(k_w)_l \to K_{2n-1}(O)_l,$$

of the natural map induced by the map $O \to k_w$. Notice, that the conditions of the corollary 3.2 are satisfied here. Let us consider the following map:

$$\Lambda'_v : K_{2n-1}(k_v)_l \to K_{2n}(Q)_l,$$

$$\Lambda'_v(\xi_v) = Tr_{E/Q}\,(\lambda(\beta_w) * s'(\xi_w))^{h^{-1}}. \tag{5.4}$$

Where h is the class number of E, $\lambda(\beta_w)$ is a generator of the principal ideal β_w^h in O, h^{-1} denotes an l - adic unit. Notice that $(h, l) = 1$ by regularity assumption and by Iwasawa theory. This map is well defined because of the discussion of the case $l|n$ following the proof of theorem 1. The proposition follows from the following lemma. QED

LEMMA 6. $\delta_Q \circ \Lambda'_v(\xi_v) = \xi_v.$ (5.5)

PROOF. The proof is done in the same way as the proof of theorem 1. Namely :

$$\delta_Q \circ \Lambda'_v(\xi_v) = N \circ \delta_E\{\lambda(\beta_w) * s'(\xi_w)^{h^{-1}}\} = N(h[k_w] * \xi_w^{h^{-1}}) = N(\xi_w) = \xi_v, \tag{5.6}$$

in $K_{2n-1}(k_v)$ where the first equality is a consequence of lemma 3, the second equality follows by diagram 4.5. Also $\delta_Q\, \Lambda'_v(\xi_v) = N(0 * \pi_u(s'(\xi_w^{h^{-1}}))) = 1$ in $K_{2n-1}(k_u)$ for any $u \neq v$ as follows in the same way as above. QED.

EXAMPLE 1. Take $n = 3$. Then $w_4(Q) = 3\times5\times16$, $\zeta_Q(-3) = (3\times5\times8)^{-1}$. Hence $w_4(Q)\zeta_Q(-3) = 2$. So proposition 1 and theorem 1 imply that up to 2-torsion:

$$K_6(Q) \cong K_6(Z) \oplus \bigoplus_v K_5(k_v). \tag{5.7}$$

EXAMPLE 2. Take $n = 5$. Then $w_6(Q) = 7\times8\times9$, $\zeta_Q(-5) = -(4\times7\times9)^{-1}$. Hence $w_6(Q)\zeta_Q(-5) = -2$. So up to 2-torsion we have:

$$K_{10}(Q) \cong K_{10}(Z) \oplus \bigoplus_v K_9(k_v). \tag{5.8}$$

The same as in example 1 and 2 is true for $n = 7$ and $n = 9$. Let $n = 11$. Then $w_{12}(Q)\zeta(-11) = 2\times691$. So for $K_{22}(Q)$ there is a problem with splitting only for one odd prime number, namely 691. It will be shown later in this presentation, that the short exact sequence does not split for 691.

CHAPTER VI
CHERN CLASS MAP

§ 1. CHERN CLASSES.

Let A be a commutative ring with identity. Let P be a finitely generated projective module. Let $\rho : G \to Aut(P)$ be a representation. We also assume that G acts on $X = specA$. The module P determines a sheaf E on X. In the notation of [18] p. 110, $E = \tilde{P}$ which is a G sheaf [13] p.195. It is clear that E is a locally free O_X module, so it determines an algebraic projective bundle $P(E)$ over X. In this way $P(E)$ is a scheme with G action. Let $O(1)$ denote the invertible sheaf $O_{P(E)}(1)$ on $P(E)$. Again $O(1)$ is a G sheaf. In addition let $Pic(Y,G)$ denote the group of isomorphisms classes of invertible G sheaves on a scheme Y on which G operates. If Φ is a sheaf for étale or Zariski topology on the scheme Y, then by [13] p. 200:

$$H^*(Y,G;\Phi) \overset{def}{=} R^*\Gamma_X^{\ G}(\Phi) \tag{6.1}$$

To avoid confusion we should write in the above formula H^*_{et} or H^*_{Zar} respectively. This cohomology group is called the G equivariant cohomology group or simply equivariant cohomology. All computational details about these groups (Zariski topology case) are in [13]. The étale topology case is treated similarly. Let $C.(G)$ denote the bar resolution, $C.(\Phi)$ denote the Godement resolution for étale or Zariski topology respectively. Let $S(Y_{et},G)$ ($S(Y_{Zar},G)$ resp.) denote the category of G sheaves on Y_{et} (Y_{Zar} resp.). Let $I.$ be an injective resolution of Φ in $S(Y,G)$.

Then we have:

$$H^*(Y,G;\ \Phi) = H^*(Hom_G(C.(G);\ \Gamma_Y(I.))) = H^*(Hom_G(C.(G);\ \Gamma_Y(C.(\Phi)))). \quad (6.2)$$

In the above formula $\Gamma_Y = \Gamma(Y,\text{-})$. If $Y = specA$ then we write Γ_A. Put $q = l^m$. If ρ is a representation as above, then the Chern class $c_i(\rho) \in H^{2i}(A,G;\mu_q^i)$ is defined in the following way.

Let r be the max of all rangs for P at all prime ideals of A. Grothendieck [14] p. 246 proves that there is a natural isomorphism:

$$H^{2r}_{\acute{e}t}(P(E),G;\ \mu_q^r) \approx \bigoplus_{0\le i\le r\text{-}1} H^{2(r\text{-}i)}_{\acute{e}t}(A,G;\ \mu_q^{r\text{-}i})\cup\xi^i. \quad (6.3)$$

This gives us the linear dependence for ξ^i with coefficients being the Chern classes:

$$\sum_{0\le i\le r} c_i(\rho)\xi^i = 0\ ,\ c_0(\rho) = 1. \quad (6.4)$$

ξ is defined in the following way.

Let $O(1)$ be as defined above for the scheme $P(E)$. Then ξ is the image of $cl(O(1)) \in Pic(specA,\ G)$ in $H^2_{\acute{e}t}(A,G;\ \mu_q)$ via maps given below.

$$Pic(specA,\ G) \to H^1_{zar}(A,G;\ G_m) \to H^1_{\acute{e}t}(A,G;\ G_m) \to H^2_{\acute{e}t}(A,G;\ \mu_q)$$

The first map from the left is the composition of natural isomorphisms:

$$Pic(specA,\ G) \to \check{H}^1_{zar}(A,G;\ G_m) \to H^1_{zar}(A,G;\ G_m),$$

the second is the edge homomorphism in the Leray spectral sequence for the map:

$$(specA,G)_{\acute{e}t} \to (specA,G)_{zar},$$

induced by the identity and the third map from left is the boundary map.

