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M-IDEAL STRUCTURES IN OPERATOR ALGEBRAS

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INTRODUCTION

Even before Alfsen and Effros [2] introduced the notion of an M-ideal in a general Banach space, it was proved by Dixmier [13] that $K(H)$, the space of all compact operators on a separable Hilbert space, is an M-ideal in $L(H)$, the space of all bounded linear operators on H. Even now only a few Banach spaces X for which $K(X)$ is an M-ideal in $L(X)$ are known; for example, c_0 and ℓ_p, $1 < p < \infty$. Some of the classical Banach spaces; for example, ℓ_1, ℓ_∞ and $L_p(0,1)$ with $1 \leq p < \infty$, $p \neq 2$ do not have the above property.

Smith and Ward [35] showed that (i) M-ideals in a C^*-algebra are precisely the closed two sided ideals, (ii) M-ideals in a commutative Banach algebra with identity are ideals and (iii) M-ideals in a Banach algebra with identity are algebras.

In chapter I, we prove that given a subspace X of ℓ_p, $1 < p < \infty$, $K(X)$ is an M-ideal in $L(X)$ if and only if X has the compact approximation property. The main result of this chapter is a strong converse to the Harmand-Lima theorem for subspaces of ℓ_p, $1 < p < \infty$. Harmand and Lima [20] showed that if X is a Banach space for which $K(X)$ is an M-ideal in $L(X)$ then there is a net in $K(X)$ so that

(i) $T_\alpha \to I$ strongly

(ii) $\|T_\alpha\| \leq 1$ for all α

(iii) $T_\alpha^* \to I$ strongly

(iv) $\lim_\alpha \|I-T_\alpha\| = 1$.

Chapter II is dovoted to proving that if X is a uniformly convex space then any M-ideal in $L(X)$ is a left ideal, and if X^* is also uniformly convex then any M-ideal in $L(X)$ is a two sided ideal in $L(X)$. This gives a partial answer to the Smith-Ward conjecture [36] that, for a uniformly convex space X , every M-ideal is a two sided ideal.

In Chapter III we will construct a space $X = (\sum \ell_p^{n_i})_r$ for which $L(X)$ contains a closed two sided ideal which is not an M-ideal in $L(X)$. Thus, the M-ideals and the closed two sided ideas in $L(X)$ are not the same in general for $X = (\sum \ell_p^{n_i})_r$.

CHAPTER I

A CHARACTERIZATION OF SUBSPACES X OF ℓ_p FOR WHICH $K(X)$ IS AN M-IDEAL IN $L(X)$

1. Introduction

A closed subspace J of a Banach space X is said to be an L-summand if there exists a closed subspace J' of X so that X is an algebraic direct sum of J and J', and if $j \in J$ and $j' \in J'$ then $\|j + j'\| = \|j\| + \|j'\|$. In this case we will write $X = J \oplus_1 J'$. Such a closed subspace J of X is called an M-summand if we have the norm condition $\|j + j'\| = \max\{\|j\|, \|j'\|\}$ in place of $\|j + j'\| = \|j\| + \|j'\|$. Here we will write $X = J \oplus_\infty J'$. A closed subspace J of a Banach space X is called an M-ideal in X if $J^\perp = \{x^* \in X^*: x^*(j) = 0$ for all $j \in J\}$, the annihilator of J in X^*, is an L-summand in X^*. By a standard duality argument we can easily see that if $X = J \oplus_\infty J'$ then $X^* = J^* \oplus_1 J'^*$. Thus M-summands are M-ideals. However, the converse is not in general true as we shall see in the next two examples. It is not difficult to see that $\ell_\infty^* = c_0^\perp \oplus_1 \ell_1$ and hence c_0 is an M-ideal in ℓ_∞. However, c_0 is not an M-summand because it is not even complemented in ℓ_∞ [11]. If H is a separable Hilbert space, $K(H)$--the space of the compact linear operators on H--is an M-ideal in $L(X)$--the space of the bounded linear operators on H. (This fact had been proved by Dixmier [13] well before the notion of an M-ideal was introduced). But $K(H)$ is not an

M-summand in $L(H)$ since $K(H)$ is not complemented in $L(H)$.

This follows from the facts that c_0 is not complemented in ℓ_∞ and that the diagonal operators in $L(H)$ can be identified with ℓ_∞ and under this identification the diagonal operators in $K(H)$ correspond to c_0.

The notion of an M-ideal was introduced by Alfsen and Effros [2]. They also characterized an M-ideal by intersection properties of open balls. We say that a closed subspace J of a Banach space X has the n-ball property if for any n open balls $B_1, B_2, \ldots, B_n$ for which $B_1 \cap \ldots \cap B_n \neq \phi$, and $B_i \cap J \neq \phi$, $i = 1,2,\ldots n$, the intersection $\bigcap_{i=1}^{n} B_i \cap J$ is nonvoid. Alfsen and Effros [2] proved that a closed subspace J of a Banach space X is an M-ideal if and only if J has the 3-ball property (equivalently, J has the n-ball property for every $n > 3$). Later Lima [27] also characterized M-ideals by various intersection properties of closed balls. In particular, he [27: Theorem 6.17] proved that J is an M-ideal in X iff for all $x \in X$, $\|x\| < 1$, for all $y_1, y_2, y_3 \in J$ with $\|y_i\| < 1$, $i = 1,2,3$, and for all $\varepsilon > 0$, there exists

$$y \in J \cap \bigcap_{i=1}^{3} \mathrm{Ball}(x+y_i, 1+\varepsilon),$$

where $\mathrm{Ball}(x,r)$ means the closed ball with center at x and radius r.

If J is an M-ideal in X, then J is a proximinal subspace of X, [2], [23], [27 : corollary 6.6]. Recall that a closed subspace J of a Banach space X is proximinal if every element in X has a best approximation in J ; that is, $\operatorname{dist}(x,J)$ is attained for each $x \in X$. Several authors have studied approximation problems in M-ideals. Perhaps the most striking approximation property of M-ideals is the following fact due to Holmes, Scranton and Ward [24]. If J is an M-ideal in a Banach space X , then for any $x \in X \setminus J$ the set of all best approximations in J to x spans J . Apparently being an M-ideal is a very strong condition. Jost [37] and some others showed that if a closed subspace J of a Banach space X satisfies the $1\frac{1}{2}$ ball property (which is weaker than the 3-ball property) then J is proximinal in X .

From the start, works on M-ideal have been closely linked to questions involving the closest compact approximants to a given non-compact operator on a Banach space. So an interesting and obviously difficult open question is to classify Banach spaces X for which $K(X)$, the space of all compact operators on X , is an M-ideal in $L(X)$, the space of all bounded linear operators on X . If X is c_0 or ℓ_p for $1 < p < \infty$, then $K(X)$ is an M-ideal in $L(X)$ [21]. However, if X is ℓ_1 or ℓ_∞ then $K(X)$ is not an M-ideal in $L(X)$ [35]. If X is $L_1(0,1)$, $L_\infty(0,1)$ or $C(0,1)$, then $K(X)$ is not an M-ideal in $L(X)$ since $K(X)$ is not even proximinal in $L(X)$ [14]. If $X = L_p(\Omega,\mu)$ for $1 < p < \infty$, $p \neq 2$ and μ a finite measure on Ω , then $K(X)$ is an M-ideal in $L(X)$ if and only if μ is purely atomic [28]. (A finite measure space (Ω,Σ,μ) is purely atomic if the complement of the union of all atoms has measure zero. An element $A \in \Sigma$ with $\mu(A) \neq 0$ is called an atom if $B \in \Sigma$ and $B \subseteq A$ can occur only for $B = A$ or $B = \emptyset$ up to measure zero. Notice that there are at most countably many atoms in a finite measure space.)

Recently Harmand and Lima [20] proved that if X is a Banach space for which $K(X)$ is an M-ideal in $L(X)$ then there is a net $\{T_\alpha\}$ in $K(X)$ so that

(i) $T_\alpha \to I$ strongly

(ii) $\|T_\alpha\| \leq 1$ for all α

(iii) $T_\alpha^* \to I$ strongly

(iv) $\lim_\alpha \|I-T_\alpha\| = 1$.

The main result of this chapter, taken from [8], is a strong converse to the Harmand-Lima theorem for subspaces of ℓ_p , $1 < p < \infty$. In theorem 9 we show that if X is a subspace of $(\sum X_n)_p$ ($\dim X_n < \infty$; $1 < p < \infty$) which has the compact approximation property, then $K(X)$ is an M-ideal in $L(X)$. (For a sequence $\{X_n\}$ of Banach spaces X_n's , $(\sum X_n)_p$ is the space of all sequences $x = (x_n)$, $x_n \in X_n$ with the norm defined by $\|x\| = (\sum_{i=1}^{\infty} \|x_i\|^p)^{1/p}$. This norm makes $(\sum X_n)_p$ a Banach space.)

Part of the proof consists of showing that such an X satisfies condition (i)-(iv) in the Harmand-Lima theorem. This result is proved for general reflexive spaces in Section 2.

Section 3 is devoted to proving the converse of the Harmand-Lima theorem for subspace of $(\sum X_n)_p$. Here we use blocking methods which have been previously used in the study of isomorphic, rather than isometric, properties of ℓ_p and a few other spaces.

2. Relations among approximation properties

If X and Y are Banach spaces, $L(X,Y)$ (respectively $K(X,Y)$) will denote the space of all bounded linear operators (respectively compact linear operators) from X to Y. If $X = Y$, then we simply write $L(X)$ (respectively $K(X)$). $\mathrm{Ball}(X)$ will denote the closed unit ball of X. Recall that a linear operator T from X to Y is bounded if $\sup\{\|Tx\| ; x \in X , \|x\| \leq 1\} < \infty$. A linear operator is bounded if and only if it is continuous. A linear operator T from X to Y is compact if the norm closure of $T(\mathrm{Ball}(X))$ is a compact subset of Y.

A Banach space X is said to have the compact approximation property (respectively, compact metric approximation property) if the identity operator in X is in the closure of $K(X)$ (respectively, $\mathrm{Ball}(K(X))$ with respect to the topology τ of uniform convergence on compact sets in X; that is, for every compact subset K in X and every $\varepsilon > 0$, there is a compact operator T (respectively, a compact operator T with $\|T\| \leq 1$) so that $\|x-Tx\| < \varepsilon$ for all $x \in K$.

