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DYNAMIC STABILITY OF A CYLINDRICAL SHELL IN AN ACOUSTIC MEDIUM

DISSERTATION

Presented in Partial Fulfillment of the Requirements for the Degree Doctor of Philosophy in the Graduate School of The Ohio State University

by

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* * * * *

The Ohio State University
1974

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*L/ma = ∞ represents results from Deng-Popelar in reference [2].

LIST OF SYMBOLS

a	=	mean radius of the circular cylindrical shell
c	=	sound velocity of the fluid medium
c_s	=	sound velocity of the solid shell material
C_f	=	damping effect of the fluid medium
C_{fmn}	=	damping effect of the fluid medium in the perturbed motion
$D_{mn}^{(k)}$	=	damping effect of the fluid medium of order k
E	=	modulus of Elasticity in tension and in compression of the shell material
h	=	thickness of the shell
$H_n^{(2)}(k_m r)$	=	Hankel's function of a second kind of order n
j	=	$-\sqrt{-1}$
$J_n(k_m r)$	=	Bessel's function of order n
K	=	$Eh/(1-\nu^2)$
$K_n(\|k_m\|r)$	=	n-th order modified Bessel function of the second kind
k	=	integer number
k_m	=	x_m/a
k_o	=	x_o/a
L	=	half-wave length of the shell

m = axial integer wave-number

M_f = virtual mass or "added mass" due to the fluid medium

$M_{mn}^{(k)}$ = virtual mass or effective mass attached to the vibrating shell, owing to the fluid medium

n = circumferential wave-number

$N_n(k_m r)$ = Neumann's function of order n and argument $(k_m r)$

N_x^1 = axial membrane force per unit length

$N_{x\theta}^1$ = membrane shear force per unit length

N_θ^1 = circumferential membrane force per unit length

p = static pressure

Q = acoustic pressure

Q_{mn} = Fourier coefficient of the acoustic pressure Q

q = dynamic or vibratory pressure

r = radial coordinate

s = nondimensional axial coordinate

t = time

u = axial displacement

u_{mn} = Fourier coefficient of the axial displacement u

v = circumferential displacement

v_{mn} = Fourier coefficient of the circumferential displacement v

w = radial displacement

w_{mn} = Fourier coefficient of the radial displacement w

x = axial coordinate

x_o = argument of the Bessel function when $m = 0$

x_m = argument of the Bessel function when $m \geq 1$

α = l/a

α_n = frequency ratios $\omega/2\omega_n$

β = $K/(a^3\delta^{\frac{1}{2}})$

γ = dimensionless dynamic pressure

δ = defined by eq.(3.12)

Θ = circumferential coordinate

Θ_{mn} = acoustic resistance for the mode (m,n)

$\Theta_{mn}^{(k)}$ = acoustic resistance for the mode (m,n) and of order k

Λ_1 = nondimensional natural frequency

$Z_{mn}(k_m r)$ = acoustic impedance of the fluid medium $= Q_{mn}/\dot{w}_{mn}$

ν = Poisson's ratio

ρ = density of the fluid medium

ρ_s = density of the solid shell material

ψ = phase angle

ϕ = velocity potential

ϕ_{mn} = Fourier coefficient of the velocity potential

χ_{mn}	= acoustic reactance
$\chi_{mn}^{(k)}$	= acoustic reactance of order k
ω	= forcing frequency
ω_o	= natural frequency of the breathing motion of an infinitely long circular cylindrical shell
Ω_1	= natural frequency of the shell
$(\dot{\ }),(\ddot{\ })$	= $\frac{d}{dt}(\ ),\ \frac{d^2}{dt^2}(\ )$ respectively The dots indicate differentiation with respect to time t.
$(\ )'$	= $\frac{d}{d(k_m r)}(\ )$
$\bar{p}_a$	= acoustic pressure vector
$\bar{u}$	= displacement vector
$\bar{y}$	= normal displacement vector
$\underset{\sim}{A}$	= stiffness matrix
$\underset{\sim}{A}_1$	= transformed stiffness matrix
$\underset{\sim}{C}$	= defined by eq.(4.17)
$\underset{\sim}{E}$	= defined by eq.(4.18)
$\underset{\sim}{F}$	= defined by eq.(4.17)
$\underset{\sim}{G}$	= modal matrix of the free vibration
$\underset{\sim}{H}_{12}$, $\underset{\sim}{H}_{11}$, $\underset{\sim}{H}_{22}$	= defined by eq.(4.33)
I	= defined by eq.(4.17)

$\underset{\sim}{M}$ = Inertia Matrix

$\underset{\sim}{P}$ = defined by eq.(4.18)

$\underset{\sim}{P}_1$ = Initial stress matrix

INTRODUCTION

The parametric buckling of a thin circular cylindrical shell vibrating under the action of a harmonically varying internal pressure has been investigated by Yao [1]* in vacuo and by Deng and Popelar [2] in an acoustic medium. In both studies, the breathing motion is considered as the primary motion of the shell. As a consequence the initial stresses are only functions of time and not of the coordinates. Yao investigated the parametric buckling of a finite circular cylindrical shell under various boundary conditions while Deng and Popelar dealt with an infinitely long circular cylindrical shell submerged in an acoustic medium.

The shell treated in ref. [2] is an infinitely long circular cylindrical shell with generators remaining straight, i.e., a plane-strain analysis. The shell is considered to be freely suspended in a compressible fluid medium and to be excited by a pressure which varies harmonically with time. A dynamic stability criterion based upon infinitesimal perturbations of the breathing motion and plane-strain of the shell was developed.

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*A number in brackets refers to the Bibliography.

In this study the generators of the shell are free to deform upon infinitesimal perturbations of the breathing motion. A stability criterion, based upon the breathing motion and a three dimensional deformation of the shell, is presented. In so far as the governing equations of the shell are concerned, the approach used is basically the one outlined by Bolotin [3]. The modal matrix of the free vibrations in vacuo is used to reduce the governing equations to the standard form.

The governing equations of the deformation of a thin circular cylindrical shell are developed by Washizu [4] on the basis of the Goldenveiser-Novozhilov theory [5] and are given by Leissa [6]. These equations are derived using the Kirkhoff-Love hypothesis. In contrast to the other sets of governing equations including the prestress forces as developed by Sanders [7], Flügge [8], Timoshenko [9] and by Herrmann-Armenakas [10], the Washizu equations are symmetric. The differences between these theories are discussed in Chapters II and III of Leissa's monograph [6].

A thin circular cylindrical shell, excited by a sinusoidal internal pressure, is said to be stable or unstable if its superimposed displacements upon the primary motion experience temporal decay or growth (respectively) when subjected to infinitesimal disturbances. The parametric buckling analysis of the circular cylindrical shell yields a set of Hill-Mathieu equations whose properties have been studied extensively in the past [11]. Only periodic solutions of period $4\pi/\omega$ are considered, where ω is the forcing frequency. The boundaries of the principal instability regions are found for some range of the shell and fluid parameters. The results will be compared with those of Deng and Popelar for a particular steel shell [2]. The coefficients of the Hill-Mathieu equations encountered in this work are frequency dependent because of the fluid effects. These types of equations have been studied by Stevens [12] and Deng and Popelar [2].

The breathing mode motion of a long circular cylindrical shell is understood to be excited by an internal pressure $p + q \cos \omega t$, where p is the static pressure and q is the dynamic or vibratory pressure. Under the breathing mode motion, the shell expands and contracts according to the sense of the vibratory pressure in such manner that its generators remain straight and its circular shape is maintained.

CHAPTER I

SHELL EQUATIONS

The shell geometry and its governing equations are defined following the Washizu theory of thin circular cylindrical shells [4].

As for all the major classical theories of thin shells, the Washizu theory of the thin circular cylindrical shells are based on the following assumptions:

1. The material of the circular cylindrical shell is homogeneous, isotropic and obeys Hooke's law.

2. The shell thickness is small in comparison with other dimensions of the shell and its radii of curvature:

$$\frac{h}{R_1} << 1 \qquad (1.1)$$

3. The displacements are small and infinitesimal. They are such that the assumption of smallness of the slopes is valid. The changes in curvature and twist are governed by linear relationships.

4. The stress component normal to the middle surface is

small compared to the other stress components and may be neglected in the stress-strain relationships

$$\sigma_n = 0 \tag{1.2}$$

5. The normal to the reference surface of the shell remains normal to it and undergoes no change in length during deformation:

$$\epsilon_{nn} = \gamma_{xn} = \gamma_{\theta n} = 0 \tag{1.3}$$

6. Rotary inertia and transverse shear in the shell are negligible.

The curvilinear coordinates x and θ are lines of principal curvatures of the middle surface of the shell before deformation as shown in Fig. 1 and Fig. 2. The strain-displacement relationships in cylindrical coordinates are given as:

$$\epsilon_x = \frac{\partial u}{\partial x}$$

$$\epsilon_\theta = \frac{1}{a} \left(\frac{\partial v}{\partial \theta} + w \right)$$

$$\gamma_{\theta x} = \frac{1}{a}\frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial x}$$

$$\xi = -\frac{\partial w}{\partial x}$$

$$\psi = \frac{1}{a}\left(v - \frac{\partial w}{\partial \theta}\right)$$

$$\kappa_x = -\frac{\partial^2 w}{\partial x^2} \qquad (1.4)$$

$$\kappa_\theta = -\frac{1}{a}\left(\frac{\partial^2 w}{\partial \theta^2} - \frac{\partial v}{\partial \theta}\right)$$

$$\kappa_{x\theta} = -\frac{1}{a}\left(\frac{\partial^2 w}{\partial x \partial \theta} - \frac{\partial v}{\partial x}\right)$$

The stress resultants and stress-moments are defined in the sense of Novozhilov [5], (Fig. 3 and Fig. 4):

$$N_x = \frac{Eh}{(1+\nu^2)}(\varepsilon_x + \nu\varepsilon_\theta)$$

$$N_\theta = \frac{Eh}{(1+\nu^2)}(\varepsilon_\theta + \nu\varepsilon_x)$$

$$N_{x\theta} = \frac{Eh}{2(1+\nu)}\gamma_{x\theta}$$

$$N_{\theta x} = \frac{Eh}{2(1+\nu)} \left(\gamma_{x\theta} + \frac{h^2}{6a} \kappa_{x\theta}\right)$$

$$M_{x\theta} = M_{\theta x} = \frac{Eh^3}{12(1+\nu)} \kappa_{x\theta}$$

$$M_x = \frac{Eh^3}{12(1-\nu^2)} \left(\kappa_x + \nu \kappa_\theta \right) \tag{1.5}$$

$$M_\theta = \frac{Eh^3}{12(1-\nu^2)} \left(\kappa_\theta + \nu \kappa_x \right)$$

The Washizu matrix operator is defined by Leissa* [6] as:

$$[L] = [L_{D-M}] + \frac{h^2}{12a^2}[L_{MOD}] + \frac{1}{K}[L_1] \tag{1.6}$$

where:

$[L_{D-M}]$ is the Donnell-Mushtari matrix operator, Leissa's eq.(2.7).

---

*The governing equations of the motion are those given by Leissa [6] in his monograph of the "Vibration of Shells".

$[L_{MOD}]$ is the modifying matrix operator, Leissa's eq.(2.9a).

$[L_1]$ is the matrix operator containing the additional terms which account for the initial stresses, Leissa's eq.(3.101c).

$$K = \frac{Eh}{1-\nu^2} \tag{1.7}$$

In the matrix operator [L], all terms in the two first rows have been multiplied by "-1" so that one can obtain a symmetric matrix when the space variation is eliminated.

The Donnell-Mushtari operator is:

$$[L_{D-M}] = \begin{bmatrix} [-\frac{\partial^2}{\partial s^2} - \frac{(1-\nu)}{2}\frac{\partial^2}{\partial \theta^2} + \rho_s\frac{(1-\nu^2)}{E}a^2\frac{\partial^2}{\partial t^2}] & -\frac{(1+\nu)}{2}\frac{\partial^2}{\partial s\partial\theta} & -\nu\frac{\partial}{\partial s} \\ -\frac{(1+\nu)}{2}\frac{\partial^2}{\partial s\partial\theta} & [-\frac{(1-\nu)}{2}\frac{\partial^2}{\partial s^2} - \frac{\partial^2}{\partial \theta^2} + \rho_s\frac{(1-\nu^2)}{E}a^2\frac{\partial^2}{\partial t^2}] & -\frac{\partial}{\partial\theta} \\ \nu\frac{\partial}{\partial s} & \frac{\partial}{\partial\theta} & 1+k\nabla^4+\rho_s\frac{(1-\nu^2)}{E}a^2\frac{\partial^2}{\partial t^2} \end{bmatrix} \tag{1.8}$$

where:

$$\nabla^4 = \nabla^2\nabla^2$$

$$\nabla^2 = \frac{\partial^2}{\partial s^2} + \frac{\partial^2}{\partial \theta^2} \tag{1.9}$$

$$\frac{\partial}{\partial s} = a\,\frac{\partial}{\partial x}$$

The modifying Washizu operator:

$$[L_{MOD}] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -2(1-\nu)\frac{\partial^2}{\partial s^2} - \frac{\partial^2}{\partial \theta^2} & (2-\nu)\frac{\partial^3}{\partial s^2 \partial \theta} + \frac{\partial^3}{\partial \theta^3} \\ 0 & -(2-\nu)\frac{\partial^3}{\partial s^2 \partial \theta} - \frac{\partial^3}{\partial \theta^3} & 0 \end{bmatrix} \tag{1.10}$$

The prestress Washizu operator is:

$$[L_1] = \begin{bmatrix} -\Delta & 0 & 0 \\ 0 & -\Delta + N^1_\theta & \left[-2N^1_\theta \frac{\partial}{\partial\theta} - 2N_{x\theta}\frac{\partial}{\partial s} - \frac{\partial N^1_{x\theta}}{\partial s}\right] \\ 0 & \left[2N^1_\theta \frac{\partial}{\partial\theta} + 2N^1_{x\theta}\frac{\partial}{\partial s} + \frac{\partial N^1_{x\theta}}{\partial s}\right] & -\Delta + N^1_\theta \end{bmatrix} \tag{1.11}$$

where

$$\Delta = \frac{\partial}{\partial s}\left(N^1_x \frac{\partial}{\partial s}\right) + N^1_\theta \frac{\partial^2}{\partial\theta^2} + N^1_{x\theta}\frac{\partial^2}{\partial s\,\partial\theta} + \frac{\partial}{\partial s}\left(N^1_{x\theta}\frac{\partial}{\partial\theta}\right) \tag{1.12}$$

When the initial motion is axisymmetric and of breathing mode, i.e., $N^1_{x\theta} = 0$, N^1_x and N^1_θ independent of the coordinates s and θ, then the Washizu prestress operator reduces to

$$[L_1] = \begin{bmatrix} -N_x^1 \frac{\partial^2}{\partial s^2} - N_\theta^1 \frac{\partial^2}{\partial \theta^2} & 0 & 0 \\ 0 & -N_x^1 \frac{\partial^2}{\partial s^2} - N_\theta^1 \frac{\partial^2}{\partial \theta^2} + N_\theta^1 & -2N_\theta^1 \frac{\partial}{\partial \theta} \\ 0 & 2N_\theta^1 \frac{\partial}{\partial \theta} & -N_x^1 \frac{\partial^2}{\partial s^2} - N_\theta^1 \frac{\partial^2}{\partial \theta^2} + N_\theta^1 \end{bmatrix} \tag{1.13}$$

The equations of the perturbed motion are then given in matrix form as:

$$[L] \begin{Bmatrix} u \\ v \\ w \end{Bmatrix} = \frac{a^2}{K} [S] \begin{Bmatrix} p_x \\ p_\theta \\ p_r \end{Bmatrix} \tag{1.14}$$

where

$$[S] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{1.15}$$

The radial forcing pressure p_r is positive outwards, i.e. consistent with the radial displacement w. The tangential components p_x, p_θ of the lateral pressure are neglected when infinitesimal deflections of the shell midsurface are considered.

