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INFERENCE FOR A BIVARIATE SURVIVAL FUNCTION INDUCED THROUGH THE ENVIRONMENT

The Ohio State University PH.D. 1986

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INFERENCE FOR A BIVARIATE SURVIVAL FUNCTION INDUCED THROUGH THE ENVIRONMENT

DISSERTATION

Presented in Partial Fulfillment of the Requirements for
the Degree Doctor of Philosophy in the Graduate
School of The Ohio State University

By

Sukhoon Lee, B. S.

* * * * *

The Ohio State University

1986

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CHAPTER I

INTRODUCTION

1.1 Objective

In this thesis, we propose a random environmental effects model for two component systems and investigate some general properties of that model. Specific models which are appropriate and useful for competing risks experiments are defined, and statistical inference procedures for the model parameters are developed. Finally, estimators' performance and properties are explored by a Monte Carlo study.

1.2 Motivation

Consider a two component system as the simplest example. Each component will have a random life length, and the life of the entire system will depend on the failure patterns of the components of the system such as serial failure pattern where failure of any component causes the system to fail or a parallel failure pattern where the system fails when all the components fail. One important practical problem is to infer the system life length from knowledge of the individual component life times.

In most of the research done in the past, the failure distributions associated with each component was assumed to be known, and the components were assumed to actindependently of each other to arrive at the system failure distribution.This common,although untestable assumption of independence, may be questionable in many practical situations. For example, in engineering systems where the individual components are subject to greater probability of failure due to their close proximity the assumption of independence may not be reasonable. In order to correctly analyze such experiments when there are dependent competing risks, a multivariate model is needed for the lifetimes of the components. The models proposed in the past have been justified on the grounds of mathematical tractability with little or no practical justification.

In this thesis, we consider a model which is motivated by the way systems are tested in the design and operating phase of development. Instead of assuming a specific multivariate model, we derive a random environmental effects model which incorporates a common environment acting on all components and then induces a dependent structure between the failure times of components.

1.3 Model

In this section, we define models which are appropriate for a two component system incorporating the knowledge of the lifetimes of the individual components and the common environment under which the system operates.

We assume that under controlled conditions, as one may encounter in the testing or design stage of development, the time to failure of the two components, to be linked in a system, are X_0 and Y_0. We suppose that under these conditions, X_0, Y_0 have survival functions F_0, G_0 on $[0,\infty)$.

Now suppose that the above two components are put into operation under usage conditions. We suppose that under such conditions the effect of the environment is to degrade or improve each component by the same random amount. That is, the effect of the environment is to select a random factor, Z, from some distribution, $H(\cdot)$, which changes the marginal survival functions of the two components to F_0^Z and G_0^Z. A value of Z less than one means that component reliabilities are simultaneously improved, while a value of Z greater than one implies a joint degradation. We assume that two components in a system under fixed conditions (i.e. given Z) function independently. Then the resulting joint survival function of the two components' lifetimes, (X,Y) in the operating environment is

$$F(x,y)=E\,(\,F_o^{Z}(x)\cdot G_o^{Z}(y)) \qquad (1.3.1)$$

The specific choice of the model is based on both theoretical support and empirical support. In Chapter 3, we discuss the general properties of this model in detail.

1.4 Inference

Statistical inference for the two component series system is carried out under the

assumption that X_o, and Y_o have exponential distributions with the hazard rates λ_1, λ_2 and Z has a gamma

distributon with density $h(z)=\dfrac{\beta^{\alpha}}{\Gamma(\alpha)}\, z^{\alpha-1}\exp(-\beta z)$. (1.4.1)

Since the parameters λ_1, λ_2 are not identifiable when only data from series systems is available, we incorporate sample information on each component under controlled conditions. Maximum likelihood and method of moments estimators are obtained and their properties are studied by Monte Carlo methods since no closed form maximum likelihood estimates are available. Also a new estimator based on the scaled total time on test transform is presented

Since this setting contains three sets of samples, two from components themselves and the other from systems, an investigation is done to find the optimal scheme for determining sample sizes subject to various cost constrain

In addition we discuss a graphical representation of this model which leads to not only checking the model itself but also testing the strength of the dependence induced by the common environment.

CHAPTER II

BIVARIATE MODELS WITH THE FIXED ENVIRONMENTAL FACTOR

2.1 Introduction

In many reliability problems for multi-component systems, the conventional methodology mentioned in section 1.2 often makes an assumption, for the sake of simplicity, that the components in the system function independently. In other words, the component lifetimes are statistically independent. However this assumption has been questioned in many practical situations. For example, we can think of a critical shock to a system which affects all the components in the system simultaneously, or of a patient with a poor prognostic indication who may be removed from the study before death.

In order to correctly analyze the data from the dependent cases multivariate models have been suggested for use in the reliability context. However most of these have been justified only on the ground of mathematical tractability. In this chapter we review the relevant literature on modeling bivariate survival functions for the two component systems, specially focusing on their practical justification. In order to separate these models from the random environmental effect models we call these models the fixed environmental effects model.

2.2 Conventional Models

In this section we shall consider bivariate survival functions which are motivated by mimicing the properties of the univariate exponential or are mathematically contrived so that the marginals are exponential. These models are commonly used since the exponential distribution has played a central role in univariate reliability studies.

Gumbel (1960) suggested three bivariate distributions with exponential margins, two of which are briefly considered here. The first is

$$F(x,y)=P(X > x, Y > y)$$

$$= \exp(- \lambda_1 x - \lambda_2 y - \delta\lambda_1\lambda_2 xy) \quad \text{for } x, y > 0. \tag{2.2.1}$$

where $0<\delta<\lambda_1 \cdot \lambda_2$. The marginal survival function of X(Y) is $\exp(-\lambda_1 x)\{\exp(-\lambda_2 y)\}$ and the correlation between X and Y is decreasing from zero to -.4837 for increasing values of δ from 0 to $\lambda_1 \cdot \lambda_2$. The second is

$$F(x,y) = \exp(-\lambda_1 x - \lambda_2 y) [1 + \alpha (1 - \exp(- \lambda_1 x)\} \{1 - \exp(- \lambda_2 y)\}], \quad \text{for } x, y \geq 0 \tag{2.2.2}$$

where $-1 \leq \alpha < 1$. He found that the correlation between X and Y is $\alpha/4$. Any practical justification in reliability context for these models is not found.

Freund (1961) presented a different bivariate extension of the exponential distribution which is designed for the life testing of a two component system, which can function with different hazard rate even after one of the components has failed. Consider a two component system. Let X and Y be the random variables denoting the component lifetimes whose distributions are exponential with $1/\alpha$ and $1/\beta$ as the mean life times respectively. If it is assumed that the failure of the one component, say A, changes the mean life time of the other component, say B, from $1/\beta$ to $1/\beta'$ then the bivariate density function f(x,y) is obtained as

$$f(x,y) = \alpha\beta'\exp\{-\beta'y - (\alpha+\beta-\beta')x\} \qquad \text{for } 0 < x < y \tag{2.2.3}$$

$$\beta\alpha'\exp\{-\alpha'x - (\alpha+\beta-\alpha')y\} \qquad \text{for } 0 < x < y$$

where $\alpha,\beta,\alpha',\beta' > 0$. First we note that the marginals are not exponential. The correlation between X and Y is shown to vary from -1/3 to 1, and estimation of the parameters of the model is discussed by Freund (1961).

Another version of the bivariate exponential is suggested by Marshall and Olkin (1967). They defined the Bivariate Exponential Distribution with parameters $\lambda_1, \lambda_2, \lambda_{12}$ {B.V.E.($\lambda_1, \lambda_2, \lambda_{12}$)} as

$F(x,y)=P(X>x, Y>y)$

$$=\exp(-\lambda_1 x - \lambda_2 y - \lambda_{12}\max(x,y)) \qquad \text{for } x>0, y>0. \tag{2.2.4}$$

The physical motivation for this model is based on occurences of "shocks" to each or both components. Consider a two component system with component lifetimes X and Y. The components are subjected to three types of fatal shocks c_1, c_2, and c_{12}. Let U_i be the time until the shock c_i occurs, i=1, 2 ,{1,2}. The random variable U_1, U_2, U_{12} are assumed to be independent exponential random variables with parameters λ_1, λ_{12} and λ_{12}, respectively. The c_1 shock "kills" component 1; the c_2 shock "kills" component 2; and the c_{12} shock "kills" both components. The B.V.E. distribution is obtained by letting $X=\min(U_1,U_{12})$ and $Y=\min(U_2,U_{12})$.

Some important properties for the B.V.E.$(\lambda_1, \lambda_2, \lambda_{12})$ are as follows:

i) The B.V.E.$(\lambda_1,\lambda_2,\lambda_{12})$ is not absolutely continuous since

$P(X=Y)=\lambda_{12}/(\lambda_1+\lambda_2+\lambda_{12})$;

ii) The marginals are exponential;

iii) The distribution has the loss of memory property (L.M.P.) in the sense that

$F(s_1+t, s_2+t) = F(s_1, s_2)\cdot F(t, t)$ for all $s_1, s_2, t>0$;

iv) The correlation between X and Y is $\lambda_{12}/(\lambda_1+\lambda_2+\lambda_{12})$.

Block and Basu (1974) showed that this distribution is the only one which has both the exponential marginals and L.M.P. Basu and Klein (1982) have reviewed some estimators of $(\lambda_1, \lambda_2, \lambda_{12})$. These are the maximum likelihood estimators obtained by Bhattacharyya and Johnson(1971), the method of moment estimator of Bemis, Bain, and Higgins (1972), an intuitive estimator of Proschan and Sullo (1976), and consistent unbiased estimators of Arnold (1968).

Marshall and Olkin also obtained a bivariate Weibull distribution as

$$F(x, y)=\exp\{-\lambda_1 x^{d_1} - \lambda_2 y^{d_2} - l_{12}\max(x^{d_1}, y^{d_2})\}, d_1, d_2 > 0. \qquad (2.2.5)$$

by transforming the bivariate exponential random variables (X,Y) into $(X^{1/d_1}, Y^{1/d_2})$. Moeschberger(1974) has explored some properties of this bivariate Weibull distribution and estimated its parameters in the competing risk framework. He has discussed maximum likelihood estimation for both the $d_1=d_2$ and $d_1 \neq d_2$ cases.

Lee and Thompson(1974) derived a bivariate Weibull distribution from the fatal shock model. Let X and Y be the lifetimes of each component in a two component system. The components are subject to three types of independent shocks, d_1, d_2, and d_{12}, with random occurence times U_1, U_2, and U_{12} respectively. The shocks d_1, d_2, d_{12} "kills" the X component, Y component, and both components, respectively. Let U_i have a Weibull distribution with survival function

$F_i(u)=\exp(-\lambda_i u^{d_i})$, λ_i, $d_i>0$, $u>0$, $(i=1,2,\{12\})$. The bivariate Weibull is obtained by letting

$X=\min(U_1, U_{12})$ and $Y=\min(U_2,U_{12})$ as

$$F(x, y)=\exp(-\lambda_1 x^{d_1} - \lambda_2 y^{d_2} - \lambda_{12}[\max(x, y)]^{d_{12}} \qquad (2.2.6)$$

for $x, y>0$. Here the marginals are not Weibull.

Block and Basu (1974) have derived, in two ways, an absolutely continuous bivariate extension of the exponential distribution (A.C.B.V.E.). They have shown(Corollary 2) that the assumptions of the L.M.P., exponential marginals, and absolute continuity yields a bivariate distribution with independent exponential marginals. By assuming that the marginals are mixtures or weighted average of exponentials instead of exponential marginals they obtained the joint survival function as

$$F(x, y)= \{\lambda/(\lambda_1+\lambda_2)\}\exp\{-\lambda_1 x-\lambda_2 y-\lambda_{12}\max(x,y)\}$$

$$-\{\lambda_{12}/(\lambda_1+\lambda_2)\}\exp\{-\lambda \max(x,y)\}, \qquad (2.2.7)$$

where $\lambda = \lambda_1 + \lambda_2 + \lambda_{12.}$ It is noted that the joint distribution, which has the L.M.P., turns out to be the absolute continuous part of B.V.E.($\lambda_1, \lambda_2, \lambda_{12}$) of Marshall and Olkin (1967). They also showed this distribution is a variant of the distribution suggested by Freund (1961). This model can also be obtained by a fatal and non-fatal shock model suggested by Friday and Patil (1977).

Downton (1970) has derived another bivariate exponential distribution whose density is

$$f(x,y)= \{\lambda_1 \cdot \lambda_2/(1-\rho)\}\exp\{-(\lambda_1 x\, \lambda_2 y)/(1-\rho)\}\cdot I_o\{2\cdot(\rho\cdot \lambda_1 \cdot \lambda_2 \cdot x \cdot y)^{1/2}(1-\rho)\} \qquad (2.2.8)$$

where $I_o(x)=\Sigma(x/2)^{2r}/(r\,!)^2$ is the modified bessel function of the first kind of order zero. The motivation for this model is based on the cumulative shock model generalized from the one component situation. Let $T_1, T_2, \cdots$ be the independent random variables representing the time between the successive shocks to a system, each having distribution function G(t) with Laplace transform ϕ (s) and let N be the random number of shocks required to cause failure with probability generating function $\Pi(z)$. Let X be the lifetime of the component. He observed that if X has an exponential distribution with mean lifetime $1/\lambda$, then the two identities, $E_x(e^{-sx})=\Pi(\phi(s))$ and $E(X)=E(N)\cdot E(T_i)$ should hold for all s. As one of the possible combinations of distributions of T_i's and N, satisfying the above two identities he considered an exponential distribution for T_i's and a geometric distribution for N. Accordingly, he assumed in a two component system that the intervals between successive shocks on each component are independent and exponentially distributed, and that the numbers of shocks required to produce failure in each component follows a bivariate geometric distribution. Then he arrived at the joint density (2.2.8) by an elegant argument.

CHAPTER III

BIVARIATE MODELS WITH RANDOM ENVIRONMENTAL FACTOR

3.1 Introduction

Instead of specifying the particular dependent model, some of which have been reviewed in the previous chapter we would like to consider a very natural situation which is encountered in the real world. As mentioned in section 1.3, the basic idea of the random environmental effect model in an engineering context is that the system, which consists of two components, is used under operating condition while the individual components are tested under controlled condition, since they are not the final product the consumers will use. Therefore the factors which are liable to affect the system survival function under operating condition may be divided into three parts, the random lifetime of each component under controlled condition, and the random stress characterizing the operating condition, and impact of this random stress on the lifetimes of the components in the system under operating condition.

In section two we present a general model and provide properties of the model. In section three the random environmental factor is assumed to follow a gamma distribution. Section four deals with the case when both components have Weibull lifetime distributions and a random environmental factor has an arbitrary distribution.

Since the joint survival function induced by the random environmental effect model provides a general dependence structure we shall discuss not only the properties of the model for system reliability but also explore the general dependence structure.

3.2 The General Model

Let X_o and Y_o be the random lifetimes of the components under controlled condition and let Z be the random stress for the operating environment. We suppose in this section that the three random variables involved follow general distributions, that is, X_o, Y_o have the survival functions $F_o(\cdot)$, $G_o(\cdot)$ respectively, and Z has a distribution $H(\cdot)$. We denote the cummulative hazard functions of X_o, Y_o as $Q_{xo}(\cdot)$ and $Q_{yo}(\cdot)$, and hazard rates as $Q_{xo}'(\cdot)=q_{xo}(\cdot)$ and $Q_{yo}'(\cdot)=q_{yo}(\cdot)$. Then we have $F_o(t)=\exp(-Q_{xo}(t))$, $G_o(t)=\exp(-Q_{yo}(t))$.

Let (X,Y) be the pair of random lifetimes of the components in a system under operating environment. The resulting joint survival function of (X,Y) is

$$F(x,y)=E\{F_o^Z(x) \cdot G_o^Z(y)\}$$

$$=E[\exp\{-(Q_{xo}(x)+Q_{yo}(y))Z\}] \qquad (3.2.1)$$

We note that this model can be introduced through Cox's regression model (1972) as described by Clayton (1978) in incidence studies. Suppose the operating environment under which the systems function allows for one covariate v, such as temperature, pressure, etc. According to Cox's model we assume the survival functions of each component are $\exp\{-Q_{xo}(x)e^{cv}\}$, and $\exp\{-Q_{yo}(y)e^{cv}\}$, and the joint survival function, conditional upon v, is $F(x,y| v)=\exp[-e^{cv}\{Q_{xo}(x)+Q_{yo}(y)\}]$. Hence the model we suggest can be thought of as being induced through a random covariate v which is commonly shared by two components in a system.

Furthermore we may incorporate fixed effect covariates, if obtainable, into our model. Let V^* be a set of variables which characterizes an environment. Since it is practically impossible in many cases to observe all the elements in V^* let us assume that we observe a subset Vof V^* and that the ommitted variables may be captured by a single random variable Z, where Z has a distribution function H(z). Let us further suppose that two components in a system function independently given V^* (i.e.,V,Z) with hazard rates $zq_{xo}(x)\exp(\underline{c}\cdot\underline{v})$ and $zq_{yo}(y)\exp(\underline{c}\cdot\underline{v})$. Thus when c is a vector of constants, we have a joint survival function for the two components in a system as

$$E[\exp\{-\exp(\underline{c}\cdot\underline{v})Z\,(Q_{xo}(x)+Q_{yo}(y))\}]. \qquad (3.2.2)$$

Although the above model with covariates present is more general, we will discuss the model without covariates since the main focus of this dissertation is on the dependent structure of the survival function induced by a random environment, and all of our results can be easily extended to the covariate model.

First we discuss positive dependence properties of the model. Before doing this, we review several alternative notions of positive dependence based on the material in Barlow and Proshan (1981).

Definition 3.1. (Barlow and Proshan.1981)

For a pair of random variables (X,Y),

1) (X,Y) is said to be TP2 dependent if the joint density function f(x,y) satisfies the condition

$$f(x_1,y_1)\cdot f(x_2,y_2) - f(x_1,y_2)\cdot f(x_2,y_1) \geq 0 \qquad (3.2.3)$$

for all $x_1<x_2$, $y_1<y_2$.

2) X is stochastically increasing in Y if

$P(X>x \mid Y=y)$ is increasing in y for all x.

3) X is right tail increasing in Y if

$P(X>x \mid Y>y)$ is increasing in y for all x.

4) X and Y are said to be associated if

for functions g_1 and g_2 nondecreasing in each variable

$$COV(g_1(X,Y), g_2(X,Y)) \geq 0 \qquad (3.2.4)$$

5) X and Y are said to be positively quadrant dependent if

$$P(X>x, Y>y) \geq P(X>x) \cdot P(Y>y) \quad \text{for all } x, y. \qquad (3.2.5)$$

Theorem 3.2 (Barlow and Proshan. 1981)

The notions 1) - 5) in the definition 3.1 can be arranged into a hierarchy as

1) => 2) => 3) => 4) => 5).

We are now in position to derive the dependence properties of the general model by showing that (X,Y) is TP2 dependent. Even if TP2 dependence is not intuitively appealing theorem 3.2 does imply that the other more appealing notions of dependence do hold and hence gives insight into the dependence structure.

Theorem 3.3 The random variables X, Y of the lifetimes of the two components in a system under the operating environment are TP2 dependent.

proof) Let $f_o(x)$ and $g_o(y)$ be the density function of X_o and Y_o respectively and f(x,y) be the joint density funtion of (X,Y). Then

$$f(x,y)=E\{Z^2F_o^{Z-1}(x)G_o^{Z-1}(y)f_o(x)g_o(y)\}$$

$$= f_o(x)g_o(y)\, E\{Z^2F_o^{Z-1}(x)G_o^{Z-1}(y)\}$$

$$= f_o(x)g_o(y) \int z^2F_o^{z-1}(x)G_o^{z-1}(y)dH(z). \qquad (3.2.6)$$

Noting that $f_o(x)=q_{xo}(x)\exp(-Q_{xo}(x))$, we have

$$f(x,y)=q_{xo}(x)q_{yo}(y)\int z^2 F_o{}^z(x)G_o{}^z(y)dH(z). \text{ Then} \tag{3.2.7}$$

$$\begin{aligned} f(x_1,y_1)\cdot f(x_2,y_2) &= \int\int_{u>v} q_{xo}(x_1)q_{yo}(y_1)\, q_{xo}(x_2)q_{yo}(y_2)\, u^2\, v^2 F_o{}^u(x_1)G_o{}^u(y_1)\, F_o{}^v(x_2)G_o{}^v(y_2)dH(u)dH(v) \\ &+ \int\int_{v>u} q_{xo}(x_1)q_{yo}(y_1)\, q_{xo}(x_2)q_{yo}(y_2)\, u^2\, v^2 F_o{}^u(x_1)G_o{}^u(y_1)\, F_o{}^v(x_2)G_o{}^v(y_2)dH(u)dH(v) \\ &= \int\int_{u>v} q_{xo}(x_1)q_{yo}(y_1)\, q_{xo}(x_2)q_{yo}(y_2)\, u^2\, v^2 F_o{}^u(x_1)G_o{}^u(y_1)\, F_o{}^v(x_2)G_o{}^v(y_2)dH(u)dH(v) \\ &+ \int\int_{u>v} q_{xo}(x_1)q_{yo}(y_1)\, q_{xo}(x_2)q_{yo}(y_2)\, u^2\, v^2 F_o{}^v(x_1)G_o{}^v(y_1)\, F_o{}^u(x_2)G_o{}^u(y_2)dH(u)dH(v). \end{aligned} \tag{3.2.8}$$

In the same manner, the other product $f(x_1,y_2)\cdot f(x_2,y_1)$ also can be written as integrals over the region $u>v$ and $f(x_1,y_1)\cdot f(x_2,y_2) - f(x_1,y_2)\cdot f(x_2,y_1)$ can be written as

$$\begin{aligned} \int\int_{u>v} u^2\, v^2 [\, & q_{xo}(x_1)\, q_{xo}(x_2)\, F_o{}^u(x_1)\, F_o{}^v(x_2) - q_{xo}(x_1)\, q_{xo}(x_2) F_o{}^v(x_1)\, F_o{}^u(x_2)]\cdot \\ [\, & q_{yo}(y_1)\, q_{yo}(y_2)\, G_o{}^u(y_1)\, G_o{}^v(y_2) \\ & - q_{yo}(y_1)\, q_{yo}(y_2) G_o{}^v(y_1)\, G_o{}^u(y_2)]\, dH(u)dH(v). \end{aligned} \tag{3.2.9}$$

Rewrite the first blanketed term in the integrand as

$$q_{xo}(x_1)\, q_{xo}(x_2)\, F_o{}^u(x_1)\, F_o{}^u(x_2)\{F_o{}^v(x_2) / F_o{}^u(x_2) - F_o{}^v(x_1) / F_o{}^u(x_1)\}. \tag{3.2.10}$$

We know that this term is always nonnegative over the region $u>v$ since

$F_o^v(x) / F_o^u(x) = \exp[-(v-u) Q_{XO}(x)]$ is increasing in x. Similarly the second blanketed term is always nonnegative over the region u > v.

Therefore the integrand is nonnegative over the region u > v so that $f(x_1,y_1) \cdot f(x_2,y_2) - f(x_1,y_2) \cdot f(x_2,y_1) \geq 0$ for all $x_1<x_2$, $y_1 < y_2$, which leads to TP2 dependence. Q.E.D.

As mentioned before, Theorem 3.3 provides useful properties of the dependence structure of the model so that one may use them to exploit related properties in the reliability context. As a simplest case the two properties 2), 3) in definition 3.1 lead us to the following corollary.

Corollary 3.3.1 Under the same setting as in theorem 3.3, the conditional hazard rates

$$q(x \mid Y=y) \text{ and } q(x \mid Y > y) \text{ are decreasing in } y.$$

Proof) Since the conditional survival function given Y=y , $F(x \mid Y=y)$ is increasing in y for all x by theorem 3.3 $q(x \mid Y=y)$ which is the derivative of $-\log F(x \mid Y=y)$, is also decreasing in y for all x. Similarly $q(x \mid Y> y)$ is decreasing. Q.E.D.

Intuitively this corollary implies that the longer one component functions, the more reliable the other component in the system is.

From a different point of view we derive an inequality in terms of the conditional hazard rates which reflects the positive dependence of the model.

Theorem 3.4 Under the same setting as in theorem 3.3, the model satisfies

$$q(x \mid Y = y) > q(x \mid Y > y).$$

Proof) Let $G_1(y)$ be the marginal survival function of y in the system exposed to the operating environment. Then

$$F(x \mid Y{=}y) = P(X > x \mid Y = y) = \frac{\partial F(x, y)}{\partial y} \Big/ \frac{dG_1(y)}{dy}$$

$$= \frac{E(ZF_o^{Z}(x) \cdot G_o^{Z-1}(y) \cdot g_o(y))}{E(Z \cdot G_o^{Z-1}(y) \cdot g_o(y))}$$

$$= \frac{E(Z \cdot F_o^{Z}(x) \cdot G_o^{Z}(y))}{E(Z \cdot G_o^{Z}(y))} , \qquad (3.2.11)$$

since $g_0(y) = q_{y0}(y) \cdot G_0(y)$.