If X and Y are Banach spaces, the space $L(X,Y)$ endowed with the topology τ of uniform convergence on compact sets in X is a locally convex space generated by the seminorms of the form $\|T\|_K = \sup\{\|Tx\| ; x \in K\}$, where K ranges over compact subset of X. (A locally convex space X is a vector space over real or complex scalar with a topology for which the vector space operations are continuous, and each point of which has a local neighborhood system consisting of convex subsets of X.) A seminorm p on a real or complex vector space X is a real valued function on X such that $p(x+y) \leq p(x) + p(y)$ and $p(\alpha x) = |\alpha| p(x)$ for all x and y in X

and all scalars α. Such a seminorm p on X induces a topology in such a way that at each point $x \in X$ the family $V(x,r) = \{y \in X , p(x-y) < r\}$ (where r ranges over all rational numbers) constitute a local base at x. Continuous linear functionals on the locally convex space $(L(X,Y),\tau)$ stated above have a nice representation in terms of sequences in X and sequences in Y^*.

The following proposition is taken from Lindenstrauss and Tzafriri's book [31: p.31].

Proposition 1. For Banach spaces X and Y, endow $L(X,Y)$ with the topology τ of uniform convergence on compact sets in X. Then the continuous linear functionals on $(L(X,Y),\tau)$ consist of all functionals ϕ of the form

$$\phi(T) = \sum_{i=1}^{\infty} y_i^*(Tx_i) , \{x_i\}_{i=1}^{\infty} \subseteq X , \{y_i^*\}_{i=1}^{\infty} \subseteq Y^* , \sum_{i=1}^{\infty} \|x_i\|\|y_i^*\| < \infty .$$

The next proposition is a relation between the compact metric approximation property and the compact approximation property, which is an analogue of the relation between the metric approximation property and the approximation property [31; p.39]. A Banach space is said to have the approximation property (respectively, the metric approximation property) if the identity operator is in the τ-closure of $F(X)$, the space of all finite rank operators (respectively, $\mathrm{Ball}(F(X))$).

Proposition 2. Let X be a Banach space. Then the following three assertions are equivalent.

(i) X has the compact metric approximation property.

(ii) $\mathrm{Ball}(K(X))$ is dense in $\mathrm{Ball}(L(X))$ in the topology τ of uniform convergence on compact sets in X.

(iii) For every choice of $\{x_n\}_{n=1}^{\infty} \subseteq X$, $\{x_n^*\}_{n=1}^{\infty} \subseteq X^*$ such that

$$\sum_{i=1}^{\infty} \|x_n\|\|x_n^*\| < \infty \quad \text{and} \quad \left|\sum_{i=1}^{\infty} x_n^*(Tx_n)\right| \le 1 \quad \text{for every } T \text{ in}$$

$\mathrm{Ball}(K(X))$, we have $\left|\sum_{i=1}^{\infty} x_n^*(x_n)\right| \le 1$.

(Proof). Let $T \in \mathrm{Ball}(L(X))$ and K a compact subset of X, then $T(K)$ is compact. So for any $\varepsilon > 0$, there is T_1 in $\mathrm{Ball}(K(X))$ so that $\|T_1Tx-Tx\| < \varepsilon$ for every $x \in K$. Since $T_1T \in \mathrm{Ball}(K(X))$, this proves the implication (i)$\Rightarrow$(ii). (iii) follows from (ii) by proposition 1. Thus it remains to prove that (iii) implies (i). Suppose X does not have the compact metric approximation property, then the identity map I on X is not in the τ-closure $\overline{\mathrm{Ball}(K(X))}^{\tau}$ of $\mathrm{Ball}(K(X))$. Since $\overline{\mathrm{Ball}(K(X))}^{\tau}$ is a closed, convex and balanced subset in a locally convex space $(L(X),\tau)$ and does not contain the identity map I on X, by the separation theorem there are positive numbers α, β and a τ-continuous linear functional ϕ on $L(X)$ such that $|\mathrm{Re}\phi(T)| < \alpha < \beta < \mathrm{Re}\phi(I)$ for all T in $\overline{\mathrm{Ball}(K(X))}^{\tau}$. Setting $\psi = \frac{1}{\alpha}\phi$, we have

$$\sup\{|\mathrm{Re}\Psi(T)| : T \in \mathrm{Ball}(K(X))\} \le 1 < \mathrm{Re}\psi(I) .$$

Viewing ψ as a linear functional on the Banach space $K(X)$ with the operator norm, we have $\|\psi\| = \|\mathrm{Re}\psi\| = \sup\{|\mathrm{Re}\psi(T)| : T \in \mathrm{Ball}(K(X))\} \le 1 < \mathrm{Re}\psi(I) \le |\psi(I)|$. Since ψ is a τ-continuous linear functional on

$L(X)$, by proposition 1 there are sequences $\{x_n\}_{n=1}^{\infty} \subseteq X$ and $\{y_n^*\} \subseteq X^*$ such that $\sum_{n=1}^{\infty} \|x_n\| \|y_n^*\| < \infty$ and for every T in $L(X)$ $\psi(T) = \sum_{n=1}^{\infty} y_n^*(Tx_n)$. Then $|\sum_{n=1}^{\infty} y_n^*(Tx_n)| \leq 1$ for all T in $\mathrm{Ball}(K(X))$ and $|\sum_{n=1}^{\infty} y_n^*(x_n)| > 1$. This contradicts to (iii). Hence (iii) implies (i).

Grothendieck [19] proved that if X is a reflexive Banach space or a separable conjugate space which has the approximation property, then X has the metric approximation property. In the case of the compact approximation property, the analogous implication is valid for reflexive Banach spaces.

Proposition 3. If X is a separable reflexive Banach space which has the compact approximation property, then X has the compact metric approximation property.

For the proof of proposition 3, we need a lemma from [31; p. 39].

Lemma 4. Let X be a separable Banach space and let $\varepsilon > 0$ be given. Then there exists a sequence of functions $\{f_i\}_{i=1}^{\infty}$ on $\mathrm{Ball}(X)$ so that for every x in $\mathrm{Ball}(X)$, $x = \sum_{n=1}^{\infty} f_i(x)$, each $f_i(x)$ is of the form $\sum_{i,j=1}^{\infty} 1_{E_{ij}}(x)x_{ij}$, where $\{E_{ij}\}_{j=1}^{\infty}$ are disjoint Borel sets of $\mathrm{Ball}(X)$, $\{x_{ij}\}_{k=1}^{\infty} \subseteq \mathrm{Ball}(X)$, $\sum_{i=1}^{\infty} \|f_i\|_{\infty} < 1 + \varepsilon$ where $\|f_i\|_{\infty} = \sup_x \|f_i(x)\| = \sup_j \|x_{ij}\|$ and $1_{E_{ij}}$ is the indicator function of E_{ij} .

(Proof of Proposition 3). Assume X is a separable reflexive Banach space. By Proposition 2 it suffices to show that if ϕ is a τ-continuous linear functional on $L(X)$ such that $|\phi(T)| \leq 1$ for every T in $\mathrm{Ball}(K(X))$, then $|\phi(T)| \leq 1$ for every T in $\mathrm{Ball}(L(X))$. For a fixed $\varepsilon > 0$, we construct a τ-continuous linear functional ψ_ε on $L(X)$ so that $\psi_\varepsilon = \phi$ on $K(X)$ and $|\psi_\varepsilon(T)| \leq (1+\varepsilon)\|T\|$ for all T in $L(X)$. Then since $K(X)$ is τ-dense in $L(X)$, $\psi_\varepsilon = \phi$ on $L(X)$ and $\|\phi(T)\| \leq (1+\varepsilon)\|T\|$. Since $\varepsilon > 0$ is arbitrary, $|\phi(T)| \leq 1$ for all T in $\mathrm{Ball}(L(X))$. Thus it remains to construct such a ψ_ε.

By reflexivity we may regard X as X^{**}. Then since X^{**} is separable X^* is also separable and $\mathrm{Ball}(X)$ with the weak topology induced by X^* is a compact metric space. Similarly $\mathrm{Ball}(X^*)$ with the weak*-topology induced by X is also a compact metric space. Thus $K = \mathrm{Ball}(X) \times \mathrm{Ball}(X^*)$ with the product topology is a compact metric space. Let $C(K)$ be the space of all continuous functions on K with the supermum norm. To each T in $K(X)$ we assign a function g_T on K defined by $g_T(x,x^*) = x^*(x)$ for all (x,x^*) in K.

We claim that g_T is continuous on K. Suppose (x_n,x_n^*) is a sequence in K which converges to a point (x,x^*) in K. Then $|g_T(x_n,x_n^*) - g_T(x,x^*)| = |x_n^*Tx_n - x^*Tx| \leq |(x_n^* - x^*)Tx| + \|x_n^*\|\|Tx_n - Tx\|$. Since $x_n^* \to x^*$ in the weak*-topology (weak topology induced by $X^{**} = X$), $(x_n^* - x^*)Tx \to 0$. Since T is compact and $x_n \to x$ weakly, $Tx_n \to Tx$ in norm. Since $\{\|x_n^*\|\}_{n=1}^\infty$ is bounded, $\|x_n^*\|\|Tx_n - Tx\| \to 0$ and hence g_T is in $C(K)$.

Next

$$\|g_T\| = \sup\{|g_T(x,x^*)| \ ; \ (x,x^*) \in K\}$$

$$= \sup\{(x^*(Tx)| \ ; \ x \in \mathrm{Ball}(X) \ , \ x^* \in \mathrm{Ball}(X^*)\}$$

$$= \sup\{\sup\{|x^*(Tx)| \ ; \ x^* \in \mathrm{Ball}(X^*)\} : x \in \mathrm{Ball}(X)\}$$

$$= \sup\{\|Tx\| : x \in \mathrm{Ball}(X)\} = \|T\|$$

Thus the map $T \to g_T$ is an isometry from $K(X)$ into $C(K)$ and we can view $K(X)$ as a subspace of $C(K)$. Observe that ϕ restricted to $K(X)$ is a norm continuous functional with norm ≤ 1. By the Hahn-Banach theorem this functional has a norm preserving extension to $L(X)$ and we still write this extension ϕ. Then by the Riesz representation theorem, there is a Borel measure μ on K of norm ≤ 1 so that

$$\phi(T) = \int_K x^*(Tx)d\mu \quad \text{for all} \quad T \in K(X) \ .$$

Then using lemma 4, for every T in $K(X)$, we have

$$\phi(T) = \sum_{i=1}^{\infty} \int_K x^*(T(f_i(x)))d\mu = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} x_{ij}^*(Tx_{ij})$$

where x_{ij}^* is the functional on X defined by $x_{ij}^*(x) = \int_{\Omega_{ij}} x^*(x)d\mu$, $\Omega_{ij} = E_{ij} \times \mathrm{Ball}(X^*)$. Obviously $\|x_{ij}^*\| \leq |\mu|(\Omega_{ij})$ and hence $\sum_{j=1}^{\infty} \|x_{ij}^*\| \leq 1$ for every i. From lemma 4, $\sum_{i=1}^{\infty} \sup_j \|x_{ij}\| \leq 1 + \varepsilon$.