The membrane forces N_x^i and N_θ^i are the stress resultants in the breathing motion which is assumed to be the primary motion in this analysis.

Eq.(1.14), in which the operator [L] is defined in eq.(1.6), represent the superimposed motion or the perturbed motion of the shell. They result from the perturbation of the initial motion. The displacements u, v, w are the additional displacements due to infinitesimal disturbances of the original motion.

To obtain the equations of motion of the free vibrations of the shell in vacuo, one has to drop out the prestress matrix operator $[L_1]$ in eq.(1.6) and to set $p_r \equiv 0$. It follows that:

$$[L_{D-M}] \begin{Bmatrix} u \\ v \\ w \end{Bmatrix} + \frac{h^2}{12a^2} [L_{MOD}] \begin{Bmatrix} u \\ v \\ w \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \tag{1.16}$$

The purpose of this work is to investigate the parametric stability of a circular cylindrical shell vibrating in an acoustic medium. The fluid effects are reflected by the generalized lateral pressure p_r in eq.(1.14). They should be known beforehand and included in the governing equations of the perturbed motion.

Therefore the fluid effects upon a vibrating shell are reviewed in Chapter II and the prestress quantities are established in Chapter III before the solution of the perturbed motion is sought.

CHAPTER II

FLUID EQUATIONS

The fluid equations are developed in cylindrical coordinates in the foregoing following Junger [13, 14]. This topic is of great importance when one considers the interaction of the fluid on the vibration of a thin shell oscillating in an acoustic medium. The fluid and the shell will be connected by the boundary conditions at their interfaces.

The infinitely long thin circular cylindrical shell is freely suspended in an homogeneous, compressible and inviscid fluid of an infinite extent. As a consequence the fluid medium will be governed by the linear Laplace equation whose only solution of interest here is an outgoing wave (see Morse and Ingard [15]).

In cylindrical coordinates the Laplace equation is:

$$\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r} \frac{\partial \phi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \theta^2} + \frac{\partial^2 \phi}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2} \qquad (2.1)$$

where ϕ and c are respectively the velocity potential and the sound velocity of the fluid medium.

The velocity of the fluid in cylindrical coordinates is the negative of the gradient of the velocity potential:

$$\vec{v} = - \vec{\text{grad}}\ \phi \tag{2.2}$$

In component form; the velocity is:

$$\begin{aligned} v_1 &= - \frac{\partial \phi}{\partial x} \\ v_2 &= - \frac{1}{r} \frac{\partial \phi}{\partial \theta} \\ v_3 &= - \frac{\partial \phi}{\partial r} \end{aligned} \tag{2.3}$$

The axial and circumferential components of the fluid velocity vector may be neglected when the displacements of the fluid and shell at their interface are small and infinitesimal. In this case the velocity of the fluid reduces to the radial component:

$$v_3 = - \frac{\partial \phi}{\partial r} \tag{2.4}$$

It follows that the boundary conditions of the fluid medium at the interface and at infinite become:

a. The radial velocity of the fluid particle is equal to the radial velocity of the shell particle at the point of contact, eq.(2.4):

$$\dot{W} = - \left.\frac{\partial \phi}{\partial r}\right|_{r=a} \tag{2.5}$$

b. The acoustic pressure exerted on the shell by the surrounding medium is the pressure at r=a along the normal of the shell. (Fig. 5).

$$Q = \rho \left.\frac{\partial \phi}{\partial t}\right|_{r=a} \tag{2.6}$$

c. The radial velocity of the fluid at infinity must vanish so that its kinetic energy remains finite there.

$$- \left.\frac{\partial \phi}{\partial r}\right|_{r=\infty} = 0 \tag{2.7}$$

The displacement patterns of an infinitely long circular cylindrical shell are those assumed by Leissa* [6]:

$$u = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} u_{mn} \cos n\theta \cos \frac{m\pi x}{L}$$

$$v = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} v_{mn} \sin n\theta \sin \frac{m\pi x}{L}$$

$$w = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} w_{mn} \cos n\theta \sin \frac{m\pi x}{L} \tag{2.8}$$

These displacements are such that

a. They are periodic of period 2π and continuous in the circumferential direction θ.

b. The origin of the circumferential coordinate θ is chosen at a particular point so that the circumferential phase angle is zero.

c. The form of solution eq. 2.8 assumes that the time and spatial variables are separable, giving rise to normal modes executing simple harmonic motion,

---

*Leissa's trial displacements for an infinitely long circular shell are defined in section 2.2 of reference [6] on page 37.

the period and phase of the motion being the same for all points on the shell.

d. The displacement eq. 2.8 is issued from a complete Fourier Series in the variable x. But here only one portion of the solution has been used because of the linearity of the equations of motions which insures the factorization of terms containing x or s, θ and t out of each equation. The equations of motion must be satisfied for all values s, θ, t allowed to vary independently.

Incidentally these displacements also satisfy the boundary conditions of the simply supported cylindrical shell.

Because of the boundary condition eq.(2.5), the velocity potential ϕ is:

$$\phi = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} A_{mn}(t)\phi_{mn}(r) \cos n\theta \sin \frac{m\pi x}{L} \qquad (2.9)$$

Subsequently, the acoustic pressure becomes because of eq.(2.6):

$$Q = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} Q_{mn} \cos n\theta \sin \frac{m\pi x}{L} \qquad (2.10)$$

In free vibration the function $A_{mn}(t)$ has the form:

$$A_{mn}(t) = \tilde{A}_{mn} e^{j\omega t} \tag{2.11}$$

Inserting eq.(2.9) and eq.(2.11) into eq.(2.1), one obtains a Bessel equation after collecting terms of the same harmonic functions since the solution must be true for all x, θ, and the time, t:

$$\frac{\partial^2 \phi_{mn}}{\partial r^2} + \frac{1}{r}\frac{\partial \phi_{mn}}{\partial r} + \left(-\frac{n^2}{r^2} + k_m^2\right)\phi_{mn} = 0 \tag{2.12}$$

in which

$$k_m^2 = \frac{\omega^2}{c^2} - \left(\frac{m\pi}{L}\right)^2 \tag{2.13}$$

The solution of eq.(2.12) leading to an outgoing wave is the Hankel's function of the second kind [16]:

$$\phi_{mn}(k_m r) = B_{mn} H_n^{(2)}(k_m r) \tag{2.14}$$

where

$$H_n^{(2)}(k_m r) = J_n(k_m r) - jN_n(k_m r)$$

$$j = -\sqrt{-1}$$

$J_n(k_m r)$ is the Bessel functions of order n.

$N_n(k_m r)$ is the Neumann functions of order n.

$$k_o = \omega/c; \quad B_{mn} = \text{constant} \tag{2.15}$$

Definitions:

$$x_m = k_m a \qquad m = 1, 2, \ldots$$

$$x_o = k_o a \qquad m = 0 \tag{2.16}$$

Upon satisfying eq.(2.5), one obtains the velocity potential as:

$$\phi = \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \frac{-\dot{w}_{mn} H_n^{(2)}(k_m r)}{k_m \dfrac{dH_n^{(2)}(x_m)}{d(k_m r)}} \cos n\theta \sin \frac{m\pi x}{L} \tag{2.17}$$

And upon inserting eq.(2.17) into eq.(2.6), one gets the acoustic pressure as:

$$Q = -\rho \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \frac{\ddot{w}_{mn} H_n^{(2)}(k_m r)}{k_m \dfrac{dH_n^{(2)}(x_m)}{d(k_m r)}} \cos n\theta \sin \frac{m\pi x}{L} \tag{2.18}$$

which may also be written as:

$$Q = \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \frac{\rho c(-j) x_o \dot{w}_{mn} H_n^{(2)}(k_m r)}{x_m \dfrac{dH_n^{(2)}(k_m r)}{d(k_m r)}} \cos n\theta \sin \frac{m\pi x}{L} \tag{2.19}$$

where

$$\omega = x_o c/a$$

$$\dot{w}_{mn} = b_{mn} \sin \omega t + c_{mn} \cos \omega t \tag{2.20}$$

Therefore the acoustic impedance is defined for each mode of vibration as the ratio of the Fourier coefficients of the acoustic pressure and radial velocity:

$$Z_{mn}(x_m) = \frac{Q_{mn}}{\dot{w}_{mn}} \tag{2.21}$$

In complex form the acoustic impedance is:

$$Z_{mn}(k_m r) = \theta_{mn}(k_m r) + j\chi_{mn}(k_m r) \tag{2.22}$$

in which

$$\theta_{mn}(k_m r) + j\chi_{mn}(k_m r) = \frac{-jx_o H_n^{(2)}(k_m r)}{x_m \dfrac{dH_n^{(2)}(x_m)}{d(k_m r)}} \tag{2.23}$$

The resistance $\theta_{mn}(k_m r)$ and the reactance $\chi_{mn}(k_m r)$ are given as:

1. k_m is real:

$$\chi_{mn}(k_m r) = \frac{-x_o[J_n(k_m r)J_n'(x_m)+N_n(k_m r)N_n'(x_m)]}{x_m \left|\dfrac{dH_n^{(2)}(x_m)}{d(k_m r)}\right|^2} \tag{2.24}$$

and

$$\theta_{mn}(k_m r) = \frac{x_o[J_n(k_m r)N_n'(x_m)-N_n(k_m r)J_n'(x_m)]}{x_m \left(\dfrac{dH_n^{(2)}(x_m)}{d(k_m r)}\right)^2} \tag{2.25}$$

2. k_m is imaginary:

$$\chi_{mn}(k_m r) = \frac{-x_o K_n(|k_m| r)}{x_m\ K_n'(|x_m|)} \tag{2.26}$$

where $K_n(|k_m|r)$ is the modified Bessel function of order n, and

$$\theta_{mn}(k_m r) = 0 \tag{2.27}$$

Upon inserting eq.(2.20) and eq.(2.22) into eq.(2.18), one obtains the real part as:

$$\begin{aligned} Q_{mn} = & -\rho c\omega[b_{mn}\chi_{mn}(x_m) + c_{mn}\theta_{mn}(x_m)] \sin \omega t \\ & -\rho c\omega[c_{mn}\chi_{mn}(x_m) - b_{mn}\theta_{mn}(x_m)] \cos \omega t \end{aligned} \tag{2.28}$$

The above function may also be written as:

$$Q_{mn} = \frac{\rho c}{\omega} \chi_{mn}(x_m)\ddot{w}_{mn} + \rho c \Theta_{mn}(x_m)\dot{w}_{mn} \tag{2.29}$$

In the light of eq.(2.29), one may easily conclude that the fluid effects on the shell are of two types:

1. One is to add a virtual mass to the vibrating shell or to decrease its stiffness. It means that the virtual mass decreases the natural frequency of the shell when submerged in an acoustic medium:

$$M_f = \frac{\rho c}{\omega} \chi_{mn}(x_m) \tag{2.30}$$

2. Another is to introduce a damping effect of viscous type. It means that the energy radiated in an acoustic medium of infinite extent is not recoverable:

$$C_f = \rho c \; \Theta_{mn}(x_m) \tag{2.31}$$

The plots of the reactance and resistance functions are given by Junger in reference [13, 14], for L/ma = small and L/ma = ∞. The diagrams of these functions taken from Junger [13, 14] for the circumferential wave numbers (n=0,2,6) are illustrated in Fig. 6 through Fig. 9.

CHAPTER III

THE AXISYMMETRIC BREATHING MOTION

The solution of the perturbed equations of Chapter I requires one to determine the prestress forces in the primary motion beforehand. For this purpose one shall consider the axisymmetric breathing motion of an infinitely long circular cylindrical shell as the primary motion. Such a motion is independent of the circumferential angle θ.

The breathing motion of an infinitely long circular cylindrical shell is characterized by " $n=0$ ":

$$
\begin{aligned}
w &= w_o(t) \\
\epsilon_\theta &= \frac{w_o}{a}
\end{aligned}
\tag{3.1}
$$

It is also assumed that there is no motion in the axial direction during the breathing motion.

The equation of the breathing motion may be deduced from eq.(1.16) in which the prestress operator is set equal to zero:

$$\rho_s h\ddot{w}_o + \frac{K}{a^2} w_o = p_{ro} \tag{3.2}$$

in which p_{ro} is the total lateral pressure acting on the shell.

The radial displacement and the lateral pressure are positive outwards (Fig. 5). Since the physical quantities are independent of the axial direction, it follows:

$$N_x^i = \frac{K\nu}{a} w(t)$$
$$N_\theta^i = \frac{K}{a} w(t) \tag{3.3}$$

The lateral pressure consists of two elements:

1. The exciting internal pressure:

$$p + q \cos \omega t \tag{3.4}$$

2. The radiated pressure due to the fluid effects:

$$Q \tag{3.5}$$

Hence the total lateral pressure is (see Fig. 5):

$$p + q \cos \omega t - Q \tag{3.6}$$

Then eq.(3.2) becomes:

$$\rho_s h\ddot{w}_o + \frac{K}{a^2} w_o = p + q \cos \omega t - Q \tag{3.7}$$

The steady state solution of eq.(3.7) is obtained by assuming a trial solution of the form:

$$w_o = A + B \cos \omega t + C \sin \omega t \tag{3.8}$$

Upon substituting eq.(3.8) into eq.(3.7) and taking in account eq.(2.22), one may write after collecting terms:

$$\begin{aligned} A &= \frac{pa^2}{K} \\ a_1 B + b_1 C &= q \\ -b_1 B + a_1 C &= 0 \end{aligned} \tag{3.9}$$

in which

$$\begin{aligned} y_o &= \frac{\omega a}{c} \\ a_1 &= -\omega^2(\rho_s h + M_f) + \frac{K}{a^2} \\ b_1 &= \omega C_f \\ M_f &= \frac{\rho c}{\omega} \chi_{oo}(y_o) \\ C_f &= \rho c \, \theta_{oo}(y_o) \end{aligned} \tag{3.10}$$

The solution of eq.(3.9) is obtained using Cramer's method:

$$B = \frac{a_1 q}{\delta}$$
$$C = \frac{b_1 q}{\delta} \qquad (3.11)$$

in which

$$\delta = a_1^2 + b_1^2 \qquad (3.12)$$

Therefore the radial displacement w becomes:

$$w_o = \frac{pa^2}{K} + \frac{q}{\delta^{\frac{1}{2}}} \cos\ (\omega t-\psi) \qquad (3.13)$$

in which

$$\psi = \tan^{-1}(\frac{b_1}{a_1})$$
$$\delta^{\frac{1}{2}} = (a_1^2 + b_1^2)^{\frac{1}{2}} \qquad (3.14)$$

It follows that the membrane forces become:

$$N_\theta^1 = pa + \frac{Kq}{a\delta^{1/2}} \cos(\omega t - \psi)$$

(3.15)

$$N_x^1 = \nu N_\theta^1$$

CHAPTER IV

STABILITY OF MATHIEU'S EQUATIONS

The equations of the perturbed motion of a circular cylindrical shell vibrating under the action of an harmonically varying lateral pressure were developed in Chapter I. The prestress forces were established in Chapter III in the case of an axisymmetric breathing motion. The fluid effects which must be included as a forcing pressure in the perturbed motion may be found following a similar approach as in Chapter II.