We write the total class as $c(\rho) = 1 + c_1(\rho) + c_2(\rho) + \ldots + c_r(\rho)$ (6.5)

The Chern classes have the following basic properties [14] p. 247, [27] p. 256.

a) (Functoriality). For a map $f : (specA',\ G') \to (specA,\ G)$, projective A–module P and a representation $\rho : G \to Aut(P)$, let $f^*(\rho)$ denote the induced representation $G' \to Aut(P\otimes_A A')$. Then $c_i(f^*(\rho) = f^*c_i(\rho)$.

b) (Normalization) $c_1(\rho) = \beta(\xi(det(\rho)))$ where $\xi(det(\rho))$ is the image in $H^1_{ét}(A,G;\, G_m)$ and β is the boundary homomorphism in the long exact sequence associated to the exact sequence: $0 \to \mu_q \to G_m \to G_m \to 0$.

c) (Additivity) If $0 \to P' \to P \to P'' \to 0$ is an exact sequence of G- modules which are projective A modules and if ρ', ρ, ρ'' are the respective representations then $c(\rho) = c(\rho') \cup c(\rho'')$.

§ 2. DEFINITION AND PROPERTIES OF THE CHERN CLASS MAP.

In this paragraph we will describe the Soulé [27] p. 256 - 261 construction of the Chern class map:

$$\overline{c}_{i,k} : K_{2i-k}(A,\, q) \to H^k_{ét}(A,\, \mu_q^i)$$

We keep the same notation as in § 1. Let us now assume that G acts trivially on $specA$. Put $\tilde{C}.(G) = C.(G) \otimes_G Z/q$. Then:

$$Hom_G(C.(G);\, \Gamma_A(C.(\mu_q^i))) = Hom(\tilde{C}.(G);\, \Gamma_A(C.(\mu_q^i))). \tag{6.6}$$

Hence:

$$H^{2i}_{ét}(A,G;\, \mu_q^i) = H^{2i}(Hom(\tilde{C}.(G);\, \Gamma_A(C.(\mu_q^i)))). \tag{6.7}$$

By [5] IV §6 prop.6.1a) there is a natural homomorphism:

$$\phi : H^{2i}(Hom(\tilde{C}.(G);\, \Gamma_A(C.(\mu_q^i)))) \to \bigoplus_{0 \le k \le 2i} Hom(H_{2i-k}(G;\, Z/q),\, H^k_{ét}(A;\, \mu_q^i))$$

Let $G = GL_r(A)$, let $P = A^r$ and $\rho = id_r$. As shown in [27] p. 257 there is the following stability property of the Chern class:

$$c_i(id_r) = i_n^*(c_i(id_{r+1})), \tag{6.8}$$

where:

$$i_n^* : H^{2i}_{ét}(A,GL_{r+1}(A);\, \mu_q^i) \to H^{2i}_{ét}(A,GL_r(A);\, \mu_q^i),$$

is the natural map. Using it and the functoriality of ϕ [5] IV §6 prop. 6.1a) in both variables of the Hom's involved, Soulé [27] p. 258 first defined the map:

$$c_{i,k}(id) : H_{2i-k}(GL(A); Z/q) \to H^k_{\acute{e}t}(A; \mu^i_q)$$

and then composed it with the Hurewicz homomorphism:

$$h_q : K_{2i-k}(A; q) \to H_{2i-k}(GL(A); Z/q),$$

to get the Chern class map:

$$\overline{c}_{ik} : K_{2i-k}(A, q) \to H^k_{\acute{e}t}(A; \mu^i_q).$$

LEMMA 7. *Let $A \to B$ be a homomorphism of commutative rings with identity. Then the following diagram commutes:*

$$\begin{tikzcd} K_{2i-k}(A;q) \arrow[r, "nat"] \arrow[d, "\overline{c}_{i,k}"] & K_{2i-k}(B;q) \arrow[d, "\overline{c}_{i,k}"] \\ H^k_{\acute{e}t}(A;\mu^i_q) \arrow[r, "nat"] & H^k_{\acute{e}t}(B;\mu^i_q) \end{tikzcd}$$

Diagram 6.1.

PROOF. The Hurewicz map h_q commutes with the natural maps of ring change hence it is enough to prove that $c_{i,k}(id)$ commutes with these maps. Consider the following map:

$$f : (specB, GL_r(A)) \to (specA, GL_r(A)).$$

Take the A–module $P = A^r$ and the representation $id_r : GL_r(A) \to GL_r(A)$. Observe that $f^*(id_r)$ equals to the natural imbedding $GL_r(A) \to GL_r(B)$. By the property a) of the Chern class and functoriality of the map ϕ in the second variable, the following diagram commutes:

$$\begin{tikzcd} H_{2i-k}(GL_r(A); Z/q) \arrow[d, "c_{i,k}"] \arrow[dr, "\phi(c_i(f^*(id_r)))"] & \\ H^k_{\acute{e}t}(A; \mu_q^i) \arrow[r, "nat"] & H^k_{\acute{e}t}(B; \mu_q^i) \end{tikzcd}$$

Diagram 6.2

On the other hand consider the following map induced by the identity map:

$$g : (specB, GL_r(A)) \to (specB, GL_r(B)).$$

Take the B-module $P = B^r$ and the representation $id_r : GL_r(B) \to GL_r(B)$. In this case $g^*(id_r)$ equals to the natural imbedding $GL_r(A) \to GL_r(B)$ too. Again by the property a) and functoriality of the map ϕ in the first variable, we have the following commutative diagram:

$$\begin{tikzcd} H_{2i-k}(GL_r(A); Z/q) \arrow[r, "nat"] \arrow[dr, "\phi(c_i(g^*(id_r)))"'] & H_{2i-k}(GL_r(B); Z/q) \arrow[d, "c_{i,k}"] \\ & H^k_{\acute{e}t}(B; \mu_q^i) \end{tikzcd}$$

Diagram 6.3.

Because $g^*(id_r) = f^*(id_r)$, we can paste diagrams 6.2 and 6.3 along the diagonals. Now the lemma follows by the stability of the Chern class. QED.

CHAPTER VII
ÉTALE COHOMOLOGY

In this chapter we assume that F/Q is totally real abelian and $l > n$.

§ 1. NOTATIONS.

We will use the following notation in this chapter.

