MULTIPOLE FIELD EXPANSIONS AND THEIR USE IN APPROXIMATE SOLUTIONS OF ELECTROMAGNETIC SCATTERING PROBLEMS

DISSERTATION

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ABSTRACT

The representation of electromagnetic fields by multipole expansions and the use of such expansions in the approximate solution of scattering problems is discussed. The problem of representing solutions of Maxwell's equations in homogeneous isotropic regions is considered in Chapter I. Several methods for obtaining multipole expansions from either a knowledge of the source distribution or the values of the tangential fields over a closed surface, or the field components and all their derivatives at a single point are described. The application of multipole fields in the approximate solution of single-body scattering problems is discussed in Chapter II. A method which obtains the best approximation to a match of tangential field components at the scatterer surface is described. The case of a perfectly conducting scatterer is considered, and it is shown that the convergence of field-matching techniques can be verified and a bound on the mean square error in the scattered field obtained if a certain inequality can be derived. Such an inequality is derived for a spherical scattering surface.

The application of approximate field matching techniques is illustrated for the perfectly conducting prolate and oblate

spheroid in Chapter III. First and second order solutions are obtained for a prolate spheroid with 0.35 and 0.28 λ axes and for an oblate spheroid of 0.42 and 0.35λ axes illuminated by a plane electromagnetic wave incident along the symmetry axis. The calculated scattering cross-sections at angles of 30°, 60°, 90° and 120° from the axis are compared with experimentally determined values and it is concluded that the approximation is accurate to within 1 decibel for these scatterers.

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A. INTRODUCTION

The interaction between electromagnetic waves and material bodies is a subject whose ramifications have engaged theoretical physicists, mathematicians, and electrical engineers over many years. In the analysis of scattering and diffraction problems, a formidable array of mathematical techniques and concepts have been assembled, but the discerning student soon learns that unless one chooses the problem to fit the method, in many cases no solution can be found. This study is concerned with the scattering of monochromatic electromagnetic waves by smooth, finite, perfectly conducting bodies. The literature on this subject is too extensive to catalogue here, but several comprehensive bibliographies have been published.[1,2] The only finite three-dimensional shape for which an exact solution has been obtained is the sphere, and this solution was obtained over fifty years ago.[3] An exact solution is defined as one in which the scattered field is given in the form of a convergent series whose coefficients are explicit functions of the scatterer geometry and the location and frequency of the source. Even in this case, computation of the scattered field may be difficult if the series is slowly convergent, as the literature on propagation around a spherical earth will attest.[4-10]

The analysis of scattering problems becomes simplified at either extreme of the source spectrum. At very low frequencies the interaction of the field and the body may be considered as a quasi-static problem, and the electrostatic and magnetostatic solutions can be used to develop a good approximation for harmonic sources. This approach was employed by Rayleigh in the classic treatment of small scatterers, and this range of scatterer size to wavelength is often called the Rayleigh region.[11] At very high frequencies, the methods of geometrical or physical optics may be used to obtain approximate solutions to scattering problems. In essence, the high frequency approximations treat the interaction of waves and bodies as a local phenomenon, and each part of the body is assumed to scatter independently of the field at other parts.

The range of scatterer size between these two extremes may be loosely called the resonance region. Here the scatterer dimensions are comparable to the wavelength, and a small change in the body dimensions or the source frequency may produce much larger oscillatory changes in the scattered field. Scattering from bodies of this size is perhaps the most difficult to approximate, although the problem can be approached from above by extensions of high frequency approximations, or from below by extension of the quasi-static approximation.[12-19]

The goal of this study is the development of methods for approximating the solution to scattering problems in finite series, when the dimensions of the scatterer are of the same order as the wavelength of the source. The problem will be considered in two parts. First, the representation of electromagnetic fields in homogeneous isotropic media by a linear combination of basic multipole fields will be discussed. The second phase of the problem will consider the application of such expansions in various scattering problems, and methods for evaluating the coefficients in a finite series approximation of the scattered field.

B. RELATIONS BETWEEN ELECTROMAGNETIC FIELDS OVER CLOSED SURFACES

A number of useful relations between pairs of electromagnetic fields of the same frequency are easily derived from Maxwell's equations. If two such fields are denoted by subscripts 1 and 2, Maxwell's equations for $e^{-i\omega t}$ time dependence are

$$
\begin{aligned}
&\nabla \times \underline{E}_1 = i\omega\mu\, \underline{H}_1 + \underline{K}_1 , \qquad && \nabla \times \underline{E}_2 = i\omega\mu\, \underline{H}_2 + \underline{K}_2 , \\
&\nabla \times \underline{H}_1 = \underline{J}_1 - i\omega\epsilon\, \underline{E}_1 , \qquad && \nabla \times \underline{H}_2 = \underline{J}_2 - i\omega\epsilon\, \underline{E}_2 .
\end{aligned}
\tag{1}
$$

The vectors $\underline{J}$ and $\underline{K}$ denote electric and "magnetic" source current densities, and it is assumed that the medium is lossless, or if lossy, that the loss currents are accounted for by the use of complex values for μ and ϵ. Forming appropriate dot products and combining, it follows that if ϵ and μ are scalar or symmetrical matrices,

$$(2) \qquad \begin{aligned} &\underline{H}_2 \cdot \nabla \times \underline{E}_1 - \underline{H}_1 \cdot \nabla \times \underline{E}_2 = \underline{H}_2 \cdot \underline{K}_1 - \underline{H}_1 \cdot \underline{K}_2 \ , \\ &\underline{E}_2 \cdot \nabla \times \underline{H}_1 - \underline{E}_1 \cdot \nabla \times \underline{H}_2 = \underline{E}_2 \cdot \underline{J}_1 - \underline{E}_1 \cdot \underline{J}_2 \ , \end{aligned}$$

and since $\nabla \cdot (\underline{A} \times \underline{B}) = \underline{B} \cdot \nabla \times \underline{A} - \underline{A} \cdot \nabla \times \underline{B}$,

$$(3) \qquad \nabla \cdot (\underline{E}_2 \times \underline{H}_1 - \underline{E}_1 \times \underline{H}_2) = -\underline{E}_2 \cdot \underline{J}_2 + \underline{E}_1 \cdot \underline{J}_2 - \underline{H}_2 \cdot \underline{K}_1 + \underline{H}_1 \cdot \underline{K}_2 \ .$$

From the divergence theorem, it follows that for a surface S bounding a finite volume V of space, with outward-pointing normal $\underline{n}$,

$$(4) \qquad \int_S (\underline{E}_2 \times \underline{H}_1 - \underline{E}_1 \times \underline{H}_2) \cdot \underline{n} \ ds = \int_V (\underline{E}_1 \cdot \underline{J}_2 - \underline{E}_2 \cdot \underline{J}_1 + \underline{H}_1 \cdot \underline{K}_2 - \underline{H}_2 \cdot \underline{K}_1) \ dv.$$

If the volume V contains no sources,

$$(5) \qquad \int_S (\underline{E}_2 \times \underline{H}_1 - \underline{E}_1 \times \underline{H}_2) \cdot \underline{n} \ ds = 0 \ .$$

On the other hand, if the sources of fields 1 and 2 are all contained in the volume V, the divergence theorem can be applied to the external region V_{ext} with S as its inner surface and a large concentric sphere S_R as the outer surface. If the radius R of the sphere S_R tends to infinity, application of the radiation condition, which requires that $R|E|$ and $R|H|$ remain bounded as R tends to infinity, and that

$$(6) \qquad \sqrt{\frac{\mu}{\epsilon}} \ \underline{H} \times \underline{n} \xrightarrow[R \to \infty]{} \underline{E}$$

on S_R, leads to zero contribution from the surface S_R. Since the volume integral over V_{ext} is zero, the contribution over S given by Eq. (3) must vanish. It follows that the surface integral of Eq. (5) vanishes provided the sources of fields 1 and 2 are entirely inside or outside of S. In addition,

$$(7) \qquad \int_{V_1} (\underline{E}_2 \cdot \underline{J}_1 + \underline{H}_2 \cdot \underline{K}_1)\, dv = \int_{V_2} (\underline{E}_1 \cdot \underline{J}_2 + \underline{H}_1 \cdot \underline{K}_2)\, dv$$

whenever V_1 contains the sources of field 1 and V_2 contains the sources of field 2. This statement is one form of the reciprocity theorem for electromagnetic fields, originally derived by Lorentz.[20]

The surface integral of Eq. (4) does not vanish when the sources of fields 1 and 2 are on opposite sides of the surface S. In this case,

$$(8) \qquad \int_S (\underline{E}_1 \times \underline{H}_2 - \underline{E}_2 \times \underline{H}_1) \cdot \underline{n}\, ds = \int_{V_1} (\underline{E}_2 \cdot \underline{J}_1 + \underline{H}_2 \cdot \underline{K}_1)\, dv = \int_{V_2} (\underline{E}_1 \cdot \underline{J}_2 + \underline{H}_1 \cdot \underline{K}_2)\, dv$$

where it is assumed that the sources of field 1 are contained in the volume V_1 entirely within S, and those of field 2 are contained in the volume V_2 entirely external to S. Since the value of the surface integral in Eq. (8) is the same for any surface S which separates the sources of fields 1 and 2, it measures a general relation between the two fields, and the name "reaction" has been suggested by Rumsey for this quantity.[21] A concise notation replaces the integral of Eq. (8) by the symbol $\langle 1, 2 \rangle_S$, or more generally, the reaction of field A with field B over the surface S is denoted by $<A, B>_S$. Richmond has shown the significance of Eq. (8) in transmission between antennas in free space or in the presence of scattering bodies, and the reaction concept has been applied and discussed by several authors.[22-26]

It is not possible to have a non-zero field A, due to sources inside a closed surface S, which has zero reaction with all fields due to sources outside S. For example, the reaction between A and the field of an external dipole cannot vanish unless the component of the field A parallel to the dipole at the exterior point vanishes. This follows from the right-hand side of Eq. (8), where the only term contributing to the volume integral over V_2 is $\underline{E}_A \cdot \underline{J}$ for an electric dipole and $\underline{H}_A \cdot \underline{K}$ for a magnetic dipole, where $\underline{E}_A$, $\underline{H}_A$ is the field of A at the dipole. If all such reaction integrals vanish, the field A must be identically zero outside S.

It will be useful to establish an uniqueness theorem which states that there is one and only one distribution of tangential electric (magnetic) field over a closed surface S corresponding to a given distribution of tangential magnetic (electric) field, produced by sources inside S. The medium outside the surface S must be specified, of course. The theorem can be shown to hold as a corollary of a more basic existence theorem. A rigorous proof of such an existence theorem is given in Reference 27. The existence theorem establishes in part that a solution to the scattering problem exists for a body with the surface S which is a perfect conductor of magnetic or electric current. This implies that any tangential distribution of $\underline{E}^i$ due to sources outside S can be matched by a field $\underline{E}^s$ produced by a collection of sources inside S, since the combination of two such fields is necessary for the solution of the scattering problem for a perfect conductor of electricity, where $(\underline{E}^i + \underline{E}^s) \times \underline{n} = 0$. Similarly for any tangential distribution of $\underline{H}^i$ set up by sources outside S, there must exist a set of sources inside S which will produce the same distribution, since for a perfect conductor of magnetic current $(\underline{H}^i + \underline{H}^s) \times \underline{n} = 0$.

Now consider two fields over S due to sources inside S which produce the same tangential electric field on S, but a different tangential magnetic field. It will be shown that this leads to a contradiction. The difference between two such fields will be a valid solution of Maxwell's equations outside S, but will have zero tangential electric field with non-zero tangential magnetic field on S. Its reaction over S with the field of an arbitrary source within S must vanish, from Eq. (5). This reaction is just

$$(9) \qquad \int_S (\underline{E}_A \times \underline{H}_D) \cdot \underline{n} \; ds = 0$$

where $\underline{H}_D$ is the difference field, and $\underline{E}_A$ is the tangential field due to an arbitrary source inside S. But from the existence of a solution for the scattering problem for a perfect conductor with surface S, $\underline{E}_A$ may equally well represent the tangential electric field of an arbitrary source outside S. It follows that the reaction of the difference field with all sources outside S must be zero, and therefore the difference field is identially zero outside S. Due to the continuity of tangential field components, the difference field is also identically zero on S. Thus it has been shown that two different distributions of tangential $\underline{H}$ cannot exist for the same distribution of tangential $\underline{E}$ over a closed surface S due to sources inside S, for a given external environment. A similar proof establishes the uniqueness of tangential $\underline{E}$, given tangential $\underline{H}$.

C. MULTIPOLE FIELDS

1. Mathematical Derivation from Debye Potentials

An important set of solutions to Maxwell's equations are the so-called multipole fields.[28-33] These correspond to a fundamental set of solutions to the wave equation in spherical coordinates, and may be derived readily from a radial Hertzian vector potential employed by Debye.[34]

In a region free of sources, the field vectors $\underline{B}$ or $\underline{D}$ can be expressed as the curl of a vector potential $\underline{\Pi}$:

(10) $$\underline{B} = -i\omega\mu\epsilon \nabla \times \underline{\Pi} ,$$

or

(11) $$\underline{D} = i\omega\mu\epsilon \nabla \times \underline{\tilde{\Pi}} .$$

If the first representation is chosen, since $\nabla \times \underline{E} - i\omega\underline{B} = 0$,

(12) $$\nabla \times (\underline{E} - k^2 \underline{\Pi}) = 0$$

for a homogeneous region, where μ and ϵ are not functions of position. The constant k is equal to $\omega\sqrt{\mu\epsilon}$, or $2\pi/\lambda$, where λ is the wavelength in the medium. The development where μ or ϵ is a function of position has been given by Tai, but will not be considered here.[35] It follows that $\underline{E}$ differs from $k^2\underline{\Pi}$ by the gradient of an arbitrary scalar U:

(13) $$\underline{E} = \nabla U + k^2 \underline{\Pi} .$$

Substituting in $\nabla \times \underline{H} + i\omega\underline{D} = 0$,

(14) $$\nabla \times \nabla \times \underline{\Pi} - \nabla U - k^2 \underline{\Pi} = 0 .$$

If $\underline{\Pi}$ is chosen to be a radial vector potential $\underline{\Pi} = \underline{r}\,\Pi$ in spherical coordinates, Eq. (14) becomes

(15) $$\underline{r} \left\{ \frac{1}{r^2 \sin\theta} \left[\frac{\partial}{\partial\theta} \left(\sin\theta \frac{\partial \Pi}{\partial\theta} \right) + \frac{1}{\sin\theta} \frac{\partial^2 \Pi}{\partial\phi^2} \right] + \frac{\partial U}{\partial r} + k^2 \Pi \right\} = 0,$$

(16) $$\underline{\theta} \left\{ \frac{1}{r} \frac{\partial^2 \Pi}{\partial r\, \partial\theta} - \frac{1}{r} \frac{\partial U}{\partial\theta} \right\} = 0 ,$$

(17) $$\underline{\phi} \left\{ \frac{1}{r\sin\theta} \frac{\partial^2 \Pi}{\partial r\, \partial\phi} - \frac{1}{r\sin\theta} \frac{\partial U}{\partial\phi} \right\} = 0,$$

where $\underline{r}$, $\underline{\theta}$, and $\underline{\phi}$ are unit spherical coordinate vectors. If the scalar U is chosen such that $U = \frac{\partial \Pi}{\partial r}$, Eqs. (16) and (17) are satisfied, and Eq. (15) becomes

$$(18) \qquad \frac{\partial^2 \Pi}{\partial r^2} + \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial \Pi}{\partial\theta}\right) + \frac{1}{r^2 \sin^2\theta} \frac{\partial^2 \Pi}{\partial\phi^2} + k^2 \Pi = 0$$

which can be reduced to the scalar wave equation in Π/r:

$$(19) \qquad \nabla \cdot \nabla(\Pi/r) + k^2(\Pi/r) = 0.$$

The potential Π is r times any solution of the scalar wave equation. Four types of solution are commonly used:

$$(20) \qquad \Pi^{(1)}_{emn} = r\, j_n(kr)\; P_n^m(\cos\theta)\, \cos m\phi\,,$$

$$(21) \qquad \Pi^{(2)}_{emn} = r\, h_n^{(1)}(kr)\; P_n^m(\cos\theta)\, \cos m\phi\,,$$

and a corresponding set $\Pi^{(1)}_{omn}$, $\Pi^{(2)}_{omn}$ with cos $m\phi$ replaced by sin $m\phi$. The radial functions $j_n(kr)$ and $h_n^{(1)}(kr)$ are the spherical Bessel and Hankel functions, and $P_n^m(\cos\theta)$ denotes the associated Legendre polynomial. The properties of these functions are discussed by Stratton, and the notation is consistent with his work.[35] The electromagnetic fields obtained from these potentials in consideration of Eqs. (10), (13) and (14) are called transverse magnetic or TM multipole fields:

$$(22) \qquad \underline{E}_{\substack{emn\\o}} = \nabla \times \nabla \times (\underline{r}\, \Pi_{\substack{emn\\o}})\,,$$

$$(23) \qquad \underline{H}_{\substack{emn\\o}} = -i\omega\epsilon \nabla \times (\underline{r}\, \Pi_{\substack{emn\\o}})\,.$$