To obtain the equations of the perturbed motion of the circular cylindrical shell, one has:

1. To insert the displacement pattern eq.(2.8) into eq.(1.14).
2. To replace the prestress membrane forces N_x^i and N_θ^i by their actual values, eq.(3.15) into eq.(1.14).
3. To introduce the lateral additional pressure into eq.(1.14) due to the fluid effects upon an infinitesimal perturbation of the breathing motion of the shell.

The radial acoustic pressure coefficient is easily obtainable from eq.(2.29) in terms of eq.(2.30), eq.(2.31), since it is proportional to the acceleration and velocity of the coefficient of the radial displacement, w_{mn}:

$$Q_{mn} = M_{fmn}\ddot{w}_{mn} + C_{fmn}\dot{w}_{mn} \tag{4.1}$$

where the quantities M_{fmn} and C_{fmn} are functions to be determined when the trial solution of the perturbed equations of motion is known. M_{fmn} is the reactance and C_{fmn} the resistance. In matrix form eq.(4.1) becomes:

$$\bar{p}_a = M_{fmn}\, \underset{\sim}{S}\, \ddot{\bar{u}} + C_{fmn}\, \underset{\sim}{S}\, \dot{\bar{u}} \tag{4.2}$$

in which

$$\bar{u} = \begin{Bmatrix} u_{mn} \\ v_{mn} \\ w_{mn} \end{Bmatrix}$$

$$\underset{\sim}{S} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{4.3}$$

$$\bar{p}_a = \begin{Bmatrix} p_{x1} \\ p_{\theta 1} \\ p_{r1} \end{Bmatrix}$$

$$\begin{Bmatrix} p_{x1} \\ p_{\theta 1} \\ p_{r1} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ Q_{mn} \end{Bmatrix}$$

Hence upon following the steps indicated above and making use of eq.(4.2), one obtains the governing equations of the perturbed motion of a circular cylindrical shell vibrating in an acoustic medium as:

$$\rho_s h \underset{\sim}{M} \ddot{\bar{u}} + \frac{K}{a^2} \underset{\sim}{A} \bar{u} + [\alpha p + \beta q \cos(\omega t - \psi)] \underset{\sim}{P}_1 \bar{u} = -M_{fmn} \underset{\sim}{S} \ddot{\bar{u}} - C_{fmn} \underset{\sim}{S} \dot{\bar{u}} \tag{4.4}$$

in which the inertia matrix $\underset{\sim}{M}$, the stiffness matrix $\underset{\sim}{A}$ and the initial stress matrix $\underset{\sim}{P}_1$ are defined as:

$$\underset{\sim}{M} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\underset{\sim}{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\underset{\sim}{P}_1 = \begin{bmatrix} p_{11} & 0 & 0 \\ 0 & p_{22} & p_{23} \\ 0 & p_{23} & p_{22} \end{bmatrix} \tag{4.5}$$

and the coefficients α and β are defined as:

$$\alpha = \frac{1}{a}$$

$$\beta = \frac{K}{a^3\delta^{\frac{1}{2}}}$$

The coefficients of the above matrices are given below. The coefficients of the stiffness matrix $\underset{\sim}{A}$ are:

$$a_{11} = \left(\frac{m\pi a}{L}\right)^2 + \frac{(1-\nu)}{2}\, n^2$$

$$a_{12} = -\frac{(1+\nu)}{2}\, n\left(\frac{m\pi a}{L}\right)$$

$$a_{13} = -\ \nu\left(\frac{m\pi a}{L}\right)$$

$$a_{21} = a_{12}$$

$$a_{22} = \frac{(1-\nu)}{2}\left(\frac{m\pi a}{L}\right)^2+n^2+\frac{h^2}{12a^2}[2(1-\nu)\left(\frac{m\pi a}{L}\right)^2+n^2]$$

$$a_{23} = n + \frac{h^2}{12a^2}\ [(2-\nu)n\left(\frac{m\pi a}{L}\right)^2+n^3]$$

$$a_{31} = a_{13}$$

$$a_{32} = a_{23}$$

$$a_{33} = 1 + \frac{h^2}{12a^2}\ [\left(\frac{m\pi a}{L}\right)^4+2n^2\left(\frac{m\pi a}{L}\right)^2+n^4] \qquad (4.6)$$

The coefficients of the initial prestress matrix $\underset{\sim}{P}_1$ are:

$$p_{11} = \left(\frac{m\pi a}{L}\right)^2\ \nu + n^2$$

$$p_{22} = 1 + n^2 + \left(\frac{m\pi a}{L}\right)^2\ \nu \qquad (4.7)$$

$$p_{23} = 2n$$

Eq.(4.4) is a system of three linear differential equations of Hill-Mathieu's type.

In view of Chapter XIV of Bolotin [3], eq.(4.4) must be reduced to the standard form before attempting any Mathieu's solution. The idea is to use the modal matrix of the free vibration of the shell in vacuo so that the inertia and stiffness matrices will become uncouple.

The eigenvalues of the free vibration of the shell in vacuo are obtained by solving the characteristic determinant:

$$\begin{bmatrix} a_{11}-\Lambda^2 & a_{12} & a_{13} \\ a_{21} & a_{22}-\Lambda^2 & a_{23} \\ a_{31} & a_{32} & a_{33}-\Lambda^2 \end{bmatrix} = 0 \tag{4.8}$$

in which:

$$\omega_o^2 = K/\rho_s h a^2 \qquad (4.8)$$

$$\Lambda^2 = (\omega/\omega_o)^2 \qquad (4.9)$$

The solution of eq.(4.8) yields three eigenvalues:

$$\Lambda_1 > \Lambda_2 > \Lambda_3 \qquad (4.10)$$

to which respectively correspond the three eigenvectors:

$$\left\{ G_1 \right\}, \left\{ G_2 \right\}, \left\{ G_3 \right\} \tag{4.11}$$

It follows that the modal matrix of the free vibration in vacuo is:

$$\underset{\sim}{G} = [\left\{ G_1 \right\} \quad \left\{ G_2 \right\} \quad \left\{ G_3 \right\}]$$

that is:

$$\underset{\sim}{G} = \begin{bmatrix} G_{11} & G_{21} & G_{31} \\ G_{12} & G_{22} & G_{32} \\ G_{13} & G_{23} & G_{33} \end{bmatrix} \tag{4.12}$$

In view of the modal matrix, the displacement is defined

$$\bar{u} = \underset{\sim}{G} \, \bar{y} \tag{4.13}$$

in which $\bar{y}$ is the principal or normal coordinate:

$$\bar{y} = \begin{Bmatrix} y_1 \\ y_2 \\ y_3 \end{Bmatrix} \qquad (4.14)$$

Eq.(4.4) may also be written as:

$$\frac{1}{\omega_o^2} \underset{\sim}{M} \ddot{\bar{u}} + \underset{\sim}{A} \bar{u} + \frac{a^2}{K} [\alpha p+\beta q \cos(\omega t-\psi)] \underset{\sim}{P}_1 \bar{u} = - \frac{a^2}{K} M_{fmn} \underset{\sim}{S} \ddot{\bar{u}} - \frac{a^2}{K} C_{fmn} \underset{\sim}{S} \dot{\bar{u}} \qquad (4.15)$$

Upon substituting eq.(4.13) into eq.(4.15) and multiplying both sides of the resulting equations from left by the transpose of the modal matrix $\underset{\sim}{G}^T$, one obtains after multiplying again each side from left by A_1^{-1}:

$$\underset{\sim}{C} \ddot{\bar{y}} + \underset{\sim}{E} \bar{y} + \frac{a^2}{K} [\alpha p+\beta q \cos(\omega t-\psi)] \underset{\sim}{P} \bar{y} = - \frac{a^2}{K} M_{fmn} \underset{\sim}{F} \ddot{\bar{y}} - \frac{a^2}{K} C_{fmn} \underset{\sim}{F} \dot{\bar{y}} \qquad (4.16)$$

in which:

$$\underset{\sim}{G} = \begin{bmatrix} G_{11} & G_{21} & G_{31} \\ G_{12} & G_{22} & G_{32} \\ 1 & 1 & 1 \end{bmatrix}$$

$$\underset{\sim}{A}_1 = \underset{\sim}{G}^T \underset{\sim}{A} \underset{\sim}{G}$$

$$\underset{\sim}{A}_1 = \begin{bmatrix} k_1 & 0 & 0 \\ 0 & k_2 & 0 \\ 0 & 0 & k_3 \end{bmatrix} \tag{4.17}$$

$$\underset{\sim}{I} = \underset{\sim}{G}^T \underset{\sim}{M} \underset{\sim}{G}$$

$$\underset{\sim}{I} = \begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix}$$

$$\underset{\sim}{C} = \frac{\rho_s h a^2}{K} \underset{\sim}{A}_1^{-1} \underset{\sim}{I}$$

$$\underset{\sim}{C} = \begin{bmatrix} \frac{1}{\Omega_1^2} & 0 & 0 \\ 0 & \frac{1}{\Omega_2^2} & 0 \\ 0 & 0 & \frac{1}{\Omega_3^2} \end{bmatrix} \tag{4.17}$$

$$\underset{\sim}{E} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\underset{\sim}{P} = A_1^{-1} \left\{ \underset{\sim}{G}^T \underset{\sim}{P}_1 \underset{\sim}{G} \right\}$$

$$\underset{\sim}{P} = \begin{bmatrix} q_{11}/k_1 & q_{12}/k_1 & q_{13}/k_1 \\ q_{21}/k_2 & q_{22}/k_2 & q_{23}/k_2 \\ q_{31}/k_3 & q_{32}/k_3 & q_{33}/k_3 \end{bmatrix} \tag{4.18}$$

$$\Lambda_i^2 = k_i/m_i \quad i=1,2,3$$

$$\Lambda_i^2 = (\omega_i/\omega_o)^2 \tag{4.18}$$

$$\Omega_i^2 = \omega_i^2 = \Lambda_i^2\ \omega_o^2$$

$$\underset{\sim}{F} = A_1^{-1} \left\{ \underset{\sim}{Q}^T\ \underset{\sim}{S}\ \underset{\sim}{Q} \right\}$$

$$\underset{\sim}{F} = \begin{bmatrix} 1/k_1 & 1/k_1 & 1/k_1 \\ 1/k_2 & 1/k_2 & 1/k_2 \\ 1/k_3 & 1/k_3 & 1/k_3 \end{bmatrix} \tag{4.19}$$

The transformed equations eq.(4.16), written in the standard form, are also a system of three linear differential equations of Hill-Mathieu's type. The theory of Floquet still applies in this case [3]. Consequently the boundaries of the instability regions correspond to periodic solutions of eq.(4.16) with period $2\pi/\omega$ and $4\pi/\omega$. Although the presence of the damping effect usually leads to the consideration of the first instability region as the region of any major significance, compared to regions of higher order, it is sometimes convenient to have this damping negligible for computational purposes in order to reduce the size of the characteristic determinant. If the arguments of the Bessel functions are imaginary, the acoustic resistance is identically zero. Therefore the first region of instability, called also principal region of instability, is not necessarily the critical one until actual evaluations of the higher regions of instability are made to assess the effect of the acoustic reactance and resistance.

The present investigation is restricted to the determination of the fluid effects upon the principal region of instability, which corresponds to a periodic solution of eq.(4.16) with period $4\pi/\omega$:

$$\bar{y} = \sum_{k=1,3,5}^{\infty} \bar{a}_k \sin \frac{k}{2}(\omega t-\psi) + \sum_{k=1,3,5}^{\infty} \bar{b}_k \cos \frac{k}{2}(\omega t-\psi) \tag{4.20}$$

At this point one may determine the Fourier coefficient of the acoustic pressure Q. To obtain this function, one makes the following changes in eq.(2.28):

a. In the harmonic function

$$\omega t \rightarrow \frac{k}{2}(\omega t-\psi)$$

b. In the other factors

$$\omega \rightarrow \frac{k}{2}\,\omega$$

The Fourier coefficient of the radial displacement w is given in view of eq.(4.13) and eq.(4.20) as:

$$w_{mn} = \sum_{k=1,3,5}^{\infty} L_{1k} \sin\frac{k}{2}(\omega t-\psi) + \sum_{k=1,3,5}^{\infty} L_{2k} \cos\frac{k}{2}(\omega t-\psi) \tag{4.21}$$

then the Fourier coefficient of the acoustic pressure Q is given as:

$$\begin{aligned} Q_{mn}(k) = &- \frac{\rho c\omega}{2} \sum_{k=1,3,5}^{\infty} [kL_{1k}X_{mn}^{(k)}(x_m) \\ &+ kL_{2k}\Theta_{mn}^{(k)}(x_m)]\sin \frac{k}{2}(\omega t-\psi) \\ &- \frac{\rho c\omega}{2} \sum_{k=1,3,5}^{\infty} [kL_{2k}X_{mn}^{(k)}(x_m) \\ &- kL_{1k}\Theta_{mn}^{(k)}(x_m)]\cos \frac{k}{2}(\omega t-\psi) \end{aligned} \tag{4.22}$$

in which

$$x_m^2(k) = \frac{k^2\omega^2a^2}{4c^2} - (\frac{m\pi a}{L})^2$$

$$x_o(k) = \frac{k\omega a}{2c} \tag{4.23}$$

In view of equation eq.(2.29), the Fourier co-efficient of the acoustic pressure eq.(4.22) may be written as:

$$Q_{mn} = \frac{\rho c}{\frac{k\omega}{2}} \chi_{mn}^{(k)}(x_m)\ddot{w}_{mn} + \rho c \Theta_{mn}^{(k)}(x_m)\dot{w}_{mn} \tag{4.24}$$

Therefore the acoustic reactance and resistance functions M_{fmn} and C_{fmn} become:

$$M_{fmn} = M_{mn}^{(k)} = \frac{2\rho c}{k\omega} \chi_{mn}^{(k)}(x_m)$$

$$C_{fmn} = D_{mn}^{(k)} = \rho c \Theta_{mn}^{(k)}(x_m) \tag{4.25}$$

The Mathieu equations eq.(4.16) may then be written because of eq.(4.25) as:

$$\left\{ \underset{\sim}{C} + M_{mn}^{(k)} \frac{a^2}{K} \underset{\sim}{F} \right\} \ddot{\bar{y}} + D_{mn}^{(k)} \frac{a^2}{K} \underset{\sim}{F} \dot{\bar{y}}$$

$$+ \left\{ \underset{\sim}{E} + \frac{a^2}{K} [\alpha p + \beta q \cos(\omega t - \psi)] \underset{\sim}{P} \right\} \bar{y} = 0 \tag{4.26}$$