μ_{l^m} = the group of all l^m power roots of unity, (7.1)

$\mu_{l^m}^i = \mu_{l^m} \otimes \mu_{l^m} \otimes \ldots \otimes \mu_{l^m}$ where μ_{l^m} is tensored with itself i-times, (7.2)

$F_\infty = F(\mu_{l^\infty})$, $G_\infty = G(F_\infty/F)$, (7.3)

μ_A, μ_A^i denote the respective sheaves on $specA$,

$W^n = Q_l/Z_l \otimes_{Z_l} \tau(n)$, (7.4)

W_k^n denotes the sheaf for the étale topology on $speck$, where k is field,

$H^k(A, \Phi) = H_{et}^k(specA, \Phi)$ where Φ denotes a sheaf for the étale topology on $specA$. (7.5)

From now on let O_l denote the ring of l integers in the field F. We can observe that:

$$O_l = \{x \in F: v(x) \geq 0 \text{ for all } v \text{ not over } l\}. \quad (7.6)$$

Let S be the finite set of primes of O_F, over a finite set of prime numbers including l. Let us put:

$$O_S = \{x \in F: v(x) \geq 0 \text{ for } v \notin S\}. \quad (7.7)$$

It is the ring of S - integers in F. From now on S will always denote such a finite set.

§ 2. APPLICATION OF THE CHERN CLASS MAP.

The following diagram 7.1 introduces notation of maps between spectra of corresponding rings.

$$\begin{tikzcd} spec\ F \arrow[r, "\alpha"] \arrow[d, "id"] & spec\ O_S \arrow[d, "\beta"] \\ spec\ F \arrow[r, "j"] & spec\ O_l \end{tikzcd}$$

Diagram 7.1.

By lemma 7 we have the following commutative diagram:

$$\begin{tikzcd} K_{2n}(O_l; l^m) \arrow[r] \arrow[d] & K_{2n}(O_S; l^m) \arrow[d] \\ H^2(O_l; j_*\mu_{l^m}^{n+1}) \arrow[r] & H^2(O_S; \alpha_*\mu_{l^m}^{n+1}) \end{tikzcd}$$

Diagram 7.2.

The upper horizontal map is the natural map in K-theory. The lower horizontal map can be identified with the edge homomorphism in the Leray spectral sequence:

$$E_2^{pq} = H^p(O_l, R^q\beta_*(\alpha_*\mu_{l^m}^{n+1})) \Rightarrow H^{p+q}(O_S, \alpha_*\mu_{l^m}^{n+1})) \qquad (7.8)$$

for the map β in the diagram 7.1. The vertical maps in diagram 7.2 are the Chern class maps. We can identify the edge homomorphism with the appropriate map for the second cohomology, induced by diagram 7.3:

$$\begin{array}{ccc} j_*\mu_{l^m}^{n+1} & \xrightarrow{id} & \beta_*\alpha_*\mu_{l^m}^{n+1} \\ \downarrow & & \downarrow \\ I. & \longrightarrow & \beta_*J. \end{array}$$

Diagram 7.3.

In the above diagram, $I.$ and $J.$ are some injective resolutions of $j_*\mu_{l^m}^{n+1}$ and $\alpha_*\mu_{l^m}^{n+1}$ respectively. It is easy to see, that there are natural isomorphisms $j_*\mu_{l^m}^{n+1} \approx (\mu_{l^m}^{n+1})_{O_l}$ and $\alpha_*\mu_{l^m}^{n+1} \approx (\mu_{l^m}^{n+1})_{O_S}$ [27] p. 267, which we used in the diagram 7.2.

Note that we can consider the following commutative diagram:

$$\begin{array}{ccc} K_{2n}(O_l;\, Z_l) & \longrightarrow & K_{2n}(O_S;\, Z_l) \\ \downarrow & & \downarrow \\ H^2(O_l;\, j_*Z_l(n+1)) & \longrightarrow & H^2(O_S;\, \alpha_*Z_l(n+1)) \end{array}$$

Diagram 7.4.

obtained from diagram 7.2 upon taking the inverse limit on coefficients. It is proven in [27] theorem 6 iii, that the vertical maps in diagram 7.4 are surjective.

The remaining paragraphs of this chapter are devoted to identification of the lower horizontal map in the diagram 7.4.

§ 3. APPLICATION OF ÉTALE COHOMOLOGY.

Observe that we have the following commutative diagram:

$$\begin{tikzcd}
H^2(O_l; j_*\mu_{l^m}^{n+1}) \arrow[r] & H^2(O_S; \alpha_*\mu_{l^m}^{n+1}) \\
H^1(O_l; j_*W_F^{n+1}) \arrow[r] \arrow[u, "\delta"] & H^1(O_S; \alpha_*W_F^{n+1}) \arrow[u, "\delta"]
\end{tikzcd}$$

Diagram 7.5.

where the vertical arrows are the boundary maps obtained in the long exact cohomology sequences associated with the short exact sequences:

$$0 \to j_*\mu_{l^m}^{n+1} \to j_*W_F^{n+1} \to j_*W_F^{n+1} \to 0$$

$$0 \to \alpha_*\mu_{l^m}^{n+1} \to \alpha_*W_F^{n+1} \to \alpha_*W_F^{n+1} \to 0$$

The lower horizontal map in the diagram 7.5 is the edge homomorphism in the Leray spectral sequence:

$$E_2^{pq} = H^p(O_l, R^q\beta_*(\alpha_*W_F^{n+1})) \Rightarrow H^{p+q}(O_S, \alpha_*W_F^{n+1}) \tag{7.9}$$

for the map β. We can describe this edge homomorphism in the same way as we did it for edge homomorphism of the spectral sequence from the § 2. We can also see this edge homomorphism arising from the morphism of Leray spectral sequences shown below on the following diagram:

$$\begin{tikzcd}
H^p(O_l; R^q j_*W_F^{n+1}) \arrow[r, phantom, "\Rightarrow"] \arrow[d] & H^{p+q}(F; W_F^{n+1}) \arrow[d, "id"] \\
H^p(O_S; R^q\alpha_*W_F^{n+1}) \arrow[r, phantom, "\Rightarrow"] & H^{p+q}(F; W_F^{n+1})
\end{tikzcd}$$

Diagram 7.6.

We will discuss the construction of the morphism later in § 4.

Let us also observe that the boundary maps in the diagram 7.5 are mappings into the projective systems:

$$\{H^2(O_l, j_*\mu_{l^m}^{n+1})\}_{l^m} \text{ and } \{H^2(O_S, \alpha_*\mu_{l^m}^{n+1})\}_{l^m}$$

respectively.