The components of these fields in spherical coordinates are given by

(24) $$\underline{E}_{\substack{e\\o}mn} \cdot \underline{r} = \left(\frac{\partial^2}{\partial r^2} + k^2\right) \Pi_{\substack{e\\o}mn} = \frac{n(n+1)}{r} z_n(kr) P_n^m(\cos\theta) \begin{Bmatrix} \cos m\phi \\ \sin m\phi \end{Bmatrix}$$

(25) $$\underline{E}_{\substack{e\\o}mn} \cdot \underline{\theta} = \frac{1}{r} \left(\frac{\partial^2}{\partial r\, \partial\theta}\right) \Pi_{\substack{e\\o}mn} = \frac{1}{r} \hat{Z}_n(kr) \frac{\partial P_n^m}{\partial\theta} \begin{Bmatrix} \cos m\phi \\ \sin m\phi \end{Bmatrix}$$

(26) $$\underline{E}_{\substack{e\\o}mn} \cdot \underline{\phi} = \frac{1}{r \sin\theta} \left(\frac{\partial^2}{\partial r\, \partial\phi}\right) \Pi_{\substack{e\\o}mn} = \mp \frac{m}{r \sin\theta} \hat{Z}_n(kr) P_n^m(\cos\theta) \begin{Bmatrix} \sin m\phi \\ \cos m\phi \end{Bmatrix}$$

(27) $$\underline{H}_{\substack{e\\o}mn} \cdot \underline{r} = 0$$

(28) $$\underline{H}_{\substack{e\\o}mn} \cdot \underline{\theta} = \frac{-i\omega\epsilon}{r \sin\theta} \left(\frac{\partial}{\partial\phi}\right) \Pi_{\substack{e\\o}mn} = \pm \frac{im\omega\epsilon}{\sin\theta} z_n(kr) P_n^m(\cos\theta) \begin{Bmatrix} \sin m\phi \\ \cos m\phi \end{Bmatrix}$$

(29) $$\underline{H}_{\substack{e\\o}mn} \cdot \underline{\phi} = \frac{i\omega\epsilon}{r} \left(\frac{\partial}{\partial\theta}\right) \Pi_{\substack{e\\o}mn} = i\omega\epsilon\, z_n(kr) \frac{\partial P_n^m}{\partial\theta} \begin{Bmatrix} \cos m\phi \\ \sin m\phi \end{Bmatrix} .$$

The generalized spherical Bessel function $z_n(kr)$ is used to denote $j_n(kr)$ for type (1) fields and $h_n^{(1)}(kr)$ for type (2) fields. The symbol $\hat{Z}_n(kr)$ denotes the derivative of the product $kr\, z_n(kr)$ with respect to kr. The use of the term TM to denote these fields is a consequence of Eq. (27). A corresponding set of transverse electric or TE fields can be obtained from the choice of vector potential given in Eq. (11). The result can be obtained from Eqs. (22-29) by replacing $\underline{E}$ by $\underline{H}$ and $\underline{H}$ by $-\underline{E}$ everywhere, as well as interchanging μ and ϵ. This set of fields also satisfies Maxwell's equations, and forms another independent set of multipole fields, which we shall denote by $\widetilde{\underline{E}}_{\substack{e\\o}mn}$ and $\widetilde{\underline{H}}_{\substack{e\\o}mn}$, using the tilda sign to distinguish the TE fields from the TM.

The multipole fields of types (1) and (2) differ in their behavior at large and small distances from the origin, because of different choices of the radial function. Fields of type (2) satisfy the radiation condition, becoming spherical waves at large distances from the origin, while those of type (1) do not. Fields of type (1) are finite at the origin, while those of type (2) become infinite as the origin is approached. In a general sense, fields of type (2) represent the effects of sources at the origin, while those of type (1) are essentially standing waves at the origin, representing fields with equal influx and efflux of energy through any surface surrounding the origin.

The multipole fields defined by Eqs. (24-29) and the corresponding TE multipole fields are complete in homogeneous, source-free regions of space between concentric spherical surfaces. That is to say, the fields in any such region due to sources outside the region can be expanded in a convergent series of multipole fields. In general, fields of type (1) and (2) will be required, but if the region includes the origin, only type (1) fields will be used.

On the other hand, if the region extends to infinity, all sources being contained inside a spherical surface of radius R, only type (2) fields will be used in the expansion outside the sphere of radius R. Instead of proceeding directly to the determination of such expansions, it is useful to consider an alternative definition of multipole fields of type (2) in terms of their sources.

2. The Sources of Multipole Fields

A distribution of currents which give rise to the various type (2) multipole fields can be constructed from infinitesimal current elements at the origin of coordinates. It is convenient to use the concept of the delta function $\delta(x)\ \delta(y)\ \delta(z)$, or in abbreviated notation $\delta(x, y, z)$, which is zero everywhere except at the origin of coordinates and is defined such that

$$(30) \qquad \int_V G(x, y, z)\ \delta(x, y, z)\ dv = G(0, 0, 0)$$

where $G(x, y, z)$ is any scalar function of the coordinates continuous at the origin and $G(0, 0, 0)$ is its value at the origin. The volume V is arbitrary, but must include the origin. In the same manner, the derivative of the delta function with respect to the coordinate x is denoted by $\delta_x(x, y, z)$, and

$$(31) \qquad \int_V G(x, y, z)\ \delta_x(x, y, z)\ dv = G_x(0, 0, 0)$$

where $G_x(0, 0, 0)$ denotes the derivative of G with respect to x at the origin.

If the current distribution giving rise to the TM multipole fields $\underset{o}{\underline{E}}_{emn}$, $\underset{o}{\underline{H}}_{emn}$ is denoted by $\underset{o}{\underline{J}}_{emn}$, the following source distributions are postulated:

$$(32)\qquad \begin{aligned} \underline{J}_{emn} + i\,\underline{J}_{omn} = {} & m(\underline{x} + i\underline{y}) \left\{ \left(\frac{\partial}{\partial ikx} + i \frac{\partial}{\partial iky} \right)^{m-1} P_n^{(m)} \left(\frac{\partial}{\partial ikz} \right) \right\} \delta(x, y, z) \\ & + \underline{z} \left\{ \left(\frac{\partial}{\partial ikx} + i \frac{\partial}{\partial iky} \right)^{m} P_n^{(m+1)} \left(\frac{\partial}{\partial ikz} \right) \right\} \delta(x, y, z) , \end{aligned}$$

where the occurrence of a differential operator to a power m implies that it is to be applied m times, and $P_n^{(m)}\left(\frac{\partial}{\partial ikz}\right)$ denotes the m^{th} derivative of the Legendre polynomial $P_n(u)$ with respect to u, the operator $\frac{\partial}{\partial ikz}$ replacing u in the resultant expression. Derivatives with respect to ikx, iky and ikz are used to make the notation compact; these can be replaced by the appropriate power of ik times the derivatives with respect to x, y, and z. The real part of the expression in Eq. (32) yields the current distribution for TM multipole fields of parity "e" and the imaginary part yields the current distribution for parity "o" TM fields. Equation 32 was deduced from a general principle first applied by Van der Pol and later extended by Erdelyi to the generation of multipole fields.[32,33]

The current distributions associated with the first few multipole fields are given by

$$\underline{J}_{eo1} = \underline{z}\,\delta(x, y, z) \;,\; \underline{J}_{e11} = \underline{x}\,\delta(x, y, z) ,$$

$$\underline{J}_{o11} = \underline{y}\,\delta(x, y, z) \;,\; \underline{J}_{eo2} = \underline{z}\,\delta_z(x, y, z) ,$$

$$J_{e12} = \underline{x}\,\delta_z(x,y,z) + \underline{z}\,\delta_x(x,y,z)\,, \quad J_{o12} = \underline{y}\,\delta_z(x,y,z) + \underline{z}\,\delta_y(x,y,z)$$

$$J_{e22} = \underline{x}\,\delta_x(x,y,z) - \underline{y}\,\delta_y(x,y,z)\,, \quad J_{o22} = \underline{y}\,\delta_x(x,y,z) + \underline{x}\,\delta_y(x,y,z)$$

where proportionality factors have been omitted to simplify the expressions. The orientation of current elements is shown schematically in Fig. 1 for these multipole sources. The arrows indicate relative instantaneous directions of current flow in infinitesimal dipoles. The multipole source distribution is obtained by shrinking this current distribution down to a point function at the origin while increasing the current in opposing dipole pairs so as to maintain a fixed moment.

The sources of TE multipole fields can be represented by the same distribution of currents, but in this case the current elements are magnetic dipoles produced by magnetic currents $\underline{K}$. With this minor change, replacing $\underline{J}_{\substack{e\\o}mn}$ by $-\underline{K}_{\substack{e\\o}mn}$, Eq. (32) holds for the sources of TE multipole fields, and Fig. 1 represents the magnetic current orientations for corresponding multipoles.

To verify that the current distributions of Eq. (32) do produce multipole fields, it is convenient to compare the fields radiated by such currents, in the neighborhood of the origin. The vector potential $\underline{A}$ associated with the current distribution $\underline{J}$ is determined by

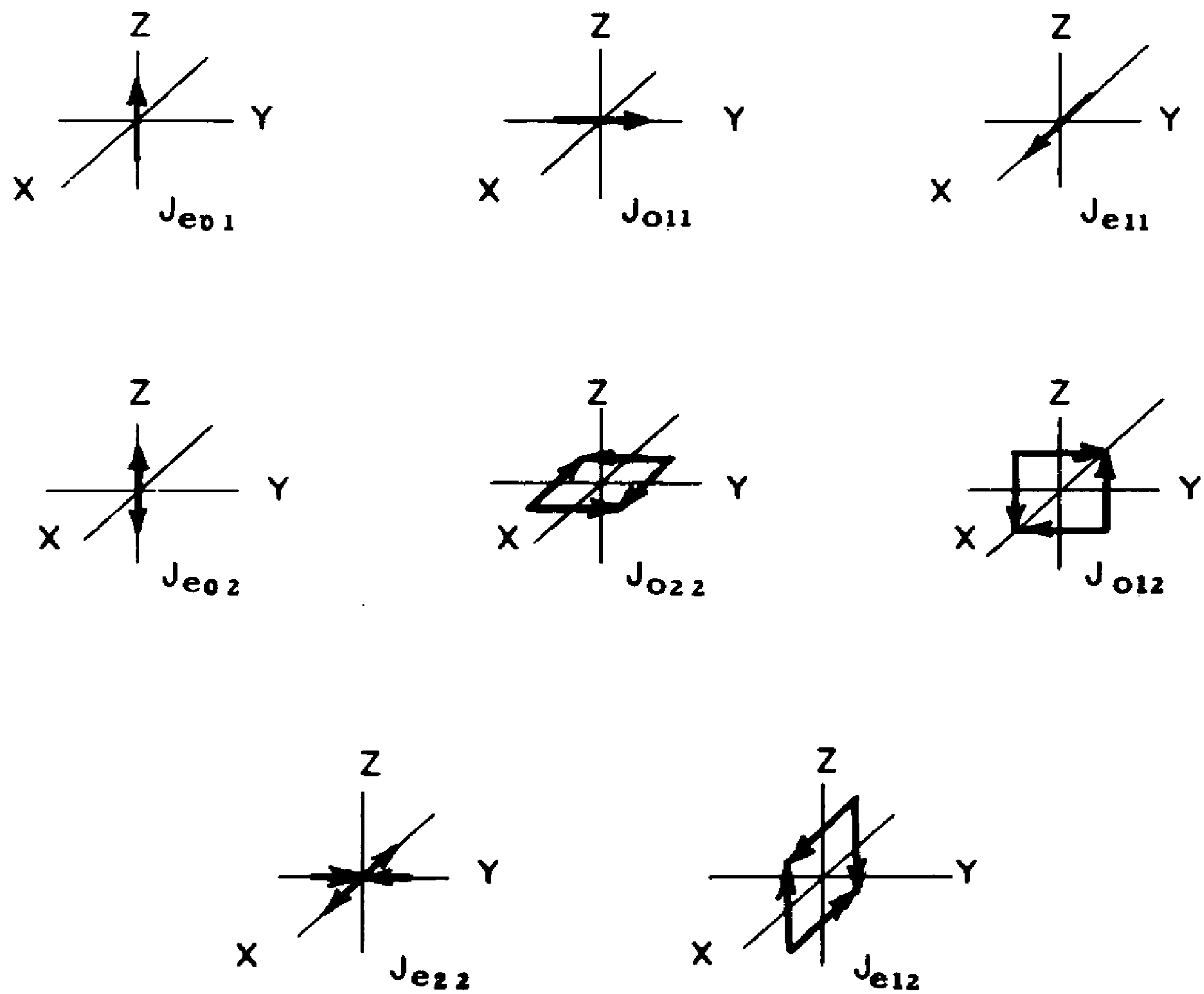


Fig. 1. Source distribution for elementary multipole fields.

$$4\pi \underline{A} = \int_V \underline{J}(x, y, z) \frac{e^{ikR}}{R} \, dv \quad ,$$

(33)

$$R = \sqrt{(x' - x)^2 + (y' - y)^2 + (z' - z)^2} \quad ,$$

where R is the distance between the field point (x', y', z') and a point (x, y, z) on the current distribution. The integration extends over the volume occupied by the currents. For the current distribution of Eq. (32), and for infinitesimal distances R from the origin,

(34)

$$4\pi(\underline{A}_{emn} + i\underline{A}_{omn}) = m(\underline{x} + i\underline{y})\left\{\left(\frac{\partial}{\partial ikx} + i\frac{\partial}{\partial iky}\right)^{m-1} P_n^{(m)}\left(\frac{\partial}{\partial ikz}\right)\right\}\frac{1}{R} + \underline{z}\left\{\left(\frac{\partial}{\partial ikx} + i\frac{\partial}{\partial iky}\right)^m P_n^{(m+1)}\left(\frac{\partial}{\partial ikz}\right)\right\}\frac{1}{R} \quad .$$

The factor e^{ikR} is omitted for sufficiently small R. The result of applying such a differential operator to $\frac{1}{R}$ is given on page 1281 of Reference 37:

(35) $$\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)^m \left(\frac{\partial}{\partial z}\right)^{n-m}\left(\frac{1}{R}\right) = (n-m)!\, R^{-n-1} P_n^m(\cos\theta)\, e^{im\phi},$$

where we have omitted a $(-1)^n$ factor due to a slight change in notation for R. The resultant value of the vector potential is obtained by considering only the leading term in $P_n^{(m)}\left(\frac{\partial}{\partial ikz}\right)$ and $P_n^{(m+1)}\left(\frac{\partial}{\partial ikz}\right)$ which give rise to the dominant terms at short distances from the origin:

(36) $$P_n^{(m)}\left(\frac{\partial}{\partial ikz}\right) \doteq \frac{(2n)!}{2^n n!(n-m)!}(ik)^{m-n}\left(\frac{\partial}{\partial z}\right)^{n-m} \quad .$$

The vector potential is

$$4\pi(\underline{A}_{emn} + i\,\underline{A}_{omn}) = (ik)^{-n+1}\frac{(2n)!}{2^n\,n!}R^{-n}\left\{(\underline{x} + i\,\underline{y})\left(mP_{n-1}^{m-1}(\cos\theta)\,e^{i(m-1)\phi}\right) + \underline{z}\,P_{n-1}^{m}(\cos\theta)\,e^{im\phi}\right\}. \tag{37}$$