Inserting eq.(4.20) into eq.(4.26), one obtains after collecting the coefficients of the same harmonic function:

$$\bar{a}_1 \left\{ -\frac{\omega^2}{4} \underset{\sim}{C} - \frac{\omega^2}{4} M_{mn}^{(1)} \frac{a^2}{K} \underset{\sim}{F} + \underset{\sim}{E} + \frac{a^2}{K} \alpha p \underset{\sim}{P} \right.$$

$$\left. - \frac{a^2}{2K} \beta q \underset{\sim}{P} \right\} - \bar{b}_1 \frac{\omega}{2} D_{mn}^{(1)} \frac{a^2}{K} \underset{\sim}{F}$$

$$+ \bar{a}_3 \frac{a^2}{K} \frac{\beta q}{2} \underset{\sim}{P} = 0 \tag{4.27}$$

$$\bar{a}_1 \frac{\omega}{2} D_{mn}^{(1)} \frac{a^2}{K} \underset{\sim}{F} + \bar{b}_1 \left\{ - \frac{\omega^2}{4} \underset{\sim}{C} \right.$$

$$- \frac{\omega^2}{4} M_{mn}^{(1)} \frac{a^2}{K} \underset{\sim}{F} + \underset{\sim}{E} + \frac{a^2}{K} \alpha p \underset{\sim}{P}$$

$$\left. - \frac{a^2}{K} \frac{\beta q}{2} \underset{\sim}{P} \right\} + \bar{b}_3 \frac{a^2}{K} \frac{\beta q}{2} \underset{\sim}{P} = 0 \tag{4.28}$$

$$\bar{a}_k \left\{ - \frac{k^2\omega^2}{4} \underset{\sim}{C} - \frac{k^2\omega^2}{4} M_{mn}^{(k)} \frac{a^2}{K} \underset{\sim}{F} + \underset{\sim}{E} \right.$$

$$\left. + \frac{a^2}{K} \alpha p \underset{\sim}{P} \right\} - \bar{b}_k \frac{k\omega}{2} D_{mn}^{(k)} \frac{a^2}{K} \underset{\sim}{F}$$

$$+ \frac{a^2}{K} \frac{\beta q}{2} \underset{\sim}{P} \bar{a}_{k-2} + \frac{a^2}{K} \frac{\beta q}{2} \underset{\sim}{P} \bar{a}_{k+2} = 0$$

$$k = 3,5,\ldots \qquad (4.29)$$

$$\bar{a}_k \frac{k\omega}{2} D_{mn}^{(k)} \frac{a^2}{K} \underset{\sim}{F} + \bar{b}_k \left\{ - \frac{k^2\omega^2}{4} \underset{\sim}{C} \right.$$

$$\left. - \frac{k^2\omega^2}{4} M_{mn}^{(k)} \frac{a^2}{K} \underset{\sim}{F} + \underset{\sim}{E} + \frac{a^2}{K} \alpha p \underset{\sim}{P} \right\}$$

$$+ \frac{a^2}{K} \frac{\beta q}{2} \underset{\sim}{P} \bar{b}_{k-2} + \frac{a^2}{K} \frac{\beta q}{2} \underset{\sim}{P} \bar{b}_{k+2} = 0$$

$$k = 3,5,\ \ldots \qquad (4.30)$$

The set of eq.(4.27) through eq.(4.30) represents a system of a double infinite homogeneous algebraic equations with double infinite unknowns $\bar{a}_k$ and $\bar{b}_k$ ($k=1,3,\ \ldots\ \infty$).

For non-trivial solution, the determinant of the coefficients must vanish. It may be shown that such infinite determinant is convergent [16].

A first approximation of the infinite determinant corresponds to the harmonic solution:

$$\bar{y} = \bar{a}_1 \sin\left(\frac{\omega t-\psi}{2}\right) + \bar{b}_1 \cos\left(\frac{\omega t-\psi}{2}\right) \tag{4.31}$$

It follows that this determinant is:

$$\begin{vmatrix} \underset{\sim}{H}_{11} & -\underset{\sim}{H}_{12} \\ \underset{\sim}{H}_{12} & \underset{\sim}{H}_{22} \end{vmatrix} = 0 \tag{4.32}$$

where

$$\underset{\sim}{H}_{12} = \frac{\omega}{2} D_{mn}^{(1)} \frac{a^2}{K} \underset{\sim}{F}$$

$$\underset{\sim}{H}_{11} = -\frac{\omega^2}{4} \underset{\sim}{C} - \frac{\omega^2}{4} M_{mn}^{(1)} \frac{a^2}{K} \underset{\sim}{F} + \underset{\sim}{E}$$

$$+ \frac{a^2}{K} \alpha p \underset{\sim}{P} - \frac{a^2}{K} \frac{\beta q}{2} \underset{\sim}{P}$$

$$\underset{\sim}{H}_{22} = -\frac{\omega^2}{4} \underset{\sim}{C} - \frac{\omega^2}{4} M_{mn}^{(1)} \frac{a^2}{K} \underset{\sim}{F} + \underset{\sim}{E}$$

$$+ \frac{a^2}{K} \alpha p \underset{\sim}{P} + \frac{a^2}{K} \frac{\beta q}{2} \underset{\sim}{P}$$

$$\tag{4.33}$$

The matrix $\underset{\sim}{H}_{12}$ represents the damping effects of the fluid medium.

The characteristic determinant corresponding to the principal regions of instability may be reduced to a determinant of lower order if the damping effects are insignificant or vanishing. In this case one obtains after neglecting the matrix $\underset{\sim}{H}_{12}$:

$$\left| \underset{\sim}{H}_{11} \right| = \left| \underset{\sim}{H}_{22} \right| = 0$$

In terms of eq.(4.33), the above determinants become:

$$\left| -\frac{\omega^2}{4} \underset{\sim}{C} - \frac{\omega^2}{4} M_{mn}^{(1)} \frac{a^2}{K} \underset{\sim}{F} + \underset{\sim}{E} + \frac{a^2}{K} \left(\alpha p \pm \frac{\beta q}{2}\right) \underset{\sim}{P} \right| = 0 \tag{4.34}$$

One defines the following:

$$\alpha_{11} = -\frac{\omega^2}{4\Omega_1^2} - \frac{\omega^2}{4} M_{mn}^{(1)} \frac{a^2}{Kk_1} + 1$$

$$\beta_{11} = \frac{a^2}{Kk_1} \alpha p q_{11} \pm \frac{a^2}{Kk_1} \frac{\beta q_{11}}{2} q$$

$$\alpha_{12} = -\frac{\omega^2}{4} M_{mn}^{(1)} \frac{a^2}{Kk_1} \tag{4.35}$$

$$\beta_{12} = \frac{a^2}{Kk_1} \alpha p q_{12} \pm \frac{a^2}{Kk_1} \frac{\beta q_{12}}{2} q$$

$$\alpha_{13} = \alpha_{12}$$

$$\beta_{13} = \frac{a^2}{Kk_1} \alpha p q_{13} \pm \frac{a^2}{Kk_1} \frac{\beta q_{13}}{2} q$$

$$\alpha_{21} = - \frac{\omega^2}{4} M_{mn}^{(1)} \frac{a^2}{Kk_2}$$

$$\beta_{21} = \frac{a^2}{Kk_2} \alpha p q_{21} \pm \frac{a^2}{Kk_2} \frac{\beta q_{21}}{2} q$$

$$\alpha_{22} = 1 - \frac{\omega^2}{4\Omega_2^2} - \frac{\omega^2}{4} M_{mn}^{(1)} \frac{a^2}{Kk_2}$$

$$\beta_{22} = \frac{a^2}{Kk_2} \alpha p q_{22} \pm \frac{a^2}{Kk_2} \frac{\beta q_{22}}{2} q$$

$$\alpha_{23} = \alpha_{21}$$

$$\beta_{23} = \frac{a^2}{Kk_2} \alpha p q_{23} \pm \frac{a^2}{Kk_2} \frac{\beta q_{23}}{2} q$$

$$\alpha_{31} = - \frac{\omega^2}{4} M_{mn}^{(1)} \frac{a^2}{Kk_3} \tag{4.35}$$

$$\beta_{31} = \frac{a^2}{Kk_3}\,\alpha p q_{31} \pm \frac{a^2}{Kk_3}\,\frac{\beta q_{31}}{2}\,q$$

$$\alpha_{32} = \alpha_{31}$$

$$\beta_{32} = \frac{a^2}{Kk_3}\,\alpha p q_{32} \pm \frac{a^2}{Kk_3}\,\frac{\beta q_{32}}{2}\,q$$

$$\alpha_{33} = 1 - \frac{\omega^2}{4\Omega_3^2} - \frac{\omega^2}{4}\,M_{mn}^{(1)}\,\frac{a^2}{Kk_3}$$

$$\beta_{33} = \frac{a^2}{Kk_3}\,\alpha p q_{33} \pm \frac{a^2}{Kk_3}\,\frac{\beta q_{33}}{2}\,q \tag{4.35}$$

If the static pressure is identically zero, the determinant eq.(4.34) becomes after making use of eq.(4.35):

$$\begin{vmatrix} \alpha_{11}+\beta_{11}q & \alpha_{12}+\beta_{12}q & \alpha_{13}+\beta_{13}q \\ \alpha_{21}+\beta_{21}q & \alpha_{22}+\beta_{22}q & \alpha_{23}+\beta_{23}q \\ \alpha_{31}+\beta_{31}q & \alpha_{32}+\beta_{32}q & \alpha_{33}+\beta_{33}q \end{vmatrix} = 0 \tag{4.36}$$

The expansion of the above determinant yields a characteristic equation in terms of the vibratory pressure q, the static pressure being identically zero:

$$P_{31}q^3+P_{32}q^2+P_{33}q+P_{34} = 0 \tag{4.37}$$

where:

$$P_{31} = \beta_{11}\beta_{22}\beta_{33}-\beta_{11}\beta_{32}\beta_{23}+\beta_{12}\beta_{23}\beta_{31}$$

$$-\beta_{12}\beta_{21}\beta_{33}+\beta_{13}\beta_{21}\beta_{32}-\beta_{13}\beta_{31}\beta_{22}$$

$$P_{32} = \alpha_{11}\beta_{22}\beta_{33}-\alpha_{11}\beta_{32}\beta_{23}+\beta_{11}\alpha_{22}\beta_{33}$$

$$+\beta_{11}\beta_{22}\alpha_{33}-\beta_{11}\alpha_{32}\beta_{23}-\beta_{11}\beta_{32}\alpha_{23}$$

$$+\alpha_{12}\beta_{23}\beta_{31}-\alpha_{12}\beta_{21}\beta_{33}+\beta_{12}\alpha_{23}\beta_{31}$$

$$+\beta_{12}\beta_{23}\alpha_{31}-\beta_{12}\alpha_{21}\beta_{33}-\beta_{12}\beta_{21}\alpha_{33}$$

$$+\alpha_{13}\beta_{21}\beta_{32}-\alpha_{13}\beta_{31}\beta_{22}+\beta_{13}\alpha_{21}\beta_{32}$$

$$+\beta_{13}\beta_{21}\alpha_{32}-\beta_{13}\alpha_{31}\beta_{22}-\beta_{13}\beta_{31}\alpha_{22}$$

$$\tag{4.38}$$

$$P_{33} = \alpha_{11}\alpha_{22}\beta_{33}+\alpha_{11}\beta_{22}\alpha_{33}-\alpha_{11}\alpha_{32}\beta_{23}$$

$$-\alpha_{11}\beta_{32}\alpha_{23}+\beta_{11}\alpha_{22}\alpha_{33}-\beta_{11}\alpha_{32}\alpha_{23}$$

$$+\alpha_{12}\alpha_{23}\beta_{31}+\alpha_{12}\beta_{23}\alpha_{31}-\alpha_{12}\beta_{21}\alpha_{33}$$

$$-\alpha_{12}\alpha_{21}\beta_{33}+\beta_{12}\alpha_{23}\alpha_{31}-\beta_{12}\alpha_{21}\alpha_{33}$$

$$+\alpha_{13}\alpha_{21}\beta_{32}+\alpha_{13}\beta_{21}\alpha_{32}-\alpha_{13}\alpha_{31}\beta_{22}$$

$$-\alpha_{13}\beta_{31}\alpha_{22}+\beta_{13}\alpha_{21}\alpha_{32}-\beta_{13}\alpha_{31}\alpha_{22}$$

$$P_{34} = \alpha_{11}\alpha_{22}\alpha_{33}-\alpha_{11}\alpha_{32}\alpha_{23}+\alpha_{12}\alpha_{23}\alpha_{33}$$

$$-\alpha_{12}\alpha_{21}\alpha_{33}+\alpha_{13}\alpha_{21}\alpha_{32}-\alpha_{13}\alpha_{31}\alpha_{22} \tag{4.38}$$

The eigenbuckling pressure of a long circular cylindrical shell under internal lateral pressure is defined as [9]:

$$P_n = \frac{Eh^3(n^2-1)}{12a^3(1-\nu^2)} \tag{4.39}$$

If the ratio γ is defined as:

$$\gamma = \frac{q}{P_n} \tag{4.40}$$

then equation eq.(4.37) becomes:

$$\gamma^3 + \frac{P_{32}}{P_{31}P_n}\gamma^2 + \frac{P_{33}}{P_{31}P_n^2}\gamma + \frac{P_{34}}{P_{31}P_n^3} = 0 \tag{4.41}$$

Since the quantities P_{31}, P_{32}, P_{33} and P_{34} are functions of the frequencies, one may solve eq.(4.41) for γ once the frequencies are given. Of course γ is a dimensionless vibratory pressure.

CHAPTER V

NUMERICAL ANALYSIS

The aim of this study from the beginning has been to determine the conditions under which a shell vibrating in its breathing motion may loose its stability. In other words, the question is to find a set of parameters for which the initial motion of the shell is stable or unstable. The governing equations of the perturbed motion developed in the previous chapters, are valid for any type of circular cylindrical shells vibrating in an acoustic medium as well as in vacuo. Of course to obtain the governing equations of the shell in vacuo, one has to set the acoustic reactances and resistances equal to zero.

The analysis undertaken is restricted to the determination of the principal regions of instability when the static pressure is identically zero. It follows that eq.(4.41) is the basis of the computations. This equation has been derived assuming that the acoustic resistance is zero or negligible. The set of parameters used here shows that the assumption is valid since the arguments of the Bessel functions are imaginary over a good range of the frequency ratio $\alpha_n = \omega/2\omega_n$, where ω and ω_n are respectively the forcing frequency of the shell vibrating in vacuo, when the circumferential wave number is n. Table V gives the range of the parameters α_n for which the Bessel arguments are imaginary. This table, when read at the same time as Fig. 10 through Fig. 14, shows that the damping effects due to the fluid medium have no influence upon the principal regions of instability. This point is discussed below. From Chapter II, it is known that the acoustic resistance is identically zero when the arguments of the Bessel functions are imaginary.

In order to determine the coefficients of the cubic algebraic equation eq.(4.41), the following steps must be executed for each given frequency ratio α_n:

a. Determine the natural frequency ω_n of the vibrating shell in vacuo. (See Table I).

b. Determine the forcing frequency ω given the frequency ratio α_n

   $\alpha_n = \omega/2\omega_n$

c. Compute the argument of the acoustic reactance and resistance functions owing to the perturbed motion using eq.(2.12) for the breathing motion and eq.(4.22) for the additional motion

   y_o, x_o, x_m

d. Determine the acoustic reactances and resistances:

$\chi_{oo}(y_o)$, $\theta_{oo}(y_o)$

$\chi_{mn}(x_m)$, $\theta_{mn}(x_m)$

$M_{mn}^{(1)}$, $D_{mn}^{(1)}$

e. Calculate the coefficients of the cubic equation

eq.(4.41)

P_{31}, P_{32}, P_{33}, P_{34}

and the eigenbuckling pressure

p_n.

f. Finally find the smallest absolute value of the nondimensional dynamic pressure

$\gamma = q/p_n$ from eq.(4.41)

A test is made to assess the validity of the equations developed in the present investigation when these are compared to those derived by Deng and Popelar in reference [2]. The results of the test are plotted in terms of the dimensionless dynamic pressure γ versus the frequency ratio α_n as shown in Fig. 10. The Deng-Popelar analysis corresponds to $L/ma = \infty$ while the present study corresponds to $L/ma = 100$. The computations are performed over the same range of shell and fluid parameters when the circumferential wave number is

$n = 6$. The data used in this chapter for a steel shell and water are:

$$a/h = 100$$

$$\rho = 1.94 \text{ slug/ft}^3$$

$$\rho_s = 15 \text{ slug/ft}^3$$

$$\nu = 0.3$$

$$D_{mn}^{(1)} \equiv 0$$

$$p \equiv 0$$

The variable functions are the circumferential wave number n and the ratios L/ma.