Hence we have the following commutative diagram:

$$\begin{tikzcd}
H^2(O_l; j_* Z_l(n+1)) \arrow[r] & H^2(O_S; \alpha_* Z_l(n+1)) \\
H^1(O_l; j_* W_F^{n+1}) \arrow[r] \arrow[u, "\approx", "\delta"'] & H^1(O_S; \alpha_* W_F^{n+1}) \arrow[u, "\approx", "\delta"']
\end{tikzcd}$$

Diagram 7.7.

where by definition:

$$H^2(O_l; j_*Z_l(n+1)) = \varprojlim_{m} H^2(O_l; j_*\mu_{l^m}^{n+1}) \tag{7.10}$$

The same for O_S.

LEMMA 8. *The vertical arrows in the diagram* 7.7 are isomorphisms.

PROOF. This is proven in [27] p. 289 but let us see the argument. We will only consider the right vertical arrow. The following diagram:

$$\begin{tikzcd}
0 \arrow[r] & \alpha_* \mu_{l^{m+1}}^{n+1} \arrow[r] \arrow[d] & \alpha_* W_F^{n+1} \arrow[r, "l^{m+1}"] \arrow[d, "l"] & \alpha_* W_F^{n+1} \arrow[r] \arrow[d, "id"] & 0 \\
0 \arrow[r] & \alpha_* \mu_{l^m}^{n+1} \arrow[r] & \alpha_* W_F^{n+1} \arrow[r, "l^m"] & \alpha_* W_F^{n+1} \arrow[r] & 0
\end{tikzcd}$$

Diagram 7.8.

gives the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & H^1(O_S; \alpha_* W_F^{n+1}) \otimes Z/l^{m+1} \arrow[r] \arrow[d] & H^2(O_S; \alpha_* \mu_{l^{m+1}}^{n+1}) \arrow[r] \arrow[d] & H^2(O_S; \alpha_* W_F^{n+1})[l^{m+1}] \arrow[d, "l"] \\
0 \arrow[r] & H^1(O_S; \alpha_* W_F^{n+1}) \otimes Z/l^{m} \arrow[r] & H^2(O_S; \alpha_* \mu_{l^{m}}^{n+1}) \arrow[r] & H^2(O_S; \alpha_* W_F^{n+1})[l^{m}]
\end{tikzcd}$$

Diagram 7.9.

But $H^2(O_S; \alpha_* W_F^{n+1}) = 0$ by theorem 5 of [27] and $H^1(O_S; \alpha_* W_F^{n+1})$ is finite. Finiteness of the last group follows by finiteness of $H^1(O_l; j_* W_F^{n+1})$ because by [20] p. 355 and [7] p. 115 $H^1(O_l; j_* W_F^{n+1}) \approx m_l(n)^{G_\infty} \approx [A \otimes_{Z_l} \tau(n)]^{G_\infty}$. The last group is finite by the theorem of Mazur and Wiles (Main conjecture in Iwasawa theory) [8] p. 225. Hence finiteness of $H^1(O_S; \alpha_* W_F^{n+1})$ follows by collapsing of the Leray spectral sequence:

$$E_2^{pq} = H^p(O_l, R^q \beta_*(\alpha_* W_F^{n+1})) \Rightarrow H^{p+q}(O_S, \alpha_* W_F^{n+1})$$

for the map β. More precisely $E_2^{pq} = 0$ for $q > 1$ [10] p. 530. QED.

§ 4. PASSAGE TO THE LIMIT.

Let B_l and B_S be the integral closures of O_l and O_S in F_∞ respectively.

Let $O_{l,m}$ and $O_{S,m}$ be the integral closures of O_l and O_S in $F(\mu_{l^m})$ respectively. Let us observe that $B_l = \varinjlim_m O_{l,m}$ and $B_S = \varinjlim_m O_{S,m}$. Let $G_m = G(F(\mu_{l^m})/F)$. The following picture introduces the notation of the maps between spectra of corresponding rings:

$$\begin{tikzcd}
 & spec\, F & & spec\, F_\infty \arrow[ll, "\gamma"'] \arrow[dd, "\tilde{j}"] \\
spec\, F \arrow[ur, "id"] \arrow[dd, "\alpha"'] & & spec\, F_\infty \arrow[ll, "\gamma"'] \arrow[ur, "id"] \arrow[dd, "\tilde{\alpha}"] & \\
 & spec\, O_l \arrow[from=uu, "j", crossing over] & & spec\, B_l \arrow[ll, "\delta"] \\
spec\, O_S \arrow[ur, "\beta"'] & & spec\, B_S \arrow[ll, "\delta'"] \arrow[ur, "\tilde{\beta}"'] &
\end{tikzcd}$$

Diagram 7.10.

Let us use the following spectral sequence for finite Galois covering [23] p. 105:

$$E_2^{p,q} = H^p(G_m;\, H^q(O_{S,m};\, \alpha_* W_F^{n+1})) \Rightarrow H^{p+q}(O_S;\, \alpha_* W_F^{n+1}). \qquad (7.11)$$

Taking the injective limit with respect to m we get the following Artin-Hochshield-Serre spectral sequence [1] p. 92, [23] p. 106:

$$E_2^{p,q} = H^p(G_\infty;\, H^q(B_S;\, \tilde{\alpha}_* W_{F_\infty}^{n+1})) \Rightarrow H^{p+q}(O_S;\, \alpha_* W_F^{n+1}). \qquad (7.12)$$

Let us take the short exact sequence of first terms. We get the following exact sequence [20] p. 355:

$$H^1(G_\infty;\, W^{n+1}) \to H^1(O_S;\, \alpha_* W_F^{n+1}) \xrightarrow{res} H^1(B_S;\, \tilde{\alpha}_* W_{F_\infty}^{n+1})^{G_\infty} \to H^2(G_\infty;\, W^{n+1}).$$

Using the spectral sequences for finite coverings we identified the middle arrow as the restriction map and in addition we know [20] p. 355 that:

$$H^1(G_\infty; W^{n+1}) = H^2(G_\infty; W^{n+1}) = 0. \tag{7.13}$$

Hence the restriction map is an isomorphism. We also have the following commutative diagram:

$$\begin{tikzcd}
H^m(O_l; j_* W_F^{n+1}) \arrow[r, "res"] \arrow[d] & H^m(B_l; \tilde{j}_* W_{F_\infty}^{n+1})^{G_\infty} \arrow[d] \\
H^m(O_S; \alpha_* W_F^{n+1}) \arrow[r, "res"] & H^m(B_S; \tilde{\alpha}_* W_{F_\infty}^{n+1})^{G_\infty}
\end{tikzcd}$$

Diagram 7.11.

