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A CONTINUOUS NONLINEAR PROGRAMMING PROBLEM

DISSERTATION

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By

Jerry Lee Silver, B.S.

* * * * *

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Approved by

Richard L. Francis

Adviser
Department of Industrial
Engineering

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VITA

June 3, 1943	Born - Cincinnati, Ohio
1962-1965	Teaching Assistant, Department of Mathematics, The Ohio State University, Columbus, Ohio
1965	B.S. in Engineering Physics The Ohio State University, Columbus, Ohio
1965-1971	Teaching Associate, Department of Mathematics The Ohio State University, Columbus, Ohio

FIELDS OF STUDY

Major Field: Industrial Engineering

Studies in Mathematical Programming. Professor Richard L. Francis

Studies in Applied Mathematics. Professor Robert C. Fisher

Studies in Operations Research. Professor Walter C. Giffin

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INTRODUCTION

This thesis investigates a continuous version of a pair of nonlinear programming problems originally formulated by Sinha [33]. Sinha's primal problem arises in stochastic linear programming [31]. One version of Sinha's pair of problems is the following:

<u>Primal</u>: $$\max_{z \in S_1} \; a^{\tau}z - \left(z^{\tau}Dz\right)^{1/2}$$

where $S_1 = \left\{z \in E^N\colon \; Bz \le c, \; z \ge 0\right\}$

<u>Dual</u>: $$\min_{w \in S_2} \; c^{\tau}w$$

where $S_2 = \left\{w \in E^M\colon \; B^{\tau}w \ge a - Dy, \; y^{\tau}Dy \le 1, \; w \ge 0, \; y \in E^N\right\}.$

Here D is an $N \times N$ positive semidefinite matrix. B is an $M \times N$ matrix, $a \in E^N$, and $c \in E^M$. Of course, if $D = 0$, we have a familiar linear programming problem:

<u>Primal</u>: $\max_{z \in S_1} a^{\tau}z, \qquad S_1 = \left\{z \in E^N\colon \; Bz \le c, \; z \ge 0\right\}$

<u>Dual</u>: $\min_{w \in S_3} c^{\tau}w, \qquad S_3 = \left\{w \in E^M\colon \; B^{\tau}w \ge a, \; w \ge 0\right\}.$

The continuous linear programming problems to be stated subsequently were first considered by Bellman [4], [5], who formulated them as bottleneck problems. Bellman presented some weak duality results; that is, conditions which guarantee that feasible solutions meeting them are optimal. However, the first strong duality results for the problem studied by Bellman were proved by Tyndall [34]. A slightly more general continuous linear programming problem, as studied by Levinson [25], Grinold [13], [14], [15], and Tyndall [35], [36] is

<u>Primal</u>: $$\max_{z \in S_4} \int_0^T a^\tau(t)z(t)dt$$

$$\text{where } S_4 = \left\{z \in L_\infty^N[0,T]: B(t)z(t) \le c(t) + \int_0^t K(s,t)z(s)ds,\right.$$
$$\left. z(t) \ge 0, \quad 0 \le t \le T\right\}$$

<u>Dual</u>: $$\min_{w \in S_5} \int_0^T c^\tau(t)w(t)dt$$

$$\text{where } S_5 = \left\{w \in L_\infty^M[0,T]: B^\tau(t)w(t) \ge a(t) + \int_t^T K^\tau(t,s)w(s)ds,\right.$$
$$\left. w(t) \ge 0, \quad 0 \le t \le T\right\}.$$

Here each entry of the $M \times N$ matrices $B(t)$ and $K(s,t)$ is a function of one and two real variables respectively.

Our work will directly generalize each of the above problems. In Chapter I, continuous versions of Sinha's non-linear problems are explicitly defined. Chapter II specifies some weak duality results while Chapter III deals with strong duality--the existence of optimal solutions under appropriate hypotheses. Concluding remarks appear in Chapter IV.

CHAPTER I

STATEMENT OF THE PROBLEM

BACKGROUND INFORMATION

We shall now formally state the continuous nonlinear problem pair which is examined herein:

1. Definition: Problem P (Primal): Find a vector-valued function $z \in L_\infty^N[0,T]$ which is optimal for

$$\max_{z \in C_P} f(z)$$

where $f(z) = \int_0^T a^\tau(t)z(t)dt - \sum_{p=1}^{r} \int_0^T [z^\tau(t)D_p(t)z(t)]^{1/2} dt$

and $C_P = \Big\{z \in L_\infty^N[0,T]\colon B(t)z(t) \leq c(t) + \int_0^t K(s,t)z(s)ds,$

$z(t) \geq 0,\ 0 \leq t \leq T\Big\}$.