The analysis of the test as depicted in Fig. 10 shows:

1. The agreement between the Deng-Popelar graph and the graph of the present study is very good in an acoustic medium as well as in vacuo.

2. The principal regions of instability emanate from $\alpha_6 = 1$ in vacuo.

3. The Deng-Popelar graph ($n=6$, $L/ma=\infty$) exhibits an area of the principal region of instability larger than the graph ($n=6$, $L/ma=100$) in vacuo since the former has a width larger than the latter at $\gamma = 1$ or $q/p_6 = 1$.

4. In an acoustic medium the principal regions of instability are shifted towards the left of $\alpha_6 = 1$. The graph (n=6, L/ma=∞) has a cut-off at $\alpha_6 = 0.57$ while the graph (n=6, L/ma=100) has one at $\alpha_6 = 0.6$ in the diagram of $\gamma - \alpha_6$ (Fig. 10).
5. The widths of the principal regions of instability are larger in vacuo than in an acoustic medium.
6. The acoustic resistance in the perturbed equations of motion is identically zero and has no effect on the principal region of instability.

The effects of the fluid medium at the parametric resonance of the principal instability regions are shown in Table II and Table III. The results indicate that the acoustic medium, here water, has a tendency to shift the principal regions of instability towards the left of the principal region of instability in vacuo, which exhibits a cut-off in the plot of $\gamma - \alpha_n$ at $\alpha_n = 1$.

Figure 11 displays the graphs (n=2, L/ma=1) beside those of Deng-Popelar (n=2, L/ma=∞). While the Deng-Popelar graph and the graph (n=2, L/ma=1) emanate from $\alpha_2 = 1$ in vacuo, the latter exhibits another branch of instability at $\alpha_2 = 0.78$, which corresponds to the ordinary resonance of the breathing motion. In fluid the graph (n=2, L/ma=1) is more shifted towards the left of $\alpha_2 = 1$ than the graph (n=2, L/ma=∞). The widths of the principal regions of instability for (n=2, L/ma=1) are smaller than that of the graph (n=2, L/ma=∞) in vacuo as well as in an acoustic medium. Similar features are exhibited in Fig. 12 for the graph (n=6, L/ma=∞) and the graph (n=6, L/ma=1) in fluid and in vacuo; the branch attached to the ordinary resonance in vacuo is out of range of the parameters used here ($\alpha_6 = 0.1$ to $\alpha_6 = 2.0$). The results of the principal regions of instability for the graph (n=2, L/ma=2) are illustrated in Fig. 13. Although the graphs in Fig. 13 present the same characteristics as the graphs (n=2, L/ma=1) in Fig. 11, the ordinary resonance of the breathing motion occurs at the right of $\alpha_2 = 1$ at $\alpha_2 = 1.53$. The graphs (n=6, L/ma=2) in Fig. 14 illustrate the same features as graph (n=6, L/ma=1) in Fig. 12.

How does the parametric buckling occur physically at the principal resonance $\alpha_n = 1$ in vacuo for instance? Since the shell particles are assumed to vibrate in their normal mode executing simple harmonic motion with the period and phase of the motion being the same for all points on the shell, the parametric buckling is explained as follows:

a. When the exciting internal pressure is compressed going from zero to its negative maximum and from this maximum to zero, the shell expands going from zero to its positive maximum.

b. When the shell starts to go from its positive maximum to zero, the exciting pressure expands going from zero to its positive maximum and from this point to zero. That is the exciting internal pressure helping the shell to quickly return from its positive maximum displacement to zero. The internal pressure is introducing new energy in the system. That is why the displacement of the shell starts to increase without bound.

Not only the mechanism of parametric buckling occurs at the principal resonance in vacuo, but also it occurs at the principal resonance in an acoustic medium as well as at any point of the instability region.

Table IV summarizes the important results at the principal regions of instability in a fluid:

a. the values of the cut-off frequency in the plot q/p_n versus α_n at $q/p_n = 0$.

b. the widths of the instability regions at $q/p_n = 1$.

With the help of Table IV, one sees that there is no clear rule between the frequency ratios and the increase of the length-radius ratios L/ma. The lack of rule again exists between the widths and the increase of the ratio L/ma.

The reason is that the acoustic reactances, called also "added mass", are functions of the ratios L/ma. The importance of the "added mass" is exemplified by larger shifts towards the left of $\alpha_n = 1$, exhibited by the principal region of instability in an acoustic medium.

To obtain the Deng-Popelar results, one has to let L/ma go to ∞. Then the arguments of the Bessel functions are real. Consequently the damping effects may become important. But if L/ma goes to zero, these arguments become imaginary and tend towards $j\infty$ where $j = -\sqrt{-1}$. Therefore the acoustic reactance may reach its maximum and then tends towards the value zero. In this case the "added mass" tending towards zero leads to no shift at all of the principal region of instability in an acoustic medium.

The motion of the shell will be stable or unstable depending upon the fact that a representative point in this space falls either outside or inside the boundary regions of instability. If it falls inside, then the displacements will grow without limit. If it falls outside, then the motion will be a steady state motion. Although the magnitude of the static pressure is the determining factor in the static stability analysis, the frequency ratios are the characteristic elements of the dynamic stability. It means that the dynamic pressure q may exceed the static buckling pressure p_n without causing a loss of stability as long as the representative point is outside the region of instability.

CONCLUSION

In summary, the discussion of the graphs of principal regions of instability has shown that:

a. The "added mass" owing to the fluid medium has the effect of shifting the principal region of instability towards the left of $\alpha_n = 1$ in the plots of q/p_n versus α_n.

b. The widths of the principal regions of instability in fluid medium are smaller than those in vacuo for the same range of parameters.

c. The effects of the ratios L/ma are significant for the shifts of the principal regions of instability in an acoustic medium.

d. The correlation of L/ma and ratios α_n or the widths of the principal region of instability cannot be established clearly for lack of sufficient information.

TABLE I

Frequency Parameters of a Circular Cylindrical Thin Shell:
$\rho_s/\rho = 7.7$, $\nu = 0.3$, $a/h = 100$, $\omega_o = 2134.798$ rad/sec.

L/ma	n	Λ_1^*	Λ_2	Λ_3
1	2	3.76	2.2678	0.6527
1	6	6.8187	4.0392	0.238696
2	2	2.7334	1.4662	0.327206
2	6	6.1353	3.564	0.10256
100	6	6.0824	3.5497	0.099655

* $\Lambda_i = (\omega_i/\omega_o)$

TABLE II

Frequency Ratios and Arguments of Bessel's Functions at the Principal Resonance: ρ_s/ρ = 7.7, ν = 0.3, a/h = 100, c/c_s = 0.282, ω_o = 2134.798 rad/sec., p $\equiv$ 0, $D_{mn}^{(1)} \equiv 0$

n	L/ma	$y_o = 2x_o$	x_o	x_m	$\alpha_n = \frac{\omega}{2\omega_n}$
6	100	0.424	0.212	0.21	0.6
2	1	1.621	0.811	-3.031j	0.35
6	1	0.593	0.297	-3.126j	0.34
2	2	0.974	0.487	-1.49j	0.42
6	2	0.128	0.064	-1.57j	0.18

TABLE III

Dimensionless Forcing Frequencies at the Principal Parametric Resonance: $\rho_s/\rho = 7.7$, $\nu = 0.3$, $c/c_s = 0.282$, $a/h = 100$, $\omega_o = 2134.798$ rad/sec., $p \equiv 0$, $D_{mn}^{(1)} \equiv 0$

n	L/ma	α_n	Λ_3	ω/ω_o *
6	100	0.6	0.099655	0.12
2	1	0.35	0.6527	0.456
6	1	0.34	0.238696	0.167
2	2	0.42	0.327206	0.275
6	2	0.18	0.10256	0.037

* $\omega/\omega_o = 2\alpha_n\Lambda_3$

TABLE IV

Frequency Ratios at the Principal Resonance and Widths of the Principal Regions of Instability at $\gamma = 1$: $\rho_s/\rho = 7.7$, $\nu = 0.3$, $c/c_s = 0.282$, $a/h = 100$, $\omega_o = 2134.798$ rad/sec., $p \equiv 0$, $D_{mn}^{(1)} \equiv 0$

n	L/ma	α_n	width at $\gamma = 1$
2	1	0.35	1.0
2	2	0.42	0.8
2	∞*	0.40	1.7
6	1	0.34	1.0
6	2	0.18	1.7
6	100	0.6	3.5
6	∞*	0.58	3.0

* $L/ma = \infty$ are results found by Deng and Popelar in Ref.[2].

TABLE V

The Range of Vanishing Acoustic Resistance

n	L/ma	$(\alpha_n)_1-(\alpha_n)_2$	$\theta_{mn}(x_m)$
6	100	0.1 - 2.0	0
2	1	0.1 - 1.3	0
6	1	0.1 - 2.0	0
2	2	0.1 - 1.3	0
6	2	0.1 - 2.0	0

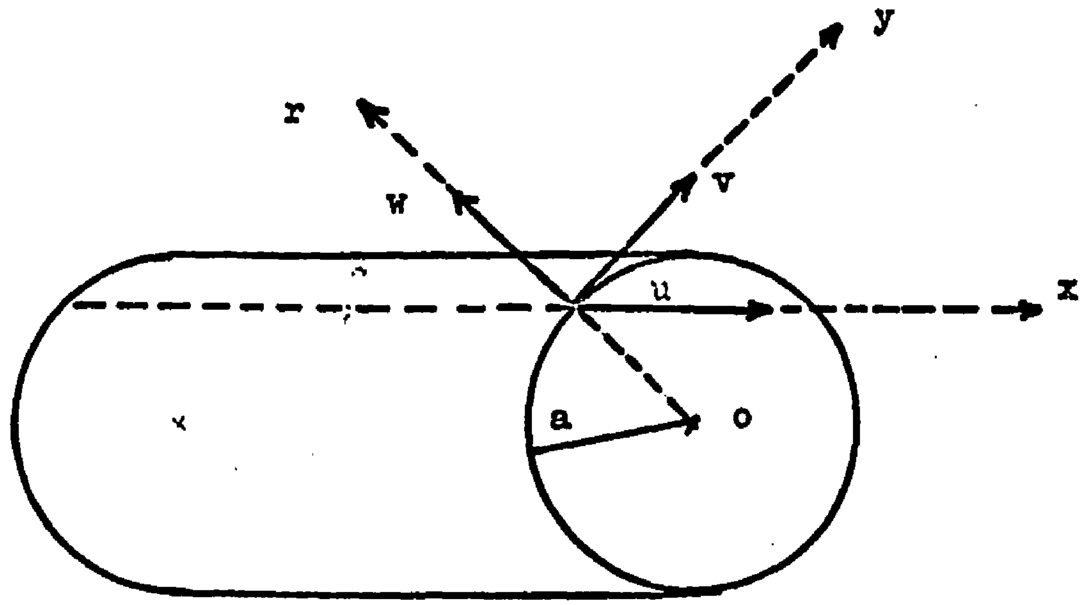


Fig.1

Shell Geometry and Coordinate system and Displacements.

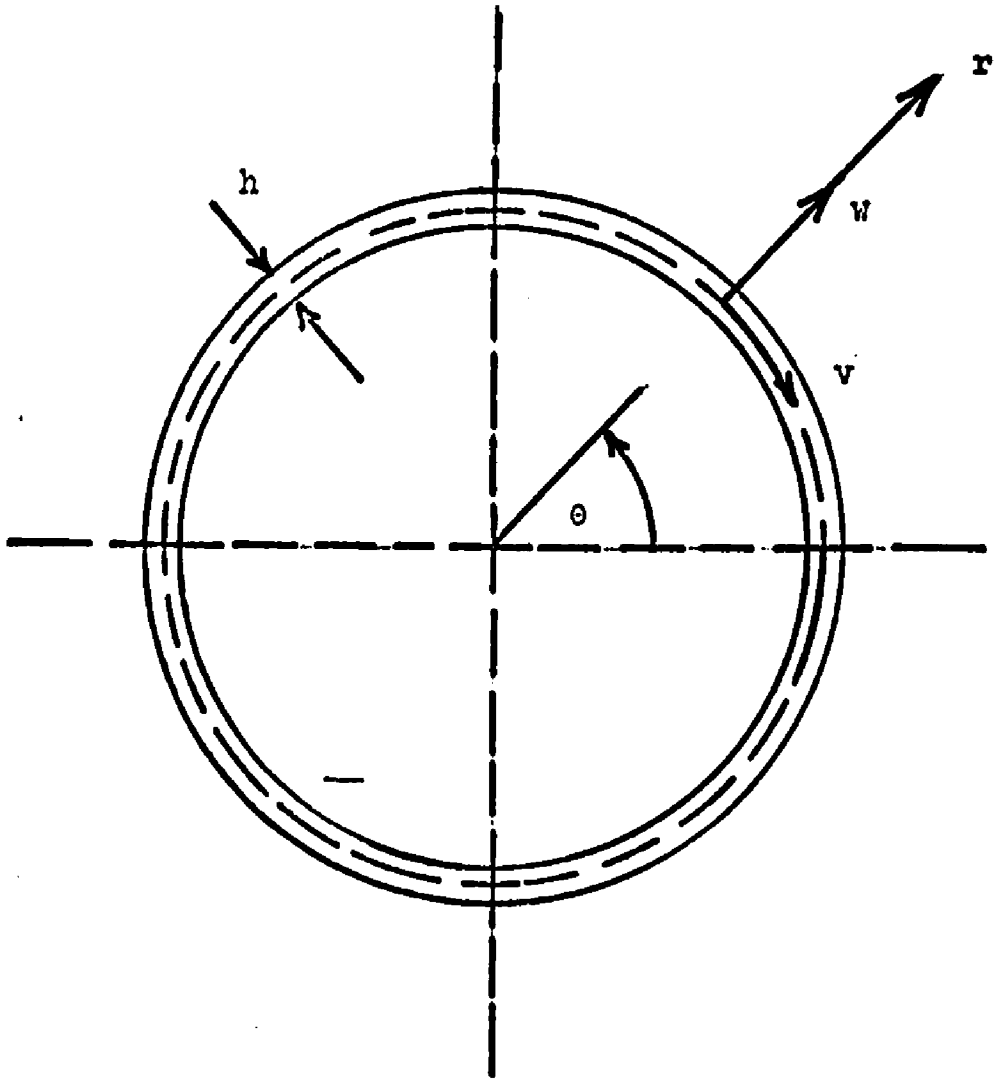


Fig. 2

Cross section of Cylindrical Shell.