In spherical coordinates, this expression becomes

$$\begin{aligned}4\pi(\underline{A}_{emn} + i\,\underline{A}_{omn}) = {} & (ik)^{-n+1}\frac{(2n)!}{2^n\,n!}R^{-n}e^{im\phi}\\ & \left\{\underline{r}\left[m\sin\theta\,P_{n-1}^{m-1}(\cos\theta) + \cos\theta\,P_{n-1}^{m}(\cos\theta)\right]\right.\\ & + \underline{\theta}\left[m\cos\theta\,P_{n-1}^{m-1}(\cos\theta) - \sin\theta\,P_{n-1}^{m}(\cos\theta)\right]\\ & \left. + \underline{\phi}\left[im\,P_{n-1}^{m-1}(\cos\theta)\right]\right\}.\end{aligned} \tag{38}$$

Calculating the resultant magnetic field $\underline{H}$ by $\underline{H} = \nabla \times \underline{A}$, after considerable reduction through use of recurrence formulas for the associated Legendre polynomials, one obtains

$$4\pi(\underline{H}_{emn} + i\,\underline{H}_{omn}) = (ik)^{-n+1}\frac{(2n)!}{2^n\,n!}R^{-n-1}\left\{\underline{\theta}\frac{im}{\sin\theta}P_n^m(\cos\theta)\,e^{im\phi} - \underline{\phi}\frac{\partial P_n^m}{\partial\theta}e^{im\phi}\right\}. \tag{39}$$

Comparing this expression with Eqs. (28) and (29), it is seen that the form of the field produced by the given current distribution is the same as the TM multipole field of like order and parity, insofar as variation with θ and ϕ is concerned. Further, the radial dependence of type (2) multipole fields in the neighborhood of the origin can be approximated by

$$(40) \qquad -i\,\frac{(2n)!}{2^n\, n!}\,(kr)^{-n-1} \approx i\, n_n(kr) \approx h_n^{(1)}(kr) \quad . \qquad r = R \to 0$$

When this expression is substituted in Eqs. (28) and (29), and the result compared with Eq. (39), it is found that the two fields are identical in R, θ and ϕ dependence close to the origin. If the current distribution is to yield the same amplitude of field as the multipole expansion of Eqs. (24-29), the current distribution given by Eq. (32) must be multiplied by a normalizing factor c_n, where

$$(41) \qquad c_n = \frac{4\pi}{k}(i)^{n+1}\sqrt{\frac{\epsilon}{\mu}} \quad .$$

A similar analysis can be made for the TE multipole fields and the associated sources. The current amplitudes must be adjusted by the same factor as given in Eq. (41), with ϵ and μ interchanged, to produce the basic TE multipole fields.

3. Expansion of Electromagnetic Fields in Multipole Series

The mathematical representation of an electromagnetic field in the form of a multipole series may serve several purposes. First, it may be convenient for computation of the field at any point of a region. This would not be the case if the determination of the multipole series required an explicit knowledge of the field at every point ab initio. We shall show how to obtain multipole expansions from a knowledge of the tangential fields over a closed surface, or from the current distribution of its sources, or in some cases from the field and all its derivatives at a single point. A second use of multipole expansions is conceptual, rather than computational. For example, the assumption that one or more fields can be expanded in the form of a multipole series with unknown coefficients may be useful in obtaining a mathematical statement of boundary conditions in terms of these coefficients, which ultimately leads to their determination. In every scattering problem, one field is known and that is the field of the source. For a plane wave, the mathematical description in rectangular coordinates is simple and explicit. Nevertheless, it may be useful to replace this expression by a more complicated multipole expansion if this is the form in which the scattered field or other associated fields are cast.

A second aspect of the multipole expansion which must be considered is its range of validity. It has been stated earlier that such expansions are complete for the representation of fields in any homogeneous source-free region contained between two concentric spherical surfaces with the origin as center. The extension of the representation throughout regions which do not have spherical boundaries may be necessary, for example. In any case, the proper combination of type (1) and type (2) fields to represent various source and scatterer combinations must be chosen before the expansions are explicitly determined.

If the expansion of a given field in terms of multipoles is to be obtained, the coefficients a_{mn} and $\tilde{a}_{mn}$ are required, where

(42)
$$\underline{E} = \sum_n \sum_m (a_{mn} \underline{E}_{mn} + \tilde{a}_{mn} \underline{\tilde{E}}_{mn})$$
$$\underline{H} = \sum_n \sum_m (a_{mn} \underline{H}_{mn} + \tilde{a}_{mn} \underline{\tilde{H}}_{mn}) \quad .$$

We shall denote the coefficients of a type (1) expansion by $a_{mn}^{(1)}$, $\tilde{a}_{mn}^{(1)}$, and those of a type (2) expansion by $a_{mn}^{(2)}$, $\tilde{a}_{mn}^{(2)}$. It is important to distinguish between two types of field expansion: those valid in the neighborhood of the origin, and those valid at infinite distances from the origin. Thus, if the source of a field is not at the origin, it can be expanded in a type (1) multipole series valid at the origin. If the source of a field is contained within a finite closed surface S surrounding the origin, it can be expanded in a type (2) multipole series outside this surface.

In general, the expansion coefficients for a type (2) series can be obtained when the tangential components of the field are known over a surface S enclosing the sources. If X denotes the source field, M_{mn} denotes a TM multipole field of order m, n and $\widetilde{M}_{mn}$ denotes the corresponding TE multipole fields, the reaction of the source field with the various multipole fields of type (1) over S leads to the following system of equations:

$$(43)\quad \begin{aligned} \langle x, M_{mn}^{(i)}\rangle_S &= \sum_i \sum_j \left\{ a_{ij}^{(2)} \langle M_{ij}^{(2)}, M_{mn}^{(i)}\rangle_S + \widetilde{a}_{ij}^{(2)} \langle \widetilde{M}_{ij}^{(2)}, M_{mn}^{(i)}\rangle_S \right\} \\ \langle x, \widetilde{M}_{mn}^{(i)}\rangle_S &= \sum_i \sum_j \left\{ a_{ij}^{(2)} \langle M_{ij}^{(2)}, \widetilde{M}_{mn}^{(i)}\rangle_S + \widetilde{a}_{ij}^{(2)} \langle \widetilde{M}_{ij}^{(2)}, \widetilde{M}_{mn}^{(i)}\rangle_S \right\} . \end{aligned}$$

From the principle stated in Eq. (8), the reaction between multipole fields is the same over any surface S enclosing the origin, so that the right-hand side of Eq. (43) can be evaluated over a sphere of large radius. Because of the orthogonal relation between tesseral harmonics, it then follows that the reaction between any two multipole fields vanishes unless they have the same indices m, n; parity (e or o), and are both TE or TM.[36] Equation (43) thus becomes

$$(44)\quad \begin{aligned} \langle x, M_{mn}^{(i)}\rangle_S &= a_{mn}^{(2)} \langle M_{mn}^{(2)}, M_{mn}^{(i)}\rangle \\ \langle x, \widetilde{M}_{mn}^{(i)}\rangle_S &= \widetilde{a}_{mn}^{(2)} \langle \widetilde{M}_{mn}^{(2)}, \widetilde{M}_{mn}^{(i)}\rangle . \end{aligned}$$

The reaction integrals on the right-hand side of Eq. (44) may be related to the radiated power in the corresponding multipole field, and are given by

(45) $$\epsilon < \widetilde{M}^{(i)}_{mn}, \widetilde{M}^{(i)}_{mn} > = -\mu < M^{(i)}_{mn}, M^{(i)}_{mn} > = \frac{\pi \epsilon_m n(n+1)\ (n+m)!}{c\ \ (2n+1)\ (n-m)!}$$.

where $\epsilon_m = 1,\ m = 0;\ \epsilon_m = 2,\ m \neq 0.$

The multipole coefficients in Eq. (42) are

(46) $$a^{(i)}_{mn} = -\frac{\epsilon_m}{4\pi}\sqrt{\frac{\mu}{\epsilon}}\ \frac{(2n+1)\ (n-m)!}{n(n+1)\ (n+m)!} < x,\ M^{(i)}_{mn} >_S$$

$$\widetilde{a}^{(i)}_{mn} = \frac{\epsilon_m}{4\pi}\sqrt{\frac{\epsilon}{\mu}}\ \frac{(2n+1)\ (n-m)!}{n(n+1)\ (n+m)!} < x,\ \widetilde{M}^{(i)}_{mn} >_S$$.

Equation (46) determines the multipole coefficients from an evaluation of the reaction integral over a surface S (enclosing the origin) on which the tangential field components of X are known. The evaluation of such integrals may be difficult in specific cases, but is greatly simplified if the surface is spherical.

An alternative approach may be used when the current distribution of the source is known. In this case, from Eq. (8), the reaction integral can be evaluated by integrating the scalar product of the multipole field and the current distribution. Thus, if the current distribution producing field X is $\underline{J}$, contained in V, inside S

(47) $$< x\ ,\ M^{(i)}_{mn} >_S = \int_V (\underline{J} \cdot \underline{E}^{(i)}_{mn})\ dv$$

$$< x,\ \widetilde{M}^{(i)}_{mn} >_S = \int_V (\underline{J} \cdot \underline{\widetilde{E}}^{(i)}_{mn})\ dv$$.

Evaluation of multipole coefficients by this means is discussed in Reference 28, where the equivalent of Eq. (47) is obtained by a different approach.

The expansion of fields in interior regions proceeds along similar lines. An expansion in terms of type (1) multipole fields is assumed:

$$\underline{E} = \sum_m \sum_n (a_{mn}^{(1)} \underline{E}_{mn}^{(1)} + \tilde{a}_{mn}^{(1)} \underline{\tilde{E}}_{mn}^{(1)})$$

(48)

$$\underline{H} = \sum_m \sum_n (a_{mn}^{(1)} \underline{H}_{mn}^{(1)} + \tilde{a}_{mn}^{(1)} \underline{\tilde{H}}_{mn}^{(1)}) \quad .$$

If the tangential components of field X are known over a surface S enclosing the origin, the coefficients are obtained by evaluating the reaction integrals of the given field with various type (2) multipole fields, and again because of the orthogonal properties of the multipole fields,

(49)

$$<x, M_{mn}^{(2)}>_S = a_{mn}^{(1)} <M_{mn}^{(1)}, M_{mn}^{(2)}>$$

$$<x, \tilde{M}_{mn}^{(2)}>_S = \tilde{a}_{mn}^{(1)} <\tilde{M}_{mn}^{(1)}, \tilde{M}_{mn}^{(2)}> \quad .$$

Substituting the values of the reaction integrals given in Eq. (45),

(50)

$$a_{mn}^{(1)} = \frac{\epsilon_m}{4\pi} \sqrt{\frac{\mu}{\epsilon}} \frac{(2n+1)\ (n-m)!}{n(n+1)\ (n+m)!} < x, M_{mn}^{(2)}>_S$$

$$\tilde{a}_{mn}^{(1)} = \frac{\epsilon_m}{4\pi} \sqrt{\frac{\epsilon}{\mu}} \frac{(2n+1)\ (n-m)!}{n(n+1)\ (n+m)!} < x, \tilde{M}_{mn}^{(2)}>_S$$

The reaction integrals in Eq. (50) can be replaced by an equivalent volume integral of X over the multipole source distribution at the origin:

(51)

$$< x, M_{mn}^{(2)}>_S = -\int_V (\underline{J}_{mn} \cdot \underline{E})\ dv$$

$$<x, \tilde{M}_{mn}^{(2)}>_S = -\int_V (\underline{K}_{mn} \cdot \underline{H})\ dv \quad .$$

If the field X and its derivatives are known at the origin, the current distributions $\underline{J}_{mn}$ and $\underline{K}_{mn}$ for the multipole sources given in Eq. (32) may be substituted in Eq. (51). When the correct normalizing factors given in Eq. (41) are used, the following formulas are obtained for the multipole coefficients:

$$\left(\frac{a_{emn}}{\lambda_{mn}}\right) + i\left(\frac{a_{omn}}{\lambda_{mn}}\right) = \left\{ m\left(\frac{\partial}{\partial ikx} + i\frac{\partial}{\partial iky}\right)^{m-1} P_n^{(m)}\left(\frac{\partial}{\partial ikz}\right)(E_x + iE_y) + \left(\frac{\partial}{\partial ikx} + i\frac{\partial}{\partial iky}\right)^{m} P_n^{(m+1)}\left(\frac{\partial}{\partial ikz}\right)E_z \right\} \tag{52}$$

$$\left(\frac{\tilde{a}_{emn}}{\lambda_{mn}}\right) + i\left(\frac{\tilde{a}_{omn}}{\lambda_{mn}}\right) = \left\{ m\left(\frac{\partial}{\partial ikx} + i\frac{\partial}{\partial iky}\right)^{m-1} P_n^{(m)}\left(\frac{\partial}{\partial ikz}\right)(H_x + iH_y) + \left(\frac{\partial}{\partial ikx} + i\frac{\partial}{\partial iky}\right)^{m} P_n^{(m+1)}\left(\frac{\partial}{\partial ikz}\right)H_z \right\} . \tag{53}$$

where

$$\lambda_{mn} = -\frac{\epsilon_m (i)^{n+1}}{k} \frac{(2n+1)\ (n-m)!}{n(n+1)\ (n+m)!} . \tag{54}$$

Equations (52), (53) and (54) yield the values of the multipole coefficients of X in terms of its derivatives evaluated at the origin. To the author's knowledge, this type of expansion has not been previously derived.

To illustrate the use of this expansion theorem, the representation of a plane wave field in a multipole series of type (1) will be derived. If the field is x-polarized, and the wave travels in the negative z direction,

$$\underline{E} = \underline{x} \; E_o \, e^{-ikz}$$

(55)

$$\underline{H} = -\underline{y} \, H_o \, e^{-ikz} \qquad .$$

Since the z components of the field vanish, as well as all derivatives with respect to x and y, the only non-zero coefficients obtained from Eqs. (52) and (53) will correspond to m = 1. Further, since $E_y = E_z = 0$ and $H_x = H_z = 0$, only a_{e1n} and $\tilde{a}_{o1n}$ coefficients are obtained:

$$a_{e1n} = -\frac{2E_o}{k}(i)^{n+1}\frac{2n+1}{[n(n+1)]^2}P_n^{(m)}\left(\frac{\partial}{\partial ikz}\right)e^{-ikz}\Big|_{x=0,\,y=0,\,z=0}$$

(56)

$$\tilde{a}_{o1n} = -\frac{2H_o}{k}(i)^{n+1}\frac{2n+1}{[n(n+1)]^2}P_n^{(m)}\left(\frac{\partial}{\partial ikz}\right)e^{-ikz}\Big|_{x=0,\,y=0,\,z=0} \; .$$

Since

$$\left(\frac{\partial}{\partial ikz}\right)^n e^{-ikz} = (-1)^n \, e^{-ikz} = (-1)^n \text{ at } x=0, y=0, z=0$$

(57)
$$P_n^{(m)}\left(\frac{\partial}{\partial ikz}\right)e^{-ikz} = P_n^{(m)}(u)\Big|_{u=-1} = \frac{n(n+1)}{2}(-1)^{n+1} \; .$$

It follows that

$$a_{e1n} = -\frac{(-i)^n}{ik}\frac{2n+1}{n(n+1)}E_o$$

(58)

$$\tilde{a}_{o1n} = \frac{(-i)^n}{ik}\frac{2n+1}{n(n+1)}H_o \qquad ,$$

which are the correct values for the expansion coefficients obtained by standard means.[36]

In summary, the multipole expansion of a field X in a homogeneous region D enclosing the origin can be determined, given (a) the tangential components of X over any closed surface in D surrounding the origin, or (b) the current distribution of the sources of X, assumed outside of D but in the same homogeneous medium, or (c) the values of the field X and all its derivatives at the origin, when D contains the origin. The expansion in type (1) and (2) multipole series was considered as two distinct problems in the derivation of Eqs. (43-51), it being assumed in the first instance that D is the unbounded region exterior to a surface S about the origin, containing the sources of X and in the second instance that D is the region within some closed surface S containing the origin, the sources of X lying outside this surface. This division is unnecessary and cannot always be made, for D may be a doubly-connected region enclosing the origin but not containing it, and the sources of X may be on both sides of this region as illustrated in Fig. 2, where the cross-hatched regions V_1 and V_2 contain sources. In every case, the tangential components of the field X over some surface S in D which encloses the origin will suffice to determine the expansion of X. In general, this will consist of a combination of type (1) and (2) fields. The coefficients of each type are determined by Eqs. (46) and (50) in this case. In a similar fashion, when the expansion coefficients are derived from the source distribution of field X, for the case shown in Fig. 2 where the sources of X are on both sides of the region D, Eqs. (46) and (47) may be used to determine the type (2) series coefficients, the integration in Eq. (47) extending over the internal source distribution in volume V_1. An extension of Eq. (47) to the evaluation of the type (1) series coefficients follows from Eq. (8), and

(59)
$$< x, M_{mn}^{(2)} >_S = - \int_{V_2} (\underline{J} \cdot \underline{E}_{mn}^{(2)})\, ds$$
$$< x, \tilde{M}_{mn}^{(2)} >_S = - \int_{V_2} (\underline{J} \cdot \tilde{\underline{E}}_{mn}^{(2)})\, ds$$

where the integration in this case extends over the external current distribution in V_2. The combination of the type (1) and type (2) expansions then represents **X** in **D**, provided the sources and **D** are in the same homogeneous medium.