Let $\psi_\varepsilon(T) = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} x^*_{ij}(Tx_{ij})$ for T $K(X)$, then by proposition 1 ψ_ε is a τ-continuous linear functional on $L(X)$ satisfying $|\psi_\varepsilon(T)| \le \|T\| \sum_{ij=1}^{\infty} \|x^*_{ij}\|\|x_{ij}\| \le (1+\varepsilon)\|T\|$.

The above proof of the proposition 3 is a modification of the Lindenstrauss-Tzafriri [31; p. 40] proof of Grothendieck's theorem.

Remarks: 1. It is a formal consequence of the proposition 3 as stated that every reflexive space with the compact approximation property also has the compact metric approximation property.

2. We do not know whether proposition 3 is true if X is only assumed to be a separable conjugate space. To apply the Lindenstrauss-Tzafriri argument one needs to prove that if Y^* is separable, then the weak*-continuous compact operators on Y^* are dense in $K(Y^*)$ when $K(Y^*)$ is given the topology of uniform convergence on compact subsets of Y^* .

A sequence $\{T_n\}_{n=1}^{\infty}$ in $L(X)$ is said to converge to T in $L(X)$ strongly if for every x in X T_nx converges to Tx in norm.

Corollary 5. If X is a separable reflexive Banach space which has the compact approximation property, then there is a sequence $\{T_n\}_{n=1}^{\infty}$ in $Ball(K(X))$ so that $T_n \to I_X$ (identity map on X) strongly and $T_n^* \to I_{X^*}$ (identity map on X^*) strongly.

(Proof). We choose a countable dense subset $D = \{x_i : i = 1,2,3..\}$ of X . For each n , let $D_n = \{x_1,\ldots,x_n\}$

and choose S_n in Ball(K(X)) so that $\|S_n x - x\| < \frac{1}{n}$ for all x in D_n. Then S_n converges to I_X (identity map on X) strongly. Since for each x in X and each x^* in X^* $(S_n^* x^*)x - x^*(x) = x^*(S_n x - x)$ converges to zero and since X is reflexive, $S_n^* x^* \to x^*$ weakly for each x^* in X^*.

We choose a countable dense subset $E = \{x_i^* : i = 1,2,3\ldots\}$ of X^* and take a sequence $\{S_{1n}\}_{n=1}^{\infty}$ so that each S_{1n} is a convex combination of $\{S_{1i}\}_{i=n}^{\infty}$ and $S_{1j}^* x_1^* \to x_1^*$ in norm. Next we take a sequence $\{S_{2n}\}_{n=1}^{\infty}$ so that S_{2n} is a convex combination of $\{S_{1i}\}_{i=n}^{\infty}$ and $S_{2i}^* x_2^* \to x_2^*$ in norm. We repeat the process in an obvious manner. Let $T_n = S_{nn}$. Then $T_n^* \to I_{X^*}$ (identity map on X^*) strongly.

<u>Proposition 6</u>. Suppose X is a reflexive subspace of a Banach space Y with the property that there exists a sequence $\{P_n\}_{n=1}^{\infty}$ in $K(Y)$ such that $\overline{\lim_n}\|I_Y - P_n\| < 1$ and $P_n \to I_Y$ (the identity map on Y) strongly, and suppose that X has the compact approximation property. Then there exists a sequence $\{T_n\}_{n=1}^{\infty}$ in Ball(K(X)) such that $\overline{\lim_n}\|I_X - T_n\| < 1$, $T_n \to I_X$ strongly and $T_n^* \to I_{X^*}$ strongly.

(Proof). Let $\{P_n\}_{n=1}^{\infty}$ be as above and for each n, $P_n|_X : X \to Y$ the restriction of P_n to X. Then $P_n|_X \to I_X$ $(X \to Y)$ strongly. By Corollary 5, there exists a sequence $\{S_n\}_{n=1}^{\infty}$ in Ball(K(X)) Ball(K(X,Y)) such that $S_n \to I_X$ strongly and $S_n^* \to I_{X^*}$ strongly. As a sequence of operators from X to Y, we have $P_n|_X - S_n \to 0$ strongly as $n \to \infty$. Since X is reflexive it follows that $P_{n|X} - S_n \to 0$ weakly in $L(X,Y)$ [32; p. 33]. Indeed, as in the proof of proposition 3, the map $S \to x^*(Sy)$ defines an isometry from $K(X,Y)$ to $C(\Omega)$, the space of continuous functions on the compact Hausdorff space $\Omega = \mathrm{Ball}(X) \times \mathrm{Ball}(Y^*)$, where $\mathrm{Ball}(X)$ has the weak topology and $\mathrm{Ball}(Y^*)$ has the weak*-topology. Observe that $\mathrm{Ball}(Y^*)$ is weak*-compact Hausdorff for any Banach space Y and $\mathrm{Ball}(X)$ is weakly compact Hausdorff since X is reflexive. Hence Ω with the product topology is a compact Hausdorff space. As a sequence in $C(\Omega)$, $\{P_{n|X}-S_n\}_{n=1}^{\infty}$ is uniformly bounded and $P_{n|X} - S_n \to 0$ pointwise on Ω. For any $\phi \in L(X,Y)^*$, the restriction of ϕ to $K(X,Y) \subseteq C(\Omega)$ is a continuous linear functional on $K(X,Y)$. By the Hahn-Banach theorem, we choose a norm preserving extension to $C(\Omega)$ and we still call this extension ϕ. Thus ϕ is a continuous linear functional on $C(\Omega)$. Then by the Riesz representation theorem, there is a regular Borel signed meaure μ on Ω such that $\phi(s) = \int_\Omega x^*(Sx)d\mu(x,x^*)$ for all $S \in K(X,Y)$. By the bounded convergence theorem, $\phi(P_{n|X}-X_n) \to 0$ as $n \to \infty$.

Since $P_{n|X} - S_n \to 0$ weakly in $L(X,Y)$, there exist sequences $\{Q_n\}_{n=1}^{\infty}$ and $\{T_n\}_{n=1}^{\infty}$ such that $Q_n = \sum_{k=a_n+1}^{a_{n+1}} \lambda_k P_{k|X}$, $T_n = \sum_{k=a_n+1}^{a_{n+1}} \lambda_k S_k$ and $\|Q_n-T_n\| \to 0$, where $\lambda_k > 0$, $\sum_{k=a_n+1}^{a_{n+1}} \lambda_k = 1$, and $\{a_n\}_{n=1}^{\infty}$ is a strictly increasing sequence of positive integers. Obviously $\|T_n\| \le 1$, $\overline{\lim_n}\|I_X-T_n\| \le \lim_n\|I_X-Q_n\| \le 1$, $T_n \to I_X$ strongly, and $T_n^* \to I_{X^*}$, strongly.

Remark. The relationship between the weak operator topology and the weak topology on the space of operators was, at least in special cases, known for a long time. The idea of using this relationship to deduce some kind of approximation condition for a subspace from the corresponding condition for the whole space is due to M. Feder [15].

3. M-ideals

Lemma 7. Suppose that $\{P_n\}_{n=1}^{\infty}$ is a sequence in $K(Y)$ for a Banach space Y which converges strongly to the identity map on Y, and that K is a weakly compact subset of Y. Given $\varepsilon > 0$ and a positive integer n, there exists an integer $m = m(n,\varepsilon) > n$ so that

$$\sup_{y\in K} \min_{n<k<m} d(P_k y, K) < \varepsilon ,$$

where $d(x,K) = \inf\{\|x-z\| : z \in K\}$ is the distance from x to the set K.

(Proof). If not, there exists a sequence $\{y_m\}_{m=n+1}^{\infty}$ in K so that for each $m = n+1, n+2, \ldots$

$$\min_{n<k<m} d(P_k y_m, K) > \varepsilon .$$

Let y be any weak cluster point of $\{y_m\}_{m=n+1}^{\infty}$ and assume $y_m \to y$ weakly by passing to a subsequence if necessary. Since each P_k is compact and $\{y_m\}_{m=1}^{\infty}$ is bounded, $\{P_k y_m\}_{m=1}^{\infty}$ has a cluster point which has to be $P_k y$ because $P_k y_m \to P_k$ weakly. Thus we infer that $\inf_{n<k<\infty} d(P_k y, K) \geq \varepsilon$. This is a contradiction, because y is in K and $\|y-P_k y\| \to 0$ as $k \to \infty$.

A Banach space X is said to have a finite dimensional Schauder decomposition $\{X_n\}_{n=1}^{\infty}$ if every $x \in X$ can be uniquely written as $x = \sum_{n=1}^{\infty} x_n$, where $x_n \in X_n$ and each X_n is a finite dimensional subspace of X. For each n, the partial sum projection P_n on X is defined by $P_n(\sum_{i=1}^{\infty} x_i) = \sum_{i=1}^{n} x_i$, where $x_i \in X_i$. It is easy to see that $\sup_n \|P_n\| < \infty$. Indeed, for every x in X, $x = \lim_n P_n x$ and hence by the Banach-Steinhaus theorem $\|P_n\| \leq M$ for all n and for some $M < \infty$.

<u>Lemma 8</u>. Let X be a reflexive Banach space which is a subspace of a Banach space Y which has a finite dimensional Schauder decomposition $\{X_n\}_{n=1}^{\infty}$ with partial sum projections $\{P_n\}_{n=1}^{\infty}$ and set $\alpha = \sup_n\{\|P_n\|\}$. Then for any $\varepsilon > 0$ and $T \in K(X)$ with $\|T\| < 2$, there exists a positive integer n such that

(i) $\|(I-P_n)Tx\| \leq \varepsilon$ for every $x \in \mathrm{Ball}(X)$,

(ii) if $x \in \mathrm{Ball}(X)$ and $\|P_n x\| \leq \frac{\varepsilon}{4}$, then $\|Tx\| \leq \varepsilon\alpha$.

(Proof). Since the closure of $T(\mathrm{Ball}(X))$ is compact, given $\varepsilon > 0$ we can choose a finite subset $\{x_1, x_2, \ldots, x_n\}$ of $\mathrm{Ball}(X)$ so that for every x in $\mathrm{Ball}(X)$, there is x_i, $1 \leq i \leq n$, with $\|x-x_i\| < \frac{\varepsilon}{4(1+\alpha)}$ and hence $\|Tx-Tx_i\| < \frac{\varepsilon}{2(1+\alpha)}$. Now for all $<$

sufficiently large n, $\|(I-P_n)Tx_i\| < \frac{\varepsilon}{2}$ for $i = 1,2,\ldots,n$. For $x \in \mathrm{Ball}(X)$, choose x_i so that $\|Tx-Tx_i\| < \frac{\varepsilon}{2}$. Then since $\|I-P_n\| < 1 + \alpha$.

$$\|(I-P_n)Tx\| \le \|(I-P_n)Tx - (I-P_n)Tx_i\| + \|(I-P_n)Tx_i\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

for all sufficiently large n.