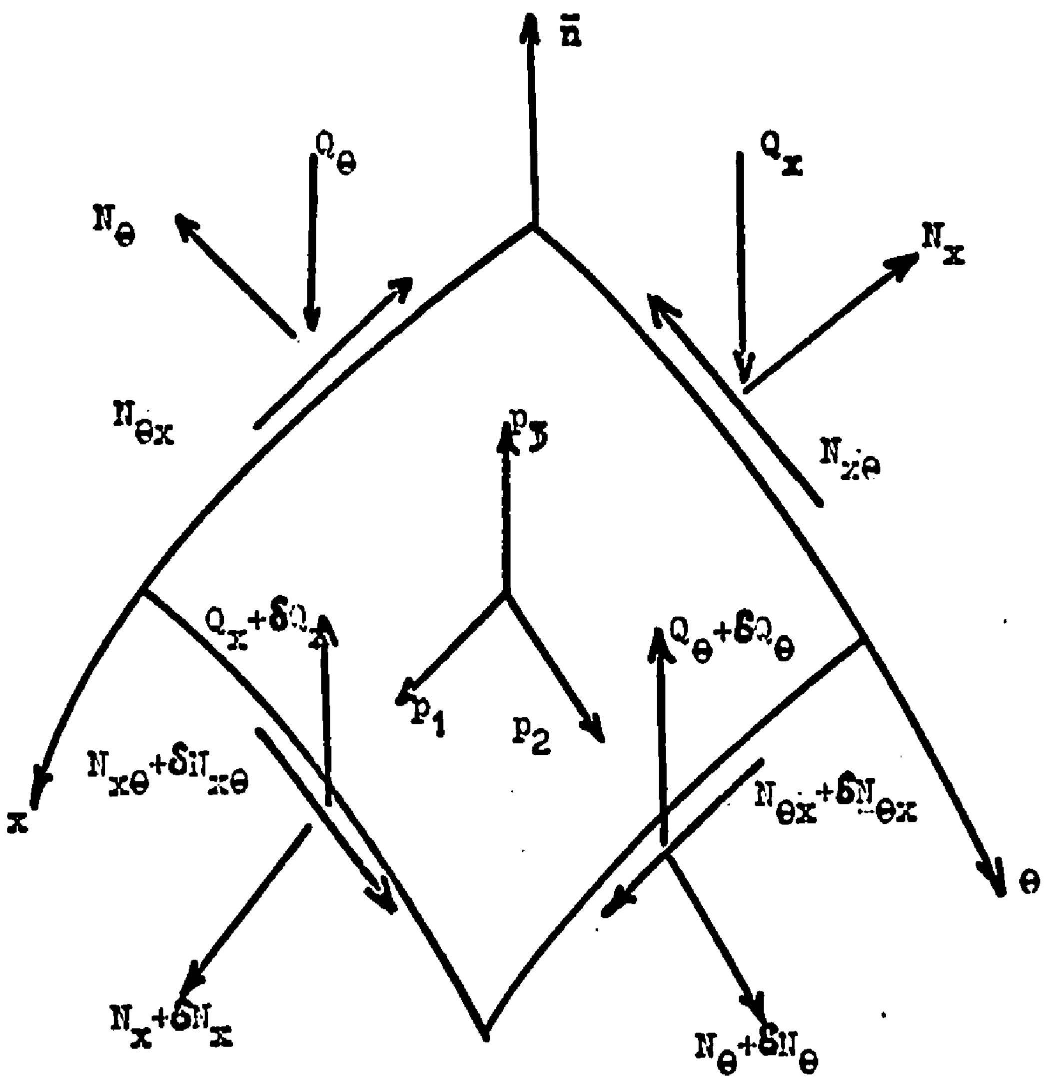


Fig. 3

Membrane Forces and Sign Convention.

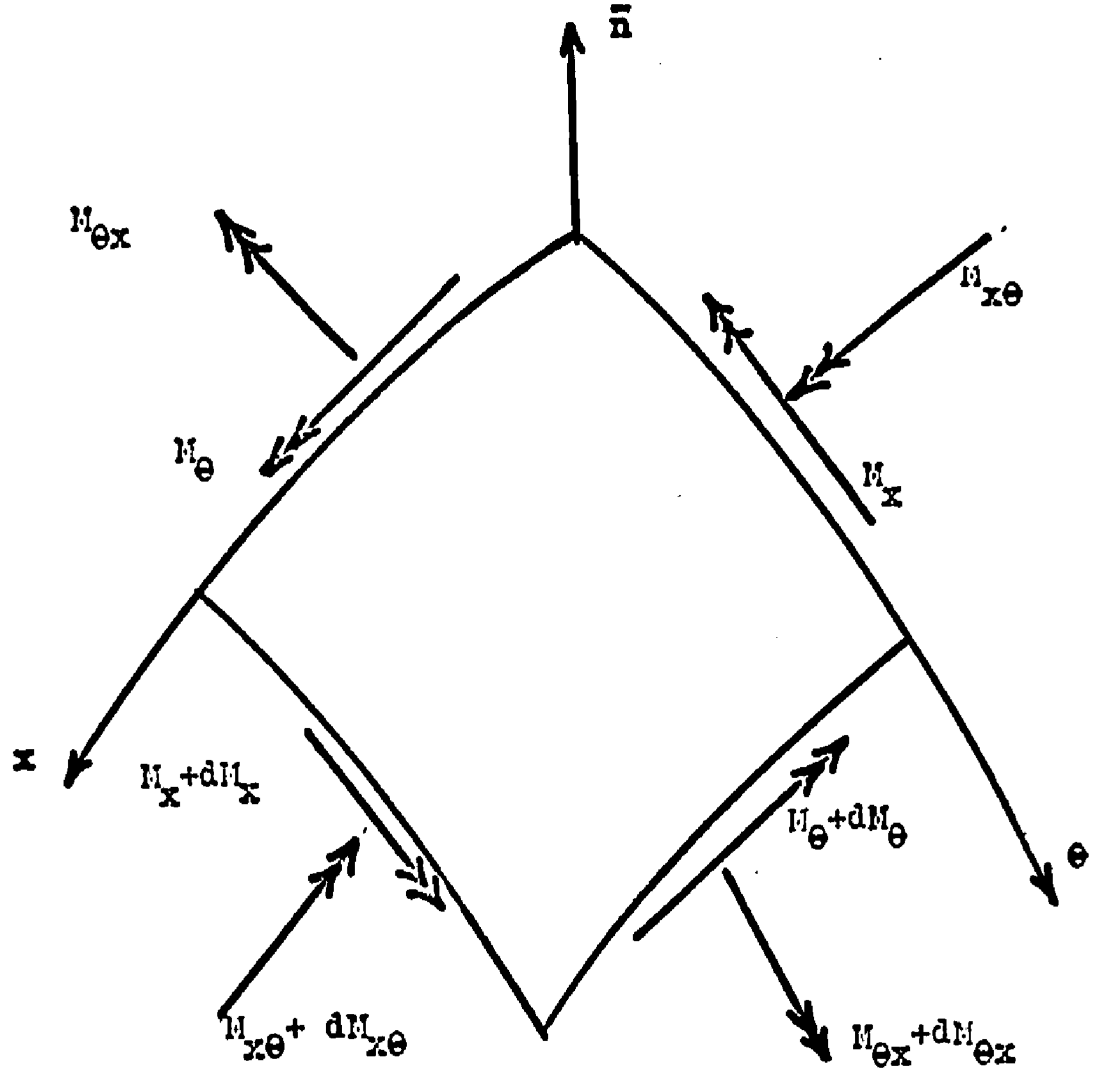
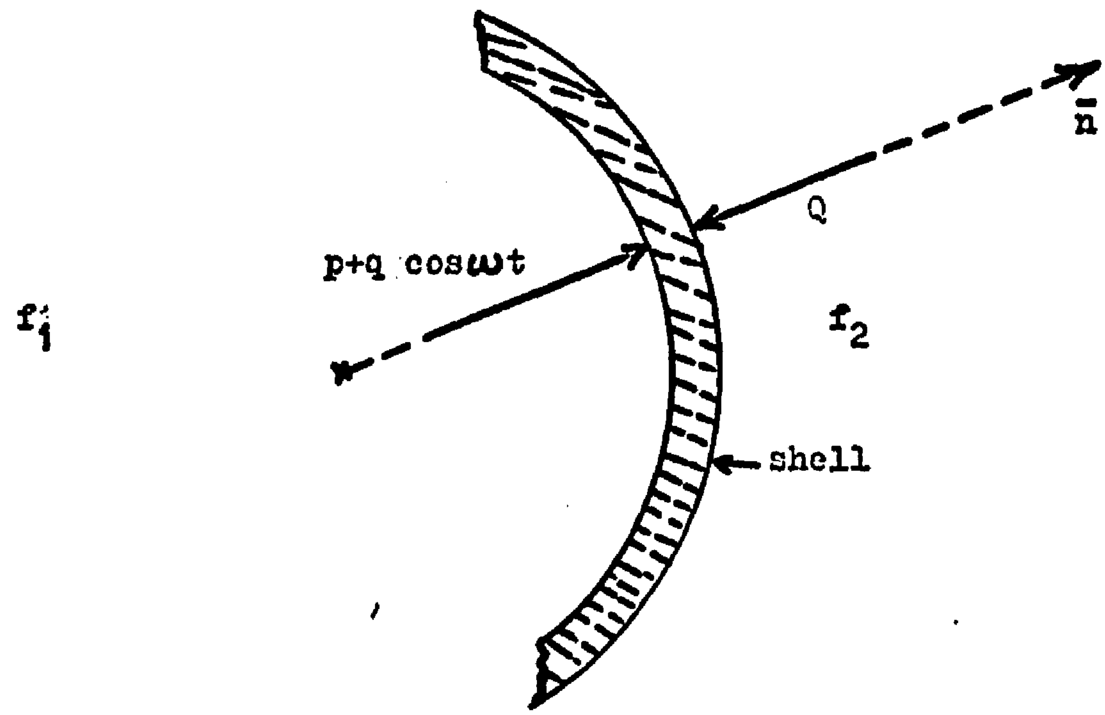


Fig. 4

Stress Moments and Sign Convention.



f_1 = internal acoustic medium.
f_2 = external acoustic medium.
p+q cosωt =internal acoustic pressure.
Q = external acoustic pressure.

Fig.5

Directions of the Internal and External Acoustic Pressures.

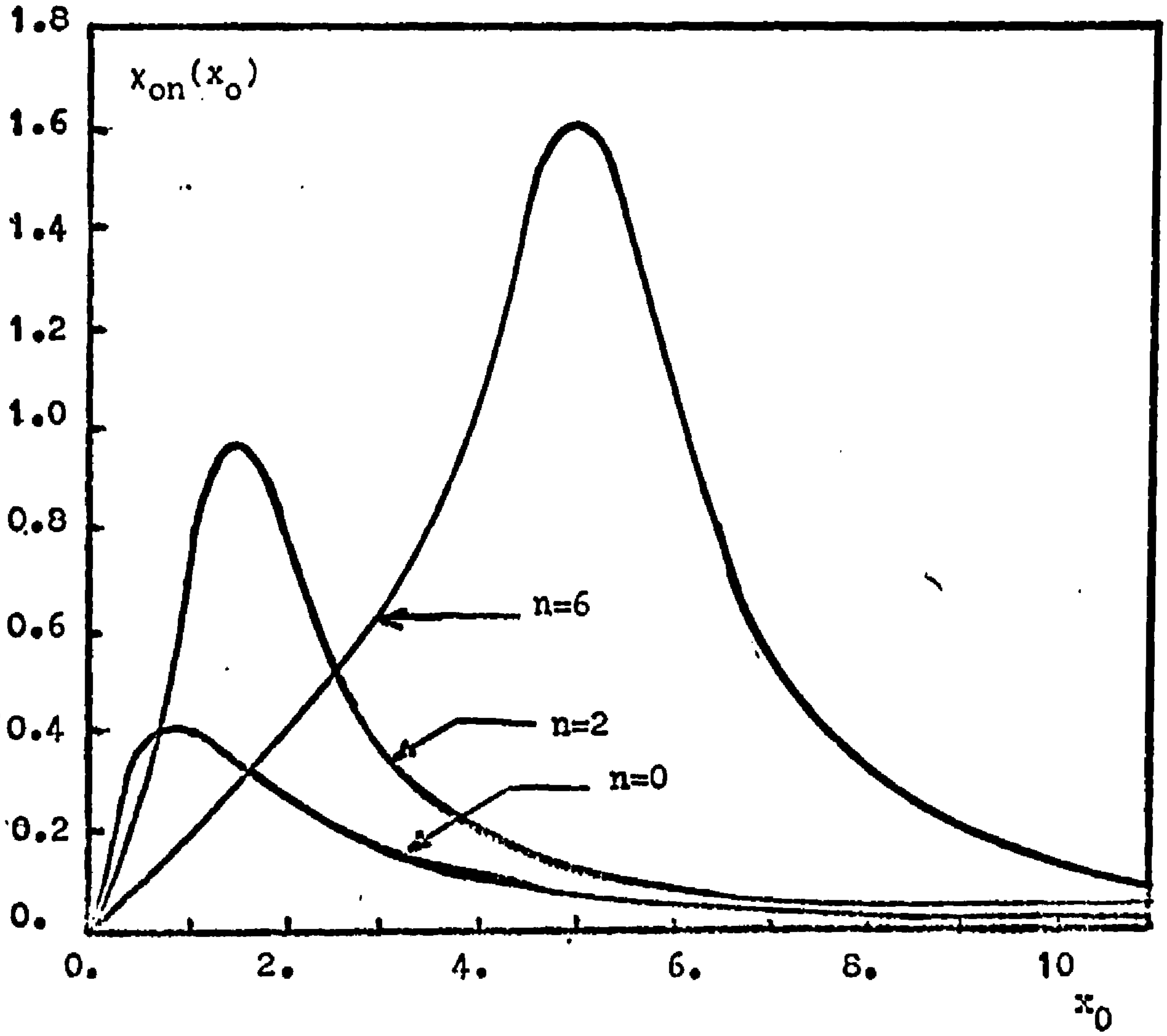


Fig.6

Acoustic Reactance Ratios for Partial Waves Radiated by a Cylindrical Surface with $L/ma = \infty$ *.