Also $F(x \mid Y > y) = P(X > x \mid Y > y) = \dfrac{E(F_o^{Z}(x) \cdot G_o^{Z}(y))}{E(G_o^{Z}(y))}$. (3.2.12)

Hence, we obtain the following inequality,

$$\frac{q(x \mid Y = y)}{q(x \mid Y > y)} = \frac{E(Z^2 \cdot F_o^{Z}(x) \cdot G_o^{Z}(y))\, E(F_o^{Z}(x) \cdot G_o^{Z}(y))}{E^2(Z \cdot F_o^{Z}(x) \cdot G_o^{Z}(y))} \geq 1, \qquad (3.2.13)$$

since $q(x \mid Y{=}y) = \partial\, [- \log F(x|Y{=}y)] \,/\, \partial x$

$= E(Z^2 F_0^{Z-1}(x) \cdot G_0^{Z}(y) \cdot f_0(x)) \,/\, E(ZF_o^{Z}(x) \cdot G_o^{Z}(y))$

$= q_{x0}(x) E(Z^2 F_0^{Z}(x) \cdot G_0^{Z}(y)) \,/\, E(ZF_o^{Z}(x) \cdot G_o^{Z}(y))$, and

$q(x|y{>}y) = q_{x0}(x) E(Z \cdot F_o^{Z}(x) \cdot G_o^{Z}(y)) \,/\, E(F_o^{Z}(x) \cdot G_o^{Z}(y))$.

The inequality in (3.2.13) is obtained by Cauchy -Schwarz inequality and equality holds if and only if the random variable Z is constant. Q.E.D.

We note that this inequality should be compared with the notion of the quasi independence, which is defined as $q(x \mid Y = y) = q(x \mid Y > y)$. Basu and Klein (1982) have reviewed this notion. We note that quasi independence is the necessary and sufficient condition that the marginal distribution under the dependent model can be recovered from the minimum of X, Y and the knowledge of which component caused the system to fail. In this case there exists a set of independent random variables which yields the same minimum and indicater of system failure as the dependent system. Futhermore these equivalent independent random variables have the same marginals as the dependent system. For example, for the Marshall and Olkin's model holds one can generate the bivariate dependent distribution through the three independent random shocks.

Up to now several properties have been explored in terms of the dependence structure induced by a random environment. Next we investigate the effects of the random environment on system reliability by comparing the reliability function with and without environmental effect. Conventional reliabilty theory commonly uses the knowledge of the component lifetimes and an assumption of independent component lifetimes in order to compute the system life distribution. In other words an investigator modeling system life, based on component information, may predict the reliability of the system, in our setting, with knowledge of $F_0(x)$ and $G_0(y)$ only by $R_{0S}(t) = F_0(t) \cdot G_0(t)$. The following theorem indicates how the two reliabilities are different in a series system.

<u>Theorem 3.5</u> Suppose a two component system is serial, i.e., the system fails if and only if any one of the two components fails. Let $R_S(t)$ and $R_{0S}(t)$ denote the system reliabilities for the cases of a random environment and of a fixed environment.

i) If $E(Z) \leq 1$ then $R_s(t) \geq R_{os}(t)$ for all t

ii) If $E(Z) > 1$ and $P(Z < 1) = 0$ then $R_s(t) < R_{os}(t)$ for all t.

iii) If $E(Z) > 1$ and $P(Z > 1) > 0$ then there exists a t^* such that

$R_s(t) < R_{os}(t)$ for all $t < t^*$ and $R_s(t) > R_{os}(t)$ for all $t > t^*$.

Statement (iii) implies that even if the average operating environment is more severe than the controlled one but if there is a chance of better environment perhaps due to highly cautious maintenance, careful users, or an effective usage the reliability under a random environment becomes more reliable beyond a certain time.

Proof) The ratio of reliabilities for variable to fixed environment $R_{os}(t)/R_s(t)$ is

$$\frac{E\{\exp(-Q_o(t)Z)\}}{\exp\{-Q_o(t)\}} \quad \text{where } Q_o(t) = Q_{xo}(t)+Q_{yo}(t).$$

In the case of $E(Z) \leq 1$, $E\{\exp(-Q_o(t)Z)\} > \exp\{-Q_o(t) \cdot E(Z)\}$ by Jensen's inequality since exp(-uz) is strictly convex function in z. Then i) follows immediately. Note that the equality holds if and only if Z is a constant, i.e. Z=1 with probability 1.The statement (ii) follows by noting that

$$E\{\exp(-Q_o(t)Z)\} = \int \exp(-Q_o(t)z)dH(z) < \exp(-Q_o(t)z) \int dH(z) = \exp\{-Q_o(t)\}.$$

To prove (iii), let

$$r(t) = \frac{E\{\exp(-Q_o(t)Z)\}}{\exp\{-Q_o(t)\}} \qquad (3.2.14)$$

Then $r'(t) = q_o(t)\ E\{\exp(-Q_o(t)Z)\}\ \exp\{-Q_o(t)\}(1-s(t))$ (3.2.15)

where $s(t) = E\{Z\exp(-Q_o(t)Z)\}/ E\{\exp(-Q_o(t)Z)\}$ and $q_o(t) = dQ_o(t)/dt$

Noting that s(0)=E(Z) since $Q_0(0) = 0$, E(Z)>1 implies that r'(0) < 0. Hence since r(t) is decreasing at t = 0 and r(0) = 1 this implies that r(t) < 1 for t in a neighborhood of t=0. To complete the proof it suffices to show that r(t) is increasing beyond a certain point, which is true if r'(t) is positive beyond that point. We claim s(t) is decreasing in t and s(t) < 1 for large t under the given condition.Let us express s(t) as

$$s(t) = \frac{E\{Z\exp(-Q_0(t)Z)\}}{E\{\exp(-Q_0(t)Z)\}} = \int z \frac{\exp(-Q_0(t)z)}{\int \exp(-Q_0(t)z)\, dH(z)} dH(z)$$

$$= \int z\, p(z\,|T>t)\, dz \qquad (3.2.16)$$

where $p(z|T>t)\, dz = \dfrac{\exp(-Q_0(t)z)dH(z)}{c(Q_0(t))}$ and $c(Q_0(t)) = \int \exp(-Q_0(t)z)dH(z)$.

Noting that p(z|T>t) is a density function, s(t) can be expressed in terms of the conditional expection E(Z |T > t). Looking at the density p(z|T>t) we see that

$$\frac{p(z|T>t)}{p(z|T>t)} = \frac{c(Q_0(t_2))}{c(Q_0(t_1))} \cdot \exp\{(Q_0(t_1) - Q_0(t_2))z\} \text{ for } t_1 < t_2 \qquad (3.2.17)$$

is decreasing in z. Then it is an immediate consequence of the following lemma, due to Lehmann (1959) that E(Z| T > t) is decreasing in t

Lemma (Lehmann(1954),pg74) Let $p_\theta(x)$ be a family of densities on the real line with monotone likelihood ratio in x. If $\psi(x)$ is nondecreasing function of x, then $E_\theta(\psi(x))$ is a nondecreasing function of θ.

Let $\theta = 1/t$. Denote $p_\theta(z) = P(z|T>t)$. Then $p_\theta(z)$ has monotone likelihood ratio in z. So

E(Z) is nondecreasing in θ, which implies that E(Z|T>t) is decreasing in t. Now it remains to be shown that s(t) < 1 for some t > 0. Let $p(z)=(z-1)\exp(-Q_0(t)\cdot z)$ and note that p(0) = -1 and p(z) has maximum $[Q_0(t)\{\exp(Q_0(t)+1)\}]^{-1}$ at $z_0=(1+Q_0(t))\cdot Q_0^{-1}(t)$ and p(z) is increasing for $z < z_0$ and decreasing $z > z_0$. Suppose $P(Z < 1) = \varepsilon > 0$.For any $0 < \delta < \varepsilon$, there exists a closed interval [u,v] contained in (0, 1) such that A(u,v) = H(v) - H(u) $\geq \delta$. Then $E(Z\exp(-Q_0(t)Z)) - E(\exp(-Q_0(t)Z))$

$$= \int_0^1 p(z)dH(z) + \int_1^\infty p(z)dH(z)$$

$$\leq \int_u^v p(z)dH(z) + [\, Q_0(t)\cdot\exp(1+Q_0(t))]^{-1}\cdot A(1, \infty)$$

$$\leq p(v)\cdot A(u,v) + [\, Q_0(t)\cdot\exp(1+Q_0(t))]^{-1}\cdot A(1, \infty)$$

$$< (v-1)\exp(-\, Q_0(t)v)\,\delta + [\, Q_0(t)\cdot\exp(1+Q_0(t))]^{-1}. \qquad (3.2.18)$$

Since the last term is negative if and only if $(1-v)\,\delta > [e\cdot Q_0(t)\cdot\exp\{(1-v)Q_0(t)\}]$ there exists a t^* such that $E(Z\exp(-Q_0(t)Z)) - E(\exp(-Q_0(t)Z)) < 0$, that is, $s(t^*) < 1$. Q.E.D.

This theorem implies that conventional methods which are based only on components' information overestimate the reliability at an earlier stage ignoring potential loss from a harsh environment which may be encountered in the beginning stage under the operating condition, while underestimating the possible gains in reliability at later stage from a better environment which meets requirement of each system's susceptibility.

The proof of the theorem yielded an interesting result about the conditional distribution H( ) of a random environmental factor.

Theorem 3.6 The mean and variance of a random environmental factor Z among system survival to a given time t, $E(Z \mid T > t)$ and $V(Z \mid T > t)$ are decreasing in t.

Proof) The results immediately follow the argument about s(t) in the proof of theorem 3.4. Q.E.D.

As one might expect, this theorem indicates that average environmental factor of the surviving systems declines with time since the systems under harsher environments tend to fail first. Also it is noted that the variability of environmental factor of the surviving systems is reduced with time.

We conclude this section by mentioning an curious phenomenon of the hazard rate. In the series system problem the life system distribution after incorporating a random environmental factor has hazard rate $q_s(t) = q_o(t) \cdot E(Z\exp(-Q_o(t)Z)) / E(\exp(-Q_o(t)Z))$. However $E(Z\exp(-Q_o(t)Z)) / E(\exp(-Q_o(t)Z))$ has been shown to be decreasing in t. Thus the lifetime distribution can often have a decreasing hazard rate which the variable environment may cause while the component hazard rates are not decreasing. One plausible explanation is that the population is subject to an early heavy selection of systems under most severe environments. This should be contrasted to reliability of system operating in a fixed environment where the systems may have a variety of shapes for the hazard rates.

3.3 The Model with a Gamma Environmental Factor Distribution

While no particular distribution has been assumed in the previous section a gamma distribution with two parameters is assumed in this section for the random environmental factor Z. The gamma distribution is chosen because it is analytically tractable, readily computable, and it is a flexible distribution that takes on a variety of shapes including exponential and bell-shaped. Since an environmental factor can not be negative the gamma distribution is one of the most commonly used to model a variable which is necessarily positive.

Some authors have proposed this model along with the specified baseline distributions which play the same role as our F_o, and G_o. Lindley and Singpurwalla (1985) have investigated ,although not in depth, some properties of the model when F_o, G_o are exponentials and discussed bounds on reliability for this model. Hutchinson (1982) proposed a similar model when

$$F_o(t) = G_o(t) = \exp(-t^{\eta}). \tag{3.3.1}$$

In order to explore the properties of this model in this section, we first investigate the relationship between this model and Oakes' model. Oakes(1982) has suggested a model

$$F(x, y) = [\, F_{ok}^{1-\theta}(x) + G_{ok}^{1-\theta}(y) - 1]^{\frac{1}{(1-\theta)}} \tag{3.3.2}$$

where F_{ok}, and G_{ok} are the marginal survivals and $\theta > 1$. A reparameterizing of this model was first introduced by Clayton (1978) to model the association in a bivariate life table. Two physical interpretations were given. The first is based on the relation between the hazard functions $q(x|\, Y = y)$ and $q(x|\, Y > y)$ for the conditional distributions of X given Y = y, and Y > y. He showed that for all x and y this model gives the identity

$q(x| Y = y) = \theta\, q(x| Y > y)$. The second is in terms of random effects. Let W be a random variable with a gamma density $h(w) = \{\Gamma(1/(1-\theta))\}^{-1}\exp(-w)w^{1/(1-\theta)-1}$. It is assumed that conditionally on W = w, X and Y are independent random variables with survival functions $\{\exp(-F_{ok}^{1-\theta}(x)+1)\}^{w}$ and $\{\exp(-G_{ok}^{1-\theta}(y)+1)\}^{w}$, respectively. Then the unconditional joint survival function is obtained as (3.3.1). He showed that as $\theta \longrightarrow 1+$, $F(x, y) \longrightarrow F_{ok}(x) \cdot G_{ok}(y)$ and $\theta \longrightarrow \infty$, $F(x, y) \longrightarrow \min(F_{ok}(x), G_{ok}(y))$, the distribution with maximal association. Also the correlation between X and Y in this model varies from 0 to 1.

We will show in this section that the gamma model is equivalent to the Oakes' model in the sense of a notion of dependence. For this purpose a nonparametric measure of dependence termed the Copula is introduced. Using this concept a partial ordering of positively quadrant dependence distributions is developed to this general case of unequal marginals in order to assertain how the joint survival function of the two components in a system under operating condition is changing according to variable parameter values of a gamma distribution

Suppose a random environmental factor follows a gamma distribution with two parameters α, β

whose density function is $h(z) = \{\Gamma(\alpha)\}^{-1}\beta^{\alpha}\exp(-z/\beta)z^{\alpha-1}$, $\alpha > 0$ $\beta > 0$. Then the joint survival function for (X, Y) is

$$F(x, y) = \frac{\beta^{\alpha}}{\{\beta + Q_{xo}(x) + Q_{yo}(y)\}^{\alpha}} \tag{3.3.3}$$

and the marginal survival functions are

$$F_1(x)=\frac{\beta^{\alpha}}{\{\beta+Q_{xo}(x)\}^{\alpha}}\ , \tag{3.3.4}$$

$$G_1(y)=\frac{\beta^{\alpha}}{\{\beta+Q_{yo}(y)\}^{\alpha}} \tag{3.3.5}$$

Here it is noted that the marginal also depends on the parameters of the random environmental factor distribution, while Oakes' model has fixed marginals.

To begin with we point out that the two properties which Oakes' model satisfies are also obtained in our gamma model. In theorem 3.4 the two conditional hazard rates are shown to have inequality $q(x|Y = y) > q(x|Y>y)$, for any $H(\cdot)$. It is easily calculated that

$q(x|Y = y) = (1+1/\alpha)q(x|Y>y)$ when $H(\cdot)$ is a gamma distribution. This relationship has been used as a key assumption to establish Oakes' model by Clayton. Another investigation leads to the same results about the coefficient of concordance that Oakes (1982) has obtained. In both case the probability of concordance is computed as

$(\alpha+1)/(2\alpha+1)$.

In order to show that both models have the same dependence structure we use a nonparametric measure, the Copula, which has been introduced by Sklar in 1959. This notion has been studied by Schweizer and Wolff (1981). They have set up the following definition and explored some properties.

Definition 3.7 (Schweizer and Wolff(1981)) A two dimensional coupla is a mapping C from the unit square [0, 1] x [0, 1] onto the unit interval [0, 1] such that

1) Domain of C is [0, 1] x [0, 1],

2) C is a two dimensional cumulative distribution,

3) One dimensional marginals are uniform over [0, 1].

Property 3.8 (Schweizer and Wolff(1981))

1) C is continous,

2) $\max(u+v-1) \leq C(u, v) \leq \min(u, v)$ for all $u, v \in [0, 1]$,

3) $\max(u+v-1)$ and $\min(u, v)$ are themselves copulas.

Theorem 3.9 (Schweizer and Wolff(1981)) Let H be a two dimensional survival function with continous marginal survival functions F, G. Then there exists a unique copula C such that $H(x, y) = C(F(x), G(y))$. Thus the copula C is given by

$$C(u, v) = H(F^{-1}(u), G^{-1}(v)). \tag{3.3.6}$$

As they noted in their paper, theorem 3.9 shows that since a copula itself is a bivariate distribution with uniform marginals the study of copulas can provide much knowledge of the joint distribution of x and y. Because our investigation is about the dependence structure, we shall discuss this matter rather than the overall properties of copulas which are of intrinsic interest.

Regarding the dependence structure we illustrate how a copula works for a joint distribution by expressing a well known measure of dependence for (U, V), Spearman's ρ, in terms of C. Suppose W_1 and W_2 have a joint survival function H with marginal survival functions F, G. Let C be the copula of (W_1, W_2). Then Spearman's ρ is

$$\rho(W_1, W_2) = 12 \int\int [H(x, y) - F(x) \cdot G(y)]\, d(1 - F(x)) \cdot d(1 - G(y)) \tag{3.3.7}$$

which can also be expressed, using the probability transformation, as

$$\rho(W_1, W_2) = 12 \int\int [C(u, v) - uv]\, du \cdot dv. \qquad (3.3.8)$$

Noting that the second expression is in terms of C alone, a nonparametric measure of dependence between W_1, and W_2 can be studied from thier copula alone. The above two authors have studied other measures as well as ρ.

We shall show that our gamma model and Oakes' model which may be applied differently according to the situation have the same dependence structure in the sense of the copula.

Property 3.10 The gamma model and Oakes' model both have the same copula,

$$\left[\frac{1}{u^{-1/\alpha} + v^{-1/\alpha} - 1}\right]^{\alpha} \qquad (3.3.9)$$

Proof) Let $u = F_1(x)$, and $v = G_1(y)$ where F_1, G_1 are the marginal survivals. Then we obtain $x = F_1^{-1}(u) = Q_{xo}^{-1}[\, \beta(u^{-1/\alpha} - 1)\,]$ and $y = G_1^{-1}(v) = Q_{yo}^{-1}[\, \beta(v^{-1/\alpha} - 1)\,]$, so that C(u, v) is computed.

We list some properties of the gamma model through the copula we have obtained in property 3.10.

1) Since the copula C(u, v) depends only on α, only the shape parameter α affects the dependence structure which is induced by the environment.

2) Since the Copula 3.3.10 is decreasing in α, and two variables are independent if and only if their Copula is $u \cdot v$, the larger the shape parameter α is, the less the dependence is induced.

3) As α goes to 0 the copula converges to min(u,v) which is the copula of maximal positive association, in other words, the copula of the two random variables one of which is a monotone function of the other.

In 2), we note that usual comparisions of the strength of dependence of two random variables require that the marginals be fixed. However we note that the copula is a surface over the unit square and that a notion of distance between a given Copula and the one of independent case, $u \cdot v$, can be used to compare the degree of dependence between distributions of arbitrary marginal structure. This idea is a generalization of the concept of partial ordering of positively quadrant dependence introduced by Ahmed, and et al (1979). They have considered a class of positively quadrant dependent bivariate distributions whose marginals are fixed. They have defined an ordering on this class by saying that one bivariate distribution is more positive quadrant dependent than another if its joint survival function is larger for any x, y. They also show that two independent random variables and two random variables of which one is the monotone function of the other one are the two extremes of this class with respect to this ordering.The Copula makes it possible to make similar comparisions of arbitrary pairs of positively quadrant random variables with respect to the degree of positively quadrant dependence.

Since we have investigated properties of the general model in the previous section, we will observe what the assumption of a gamma distribution as a random environmental factor distribution yields.

Property 3.11 The random environmental factor for those systems for which component A has functioned more than x time units and component B has functioned y time units also follows a gamma distribution with same shape parameter

α and scale parameter $Q_{xo}(x) + Q_{yo}(y) + \beta$. While for the population of the systems whose components failed at time X=x, Y=y the environmental factor follows a gamma distribution with shape parameter $\alpha+2$ and scale parameter $Q_{xo}(x) + Q_{yo}(y) + \beta$.

Proof) The conditonal density of Z given X > x, Y > y, h(z | X > x, Y > y), is

$$\frac{P(X > x, Y > y \mid Z = z)\cdot h(z)}{P(X > x, Y > y)}$$

$$= \frac{\{ Q_{xo}(x) + Q_{yo}(y) + \beta\}^{\alpha} \exp[-\{ Q_{xo}(x) + Q_{yo}(y) + \beta\} z]\cdot z^{\alpha-1}}{\Gamma(\alpha)} \qquad (3.3.10)$$

which is a gamma density with parameters α, $\{ Q_{xo}(x) + Q_{yo}(y) + \beta\}$.

On the other hand the conditional density given X = x and Y = y, h(z | X = x Y = y) is

$$\frac{h(z, x, y)}{h(x, y)} = \frac{h(x, y \mid z)\cdot h(z)}{h(x,y)}$$

$$= \frac{\{ Q_{xo}(x) + Q_{yo}(y) + \beta\}^{\alpha+2}\cdot \exp[-\{ Q_{xo}(x) + Q_{yo}(y) + \beta\} z]\cdot z^{\alpha+1}}{\Gamma(\alpha+2)}, \qquad (3.3.11)$$

a gamma density with parameters $\alpha+2$, $\{ Q_{xo}(x) + Q_{yo}(y) + \beta\}$. Q.E.D.

Property 3.11 indicates that the mean of the environmental factor for the population of systems whose components are functioning at time t is a decreasing function of t. Another point to be noted from this property is that the density of the environmental factor for the population of the systems whose components' lifetimes have X > x, Y > y has the shape paramater α, which is identical to that in the unconeditional density of Z. It can be interpreted that the dependence structure between the components of all the functioning system beyond a certain time t is of the same as the dependence structure between components of a system operating at time 0.

We will conclude this section by presenting an interesting relationship between a component's survival function $F_o(\cdot)$, and the reliability of the series system $R_s(\cdot)$. Klein and Moeschberger (1985) have obtained an expression for the marginal survival function of X in terms of the survival function of T, where T = min (X, Y), under the assumption that (X, Y) follows Oakes' bivariate survival function. In our gamma model, consider the crude density function $p_1(\cdot)$ associated with one component, say A, whose lifetime is denoted by X_o. Now,

$$p_1(t) = \frac{d}{dt} P(T < t, X < Y)$$

$$= \frac{d}{dt} \int_0^t \int_u^\infty f(u, v)\, dv\, du$$

$$= \int_t^\infty f(t, v)\, dv = - \frac{\partial F(u, v)}{\partial u} \Big|_{u=t,\, v=t}$$

$$= \alpha \cdot [R_s(t)]^{(\alpha+1)/\alpha} \cdot \frac{f_o(t)}{\beta \cdot F_o(t)} \quad . \qquad (3.3.12)$$

Consider the differential equation

$$\frac{f_o(t)}{F_o(t)} = \frac{\beta}{\alpha}\, p_1(t) \cdot [R_s(t)]^{-(1+\alpha)/\alpha} \, . \qquad (3.3.13)$$

Then the solution of the above equation for $F_o(t)$ is

$$F_o(t) = \exp\Big[- \frac{\beta}{\alpha} \int_0^t p_1(t) \cdot [R_s(t)]^{-(1+\alpha)/\alpha}\, dt \Big] \, . \qquad (3.3.14)$$

As the above authors have done, this expression may be used to obtain bounds for the survival function of a component of interest based on the data one can obtain in a competing risks experiment. Let $T_1, \cdots, T_n$ denote the observed test times of n two component systems and let δ_i, $i = 1, \cdots, n$, be 1 or 0 according to whether the T_i was an observation on X_i or Y_i, respectively. Then the usual estimators of $R_s(t)$ and $p_1(t)$ enable us to have an estimator of $F_0(t)$ provided that α, β are known.

3.4 The Model When Both Components Have Weibull Lifetime Distribution

In this section both components are assumed to have a Weibull form with parameters (η_1, λ_1) and (η_2, λ_2), respectively. That is, $F_0(x) = \exp(-\lambda_1 x^{\eta_1})$. The Weibull distribution, which may have increasing ($\eta > 1$), decreasing ($\eta < 1$) or constant failure rate ($\eta = 1$) has been shown experimentally to provide a reasonable fit to many different types of survival data. (See Bain (1978)). The resulting joint reliability of the two components' lifetimes, (X,Y) in the operating environment is $F(x,y) = E[\exp(-Z(\lambda_1 x^{\eta_1} + \lambda_2 y^{\eta_2}))]$. (3.4.1)

The model described above for a general distribution of the environmental stress has a particular dependence structure which we summarize in the following lemmas.

Lemma 1. Let (X,Y) follow the model (3.4.1) where Z is a positive random variable with finite $\left(\frac{r}{\eta_1} + \frac{s}{\eta_2}\right)^{th}$ inverse moment. Then

$$E(X^r Y^s) = \lambda_1^{-r/\eta_1} \lambda^{-s/\eta_2} \Gamma(1 + r/\eta_1)\ \Gamma(1 + s/\eta_2)\ E(Z^{-(r/\eta_1 + s/\eta_2)}) \tag{3.4.2}$$

The proof follows by noting that, given Z = z, (X,Y) are independent Weibulls with parameters $(\eta_1, \lambda_1 z)$ and $(\eta_2, \lambda_2 z)$, respectively and $E(X^r|Z{=}z) = \lambda_1^{-r/\eta_1} z^{-r/\eta} \Gamma(1+ r/\eta_1)$ with a similar expression for Y^s. When the appropriate moments exist, we have

(A) $E(X) = E(X_0)\, E(Z^{-1/\eta_1})$,

(B) $V(X) = E(X_0^2)\, Var(Z^{-1/\eta_1}) + E(Z^{-1/\eta_1})^2\, Var(X_0)$,

(C) $Cov(X,Y) = E(X_0)\, E(Y_0)\, Cov(Z^{-1/\eta_1}, Z^{-1/\eta_2})$ which is greater than 0.