As an illustration of the source distribution method, we shall derive the expansion coefficients for a radial electric dipole located at the point (r_o, θ_o, ϕ_o) or (x_o, y_o, z_o) in a homogeneous medium. For points within a sphere of radius r_o about the origin, a type (1) expansion must be used, and by Eqs. (50) and (59), the value of the coefficients are

(60)
$$a_{\substack{e\\o}mn}^{(1)} = - \frac{\epsilon_m}{4\pi} \sqrt{\frac{\mu}{\epsilon}} \frac{(2n+1)\ (n-m)!}{n(n+1)\ (n+m)!} \int_V (\underline{J}_D \cdot \underline{E}_{\substack{e\\o}mn}^{(1)})\, dv ,$$

where $\underline{J}_D$ is the current distribution of the dipole, and the volume V encloses the point (x_o, y_o, z_o). In consideration of Eq. (47), if the dipole is to produce a unit multipole field when placed at the origin, the current distribution $\underline{J}_D$ is

(61) $$\underline{J}_D = -\frac{4\pi}{k}\sqrt{\frac{\epsilon}{\mu}}\ \delta(x-x_o)\ \delta(y-y_o)\ \delta(z-z_o)\,.$$

The volume integral in Eq. (60) thus yields the value of the radial component of $\underline{E}^{(1)}_{\substack{e\\o}mn}$ at the dipole locations, and from Eq. (24)

(62) $$a^{(1)}_{\substack{e\\o}mn} = \frac{\epsilon_m}{k r_o}\frac{(2n+1)\ (n-m)!}{(n+m)!}\ h^{(1)}_n\,(kr_o)\ P^m_n(\cos\theta_o)\begin{Bmatrix}\cos m\phi_o\\ \sin m\phi_o\end{Bmatrix}.$$

In the same fashion, the expansion coefficients of the dipole field in a multipole series valid outside the sphere of radius r_o about the origin are in consideration of Eqs. (24), (46), (47) and (61),

(63) $$a^{(2)}_{\substack{e\\o}mn} = \frac{\epsilon_m}{k\ r_o}\ \frac{(2n+1)\ (n-m)!}{(n+m)!}\ j_n\,(kr_o)\ P^m_n(\cos\theta_o)\begin{Bmatrix}\cos m\phi_o\\ \sin m\phi_o\end{Bmatrix}.$$

Equations (61) and (62) are special cases of a general addition formula for multipole fields. Using Eqs. (32) and (41) for the current distribution associated with higher order multipole fields, the expansion of an arbitrary type (2) multipole located at (r_o, θ_o, ϕ_o) in terms of multipole fields about the origin can be determined by the same method. In every case, a type (1) expansion is obtained for $r \leq r_o$, and a type (2) expansion for $r \geq r_o$.

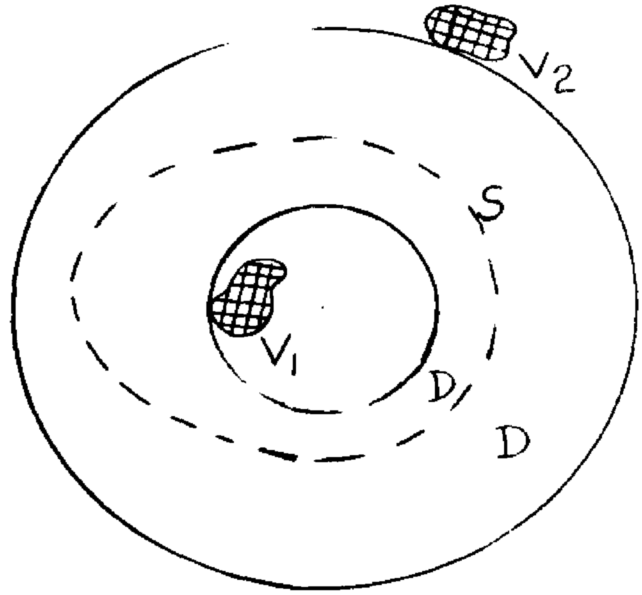


Fig. 2. Multipole expansions in a doubly-connected region D.

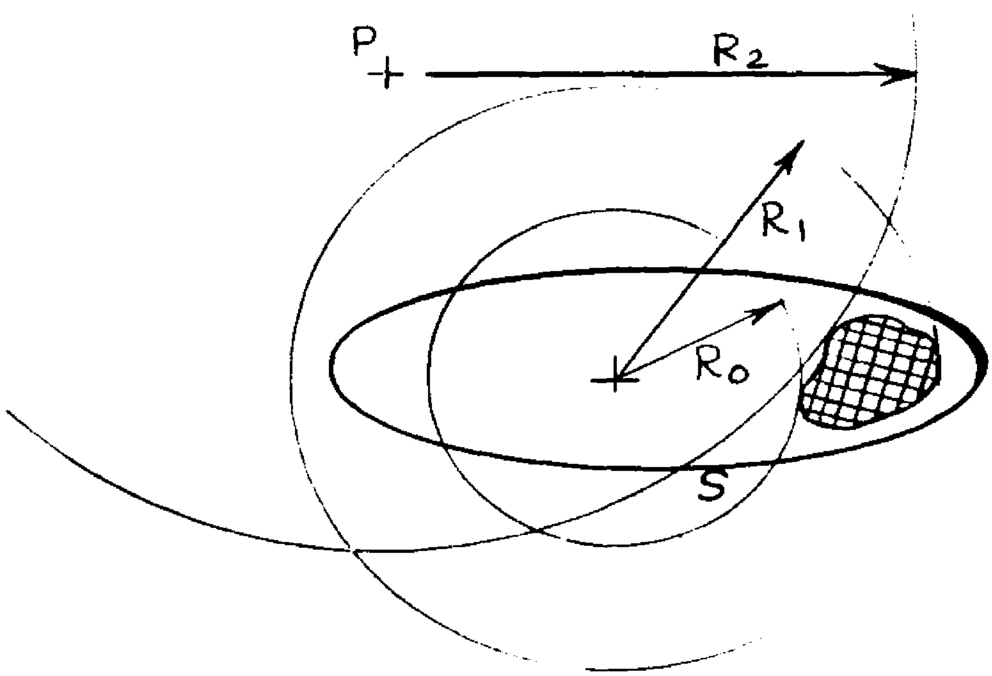


Fig. 3. Regions of convergence for multipole expansions.

4. Convergence of Multipole Expansions

Multipole expansions of types (1) and (2) are in many ways analogous to Taylor and negative power Laurent series expansions. Just as the Taylor and Laurent series have associated circles of convergence, which define the regions in which the series converge, the multipole expansions have associated spherical boundaries which separate regions within which the expansions converge. It is easily shown from the expansion of the field of a dipole at a point (R, θ, ϕ) that the type (1) multipole expansion converges inside the sphere of radius R about the origin, while a type (2) expansion converges outside this sphere. In general, therefore, the type (1) multipole expansion of a field will converge only within the largest sphere about the origin excluding all sources, and the type (2) multipole expansion will converge only outside the smallest sphere about the origin enclosing all sources.

The situation is illustrated in Fig. 3. A distribution of currents confined to the cross-hatched volume produces a field **X**. If the values of the tangential components of **X** are known over the elongated surface S, the coefficients for a type (2) multipole expansion may be determined by Eq. (46). However the expansion obtained will converge only outside the sphere of radius R_I which excludes most of the surface S. An expansion in type (1) multipole fields will converge only inside the sphere of radius R_O.

Although the multipole series of type (2) does not converge everywhere outside or on the surface S, in a broader sense the multipole coefficients may be said to represent the field X there, since the multipole expansion could be used to evaluate the derivatives of X at some external point such as P, and the type (1) multipole expansion of X about P obtained from Eqs. (52) and (53) would converge inside the sphere of radius R_2 about P. Such a process is analogous to analytic continuation of Taylor or Laurent series beyond the circle of convergence to obtain a monogenic analytic function limited only by the natural boundary.[38] In this case the natural boundary would be the cross-hatched volume.

The coefficients $a^{(2)}_{mn}$, $\tilde{a}^{(2)}_{mn}$ of a multipole expansion of type (2) must be square-summable when properly normalized. That is,

$$N = \sum_m \sum_n \epsilon_m \frac{n(n+1)\ (n+m)!}{(2n+1)\ (n-m)!} \left\{ \left|a^{(2)}_{mn}\right|^2 + \left|\tilde{a}^{(2)}_{mn}\right|^2 \right\} \tag{64}$$

must be finite if the radiated power in the represented field is finite. However, a similar restriction on the coefficients in a type (1) expansion does not exist. If field A is expanded in a type (2) multipole series and field B is a type (1) series about the same origin, the reaction over a surface S enclosing the origin is given by the product

$$\langle A, B\rangle_S = \pi \sum_m \sum_n \epsilon_m \frac{n(n+1)\,(n+m)!}{(2n+1)\,(n-m)!} \left\{ \sqrt{\frac{\epsilon}{\mu}}\, \tilde{a}^{(2)}_{mn}\, \tilde{a}^{(1)}_{mn} - \sqrt{\frac{\mu}{\epsilon}}\, a^{(2)}_{mn}\, a^{(1)}_{mn} \right\} \tag{65}$$

whenever this sum converges. The reaction between fields of the same type vanishes, of course.

CHAPTER II
THE SCATTERING OF ELECTROMAGNETIC WAVES BY FINITE SMOOTH BODIES

A. INTRODUCTION

The scattering problems to be considered in this work involve a single scattering body. In such problems, a primary electromagnetic field F is specified in free space or any other unbounded isotropic homogeneous medium with constitutive parameters ϵ_1 and μ_1. This field is called the incident field F, with electric and magnetic field vectors $\underline{E}^i$ and $\underline{H}^i$ defined everywhere in the homogeneous medium. A finite scattering body is now introduced into the medium. This body consists of a singly connected region D composed of a homogeneous isotropic medium with constitutive parameters ϵ_2 and μ_2, where $\epsilon_2 \neq \epsilon_1$, and (or) $\mu_2 \neq \mu_1$. The effect of introducing this body into the primary field can be described by an additional field X, with electric and magnetic field vectors $\underline{E}^s$ and $\underline{H}^s$ defined everywhere outside the body and called the scattered field. The sum of the incident and scattered field then gives the external field everywhere in the presence of the body. An internal field Y is used in the interior region D to replace the incident field F. The electric and magnetic field vectors associated with Y are $\underline{E}^t$ and $\underline{H}^t$, defined in D.

The scattering problem consists of finding a representation for fields X and Y such that each satisfy Maxwell's equations in their region of definition, and

$$(66)\qquad \begin{aligned} \underline{n} \times (\underline{E}^i + \underline{E}^s - \underline{E}^t) &= 0 \\ \underline{n} \times (\underline{H}^i + \underline{H}^s - \underline{H}^t) &= 0 \end{aligned}$$

on the boundary surface S of the scatterer, where $\underline{n}$ is the normal at each point. If the scatterer is a perfect conductor of electricity, the internal field Y is zero and the single boundary condition $\underline{n} \times (\underline{E}^i + \underline{E}^s) = 0$ is used. For a perfect conductor of magnetic current, field Y is also zero and the single boundary condition $\underline{n} \times (\underline{H}^i + \underline{H}^s) = 0$ is used.

In each case, the scattered field X is defined in an exterior region extending to infinity, while the internal field Y (if non-zero) is defined in an interior region. If multipole expansions of fields X and Y are attempted, it is customary to choose an origin inside D, in which case a type (2) expansion must be used for the scattered field X and a type (1) expansion must be used for the internal field Y. The incident field F will be assumed to have its sources outside D, so that it will be represented on the boundary of D by a type (1) expansion.

When the boundary S of the scattering body is a coordinate surface in a coordinate system for which the vector wave equation is separable, all fields can be expanded in series of eigenfunctions appropriate to the boundary, and these are the natural expansions to use. Indeed, it is such an expansion for spherical boundaries which leads to the multipole fields we have considered. In this case, Eq. (66) reduces to an equation relating corresponding multipole terms in the expansions of **X**, **Y** and **F**. The coefficients for fields **X** and **Y** can thus be determined explicitly from those of **F**. However, if S is not one of these very special surfaces, the expression for the fields on the boundary becomes complicated in any eigenfunction expansion, and the resulting series cannot be equated term by term.

Two methods of obtaining approximate solutions to the general problem will be considered. A direct or "brute force" method consists of approximating the fields **X** and **Y** by multipole series with a finite number of terms and determining the coefficients so as to minimize the left-hand side of Eq. (66). This is the method of approximate tangential field matching. A second method to be considered employs a set of exact solutions to the scattering problem, obtained by choosing the internal field **Y** and determining the multipole expansion of fields **F** and **X** from the value of Y on the boundary. By combining a finite number of such solutions, an approximate match to the given incident field F can be obtained as well as an approximate scattered field X.

The method of approximate field matching will be considered first. The only flaw in this method lies in the fact that the error between the approximate solution and the true solution cannot be estimated, however small the quantity on the left-hand side of Eq. (66) becomes. In the case of the perfectly conducting scatterer, the error between the approximate and exact tangential electric field can be determined, but we are unable at present to obtain a bound on the error in the scattered field from a knowledge of the tangential error field alone, although one intuitively feels that such a bound exists. In the case of a spherical boundary, the desired bound can be obtained, but the exact solution is available in this case.

B. SOLUTIONS OBTAINED BY TANGENTIAL FIELD MATCHING

An approximate solution to the scattering problem can be obtained by adjusting the amplitude of N independent internal fields and M scattered fields such that the mean-square deviation from match of tangential fields at the body surface is minimized. A tangential difference field $\underline{E}_t^d$, $\underline{H}_t^d$ is defined on S by

$$\left[\underline{n} \times \left(\underline{E}^i + \sum_{n=1}^{M} a_n \underline{E}_n^s - \sum_{n=1}^{N} b_n \underline{E}_n^t \right) \right] \times \underline{n} = \underline{E}_t^d \tag{67}$$

$$\left[\underline{n} \times \left(\underline{H}^i + \sum_{n=1}^{M} a_n \underline{H}_n^s - \sum_{n=1}^{N} b_n \underline{H}_n^t \right) \right] \times \underline{n} = \underline{H}_t^d$$

where $\underline{E}^i$, $\underline{H}^i$ is the incident field; $\underline{E}_n^s$, $\underline{H}_n^s$ the n^{th} choice of exterior field with amplitude a_n; and $\underline{E}_n^t$, $\underline{H}_n^t$ are the n^{th} choice of internal field with amplitude b_n. To obtain a least-square approximate solution, an appropriate squared norm of the difference field over S is defined. For example,

$$W = \int_S \left\{ (\underline{E}_t^d) \cdot (\underline{E}_t^d)^* + (\underline{H}_t^d) \cdot (\underline{H}_t^d)^* \right\} ds, \tag{68}$$

where the asterisk denotes the complex conjugate of a quantity, may be minimized with respect to the choice of coefficients a_n, b_n. If W can be made to vanish, an exact solution is obtained, so it is reasonable to infer that for small values of W, the approximation will be close to the true solution. To obtain a quantitative definition of closeness, a bound must be established between the exact and approximate solutions in terms of W. Since no component of the exact scattered field $\underline{E}^s$, $\underline{H}^s$ or internal field $\underline{E}^t$, $\underline{H}^t$ is known, such a bound is difficult to obtain.