*These curves are taken from reference 13 (Junger)

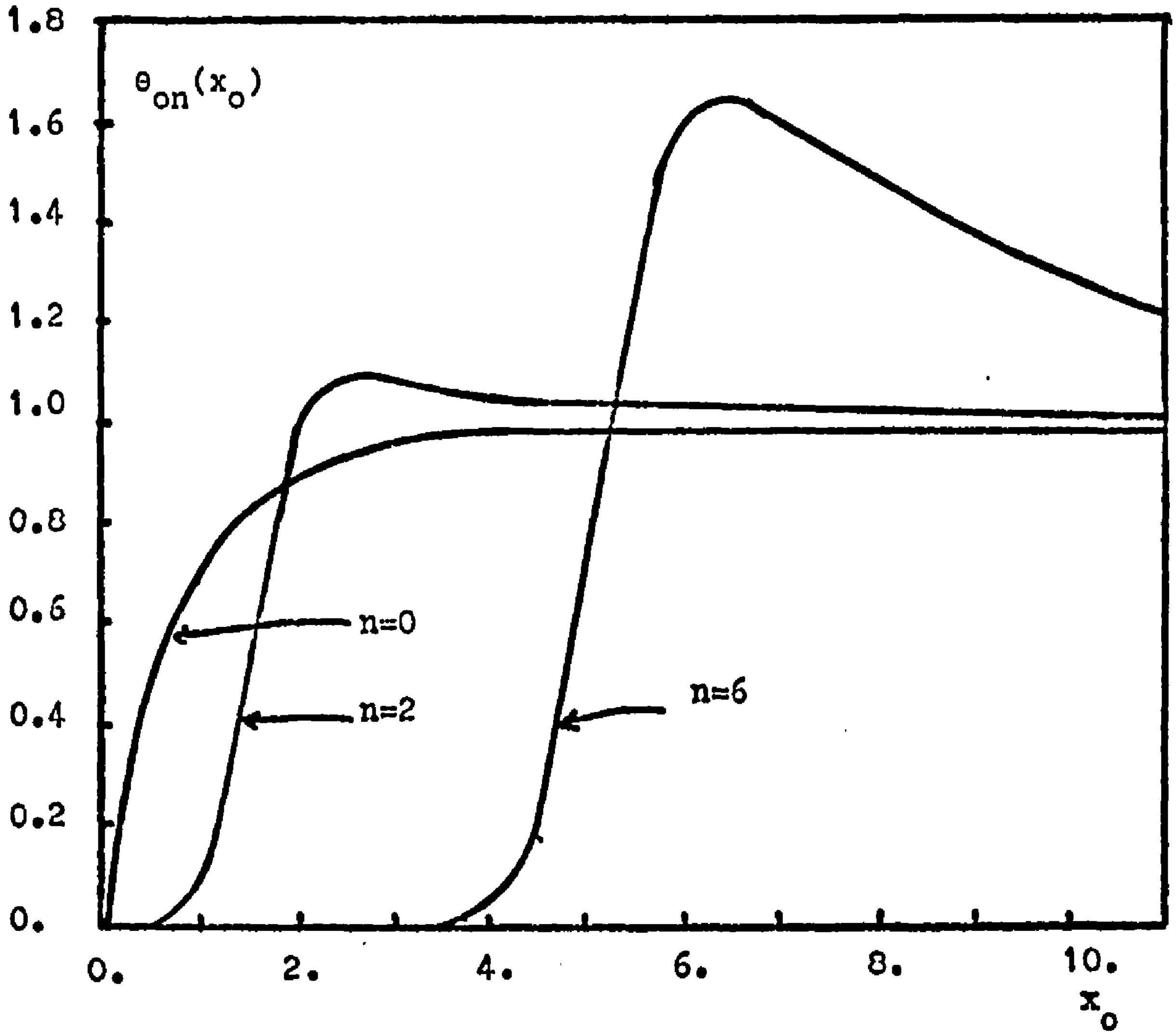


Fig.7

Acoustic Resistance Ratios for Partial Waves Radiated by a Cylindrical Surface with $L/ma = \infty$ *.

*These curves are taken from reference 13 (Junger).

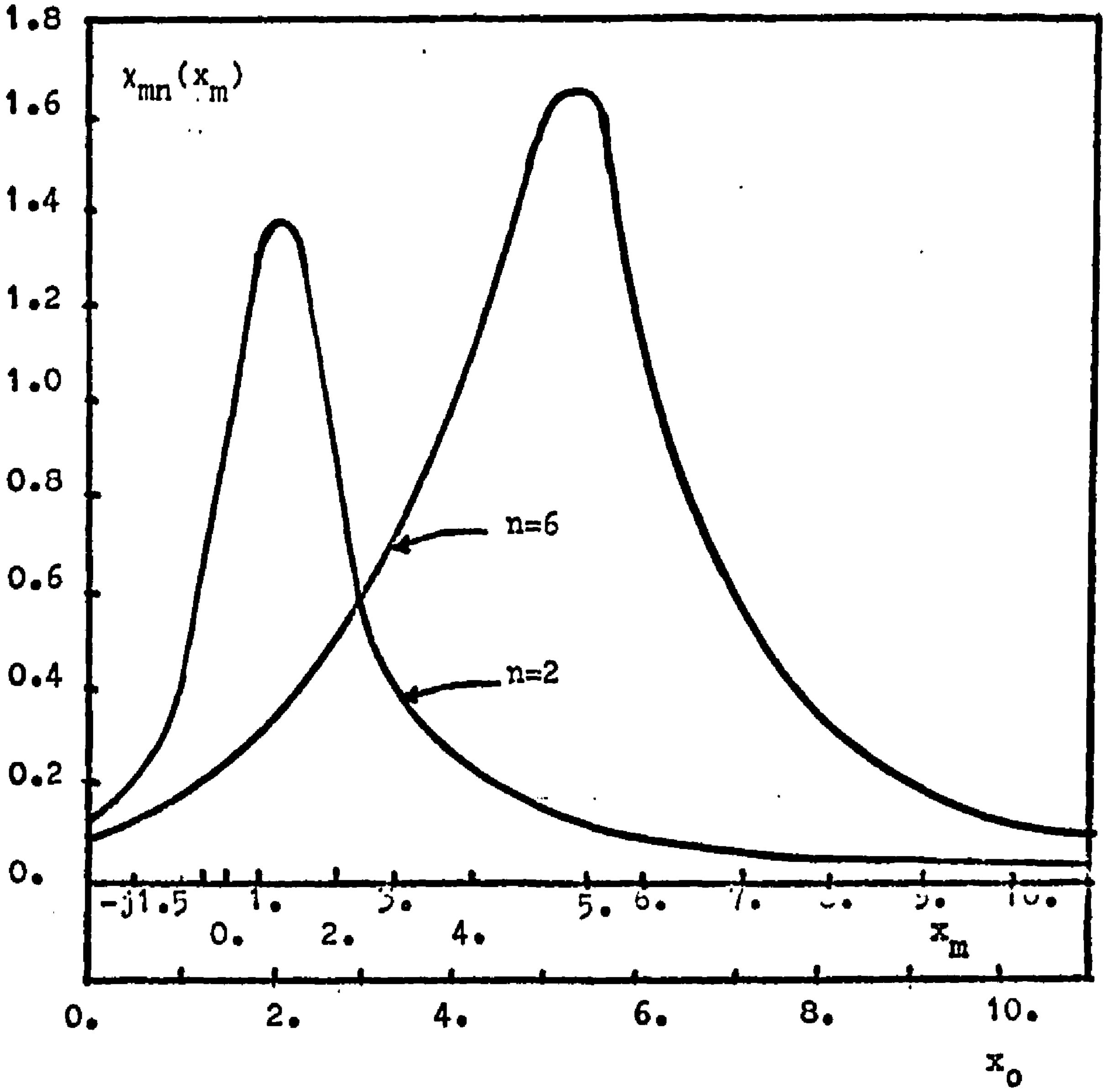


Fig.8

Acoustic Reactance Ratios for L/ma =2 , L/a = 4 , m = 2.*

* These curves are taken from reference 14 (Junger).

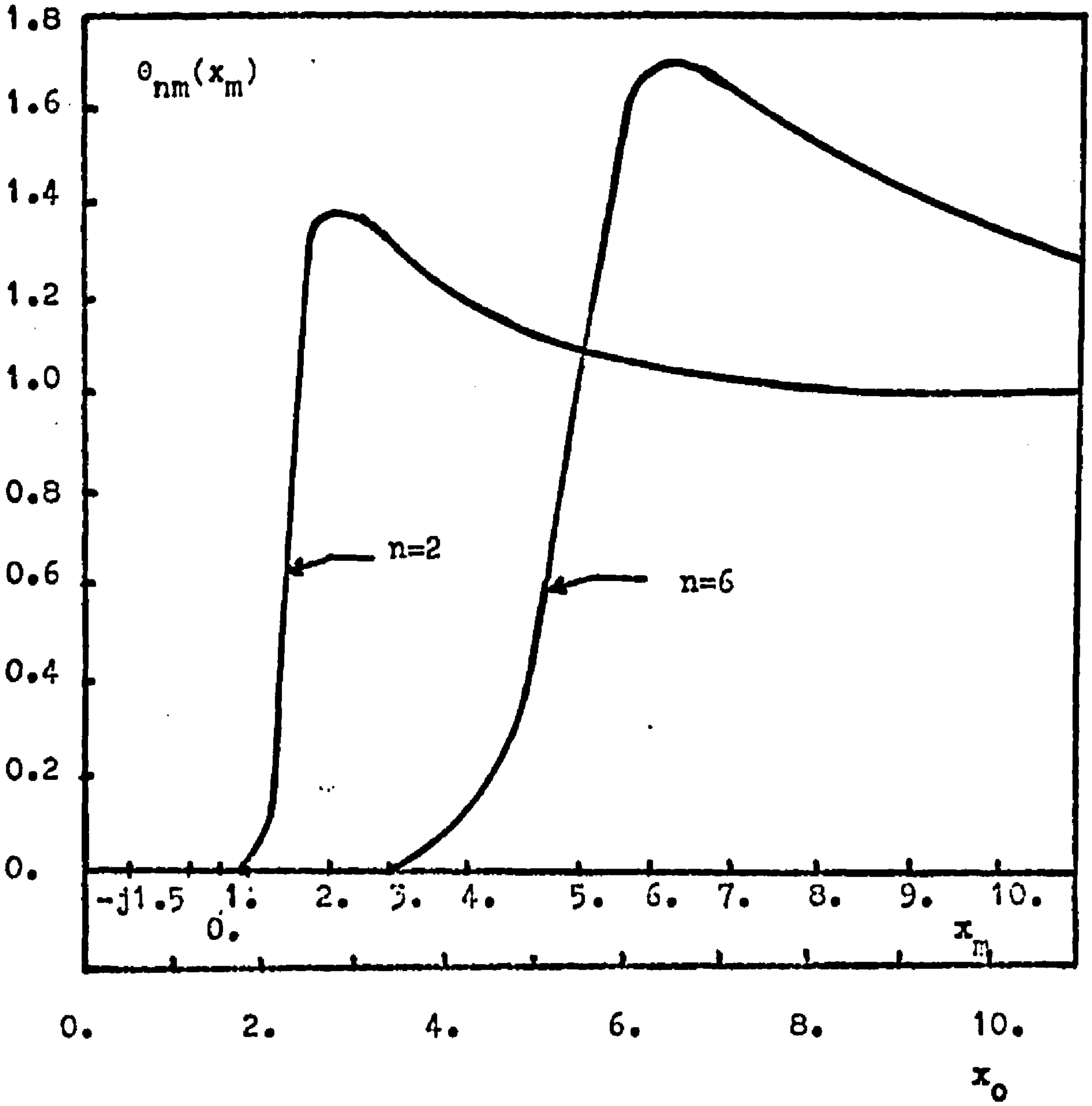


Fig.9

Acoustic Resistance Ratios for L/ma =2, L/a =4 ,m=2.*

*These curves are taken from reference 14 (Junger).

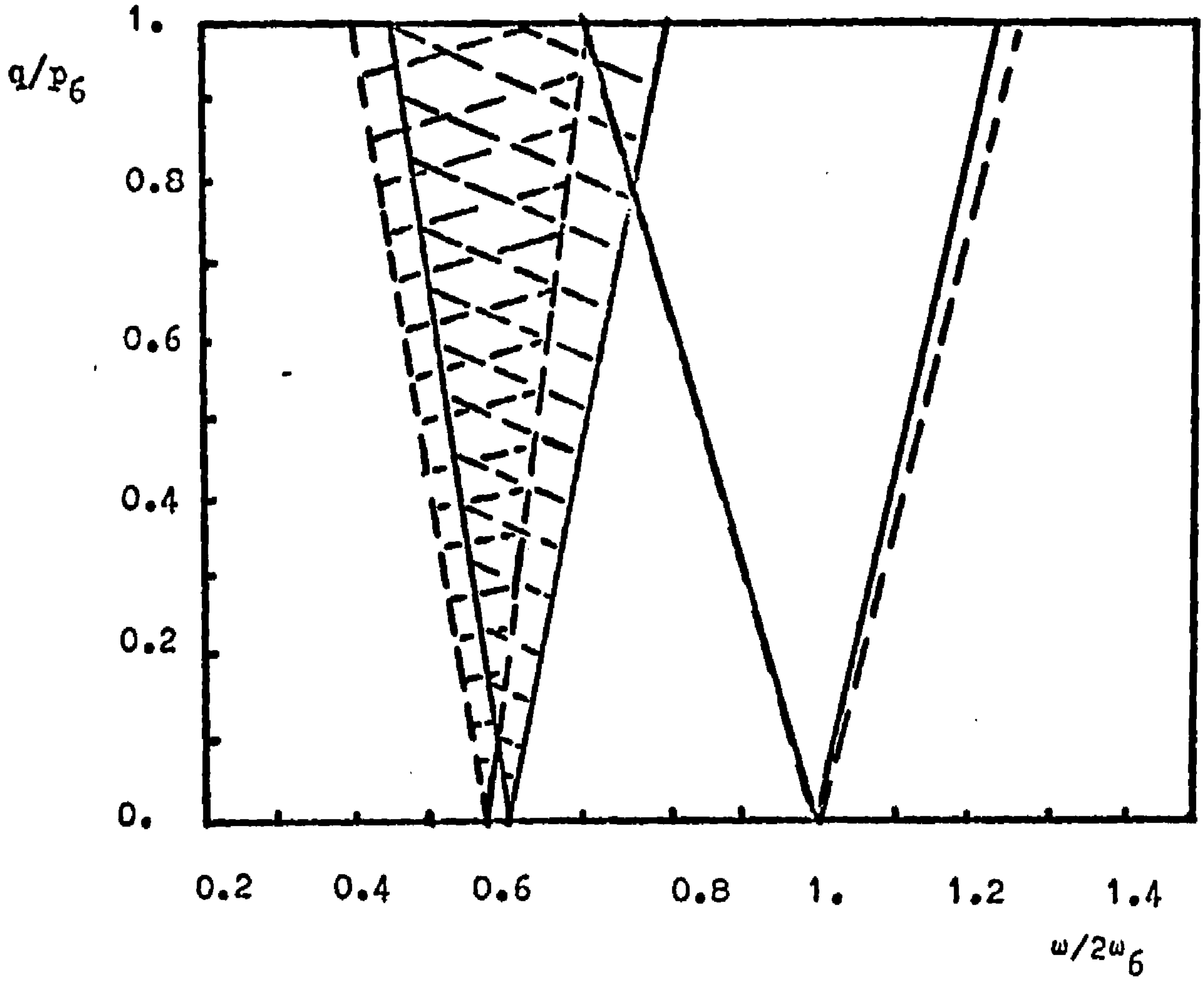


Fig.10

Principal Instability Region for a steel Shell:
-------- Deng-Popelar $L/ma = \infty$ *, ______ $L/ma = 100$,
$\rho_s/\rho = 7.7$, $\nu = 0.3$, $a/h = 100$, $c/c_s = 0.282$, $n=6$,
$p \equiv 0$, $D^{(1)}_{mn} \equiv 0$, or in water ,
in vacuo.

*The Deng -Popelar curves are taken from reference 2.