If $\eta_1 = \eta_2 = \eta$ then the correlation between (X,Y) is

$$\rho = \frac{\Gamma(1+ 1/\eta)^2\, Var(Z^{-1/\eta})}{Var(Z^{-1/\eta})\, \Gamma(1+2/\eta) + (\Gamma(1+2/\eta) - \Gamma(1+1/\eta)^2)\, E(Z^{-1/\eta})^2}. \tag{3.4.3}$$

In this case the correlation is bounded above by $\Gamma(1+1/\eta)^2 / \Gamma(1+2/\eta)$. Figure 1 shows the maximal correlation as a function of η for $\eta \in (0, 10)$. Note that this maximal correlation is an increasing function of η.

Exact expressions for competing risks quantities of interest can be computed when a particular model is assumed for the distribution of Z. We shall consider the gamma and uniform models. Consider first the gamma model with $h_z(z) = \beta^\alpha z^{\alpha-1} \exp(-\beta z)/\Gamma(\alpha)$, $z > 0$. For this model, the joint survival function is

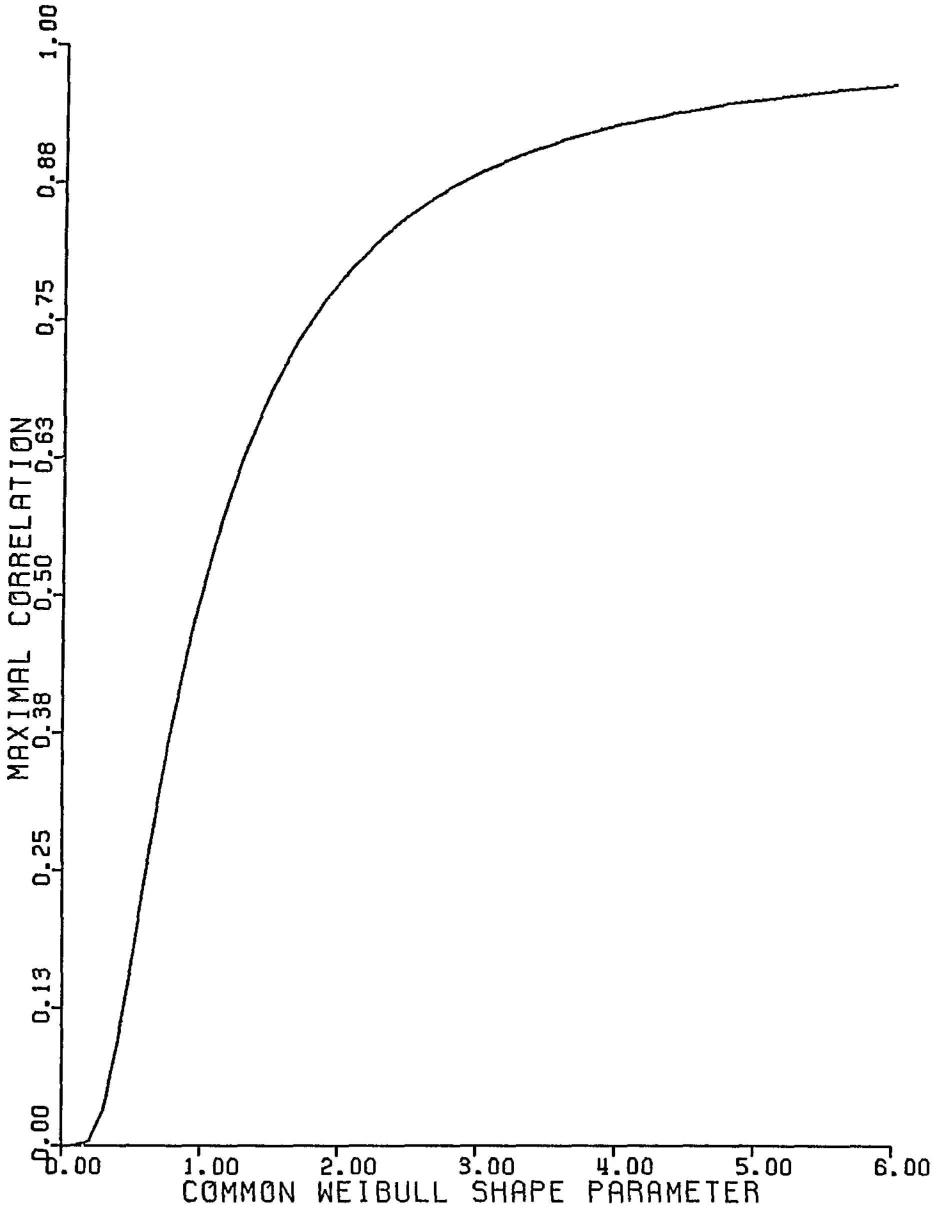

FIGURE 1

UPPER BOUND ON MAXIMAL CORRELATION FOR RANDOM ENVIORNMENT MODEL.

$$F(x,y) = \frac{\beta^{\alpha}}{[\beta + \lambda_1 x^{\eta_1} + \lambda_2 y^{\eta_2}]^{\alpha}} \qquad (3.4.4)$$

which is a bivariate Burr Distribution (see Takahasi (1965)), the marginal distributions are univariate Burr distributions with

$$E(X) = (\lambda_1/\beta)^{-1/\eta_1}\,\Gamma(1+1/\eta_1)\Gamma(\alpha - 1/\eta_1)\,\Gamma(\alpha), \text{ if } \alpha > 1/\eta,$$

$$Var(X) = (\lambda_1/\beta)^{-2/\eta_1}\left\{\frac{\Gamma(1+2/\eta_1)\Gamma(\alpha-2/\eta_1)}{\Gamma(\alpha)} - \left[\frac{\Gamma(1+1/\eta_1)\Gamma(\alpha-1/\eta_1)}{\Gamma(\alpha)}\right]^2\right\}, \text{ if } \alpha > 2/\eta_1$$

with similar expressions for E(Y), Var(Y). The covariance of (X,Y) is

$$Cov(X,Y) = (\lambda_1/\beta)^{-1/\eta_1}(\lambda_2/\beta)^{-1/\eta_2}\left\{\frac{\Gamma(\alpha-1/\eta_1-1/\eta_2)}{\Gamma(\alpha)} - \frac{\Gamma(\alpha-1/\eta_2)\,\Gamma(\alpha-1/\eta_2)}{\Gamma(\alpha)}\right\}$$

$$\Gamma(1+1/\eta_1)\,\Gamma(1+1/\eta_2)$$

for $\alpha > 1/\eta_1 + 1/\eta_2$. For the gamma model, the reliability function for a bivariate series system is given by

$$R_s(t) = (1+(\lambda_1/\beta)t^{\eta_1} + (\lambda_2/\beta)t^{\eta_2})^{-\alpha}, \qquad (3.4.5)$$

and for a parallel system by

$$R_p(t) = (1+(\lambda_1/\beta)t^{\eta_1} + (1+(\lambda_2/\beta)t^{\eta_2})^{-\alpha} - (1+(\lambda_1/\beta)t^{\eta_2} + (\lambda_2/\beta)t^{\eta_2})^{-\alpha} \qquad (3.4.6)$$

Figures 2-5 are plots of the series system reliability for $\lambda_1 = 1$, $\lambda_2 = 2$ and several combinations of η_1, η_2. Each figure shows the reliablity for $\alpha = 1/2, 1, 2, 4$, and the independent Weibull model. In all cases, $\beta = 1$. For these figures we note that for fixed

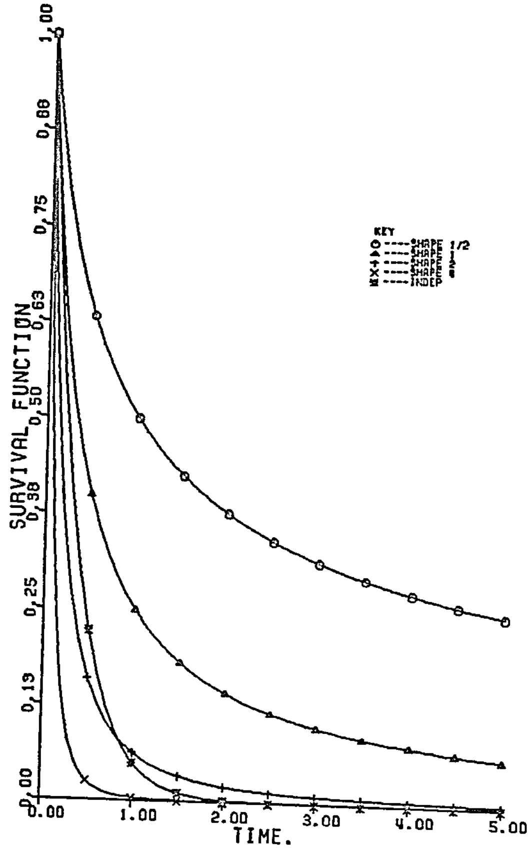

FIGURE 2

SERIES SYSTEM RELIABILITY UNDER GAMMA MODEL
FOR THE ENVIRONMENTAL STRESS
$\lambda_1=1.0,\ \lambda_2=2.0,\ \eta_1=1.0,\ \eta_2=1.0.$

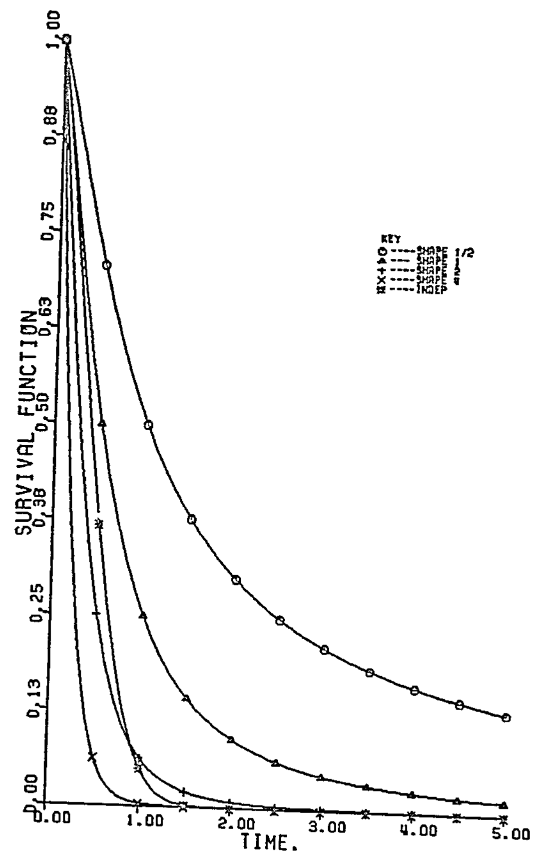

FIGURE 3

SERIES SYSTEM RELIABILITY UNDER GAMMA MODEL
FOR THE ENVIRONMENTAL STRESS
$\lambda_1=1.0,\ \lambda_2=2.0,\ \eta_1=1.0,\ \eta_2=2.0.$

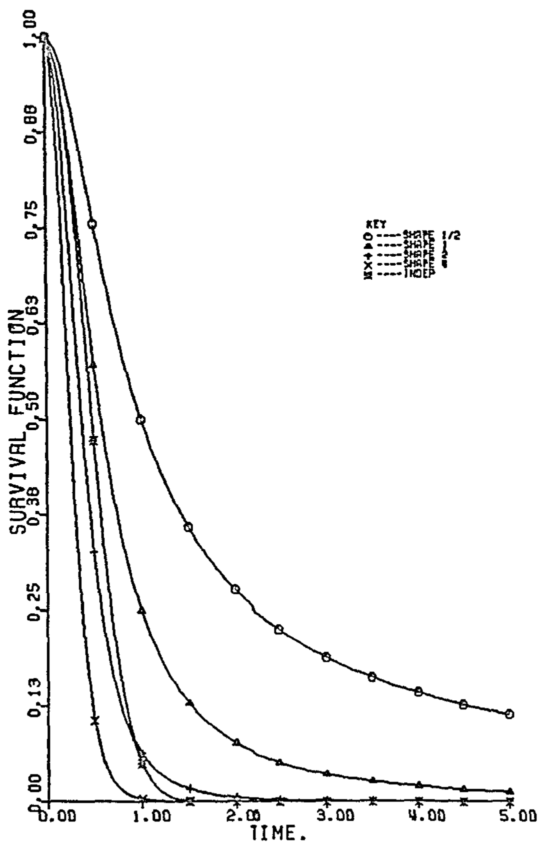

FIGURE 4

SERIES SYSTEM RELIABILITY UNDER GAMMA MODEL
FOR THE ENVIRONMENTAL STRESS.
$\lambda_1=1.0$, $\lambda_2=2.0$, $\eta_1=2.1$, $\eta_2=2.0$.

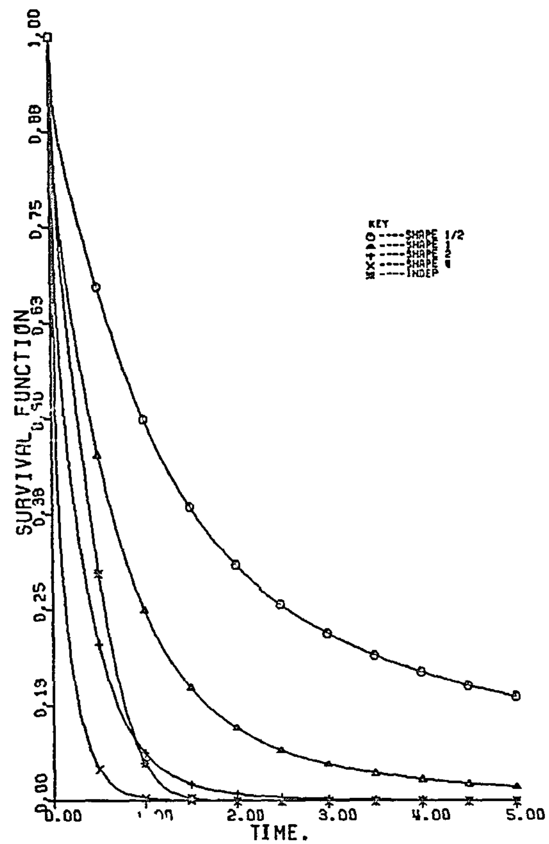

FIGURE 5

SERIES SYSTEM RELIABILITY UNDER GAMMA MODEL
FOR THE ENVIRONMENTAL STRESS.
$\lambda_1=1.0$, $\lambda_2=2.0$, $\eta_1=1/2$, $\eta_2=2.0$.

$\lambda_1, \lambda_2\, \eta_1, \eta_2$, t, the series system reliability is a decreasing function of the shape parameter α. Figures 6-9 are plots of the parallel system reliability (3.4.6) for the above parameters. Again, the reliability is a decreasing function of α. Also in both the series and parallel system reliability, the shape of the reliability function is quite different from that encountered under independence.

The gamma model is a reasonable model for the environmental stress due to its flexibility and the tractability of the model in obtaining close form solutions for the relevant quantities and in estimating parameters. However, in some cases, such as when the operating environment is always more severe than the laboratory environment, the support of H may be restricted to some fixed interval. A possible model for such an environmental stress is the uniform distribution over [a,b]. For this model, the joint survival function is

$$F(x,y) = \frac{[\exp(-b(\lambda_1 x^{\eta_1} + \lambda_2 y^{\eta_2})) - \exp(-a(\lambda_1 x^{\eta_1} + \lambda_2 y^{\eta_2}))]}{(b-a)(\lambda_1 x^{\eta_1} + \lambda_2 y^{\eta_2})} \qquad (3.4.7)$$

$$E(X) = \lambda_1^{-1/\eta_1}\, \Gamma(1+1/\eta_1)\, \eta_1 \,(b^{(\eta_1-1)/\eta_1} - a^{(\eta_1-1)/\eta_1}) / ((\eta_1-1)(b-a)] \quad \text{if } \eta_1 \neq 1,$$

$$= \log(b/a)/[\lambda_1(b-a)] \quad \text{if } \eta_1 = 1,$$

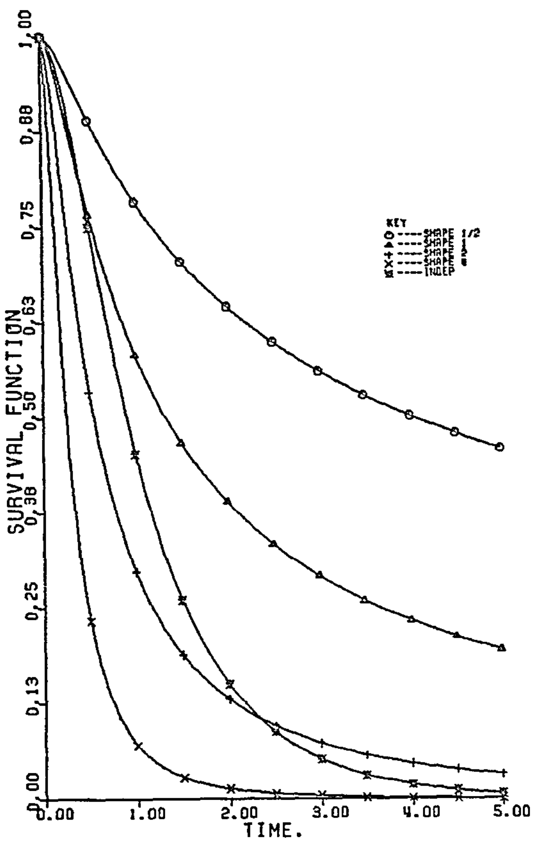

FIGURE 6

PARALLEL SYSTEM RELIABILITY UNDER GAMMA MODEL
FOR THE ENVIRONMENTAL STRESS.
$\lambda_1=1.0$, $\lambda_2=2.0$, $\eta_1=1.0$, $\eta_2=1.0$.

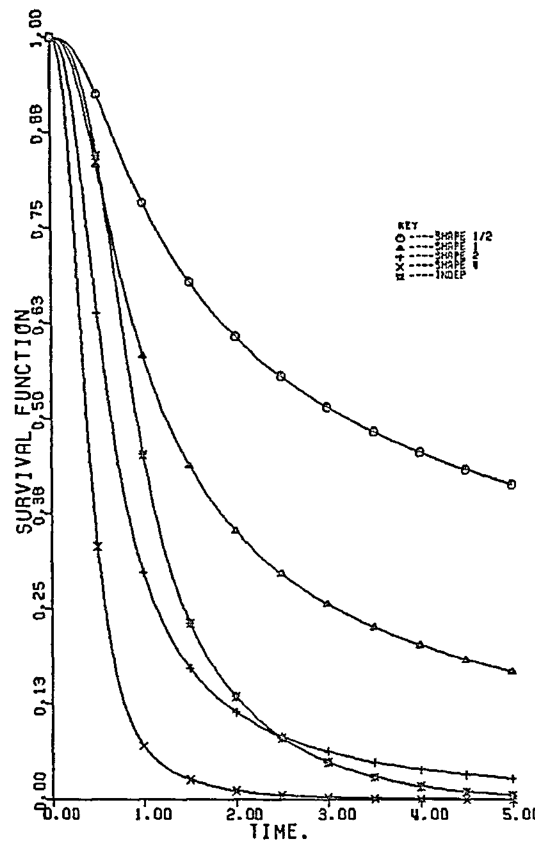

FIGURE 7

PARALLEL SYSTEM RELIABILITY UNDER GAMMA MODEL
FOR THE ENVIRONMENTAL STRESS.
$\lambda_1=1.0$, $\lambda_2=2.0$, $\eta_1=1.0$, $\eta_2=2.0$.

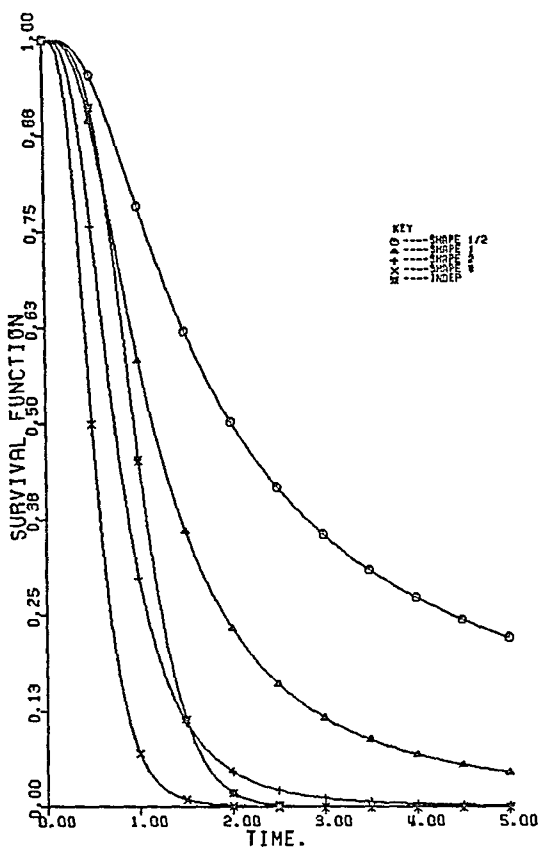

FIGURE 8

PARALLEL SYSTEM RELIABILITY UNDER GAMMA MODEL
FOR THE ENVIRONMENTAL STRESS.
$\lambda_1=1.0$, $\lambda_2=2.0$, $\eta_1=2.0$, $\eta_2=2.0$.

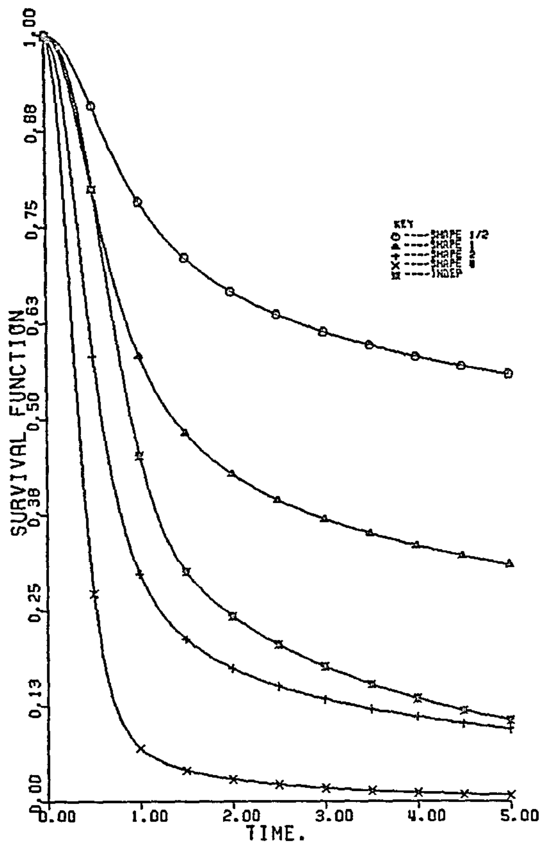

FIGURE 9

PARALLEL SYSTEM RELIABILITY UNDER GAMMA MODEL
FOR THE ENVIRONMENTAL STRESS.
$\lambda_1=1.0$, $\lambda_2=2.0$, $\eta_1=1/2$, $\eta_2=2.0$.

$$Var(X) = \frac{\eta_1 \lambda_1^{-2/\eta_1}}{(b-a)} \{ \Gamma(1+2/\eta_1)\, \eta_1 \,(b^{(\eta_1-2)/\eta_1} - a^{(\eta_1-2)/\eta_1})$$

$$\frac{- \Gamma(1+1/\eta_1)^2 \eta_1 (b^{(\eta_1-1/\eta_1)} - a^{(\eta_1-1/\eta_1)})^2}{(\eta_1-1)^2 (b-a)} \} \qquad \text{if } \eta_1 \neq 1,2,$$

$$= 2/(\lambda_1^2 ab) - \log(b/a)^2/[(b-a)\lambda_1]^2 \qquad \text{if } \eta_1 = 1,$$

$$= \lambda_1^{-1}[\frac{\log(b/a)}{(b-a)} - \frac{\pi}{(b^{1/2} + a^{1/2})^2}] \qquad \text{if } \eta_1 = 2.$$

For this model, the reliability function for a series system is

$$R_s(t) = \frac{[\exp(-b(\lambda_1 t^{\eta_1} + \lambda_2 t^{\eta_2})) - \exp[-a(\lambda_1 t^{\eta_1} + \lambda_2 t^{\eta_2})]]}{(b-a)(\lambda_1 t^{\eta_1} + \lambda_2 t^{\eta_2})} \qquad (3.4.8)$$

and for a parallel system is

$$R_p(t) = \frac{[\exp(-b(\lambda_1 t^{\eta_1}) - \exp(-a\lambda_1 t^{\eta_1})}{(b-a)\lambda_1 t^{\eta_1}} + \frac{[\exp(-b\lambda_2 t^{\eta_2}) - \exp(-a\lambda_2 t^{\eta_2})}{(b-a)\lambda_2 t^{\eta_2}} - Rs(t) \qquad (3.4.9)$$

Figures 10-13 show the reliability for a series system and figures 14-17 for a parallel system under the uniform model for various combinations of $\lambda_1, \lambda_2, \eta_1, \eta_2$ a,b. Notice that when A = .25, B = .75, which corresponds to an operating environment which is less severe than the test environment, the system reliability is greater than that expected under independence, while when (a,b) = (1.25, 1.75) or (1., 2), which

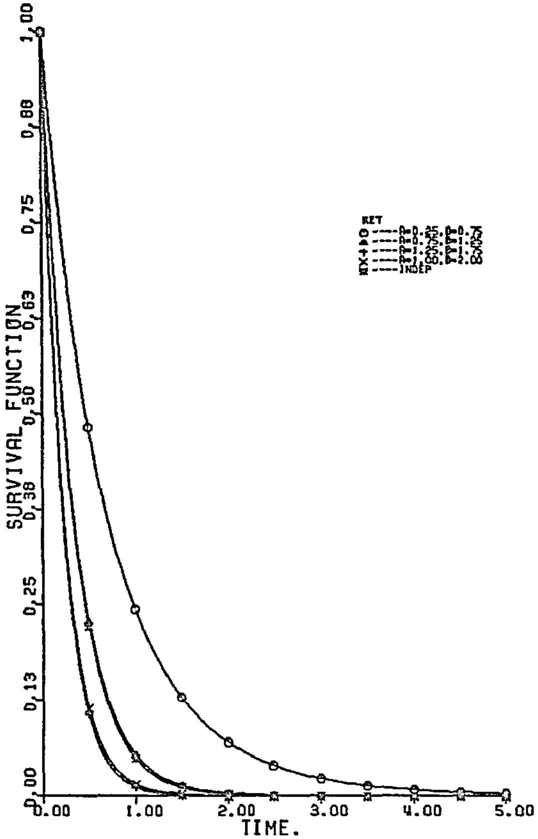

FIGURE 10

SERIES SYSTEM RELIABILITY UNDER UNIF(A,B) MODEL
FOR THE ENVIRONMENTAL STRESS.
$\lambda_1=1.0$, $\lambda_2=2.0$, $\eta_1=1.0$, $\eta_2=1.0$.