In the case of a perfectly conducting scatterer, the situation is simpler, since the internal field vanishes and the exact tangential $\underline{E}^s$ is known. Equations (67) and (68) are replaced by

$$\left[\underline{n} \times \left(\underline{E} + \sum_{n=1}^{M} a_n \underline{E}_n^s \right) \right] \times \underline{n} = \underline{E}_t^d \tag{69}$$

$$W = \int_S (\underline{E}_t^d) \cdot (\underline{E}_t^d)^* \, ds \quad . \tag{70}$$

The coefficients a_n are chosen so as to minimize W, and an approximate scattered field $\underline{E}^s$ is obtained. In this case the field $\underline{E}_t^d$ measures the difference between exact and approximate tangential scattered electric field. The difference field is a valid solution of Maxwell's equations in the exterior region and measures everywhere the difference between exact and approximate solutions. Since the tangential component is known, it may be possible to obtain a bound on the power radiated by the difference field.

From the uniqueness theorem, it is clear that a distribution of tangential $\underline{E}$ over a closed surface S due to sources inside S determines the tangential $\underline{H}$ field over S, and the two produce a definite radiation field outside S. Although determining the radiation pattern from a knowledge of tangential $\underline{E}$ alone is merely another statement of the scattering problem for a perfectly conducting body, just as is the determination of tangential $\underline{H}$ from a knowledge of tangential $\underline{E}$, it is in some respects simpler to obtain a bound for the average radiated power in terms of a norm of tangential $\underline{E}$.

To illustrate, such a bound will be determined for a spherical surface. Any external field due to sources inside the sphere can be represented by a multipole series of type (2):

$$\underline{E} = \sum_n \sum_m \left(a_{mn}^{(2)} \underline{E}_{mn}^{(2)} + \tilde{a}_{mn}^{(2)} \underline{\tilde{E}}_{mn}^{(2)} \right) \tag{71}$$

$$\underline{H} = \sum_n \sum_m \left(a_{mn}^{(2)} \underline{H}_{mn}^{(2)} + \tilde{a}_{mn}^{(2)} \underline{\tilde{H}}_{mn}^{(2)} \right) .$$

The average radiated power is given by

$$P = \frac{1}{4} \int_S \left(\underline{E} \times \underline{H}^* + \underline{E}^* \times \underline{H} \right) \cdot \underline{n} \, ds \tag{72}$$

and in terms of multipole coefficients, this becomes

$$P = \pi \sum_n \sum_m \epsilon_m \frac{n(n+1)\,(n+m)!}{(2n+1)\,(n-m)!} \left\{ \sqrt{\frac{\varepsilon}{\mu}} \left| a_{mn}^{(2)} \right|^2 + \sqrt{\frac{\mu}{\epsilon}} \left| \tilde{a}_{mn}^{(2)} \right|^2 \right\} . \tag{73}$$

The quantity W in Eq. (70), in this case

$$W = \int_S \left(\underline{E} \times \underline{n} \right) \cdot \left(\underline{E} \times \underline{n} \right)^* \, ds \tag{74}$$

can be reduced, because of the orthogonality of multipole fields on a spherical surface, to

$$(75) \qquad W = \sum_n \sum_m \left\{ |a_{mn}^{(2)}|^2 \int_S (\underline{E}_{mn}^{(2)} \times \underline{n}) \cdot (\underline{E}_{mn}^{(2)} \times \underline{n})^* ds + |\tilde{a}_{mn}^{(2)}|^2 \int_S (\underline{\tilde{E}}_{mn}^{(2)} \times \underline{n}) \cdot (\underline{\tilde{E}}_{mn}^{(2)} \times \underline{n})^* ds \right\} .$$

Evaluating the integrals over a sphere of radius r, where $\rho = kr$, one obtains

$$(76) \qquad W = 2\pi \sqrt{\frac{\mu}{\epsilon}} \sum_n \sum_m \epsilon_m \frac{n(n+1)\ (n+m)!}{(2n+1)\ (n-m)!} \left\{ \sqrt{\frac{\epsilon}{\mu}}\ |a_{mn}^{(2)}|^2\ |\hat{H}_n(\rho)|^2 + \sqrt{\frac{\mu}{\epsilon}}\ |\tilde{a}_{mn}^{(2)}|^2\ |\rho h_n(\rho)|^2 \right\} .$$

If the greatest lower bound for $|\hat{H}_n(\rho)|^2$ and $|\rho h_n(\rho)|^2$ is denoted by L,

$$(77) \qquad P \leq \frac{1}{2} \sqrt{\frac{\epsilon}{\mu}}\ \frac{W}{L} .$$

The bound L is determined by $|\hat{H}_n(\rho)|^2$, since it can be shown that $|\rho h_n(\rho)|^2 \geq 1$. The variation of L with ρ is shown in Fig. 4. For large ρ, L varies inversely as the one-third power of ρ. It is seen that an upper bound on the radiated power in terms of W can be given for a spherical surface. This bound increases with the radius of the sphere in terms of wavelength, and it seems reasonable to choose as an upper bound for a smooth non-spherical surface the corresponding bound for the sphere which just encloses the surface. However, this choice is not rigorously established.

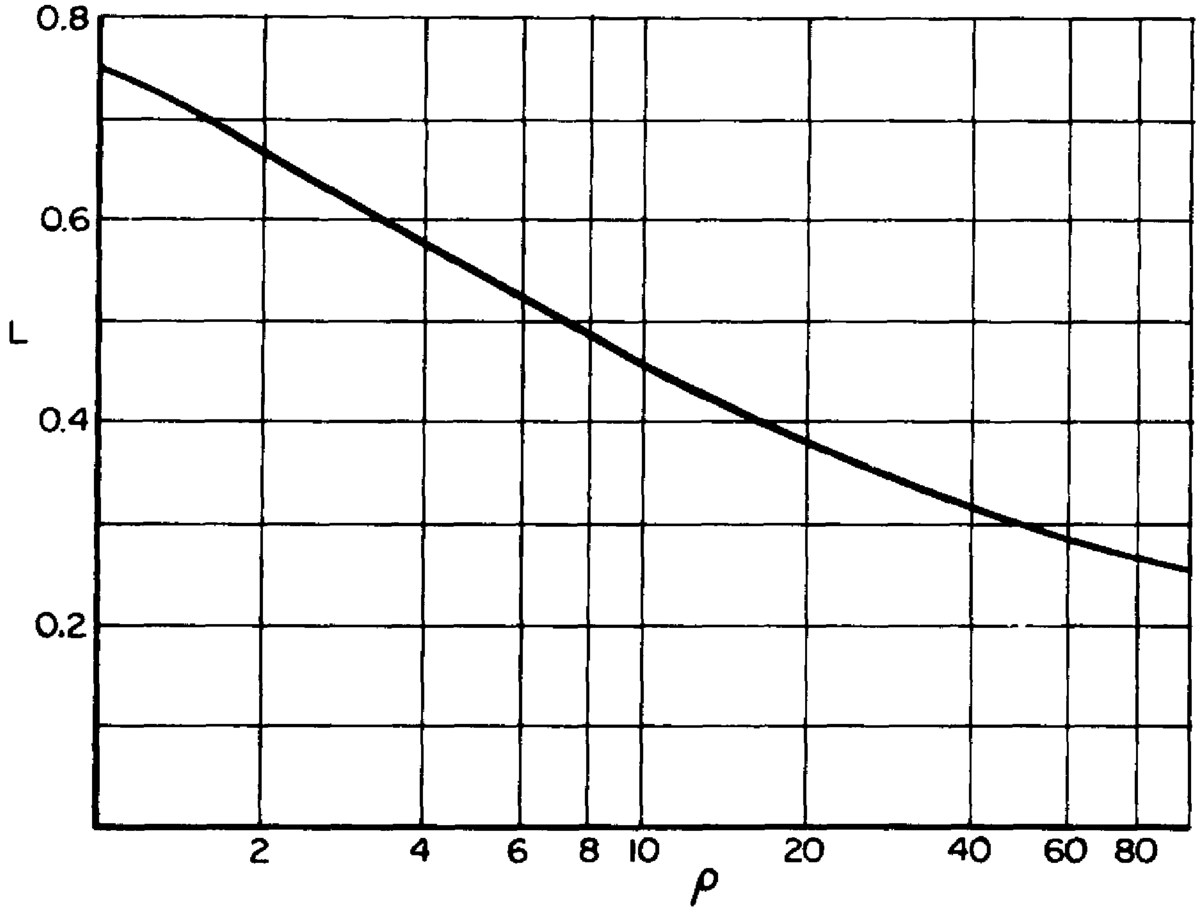


Fig. 4. A lower bound on $|\hat{H}_n(\rho)|^2$ as a function of ρ.

Returning to the approximate field matching technique of Eqs. (67-70), the particular choice of N independent scattered fields X_n and M internal fields Y_m has not been specified. In general, these could be any set of independent solutions to Maxwell's equations in the region of definition, but we shall consider the use of multipole fields within a fixed origin in the scattering body. The number of terms required for a close approximation to the correct field will naturally depend upon the nature of the incident field F, the boundary shape, and the choice of origin. When the body has one or more planes of symmetry, it is simplest to choose an origin common to these. If all the multipole fields M_{emn}, $\widetilde{M}_{emn}$ of one type up to a maximum value n_o of the index n are used, $(n_o + 1)$ $(n_o + 2)$ terms are involved in the associated multipole series, since such fields with $m > n$ do not exist. For the multipole fields M_{omn}, $\widetilde{M}_{omn}$, (n_o) $(n_o + 1)$ terms are involved since these fields with $m \geq n$ do not exist. In many cases, the symmetry of the incident field or the body greatly restricts the choice of the parameters m, n, e and o. The case for a rotationally symmetric body will be described later. It is clear that a multipole expansion of fields X and Y will be exact for a spherical boundary, and rapidly convergent for boundaries which are nearly spherical. In this respect, the method is similar to one described by Mushiake, in which small perturbations in the boundary surface from a sphere were treated analytically.[39]

The convergence of the multipole expansions in regions with non-spherical boundaries may be questioned. In regions with boundaries such as the surface S in Fig. 3, multipole series used to represent a tangential field on the boundary converge in the mean to the given field, and therefore it is possible to minimize the quantity W given by Eqs. (68) and (70) without uniform convergence of the approximation at every point of the boundary. Calderon has considered the multipole representation of tangential fields over an arbitrary surface S with a continuously turning tangent plane. Using methods of functional analysis, it can be shown that any continuous bounded tangential electric field over S can be approximated as close as desired by a type (2) multipole field expansion about an interior point.[40]

The minimization of the parameter W is in some respects similar to a variational approximation to the solution of scattering problems described by Kouyoumjian.[41] In the application of the variational method to perfectly conducting scatterers, for example, the surface current may be expanded in a finite series of independent current distributions whose amplitudes are chosen so as to render an expression for the far-zone scattered field stationary. Unlike the method of approximate field matching, however, the amplitudes obtained will change with the direction in which the scattered field is evaluated. In both methods, it is difficult to obtain bounds on the error incurred in a finite expansion.

C. MULTIPOLE EXPANSIONS OF A SET OF EXACT SOLUTIONS

As an alternative to the approximate field matching method, a technique which obtains a set of exact solutions to the scattering problem will be considered. Since it is difficult to obtain the scattered field X and the internal field Y for a given incident field F, the internal field Y will be chosen and X and F determined. Let us assume $Y = Y_o$ has been chosen and is represented by a particular type (1) multipole series in the interior of the body. This series may consist of a single term. The tangential components of Y_o on the scatterer surface can be obtained explicitly. A multipole expansion in type (1) and (2) terms valid in the exterior region can then be determined from Eqs. (46) and (50), using only the tangential components of Y_o on S. The series of type (1) thus obtained is identified as the incident field F_o and the series of type (2) is identified as the scattered field X_o. A particular solution X_o, Y_o, F_o of the scattering problem is obtained. The process can be repeated with other choices of the internal field $Y = Y_1$, Y_2, etc. A set of solutions X_i, Y_i, F_i are thus generated, in principle, although their exact determination may require an infinite number of multipole terms in each case for X_i and F_i. An approximate solution to the scattering problem is now obtained by expanding the true incident field F in a finite combination of the F_i and the associated scattered field X and internal field Y by the same finite combination of the X_i and Y_i. If all fields are expressed in multipole series and a finite number of terms retained, the expansion of F in terms of the F_i reduces to the solution of a system of linear equations.

When applied to a perfectly conducting scatterer, this method reduces to the choice of the tangential field on the surface of the scatterer. Since $\underline{n} \times \underline{E} = 0$ on such a surface, only tangential $\underline{H}$ must be chosen. The determination of incident and scattered field multipole expansions follows as before from Eqs. (46) and (50). The set of fields F_i, X_i obtained are then used to obtain an approximate representation of the true incident field and the associated scattered field.

In many respects, this method is similar to one proposed by Rumsey, in which the reaction between the approximate scattered field and various test sources is minimized and the associated expansion coefficients for the scattered field thereby determined.[illegible] It is believed that the use of the multipole representation for the fields involved will enable this procedure to be systematized for high-speed numerical computation.

CHAPTER III
APPLICATION TO PLANE WAVE SCATTERING BY A PERFECTLY CONDUCTING SPHEROID

A. INTRODUCTION

The "perfectly" conducting scatterer is a convenient mathematical abstraction which is approximated closely in the microwave region by most metallic bodies. For scatterers of this type, the solution of the boundary value problem only requires the determination of a scattered field defined in the external region whose tangential $\underline{E}$ components reduce to the negative of those for the incident field at every point of the conducting surface. The scattering problem is further simplified in the case of rotationally symmetric bodies, where incident fields with a given azimuthal dependence produce scattered fields with the same azimuthal dependence. If the incident field is a plane wave along the symmetry axis, the multipole expansion of incident and scattered fields requires the use of $\underline{E}_{e1n}$, $\underline{H}_{e1n}$, $\underline{\tilde{E}}_{o1n}$ and $\underline{\tilde{H}}_{o1n}$ type multipole fields only. The restriction to the single eigenvalue m = 1 reduces the determination of the expansion coefficients to a single parameter family rather than a double parameter set in m and n.