Thus (i) is true for all large n.

If no n satisfies (ii), then there is a sequence $\{x_k\}_{k=1}^{\infty}$ in $\mathrm{Ball}(X)$ such that $\|P_k x_k\| \le \frac{\varepsilon}{4}$ and $\|Tx_k\| > \varepsilon\alpha$. Since $\mathrm{Ball}(X)$ is weakly compact, by passing to a subsequence if necessary, we may assume that $x_k \to x \in X$ weakly. We claim that $\|x\| \le \frac{\varepsilon\alpha}{3}$. If not, $\|P_\ell x\| > \frac{\varepsilon\alpha}{3}$ for all large ℓ. Since P_ℓ is a compact operator and a compact operator carries a weakly convergent sequence to a norm convergent sequence, $P_\ell x_k \to P_\ell x$ in norm as $k \to \infty$ and hence $\|P_\ell x_k\| \to \|P_\ell x\| > \frac{\varepsilon\alpha}{3}$. This is impossible since for $k > \ell$, $\|P_\ell x_k\| \le \alpha\|P_k x_k\| < \frac{\varepsilon\alpha}{4}$. Thus we have $\|x\| \le \frac{\varepsilon\alpha}{3}$.

Since T is compact and $x_k \to x$ weakly, by the same reason as above, $\|Tx_k\| \to \|Tx\|$ as $k \to \infty$. This is a contradiction because $\|Tx_k\| > \varepsilon\alpha$ for all k and $\|Tx\| \le \|T\|\|x\| < \frac{2\varepsilon\alpha}{3} < \varepsilon\alpha$.

The above proof actually shows that there are infinitely many n satisfying the conditions (i) and (ii).

Now we are ready to prove our main theorem. In this proof we use the following characterization of M-ideals due to Lima [27; Theorem 6.17]. A closed subspace J of a Banach space X is an M-ideal of X

if and only if for any $\varepsilon > 0$, for any x in $\mathrm{Ball}(X)$ and for any y_i in $\mathrm{Ball}(J)$ $(i=1,2,3)$, there exists $y \in J$ such that $\|x+y_i-y\| < 1+\varepsilon$ for $i = 1,2,3$.

Theorem 9. If X is a closed subspace of $Y = (\sum X_n)_p$ ($\dim X_n < \infty$, $1<p<\infty$) which has the compact approximation property, then $K(X)$ is an M-ideal in $L(X)$.

(Proof). Let $S_1, S_2, S_3 \in \mathrm{Ball}(K(X))$ and $T \in \mathrm{Ball}(L(X))$. We will show that for any $\eta > 0$, there exists $K \in K(X)$ such that $\|S_i + T - K\| < 1 + \eta$ $(i = 1,2,3)$.

By Propositon 6, we can choose a sequence $\{T_n\}_{n=1}^{\infty}$ in $\mathrm{Ball}(K(X))$ so that $\overline{\lim_n} \|I_X - T_n\| < 1$, $T_n \to I_X$ strongly, and $T_n^* \to I_{X^*}$ strongly. Let $\{P_n\}$ denote the partial sum projections associated with the natural finite dimensional decomposition $\{X_n\}_{n=1}^{\infty}$ of Y. Using Lemma 8, with this choice of P_n's (so that $\alpha = 1$), for a fixed $0 < \varepsilon < 1$, choose M so that for $i = 1,2,3$

(i) if $x \in \mathrm{Ball}(X)$, then $\|(I-P_M)S_i x)\| < \varepsilon$ where $I = I_X$

(ii) if $x \in \mathrm{Ball}(X)$ and $\|P_M x\| < \frac{\varepsilon}{4}$, then $\|S_i x\| < \varepsilon$.

By lemma 7, we can choose $N > M$ so that for every $x \in X$, there is $k = k(x)$ $(M < k < N)$ such that $d(P_k x, X) < \varepsilon\|x\|$.

Given $x \in X$ with $\|x\| = 1$, let $k = k(x)$ and pick $y_1 \in X$ so that $\|P_k x - y_1\| < \varepsilon$. Setting $y_2 = x - y_1$, we have

(iii) $\|y_2-(I-P_k)x\| = \|P_k x-y_1\| < \varepsilon$, $\|(I-P_k)y_1\| < \varepsilon$ and $\|P_k y_2\| < \varepsilon$.

Finally, choose r large enough so that

(iv) $\|(I-T_r)Ty\| < 8\varepsilon$ for every y in the set

$$A = \{y \in X : \|y\| < 2 \text{ and } \|(I-P_N)y\| < \varepsilon\}$$

(v) $\|P_M(I-T_r)T\| = \|T^*(I-T_r^*)P_M^*\| < \varepsilon$ and $\|I-T_r\| < 1 + \varepsilon$.

This is possible because the set A above has a 3ε-net and $T_n \to I$ strongly. (If B is a subset of a metric space and $\varepsilon > 0$, then a subset A of B is called an ε-net of B if for every x in B , there is $y \in A$ such that $d(x,y) < \varepsilon$.)

For $x \in X$ with $\|x\| = 1$, write $x = y_1 + y_2$ as in (iii). Then for $i = 1,2,3$, we have

$$\|S_i x + (I-T_r)Tx\|^p$$

$$\leq (\|P_M(S_i x) + (I-P_M)(I-T_r)Tx\| + \|(I-P_M)S_i x\| + \|P_M(I-T_r)Tx\|)^p$$

$$\leq (\|P_M(S_i x) + (I-P_M)(I-T_r)Tx\| + \varepsilon + \varepsilon)^p \quad \text{(by (i) and (v))}$$

$$= (\|P_M(S_i x)\|^p + \|(I-P_M)(I-T_r)Tx\|^p + f(\varepsilon) \quad (f(\varepsilon) \to 0 \text{ as } \varepsilon \to 0)$$

$$\leq (\|P_M(S_i y_1)\| + \|P_M S_i y_2\|)^p + (\|(I-P_M)(I-T_r)Ty_1\| + \|(I-P_M)(I-T_r)Ty_2\|)^p + f(\varepsilon)$$

$\leq (\|y_1\| + 4\varepsilon)^p + (8\varepsilon + (1+\varepsilon)\|y_2\|)^p + f(\varepsilon)$ (since $\|y_2\| < 2$ and $\|P_M y_2\| < \varepsilon$, $\|S_i y_2\| < 4\varepsilon$ by (ii) . Since $y_1 \varepsilon A$ by (iii) , $\|(I-T_r)Ty_1\| < 8\varepsilon$. $\|(I-T_r)T\| \leq 1 + \varepsilon$ by (v))

$$< (\|P_k x\| + 5\varepsilon)^p + (\|(I-P_k)x\| + 11\varepsilon)^p + f(\varepsilon) \quad \text{(by (iii))}$$

$$< \|P_k x\|^p + \|(I-P_k)x\|^p + g(\varepsilon) \quad (g(\varepsilon) \to 0 \text{ as } \varepsilon \to 0) .$$

$$= 1 + g(\varepsilon) .$$

Thus for $i = 1,2,3$, $\|S_i + T - T_rT\| = \|S_i + (I-T_r)T\| < (1+g(\varepsilon))^{1/p}$. Choose ε so that $(1+g(\varepsilon))^{1/p} < 1 + r$ and let $K = T_rT$.

Combining Theorem 9 with the Harmand-Lima Theorem, we get the following.

Corollary 10. If X is a closed subspace of $(\sum X_n)_p$ $(\dim X_n < \infty,\ 1 < p < \infty)$, then $K(X)$ is an M-ideal in $L(X)$ if and only if X has the compact approximation property.

CHAPTER II

M-IDEALS AND IDEALS IN $L(X)$

1. Introduction

As has already been stated in the introduction to chapter I, many authors have studied M-ideal structures in operator algebras with a view toward characterizing those Banach spaces X for which $K(X)$, the space of compact operators on X, is an M-ideal in $L(X)$, the space of continuous operators in X. An extensive study of the M-ideal structure of a general complex Banach algebra with identity was done by Smith and Ward [35].

Originally an M-ideal was defined in a real Banach space [2]. However, this notion can be extended to a complex Banach space [22].

Smith and Ward [35] proved that M-ideals in a complex Banach algebra with identity are subalgebras and that they are two sided (algebraic) ideals if the algebra is commutative. However, M-ideals in a general complex Banach algebra need not to be either left ideals or right ideals as the following two examples from [35] show.

Example 1. Let A be the Banach algebra of 2×2 matrices with the usual multiplication and norm defined by

$$\left\| \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \right\| = \max\{ |\alpha| + |\gamma| , |\beta| + |\delta| \} .$$

Let J_1 and J_2 be respectively the sets of elements of the form

$$\begin{pmatrix} \alpha & 0 \\ \gamma & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & \beta \\ 0 & \delta \end{pmatrix} .$$

Then it is easily verified that J_1 and J_2 are complementary M-summands in A. However the equation

$$\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$$

shows that J_1 is not closed under multiplication from the right, though both M-summands are easily shown to be left ideals.

By the standard duality it is easy to show that M-summands are M-ideals [4].

Examples 2. Let A be the Banach algebra defined in Example 1. Let B be the Banach algebra $A \times A$ with multiplication defined for $a , b , c , d \in A$ by

$$(a,b)(c,d) = (ac,bd) ,$$

and norm defined by

$$\|(a,b)\| = \max\{\|a\|,\|b\|\} .$$

Let J_1 and J_2 denote respectively the sets of elements of the form

$$\left(\begin{pmatrix}\alpha & 0\\ \beta & 0\end{pmatrix}, \begin{pmatrix}0 & \gamma\\ 0 & \delta\end{pmatrix}\right)$$

and

$$\left(\begin{pmatrix}0 & \alpha\\ 0 & \beta\end{pmatrix}\begin{pmatrix}\gamma & 0\\ \delta & 0\end{pmatrix}\right) .$$

Then J_1 and J_2 are complementary M-summands. However the equations

$$\left(\begin{pmatrix}0 & 1\\ 0 & 1\end{pmatrix}, \begin{pmatrix}1 & 0\\ 1 & 0\end{pmatrix}\right)\left(\begin{pmatrix}1 & 0\\ 1 & 0\end{pmatrix}, \begin{pmatrix}0 & 1\\ 0 & 1\end{pmatrix}\right) = \left(\begin{pmatrix}1 & 0\\ 1 & 0\end{pmatrix}, \begin{pmatrix}1 & 0\\ 1 & 0\end{pmatrix}\right)$$

and

$$\left(\begin{pmatrix}1 & 0\\ 1 & 0\end{pmatrix}, \begin{pmatrix}0 & 1\\ 0 & 1\end{pmatrix}\right)\left(\begin{pmatrix}0 & 1\\ 0 & 1\end{pmatrix}, \begin{pmatrix}1 & 0\\ 1 & 0\end{pmatrix}\right) = \left(\begin{pmatrix}0 & 1\\ 0 & 1\end{pmatrix}, \begin{pmatrix}0 & 1\\ 0 & 1\end{pmatrix}\right)$$

show that neither M-summand is either a left or right ideal.