Problem D (Dual): Find a function $w \in L_\infty^M[0,T]$ which is optimal for

$$\min_{w \in C_D} g(w)$$

where $g(w) = \int_0^T c^\tau(t)w(t)dt$

$$\text{and}\quad C_D = \Big\{ w \in L_\infty^M[0,T] : B^T(t)w(t) \geq a(t) + \int_t^T K^T(t,s)w(s)ds$$

$$- \sum_{p=1}^{r} D_p(t)y_p(t), w(t) \geq 0\,,\; y_p^T(t)D_p(t)y_p(t) \leq 1$$

$$y_p \in L_\infty^N[0,T], \qquad p = 1,\ldots,r,\; 0 \leq t \leq T \Big\}.$$

We require that $a(t) \in L_\infty^N[0,T]$, $c(t) \in L_\infty^M[0,T]$, $B(t) \in L_\infty^{MN}[0,T]$, $K(s,t) \in L_\infty^{MN}[0,T] \times [0,T]$, and $D_p(t) \in L_\infty^{NN}[0,T]$ for $p = 1,\ldots,r$. Further, we require that each $D_p(t)$ be positive semidefinite for every $t \in [0,T]$, $p = 1,\ldots,r$.

Since the notation becomes more awkward in succeeding chapters, hereafter we shall assume $r = 1$. In Appendix A we indicate how the proofs need to be modified to obtain the more general results.

It is easily seen that this problem pair includes the continuous, linear problems of Tyndall [34], Levinson [25], and Grinold [14] as special cases by setting each $D_p = 0$. Our problems also include the nonlinear problems of Sinha by setting $K(s,t) = 0$ and taking all remaining functions to be constant functions over $[0,T]$.

2. Remarks: Nonlinearity.

Sinha [31] formulated a stochastic linear programming problem which leads to a deterministic nonlinear programming problem. The nonlinearity occurs in the objective function as a sum of square roots of positive semidefinite quadratic forms. Although the problem is a concave programming problem, the nondifferentiability of the square root terms in the objective function precludes the use of many standard methods for solving concave programming problems. With this motivation, a result of Eisenberg [9] was extended by Sinha [32] and then used by Sinha [33] to formulate a dual problem where the nonlinearity occurs in the constraints. Further, the constraint functions are differentiable.

Eisenberg's work [9] is a nonlinear version of the Farkas Lemma which is fundamental to the theory of linear programming. Related work has since been done by Mehndiratta [27], Kaul [22], and Mond [28], [29]. Kaul's results extend directly those of Eisenberg to complex space, while the work of Mond extends these results to convex polyhedral cones in complex space.

Of course, Sinha based his duality theorems on his extension of Eisenberg's work. In order to establish the existence of optimal solutions, he required the primal constraint set to be a closed bounded subset of E^N. Later, Bhatia [7] relaxed this requirement in some of Sinha's programming theorems.

Bhatia and Kaul [8] have expanded Sinha's programming results to complex space, and Mond [29] has extended these results to convex polyhedral cones in complex space.

3. Remarks: Continuous Variables.

Tyndall [34] was the first to publish a strong duality theorem for the continuous linear programming problem. This work assumed constant matrices for B and K, stronger continuity conditions, and, of course, $D = 0$. In addition, he assumed certain algebraic conditions which implied the existence of optimal solutions. These conditions are [34, p. 646]:

(i) $\{z \in E^N : B z \leq 0 , z \geq 0\} = \{0\}$;

(ii) $B \geq 0$, $K \geq 0$, and $c \geq 0$.

He also demonstrated that neither hypothesis (i) nor (ii) is sufficient to insure existence [34, p. 649]. In addition, he observed another interesting fact -- namely, the condition that both primal and dual be feasible, a sufficient condition for the finite linear programming duality theorem, is <u>not</u> sufficient for the continuous linear programming problem [34, p. 648]. Of course, this is true for our problems as well, since they are a direct generalization of Tyndall's problems.

Levinson [25] extended Tyndall's results by allowing "time-varying" matrices for K and B. Because of this, an additional hypothesis was added to Tyndall's. It is this set of hypotheses, used in the linear problem, that we shall use in Chapter 3 in our nonlinear problem. Tyndall also used these hypotheses in later papers [35] and [36] which extended previous results and weakened the regularity conditions to continuity almost everywhere.

Grinold was attracted by the symmetry of the Tyndall's primal and dual problems, but was disturbed that the algebraic conditions did not possess a similar symmetry. In [13], [14] and [15] he exploited the intrinsic symmetry of the problems and was able to obtain duality theorems using weaker symmetric hypotheses.