The vertical arrows are the edge homomorphisms in the appropriate Leray spectral sequences. More precisely the right vertical arrow is the edge homomorphism restricted to the G_∞ invariant subgroups.

Now we discuss the construction of the morphism of spectral sequences from the diagram 7.6. The upper (resp. the lower) spectral sequence is constructed by taking the normal injective resolution $I..$ ($J..$ resp.) [4] p. 178, [19] p. 301 of the complex:

$$j_* W_F^{n+1} \to j_* I_0 \to j_* I_1 \to$$

$$(\text{resp. } \alpha_* W_F^{n+1} \to \alpha_* I_0 \to \alpha_* I_1 \to \,)$$

where

$$W_F^{n+1} \to I_0 \to I_1 \to$$

is an injective resolution. The natural map:

$$\beta^* j_* I. \to \alpha_* I.$$

induces naturally a homomorphism [4] p. 183:

$$\begin{tikzcd} \beta^* j_* I. \arrow[r] \arrow[d] & \alpha_* I. \arrow[d] \\ \beta^* I.. \arrow[r] & J.. \end{tikzcd}$$

Diagram 7.12.

between these double complexes. It gives us the homomorphism:

$$\begin{tikzcd} j_* I. \arrow[r, "id"] \arrow[d] & \beta_* \alpha_* I. \arrow[d] \\ I.. \arrow[r] & \beta_* J.. \end{tikzcd}$$

Diagram 7.13.

which leads immediately to the morphism of spectral sequences. We want to use the above construction to build the following morphism of spectral sequences of G_∞ modules:

$$\begin{tikzcd} H^p(B_l;\ R^q \tilde{j}_* W^{n+1}_{F_\infty}) \arrow[r, phantom, "\Rightarrow"] \arrow[d] & H^{p+q}(F_\infty;\ W^{n+1}_{F_\infty}) \arrow[d, "id"] \\ H^p(B_S;\ R^q \tilde{\alpha}_* W^{n+1}_{F_\infty}) \arrow[r, phantom, "\Rightarrow"] & H^{p+q}(F_\infty;\ W^{n+1}_{F_\infty}) \end{tikzcd}$$

Diagram 7.14.

To obtain 7.14 we need to apply the functor δ^* (see the diagram 7.10) to 7.13. By [1] chap. III. §3 we know that δ^* is exact, preserves flasque sheaves and if Φ is a sheaf on $(spec\ O_l)_{\acute{e}t}$, then for an étale open U on $spec\ O_{l,n}$:

$$\delta^* \Phi(U \times_{O_{l,n}} spec\ B_l) \approx \varinjlim_m \Phi(U \times_{O_{l,n}} spec\ O_{l,m}) \tag{7.14}$$

We also have the natural isomorphisms:

$$\delta^* j_* \approx \tilde{j}_* \gamma^* \ , \ \delta^* \beta_* \approx \tilde{\beta}_* \delta'^* \ , \ \gamma^* W_F^{n+1} = W_{F_\infty}^{n+1} \tag{7.15)}$$

All these data give us the morphism 7.14. This morphism of spectral sequences gives us the following commutative diagram of G_∞ modules with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & H^1(B_l;\, \tilde{j}_* W_{F_\infty}^{n+1}) \arrow[r] \arrow[d] & H^1(F_\infty;\, W_{F_\infty}^{n+1}) \arrow[r] \arrow[d, "id"] & H^0(B_l;\, R^1\tilde{j}_* W_{F_\infty}^{n+1}) \arrow[d] \\
0 \arrow[r] & H^1(B_S;\, \tilde{\alpha}_* W_{F_\infty}^{n+1}) \arrow[r] & H^1(F_\infty;\, W_{F_\infty}^{n+1}) \arrow[r] & H^0(B_S;\, R^1\tilde{\alpha}_* W_{F_\infty}^{n+1})
\end{tikzcd}$$

Diagram 7.15.

§ 5. IDENTIFICATION OF SOME MAPS VIA CONTINUOUS COHOMOLOGY.

Consider the exact sequence [7] p. 101:

$$0 \to m_S \to F_\infty^* \otimes Q_l/Z_l \to J_S \otimes Q_l/Z_l \to 0,$$

where J_S is the ideal group of B_S. Twisting with $\tau(n)$ we get:

$$0 \to m_S(n) \to F_\infty^* \otimes W^n \to \bigoplus_w W^n \to 0,$$

where the direct sum is over all places w of B_S. Taking G_∞ invariants we get the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & m_l(n)^{G_\infty} \arrow[r] \arrow[d] & (F_\infty^* \otimes W^n)^{G_\infty} \arrow[r] \arrow[d, "id"] & \bigoplus_{v \nmid l} W^n(F_v) \arrow[d, "proj"] \\
0 \arrow[r] & m_S(n)^{G_\infty} \arrow[r] & (F_\infty^* \otimes W^n)^{G_\infty} \arrow[r] & \bigoplus_{v \notin S} W^n(F_v)
\end{tikzcd}$$

Diagram 7.16.

PROPOSITION 3. *The map*:

$$H^1(F_\infty; W^{n+1}) \to H^0(B_S; R^1\tilde{\alpha}_* W^{n+1}_{F_\infty}),$$

can be identified (after composition with an imbedding on the right) with the natural map:

$$F^*_\infty \otimes W^n \to \bigoplus_w W^n,$$

where w runs through all points of $specB_S$ but the generic one (or in other words w runs over all places of F_∞ not over l.). Similarly for the map with $\tilde{j}$

PROOF. Investigating the bicomplex for the Leray spectral sequence for the map $\tilde{\alpha}$ we find out that the map can be seen as the map:

$$Z_1(specB_S)/im\ d_0 \to Z_1/B_0(specB_S)$$

The notation in the above formula can be described as follows:

$I.$ is the flasque resolution of $W^{n+1}_{F_\infty}$. Applying $\tilde{\alpha}_*$ to this resolution we get a complex whose 1- cocycles are Z_1 and zero coboundary are B_0. Evaluating the complex $\tilde{\alpha}_*I$ on $specB_S$ we get another complex whose zero differential is d_0. If w is as above, let $\bar{w}$ be the corresponding geometric point. Let V be an étale neighborhood of $\bar{w}$ over $specB_S$. If we evaluate $\tilde{\alpha}_*I$ on V, we will get the following commutative diagram:

$$\begin{tikzcd}
Z_1(specB_S)/im\,d_0 \arrow[r] \arrow[d, "res"] & Z_1/B_0(specB_S) \arrow[d, "res"] \\
Z_1(V)/im\,d_0^V \arrow[r] & Z_1/B_0(V)
\end{tikzcd}$$