Smith and Ward [35] also proved that the M-ideals in a C^*-algebra are exactly the two sided ideals.

Later Flinn [17], and Smith and Ward [36] showed that for $1 < p < \infty$, $K(\ell_p)$ is the only nontrivial ideal in $L(\ell_p)$, and since 0 and $L(\ell_p)$ are both ideals and M-ideals, the M-ideals in $L(\ell_p)$ are exactly the two sided ideals in $L(\ell_p)$.

In this chapter we will prove that for a uniformly convex space X , every M-ideal in $L(X)$ is a left ideal. Moreover if X^* is also uniformly convex, then every M-ideal in $L(X)$ is a two sided ideal. This verifies a special case of the conjecture of Smith and Ward [35] that if X is a uniformly convex space then every M-ideal in $L(X)$

is a two sided ideal.

Expressed in geometrical terms, a Banach space X is uniformly convex if the mid point of a variable chord of the closed unit ball of the space X cannot approach the boundary of the ball unless the length of the chord goes to zero. Formally, for any Banach space X with $\dim X \geq 2$, the modulus of convexity $\delta_X(\varepsilon)$, $0 < \varepsilon \leq 2$, of X is defined by

$$\delta_X(\varepsilon) = \inf\{1 - \frac{\|x+y\|}{2} \ ; \ x,y \in X \ , \ \|x\| = \|y\| = 1 \ , \ \|x-y\| = \varepsilon\} \ .$$

A Banach space X is said to be uniformly convex if $\delta_X(\varepsilon) > 0$ for every $0 < \varepsilon \leq 2$. In the definition of $\delta_X(\varepsilon)$ we can also take the infimum over all vectors x, $y \in X$ with $\|x\|$, $\|y\| \leq 1$ and $\|x-y\| \geq \varepsilon$ [32; p. 60].

For $1 < p < \infty$, ℓ_p and L_p are uniformly convex [10], but obviously ℓ_∞ and L_∞ are not. We can easily see from the definition that a Banach space X is not uniformly convex if the boundary of the unit ball contains a line segment.

2. Some results related with M-ideals in complex Banach algebras.

This section contains some background material and several facts due to Smith and Ward [35], [36], which will be needed in the proof of the main theorem in section 3.

A Banach algebra A is a real or complex Banach space A with a multiplication having the following properties:

(i) $x(y+z) = xy + xz$ and $(y+z)x = yx + zx$ for $x, y, z \in A$

(ii) $\alpha(xy) = (\alpha x)y = x(\alpha y)$ for any scalar α and $x, y \in A$.

(iii) $\|xy\| \leq \|x\|\|y\|$ for $x, y \in A$.

A Banach algebra is unital if it contains the identity element e with respect to the multiplication and $\|e\| = 1$.

If A is a Banach algebra, then the second dual A^{**} of A becomes a Banach algebra with respect to the Arens multiplication which is defined in the following fashion [6]. If $y \in A$, $f \in A^*$ and $F, G \in A^{**}$, then linear functionals f_y, $F_f \in A^*$ are defined by $f_y(x) = f(yx)$ and $F_f(x) = F(f_x)$ for $x \in A$. Then Arens multiplication $GF \in A^{**}$ is defined by

$$(GF)(f) = G(F_f) \quad \text{for all} \quad f \in A^* .$$

The canonical embedding of A into A^{**} is an isometric algebra isomorphism of A into A^{**}. Moreover, if A has identity e then the canonical image of e in A^{**} is the identity element of A^{**}.

In the rest of this section, A will denote a complex Banach algebra with unit e. In the dual space A^* of A, the state space S is defined to be $\{f \in A^* : f(e) = \|f\| = 1\}$. Obviously, this is weak*-closed and it is known [34] that A^* is algebraically spanned by S. If J is an L-summand in A^* with the complementary subspace J'; that is, $A^* = J \oplus_1 J'$, then J and J' are algebraically spanned by $F = J \cap S$ and $F' = J' \cap S$, respectively. More specifically, Proposition 1 [34]. F and F' is a pair of complementary split faces of S, and J and J' are algebraically spanned by F and F', respectively.

An element $h \in A$ is said to be hermitian if $f(h)$ is real for each f in the state space S. It is known [6; p. 46] that $h \in A$ is hermitian if and only if $\|e^{ith}\| = 1$ for all real numbers t. Of course, e^{ith} is defined by $\sum_{n=0}^{\infty} \frac{(ith)^n}{n!}$.

Proposition 2 [35]. Suppose $A = J_1 \oplus_\infty J_2$, $J_i \neq \{0\}$ $(i = 1,2)$, $P : A \to J_1$ is the natural projection onto J_1 and $z = P(e)$. Then z is hermitian and $z^2 = z$.

For a Banach space X, a projection $P : X \to X$ (continuous linear operator on X with $P^2 = P$) is called an L-projection (respectively M-projection) if $\|x\| = \|Px\| + \|x - Px\|$ (respectively $\|x\| = \max\{\|Px\|, \|x - Px\|\}$) for every $x \in X$. If P is an L-projection on X then range P and ker P are L-summands and $P \to$ range P gives one to one correspondence between the set of all L-projections on X and the set of all L-summands in X [4; p. 12]. The same relation holds between M-projections on X and M-summands in X.

If P is a projection on a Banach space, then P is an L-projection (respectively M-projection) if and only if its adjoint P^* on X^* is M-projection (respectively L-projection([4; prop. 1.5].

If J is an M-ideal in a complex unital Banach algebra A, then $A^* = J^\perp \oplus_1 J^{\perp'}$ for some closed subspace $J^{\perp'}$ of A^* and it is easy to show that $A^{**} = (J^\perp \oplus_1 J^{\perp'})^* = J^{\perp\perp} \oplus_\infty J^{\perp\perp'}$, where $J^{\perp\perp} = (J^{\perp'})^*$ and $J^{\perp\perp'} = (J^{\perp'})^\perp = (J^\perp)^*$ up to isometry. Let $P : A^{**} \to J^{\perp\perp}$ be the M-projection onto $J^{\perp\perp}$ and let $z = P(e)$; then by proposition 2, z is a hermitian projection in A^{**}, that is, z is hermitian in A^{**} and satisfies $z^2 = z$. We shall need the following theorem of Smith and Ward in the next section.

Theorem 3. [36] Let z be a hermitian projection in A^{**} associated with an M-ideal J in A. Then, given $\varepsilon > 0$, z is the weak* limit of a net (e_α) in A such that

$$\|e_\alpha\| , \|e-e_\alpha\| , \|e-e_\alpha\| < 1 + \varepsilon .$$

The following lemma is essentially due to Smith and Ward [36], although they restricted attention to right multiplication by a hermitian projection z associated with an M-ideal J in A.

Lemma 4. In the Banach algebra A^{**}, right multiplication by every element y in A^{**} is a weak*-continuous function on A^{**} and if $u \in A^{**}$ is the weak*-limit of a net $\{u_\alpha\}$ in A then, for every x in A, xu is the weak*-limit of $\{xu_\alpha\}$.

(Proof.) To prove the first statement, let $\{v_\alpha\}$ be a net in A^{**} with the weak*-limit v in A^{**}. If $f \in A^*$ and $y \in A^{**}$, then, by the definition of Arens multiplication,

$$(vy)(f) = v(y_f) = \lim_\alpha v_\alpha(y_f) = \lim_\alpha (v_\alpha y)(f) .$$

Thus $v_\alpha y \to vy$ in the weak*-topology and hence right multiplication by

$y \in A^{**}$ is weak*-continuous.

To prove the second statement, let u and $\{u_\alpha\}$ be as above and $f \in A^*$. If $x \in A$, then

$$(xu)(f) = x(u_f) = u_f(x) = u(f_x) = \lim_\alpha u_\alpha(f_x) = \lim_\alpha f(xu_\alpha) = \lim_\alpha (xu_\alpha)(f)$$

and hence xu is the weak*-limit of $\{xu_\alpha\}$.

3. M-ideals and ideals in $L(X)$.

It is known [32: p. 66] that for every Banach space X, $\frac{\delta_X(\varepsilon)}{\varepsilon}$ is a nondecreasing function on $(0,2]$. Thus if X is uniformly convex, then $\delta_X(\varepsilon)$ is strictly increasing function on $(0,2]$ and its inverse δ_X^{-1} is also a strictly increasing function on $(0, \delta_X(2)]$.

Lemma 5. Let X be a uniformly convex space. Then there is a nonnegative real valued function f on $(0,2] \times (0,\infty)$ such that $\lim_{\lambda \to 0} \lim_{\varepsilon \to 0} f(\varepsilon,\lambda) = 0$, and for every A, T in $L(X)$ with $\|T\|$, $\|I-T\|$, $\|I-2T\| < 1 + \varepsilon$, $\|A\| \leq 1$, we have $\|T+\lambda A(I-T)\| \leq 1 + \varepsilon + \lambda f(\varepsilon,\lambda)$, where I is the identity map on X.

(Proof). Fix ε, $\lambda > 0$ and $y \in X$ with $\|y\| = 1$. If $\|Ty\| \leq 1 - \lambda(1+\varepsilon)$, then $\|(T+\lambda A(I-T)y\| \leq 1 + \varepsilon$. So we assume that $\|Ty\| \geq 1 - \lambda(1+\varepsilon)$. Set $u = \frac{Ty}{1+\varepsilon}$ and $v = \frac{y - Ty}{1+\varepsilon}$, then $\|u+v\| = \left\|\frac{y}{1+\varepsilon}\right\| \leq 1$ and $\|u-v\| = \left\|\frac{y-2Ty}{1+\varepsilon}\right\| \leq 1$. Since $u = \frac{1}{2}\{(u+v) + (u-v)\}$ and $2v = (u+v) - (u-v)$, we have

$\delta_X(\|2v\|) < 1 - \|u\|$. Hence $\|u\| < 1 - \delta_X(2\|v\|)$. By assumption, $\frac{1 - \lambda(1+\varepsilon)}{1 + \varepsilon} < \|u\|$. Combining the last two inequalities, we have $\frac{1 - \lambda(1+\varepsilon)}{1 + \varepsilon} < 1 - \delta_X(2\|v\|)$ and hence $\delta_X(2\|v\|) < (1 - \frac{1}{1 + \varepsilon}) + \lambda$. Since δ_X^{-1} is an increasing function, $\|v\| < 2\|v\| < \delta_X^{-1}(1 - \frac{1}{1 + \varepsilon} + \lambda)$. Then $\|(T+\lambda A(I-T))y\| < \|Ty\| + \lambda\|(I-T)y\| < 1 + \varepsilon + \lambda(1+\varepsilon)\delta_X^{-1}(1 - \frac{1}{1 + \varepsilon} + \lambda)$. Hence $\|T+\lambda A(I-T)\| < 1 + \varepsilon + \lambda(1+\varepsilon)\delta_X^{-1}(1 - \frac{1}{1 + \varepsilon} + \lambda)$. Now let $f(\varepsilon,\lambda) = (1+\varepsilon)\delta_X^{-1}(1 - \frac{1}{1 + \varepsilon} + \lambda)$.