To best display this symmetry, his algebraic conditions were phrased geometrically. For example, $c(t)$ must lie in the convex polyhedral cone generated by the matrix $[B(t), I]$. By definition this means that $B(t)z + Ix = c(t)$ for some $z,x \geq 0$; equivalently, $B(t)z \leq c(t)$, $z \geq 0$ has a feasible solution. In addition to four such algebraic conditions, he also imposed the same regularity conditions and boundedness conditions. The boundedness conditions are needed for the case $B = B(t)$ and are satisfied by Levinson's additional requirement.

Grinold [13], [16] has also obtained some more general results involving nonlinearity in the objective functions. He considers a primal problem and a Lagrangian function to prove some rather general saddle-point theorems. He does not consider any dual problem as such. Further, since our dual constraints have a different form than the constraints of the primal, his work in this area is not directly applicable to our problems.

Hanson and Mond [20] and Hanson [18] have considered a class of continuous nonlinear programming problems which incorporate the same primal constraint set as our problem. However, they require a twice differentiable concave function as the integrand in the objective function to obtain their results.

Gogia and Gupta [12] have investigated a continuous quadratic programming problem. Essentially, they were able to extend Levinson's results by linearizing the quadratic problem.

CHAPTER II

WEAK DUALITY

In this chapter we shall develop the basic inequality relating the objective functions of P and D, and explore the implications of equality holding for the two objective functions. Conditions which insure that equality will hold are described in Chapter 3.

The results in this chapter are immediate consequences of the problem statement and basic concepts. They are straightforward extensions or repetitions of the earlier works in this area; we have included them here for completeness.

The first result is easily established using Fubini's Theorem.

1. Lemma: If $z \in L_\infty^N[0,T]$ and $w \in L_\infty^M[0,T]$, then

$$\int_0^T w^\tau(t) \left[\int_0^t K(s,t)z(s)ds \right] dt = \int_0^T z^\tau(t) \left[\int_t^T K^\tau(t,s)w(s)ds \right] dt .$$

Integrating Sinha's result (Lemma 1 in [32]) over $[0,T]$ yields

2. Lemma: If $D(t)$ is positive semidefinite for $t \in [0,T]$, and if $y,z \in L_\infty^N[0,T]$

$$\int_0^T y^\tau(t)D(t)z(t)dt \leq \int_0^T \left[y^\tau(t)D(t)y(t) \right]^{1/2} \left[z^\tau(t)D(t)z(t) \right]^{1/2} dt$$

Of course, Lemma (2) is just the Cauchy-Schwartz inequality for the pseudo-inner product $(y,z) = \int_0^T y^\tau(t)D(t)z(t)dt$.

We are now prepared to prove the fundamental inequality. Recall, from Definition (1:1) that C_P and C_D are the primal and dual constraint sets and that f and g are the primal and dual objective functions respectively.

3. Theorem: $\sup\limits_{z \in C_P} \{f(z)\} \leq \inf\limits_{w \in C_D} \{g(w)\}$

Proof: The sup and inf over the empty set are $-\infty$ and ∞ respectively. Thus it suffices to show that $f(z) \leq g(w)$ for every feasible z and w.

Let z and w be feasible for P and D. Then

$$z^\tau(t) \left[a(t) + \int_t^T K^\tau(t,s)w(s)ds - D(t)y(t) \right]$$

$$\leq z^\tau(t)B^\tau(t)w(t) = w^\tau(t)B(t)z(t)$$

$$\leq w^\tau(t) \left[c(t) + \int_0^t K(s,t)z(s)ds \right].$$

Integrating the above inequality over $[0,T]$, we have

$$\int_0^T a^\tau(t)z(t)dt + \int_0^T z^\tau(t)\left[\int_t^T K^\tau(t,s)w(s)ds\right]dt - \int_0^T z^\tau(t)D(t)y(t)dt$$

$$\leq \int_0^T c^\tau(t)w(t)dt + \int_0^T w^\tau(t)\left[\int_0^t K(s,t)z(s)ds\right]dt .$$

Using Lemma (1), this last inequality becomes

$$\text{(i)} \quad \int_0^T a^T(t)z(t) - \int_0^T z^T(t)D(t)y(t)dt \leq g(w) \ .$$

But Lemma (2) and the constraint

$$y^T(t)D(t)y(t) \leq 1$$

then imply

$$\text{(ii)} \quad \int_0^T z^T(t)D(t)y(t)dt \leq \int_0^T \left[z^T(t)D(t)z(t)\right]^{1/2} dt \ .$$

Thus, combining (i) and (ii) yields

$$\text{(iii)} \quad f(z) = \int_0^T a^T(t)z(t)dt - \int_0^T \left[z^T(t)D(t)z(t)\right]^{1/2} dt$$

$$\leq \int_0^T a^T(t)z(t)dt - \int_0^T y^T(t)D(t)z(t)dt \leq g(w)$$

and the result follows.