The choice of coordinates for this problem are shown in Fig. 5. The incident plane wave is polarized in the x-direction, traveling

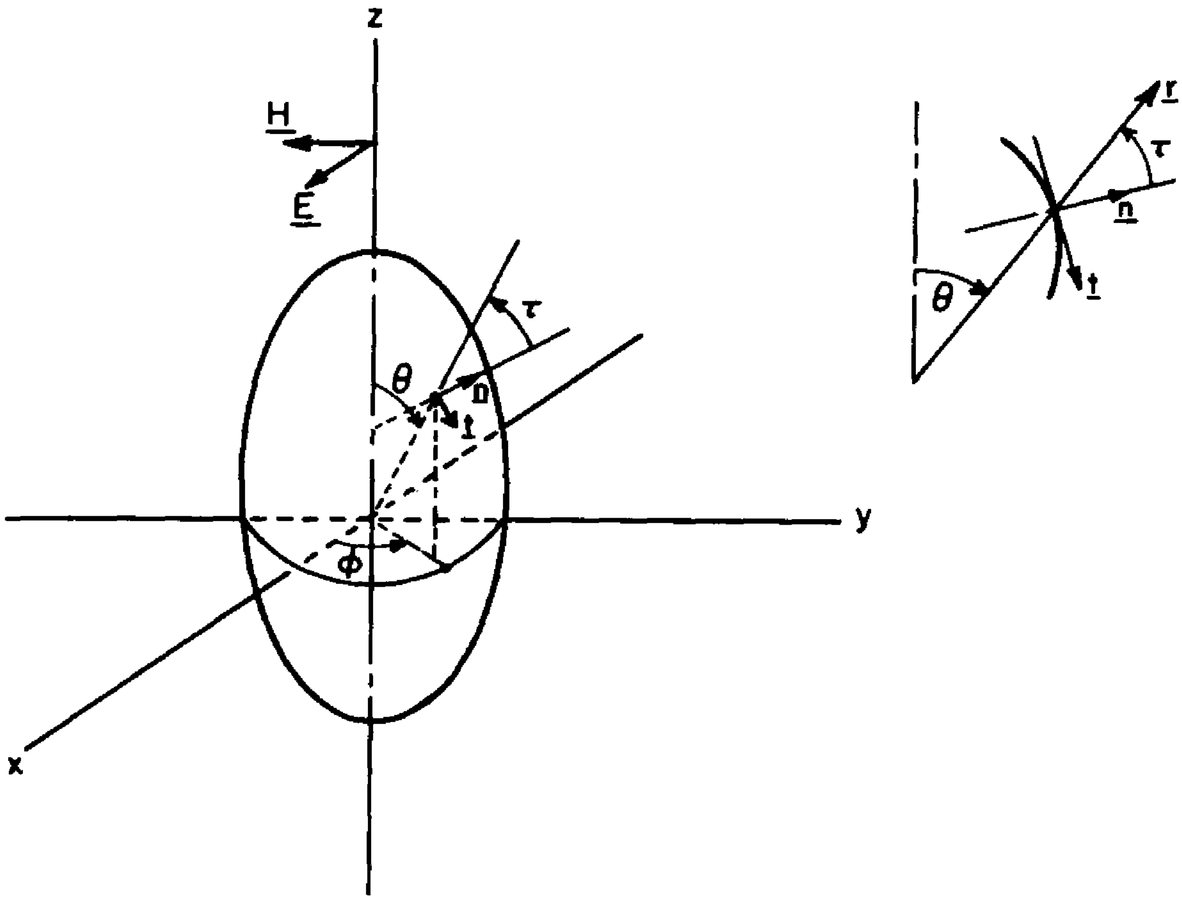


Fig. 5. Coordinate System for Rotationally Symmetric Scatterer.

in the negative z-direction, which coincides with the axis of symmetry of the scatterer. Unit tangential vectors $\underline{t}$ and $\underline{\phi}$ are defined on the surface, with $\underline{t}$ the analogue of $\underline{\theta}$ in spherical coordinates, and $\underline{\phi}$ identical with the same vector in spherical coordinates. The unit normal is $\underline{n}$ and the angle between the unit normal and the spherical coordinate vector $\underline{r}$, measured in the direction of increasing $\underline{\theta}$, is denoted by τ. The components of the incident plane wave in this coordinate system are

(78) $$\underline{E}^i = E_n^i\, \underline{n} + E_t^i\, \underline{t} + E_\phi^i\, \underline{\phi}$$

$$E_n^i = \sin(\theta + \tau)\, e^{ikr\cos\theta} \cos\phi$$

$$E_t^i = \cos(\theta + \tau)\, e^{ikr\cos\theta} \cos\phi$$

$$E_\phi^i = -e^{ikr\cos\theta} \sin\phi \quad .$$

The tangential field components produced by a collection of multipoles at the origin are given by

(79) $$E_t = \sum_n (a_n x_n + b_n y_n) \cos\phi$$

$$E_\phi = \sum_n (a_n' x_n + b_n' y_n) \sin\phi$$

$$x_n = k a_{eln}^{(2)} , \quad y_n = -ik\sqrt{\frac{\mu}{\epsilon}}\, \tilde{a}_{oln}^{(2)}$$

where x_n and y_n are related to the coefficients of the n^{th} order electric and magnetic multipoles, respectively and where

$$a_n \cos\phi = \underline{E}_{eln}^{(2)} \cdot \underline{\tau} = \left\{ \frac{\cos\tau}{\rho} \hat{H}_n^{(1)}(\rho) \frac{\partial P_n^1}{\partial\theta} - \frac{n(n+1)\sin\tau}{\rho} h_n^{(1)}(\rho)\, P_n^1(\cos\theta) \right\} \cos\phi$$

(80) $$b_n \cos\phi = \underline{\tilde{E}}_{oln}^{(2)} \cdot \underline{\tau} = \left\{ \frac{-\cos\tau}{\sin\theta} h_n^{(1)}(\rho)\ P_n^1(\cos\theta) \right\} \cos\phi$$

$$a_n' \sin\phi = \underline{E}_{eln}^{(2)} \cdot \underline{\phi} = \left\{ -\frac{1}{\rho\sin\theta} \hat{H}_n^{(1)}(\rho)\ P_n^1(\cos\theta) \right\} \sin\phi$$

$$b_n' \sin\phi = \underline{\tilde{E}}_{oln}^{(2)} \cdot \underline{\phi} = \left\{ h_n^{(1)}(\rho)\ \frac{\partial P_n^1}{\partial\theta} \right\} \sin\phi$$

$$\rho = kr \quad .$$

To obtain an exact solution to the scattering problem would require that the tangential components of $\underline{E}$ produced by the multipole fields be the negative of the tangential components of $\underline{E}^i$ over the surface of the scatterer, or that

$$(81)\qquad \begin{aligned} &\sum_n a_n(r,\theta)\ x_n + b_n(r,\theta)\ y_n = -\cos\{\theta+\tau(r,\theta)\}\ e^{ikr\cos\theta} \\ &\sum_n a'_n(r,\theta)\ x_n + b'_n(r,\theta)\ y_n = e^{ikr\cos\theta} \end{aligned}$$

where $r = f(\theta)$ and $\tau(r,\theta)$ are specified by the scatterer surface. To obtain a solution in finite terms, the continuous equations above are replaced by a set of equations holding at M points on the surface and involving multipole fields up to a maximum order N. If N = M, this becomes a system of 2N equations in 2N unknowns, but in general, M will be chosen greater than N and the following system is obtained:

$$(82)\qquad \begin{aligned} &\sum_{n=1}^{N} a_{mn}\,x_n + b_{mn}\,y_n = -\cos(\theta+\tau)_m\ e^{ikr_m\cos\theta_m} \\ &\sum_{n=1}^{N} a'_{mn}\,x_n + b'_{mn}\,y_n = e^{ikr_m\cos\theta_m} \\ &a_{mn} = a_n(r_m,\theta_m) \qquad\qquad b_{mn} = b_n(r_m,\theta_m) \\ &a'_{mn} = a'_n(r_m,\theta_m) \qquad\qquad b'_{mn} = b'_n(r_m,\theta_m) \end{aligned}$$

where the M points (r_m, θ_m) are on the scatterer surface. The system of equations can be solved in the sense of least squares, which is to say the unknowns x_n and y_n are chosen so as to minimize the mean square error between the approximate multipole field and the exact tangential scattered field $\underline{E}^s$ at the M points of the surface. This procedure essentially replaces the parameter W previously defined as the surface integral of the absolute square of the tangential electric error field by the average of a finite sum. For fields which are not rapidly varying with position on the scatterer surface this approximation is quite accurate for a relatively small number of points. In any event, the exact value of W as an integral can be computed when the coefficients x_n, y_n have been determined by this method. The accuracy of the approximation can then be evaluated. For a linear system of M equations in N unknowns

$$(A)\,X = C \tag{83}$$

the resulting N X N system is

$$(A^{CT})\,(A)X = (A^{CT})\,C \tag{84}$$

where CT denotes the complex conjugate transposed matrix. This type of problem can be handled easily by modern digital computers for N up to 20 or 30, and for much larger values of M.

B. CALCULATED SCATTERING CROSS-SECTIONS

The method described above was used to approximate the scattering of a plane wave incident along the symmetry axis of a prolate and an oblate spheroid. The major and minor axes of the prolate spheroid were 0.35 and 0.28 wavelength, and those of the oblate spheroid were 0.42 and 0.35 wavelength.

Using 21 evenly spaced points on the surface, approximate solutions in the form of multipole expansions were obtained by least square matching of the tangential fields. For a first order solution, two electric and two magnetic multipole terms were used, including dipole and quadrupole terms. A second order solution was also obtained, using four electric and four magnetic multipoles. The coefficients in the associated system of linear equations were obtained with a desk computer, and an existing IBM 650 program for the least squares solution was utilized after the system was reduced to a system of linear equations with real coefficients. Further simplification was obtained because of the symmetry of the body about the xy-plane, and the resulting system of equations was 21 by 4 for the first order solution and 21 by 8 for the second order solution. In each case, two such systems were solved to determine the complete set of multipole coefficients.

The least squares fit to the exact tangential $\underline{E}$ distribution obtained for the first order approximation is shown in Fig. 6 for the prolate spheroid and in Fig. 7 for the oblate spheroid. The incident plane wave is assumed of unit amplitude and the graphs show the variation of real and imaginary parts of the complex tangential field components E_ϕ and E_t versus the angle θ from the symmetry axis of the spheroid. Figures 8 and 9 present the same data for the second order approximation. The improvement in field matching obtained with the second order approximation is significant, the mean square error decreasing by a factor of approximately eight in both cases.

The values for the normalized multipole coefficients x_n and y_n of Eq. (79) obtained from the first and second order approximations are given in Table I. The change in the values of the coefficients for the second order approximation does not exceed 5% of the largest coefficient for the prolate spheroid, nor does it exceed 11% of the largest coefficient for the oblate spheroid. The calculated scattering cross sections for the two approximations are given in Fig. 10 for the prolate spheroid and in Fig. 11 for the oblate spheroid. The scattering cross-section σ in square wavelengths is plotted as a function of the bistatic angle between the spheroid symmetry axis and the receiver direction. A bistatic angle of 0° corresponds to back scattering along the symmetry axis and a bistatic angle of 180° corresponds to forward scatter along the same axis. When the polarization of incident plane wave and the receiving antenna are perpendicular to the plane of scattering, the curves labeled H-plane apply; when the polarizations are parallel to the scattering plane, the E-plane curves apply.

TABLE I
VALUES OF MULTIPOLE COEFFICIENTS x_n and y_n

PROLATE SPHEROID

Coefficient	First order solution	Second order solution
x_1	-0.27810 + i 0.53526	-0.27465 + i 0.55683
x_2	-0.02389 - i 0.00005	-0.02618 + i 0.00002
x_3		-0.00049 + i 0.00058
x_4		-0.00005
y_1	-0.29381 + i 0.06250	-0.27246 + i 0.05451
y_2	0.00093 - i 0.01582	0.00244 - i 0.01868
y_3		-0.00039 + i 0.00011
y_4		-0.00001 - i 0.00003

OBLATE SPHEROID

Coefficient	First order solution	Second order solution
x_1	-0.70282 + i 0.69165	-0.82232 + i 0.78372
x_2	-0.03381 - i 0.00529	-0.05557 - i 0.00755
x_3		0.00152 - i 0.00233
x_4		0.00015 + i 0.00002
y_1	-0.48201 + i 0.16471	-0.49990 + i 0.17382
y_2	-0.00025 - i 0.03117	-0.01265 - i 0.02598
y_3		0.00134 - i 0.00042
y_4		i 0.00015

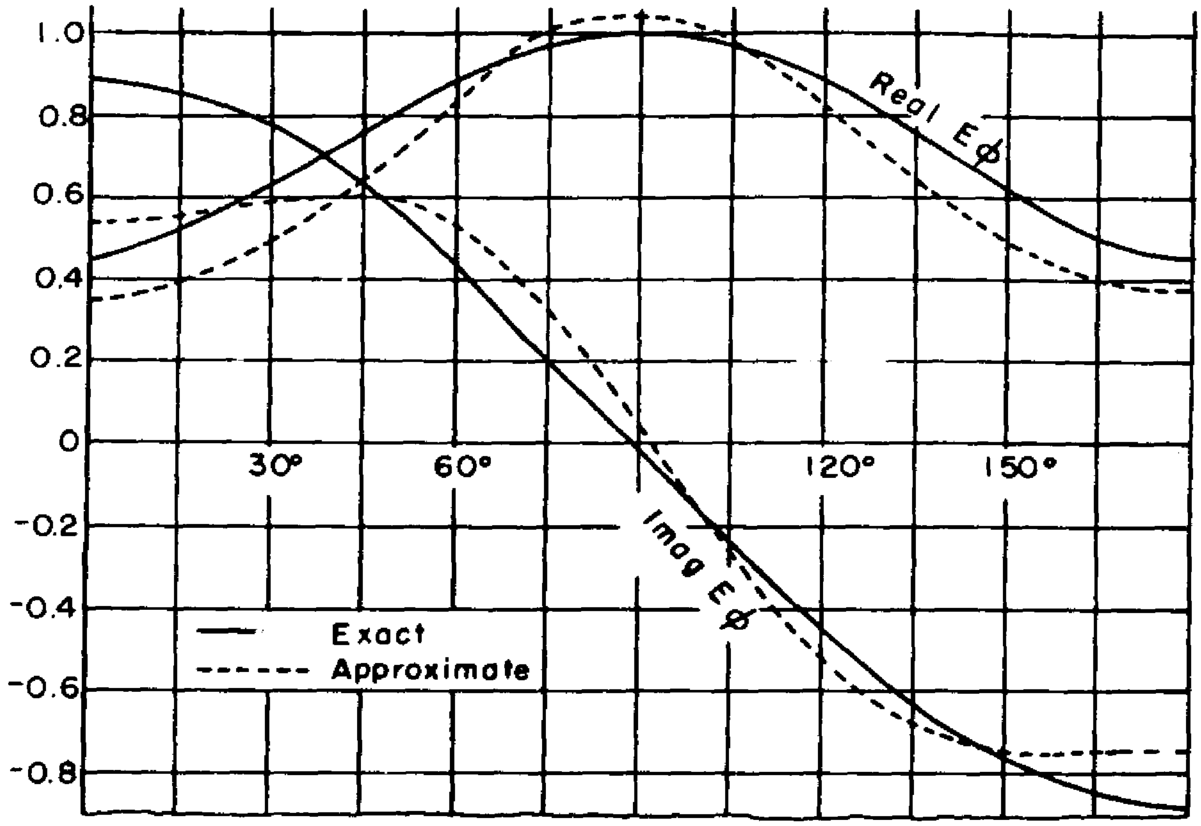
C. EXPERIMENTAL RESULTS

Values of the scattering cross section for the two spheroids were obtained experimentally using a microwave reflection-measuring system in an anechoic chamber. The measurements were obtained at a source frequency of 9380 ± 20 megacycles, using properly scaled aluminum spheroids. The scattered signal at bistatic angles of 30°, 60°, 90°, and 120° was recorded as a function of spheroid rotation in a horizontal plane, and calibrated by comparison with the scattered signal from 0.69, 0.51 and 0.33 wavelength diameter aluminum spheres at the same bistatic angles. The scattering cross section of the spheres can be computed for all bistatic angles, so that any sphere can be used for calibration. The use of three spheres permits an estimate of experimental error to be made, however.

A comparison of the measured and calculated scattering cross sections for the spheroids is given in Figs. 12 and 13. The deviation does not exceed one decibel, which is within the usual limits of experimental error for a system of this type. The E-plane results agree remarkably well with the calculated values, H-plane results showing somewhat larger deviations between theory and experiment.

Exact and measured scattering cross sections for the spherical standards are compared in Figs. 14, 15 and 16. At each bistatic angle the value of the measured cross section of one sphere can be arbitrarily set, and the measured values of the other two are then determined by the relative level of the recorded signals. In the experiment, this arbitrary value was chosen so as to yield the minimum average deviation between theoretical and measured values for the three spheres. The average minimum deviation obtained is less than one-half decibel, as shown in Figs. 14-16.

It is concluded from the experimental data that the calculated values of scattering cross section obtained from the second order approximate solution for oblate and prolate spheroid are within one decibel of the true value everywhere, with an average deviation of the order of one-half decibel.