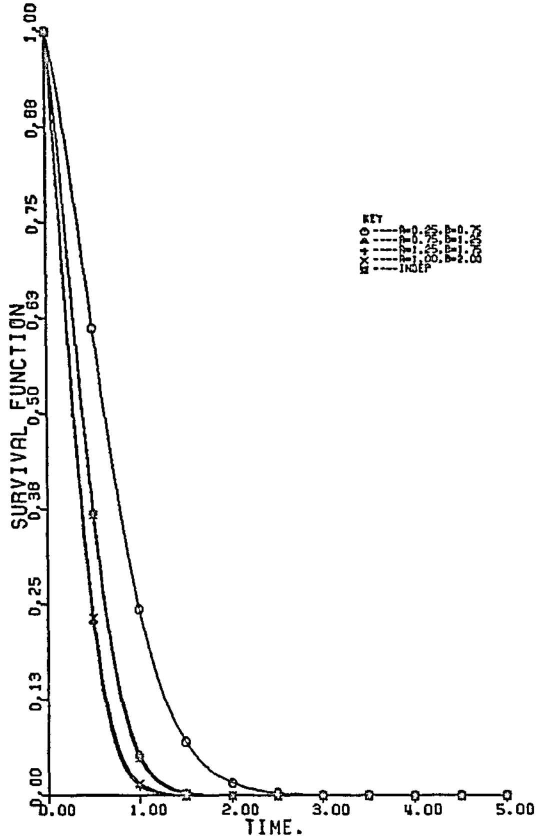

FIGURE 11

SERIES SYSTEM RELIABILITY UNDER UNIF(A,B) MODEL
FOR THE ENVIRONMENTAL STRESS.
$\lambda_1=1.0$, $\lambda_2=2.0$, $\eta_1=1.0$, $\eta_2=2.0$.

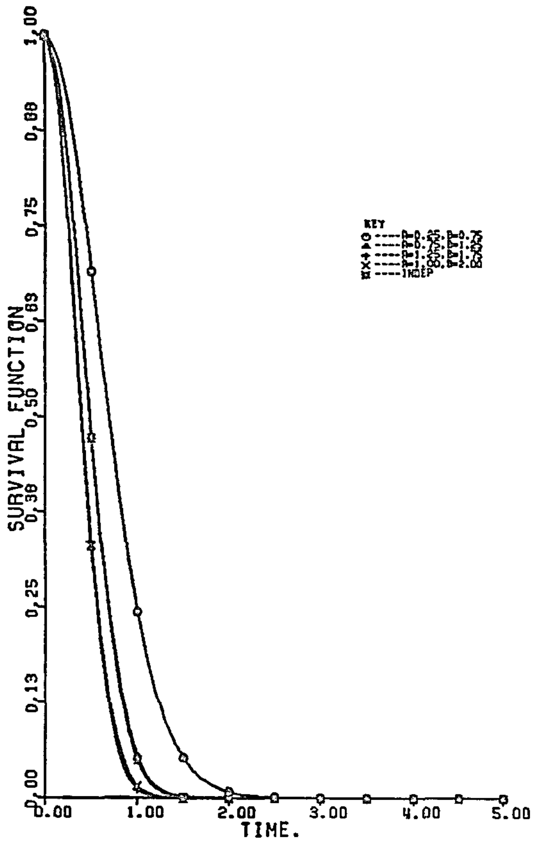

FIGURE 12

SERIES SYSTEM RELIABILITY UNDER UNIF (A,B) MODEL
FOR THE ENVIRONMENTAL STRESS.
$\lambda_1=1.0$, $\lambda_2=2.0$, $\eta_1=2.0$, $\eta_2=2.0$.

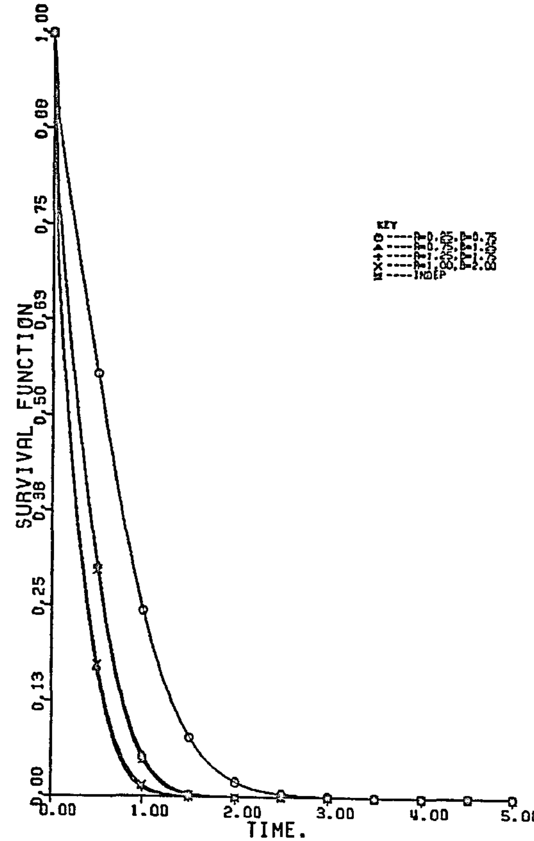

FIGURE 13

SERIES SYSTEM RELIABILITY UNDER UNIF (A,B) MODEL
FOR THE ENVIRONMENTAL STRESS.
$\lambda_1=1.0$, $\lambda_2=2.0$, $\eta_1=1/2$, $\eta_2=2.0$.

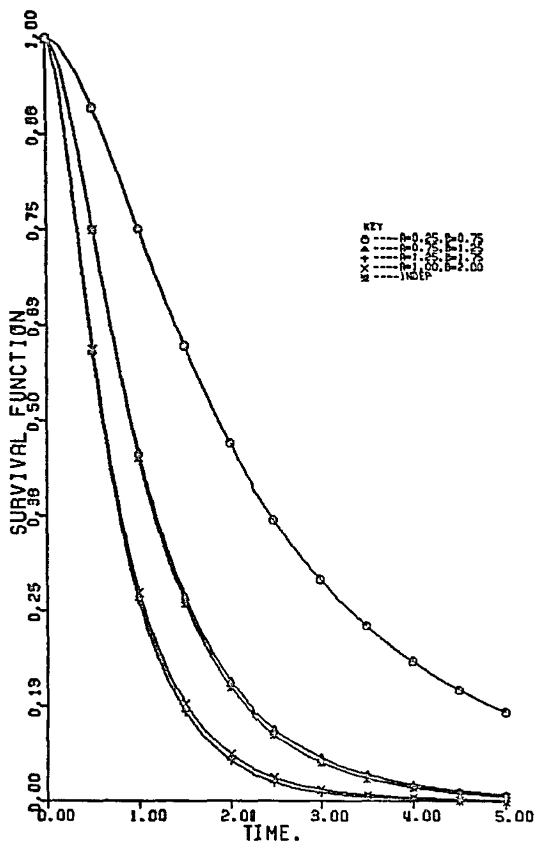

FIGURE 14

PARALLEL SYSTEM RELIABILITY UNDER UNIF(A,B) MODEL
FOR THE ENVIRONMENTAL STRESS.
$\lambda_1=1.0$, $\lambda_2=2.0$, $\eta_1=1.0$, $\eta_2=1.0$.

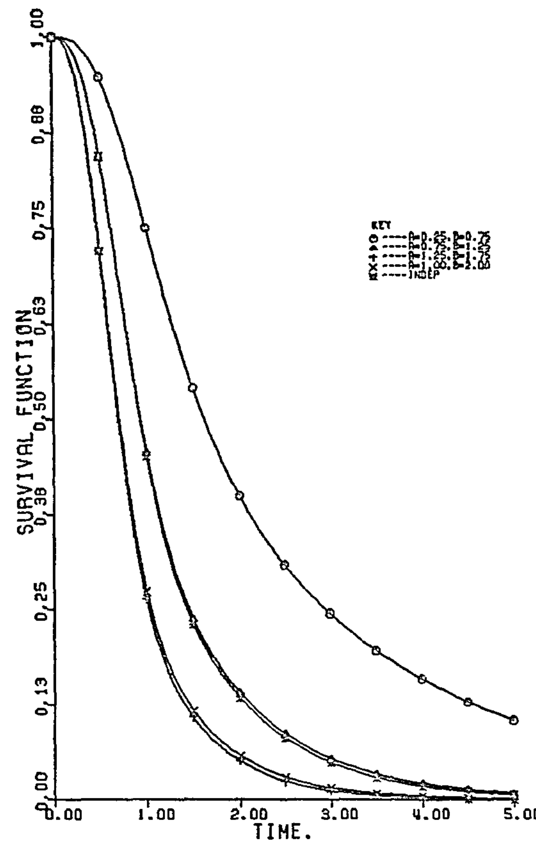

FIGURE 15

PARALLEL SYSTEM RELIABILITY UNDER UNIF(A,B) MODEL
FOR THE ENVIRONMENTAL STRESS.
$\lambda_1=1.0$, $\lambda_2=2.0$, $\eta_1=1.0$, $\eta_2=2.0$.

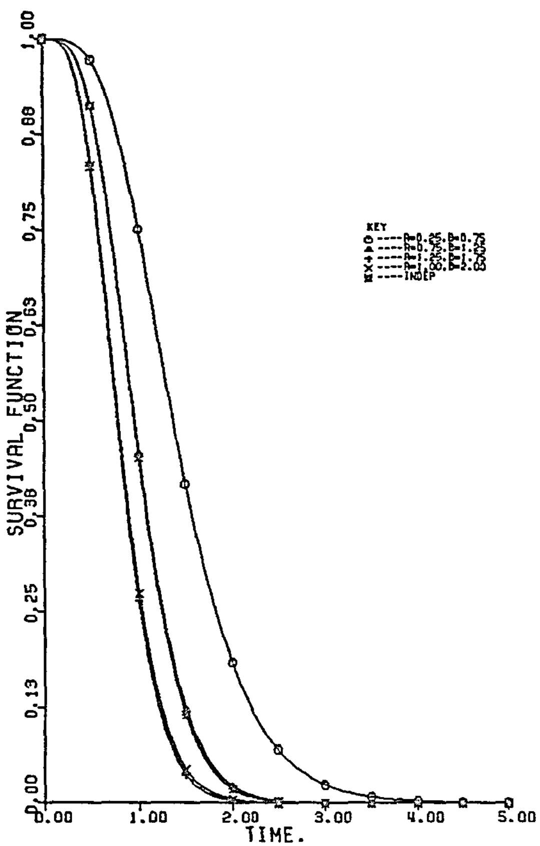

FIGURE 16

PARALLEL SYSTEM RELIABILITY UNDER UNIF (A,B) MODEL
FOR THE ENVIRONMENTAL STRESS.
$\lambda_1=1.0,\quad \lambda_2=2.0,\quad \eta_1=2.0,\quad \eta_2=2.0.$

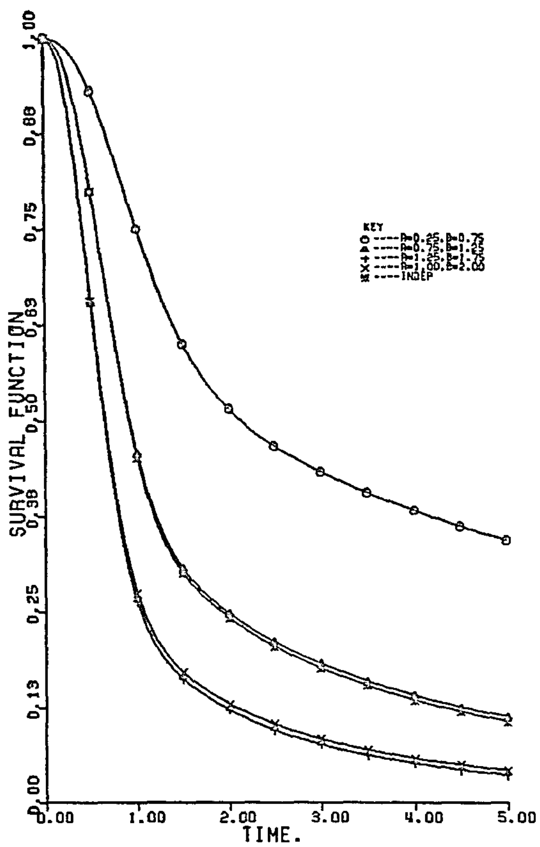

FIGURE 17

PARALLEL SYSTEM RELIABILITY UNDER UNIF (A,B) MODEL
FOR THE ENVIRONMENTAL STRESS.
$\lambda_1=1.0,\quad \lambda_2=2.0,\quad \eta_1=1/2,\quad \eta_2=2.0.$

corresponds to an environment more severe than the test environment, the system reliability is smaller. Also when (a,b) contains 1, which corresponds to an environment which incurs the possibility of no differential effect from that found in the laboratory, there is little difference in the dependent and independent system reliability.

Up to this point we have proposed a practically motivated random environmental effect model and have investigated this model from the points of view of both system reliability and the induced dependence structure. Even if the results are not as tractable as those of the other dependent models used in reliability studies and survival analysis the basis of our assumptions are so realistic that any one applying reliability methods in the practice may consider this model to detect the environmental effect and to resolvediscrepancies in estimating reliability under more standard models. Finally, we finish this chapter by noting that this model and its properties, which we explored from the point of view of engineering application, should be investigated in depth for biological application.

CHAPTER IV

INFERENCE

4.1 Introduction

In this chapter the problem of analyzing life tests of two component series systems which are assumed to follow the random environmental effects model described in the chapter 3 is discussed. We will focus on the model which assumes that the lifetimes of the components follow the Weibull lifetime distribution with same shape parameters, that is, $\eta_1 = \eta_2 = \eta$ under laboratory condition. If η is assumed to be known, then the lifetimes, after a suitable transformation, may be assumed to follow an exponential distribution. In section 4.2 we will briefly discuss the case when the random environmental factor follows an arbitrary distribution. In the remainder of this chapter a gamma distribution is assumed for the random environmental factor. Maximum likelihood estimators of the parameters are obtained in section 4.3 and we propose an optimal scheme for determining sample sizes subject to various cost constraints in section 4.4. In section 4.5 the method of moments estimators of the parameters associated with the random environmental factor are discussed together with a modified estimator. We present several new estimators in section 4.6 which are based on a graphical approach to the analysis of such experiment. In section 4.7 a comparision of the estimators obtained will be made through a small scale Monte Carlo study. Finally in section 4.8 we discuss how to test the dependence induced by the common environment under the Weibull - gamma model.

Before describing the analysis we shall discuss the experimental design and some notation. Consider two components, say component A and B, which are linked into a series system, say S. The whole experiment consists of three distinct parts. One experiment is done on component A under controlled condition, such as found in the laboratory and another independent experiment is performed on component B under controlled conditions. The third experiment is carried out on the series systems S under operating conditions which allow for introducing the common environmental effects.Sample data from the first two parts consist of times to failure of each component. The last part consists of the failure times of the system and an indicator variable which tells us which component causes the system to fail.

Now let us explain the following notations: Let

$X_{o,i}$ = Lifetime of the i-th component A in part I, $i = 1, 2, \cdots, n$;

$Y_{o,j}$ = Lifetime of the j-th component B in part II, $j = 1, 2, \cdots, m$;

n = Number of component A's put on the test under contorlled conditions;

m = Number of component B's put on the test under controlled conditions;

X_i = The potential lifetime of component A of the i-th system under operating conditions;

Y_i = The potential lifetime of component B of the i-th system under operating conditions;

δ_i = An indicator variable whose value is equal to 1 if $X_i < Y_i$ and otherwise equal to 0;

T_i = Lifetime of the i-th system, ($T_i = \min [X_i , Y_i]$);

s = Number of the systems put on the test under operating conditions;

λ_1 = Hazard rate of the component A under controlled conditions;

λ_2 = Hazard rate of the component B under controlled conditions;

$H(\cdot)$ = Cummulative distribution function of the random environmental factor, Z;

α = The shape parameter of the gamma distribution assumed for $H(\cdot)$;

β = The scale parameter of the gamma distribution assumed for $H(\cdot)$.

4.2 The Model with a General Environmental Factor Distribution

In this section, we consider the maximum likelihood estimators (M.L.E.) of the hazard rates of both components, λ_1, and λ_2 and the distribution function $H(\cdot)$ given three independent samples,

$(X_{o,1}, X_{o,2}, \cdots, X_{o,n})$, $(Y_{o,1}, Y_{o,2}, \cdots, Y_{o,m})$, and

$(T_1, \delta_1, T_2, \delta_2, \cdots, T_s, \delta_s)$. Let

$S_{o,1} = \sum_{i=1}^{n} X_{o,i}$ and $S_{o,2} = \sum_{j=1}^{m} Y_{o,j}$ be the total times on test of both components in the laboratory testing experiments, and $M = \sum_{k=1}^{s} \delta_k$ be the number of systems whose failures are due to component A.Since the observations from different samples are independent the relevant likelihood is $L = L_1 \cdot L_2 \cdot L_3$ where

$L_1 = \lambda_1^{n} \cdot \exp(-\lambda_1 \cdot S_{o,1})$, $L_2 = \lambda_2^{m} \cdot \exp(-\lambda_2 \cdot S_{o,2})$, and

$$L_3 = {}_sC_M \cdot \lambda_1^{M} \cdot \lambda_2^{(s-M)} \cdot \prod_{k=1}^{s} \int z \cdot \exp\{-(\lambda_1+\lambda_2) z T_k\}\, dH(z),$$

where ${}_sC_M = \dfrac{s!}{M!\,(s - M)!}$.

Hence the loglikelihood after a slight modification is

$$\log L = n \cdot \log\lambda_1 + m \cdot \log\lambda_2 - \lambda_1 S_{o,1} - \lambda_2 S_{o,2} + \log({}_sC_M) + M\log\lambda_1 + (s - M)\log\lambda_2$$
$$- s \log(\lambda_1 + \lambda_2) + \sum_{k=1}^{s} \log \int (\lambda_1+\lambda_2)\, z \cdot \exp\{-(\lambda_1+\lambda_2) z T_k\}\, dH(z). \qquad (4.2.1)$$

Since the last term of logL depends on λ_1, λ_2, and $H(\cdot)$ the usual technique of finding M.L.E.'s can not be directly applied here. One approach we suggest here is due to Jewell (1982). He has shown the existence of the M.L.E. of a mixing measure μ given a sample of n independent observations from the mixture distribution $F(t) = \int (1- \exp(-zt))\, d\mu(z)$ and suggested an algorithm for computation of the M.L.E.

Following Jewell, we suggest first obtaining the M.L.E. of $H^*(\cdot)$. That is, the distribution function of $(\lambda_1 + \lambda_2)Z = Z^*$. From the last term of logL we maximize

$$\sum_{k=1}^{s} \log \int z^* \exp(-z^* T_k)\, dH^*(z), \qquad (4.2.2)$$

and then obtain the M.L.E.'s of λ_1, λ_2 from the remaining terms solving the two likelihood equations,

$$\frac{n}{\lambda_1} - S_{o,1} + \frac{M}{\lambda_1} - \frac{s}{\lambda_1+\lambda_2} = 0, \quad (4.2.3)$$

$$\frac{m}{\lambda_2} - S_{o,2} + \frac{s-M}{\lambda_2} - \frac{s}{\lambda_1+\lambda_2} = 0 \quad (4.2.4)$$

Since the M.L.E. of $H^*(\cdot)$ depends only on T_k's, the M.L.E. of $H(\cdot)$ can be obtained by the invariance property of the M.L.E.

Since the estimators of λ_1 and λ_2 will be discussed in depth in the next two sections, we consider only the estimator of $H^*(\cdot)$ in this section. Jewell, in his paper, has shown that the M.L.E. of $H^*(\cdot)$ has finite support, containing at most s distinct points which are lying between the two extremes, the inverses of the largest and smallest T_k's. Thus he proposed a minor adjustment to an algorithm, which has been suggested by Hasselblad (1969) for maximizing the likelihood of a finite mixture of exponentials with the number of mass points known. The algorithm proceeds as follows. For each $q = 1, \cdots, s$ we estimate the distribution of Z^* by finding the values of $p_{i,q}$ and $z_{i,q}$, $i = 1, 2, \cdots, q$ where $p_{i,q} = P(Z^* = z_{i,q})$, which maximize (4.2.2). For fixed q these points are found iteratively as follows. Let $p_i^{(0)}$, $z_i^{(0)}$ be an initial guess at $p_{i,q}$, $z_{i,q}$. Set $g^{(k)}(t) = \sum_{j=1}^{q} p_j^{(k)} . z_j^{(k)}.\exp(-z_j^{(k)}t)$ and $f_j^{(k)}(t) = z_j^{(k)}.\exp(-z_j^{(k)}t)$,

$j = 1, 2, \cdots, q$.

Then $p_j^{(k+1)} = p_j^{(k)} [\sum_{i=1}^{s} f_j^{(k)}(T_i) / g^{(k)}(T_i)] / s$, and

$$z_j(k+1) = \frac{\sum_{i=1}^{s} f_j^{(k)}(T_i) / g^{(k)}(T_i)}{\sum_{i=1}^{s} [\, T_i f_j^{(k)}(T_i) / g^{(k)}(T_i)]} \quad , \quad j = 1,2,\cdots,q.$$

We continue untill convergence is obtained. As noted in Jewell this algorithm suffers from the limitations of a large variance and too many iterations. The relatively poor behavior of this estimation procedure was also noted by Heckman and Singer (1982) in somewhat differnt context. In the sequel we shall describe improved estimators based on assumed parametric model for $H(\cdot)$.