Diagram 7.17.

d_0^V denotes the zero differential of the complex $\tilde{\alpha}_*I.(V)$. Diagram 7.17 gives us the following commutative diagram:

$$\begin{tikzcd} H^1(F_\infty; W_{F_\infty}^{n+1}) \arrow[r] \arrow[d, "res"] & H^0(B_S; R^1\tilde{\alpha}_* W_{F_\infty}^{n+1}) \arrow[d, "res"] \\ \varinjlim_{V} H^1(spec\, F_\infty \times_{B_S} V; W_{F_\infty}^{n+1}) \arrow[r, "="] & (R^1\tilde{\alpha}_* W_{F_\infty}^{n+1})_{\bar{w}} \end{tikzcd}$$

Diagram 7.18.

Observe that the left vertical arrow in the diagram 7.18 equals to:

$$H^1(specF_\infty; W_{F_\infty}^{n+1}) \xrightarrow{res} H^1(specF_{\infty w}^{sh}; W_{F_{\infty w}^{sh}}^{n+1}),$$

because by [1] Chap. III. §3:

$$\varinjlim_{V} H^1(specF_\infty \times_{B_S} V; W_{F_\infty}^{n+1}) = H^1(specF_{\infty w}^{sh}; W_{F_{\infty w}^{sh}}^{n+1}). \tag{7.16}$$

In the above formulas $F_{\infty w}^{sh}$ denotes the strong henselization of F_∞ at w. More precisely $F_{\infty w}^{sh} = F_\infty \otimes_{B_S} B_{Sw}^{sh}$, where B_{Sw}^{sh} is the strong henselization of B_S at w. We can also see that $F_{\infty w}^{sh}$ is the inertia field for a place of $\bar{F}_\infty$ over w. Hence the strong henselization equals to $F_{\infty w}^{ur} \cap \bar{F}_\infty$, where $F_{\infty w}$ is the completion of F_∞ at w. On the other hand we see that the right vertical map in the diagram 7.18 factors naturally through:

$$H^0(B_S; R^1\tilde{\alpha}_* W_{F_\infty}^{n+1}) \to i_w^* R^1\tilde{\alpha}_* W_{F_\infty}^{n+1}(speck_w),$$

where k_w is the residue field of B_S for the place w.

But by [15] p. 31 we have the natural isomorphism of sheaves on $specB_S$:

$$R^1\tilde{\alpha}_* W_{F_\infty}^{n+1} \to \bigoplus_w i_{w*} i_w^* R^1\tilde{\alpha}_* W_{F_\infty}^{n+1}.$$

We used i_w to be the natural map $i_w : speck_w \to specB_S$. Hence evaluating this isomorphism on $specB_S$, utilizing the above explanation and the diagram 7.18 for each w, we can identify our map from the lemma with the map:

$$H^1(specF_\infty; W_{F_\infty}^{n+1}) \xrightarrow{res} \bigoplus_w H^1(specF_{\infty w}^{sh}; W_{F_{\infty w}^{sh}}^{n+1})$$

We used above the following fact. If K is a field and Φ is a sheaf on $specK$ than the natural map $\Phi\,(specK) \to \Phi_{\bar{x}}$ is an imbedding. We denote $\bar{x} = spec\bar{K}$. It is the place in the proof where we comply to identify our map up to an imbedding. It is enough for our purpose because we are interested in the kernel of the map. Let us observe that the map:

$$H^1(specF_\infty;\, W_{F_\infty}^{n+1}) \xrightarrow{res} H^1(specF_{\infty w}^{sh};\, W_{F_{\infty w}^{sh}}^{n+1}),$$

is given by the following map of complexes:

$$(I._{\bar{w}})^{G(\bar{F}_\infty/F_\infty)} \to (I._{\bar{w}})^{G(\bar{F}_\infty/F_{\infty w}^{sh})}$$

To finish we need to check that for each $n \geq 0$, $I_{n\bar{w}}$ is $G(\bar{F}_\infty/F_\infty)$ as well as $G(\bar{F}_\infty/F_{\infty w}^{sh})$ – acyclic in the sense of continuous cohomology. But the continuous cohomology of a profinite group G and a discrete G - module can be computed as the right derived functor of Γ^G (G–invariants functor) in the category of discrete G – modules. Hence putting $G = G(\bar{F}_\infty/F_\infty)$ and taking $m \geq 1$ we get:

$$0 = H^m(specF_\infty;\, I.) = R^m\Gamma^G(I._{\bar{w}}) = H^m_{cts}(G;\, I._{\bar{w}}), \tag{7.17}$$

because $I.$ is a flasque sheaf. The middle equality follows from theorem 1.9 [23] p. 53 or proposition (4.4) [16] p. 25. We have the same explanation for the group $G(\bar{F}_\infty/F_{\infty w}^{sh})$. We need to pull back the resolution $I.$ to $specF_{\infty w}^{sh}$ to obtain again a flasque resolution. It is so because $F_{\infty w}^{sh}$ is a sum of finite extensions of F_∞ so the passage to the limit theorem [1] p. 80 applies. The proof of the proposition is finished by the following lemma. QED.

Before stating the lemma let us introduce some notation. For a profinite group G and a G discrete module M let:

$$Hom_G^{cts}(C.(G);\, M) \stackrel{def}{=} \varinjlim_U Hom_{G/U}(C.(G/U);\, M^U), \tag{7.18}$$

where as usual $\{U\}$ is the injective system of all normal, open subgroups of G and $C.(G/U)$ denotes the standard resolution for the finite group G/U. We observe that $Hom_G^{cts}(C_n(G);\, M) = C^n(G, M)$ (identified as sets) where the last is the set of all continuous functions from G^n to M.

LEMMA 9. *Let $H \subset G$ be two profinite groups. Let $I.$ be an exact complex of G as well as H discrete and acyclic modules (in the continuous cohomology sense) with an augmentation map $M \to I_0$. In addition assume that cohomology of the complex $I.^G$ ($I.^H$ resp.) computes the continuous G (H resp.) cohomology. Then the following diagram commutes and the horizontal maps induce isomorphisms on cohomology:*

$$\begin{array}{ccccc}
Hom_G^{cts}(C.(G); M) & \longrightarrow & Hom_G^{cts}(C.(G); I.) & \longleftarrow & Hom_G(Z; I.) \\
\downarrow & & \downarrow & & \downarrow \\
Hom_H^{cts}(C.(H); M) & \longrightarrow & Hom_H^{cts}(C.(H); I.) & \longleftarrow & Hom_H(Z; I.)
\end{array}$$

Diagram 7.19.