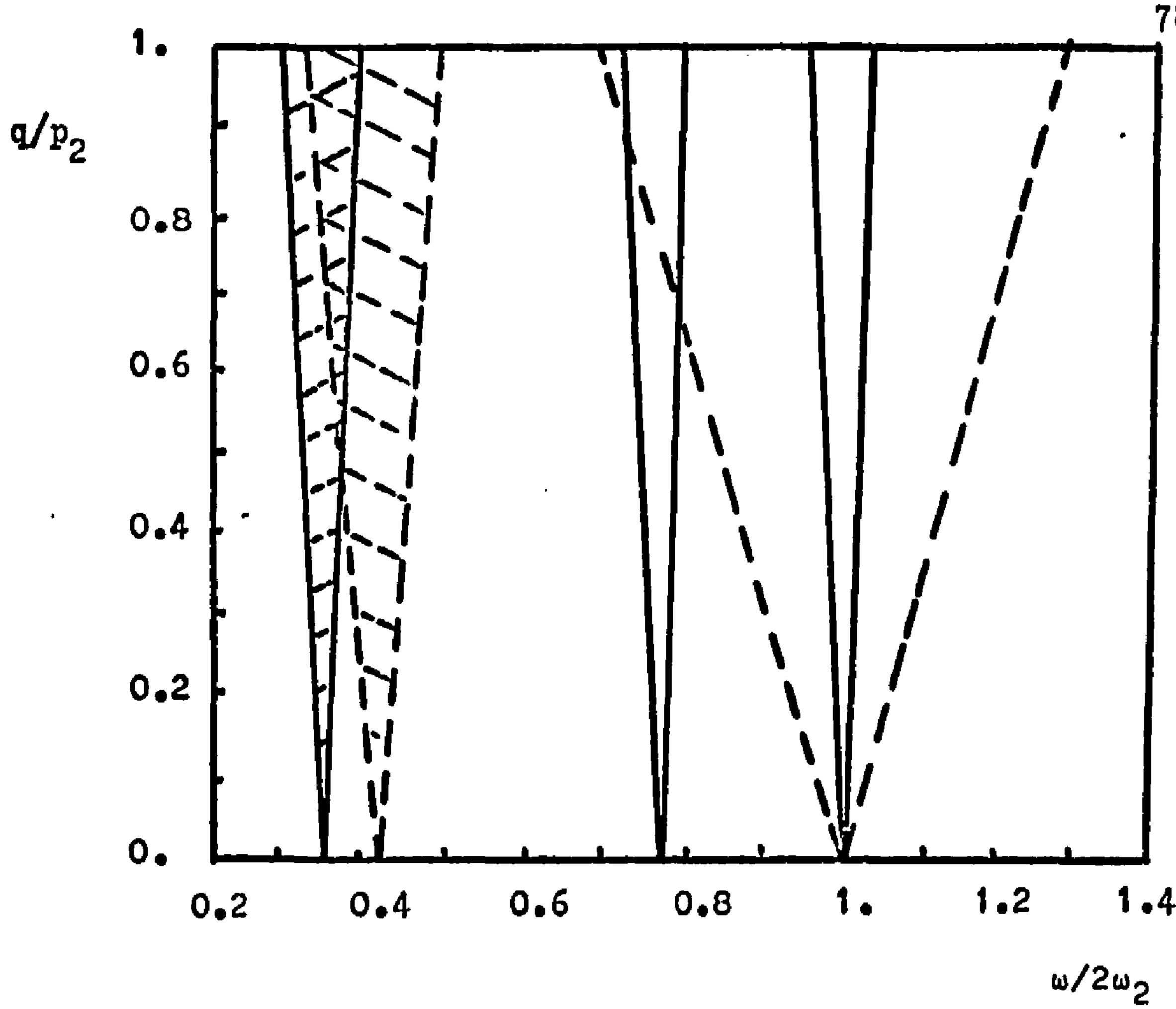


Fig.11

Principal Instability Region for a steel Shell:

------ Deng-Popelar $L/ma = \infty$ *, ______ $L/ma = 1.$,

$\rho_s/\rho = 7.7$, $\nu = 0.3$, $a/h = 100$, $c/c_s = 0.282$,

$n=2$, $p \equiv 0$, $D_{mn}^{(1)} \equiv 0$. or

in water , in vacuo.

* The Deng-Popelar curves are taken from reference 2.

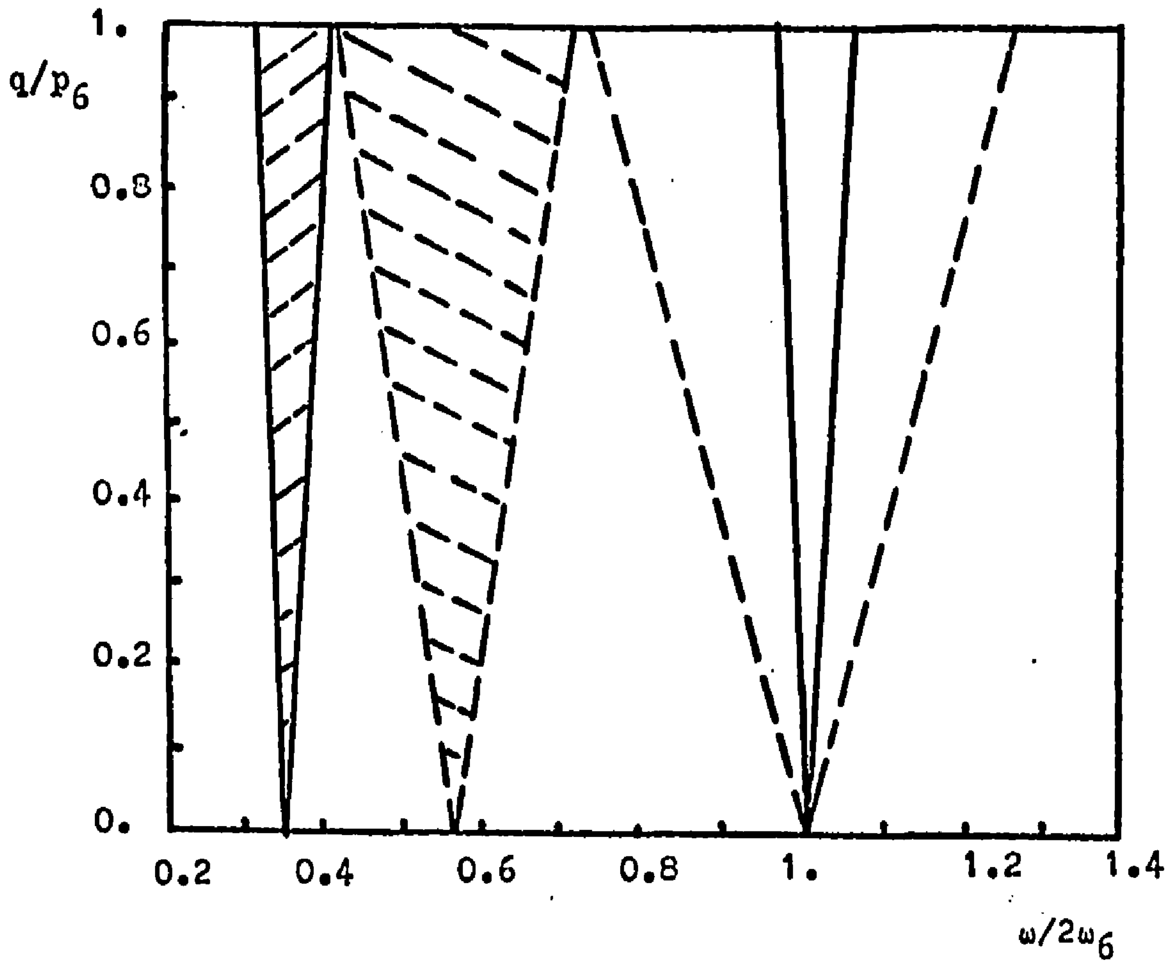


Fig.12

Principal Instability Region for a steel Shell :
------ Deng-Popelar $L/ma = \infty$ * , ______ $L/ma = 1.$,
$\rho_s/\rho = 7.7$, $\nu = 0.3$, $a/h = 100$, $c/c_s = 0.282$,
$n=6$, $p \equiv 0$, $D_{mn}^{(1)} \equiv 0$, [hatched box] or [hatched box]
in water , [empty box] in vacuo .

*The Deng -Popelar curves are taken from reference 2.

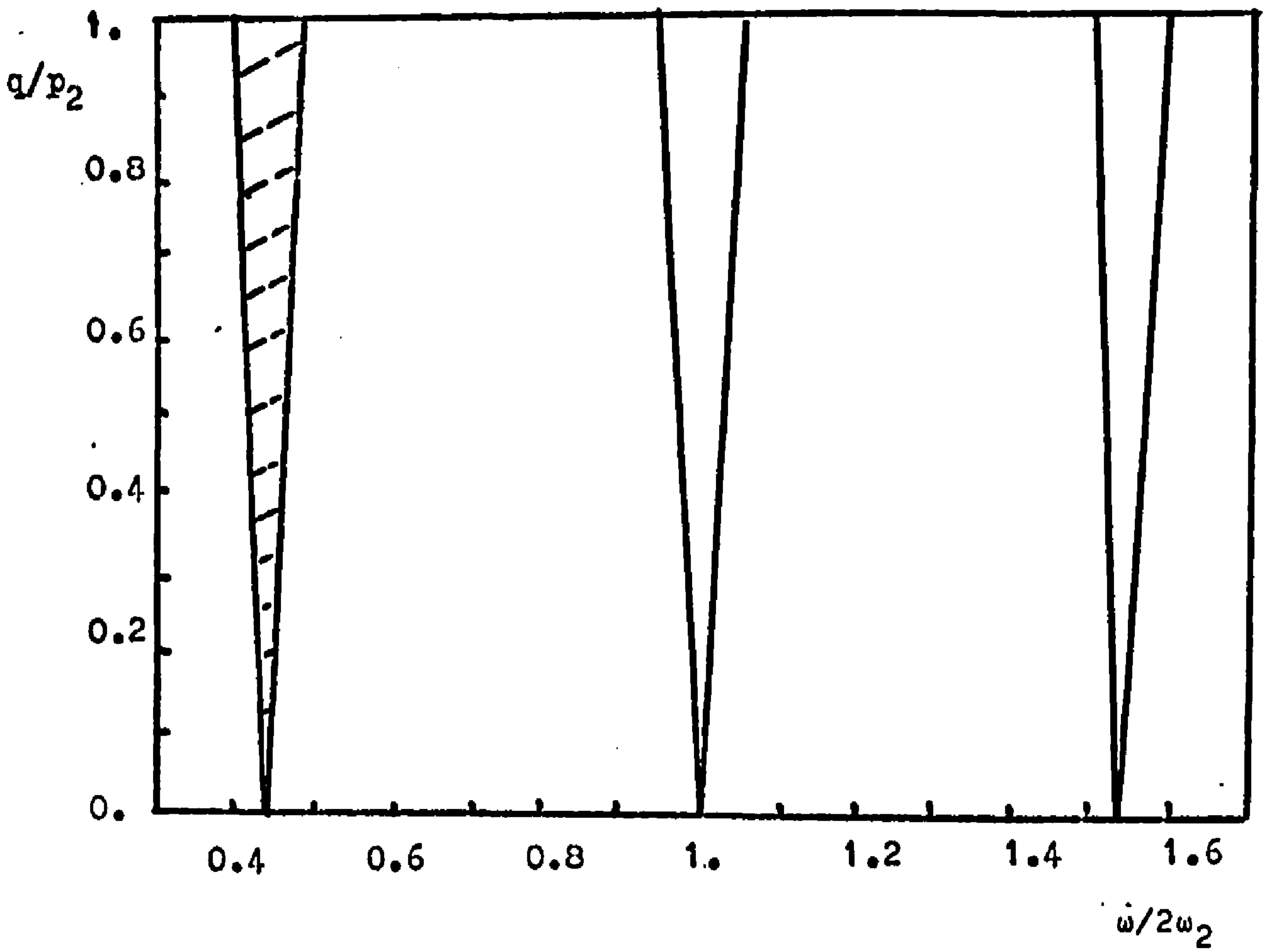


Fig.13

Principal Instability Region for a steel Shell:

$L/ma = 2$, $\rho_s/\rho = 7.7$, $\nu = 0.3$, $a/h = 100$,

$p \equiv 0$, $D^{(1)}_{mn} \equiv 0$, $c/c_s = 0.282$, $n = 2$,

in water , in vacuo .

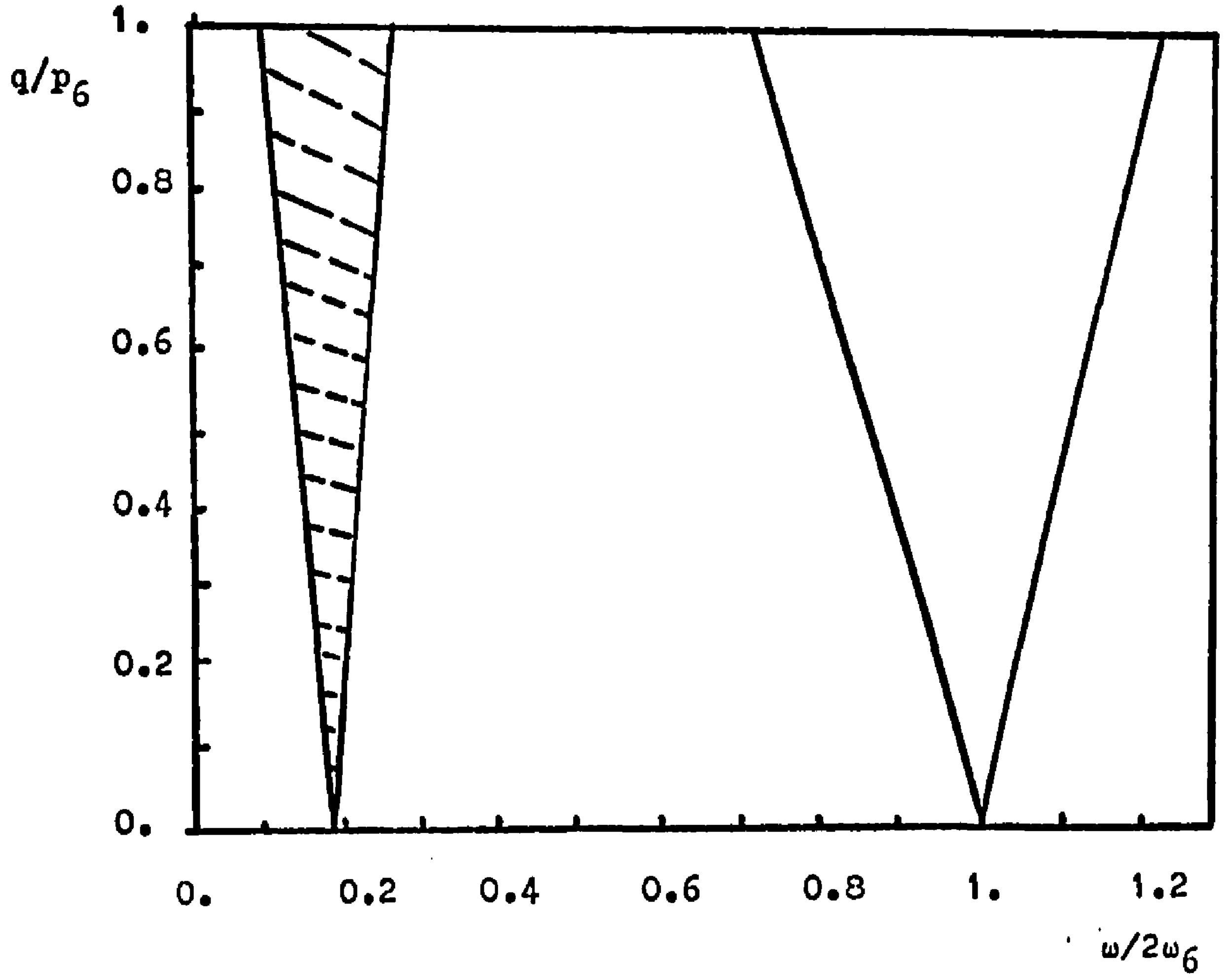


Fig.14

Principal Instability Region for a steel Shell:
$L/ma = 2$, $\rho_s/\rho = 7.7$, $\nu = 0.3$, $a/h = 100$,
$p \equiv 0$, $D_{mn}^{(1)} \equiv 0$, , $c/c_s = 0.282$, $n=6$,
in water , in vacuo .

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