4.3 Maximum Likelihood Estimators for the Model with a Gamma Environmental Factor Distribution

In this section we assume that the random environmental factor has a gamma distribution, whose distribution function is $H(z) = \int \beta^{\alpha} (\Gamma(\alpha))^{-1} u^{\alpha-1} e^{-u\beta} du$, with finite mean and study the method of maximum likelihood estimation for the parameters $\lambda_1, \lambda_2, \alpha$, and β. As noted in section 3.4, the reliability for this series system is $R_s(t) = (1 + (\lambda_1+\lambda_2)\, t / \beta)^{-\alpha}$. Based on the three independent samples described in section 4.1, the relevant loglikelihood is

$$\log L = n\cdot\log\lambda_1 + m\cdot\log\lambda_2 - \lambda_1 S_{o,1} - \lambda_2 S_{o,2} + \log({}_sC_M) + M\log\lambda_1 + (s - M)\log\lambda_2$$
$$+ s\log\alpha - s\log\beta - (\alpha + 1) \sum_{k=1}^{s} \log \{1 + (\lambda_1+\lambda_2)\, T_k / \beta\} . \qquad (4.3.1)$$

The derivatives in λ_1, λ_2, α, and β are

$$\frac{\partial \log L}{\partial \lambda_1} = \frac{n}{\lambda_1} - S_{o,1} + \frac{M}{\lambda_1} - \frac{(\alpha+1)}{\beta} \sum_{k=1}^{s} \frac{T_k}{\{1 + (\lambda_1+\lambda_2) T_k / \beta\}} \tag{4.3.2}$$

$$\frac{\partial \log L}{\partial \lambda_2} = \frac{m}{\lambda_2} - S_{o,2} + \frac{s-M}{\lambda_2} - \frac{(\alpha+1)}{\beta} \sum_{k=1}^{s} \frac{T_k}{\{1 + (\lambda_1+\lambda_2) T_k / \beta\}} \tag{4.3.3}$$

$$\frac{\partial \log L}{\partial \alpha} = \frac{s}{\alpha} - \sum_{k=1}^{s} \log \{1 + (\lambda_1+\lambda_2) T_k / \beta\} \tag{4.3.4}$$

$$\frac{\partial \log L}{\partial \beta} = \frac{1}{\beta} \left[(\alpha + 1) \sum_{k=1}^{s} \frac{(\lambda_1+\lambda_2) T_k / \beta}{1 + (\lambda_1+\lambda_2) T_k / \beta} - s \right]. \tag{4.3.5}$$

The usual method of maximum likelihood is to set the above derivatives to 0 and to solve the system of four nonlinear equations which after some manipulation are:

i) $$\frac{n}{\lambda_1} - S_{o,1} + \frac{M}{\lambda_1} - \frac{s}{\lambda_1+\lambda_2} = 0$$

ii) $$\frac{m}{\lambda_2} - S_{o,2} + \frac{s-M}{\lambda_1} - \frac{s}{\lambda_1+\lambda_2} = 0$$

iii) $$\frac{s}{\alpha} - \sum_{k=1}^{s} \log \{1 + (\lambda_1+\lambda_2) T_k / \beta\} = 0$$

iv) $$\sum_{k=1}^{s} \frac{T_k}{1 + (\lambda_1+\lambda_2) T_k / \beta} - \frac{s\beta}{(\alpha+1)(\lambda_1+\lambda_2)} = 0 .$$

The first two equations are to be solved for λ_1, λ_2 to obtain $\lambda_{11}, \lambda_{21}$. We may obtain $\alpha_{mle}, \beta_{mle}$ by solving the last two equations. More precisely, we separate the loglikelihood (4.3.1) into two parts after the one to one transformation,

$(\lambda_1, \lambda_2, \alpha, \beta) \longrightarrow (\lambda_1, \lambda_2, \alpha, \theta)$ where $\theta = (\lambda_1 + \lambda_2) / \beta$. That is,

$\log L = \log L_1 + \log L_2$ where

$$\log L_1 = n \cdot \log\lambda_1 + m \cdot \log\lambda_2 - \lambda_1 S_{o,1} - \lambda_2 S_{o,2} + \log({}_sC_M) + M\log\lambda_1 + (s - M)\log\lambda_2 - s\log(\lambda_1 + \lambda_2), \tag{4.3.6}$$

and,

$$\log L_2 = s\log\alpha + s\log\theta - (\alpha + 1) \sum_{k=1}^{s} \log(1 + \theta T_k). \tag{4.3.7}$$

Thus the problem of finding the parameter values maximizing logL consists of the two parts: the first to find the values of λ_1, λ_2 maximizing $\log L_1$ and the second to find those values of α and θ maximizing $\log L_2$.

It should be noted that the M.L.E's of λ_1, λ_2 depend on the samples of the components under controlled conditions and M, the number of systems which fail due to the component A only, while the actual system lifetimes are used to estimate α and θ. This is somewhat obvious if we recall our assumption that the two components in a system under any fixed environment are functioning independently.

Going back to the estimation problem, the M.L.E.'s of λ_1, λ_2 are easily calculated by solving the quadratic equation

$$r_2 \lambda_1^2 + r_1\lambda_1 - r_o = 0 \qquad (4.3.8)$$

where $r_2 = S_{o,1} \cdot S_{o,2} - S_{o,1}^2$,

$r_1 = (n_c + n_A)\, S_{o,1} + (s - n_A)\, S_{o,2}$,

$r_o = n_C \cdot n_A$,

$n_C = n + m$; number of both components put on tests

$n_A = n + M$; number of component A's used in the whole experiment.

The estimator $\lambda_{10} = \{-r_1 + (r_1^2 + 4r_o r_2)^{1/2}\} / 2r_2$ if $r_2 > 0$,and

$\{-r_1 - (r_1^2 + 4r_o r_2)^{1/2}\} / 2r_2$ otherwise. The other estimator λ_{20} is computed as

$(n_C - S_{o1}\lambda_{10}) / S_{o2}$.

Looking at how the M.L.E.'s were obtained a natural question is " How much are these estimators improved by adding information from the system experiment ?" Since this question is of independent interest we will discuss this problem in the next section.

Prior to calculating the M.L.E.'s of α and θ, we would like to check their existence in the allowable parameter space ($\alpha > 0$, $\theta > 0$) since some authors (see Harris and Singpurwalla (1969)) have found M.L.E. in similar settings without a thorough investigation. Noting that $\log L_2$ is a function of α and θ, the two likelihood equations are

$$\frac{\partial \log L_2}{\partial \alpha} = \frac{s}{\alpha} - \sum_{k=1}^{s} \log\{1 + \theta T_k\} = 0\text{, and} \qquad (4.3.9)$$

$$\frac{\partial \log L_2}{\partial \theta} = \frac{s}{\theta} - (\alpha + 1) \sum_{k=1}^{s} \frac{T_k}{1 + \theta T_k} = 0\,. \qquad (4.3.10)$$

Solving 4.3.9 for α we obtain $\alpha_{mle}=s/\{\Sigma \log\{1 + \theta T_k\}\}$. Substituting this value into 4.3.10 we obtain the following equation which is to be solved for θ:

$$\frac{s}{\theta} - f_2(\theta)\left[\frac{s}{f_1(\theta)} + 1\right] = 0 \qquad (4.3.11)$$

where

$$f_1(\theta) = \sum_{k=1}^{s} \log(1 + \theta T_k), \qquad f_2(\theta) = \sum_{k=1}^{s} \frac{T_k}{(1 + \theta T_k)}\,.$$

This equation can not be shown to have a positive and finite root. Thus we provide a sufficient condition under which there is a M.L.E. of θ.

Let $\log L_\theta = s \log s - s + s \log \theta - s \log(\Sigma \log(1 + \theta T_k)) - \Sigma \log(1 + \theta T_k)$ be the loglikelihood of (θ,α) evaluated at $\alpha = \alpha_{mle}$. Then

$$\lim_{\theta \to 0} \log L_\theta = s\log s - s - s\log\left(\sum_{k=1}^{s} T_k\right)$$

$$\lim_{\theta \to \infty} \log L_\theta = -\infty$$

$$\lim_{\theta \to 0} \frac{d \log L_\theta}{d\theta} = \frac{s\, \Sigma T_k^2 - 2(\Sigma T_k)^2}{2(\Sigma T_k)}$$

$$\lim_{\theta \to \infty} \frac{d \log L_\theta}{d\theta} = 0\,,$$

We note that if the observation T_k's satisfies

$$s \sum_{k=1}^{s} T_k^2 - 2 (\sum_{k=1}^{s} T_k)^2 > 0 \qquad (4.3.12)$$

then $\log L_\theta$ is increasing in a neighborhood of 0. Noting that $\lim_{\theta \to \infty} \log L_\theta = -\infty$, and

$\lim_{\theta \to 0} \log L_\theta$ is

finite, it follows that the smallest root of the equation (4.3.11) is a M.L.E. of θ.

Thus if the sample satisfies (4.3.12), a M.L.E. θ_{mle} of θ is obtained by solving the

equation (4.3.11) numerically and a M.L.E. of α is computed as $s/[\Sigma \log(1 + \theta_{mle} t_k)]$.

In the case that the data does not satisfy the condition (4.3.12) we would have a M.L.E.

of θ at $\theta = 0$ which leads to ∞ as a M.L.E. of α. In such a case the reliability for the series system becomes

$$\lim_{\substack{\alpha \to \infty \\ \alpha/\beta \to \mu}} R_s(t) = \lim_{\substack{\alpha \to \infty \\ \alpha/\beta \to \mu}} (1 + (\lambda_1+\lambda_2) / t)^{-\alpha} = \exp(-\mu (\lambda_1+\lambda_2) t) \qquad (4.3.13)$$

so that we conclude that the series system has constant hazard rate and it seems reasonable to carry out the inference procedure accordingly.

A practical meaning can be given if the condition (4.3.12) is expressed as

$$[\sum_{k=1}^{s} T_k^2 - T^2] / s > T^2 \quad \text{where} \quad T = \{\sum_{k=1}^{s} T_k\} / s . \qquad (4.3.14)$$

The above expression shows that the condition is satisfied if the sample deviation is larger than the sample mean.

In addition the existence problem of maximum likelihood estimation we note that the estimate of α can be less than one, which implies that the mean system reliability is infinite. To study the properties of the estimators as well as some others which will be discussed in the next sections a small scale Monte Carlo study will be presented in section 4.7.

4.4 A Note on the Estimation of the Components' Hazard Rates

As discussed in the previous section the M.L.E.'s of λ_1, λ_2, the components' hazard rates under controlled condition are obtained through the likelihood function $\log L_1$ which is constructed from three independent samples, one based on each component tested separately and one based on system data. However the only contribution from the system data to the likelihood function for λ_1, λ_2 is the information as to which component has caused the system failure, while the contribution from the component data consists of their lifetimes.

The framework of this problem is combining component and system information. Eastering and Prairie (1971) have discussed this problem when there is data from both components and systems consisting of several identical, but independent components in series or in parallel. In case of attribute testing and life testing they have obtained estimators of the components hazard rates and have investigated how much information about the hazard rate of the components is obtained from the system sample using the asymptotic variances of the maximum likelihood estimators. Mastran (1975) has considered this problem from a Bayesian point of view. Miyamura (1982) has investigated the systems of independent but not-identically distributed components and suggested a maximum likelihood estimation procedure. In the framework discussed in this section the components are not identically distributed and we combine lifetimes of both components and information of the cause of system failures to estimate component parameters.

First, we compute the asymptotic variances of M.L.E.'s of λ_1, λ_2 obtained in the previous section and compare them with the variances of the M.L.E. computed only from the component samples. Second, we investigate some possible strategies for determining sample sizes under cost constraints which may occur if it costs to check which component caused the system to fail.

We assume that the sample sizes of both components are same, that is, $n = m = N$. Suppose
that the ratio of the component sample size to the total sample size, $N/(2N + s)$, goes to c as both N and s go to ∞. Now

$$I(\lambda) = c\, I_1(\lambda) + c\, I_2(\lambda) + (1-2c)\, I_3(\lambda) \qquad (4.4.1)$$

where $\lambda = (\lambda_1, \lambda_2)$ and $I_i(\lambda)$ is the information matrix based on the density function corresponding to i-th sample.

To obtain $I_i(\lambda)$ the loglikelihood $\log L_1$ should be divided into the three different parts corresponding to a sample of size one from each phase of the experiment. Let

$$\log L_{11} = \log\lambda_1 - X_{o,1}\, \lambda_1$$

$$\log L_{12} = \log\lambda_2 - Y_{o,2}\, \lambda_2$$

$\log L_{13} = \delta \log \lambda_1 + (1-\delta) \log \lambda_2 - \log(\lambda_1 + \lambda_2).$

Noting that $I_i(\lambda) = -E\left[\frac{\partial^2 \log L_{1i}}{\partial \lambda_1 \partial \lambda_2}\right]$, $(i = 1, 2, 3)$ and $E(M) = \frac{\lambda_1}{\lambda_1 + \lambda_2}$,

we obtain

$$I_1(\lambda) = \begin{pmatrix} 1/\lambda_1^2 & 0 \\ 0 & 0 \end{pmatrix}, \qquad I_2(\lambda) = \begin{pmatrix} 0 & 0 \\ 0 & 1/\lambda_2^2 \end{pmatrix},$$

$$I_3(\lambda) = \begin{pmatrix} \frac{1}{\lambda_1(\lambda_1+\lambda_2)} - \frac{1}{(\lambda_1+\lambda_2)^2} & -\frac{1}{(\lambda_1+\lambda_2)^2} \\ -\frac{1}{(\lambda_1+\lambda_2)^2} & \frac{1}{\lambda_2(\lambda_1+\lambda_2)} - \frac{1}{(\lambda_1+\lambda_2)^2} \end{pmatrix}, \text{ so}$$

$$I(\lambda) = \begin{pmatrix} \frac{c(\lambda_1^2+\lambda_2^2)+\lambda_1 \cdot \lambda_2}{\lambda_1^2(\lambda_1+\lambda_2)^2} & \frac{-(1-2c)}{(\lambda_1+\lambda_2)^2} \\ \frac{-(1-2c)}{(\lambda_1+\lambda_2)^2} & \frac{c(\lambda_1^2+\lambda_2^2)+\lambda_1 \cdot \lambda_2}{\lambda_2^2(\lambda_1+\lambda_2)^2} \end{pmatrix}, \tag{4.4.2}$$

$\det(I(\lambda)) = |I(\lambda)|$

$$= \frac{c^2(\lambda_1^2 - \lambda_2^2)^2 + 2c\lambda_1\lambda_2 \; (\lambda_1+\lambda_2)^2}{\lambda_1^2\lambda_2^2(\lambda_1+\lambda_2)^4}. \tag{4.4.3}$$

Thus the variance - covariance matrix $I^{-1}(\lambda)$ of M.L.E.s of (λ_1, λ_2) is

$$\begin{pmatrix} \dfrac{\lambda_1^2 (\lambda_1+\lambda_2)^2\{ c(\lambda_1^2+\lambda_2^2)+\lambda_1\lambda_2\}}{c^2(\lambda_1^2-\lambda_2^2)^2+2c\lambda_1\lambda_2(\lambda_1+\lambda_2)^2} & \dfrac{\lambda_1^2\lambda_2^2(\lambda_1+\lambda_2)^2(1-2c)}{c^2(\lambda_1^2-\lambda_2^2)^2+2c\lambda_1\lambda_2(\lambda_1+\lambda_2)^2} \\ \dfrac{\lambda_1^2\lambda_2^2(\lambda_1+\lambda_2)^2(1-2c)}{c^2(\lambda_1^2-\lambda_2^2)^2+2c\lambda_1\lambda_2(\lambda_1+\lambda_2)^2} & \dfrac{\lambda_2^2(\lambda_1+\lambda_2)^2\{c(\lambda_1^2+\lambda_2^2)+\lambda_1\lambda_2\}}{c^2(\lambda_1^2-\lambda_2^2)^2+2c\lambda_1\lambda_2(\lambda_1+\lambda_2)^2} \end{pmatrix}. \tag{4.4.4}$$

We obtain, by Theorem 6.1 of Lehmann (1983),

$$\sqrt{2N+s}\,(\lambda_{11}-\lambda_1) \longrightarrow N(0,\ I_{11}^{-1}), \tag{4.4.5}$$

$$\sqrt{2N+s}\,(\lambda_{21}-\lambda_2) \longrightarrow N(0,\ I_{22}^{-1}), \tag{4.4.6}$$

where I_{ij}^{-1} is the i-jth entry in the matrix $I^{-1}(\lambda)$.

Next we study how much the estimator λ_{11}, incorporating the information from the system improves on the estimator λ_{10} of λ_1 computed from the component sample by computing the asymptotic relative efficiency (A.R.E.) of λ_{10} of λ_1 to λ_{11}. Since

$\sqrt{N}(\lambda_{10}-\lambda_1) \longrightarrow N(0,\ \lambda_1^2)$, we obtain the A.R.E. e_1 as

$$e_1 = c\frac{I_{11}^{-1}}{\lambda_1^2} = \frac{(\lambda_1+\lambda_2)^2\lambda_1\lambda_2 + c(\lambda_1+\lambda_2)^2(\lambda_1^2+\lambda_2^2)}{2(\lambda_1+\lambda_2)^2\lambda_1\lambda_2 + c(\lambda_1+\lambda_2)^2(\lambda_1-\lambda_2)^2}. \tag{4.4.7}$$

If we assume that $\lambda_2 = k\lambda_1$, then

$$e_1 = \frac{k + c\,(k^2+1)}{2k + c\,(k-1)^2}. \tag{4.4.8}$$

Figure 18 shows the A.R.E. of λ_{11} to λ_{10} at varying ratios of component sample size to total sample size for five different k values. Notice that for fixed ratio the more similar the hazard rates of the two components are, the smaller the A.R.E. is. That is, the more the information from the system sample contributes to reducing the variance. Figure 19 shows the A.R.E. at varying k's for given ratio of sample sizes.Note that c=0 corresponds to the situation where N is very small compared to s, c=1/6 to the case where s is of the order 4 times N, c=1/3 to the case where s is almost same as N, c=2/5 to the case where N of the order 1/2 times N and c=1/2 to the case where s is very small compared to N. The above result is also valid for the estimators, λ_{21} and λ_{20} since all the formulae are symmetric in λ_1 and λ_2.

In the previous section we saw that estimating the scale parameter β of the gamma distribution involves estimating $\lambda_1 + \lambda_2$ rather than λ_1 or λ_2 themselves so that we shall turn to comparision of the variances of $\lambda_{11} + \lambda_{21}$ and $\lambda_{10} + \lambda_{20}$. Using the same procedure as before we are led to the A.R.E. of $\lambda_{10} + \lambda_{20}$ to $\lambda_{11} + \lambda_{21}$ which is

$$e_2 = \frac{(\lambda_1 + \lambda_2)^2 \; \{ \lambda_1\lambda_2 + c \; (\lambda_1 - \lambda_2)^2 \}}{(\lambda_1{}^2 + \lambda_2{}^2) \; \{2\,\lambda_1\lambda_2 + c \; (\lambda_1 - \lambda_2)^2 \}}. \tag{4.4.9}$$

Setting again $\lambda_1 = k\,\lambda_2$,

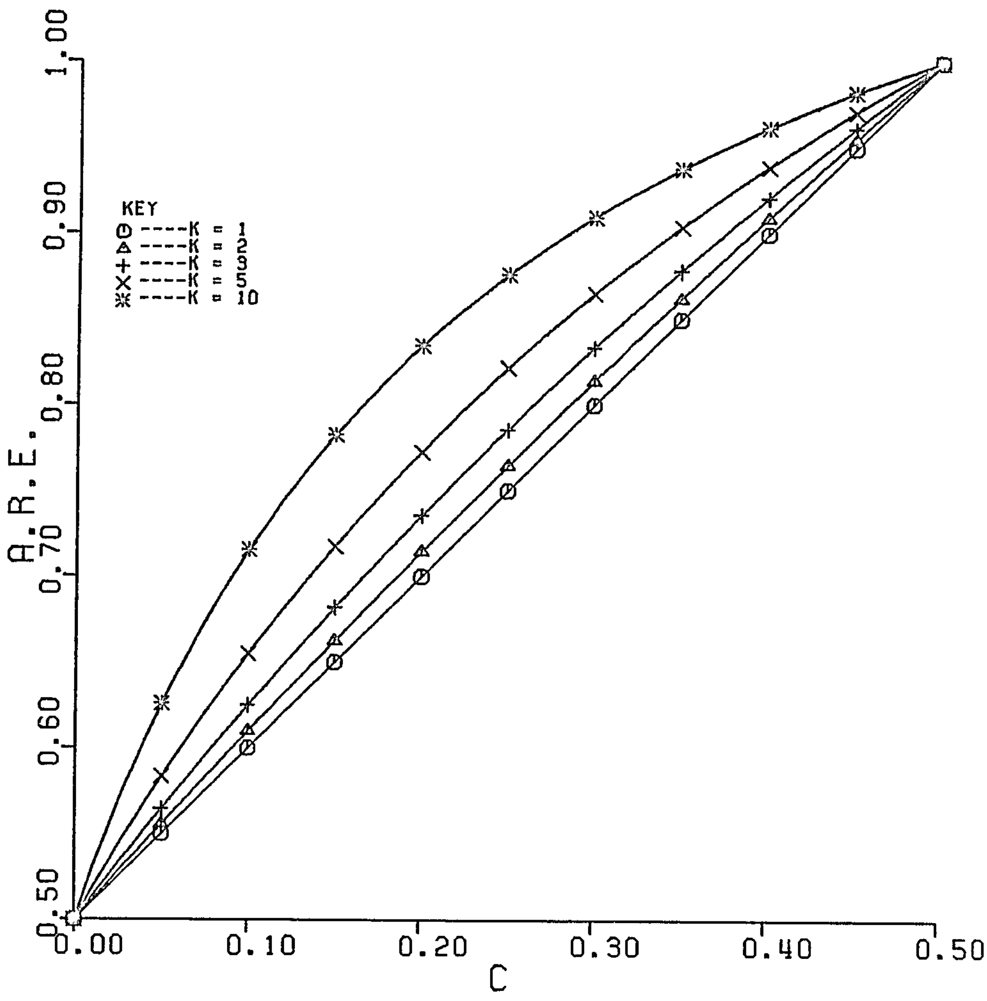

FIGURE 18

A.R.E. OF λ_{11} TO λ_{10} AS A FUNCTION OF C

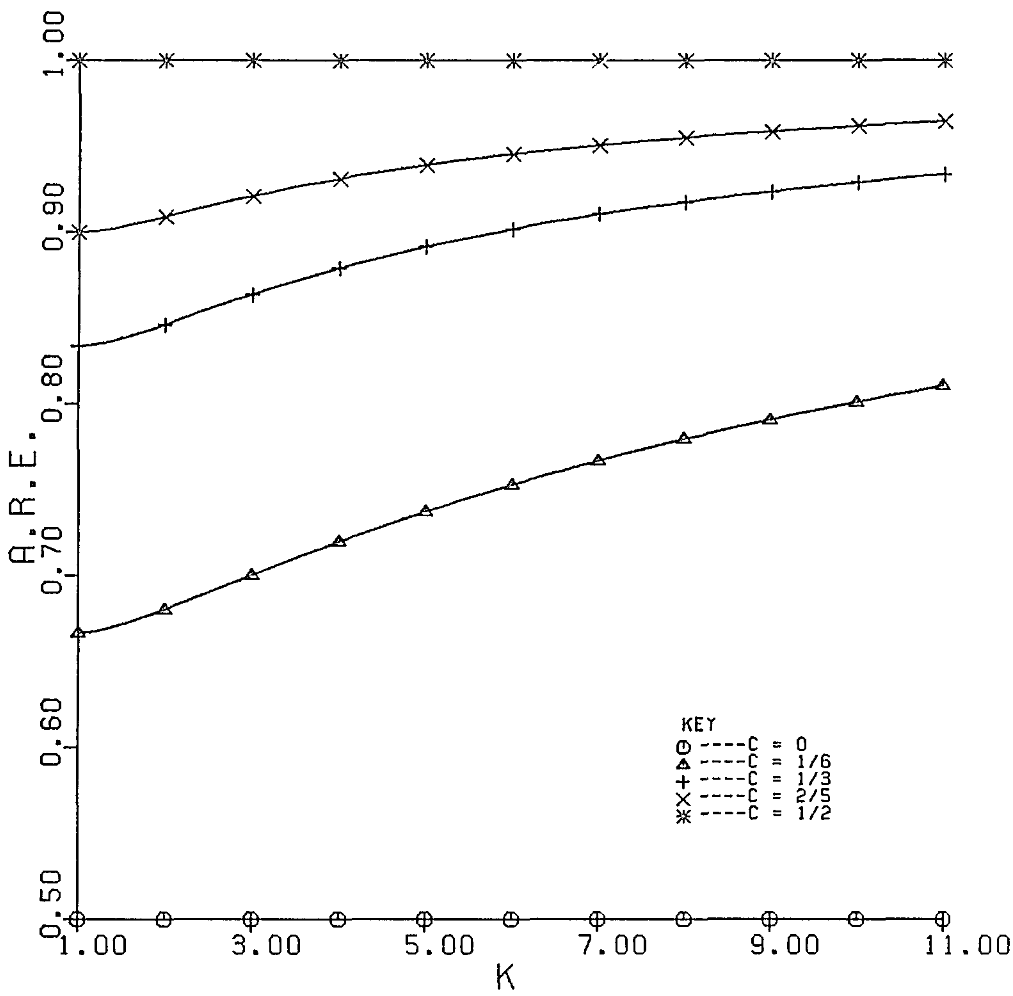

FIGURE 19

A.R.E. OF λ_{11} TO λ_{10} AS A FUNCTION OF K

$$e_2 = \frac{(k+1)^2 \{c(k-1)^2 + k\}}{(k^2+1)\{c(k-1)^2 + 2k\}},$$

$$= \left(1 + \frac{2k}{(k^2+1)}\right)\left(1 - \frac{k}{c(k-1)^2 + 2k}\right). \quad (4.4.10)$$

The plots of the A.R.E.'s are shown for various k's and c's in figure 20 and 21. Surprisingly it is found that when the hazard rates of two components are identical or very similar the information from the system does not contribute to a reduction of the asymptotic variance. Since λ_{11} and λ_{21} are correlated it is not easy to explain this finding analytically. However, an intuitive explanation is that when the two hazard rates are similar the information from the system, which only contributes information on the amount of difference between the hazard rates through the numbers of systems which fail from each type of component failure, contributes least to the inference on the relation between the two components in the system.