PROOF. It is enough to observe, that the functor $Hom_G^{cts}(C.(G); -)$ is exact for any profinite group G. Hence by assumptions both bicomplexes $Hom_G^{cts}(C.(G); I.)$ and $Hom_H^{cts}(C.(H); I.)$ have vertical and horizontal differentials exact. QED.

We also observe, that we can do the identifications in proposition 3 in such a way, that when we put the identified maps to the diagram 7.15, all maps in this diagram will be G_∞ equivariant. We observe also that the right vertical arrow in the diagram 7.11 for $m = 1$ is just the left vertical arrow in the diagram 7.15 after the identification and taking G_∞ invariants, hence it equals to the left vertical arrow in the diagram 7.16.

§ 6. GOING BACK TO THE K-THEORY.

It is easy to observe that in the paragraphs 1-5 we have proven the following lemma.

LEMMA 10. *The following diagram commutes:*

$$\begin{tikzcd}
K_{2n}(O_l)_l \arrow[r, "nat"] \arrow[d, "\approx"] & K_{2n}(O_S)_l \arrow[d, "\approx"] \\
K_{2n}(O_l; Z_l) \arrow[r, "nat"] \arrow[d] & K_{2n}(O_S; Z_l) \arrow[d] \\
m_l(n)^{G_\infty} \arrow[r] & m_S(n)^{G_\infty}
\end{tikzcd}$$

Diagram 7.20.

We observe that by computation in this chapter and by [27] theorem 6, the compositions of the vertical maps in the diagram 7.20 are surjective. These compositions are (up to isomorphisms which we saw before) the Chern class maps which in [27] p. 261 are denoted $c_{n+1,2}$. Moreover by Quillen localization theorem [24] (corollary of theorem 5) and by diagram 7.16 we have:

$$\varinjlim_S K_{2n}(O_S)_l = K_{2n}(F)_l \text{ and } \varinjlim_S m_S(n)^{G_\infty} = (F_\infty^* \otimes W^n)^{G_\infty}. \qquad (7.19)$$

Hence the diagram 7.20 and the above equalities give us the following surjective map upon taking the injective limit with respect to S:

$$\tilde{c}_{n+1,2} : K_{2n}(F)_l \to (F_\infty^* \otimes W^n)^{G_\infty}.$$

Observe that by [28] prop. 2.3:

$$H^1(F;\, W^{n+1}) \approx H^2_{cts}(F;\, Z_l(n+1)). \qquad (7.20)$$

Hence we have constructed a surjective map:

$$\tilde{c}_{n+1,2} : K_{2n}(F)_l \to H^2_{cts}(F;\, Z_l(n+1)).$$

Let us now apply some Galois cohomology. Consider the following commutative diagram:

$$\begin{tikzcd}
(F_\infty^* \otimes W^n)^{G_\infty} \arrow[r] & \bigoplus_{v \nmid l} (W^n)^{G_{v\infty}} \\
H^1(F_\infty; W^{n+1})^{G_\infty} \arrow[r] \arrow[u, "\approx"] & \bigoplus_{v \nmid l} H^1(F_{v\infty}; W^{n+1})^{G_{v\infty}} \arrow[u, "\approx"] \\
H^1(F; W^{n+1}) \arrow[r] \arrow[u, "\approx"', "res"] & \bigoplus_{v \nmid l} H^1(F_v; W^{n+1}) \arrow[u, "\oplus res"]
\end{tikzcd}$$

Diagram 7.21.

The upper vertical isomorphisms are given by Kummer pairings. In the above diagram $G_{v\infty} = G(F_{v\infty}/F_v)$ where $F_{v\infty} = F_v(\mu_{l^\infty})$. We also have the following commutative diagram:

$$\begin{tikzcd}
 & & H^1(F; W^{n+1}) \arrow[r] & \bigoplus_{v \nmid l} (H^1(F_v; W^{n+1})/Div) \\
0 \arrow[r] & D_{n+1}(F) \arrow[r] & H^1(F; W^{n+1}) \arrow[r] \arrow[u, "id"] & \bigoplus_{v} (H^1(F_v; W^{n+1})/Div) \arrow[u]
\end{tikzcd}$$

Diagram 7.22.

In the lower horizontal exact sequence $D_{n+1}(F)$ denotes the group of all divisible elements in $H^1(F; W^{n+1})/Div$, [26] §4 satz 8. Div denotes the maximal divisible subgroup in a respective group. The middle horizontal, exact sequence in the diagram 7.23 below shows that Div in $H^1(F; W^{n+1})$ is trivial. It is so, because

$m_l(n)^{G_\infty} \approx [A \otimes_{Z_l} \tau(n)]^{G_\infty}$ [7] p. 115 which is finite by the theorem of Mazur and Wiles (main conjecture in Iwasawa theory). For reference see [8] p. 222 and 225 or [22].

$$\begin{tikzcd}
0 \arrow[r] & K_{2n}(O_l)_l \arrow[r, "nat"] \arrow[d, "c_{n+1,2}"] & K_{2n}(F)_l \arrow[d, "\tilde{c}_{n+1,2}"] & \\
0 \arrow[r] & m_l(n)^{G_\infty} \arrow[r] & (F_\infty^* \otimes W^n)^{G_\infty} \arrow[r] & \bigoplus_{v \nmid l} W^n(F_v) \\
0 \arrow[r] & D_{n+1}(F) \arrow[r] \arrow[u, hook] & H^1(F; W^{n+1}) \arrow[r] \arrow[u, "\approx"] & \bigoplus_{v} (H^1(F_v; W^{n+1})/Div) \arrow[u]
\end{tikzcd}$$

Diagram 7.23.

CHAPTER VIII
WILD KERNEL

In this chapter we still assume that F/Q is totally real abelian and $l > n$

§ 1. TWO LEMMAS.

LEMMA 11. *The kernel of the map* $\tilde{c}_{n+1,2}$ *is finite.*

PROOF. We have the following commutative diagram:

$$\begin{array}{ccc} K_{2n}(F)_l & \xrightarrow{id} & K_{2n}(F)_l \\ \downarrow{\scriptstyle \varinjlim_S} & & \downarrow{\scriptstyle c_{n+1,2}} \\ \varinjlim_S H^2(O_S;\alpha_* Z_l(n+1)) & \longrightarrow & H^2(F; Z_l(n+1)) \end{array}$$

Diagram 8.1.