Now we are ready to prove the main theorem by using the Smith-Ward argument [36], but by replacing Clarkson's inequalities in ℓ_p , $1 < p < \infty$, by the inequality in lemma 5.

Theorem 6. Let X be a uniformly convex space and J an M-ideal in $L(X)$. Then J is a left ideal in $L(X)$ and if X^* , the dual of X , is also uniformly convex then J is a two sided ideal in $L(X)$.

(Proof). Let $J^{\perp'}$ be the complementary subspace of the L-summand $J^{\perp}$ in $L(X)^*$; that is , $L(X)^* = J^{\perp} \oplus_1 J^{\perp'}$, and let $F = J^{\perp} \cap S$ and $F' = J^{\perp'} \cap S$ where S is the state space in $L(X)^*$. Let P be the M-projection of $L(X)^{**} = J^{\perp\perp} \oplus_\infty J^{\perp\perp'}$ onto $J^{\perp\perp}$ and $z = P(e)$ where e is the identity operator on X . Then z vanishes on $J^{\perp}$ and hence on F . Similarly $e - z$ vanishes on F' . For each $\phi \in F'$, $1 = \phi(e) = \phi(e-z) + \phi(z) = \phi(z)$ and hence $z = 1$ on F' .

First we will show that $L(X)(e-z) \subseteq J^{\perp\perp'}$. In view of proposition 1 and equations $L(X)^{**} = (J^{\perp} \oplus_1 J^{\perp'})^* = (J^{\perp} \oplus_1 J^{\perp'})^* = (J^{\perp})^* \oplus_\infty (J^{\perp'})^* = J^{\perp\perp'} \oplus_\infty J^{\perp\perp}$, it suffices to show that if A $L(X)$ with $\|A\| \leq 1$ then $\phi(A(e-z)) = 0$ for all $\phi \in F'$. Suppose there is $\phi \in F'$ and $A \in L(X)$ with $\|A\| \leq 1$ such that $\phi(A(e-z)) \neq 0$. By multiplying A by a scalar we may assume that $\phi(A(e-z)) = \lambda$, $0 < \lambda < 1$. Let $A_n = z + \lambda^n A(e-z) \in L(X)^{**}$. Then by theorem 3 and lemma 4, A_n is the weak*-limit of a net $\{e_\alpha + \lambda^n A(e-e_\alpha)\}_\alpha$ in $L(X)$ with $\|e_\alpha\|$, $\|e-e_\alpha\|$, $\|e-e_\alpha\| < 1 + \varepsilon$. By lemma 5, $\|e_\alpha + \lambda^n A(e-e_\alpha)\| \leq 1 + \varepsilon + \lambda^n f(\varepsilon,\lambda^n)$ and hence we have

$$\|A_n\| \leq 1 + \varepsilon + \lambda^n f(\varepsilon,\lambda^n) .$$

Since $\|\phi\| = 1$ and $\phi(z) = 1$, $1 + \lambda^{n+1} = \phi(A_n) \leq \|A_n\| \leq 1 + \varepsilon + \lambda^n f(\varepsilon,\lambda^n)$. Letting $\varepsilon \to 0$, we have $\lambda \leq \lim_{\varepsilon\to 0} f(\varepsilon,\lambda^n)$, and letting $n \to \infty$, $0 < \lambda \leq \lim_{n\to\infty} \lim_{\varepsilon\to 0} f(\varepsilon,\lambda^n) = 0$. This is a contradiction. Hence $\phi(A(e-z)) = 0$ for all $A \in L(X)$ and all $\phi \in F'$, and we get that

$$L(X)(e-z) \subseteq J^{\perp\perp'} .$$

Since $J^{\perp\perp'}$ is weak*-closed and by lemma 4 right multiplication by $e - z$ is a weak*-continuous function on $L(X)^{**}$, we have $L(X)^{**}(e-z) \subseteq J^{\perp\perp'}$ by weak*-denseness of $L(X)$ in $L(X)^{**}$. Notice that if I is the identity map on $L(X)^{**}$ then $I - P$ is the

M-projection of $L(X)^{**} = J^{\perp\perp} \oplus_\infty J^{\perp\perp'}$ onto $J^{\perp\perp'}$ and $(I-P)e = e - Pe = e - z$. Thus by replacing z by $e - z$ in the above argument we get that

$$L(X)^{**} z \subseteq J^{\perp\perp} .$$

From the above two inclusion, we have $L(X)^{**} z = J^{\perp\perp}$ and $L(X)^{**}(e-z) = J^{\perp\perp'}$. Since $J^{\perp\perp} = L(X)^{**} z$ is a left ideal in $L(X)^{**}$, $J = J^{\perp\perp} \cap L(X)$ is a left ideal in $L(X)$.

Next suppose that X and X^* are uniformly convex. Let $\sigma : L(X) \to L(X^*)$ be defined by $\sigma(A) = A^*$, the adjoint of A. Then σ is an isometry and $\sigma(AB) = \sigma(B)\sigma(A)$ for every $A , B \in L(X)$. If J is an M-ideal in $L(X)$, then $\sigma(J)$ is an M-ideal in $L(X^*)$ and hence is a left ideal in $L(X^*)$ by the above result. Then $J = \sigma^{-1}\sigma(J)$ is a right ideal and hence a two sided ideal in $L(X)$.

CHAPTER III

AN EXAMPLE OF A SPACE $X = (\sum \ell_p^{n_i})_r$ FOR WHICH $L(X)$ CONTAINS A CLOSED TWO SIDED IDEAL WHICH IS NOT AN M-IDEAL IN $L(X)$

1. Introduction

As stated in the introduction to chapter II, M-ideals in $L(\ell_p)$ for $1 < P < \infty$ are exactly closed two sided ideals in $L(\ell_p)$. For $X = (\sum \ell_p^{n_i})_r$ with $1 < p$, $r < \infty$ and $\{n_i\}$ a bounded sequence of positive integers, $K(X)$ is an M-ideal in $L(X)$ by theorem 9 of chapter I or [8]. Since both X and X^* are uniformly convex [12], by theorem 6 of chapter II or [9] M-ideals in $L(X)$ are closed two sided ideals. Since X is isomorphic to ℓ_r and $K(\ell_r)$ is the only nontrivial two sided ideal in $L(\ell_r)$ [18] , $K[X]$ is the only non-trivial two sided ideal in $L(X)$ and hence M-ideals in $L(X)$ are exactly two sided ideals in $L(X)$.

In this chapter we will construct a space $X = (\sum \ell_p^{n_i})_r$ for which $L(X)$ contains a closed two sided ideal which is not an M-ideal.

The construction of our space X was motivated by Benyamini and Lin's paper [5]. In fact, X will be constructed so that the Benyamini-Lin argument can be applied to a certain ideal in $L(X)$. We will prove that for this space X the closure $\overline{S_r(X)}$ of $S_r(X)$, the ideal of all operators in $L(X)$ which factor through a subspace of an L_r-space, is not proximinal in $L(X)$. As stated before, since an M-ideal in $L(X)$ is proximinal, $\overline{S_r(X)}$ is not an M-ideal in $L(X)$.

2. Preliminary

For a Banach space X and $1 < r < \infty$, $S_r(X)$ will denote the space of all operators in $L(X)$ which factor through a subspace of an L_r-space. Thus an operator T in $L(X)$ belongs to $S_r(X)$ if there exists a subspace E of an $L_r(\Omega)$ and bounded linear operators $A : X \to E$, $B : E \to X$ such that $T = BA$. It is easy to see that $S_r(X)$ is a two sided ideal in $L(X)$. Since the closure of any two sided ideal is also a two sided ideal in $L(X)$, the closure $\overline{S_r(X)}$ of $S_r(X)$ is a two sided ideal in $L(X)$.

On $S_r(X)$, we put a norm which is defined for T in $S_r(X)$ by

$$S_r(T) = \inf\{\|A\|\|B\| : T = BA , A \in L(X,E) , B \in L(E,X) \text{ and } E \text{ is a closed subspace of an } L_r\text{-space}\}$$

where the infimum is taken over all possible factorizations of T through subspaces of L_r-spaces.

In this chapter we will heavily use the following lemma which is due to Figiel and Johnson.

Lemma 1 [16]. Suppose $2 < p < \infty$, $T : \ell_p^k \to \ell_p^{2k}$ with $\|T\| < 1$ and $\operatorname{Average}_{\pm}\{\|\sum_{i=1}^{k} \pm Te_i\|\} > \delta k^{1/p}$, $\delta > 0$, where the average is taken over all choices of $+$ and $-$ signs. Then there exist positive constants $c = c(p,r,\delta)$ and $\alpha = \alpha(p,r)$ such that $S_r(T) \geq ck^{\alpha}$.

Lemma 2. Suppose $1 < p \neq r < \infty$ and it is false that $1 < r < p < 2$. Then we have $d(I\ ,\ S_r(X)) = \inf\{\|I-T\| : T \in S_r(X)\} \geq 1$, where I is the identity map on $X = (\sum_{k=1}^{\infty} \ell_p^k)_r$.

(Proof). If $d(I\ ,\ S_r(X)) < 1$ then there is $F \in S_r(X)$ such that $\|F-I\| = 1 - \varepsilon$, $\varepsilon > 0$ and F factors through a subspace of L_r

$$\begin{tikzcd}
 & E \subseteq L_r \arrow[dr, "S"] & \\
(\sum_{k=1}^{\infty} \ell_p^k)_r \arrow[ur, dashed, "T"] \arrow[rr, dashed, "F"] & & (\sum_{k=1}^{\infty} \ell_p^k)_r \ .
\end{tikzcd}$$

Thus $\|ST-I\| = 1 - \varepsilon$.