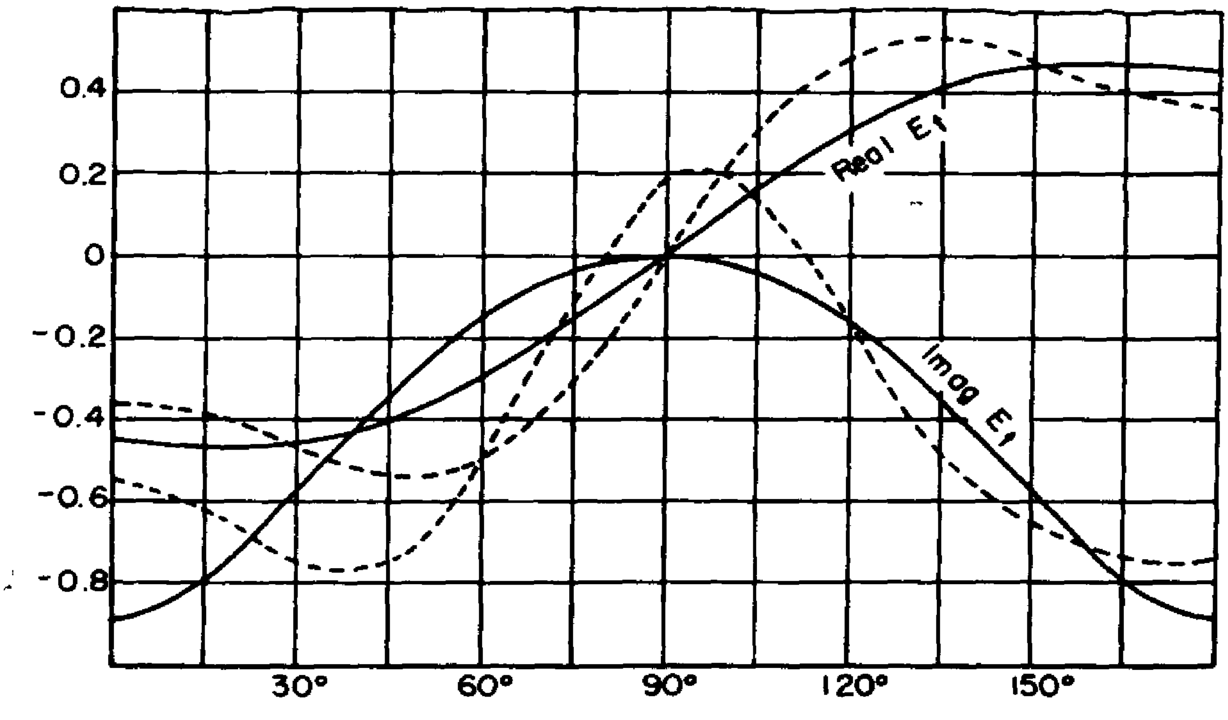


Fig. 6. Comparison of exact and first order approximate tangential electric field, prolate spheroid.

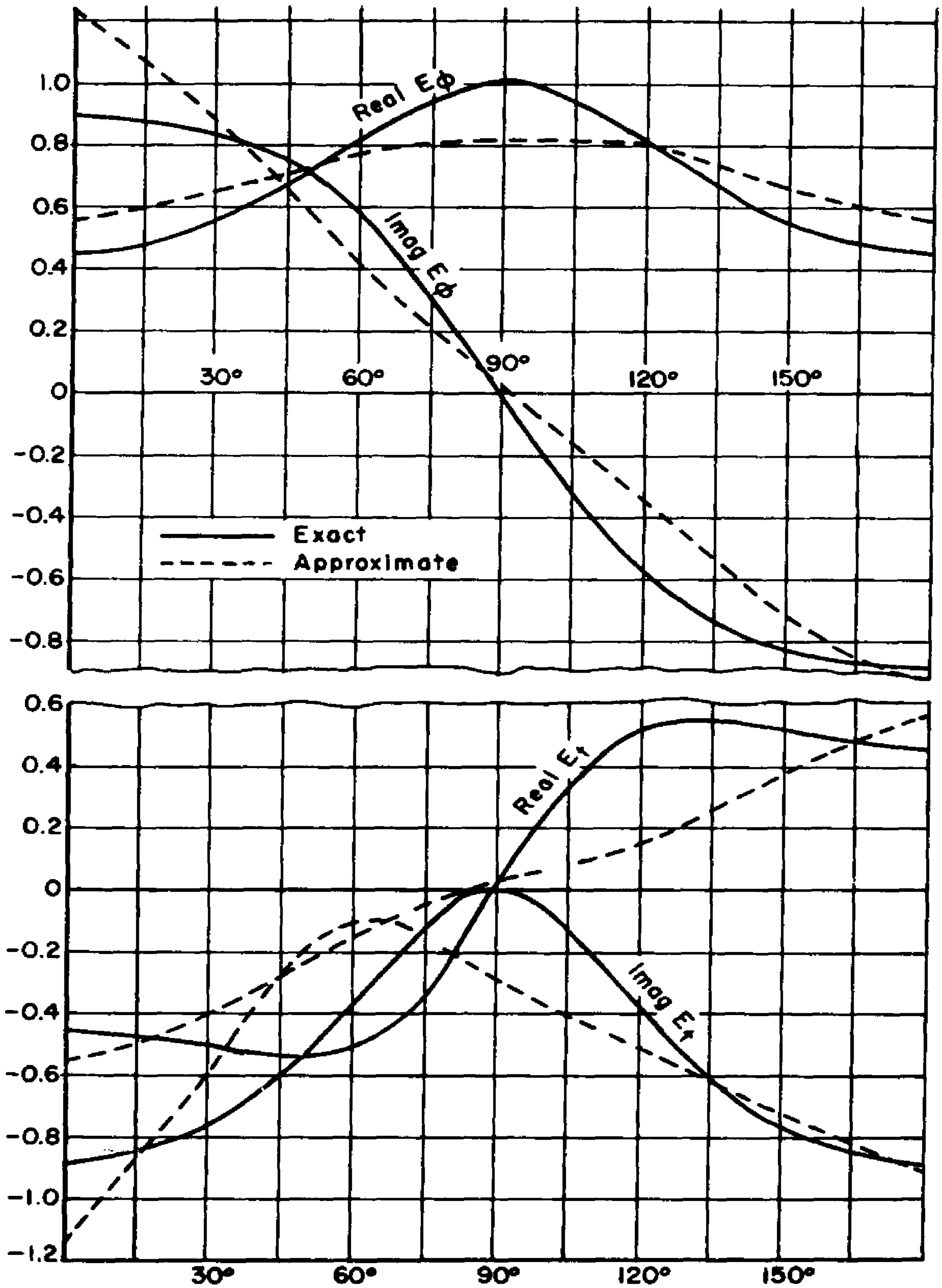


Fig. 7. Comparison of exact and first order approximate tangential electric field, oblate spheroid.

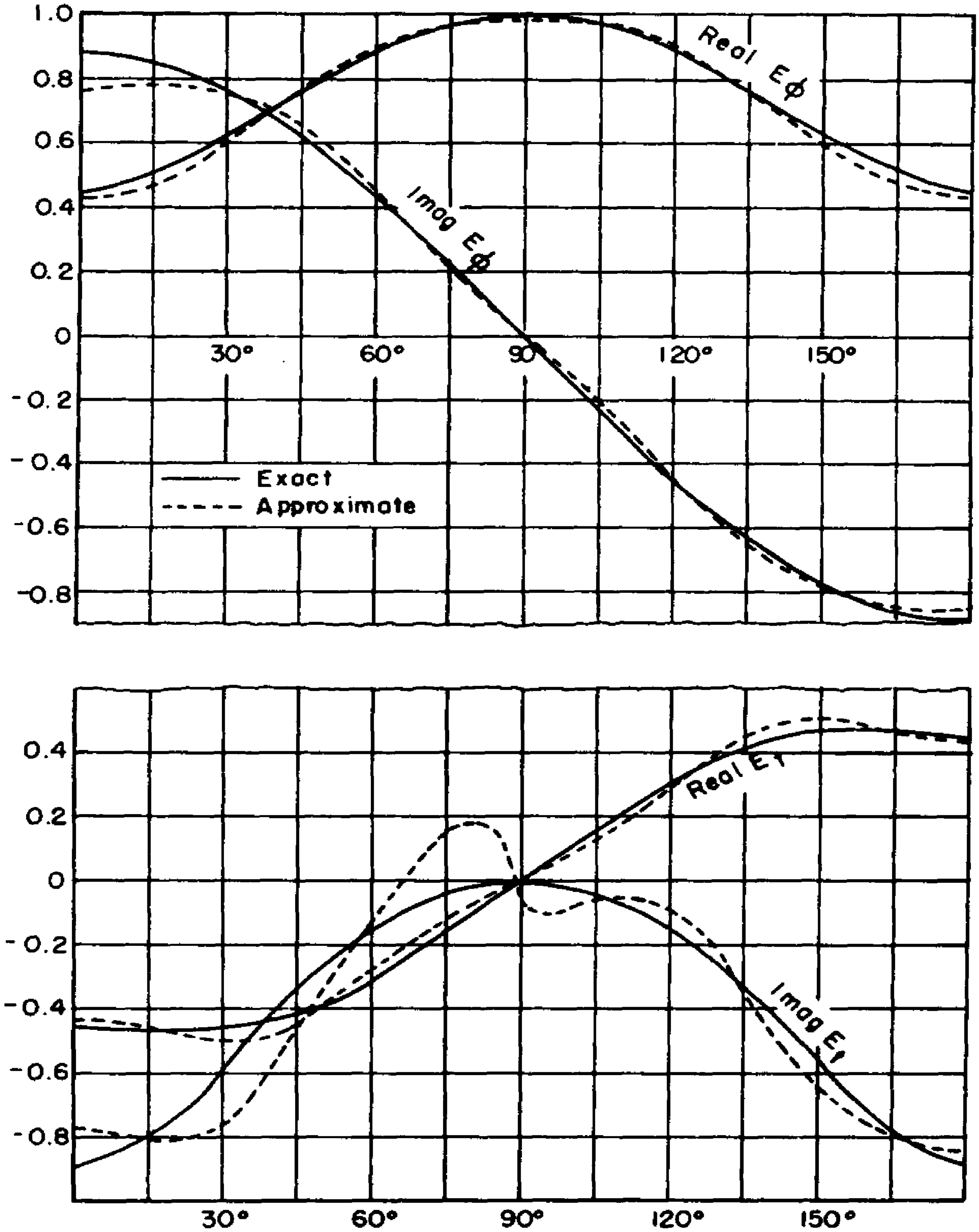


Fig. 8. Comparison of exact and second order approximate tangential electric field, prolate spheroid.

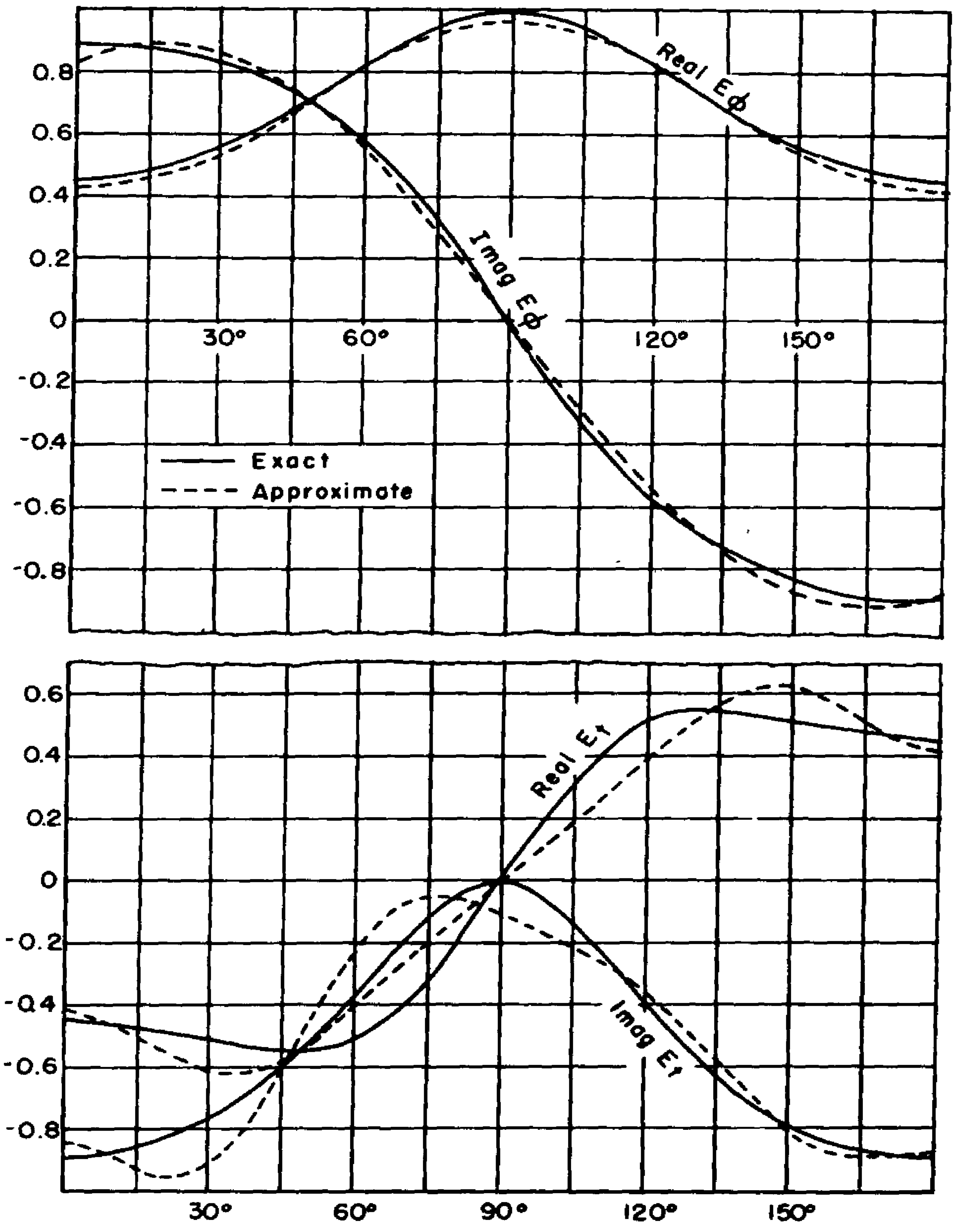


Fig. 9. Comparison of exact and second order approximate tangential electric field, oblate spheroid.

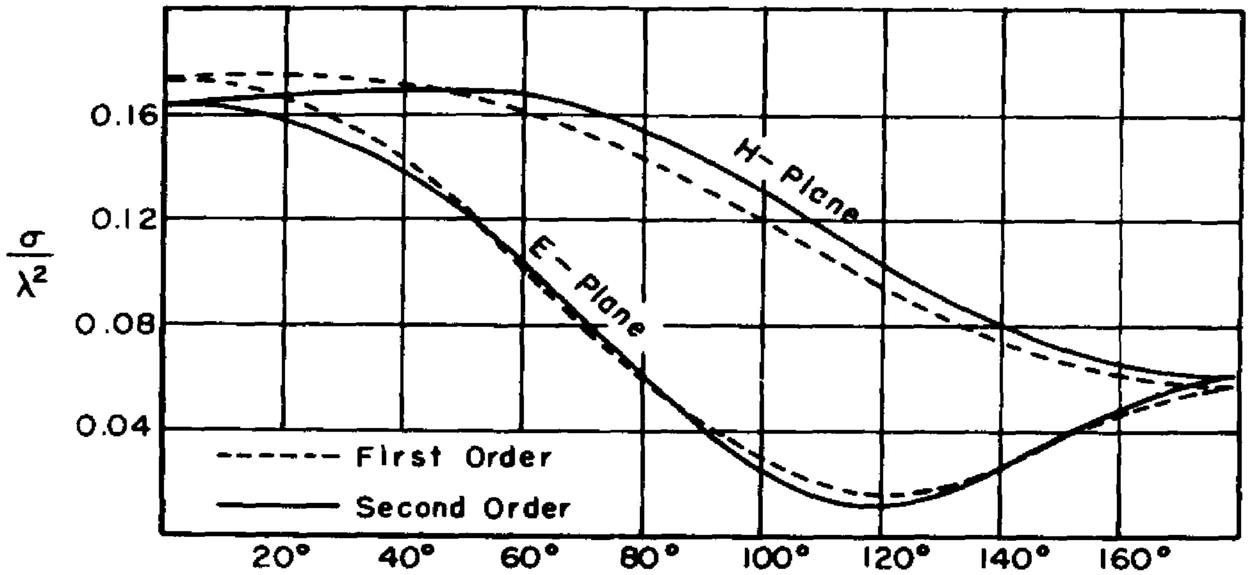


Fig. 10. Calculated scattering cross section of prolate spheroid, first and second order solutions.