In order to check the above results for finite sample sizes, as encountered in practice, we have investigated the ratios of the mean square errors of estimates,

$$e_{1hat} = \frac{MSE(\lambda_{11})}{MSE(\lambda_{10})}, \text{ and } e_{2hat} = \frac{MSE(\lambda_{11}+\lambda_{21})}{MSE(\lambda_{10}+\lambda_{20})},$$

through a Monte Carlo study. The sufficient statistics for a random sample from the above sampling scheme consist of three random deviates: one from a gamma distribution with parameters (N, λ_1), another from a gamma distribution with parameters (N, λ_2), and the other from a binomial distribution with parameters $[s, \lambda_1 / (\lambda_1+\lambda_2)]$. 1000 sets of sufficient statistics were generated to obtain the estimates of e_{1hat}, and e_{2hat} for each

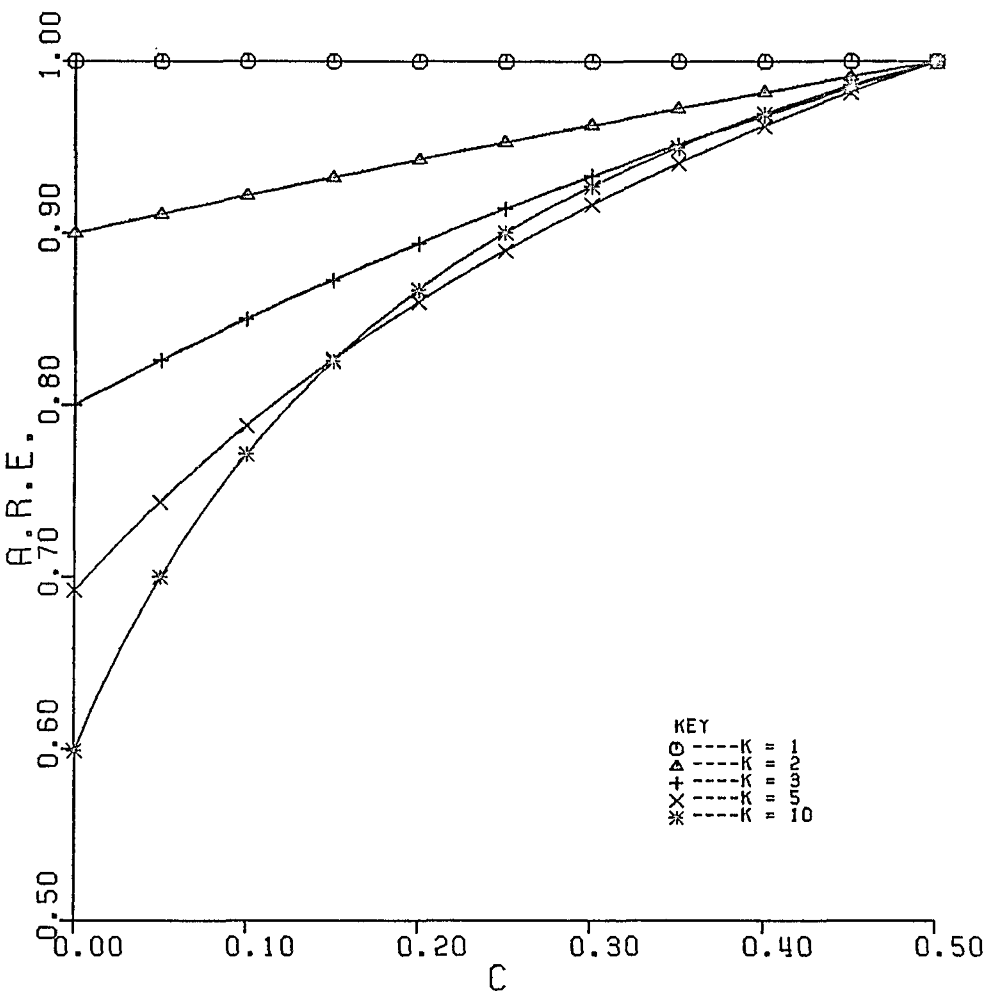

FIGURE 20

A.R.E. OF $\lambda_{11} + \lambda_{21}$ TO $\lambda_{10} + \lambda_{20}$ AS A FUNCTION OF C.

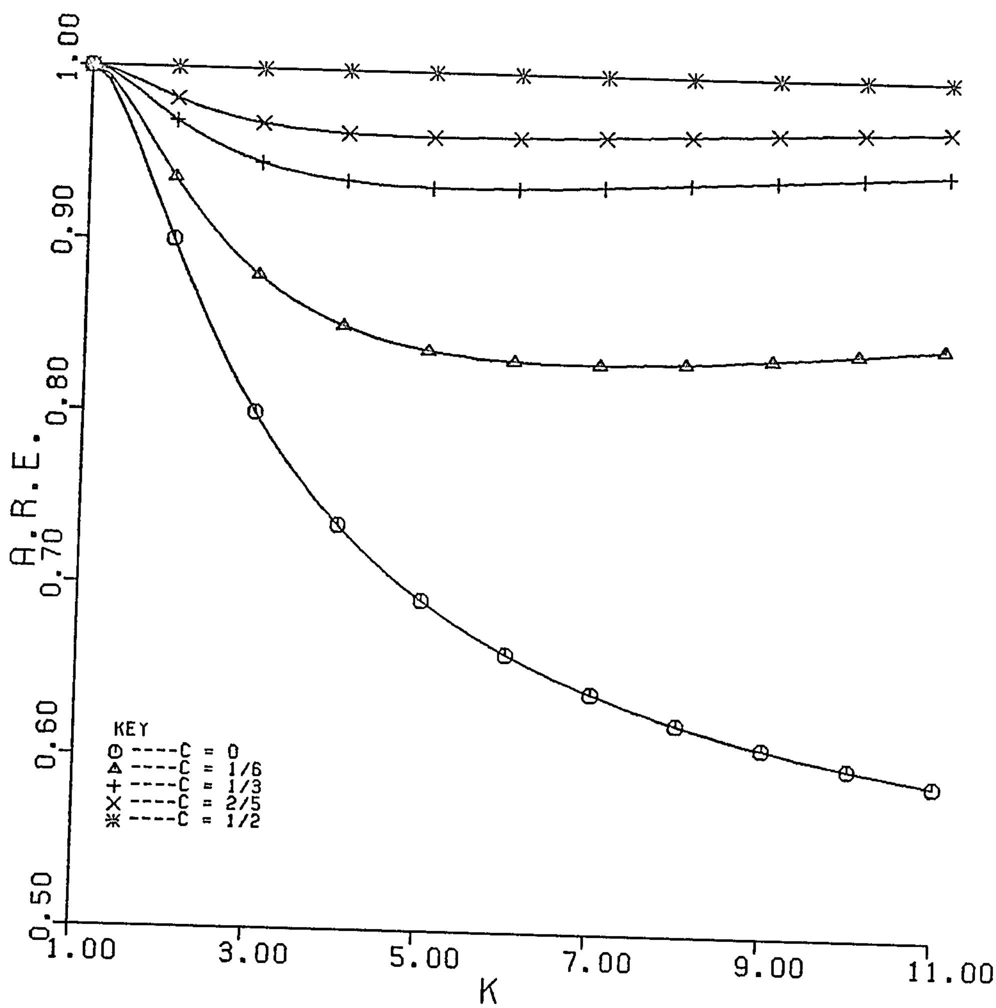

FIGURE 21

A.R.E. OF $\lambda_{11} + \lambda_{21}$ TO $\lambda_{10} + \lambda_{20}$ AS A FUNCTION OF K.

combination of the component sample size N=20,50,100, 500, and the sample fraction $c^* = N / (2N+s) = .1, .2, .3, .4$ with $\lambda_1 = 1$ and $\lambda_2 = 1, 2, 10$. The estimates for each case are reported in the following table 1 for e_{1hat}, and table 2 for e_{2hat}. From these tables we can see generally agreement with the asymptotic results for these combinations of moderate sample sizes. Also it is confirmed that the information from the system sample contributes very little to the reduction of the mean square error of $\lambda_{11} + \lambda_{21}$ since the first line of each box(i.e. $\lambda_2 = 1$) in the table 2 is about 1.0.

There are several problems which may be considered in the light of the above results. We will discuss two of these:

i) If we suppose that it costs to check the cause of system failure, when is it resonable to check the systems?

ii) If we are allowed to check randomly some of systems how many systems should be checked to achieve minimum variance under some constraints?

To investigate the above two problems we assume that the sample size of the system sample is fixed at s and that the sample sizes and the unit price of testing both components are the same. Let P_T be the total remaining allowable cost after administrative costs and the costs of collecting system life data are removed and, let P_U be the cost of testing a component, and let P_c be the additional cost of checking a system to determine its failure mode. Suppose these costs P_T , P_U, and P_c are predetermined.

Table 1: Monte Carlo Estimation of the efficiency of λ_{10} to λ_{11} based on 1000 samples

N	λ_2	c^* = .1	.2	.3	.4
20	1	.56534	.67647	.82189	.90085
	2	.56225	.67686	.79254	.93825
	10	.62437	.76877	.87676	.92083
50	1	.58265	.67777	.76197	.88336
	2	.60692	.70019	.78153	.90748
	10	.70346	.82000	.91630	.96813
100	1	.59245	.69637	.81654	.89781
	2	.59418	.70775	.78777	.91425
	10	.69286	.83290	.90202	.95081
500	1	.63320	.72681	.81371	.92221
	2	.61353	.69582	.81650	.91505
	10	.68270	.81903	.90938	.97090

Table 2: Monte Carlo Estimation of the efficiency of $\lambda_{10}+\lambda_{20}$ to $\lambda_{11}+\lambda_{21}$ based on 1000 samples

N	λ_2	c^* = .1	.2	.3	.4
20	1	.89377	.91940	.95326	.98167
	2	.85409	.88025	.95558	.97752
	10	.78360	.87069	.89873	.96726
50	1	.93494	.95313	.96733	.98379
	2	.91066	.94605	.97147	.99287
	10	.81203	.88544	.92727	.99362
100	1	1.01925	1.02397	1.03229	1.03585
	2	.97199	.98356	1.02168	1.01971
	10	.76529	.88264	.95298	.97951
500	1	1.04136	1.03872	1.03979	1.04268
	2	.94269	.97442	.96699	1.00756
	10	.80495	.87426	.93227	.96923

The sample sizes are assumed to be reasonably large so that asymptotic variance results hold.

We consider question (i). We see that

$N_c = P_T / 2P_U$: The maximum number of each component we could test when systems are not checked.

Let

$R = P_c / P_U$: The ratio of the costs,

$Q = N_c / s$: The ratio of the component sample size if systems are not checkedto the system sample size.

Then

$$P_T = 2P_U N + s\,Pc\,, \quad \text{and} \quad N = N_c - R\cdot s / 2.$$

Our goal is to find the maximum value of R such that

$$V(\lambda_{11,N} + \lambda_{21,N}) < V(\lambda_{10,N_c} + \lambda_{20,N_c}) \tag{4.4.11}$$

where the subscripts N, N_c denote the component sample sizes when the estimators are computed.

Noting that $V(\lambda_{11,N} + \lambda_{21,N})$ is approximated by

$$\frac{1}{2N+s} \quad \frac{(\lambda_1 + \lambda_2)^2}{c_f} \left[\frac{c_f(\lambda_1 - \lambda_2)^2 + \lambda_1\lambda_2}{c_f(\lambda_1 - \lambda_2)^2 + 2\lambda_1\lambda_2} \right] \text{ where } c_f = N/(2N+s), \text{ and}$$

$V(\lambda_{10,N} + \lambda_{20,N})$ by $(\lambda_1^2 + \lambda_2^2) / N_c$, we obtain the following approximate ratio

$$\frac{V(\lambda_{11,N}+\lambda_{21,N})}{V(\lambda_{10,N_c}+\lambda_{20,N_c})} = \frac{Q}{Q - 0.5R}\left[\frac{(k+1)^2}{k^2+1}\left(\frac{\frac{Q-0.5R}{2Q+(1-R)}(k-1)^2+k}{\frac{Q-0.5R}{2Q+(1-R)}(k-1)^2+2k}\right)\right].$$

(4.4.12)

If we let r(R, Q, k) denote the above ratio of variances, after some algebraic manipulation we see that

$r(R,Q,k) < 1$ implies

$$R^2(k+1)^2 - R\,(2Qk^2 + 4Qk + 4k + 2Q) + \{(k-1)^2 / (k^2+1)\}4Qk > 0. \quad (4.4.13)$$

The left hand side of the inequality has positive value for small R and then a negative value for large value of R. R must be less than 2Q to assure a positive value of N. The smaller root of the corresponding quadratic equation is the maximum value of R for which $V(\lambda_{11,N} + \lambda_{21,N})$ is smaller than $V(\lambda_{10,N} + \lambda_{20,N})$ and hence determination of the cause of system failure is advisable. That is, the maximum value of R is

$$\frac{1}{(k+1)^2}\left(B - \sqrt{B^2 - 4Qk(k-1)^2}\right), \text{ where } B = Qk^2 + 2Qk + 2k + Q$$

(4.4.14)

Figure 22 shows the maximum value of R for each Q at different k's. For example, suppose we have the idea that Q is equal to 10 computed using the predetemined values s, P_T, and P_U. We also assume that the ratio of the hazard rates is ,3 say, which might be guessed through past experience. Then this figure tells that if the relative cost, R, is less than 0.1 then it is recommended to check the systems. As discussed before, if the two components have the same hazard rates it is not recommended to do it.

Considering the question (ii), we find the number of systems to be checked to

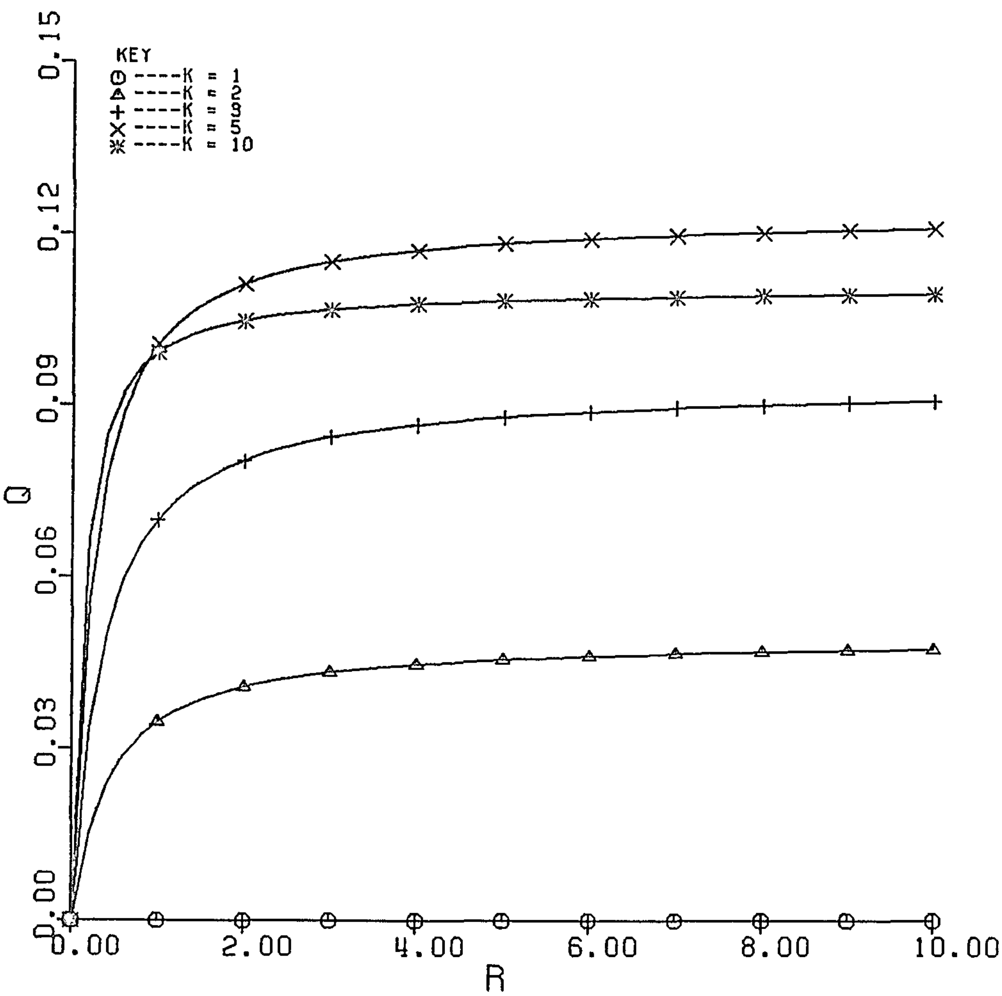

FIGURE 22

MAXIMUM RELATIVE COST, R, AT WHICH SYSTEM FAILURE MODE INFORMATION IS WORTH COLLECTING.

under the given constraints. Let s^* be the number of systems to be checked among the s systems, and set $y = s^* / N_c$. Noting that

$N = N_c - 0.5 R s^*$, and recalling that the ratio of component sample to complete

sample size c is $N/(2N+s^*)$, we obtain the asymptotic variance of $V(\lambda_{11,N}+\lambda_{21,N})$ as

$$\frac{1}{2N+s^*}\left[\frac{(k+1)^2 \lambda_1^2 \{c(k-1)^2+k\}}{c\{c(k-1)^2+2k\}}\right],$$

$$= \frac{1}{N}\left[(k+1)^2 \lambda_1^2 \frac{\{c(k-1)^2+k\}}{\{c(k-1)^2+2k\}}\right],$$

$$= \frac{1}{N_c}(k+1)^2 \lambda_1^2 \frac{1}{1-0.5Ry}\left[\frac{\{c(k-1)^2+k\}}{\{c(k-1)^2+2k\}}\right]. \qquad (4.4.15)$$

Before calculating the value of R we note the followings:

(i) Since $1 - 0.5Ry \geq 0$, $y \leq \min(s, 2/R)$.

(ii) $c = N/(2N + s^*) = [2 + \{y/(1-0.5Ry)\}]^{-1}$

(iii) $[\{c(k-1)^2+k\}/\{c(k-1)^2+2k\}] > 0.5$.

Ignoring $\{(k+1)^2 \lambda_1^2\}/N_c$ term in (4.4.15) we let

$$q(y,R,k) = \frac{1}{1-0.5Ry}\left[\frac{\{c(k-1)^2+k\}}{\{c(k-1)^2+2k\}}\right]. \qquad (4.4.16)$$

The derivative of q with respect to y after substituting (ii) for c is,

$$\frac{dq}{dy} = \frac{q_1(y)}{(1-0.5Ry)^2[(k+1)^2+y(2k-kR-0.5k^2R-0.5R)]^2} \qquad (4.4.17)$$

where $q_1(y) = y^2 [0.5R(k^2+1) - k][0.5R(k+1)^2 - 2k](R/2)$

$$+ y[R(k^2+1)(2k - 0.5R(k+1)^2)]$$

$$+ [0.5(k+1)^2(k^2+1)]R - k(k-1)^2 .$$

In order to minimize $q(y, R, k)$, it is necessary to study the function $q_1(y)$ in detail. Let

$$d_1 = \frac{2k(k-1)^2}{(k+1)^2(k^2+1)} , \quad d_2 = \frac{2k}{(k^2+1)} , \quad d_3 = \frac{4k}{(k+1)^2} ,$$

and note that $d_1 < d_2 < d_3$.

For R in the interval $(0, d_1)$ the coefficient of y^2 assumes positive value and the constant term of $q_1(y)$ is negative, and the discriminant function, $D = R[2k - 0.5R(k+1)^2][2k^2(k-1)^2]$ is positive. So the larger root of the equation obtained by setting $q_1(y)$ to be zero is what minimizes $q(y, R, k)$, and hence the asymptotic variance of $\lambda_{11,N} + \lambda_{21,N}$. Additionally it is found that this root is less than 1/R. For R larger than d_1, the same steps as the above show that $q(y, R, k)$ is minimized at $y = 0$. This means that no contribution is made by checking the system failure mode if the relative cost P_c / P_U is larger than $[2k(k-1)^2] / [(k+1)^2(k^2+1)]$. Finally we conclude that if the relative cost is smaller than the above ratio the optimal number of systems to be checked is,

$$N_c \left[\frac{\sqrt{2}\,k(k-1)}{(k - 0.5R(k^2+1))\sqrt{R(2k - 0.5R(k+1)^2)}} - \frac{(k^2+1)}{k - 0.5R(k^2+1)} \right] . \quad (4.4.18)$$

Figure 23 shows the optimal fraction $y = s^*/N_c$ at the allowable R's for k = 2, 3, and 5. For example, suppose we have the idea that the ratio of the hazard rates, k is equal to 5, and that the relative cost R is equal to .1. Then this figure tells that the optimal number of the systems to be checked is 1.5 times as much as N_c.

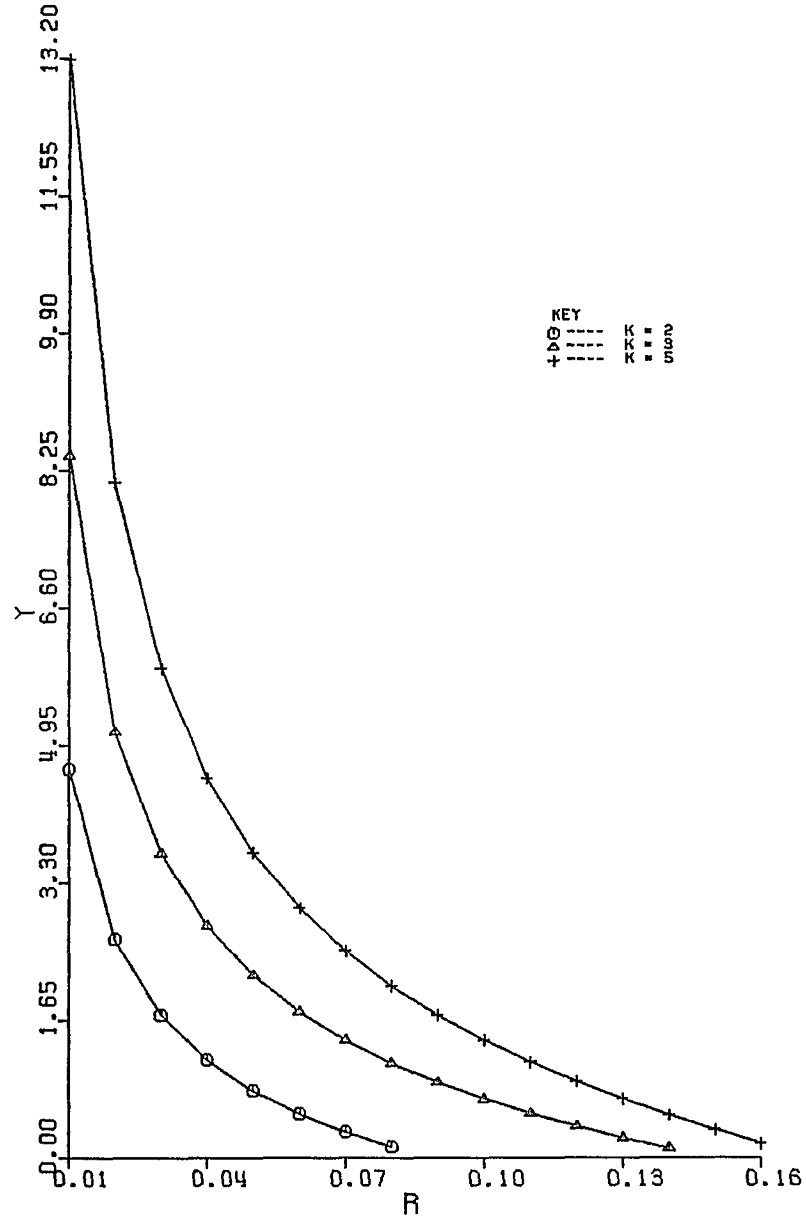

FIGURE 23

OPTIMAL FRACTION Y, OF THE SYSTEMS TO MAKE FAILURE MODE DETERMINATIONS ON FOR A FIXED COST

4.5 Other Conventional Estimators.

In section 4.3 and 4.4 we have discussed estimation schemes for the component hazard rates λ_1, λ_2 as well as the problem of experimental design. Also we have studied M.L.E.'s of the parameters α and θ in section 4 and have noted that these estimators depend only on the system failure times. In this section we describe two other estimators of α and θ, one of which is the Method of Moment Estimators (M.M.E.). These estimators may be used as an initial values for the iterative solution of the likelihood equations.

Recall that the system reliability, the mean and variance of system lifetime are

$$R_s(t) = (1 + \theta t)^{-\alpha}$$

$$E(T) = [(\alpha - 1)\theta]^{-1} \quad \text{if} \quad \alpha > 1,$$

$$V(T) = \alpha / [(\alpha - 1)(\alpha - 2)\theta^2] \quad \text{if} \quad \alpha > 2.$$

The M.M.E.s, found by equating the first two sample and theoretical moments,

$$\alpha_{mme} = 1 + \frac{sE_2}{s E_2 - 2 E_1^2} \qquad \text{if} \quad s E_2 > 2 E_1^2 , \quad \text{and} \qquad (4.5.1)$$

$$\theta_{mme} = \frac{s E_2 - 2 E_1^2}{E_1 \cdot E_2} , \qquad \text{where } E_1 = \sum_{k=1}^{s} T_k , \qquad E_2 = \sum_{k=1}^{s} T_k^2 . \qquad (4.5.2)$$

Hui and Berger (1983) have suggested estimators in a different context. To avoid difficulties of maximizing the loglikelihood function with two unknown parameters they have suggested a modified method of moments estimator. From the mean of system lifetime, we have $[(\alpha - 1)\theta]^{-1} = E_1 / s$. Solving this equation with respect to θ, and replacing $(\alpha - 1)$ by α to make θ positive we obtain $\theta = [\alpha E_1 / s]^{-1}$, which is used as the true value of θ in the likelihood function (4.3.7) so that the estimator of α is the solution to

$$- \sum_{i=1}^{s} \log \left(1 + \frac{s}{E_1 \alpha} T_i \right) + \frac{s (\alpha + 1)}{E_1 \alpha^2} \sum_{i=1}^{s} \frac{T_j}{1 + sT_i / (E_1 \cdot \alpha)} = 0. \quad (4.5.3)$$

It is possible that there is no finite solution to this equation. With an argument similar to that used in M.L.E. case it can be shown that a sufficient condition for a finite solution to (4.3.7) is that $s E_2 > 2 E_1^2$, (4.5.4)

which is the same one for M.L.E. and M.M.E. cases.