Let Π_k be the projection from $(\sum_{k=1}^{\infty} \ell_p^k)_r$ onto ℓ_p^k , then $\Pi_k(F|_{\ell_p^k})$ has a factorization

$$\begin{tikzcd}
 & E \subseteq L_r \arrow[dr, dashed, "S_k"] & \\
\ell_p^k \arrow[ur, "T_k"] \arrow[rr, "\Pi_k(F|\ell_p^k)"] & & \ell_p^k
\end{tikzcd}$$

where $T_k = T|_{\ell_p^k}$ and $S_k = \Pi_k S$. Then $\|S_kT_k - I_k\| < 1 - \varepsilon$, where I_k is the identity map on ℓ_p^k. Thus S_kT_k is invertible and by the Neumann series expansion of $(S_kT_k)^{-1}$, we have the estimates $\|(S_kT_k)^{-1} < \frac{1}{1-(1-\varepsilon)} = \frac{1}{\varepsilon}$ and

$$S_r(I_k) < \|S_k\|\|T_k(S_kT_k)^{-1}\| < \|S\|\|T\|\frac{1}{\varepsilon}.$$

To draw a contradiction, we will show that $\sup_k S_k(I_k) = \infty$. It is known that ℓ_p cannot be (isomorphically) embedded in L_r under the hypothesis on p and r [3; p. 206], [26]. So ℓ_p^k's cannot be uniformly embedded in L_r.

Indeed, if ℓ_p^k's can be uniformly embedded in $L_r(\mu)$ for some measure μ then there exist positive numbers a, $b > 0$ and embeddings $T_k : \ell_p^k \to L_r(\mu)$ such that for every $k = 1,2,3\ldots$

$$a\|x\| < \|T_kx\| < b\|x\| \quad \text{for any } x \in \ell_p^k$$

By taking ultra product of $\{T_k\}_{k=1}^{\infty}$, we have [32; p. 120] that

$$T = (T_k)_{\mathcal{U}} : (\ell_p^k)_{\mathcal{U}} \to (L_r(\mu))_{\mathcal{U}}$$

where $\mathcal{U}$ is an ultrafilter on N, the set of all positive integers.

By definitions, for any $(x_k)_{\mathcal{U}}$ in $(\ell_p^k)_{\mathcal{U}}$, $(T_k)_{\mathcal{U}}((x_k)_{\mathcal{U}}) = (T_kx_k)_{\mathcal{U}}$, $\|(x_k)_{\mathcal{U}}\| = \lim_{\mathcal{U}} \|x_k\|$ and $\|(T_kx_k)_{\mathcal{U}}\| = \lim_{\mathcal{U}}\|T_kx_k\|$. Hence we have

$$a\|(x_k)_{\mathcal{U}}\| \le \|T(x_k)_{\mathcal{U}}\| \le b\|(x_k)_{\mathcal{U}}\| ,$$

that is, T is an embedding of $(\ell_p^k)_{\mathcal{U}}$ in $(L_r(\mu))_{\mathcal{U}}$. Since $(L_r)_{\mathcal{U}}$ is also an L_r-space [32; p. 271] and $(\ell_p^k)_{\mathcal{U}}$ contains an isometric copy of ℓ_p, T yields an embedding of ℓ_p into L_r, which is a contradiction.

Going back to the main stream, for any $\delta > 0$ and each k, there is a factorization

$$\begin{tikzcd} & E \subseteq L_r \arrow[dr, dashed, "B_k"] & \\ \ell_p^k \arrow[ur, dashed, "A_k"] \arrow[rr, "I_k"] & & \ell_p^k \end{tikzcd}$$

so that $\|A_k\|\|B_k\| \le S_r(I_k) + \delta$ and $\|B_k\| = 1$. Since A_k is an embedding $\sup_k \|A_k\| = \infty$, and hence $\sup_k S_r(I_k) = \infty$.

3. Main theorem

For $k = 1,2,3,\ldots$, $m = 1,2,3,\ldots$, and $1 \le i \le m$, let $\Omega_{k,m,i} = \{(s,t) : 1 \le s \le km ,\ 1 \le t \le k ,\ s$ and t are integers$\} \cup \{(1,0)\}$ be a measure space with $\mu\{(1,0)\} = \frac{1}{m}$ and $\mu\{(s,t)\} = \frac{1}{k^2 m}$ if $(s,t) \neq (1,0)$.

For notational convenience we denote $L_p(\Omega_{k,m,i})$, $p > 2$ by $X(k,m,i)$, the indicator function of $\{(1,0)\}$ by $e(k,m,i)$ and the indicator function of $\{(s,t)\}$ by $e_{s,t}(k,m,i)$ if $(s,t) \neq (1,0)$.

So $\{e_{s,t}(k,m,i) : 1 \leq s \leq km , 1 \leq t \leq k\} \cup \{e(k,m,i)\}$ is the natural basis of $X(k,m,i)$. Usually the dependence on k, m and i will be supressed.

Let $P_{k,m,i}$ be the projection on $X(k,m,i)$ defined by $P_{k,m,i}(e) = 0$ and $P_{k,m,i}(e_{s,t}) = \frac{1}{km} \sum_{u=1}^{km} e_{u,t}$. We define a linear map $S_{k,m,i}$ on $X(k,m,i)$ by $S_{k,m,i}(e) = \sum_{t=1}^{k} \sum_{s=1}^{k} e_{s,t}$ and $S_{k,m,i}(e_{s,t}) = 0$. We can easily see that both $P_{k,m,i}$ and $S_{k,m,i}$ have norm one.

Let $\tilde{X}(k,m) = (\sum_{i=1}^{m} X(k,m,i))_p$ and $X = (\sum_{k,m=1}^{\infty} \tilde{X}(k,m))_r$, $2 < p < \infty$, $1 < r \neq p < \infty$. Let $P : X \to X$ and $S : X \to X$ be the direct sum of families $\{P_{k,m,i}\}$ and $\{S_{k,m,i}\}$ respectively. Since each $\tilde{X}(k,m,)$ is isometric to $\ell_p^{k^2m^2+m}$, X is isometric to $(\sum_{k,m=1}^{\infty} \ell_p^{k^2m^2+m})_r$.

Our goal is proving that $\overline{S_r(X)}$ is not proximinal in $L(X)$ by showing that $P + S$ does not have a best approximant in $\overline{S_r(X)}$.

<u>Proposition 3</u>. $d(P + S , \overline{S_r(X)}) = 1$.

(Proof). It suffices to show that $d(P + S , S_r(X)) = 1$. For a fixed n, define an operator S_n on X so that S_n is the direct sum of operators $T_{k,m,i}$ on $X(k,m,i)$ where $T_{k,n,i} = S_{k,n,i}$ and $T_{k,m,i} = 0$ if $m \neq n$.

From the definition of S_n, it is easy to see that the range of $S_n|\tilde{X}(k,n)$ (S_n restricted to $\tilde{X}(k,n)$) is isometric to ℓ_p^n. Since ℓ_p^n is isomorphic to ℓ_r^n, $S_n|\tilde{X}(k,n)$ factors through ℓ_r^n.

Thus it follows that S_n factors through $\ell_r = (\sum \ell_r^n)_r$, ℓ_r-sum of infinitely many copies of ℓ_r^n , and hence $\tilde{S}_N = \sum_{n=1}^{N} S_n \in S_r(X)$.

Now we claim that $\|P + S - \tilde{S}_N\| \leq 1 + (\frac{1}{N})^{1/p}$. To prove the claim, observe that

$$\|P + S - \tilde{S}_N\| = \sup\{\|(P + S - \tilde{S}_N)|X(k,m,i)\|\}$$

where the supremum is taken over all k , $m = 1,2,3,\ldots,$ and $1 \leq i \leq m$, and

$$\|(P + S - \tilde{S}_N)|X(k,m,i)\| = \begin{cases} \|P_{k,m,i}\| & \text{if } m \leq N \\ \|P_{k,m,i} + S_{k,m,i}\| & \text{if } m > N \end{cases} .$$

To prove that $\|P_{k,m,i} + S_{k,m,i}\| \leq 1 + (\frac{1}{N})^p$ for all $m > N$, let $B = \{(1,0)\}$ and $A = \{(s,t) \in \Omega_{k,m,i} : 1 \leq s , t \leq k\}$. For $f \in X(k,m,i)$, let $f_1 = f|_B$ and $f_2 = f - f_1$. Then $P_{k,m,i}f_1 = 0$ and $S_{k,m,i}f_2 = 0$.

Since $\|P_{k,m,i}\| = 1$ and $P_{k,m,i}f_2$ is constant on each column of $\Omega_{k,m,i}\setminus B$, $\|1_A P_{k,m,i}f_2\| = (\frac{1}{m})^{1/p}\|P_{k,m,i}f_2\| \leq (\frac{1}{m})^{1/p}\|f\|$, where 1_A is the indicator function of A .

Since $S_{k,m,i}f_1$ and $(1 - 1_A)P_{k,m,i}f_2$ have disjoint supports, $\|S_{k,m,i}\| = 1$ and $\|P_{k,m,i}\| = 1$, we have

$$\|S_{k,m,i}f_1 + (1-1_A)P_{k,m,i}f_2\| = (\|S_{k,m,i}f_1\|^p + \|(1-1_A)P_{k,m,i}f_2\|^p)^{1/p} \le \|f\| .$$

Hence, for $f \in X(k,m,i)$,

$$\begin{aligned}\|(P_{k,m,i} + S_{k,m,i})f\| &\le \|P_{k,m,i}f_1\| + \|1_A P_{k,m,i}f_2\| + \\ &\quad \|S_{k,m,i}f_1 + (1-1_A)P_{k,m,i}f_2\| + \|S_{k,m,i}f_2\| \\ &\le 0 + (\tfrac{1}{m})^{1/p}\|f\| + \|f\| + 0 \\ &= (1 + (\tfrac{1}{m})^{1/p})\|f\| .\end{aligned}$$

Thus for $m > N$, $\|P_{k,m,i} + S_{k,m,i}\| \le 1 + (\frac{1}{m})^{1/p} \le 1 + (\frac{1}{N})^{1/p}$ and the proof of the claim is complete.

Since $\tilde{S}_N \in S_r(X)$, by letting $N \to \infty$, we infer that $d(P + S , S_r(X)) \le 1$.

To prove the reverse inequality, notice that $P + S$ restricted to the $\operatorname{span}\{\sum_{s=1}^{km} e_{s,t} : t = 1,2,\ldots k\}$, which is isometric to ℓ_p^k , acts as the identity operator. Thus $P + S$ acts as the identity operator on and isometric copy of $(\sum_{k=1}^{\infty} \ell_p^k)_r$. So by lemma 2 we have $d(P + S , S_r(X)) \ge 1$ and the proof of proposition 3 is complete.

<u>Lemma 4</u>. Let $Q_{k,m,i}$ be the projection of X onto $\operatorname{span}\{\sum_{s=1}^{k} e_{s,t}(k,m,i) , \sum_{s=k+1}^{km} e_{s,t}(k,m,i) \}_{i=1}^{k}$. If T is in $\overline{S_r(X)}$, then $\limsup_{m\to\infty} \sup_k \min_{1\le i\le m} \|m^{1/p} Q_{k,m,i}Te(k,m,i)\| = 0$.