Hence it is clear by the previous chapter that the map $\tilde{c}_{n+1,2}$ is a composition of the left vertical arrow in the diagram 8.1 and some isomorphisms. So it is enough to prove that the left vertical map has finite kernel and eventually it suffices to prove that the right vertical arrow has finite kernel.

Let us consider the following diagram, which commutes up to homotopy:

$$\begin{tikzcd} S^{2n-1} \arrow[r, "l^m"] \arrow[d, "id"] & S^{2n-1} \arrow[r] \arrow[d, "l"] & Y^{2n}_{l^m} \arrow[d] \\ S^{2n-1} \arrow[r, "l^{m+1}"] & S^{2n-1} \arrow[r] & Y^{2n}_{l^{m+1}} \end{tikzcd}$$

Diagram 8.2.

It gives the following commutative diagram:

$$\begin{tikzcd} K_{2n}(F) \arrow[r, "l^{m+1}"] \arrow[d, "l"] & K_{2n}(F) \arrow[r] \arrow[d, "id"] & K_{2n}(F; l^{m+1}) \arrow[d] \\ K_{2n}(F) \arrow[r, "l^m"] & K_{2n}(F) \arrow[r] & K_{2n}(F; l^m) \end{tikzcd}$$

Diagram 8.3.

The right vertical arrow in the diagram 8.1 factors by definition in the following way:

$$K_{2n}(F)_l \to K_{2n}(F; Z_l) \to H^2(F; Z_l(n+1))$$

We see by the diagram 8.3 that the kernel of the map:

$$K_{2n}(F)_l \to K_{2n}(F; Z_l)$$

is equal to:

$$(\bigcap_{m\geq 1}(K_{2n}(F))^{l^m})_l = \bigcap_{m\geq 1}(K_{2n}(F)_l)^{l^m} \tag{8.1}$$

which is finite because it is contained in $K_{2n}O_F$. So we need to prove that the kernel of the map

$$\overleftarrow{c}_{n+1,2} : K_{2n}(F; Z_l) \to H^2(F; Z_l(n+1))$$

is finite. But $\overleftarrow{c}_{n+1,2}$ is the inverse limit on coefficients of the maps:

$$\bar{c}_{n+1,2} : K_{2n}(F; l^m) \to H^2(F; \mu_{l^m}^{n+1})$$

Now we can use the following commutative diagram [27] p. 288, with exact rows:

$$\begin{tikzcd}
K_{2n}(O_l;l^m) \arrow[r] \arrow[d, "\bar{c}_{n+1,2}"] & K_{2n}(F;l^m) \arrow[r] \arrow[d, "\bar{c}_{n+1,2}"] & \bigoplus_{v\nmid l} K_{2n-1}(k_v;l^m) \arrow[d, "-n\bar{c}_{n,1}"] \\
H^2(O_l;\mu_{l^m}(n+1)) \arrow[r] & H^2(F;\mu_{l^m}(n+1)) \arrow[r] & \bigoplus_{v\nmid l} H^1(k_v;\mu_{l^m}(n))
\end{tikzcd}$$

Diagram 8.4.

Because $l > n$, the right vertical arrow is an isomorphism [27] proposition 5. In this case simple diagram chasing shows that the kernel of $\bar{c}_{n+1,2}$ (the middle one) is contained in the image of $K_{2n}(O_F;\ l^m)$. The Bockstein exact sequence gives us the following short exact sequence:

$$0 \to K_{2n}(O_F)/l^m \to K_{2n}(O_F;\ l^m) \to K_{2n-1}(O_F)_{l^m} \to 0$$

The groups $K_{2n}(O_F;\ l^m)$ have bounded orders independently of l^m because the group $K_{2n}(O_F)$ is finite and $K_{2n-1}(O_F)$ is finitely generated abelian group. Hence the kernels of the maps $\bar{c}_{n+1,2}$ have bounded orders independently of l^m. Since the inverse limit is left exact, the kernel of $\overleftarrow{c}_{n+1,2}$ is equal to the inverse limit of the kernels of the maps $\bar{c}_{n+1,2}$. Such an inverse limit must be finite. QED.

LEMMA 12. *Let A and B be l-torsion groups. In addition let the following homomorphism f: $A \to B$ be surjective and have finite kernel. Then for every divisible element b in B there is at least one divisible element a in A such that $f(a) = b$.*

PROOF. Observe that it is enough to check only l-divisibility. For any m, $b = b_m^{l^m}$ for some element b_m. Because f is surjective, there is for each m, an element a_m in B such that $f(a_m) = b_m$. But $f(a_m^{l^m}) = b_m^{l^m} = b$. Hence $a_1^{-l}a_m^{l^m} \in ker(f)$ for each m. But the kernel is finite so infinitely many elements of the form $a_1^{-l}a_m^{l^m}$ must be equal to each other. Hence infinitely many elements of the form $a_m^{l^m}$ must be equal to each other and it means that they define a divisible element a. Clearly $f(a) = b$. QED.

§ 2. DIVISIBLE ELEMENTS IN K-THEORY.

THEOREM 2. *If F/Q is totally real abelian and $l > n$ then:*

$$\#\Big(\bigcap_{r\geq 1} K_{2n}(F)^r\Big)_l \;\geq\; \left| \frac{w_{n+1}(F)\zeta_F(-n)}{\prod_{v|l} w_n(F_v)} \right|_l^{-1} \tag{8.2}$$

PROOF. It follows by lemma 11, diagram 8.1 and [27] theorem 6 iii, that the map $\tilde{c}_{n+1,2}$ satisfies all conditions of lemma 12. Hence it follows by the bottom part of the diagram 7.23 and the last lemma that:

$$\#\Big(\bigcap_{r\geq 1} K_{2n}(F)^r\Big)_l \geq \# D_{n+1}(F) \tag{8.3}$$

But by [26] §8 and §5 satz 5:

$$\# D_{n+1}(F) \;=\; \left| \frac{w_{n+1}(F)\zeta_F(-n)}{\prod_{v|l} w_n(F_v)} \right|_l^{-1} \tag{8.4}$$

because the following Lichtenbaum conjecture:

$$|\zeta_F(-n)|_l^{-1} = \frac{\# H^1(O_l;\, j_* W_F^{n+1})}{\# H^0(O_l;\, j_* W_F^{n+1})} \tag{8.5}$$

follows from the theorem of Mazur and Wiles [8] p. 222 and 225, [22].QED

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