4.6 Graphical Inference

In the previous sections conventional estimating procedures for the parameters of interest have been considered. While discussing related problems we have learned that the existence of these estimators in the allowable parameter space depends on the data collected. Accordingly, it is not so simple to have an estimator of the degree of dependence induced by a random environmental factor. In this section, we discuss a graphical approach which is helpful in visualizing the condition of existence of the estimators and also the degree of dependence as well as in checking feasibility of the model. Later in this section we suggest estimators based on this graphical approach. Throughout this section we assume the same model as in the section 4.2. However we shall handle the model as if the component hazard rates λ_1 and λ_2 were known, based on data from the laboratory experiment, since the estimation of λ_1 and λ_2 presents little difficulty and has been discussed in detail in section 4.4. Thus, the model in this section is that the lifetime, T, of a system in the operating environment has a survival function

$$R_s(t) = (1 + \theta t)^{-\alpha}. \qquad (4.6.1)$$

The method we present in this section is based on the scaled total time on test (STTOT) plot of Barlow and Campo (1975). They have presented a graphical approach to failure data analysis for arbitrary distributions, using the total time on test transforms introduced and discussed in chapter 5 and 6 of Barlow, Bartholomew, Brenner, and Bunk(BBBB) (1972). To begin with, we review the concept of total time on test transforms. Suppose that we have an ordered sample

$0 = U_{n,o} < \cdots < U_{n,n}$ from a distribution function A (·) with the finite mean.

Then the total time on test up to the r-th failure is defined as

$$V_{n,r} = n\,U_{n,1} + (n-1)(U_{n,2} - U_{n,1}) + \cdots + (n-r+1)(U_{n,r} - U_{n,r-1}) \qquad (4.6.2)$$

and the total time on test transform is defined as

$$TOT_A(t) = \int_0^{A^{-1}(t)} [1 - A(u)]\,du, \quad 0 \le t \le 1. \qquad (4.6.3)$$

The relation ships between these two can be shown through

$$TOT_{A_n}(r/n) = \int_0^{A_n^{-1}(r/n)} [1 - A_n(u)]\,du$$

$$= \sum_{j=1}^{r} \left(1 - \frac{(j-1)}{n}\right)(U_{n,j} - U_{n,j-1})$$

$$= \frac{V_{nr}}{n} \qquad (4.6.4)$$

where A_n is the emprical distribution of U's and

$$A_n^{-1}(u) = \inf\{ x \mid A_n(x) < u \}.$$

Note that $$\lim_{n \to \infty, r/n \to t} TOT_A(r/n) = \int_0^{A^{-1}(t)} (1 - A(u))\, du = TOT_A(t). \qquad (4.6.5)$$

Since $TOT_A(1)$ is just the mean of H, $ST_A(t)$ which is the standardized total time on test, i.e., $\frac{TOT_A(t)}{TOT_A(1)}$ is termed scaled total time on test transform and $\frac{V_{n,r}}{V_{n,n}}$ is called the emperical scaled total time on test (STTOT).

Let $T_1, T_2, \cdots, T_s$ be the system failure times collected in the operating environment. Suppose $A(t)$ is the cummulative distribution function of T, that is,

$$A(t) = 1 - R_s(t)$$

$$= 1 - (1 + \theta t)^{-\alpha}. \qquad (4.6.6)$$

The STTOT transform for T is computed as

$$ST_A(t) = \frac{\int_0^{A^{-1}(t)} R_s(u)\, du}{\int_0^{A^{-1}(1)} R_s(u)\, du}$$

$$= 1 - (1 - t)^{(\alpha - 1)/\alpha} \quad \text{for } \alpha > 1, \qquad (4.6.7)$$

where $A^{-1}(t) = \theta^{-1}(-1 + (1 - t)^{-1/\alpha})$. Here we note that $ST_A(t)$ depends only on the shape parameter α. Figure 24 shows the form of the STTOT transform for several

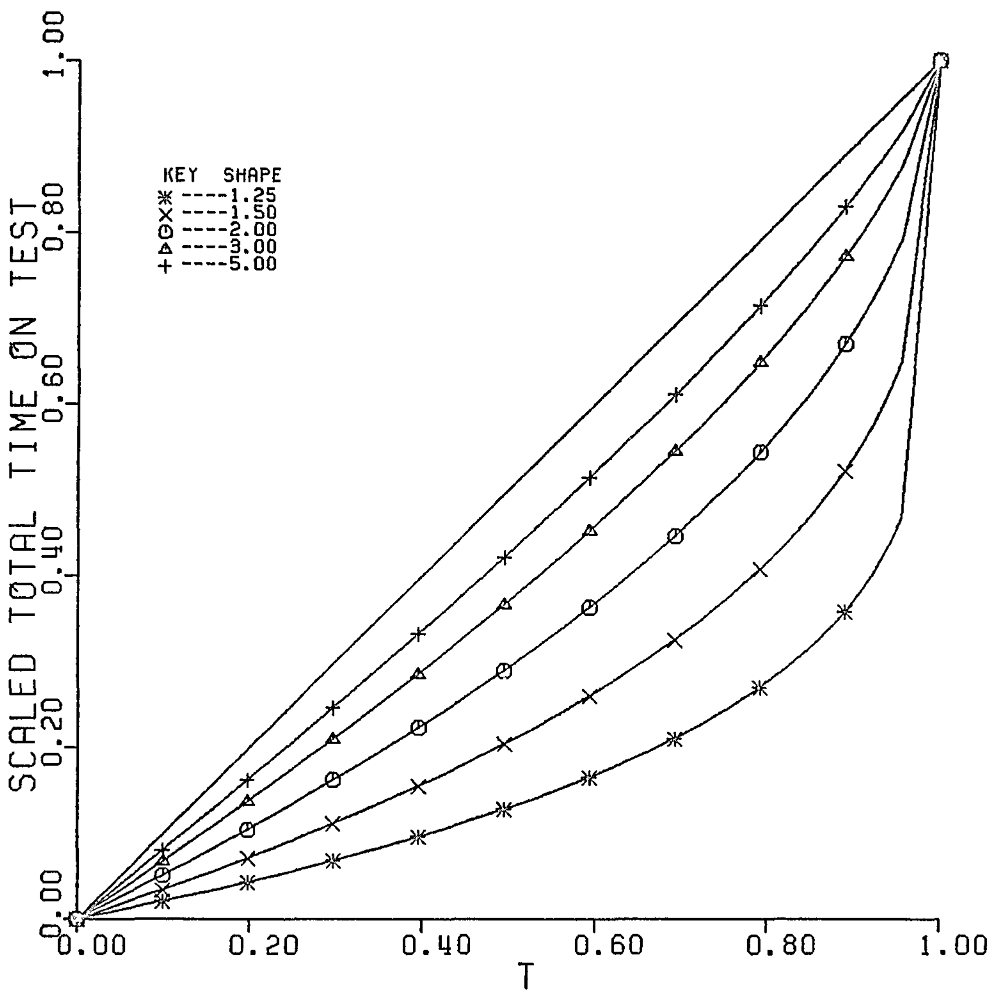

FIGURE 24

SCALED TØTAL TIME ØN TEST TRANSFØRM
FØR GAMMA MØDEL.

values of α. Notice that for all α, the STTOT transform is below the 45 ^{o} line (which corresponds to exponential system life) since the hazard rate of the series system is decreasing. Regarding the dependence structure induced by a random environmental factor this figure tells us that the smaller the shape parameter is, the more depndence is induced, which has also been mentioned in terms of copula in section 3.3.

Computing the total time on test, and plotting $V_{s,r}/V_{s,s}$ versus r/s for r = 1, 2, ···,s, we obtain so called the emprical STTOT plot. Since $V_{s,r}/V_{s,s}$ converges to $ST_A(t)$ with probability one and uniformly in $0 \leq t \leq 1$ as s —> ∞ and r/s —> t, the STTOT plot can be compared to the figure 24 for a graphical check of the model's validity.

We can also obtain crude estimators of the shape parameter α comparing the empirical and theoretical STTOT plots. Let $C_i = \log(1\text{-}i/s)$ and $D_i = \log(1 - V_{s,i} / V_{s,s})$, i = 1, ..., s-1. From $ST_A(t) = 1\ (1 - t)^{(\alpha-1)/\alpha}$ we have $\log(1 - ST_A(t)) = (1 - 1/\alpha)$ $\log(1 - t)$ so that

$$D_i \approx (1\text{-}1/\alpha)C_i, \quad i = 1, ..., s\text{-}1, \tag{4.6.8}$$

First we consider , as a reasonable estimator of α , the value of α which minimizes the squared distances between D_i and $(1 - 1/\alpha)\, C_i$. That is,

$$\sum_{i=1}^{s-1} (D_i - (1-1/\alpha)\, C_i)^2 .$$

The resulting estimator is $\alpha_{ls} = \dfrac{\Sigma C_i^2}{\Sigma C_i^2 - \Sigma C_i D_i}$ (4.6.9) which is in the parameter space if $\Sigma C_i^2 > \Sigma C_i D_i$. A better estimator should be obtained by weighting the D_i's differently since for $i < j$, Var $(D_i) <$ Var (D_j). The variance of D_1 depends on the unknown parameter α so we weight by the variance of D_i computed under an assumed exponential distribution. If $T_1, T_2, \cdots, T_s$ are assumed to follow an exponential distribution, then $[1 - V_{s,r} / V_{s,s}]$ follows a beta distribution with parameters s-r and r for r = 1,2, ...,s-1.Noting that the r-th order statistics of a sample of size s-1 from a uniform distribution follows a beta distribution with parameters r and s-r one can show that $- D_i$ is the i-th order statistics of a sample of size s-1 from a standard exponential distribution. Hence the variance of D_i in that case is

$$V_i = \sum_{j=1}^{i} \frac{1}{(s-j)^2}, \qquad i = 1, \ldots, s-1 \tag{4.6.10}$$

so that the weighted least squares estimator of α is

$$\alpha_{wls} = \frac{\Sigma C_i^2/V_i}{(\Sigma \frac{C_i^2}{V_i} - \Sigma \frac{C_i D_i}{V_i})} \qquad \text{if } \Sigma C_i^2/V_i > \Sigma C_i D_i/V_i. \tag{4.6.11}$$

Once we have obtained an estimator of α by either of the two least squares estimators, we substitute this value into (4.3.7) and solve this equation numerically for θ_{ls} or θ_{wls}. We note that the unique root of the equation lies between $1/(\alpha T_{s,s})$ and $1/(\alpha T_{s,1})$.

Due to the computational complexity of these estimators the analytic properties of these estimators are not available so a small scale Monte Carlo study was performed in the next section to compare these estimators with the other three estimators, M.L.E. , and M.M.E., and one suggested by Hui and Berger.

4.7 Monte Carlo Study

In this section we compare the estimators of the shape parameter α and the scale parameter θ through a small scale Monte Carlo study. Before describing the main study we present graphically some data simulated from our model which do or do not meet the sufficient condition for reasonable parameter values based on scaled total time on test plots.

Figures 25 and 26 are scaled total time on test plots from two simulated samples of size 30 from the model $R_s(t) = (1 + \theta t)^{-\alpha}$ with $\alpha = 3$, $\theta = 1$. Looking at figure 25, we see that the estimated scaled total on test doesn't look too different from the 45° so that an exponential model might not be unreasonable. For this data set only the weighted least squares estimator exists and it yields $\alpha_{wls} = 45.33$ and $\theta_{wls} = .0567$. For the data in figure 26 all estimates exist, and we have

$\theta_{mle} = .93$ $\alpha_{mle} = 2.98$

$\theta_{mme} = .491$ $\alpha_{mme} = 4.86$

$\theta_{ber} = .720$ $\alpha_{ber} = 7.02$

$\theta_{ls} = .739$ $\alpha_{ls} = 3.58$

$\theta_{wls} = .970$ $\alpha_{wls} = 2.89$

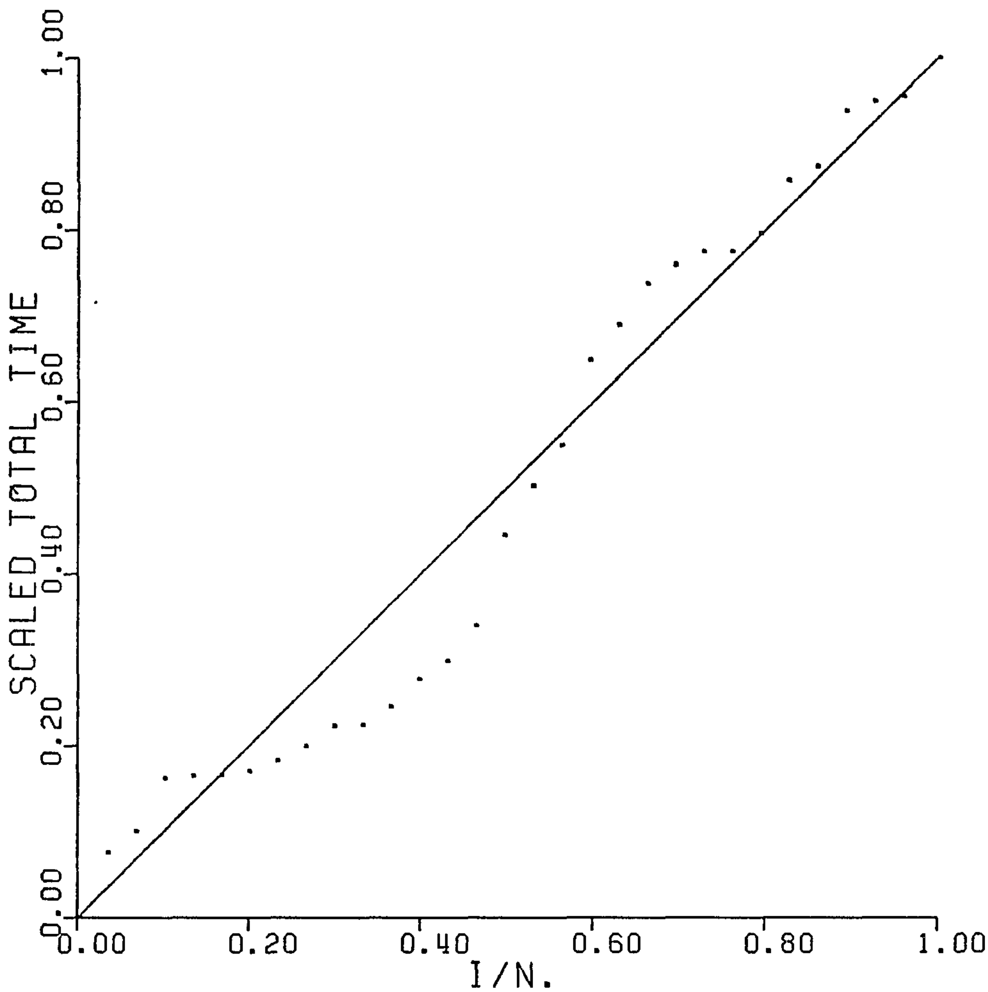

FIGURE 25

SCALED TOTAL TIME ON TEST PLOT
FOR SIMULATED DATA.

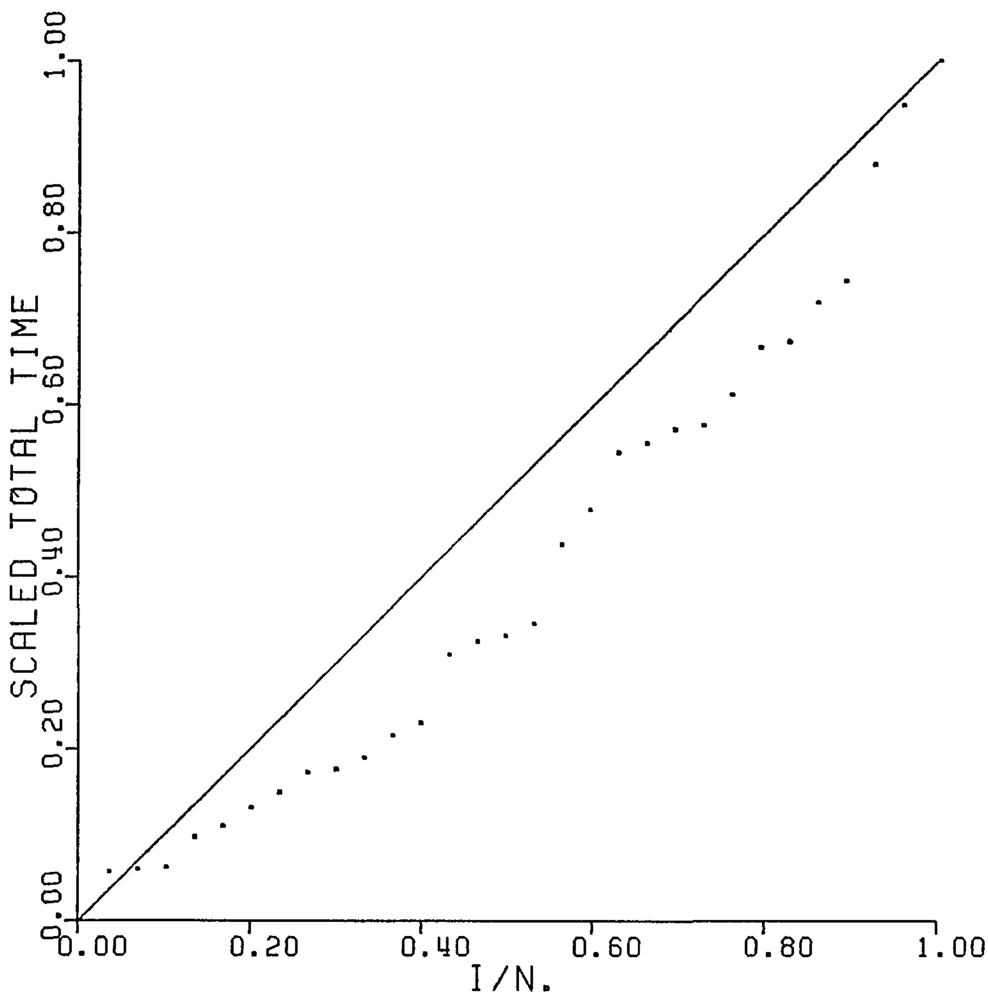

FIGURE 26

SCALED TØTAL TIME ØN TEST PLØT
FØR SIMULATED DATA.

We notice that the first sample of figure 25, which satisfies the condition $\Sigma C_i^2/V_i > \Sigma C_i D_i/V_i$ but fail to meet the condition (4.2.12) for other estimators, yields a very large estimate of α. Since a reasonable model for T when θ and α are not estimable is the independent exponential series system which has system reliability very close to that of our random effects model when α is very large, this is not a problem.

The main comparisions of interest are done in terms of the bias, standard deviation of the estimates of α and θ, and the number of samples where the reasonable estimators exist. Also the estimators of the system reliabilty at $t_o = 0.1006$ are compared. Random samples of size s = 15, 30, 50, 75, or 100 were generated with $\lambda_1 + \lambda_2 = 3$, $\beta = 3$, so $\theta = 1$ and $\alpha = 2, 3, 5$. 1000 samples were generated for each combination of s and α. The bias, standard deviation of the estimates and NS, the number of samples where the estimator exists is reported in table 3 for α, table 4 for θ, and in table 5 for an estimator of the system reliability obtained from (4.6.1) at $t_0 = 0.1006$. The true system reliability at t_0 is .8255 when $\alpha = 2$, .75 when $\alpha = 3$, and .619 when $\alpha = 5$. Also reported in each table is the bias and standard deviation of the weighted least square estimators when they are restricted to those samples where the other estimators exist.

From these tables we note that Berger's modified estimator performs very poorly. Also the weighted least squares estimator allows for estimation of parameters in many more samples when s is small. In general the maximum likelihood estimator outperforms the other estimators, however, when the weighted least squares estimator is

Table 3: Bias and Standard Deviation (SD) of Estimators of α

Sample	Esti-		α = 2			α = 3			α =5	
Size	mator	NS	Bias	SD	NS	Bias	SD	NS	Bias	SD
15	mle	769	4.5	29.	642	7.3	39.	522	38.7	573.
	wls	852	4.8	41.	753	36.4	843.	665	8.2	53.
	*	766	1.3	5.	636	1.3	9.	516	-0.7	5.
	ls	762	6.8	49.	653	13.1	149.	558	30.3	493.
	mme	770	9.1	37.	643	16.8	77.	522	69.8	925.
	ber	770	14.4	65.	643	26.0	114.	522	112.9	1505.
30	mle	916	2.8	20.	809	5.7	30.	674	20.4	148.
	wls	953	4.7	37.	870	13.8	141.	752	9.3	68.
	*	912	1.1	3.	804	1.8	6.	660	3.2	29.
	ls	877	6.4	52.	768	11.9	100.	669	13.7	109.
	mme	916	6.1	32.	809	9.9	104.	674	31.7	202.
	ber	916	10.0	52.	809	17.7	68.	674	56.3	347.
50	mle	979	5.8	114	916	3.6	18.	801	7.6	39.
	wls	981	1.7	10.	935	6.9	65.	850	9.0	97.
	*	976	1.0	3.	912	2.9	29.	787	2.0	10.
	ls	956	4.0	16.	864	6.4	33.	756	13.4	88.
	mme	979	8.5	131.	916	6.6	25.	801	11.4	52.
	ber	979	15.4	241.	916	12.5	42.	801	23.4	91.
75	mle	996	0.9	4.	963	2.5	14.	893	12.8	139.
	wls	998	1.0	5.	977	2.8	17.	915	8.0	94.
	*	996	1.0	5.	958	1.3	5.	878	2.9	16.
	ls	974	2.4	12.	925	11.6	144.	823	6.6	22.
	mme	996	2.2	4.	963	4.7	24.	893	15.3	122.
	ber	996	4.5	8.	963	9.6	38.	893	32.6	260
100	mle	999	0.5	3.	978	1.7	7.	892	9.5	84.
	wls	1000	1.7	35.	989	2.1	12.	913	19.6	307.
	*	999	0.6	2.	978	1.3	5.	879	13.7	273.
	ls	989	1.5	9.	956	3.7	19.	835	11.0	81.
	mme	999	1.7	5.	978	3.0	9.	892	13.0	120
	ber	999	3.7	8.	978	7.2	15.	892	27.1	203.

'*' represents the weighted least squares estimator restricted to those samples where all estimators exist.

Table 4: Bias and Standard Deviation (SD) of θ

Sample Size	Estimator	α = 2 NS	α = 2 Bias	α = 2 SD	α = 3 NS	α = 3 Bias	α = 3 SD	α =5 NS	α =5 Bias	α =5 SD
15	mle	769	0.356	1.702	642	0.691	1.900	522	1.352	3.109
	wls	852	-.102	0.742	753	0.210	1.049	665	0.715	1.609
	*	766	-.027	0.025	636	0.390	1.040	516	1.084	1.599
	ls	782	-.192	0.729	653	0.119	1.031	558	0.578	1.536
	mme	770	-.683	0.205	643	-.513	0.348	522	-.238	0.601
	ber	770	-.803	0.122	643	-.705	0.203	522	-.546	0.348
30	mle	916	0.112	0.919	809	0.175	1.049	674	0.558	1.609
	wls	953	-.135	0.580	870	0.000	0.757	752	0.366	1.118
	*	912	-.100	0.567	804	0.074	0.740	660	0.523	1.104
	ls	877	-.254	0.586	769	-.096	0.745	669	0.180	1.026
	mme	916	-.623	0.192	809	-.469	0.338	674	-.199	0.576
	ber	916	-.798	0.095	809	-.725	0.160	674	-.584	0.282
50	mle	979	0.016	0.648	916	0.075	0.766	801	0.256	1.559
	wls	989	-.126	0.492	935	-.012	0.618	850	0.184	0.869
	*	976	-.115	0.486	912	0.011	0.609	787	0.267	0.850
	ls	956	-.263	0.514	864	-.112	0.663	756	0.105	0.863
	mme	979	-.575	0.184	916	-.404	0.333	801	-.193	0.541
	ber	979	-.792	0.079	916	-.718	0.135	801	-.615	0.232
75	mle	996	-.025	0.522	963	0.030	0.624	893	0.128	0.817
	wls	998	-.125	0.432	977	-.027	0.555	915	0.112	0.728
	*	996	-.124	0.431	958	-.010	0.546	878	0.153	0.715
	ls	974	-.247	0.275	925	-.144	0.603	827	0.014	0.747
	mme	996	-.541	0.174	963	-.375	0.322	893	-.189	0.535
	ber	996	-.790	0.065	963	-.717	0.120	893	-.628	0.210
100	mle	999	-.019	0.437	978	-.028	0.515	892	0.033	0.683
	wls	1000	-.101	0.381	989	-.075	0.472	913	0.020	0.628
	*	999	-.100	0.401	978	-.055	0.465	879	0.055	0.615
	ls	989	-.216	0.423	956	-.165	0.511	835	-.064	0.666
	mme	999	-.508	0.153	978	-.345	0.297	892	-.206	0.494
	ber	999	-.785	0.052	978	-.716	0.104	892	-.644	0.185

'*' represents the weighted least squares estimator restricted to those samples where all estimators exist.