(Proof). Obviously it suffices to prove the lemma for T in $S_r(X)$ with $\|T\| < 1$. If the statement is false then there is $\delta > 0$ such that $\sup_k \min_{1 \le i \le m} \|m^{1/p} Q_{k,m,i} Te(k,m,i)\| > 2\delta$ for infinitely many m. Fix such an m and choose $k = k(m)$ so that

$$\|m^{1/p} Q_{k,m,i} Te(k,m,i)\| > \delta \quad \text{for all} \quad i = 1,2,\dots,m .$$

The map $\psi : \ell_p^m \to \operatorname{span}\{e(k,m,i)\}_{i=1}^m$ defined by $e_i \to m^{1/p} e(k,m,i)$ $(i = 1,2,\dots,m)$ is an isometry onto where $\{e_i\}_{i=1}^m$ is the unit vector basis of ℓ_p^m.

Since vectors in $\{Q_{k,m,i} Te(k,m,i)\}_{i=1}^m$ have disjoint supports, $\operatorname{span}\{Q_{k,m,i} Te(k,m,i)\}_{i=1}^m$ is isometric to a subspace of ℓ_p^m and hence the map S defined by $e_i \to Q_{k,m,i} T\psi(e_i)$ can be viewed to have values in ℓ_p^m. Since $\|Se_i\| = \|m^{1/p} Q_{k,m,i} Te(k,m,i)\| > \delta$ for each $i = 1,2,\dots,m$ and since L_p has cotype p with constant 1 [31; p. 73], we have

$$\operatorname*{Averag}_{\pm}\{\|\sum_{i=1}^m \pm Se_i\|\} \ge (\|\sum_{i=1}^m \|Se_i\|^p)^{1/p} \ge \delta m^{1/p} .$$

Since $\|S\| \le 1$, we conclude by the Figiel-Johnson lemma that there exist positive constants $c = c(p,r,\delta)$ and $\alpha = \alpha(p,\alpha)$ such that

$$S_r(S) \ge cm^{\alpha} .$$

Since $S = (\sum_{i=1}^n Q_{k,m,i}) T\psi$ and ψ is an isometry, we get that

$S_r(S) = S_r(\sum_{i=1}^{n} Q_{k,m,i}T) < S_r(T)$ and so $S_r(T) > S_r(S) > cm^{\alpha}$ for infinitely many m. Since $T \in S_r(X)$ this is a contradiction.

Theorem 5. $P + S$ has no best approximant in $\overline{S_r(X)}$.

(Proof). Suppose $P + S$ has a best approximant T in $\overline{S_r(X)}$, then by proposition 3 $\|P + S - T\| = 1$. In view of lemma 4, we don't lose anything by assuming, for notational convenience, that for all k and all m,

$$\|m^{1/p} Q_{k,m,i} Te(k,m,1)\| < \frac{1}{4}.$$

In the sequel we will write $Q_{k,m,1}$ as $Q_{k,m}$. So the above inequality is

$$\|Q_{k,m} Te(k,m,1)\| < \frac{1}{4}\left(\frac{1}{m}\right)^{1/p}. \tag{*}$$

For each k, m and $\varepsilon = (\varepsilon_i)_{i=1}^{k}$ with $\varepsilon_i = \pm 1$, we consider a Rademacher function $r_{\varepsilon,k,m}$ in the range of $Q_{k,m}$ defined by $r_{\varepsilon,k,m} = \sum_{t=1}^{k} \varepsilon_t \sum_{s=1}^{km} e_{s,t}(k,m,1)$. Since the rank of $Q_{k,m}$ is $2k$, by the Figiel-Johnson lemma and an approximation argument, we get that for any $\delta > 0$, there is a $k(\delta)$ such that

$$\frac{1}{2^k} \sum_{\varepsilon \in \{-1,1\}^k} \|Q_{k,m} T r_{\varepsilon,k,m}\| < \delta \tag{**}$$

for all $k \geq k(\delta)$ and all m .

Indeed, there exists $\tilde{T}$ in $S_r(X)$ such that $\|T - \tilde{T}\| < \frac{\delta}{2}$. So $\|Q_{k,m}T - Q_{k,m}\tilde{T}\| < \frac{\delta}{2}$ and $S_r(Q_{k,m}\tilde{T}) \leq S_r(\tilde{T})$ for any k and m . If (**) is violated then we get that

$$\frac{1}{2^k} \sum_{\varepsilon\in\{-1,1\}^k} \|Q_{k,m}\tilde{T}r_{\varepsilon,k,m}\| > \frac{\delta}{2}$$

for infinitely many k and some $m = m(k)$. Notice that the map $e_i \to k^{1/p} \sum_{s=1}^{km} e_{s,i}(k,m,1)$ defines an isometry from ℓ_p^k onto $\operatorname{span}\{\sum_{s=1}^{km} e_{s,i}(k,m,1)\}_{i=1}^k$. Since the range of $Q_{k,m}$ is isometric to ℓ_p^{2k} , applying the Figiel-Johnson lemma as in lemma 4 we get that

$$S_r(\tilde{T}) \geq S_r(Q_{k,m}\tilde{T}) \geq ck^{\alpha}$$

for infinitely many k . This is a contradiction and so (**) is ture.

For a fixed vector $x \in X(k,m,1)$ with the expansion $x = \sum x_{s,t}e_{s,t} + x_0e(k,m,1)$ with respect to the natural basis for $X(k,m,1)$, we have $\langle r_{\varepsilon,k,m} , x\rangle = \int r_{\varepsilon,k,m}x \, d\mu = \frac{1}{k^2m} \sum_{t=1}^{k} \varepsilon_t \sum_{s=1}^{km} x_{s,t}$ and so

$$\left(\underset{\varepsilon}{\text{Average}}|\langle r_{\varepsilon,k,m} , x\rangle|^2\right)^{\frac{1}{2}}$$

$$= \left(\frac{1}{2^k} \sum_{\varepsilon\in\{-1,1\}^k} |\langle r_{\varepsilon,k,m} , x\rangle|^2\right)^{\frac{1}{2}}$$

$$= \frac{1}{k^2m}\left(\frac{1}{2^k} \sum_{\varepsilon\in\{-1,1\}^k} \left|\sum_{t=1}^{k} \varepsilon_t \sum_{s=1}^{km} x_{s,t}\right|^2\right)^{\frac{1}{2}}$$

$$= \frac{1}{k^2m}\left(\sum_{t=1}^{k} \left|\sum_{s=1}^{km} x_{s,t}\right|^2\right)^{\frac{1}{2}}$$

$$\leq \frac{1}{k^2m}\left(\sum_{t=1}^{k} \left(\sum_{s=1}^{km} |x_{s,t}|^2\right) km\right)^{\frac{1}{2}} \text{ by Hölder's inequality}$$

$$= \frac{1}{\sqrt{k}}\left(\sum_{t=1}^{k} \sum_{s=1}^{km} \frac{1}{k^2m} |x_{s,t}|^2\right)^{\frac{1}{2}} = \frac{1}{\sqrt{k}}\|x\|_2$$

where $\|x\|_2$ is the L_2-norm of x .

It is easy to see that $\langle r_{\varepsilon,k,m} , x\rangle = \langle r_{\varepsilon,k,m} , Q_{k,m}x\rangle$ and hence from the above inequality we have

$$\left(\underset{\varepsilon}{\text{Average}}|\langle r_{\varepsilon,k,m} , Q_{k,m}x\rangle|^2\right)^{1/2} \leq \frac{1}{\sqrt{k}}\|x\|_2 \qquad (***)$$

Now we will show that $\|P + S - T\| > 1$ to finish the proof of theorem 5.

For each positive integer n , choose a positive integer $k(n)$ such that $k(n+1) > k(n)$ and for $k = k(n)$ the left hand sides of (**) and (***) are smaller than $(4n)^{-1}$ and $(4n^2\|x\|_2)^{-1}$ respectively.

Then we have

$$\|Q_{k(n),m}Tr_{\varepsilon,k(n),m}\| < \frac{1}{n}$$

and $|\langle r_{\varepsilon,k(n),m} , Q_{k(n),m}x\rangle| < \frac{1}{n}\|x\|_2$ for all m and some $\varepsilon = \varepsilon(n,m)$.

If we set $g = r_{\varepsilon,k,m} + \lambda e(k,m,1)$, $\lambda > 0$, then

$$\|g\| = (1 + \lambda^p \tfrac{1}{m})^{1/p} \quad \text{and} \quad \|Q_{k,m}(P + S - T)g\| =$$

$$\|r_{\varepsilon,k,m} - Q_{k,m}Tr_{\varepsilon,k,m} + \lambda Q_{k,m}(S - T)e(k,m,1)\|$$

$$> \|r_{\varepsilon,k,m} + \lambda Q_{k,m}(S - T)e(k,m,1)\| - \|Q_{k,m}Tr_{\varepsilon,k,m}\|$$

$$> \|r_{\varepsilon,k,m} + \lambda Q_{k,m}(S - T)e(k,m,1)\|_2 - \|Q_{k,m}Tr_{\varepsilon,k,m}\|$$

$$> (1 + \lambda^2\|Q_{k,m}(S - T)e(k,m,1)\|_2^2 - \lambda\langle r_{\varepsilon,k,m} , Q_{k,m}(S - T)e(k,m,1))\rangle)^{1/p}$$

$$- \|Q_{k,m}Tr_{\varepsilon,k,m}\| .$$

Since $|Q_{k,m}Te(k,m,1)\| < \frac{1}{4}(\frac{1}{m})^{1/p}$ by (*) , by Chebyshev's inequality we have

$$\mu(\{(s,t) \in \Omega_{k,m,1} : |Q_{k,m}Te(k,m,1)| > \tfrac{1}{2}\})$$

$$< 2^p\|Q_{k,m}Te(k,m,1)\|^p < 2^p \cdot \frac{1}{4^p} \cdot \frac{1}{m} < \frac{1}{4m} .$$

Since $Q_{k,m}Se(k,m,1) = 1_A$, the indicator function of A , if we set

$C = \{(s,t) \in A : |Q_{k,m}Te(k,m,1)| < \frac{1}{2}\}$ then $\mu(C) > \mu(A) - \frac{1}{4m} = \frac{3}{4m}$ and hence we have

$$\|Q_{k,m}(S - T)e(k,m,1)\|_2^2 > \int_C |1 - Q_{k,m}Te(k,m,1)|^2 d\mu$$

$$> \int_C (\tfrac{1}{2})^2 d\mu = \frac{3}{16}\,\frac{1}{m} \, .$$

Thus

$$\|P + S - T\| > \overline{\lim_n}\, \|Q_{k(n),m}(P + S - T)\|$$

$$> \left(\frac{1 + \lambda^2 \frac{3}{16} \cdot \frac{1}{m}}{1 + \lambda^p \frac{1}{m}} \right)^{\frac{1}{p}}$$

$$> 1 \quad \text{for small } \lambda \, .$$

This is a contradiction and the proof of theorem 5 is complete.

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