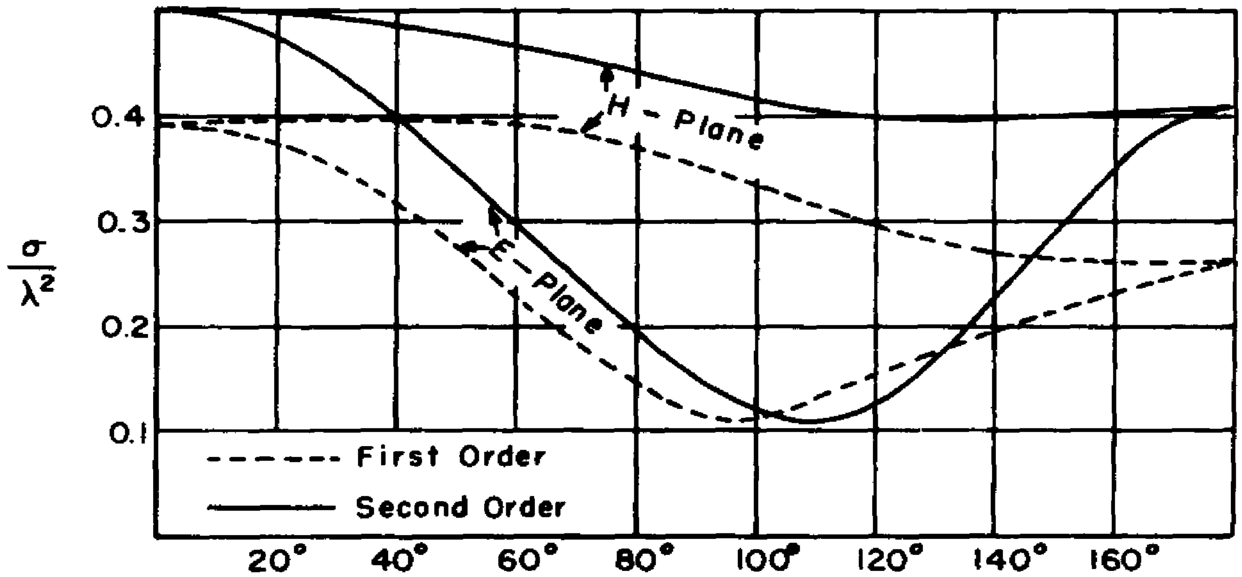


Fig. 11. Calculated scattering cross section of oblate spheroid, first and second order solutions.

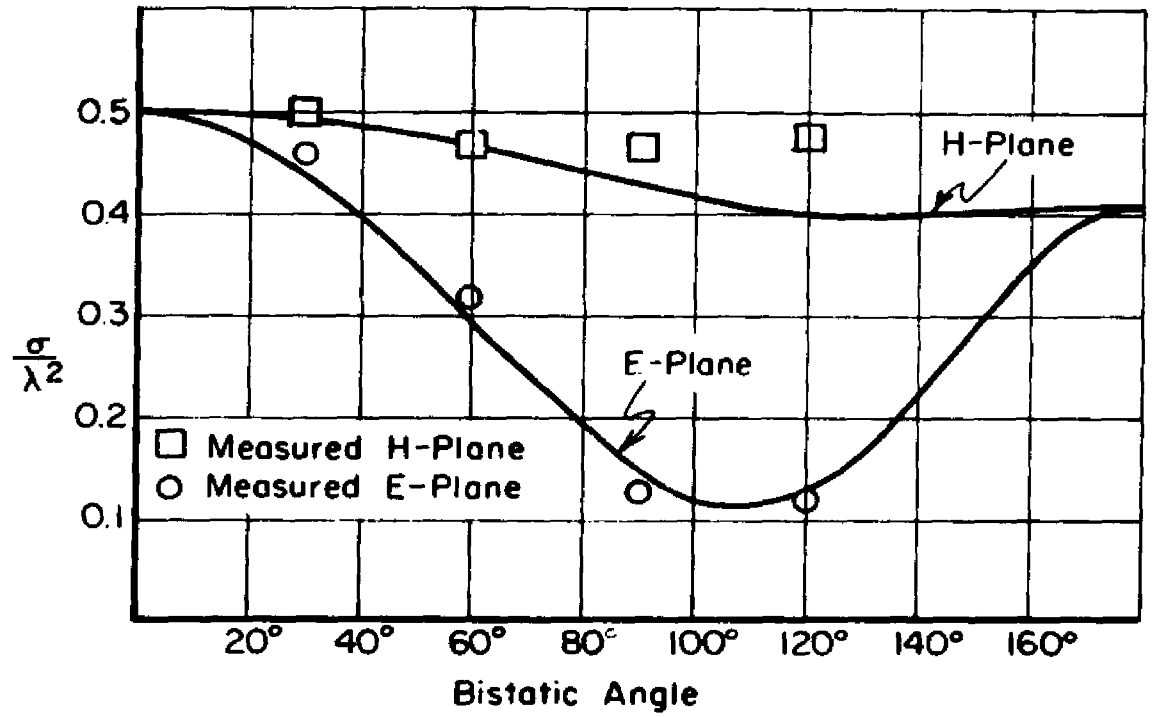


Fig. 12. Comparison of calculated and measured scattering cross sections, oblate spheroid.

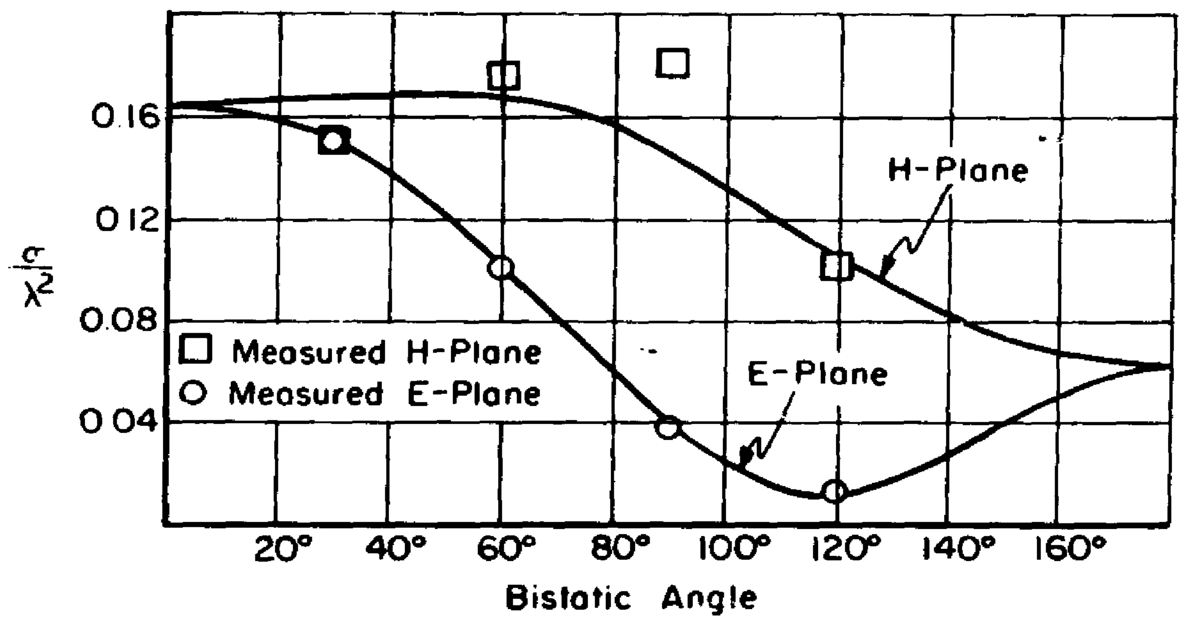


Fig. 13. Comparison of calculated and measured scattering cross sections, prolate spheroid.

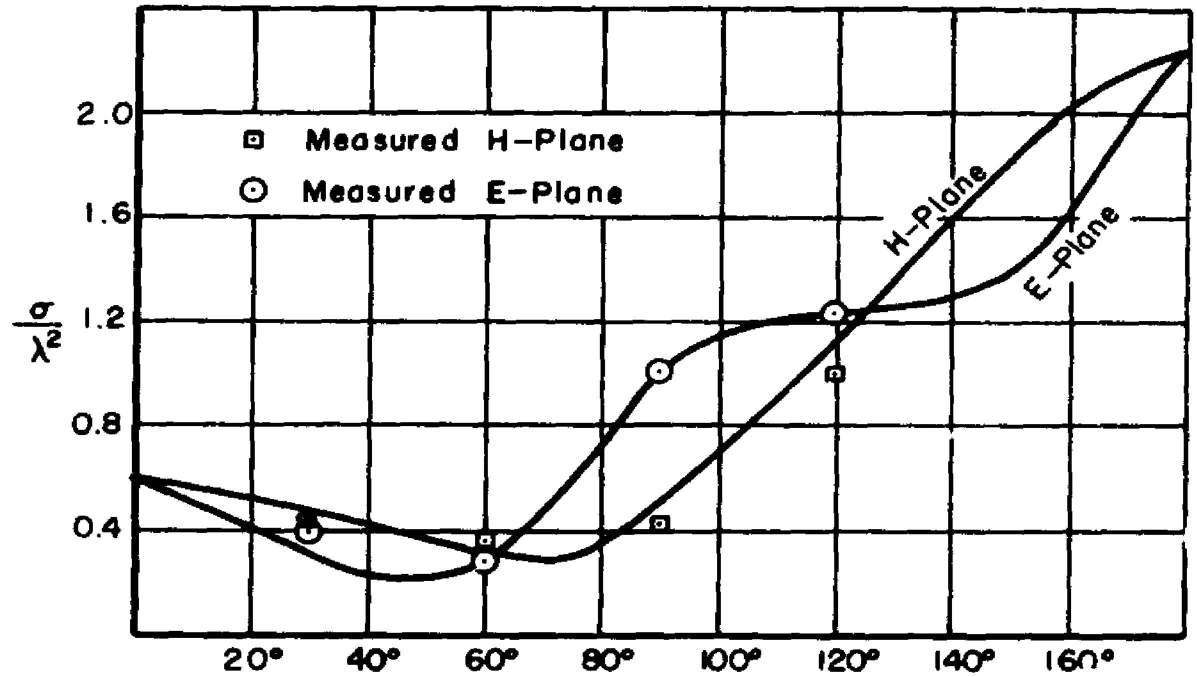


Fig. 14. Scattering cross section of 0.69λ diameter sphere.

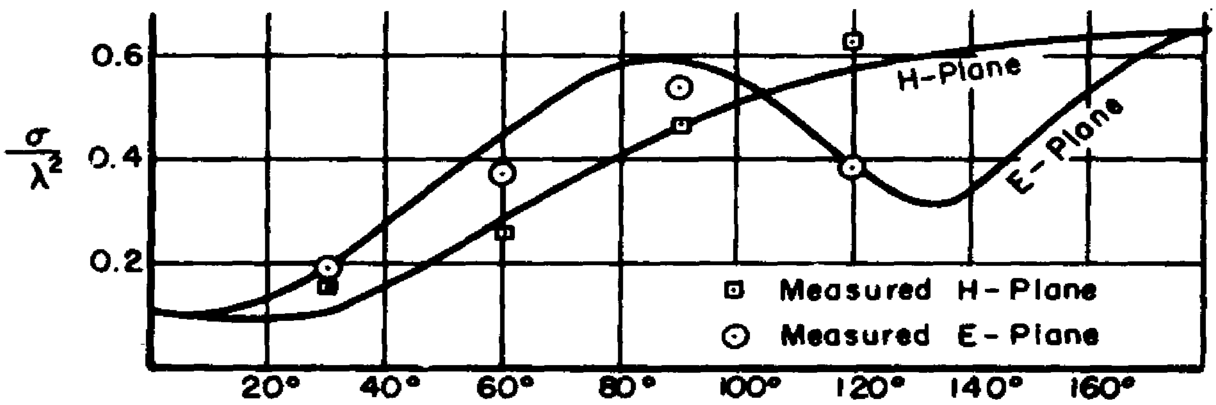


Fig. 15. Scattering cross section of 0.51λ diameter sphere.

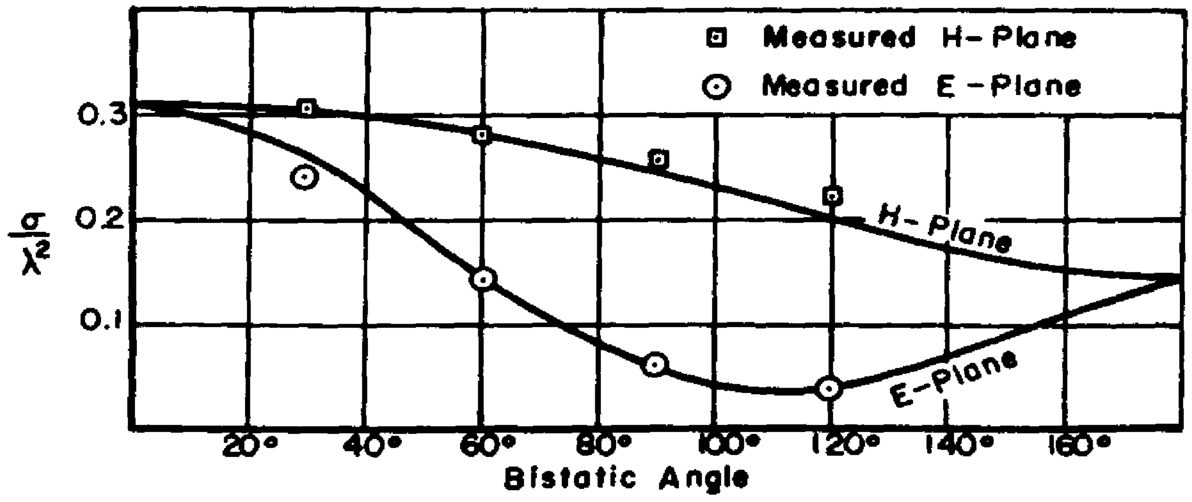


Fig. 16. Scattering cross section of 0.33λ diameter sphere.

CHAPTER IV
CONCLUSIONS

Solutions to electromagnetic scattering problems are obtained by choosing a mathematical representation of fields valid in interior and exterior domains and imposing boundary conditions at the surfaces of separation. The use of multipole expansions as a field representation is desirable because the expansion coefficients can be determined from a knowledge of the tangential field over any closed surface or, in interior regions, from a knowledge of the field and its derivatives at a point. The multipole expansion is also well-suited for the representation of fields at great distances from the source, where the wavefront is spherical.

Given a representation of fields in the form of series with unknown coefficients, an approximate solution to scattering problems in finite terms can be obtained by requiring that the least square deviation from match of tangential fields be obtained on all boundary surfaces. In the case of a perfectly conducting scatterer, the mean square deviation from the correct tangential electric field over the bounding surface can be determined for each approximate solution. The minimum deviation obtained can be used to estimate the mean square error in the scattered field at large distances. Fundamentally, the existence of a bound on the radiated power from any tangential distribution of E over a closed surface S in terms of its mean square value is required to prove that tangential field-matching methods converge to the true solution. If a value of the bound can be determined, the error in a finite approximation can be estimated even when the exact solution cannot be found.

The application of field-matching techniques to smooth perfectly-conducting scatterers has been demonstrated for prolate and oblate spheroids of low eccentricity. Calculated scattering cross-sections have been verified experimentally, for spheroids close to the first resonance peak in back scattering. It is concluded that field matching techniques are useful for scatterers whose major dimensions are of the order of the wavelength, although computation of scattering cross sections by this method will require the use of high speed computers. A general computer program to obtain a sequence of approximate solutions from a mathematical description of the scatterer surface and the direction of incidence for a plane wave source should be developed.

Although the method has not been applied to bodies of high eccentricity or dimensions larger than the wavelength, these conditions could at worst increase the number of terms in the multipole series required to give a good match at the boundary. Since the examples considered were computed on an IBM 650 in less than ten minutes, it is believed that an IBM 704 would suffice for most smooth bodies in the resonance region. At any step in the computation, the error in a finite approximation could be estimated by a comparison of the exact and approximate tangential electric fields.

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I, Edward Morton Kennaugh ,was born in New York,New York, on October 3, 1922. I received my secondary education in the public schools of that city, and my undergraduate training at the Ohio State University, which granted me the Bachelor of Electrical Engineering degree in 1947, and the degree Master of Science (Physics) in 1952. Since 1949 I have been engaged in research in the field of microwave theory, antennas and radiation at the Antenna Laboratory, Department of Electrical Engineering, The Ohio State University. In 1954 I was appointed an Associate Supervisor of the Antenna Laboratory. I held this position for five years while completing the requirements for the degree Doctor of Philosophy.