Table 5: Bias and Standard Deviation (SD) of Estimators of System Reliability at t = .1006

Sample	Esti-		α = 2			α = 3			α =5	
Size	mator	NS	Bias	SD	NS	Bias	SD	NS	Bias	SD
15	mle	769	-.012	.0647	642	-.018	.0815	522	-.029	.1011
	wls	852	-.004	.0586	753	-.010	.0767	665	-.024	.0967
	*	766	-.006	.0577	636	-.015	.0764	516	-.030	.0968
	ls	762	0.002	.0588	653	-.006	.0748	558	-.020	.0977
	mme	770	0.037	.0503	643	0.031	.0661	522	0.012	.0926
		769	0.064	.0463	643	0.067	.0616	522	0.054	.0921
30	mle	916	-.005	.0473	809	-.007	.0577	674	-.022	.0691
	wls	953	0.002	.0424	870	-.007	.0552	752	-.020	.0674
	*	912	0.001	.0426	804	-.006	.0544	660	-.027	.0665
	ls	877	0.010	.0434	769	0.002	.0551	669	-.013	.0651
	mme	916	0.037	.0357	809	0.024	.0490	674	0.001	.0613
	ber	916	0.069	.0335	809	0.062	.0472	674	0.045	.0608
50	mle	979	-.001	.0372	916	-.003	.0429	801	-.006	.0545
	wls	989	0.003	.0349	935	-.001	.0412	850	-.006	.0530
	*	976	0.003	.0348	912	-.002	.0411	787	-.008	.0528
	ls	956	0.010	.0359	864	0.005	.0431	756	-.002	.0532
	mme	979	0.035	.0300	916	0.022	.0366	801	0.010	.0494
	ber	979	0.071	.0233	916	0.066	.0340	801	0.055	.0485
75	mle	996	0.000	.0290	963	-.001	.0372	893	-.005	.0442
	wls	998	0.004	.0274	977	0.000	.0356	915	-.005	.0436
	*	996	0.003	.0274	958	0.000	.0357	878	-.006	.0430
	ls	974	0.010	.0292	925	0.007	.0374	827	-.002	.0431
	mme	996	0.034	.0244	963	0.021	.0327	893	0.007	.0406
	ber	996	0.072	.0238	963	0.067	.0313	893	0.051	.0380
100	mle	999	0.001	.0243	978	0.001	.0309	892	-.002	.0375
	wls	1000	0.004	.0234	984	0.002	.0299	913	-.001	.0392
	*	999	0.004	.0233	978	0.002	.0299	879	-.002	.0390
	ls	989	0.010	.0248	956	0.008	.0311	835	0.003	.0380
	mme	999	0.034	.0223	978	0.019	.0267	892	0.008	.0353
	ber	999	0.075	.0216	978	0.067	.0261	892	0.052	.0345

'*' represents the weighted least squares estimator restricted to those samples where all estimators exist.

restricted to those samples where the maximum likelihood estimator exists, this estimator performs much better when s is small. The somewhat better performance of the M.L.E in terms of bias is deceptive since some of the estimates of α are less than one, which implies that the mean system reliability is infinite. Also the weighted least squares estimator of system reliability seems to outperform the other estimators of the system reliability in spite of its relatively poor performance as an estimator of θ. Our recommendation is to use the weighted least squares estimator since it more often provides estimators of the relevant parameters and is somewhat easier to compute.

4.8 Test for Dependence Induced by a Common Environmental Factor

In this section we discuss the problem of determining whether there is a dependence structure induced by an environmental factor. In our setting, we observe only the system failure times T_i with the assumption that the survival function of T_i is $R_s(t) = (1+\theta t)^{-\alpha}$. As pointed out in section 4.6, the graphical presentation as well as the copula indicates that the shape parameter, α only affects the dependence structure. This idea is supplemented by looking at the correlation computed in section 3.4. Accordingly we will call the quantity $\gamma = 1/\alpha$ a measure of dependence induced by the environmental factor. Since our model assumes finite mean system lifetime, that is, α is assumed to be greater than 1, γ varies from 0 to 1. If there is no dependence induced γ is equal to 0 and the more dependence is induced the closer to 1 the value of γ is.

One possible statistics is constructed from the weighted least square estimator,

$$\alpha_{wls} = \frac{\Sigma C_i^2/V_i}{(\Sigma \frac{C_i^2}{V_i} - \Sigma \frac{C_iD_i}{V_i})} \text{ if } \Sigma C_i^2/V_i > \Sigma C_iD_i/V_i.$$

From this statistic we consider

$$Q_s = \frac{\Sigma \frac{C_iD_i}{V_i}}{\Sigma \frac{C_i^2}{V_i}} \quad . \tag{4.8.1}$$

Under the null hypothesis of independence, - D_i follows the distribution of the i-th order statistics among the sample of size s-1 from an exponential distribution so that Q_s is just a linear combination of order statistics from exponentials. Hence Q_s has the same distribution as a linear combination, $Q_s(z)$, of identically independent exponential random variables since the i-th exponential order statistics can be expressed as a linear function of s-1 independent standard exponentials. Correspondingly we have

$$Q_s(z) = \sum_{i=1}^{s-1} p_i Z_i \tag{4.8.2}$$

where Z_i is a random variable following the standard exponential distribution,

$$\text{and } p_i = \frac{1}{s-i} \sum_{j=i}^{s-1} \frac{C_j / V_j}{\sum_{k=1}^{s-1} C_k^2 / V_k} \quad . \tag{4.8.3}$$

The exact distribution of $Q_s(z)$ is found in David (1981) as the mixture of exponentials,

$$f_{Q_s(z)}(t) = \sum_{i=1}^{s-1} \frac{w_i}{p_i} \exp\left(\frac{-t}{p_i}\right), \tag{4.8.4}$$

$$\text{where } w_i = \frac{p_i^{s-2}}{\prod_{h \neq i}(p_h - p_i)} .$$

On the other hand we can note that if γ goes to 1, Q_s tends to have smaller value. Table 6 shows the critical values of the standardized Q_s for different sample sizes with the type one error probability $\alpha = .01, .05, .1$. Since the distribution under alternatives is hard to obtain a simulation study has been constructed to study the tests power which is discussed later.

Table 6: Critical Values of the StandardizedStatistics of Q_s

Sample Size	1 %	5 %	10 %
s = 15	-1.8880	-1.4540	-1.1968
s = 20	-1.9382	-1.4815	-1.2106
s = 25	-1.9796	-1.5000	-1.2194
s = 30	-2.0010	-1.5140	-1.2260
s = 35	-2.0334	-1.5243	-1.2307
s = 40	- 2.0526	-1.5322	-1.2345
s = 50	-2.0816	-1.5453	-1.2404

We have tried to prove the asymptotic normaity of the test statistic Q_s analytically in vain. However we are still very sure of its asymptotic normality with the following arguments. Consider the characteristic function φ_s (t) of

$$\frac{\Sigma p_i Z_i - \mu_s}{\sigma_s} \quad \text{where } \mu_s = \Sigma p_i \text{, and } \sigma_s = \sqrt{\Sigma p_j^2}.$$

$$\varphi_s(t) = E\left[\exp\left\{ it\left(\frac{\Sigma p_i Z_j - \mu_s}{\sigma_s}\right)\right\}\right]$$

$$= \prod_{j=1}^{s-1} E\left(\exp\left(it\frac{p_j}{\sigma_s}Z_j\right)\cdot \exp\left(-it\frac{\mu_s}{\sigma_s}\right)\right.$$

$$= \prod_{j=1}^{s-1}\left(1 - it\frac{p_j}{\sigma_s}\right)^{-1} \exp\left(-it\frac{\mu_s}{\sigma_s}\right) \qquad (4.8.5)$$

Taking the log on the both sides we obtain

$$\log \varphi_s(t) = -\sum_{j=1}^{s-1} \log\left(1 - it\frac{p_j}{\sigma_s}\right) - it\frac{\mu_s}{\sigma_s}. \qquad (4.8.6)$$

Noting that

$$\sum_{j=1}^{s-1} \log\left(1 - it\frac{p_j}{\sigma_s}\right) = -it\sum_{j=1}^{s-1}\frac{p_j}{\sigma_s} + \frac{t^2}{2} + \rho_s(t), \text{ where}$$

$$|\rho_s(t)| \le \frac{\Sigma p_j^3 t^3}{3\sigma_s^3} \le \frac{\max p_j}{3\sigma_s}\,\frac{\Sigma p_j^2 t^3}{\sigma_s^2} \le \frac{\max p_j}{3\sigma_s}\,t^3 \qquad (4.8.7)$$

we have $\varphi_s(t) \longrightarrow \exp(-t^2/2)$ if and only if $[\max p_j / \sigma_s] \longrightarrow 0$.

From the above condition it suffices to show that $[\max p_j / \sqrt{\Sigma p_j^2}] \longrightarrow 0$. However p_j's are so complicated that the convergence has not been proved analytically. Instead of doing this we try to show the performance of p_j's and $[\max p_j / \sqrt{\Sigma p_j^2}]$ by computation for various s's. First it seems that $p_1 > p_2 > \cdots > p_{s-1}$, that is, $\max p_j = p_1$. Second it seems that

$$\frac{p_2^2}{p_1^2} + \frac{p_3^2}{p_1^2} + \cdots + \frac{p_{s-1}^2}{p_1^2} + 1$$ diverges to infinite faster than log s. In table 7 the divergence rate is compared with that of log s.

Table 7: Divergence Rate of $\sqrt{\Sigma\ p_j^2}\ /\ \max p_j$

s	$\sqrt{\Sigma p_j^2}/\max p_j$	log s
200	9.54030	5.29832
220	10.01179	5.39363
240	10.46207	5.48064
260	10.89376	5.56068
280	11.30900	5.63479
300	11.70953	5.70378
320	12.09682	5.76832
340	12.47209	5.82895
360	12.83640	5.88610
380	13.19065	5.94017
400	13.53564	5.99146
420	13.87205	6.04025
440	14.20050	6.08677
460	14.52152	6.13123
480	14.83561	6.17379
500	15.14317	6.21461

A second test is based on the cummlative total time on test statistic, which has been introduced by (BBBB). The cummulative total time on test statistics (CTTS) which they have defined is

$$B_s = \sum_{r=1}^{s} \frac{V_{s,r}}{V_{s,s}}$$ where $V_{s,r}$ is the total time on test defined in section 4.7.

They have shown, in the chapter 6 of their book, that if the underlying distribution A has the finite 2nd moment then

$$s^{1/2} \left[s^{-1} \Sigma \frac{V_{s,r}}{V_{s,s}} - k(A) \right] \longrightarrow N\left(0, \frac{\sigma^2(A)}{\mu^2}\right) \qquad (4.8.9)$$

where $k(A) = \int_0^1 ST_A(u)\,du$, and $\mu = \int_0^\infty x\,dA(x)$,

and $\sigma^2(A) = 2 \int\int_{u<v} \{2[1 - A(u)] - k(A)\}\{2[1 - A(v)] - k(A)\} A(u)[1 - A(v)]\,du\,dv.$

They have suggested using this statistic for the problem of testing the null hypothesis that the underlying distribution A is exponential versus the alternative that A has increasing hazard rate or decreasing hazard rate. Since the dependence structure of our setting causes the system hazard rate to be decreasing, it might be reasonable to use B_s as a test statistic. Some computation leads us to

$$k(A) = \frac{\alpha - 1}{2\alpha - 1}$$

$$\frac{\sigma^2(A)}{\mu^2} = 2\left(\frac{\alpha - 1}{2\alpha - 1}\right)^2 \left[\frac{1}{2} + \frac{\alpha - 1}{\alpha - 2} - 2\left(\frac{\alpha - 1}{3\alpha - 2}\right)\right] + \left(\frac{\alpha - 1}{2\alpha - 1}\right)\left[3 - 2\frac{\alpha - 1}{3\alpha - 2}\right]$$
$$- 4\left(\frac{\alpha - 1}{3\alpha - 2}\right).$$

We note that under the null hypothesis, $\gamma = 0$, k (A) = 1/2, and [$\sigma^2(A) / \mu^2$] = 1/12 so that the exact test statistics R_s is

$$(12\, s)^{1/2} \left[s^{-1} \sum_{r=1}^{s} \frac{V_{s,r}}{V_{s,s}} - \frac{1}{2} \right], \qquad (4.8.10)$$

which has smaller values as γ goes to 1.

A third test statistic was introduced by Klefsjo (1983). He has used the property of convexity of the STTOT, $ST_A(t)$ and obtained the statistic,

$$K_s = \sum_{j=1}^{s} a_j \frac{(s - j + 1)(T_{s,j} - T_{s,j-1})}{V_{s,s}} \qquad (4.8.11)$$

where $a_j = \dfrac{\{ (s-1)^3 j - 3(s+1)^2 j^2 + 2(s+1) j^3 \}}{6}$, which has smaller values as γ goes to 1.

Using the asymptotic properties of linear combination of order statistics he has shown that under the null hypothesis, the test statistic K_s is asymptotically normally distributed. In addition to that, he has constructed a list of critical values from the exact distribution of K_s under the null hypothesis.

We have, by simulation, estimated the powers for the various shape parameter values. Sample sizes 20 and 50 have been studied.In figure 27 and figure 28 the estimated power curves for the three tests mentioned above are obtained by the following scheme. The total number of replication for each investigated γ-value, measure of dependence, which increases from .00 to .75 is 1000. The significance levels are equal to .05. The three powers at each γ-value have been estimated from the same set of data.

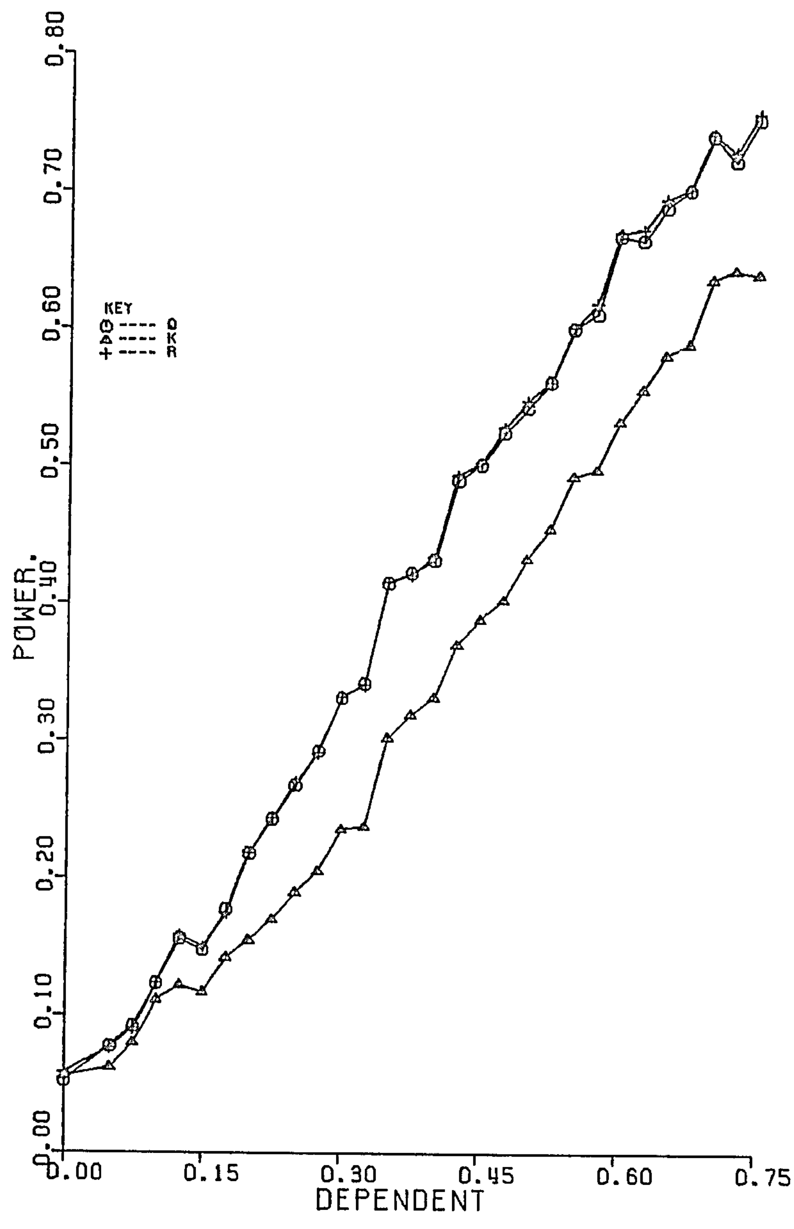

FIGURE 27

ESTIMATED POWERS OF TESTS FOR
INDEPENDENCE FOR S = 20

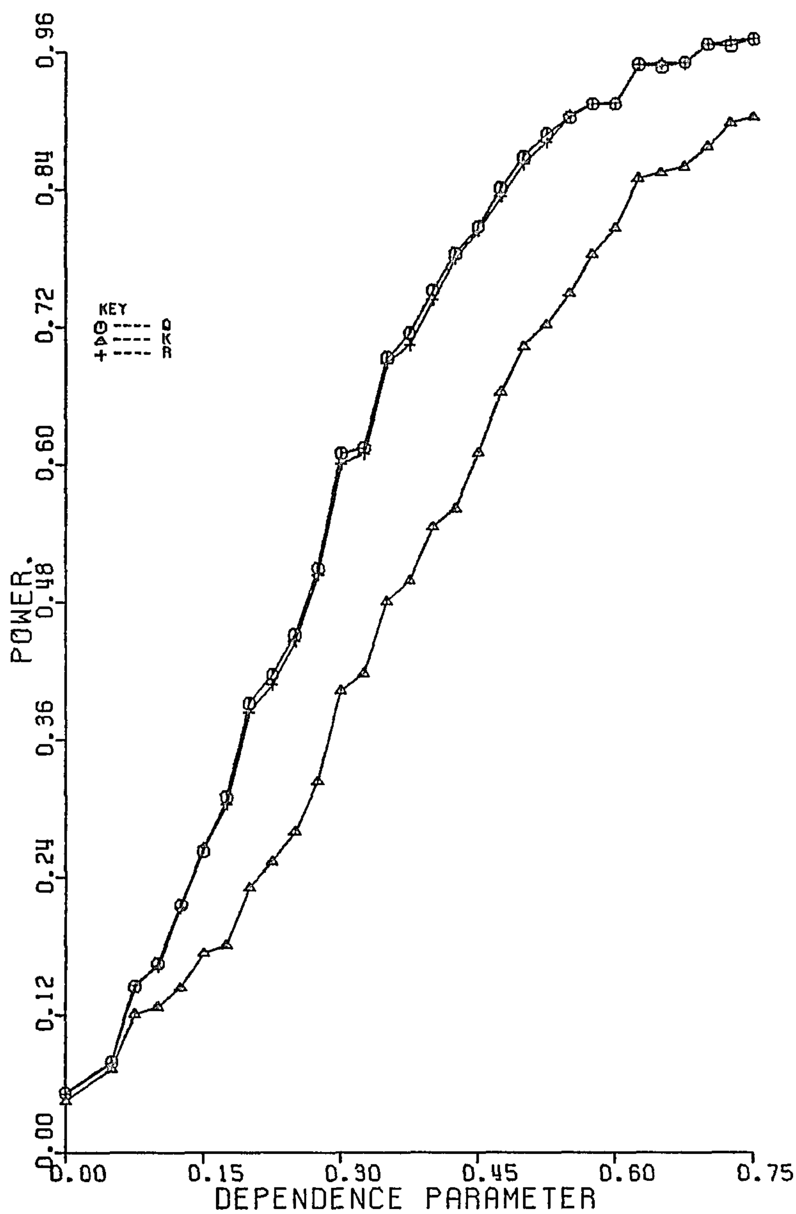

FIGURE 28

ESTIMATED POWERS OF TESTS FOR
INDEPENDENCE FOR S = 50

Both figures indicate that the two test statistics R_s and Q_s perform better than the other test statistics K_s in terms of the power at all positive γ. Since the statistics R_s, and Q_s seem to have almost the same power at most γ levels we have tested the equality of the two statistics with respect to their powers. Let M_i denote the i-th sample and let IT be thetotal number of replication. We define the rejection rules of both tests correspondingto the statistics R_s and Q_s, $R(M_i)$ and $Q(M_i)$, as $R(M_i) = 1$ if rejected and 0 if not rejected, and $Q(M_i) = 1$ if rejected and 0 if not rejected. Thus we obtain estimators $\dot{p}_R$ and $\dot{p}_Q$ of the powers p_R and p_Q as

$$\dot{p}_R = \sum_{i=1}^{IT} \frac{R(M_i)}{IT}, \qquad \text{and} \qquad \dot{p}_Q = \sum_{i=1}^{IT} \frac{Q(M_i)}{IT}. \tag{4.8.12}$$

In order to test H_0: $p_R = p_Q$ versus H_1: $p_R \neq p_Q$, we have used the usual Z - statistics,

$$z_{RQ} = \frac{\dot{p}_R - \dot{p}_Q}{\sqrt{V(\dot{p}_R - \dot{p}_Q)}}. \tag{4.8.13}$$

Noting that $R(M_i)$ and $Q(M_i)$ are not independent we need to compute an estimator of $COV(\dot{p}_R, \dot{p}_Q)$ to obtain $V(\dot{p}_R - \dot{p}_Q)$.

$$E(\dot{p}_R \cdot \dot{p}_Q) = \frac{1}{IT^2} E\left[\sum_{i=1}^{IT} R(M_i) \cdot \sum_{j=1}^{IT} Q(M_j)\right]$$

$$= [\, IT \cdot E\{ R(M_1)\, Q(M_1) \} + IT\,(IT - 1)\, E(R(M_1))\, E(Q(M_1)) \,] \,/\, IT^2. \tag{4.8.14}$$

Letting $COM = \sum_{i=1}^{IT} R(M_i)\, Q(M_i)$ be denote an estimator of $IT \cdot E\{ R(M_1)\, Q(M_1)\}$, we take

$\frac{1}{IT} \left(\frac{COM}{IT} - \dot{p}_R \cdot \dot{p}_Q \right)$ as an estimator of $COV(\dot{p}_R, \dot{p}_Q)$. Thus we obtain an estimator of $V(\dot{p}_R - \dot{p}_Q)$ as $2[\, p \cdot q + p^2 - COM/IT \,] / IT$, (4.8.15)

where $p = (\dot{p}_R + \dot{p}_Q) / 2$ and $q = 1 - p$.

The powers and the Z - values at some γ levels for sample sizes 20, 50 are reported in the table 8. Our investigation leads us to the conclusion that the test statistics Q_S, which has been developed only for this specific model, is not better than R_S which can detect more general alternatives than Q_S. However, the test statistics Q_S, is simple to obtain while we are making a graphical inference on the shape parameter and is guaranteed to keep power as high as R_S.

4.9 Future Study

In this thesis, we have considered a dependent reliability model which is induced through a random environment under which the systems are operated. We have discussed the properties of the model and its infernce procedures only for a two component system in terms of engineering application.

Since the dependent structure introduced by the model is expected to be found not only in an engineering setting but also in biological, medical and demographic settings as well, we plan a thorough investigation of this structure with the experts in each special field. Also a natural generalization of the model can be to a multi- component system rather than only a two component system. We are currently investigating how to apply this model to accelerated life tests. Accelerated life tests are often used to obtain information on item's performance under normal operating conditions from the test data collected under condition more severe than usually encountered in normal usage. This model is suited to explaining the different environmental effects that occur at different

Table 8: Comparision of Powers of Test Statistics, Q_s and R_s

s	γ	$\dot{p}_Q$	$\dot{p}_R$	| Z | - value
20	.05	.07550	.07550	0.00000
	.10	.12400	.12350	0.10310
	.15	.16800	.16450	0.17636
	.20	.22150	.22000	0.23079
	.25	.27050	.27050	0.28043
	.30	.33750	.34000	0.87040
	.35	.39050	.39000	0.40141
	.40	.43800	.44200	0.44909
	.45	.51150	.50850	0.52268
	.50	.54300	.54900	0.55424
	.55	.60200	.60700	0.62295
	.60	.66145	.66300	0.67208
	.65	.69150	.70000	0.70183
	.70	.72300	.72900	0.73301
	.75	.76800	.77300	0.77744
50	.05	.10400	.10100	0.71573
	.10	.17000	.17500	0.18188
	.15	.26300	.25900	0.27692
	.20	.35700	.35300	0.37215
	.25	.50200	.49500	0.51781
	.30	.58400	.58000	0.59959
	.35	.69100	.68400	0.70561
	.40	.76000	.75800	0.77351
	.45	.80100	.79800	0.81363
	.50	.87100	.86800	0.88160
	.55	.89700	.89800	0.90661
	.60	.92600	.92500	0.93428
	.65	.95100	.95300	0.95783
	.70	.96900	.96600	0.97448
	.75	.96800	.96900	0.97357

stress levels.

With regards to inference, we first plan to extend the results to censored data. Since the inference we have done strongly depends on the parametric distribution we feel that robustness studies of the environmental factor distribution may be appropriate for the next stage of this problem.

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