4. Remark: Notice that $f(z) = g(w)$ if, and only if, expression (3-iii) (or equivalently (3-i) and (3-ii)) hold as equalities. An immediate consequence of Theorem (3) is the following:

5. Theorem: If z and w are feasible for P and D respectively, and if $f(z) = g(w)$, then z and w are optimal solutions.

Another property of linear programming problems which has an immediate analogue for our problems involves the concept of complementary slackness.

6. Definition: Let $z \in C_P$ and $w \in C_D$. Then z and w are said to be equilibrium solutions, or to satisfy the complementary slackness conditions, if the following equations hold for almost every $t \in [0,T]$.

$$\text{(i)}\quad w^T(t)\left[c(t) + \int_0^t K(s,t)z(s)ds - B(t)z(t)\right] = 0,$$

$$\text{(ii)}\quad z^T(t)\left[B^T(t)w(t) - a(t) - \int_t^T K^T(t,s)w(s)ds + D(t)y(t)\right] = 0,$$

and

$$\text{(iii)}\quad z^T(t)D(t)y(t) = \left[z^T(t)D(t)z(t)\right]^{1/2}.$$

7. Remark: Equivalently, we could rewrite (6-i) and (6-ii) as

$$\text{(i)}\quad w_i(t)\left[c(t) + \int_0^t K(s,t)z(s)ds - B(t)z(t)\right]_i = 0,\ i = 1,\ldots,M;$$

$$\text{(ii)}\quad z_j(t)\left[B^T(t)w(t) - a(t) - \int_t^T K^T(t,s)w(s)ds - D(t)y(t)\right]_j = 0,$$

$$j = 1,\ldots,N$$

where the subscript $i(j)$ denotes the $i^{th}(j^{th})$ component of a vector in $E^M(E^N)$. The equivalency is demonstrated by noticing that (6-i) and (6-ii) are just the summations of the expressions (i) and (ii) respectively, and that the factors of (i) and (ii) are non-negative since z and w are feasible solutions.

8. Theorem: Let $z \in C_P$ and $w \in C_D$. Then z and w are

equilibrium solutions for P and D if, and only if, $f(z) = g(w)$.

Proof: If z and w are equilibrium solutions, then adding Equations (6-i) and (6-ii) and integrating this sum over $[0,T]$ yields:

$$\text{(i)}\quad 0 = \int_0^T w^\tau(t)\Big[c(t) + \int_0^t K(s,t)z(s)ds - B(t)z(t)\Big]\,dt$$

$$+ \int_0^T z^\tau(t)\Big[B^\tau(t)w(t) - a(t) - \int_t^T K^\tau(t,s)w(s)ds + D(t)y(t)\Big]dt$$

$$= g(w) - \int_0^T a^\tau(t)z(t)dt + \int_0^T z^\tau(t)D(t)y(t)dt$$

$$= g(w) - \int_0^T a^\tau(t)z(t)dt + \int_0^T [z^\tau(t)D(t)z(t)]^{1/2}dt$$

$$= g(w) - f(z).$$

using Lemma (1) and Equation (6-iii).

Conversely, if we reverse the steps in the above argument, Remark (4), Lemma (1), and Definition (6) imply Equation (i) above. But the fact that $z \in C_P$ and $w \in C_D$ insures that both integrands of (i) are non-negative, and so the two integrands must each equal zero. Thus z and w are equilibrium solutions.

9. Corollary: If z and w are equilibrium solutions, then z and w are optimal solutions.

Proof: An immediate consequence of Theorems (6) and (8).

10. Remark: Perhaps a word of caution is in order. It is possible to have both primal and dual problems feasible such that there exists an optimal solution to the primal but there exists no optimal solution to the dual [13, App. A]. This contrasts with the finite linear case where the converse of Corollary (9) is true [11, p.19].

CHAPTER III

STRONG DUALITY

In this section we shall establish the main duality theorem. We do this by considering a sequence of discrete problems which are, in fact, equivalent to Sinha's programming problems. For each of these problem pairs, we establish the existence of uniformly bounded optimal solutions. We then show that these discrete solutions converge to optimal solutions for the original problems P and D. This technique was used by Tyndall [34] in his original work on continuous linear programming.

In (5) through (9) of this chapter we define discrete problems and show that they are equivalent to Sinha's dual problems. In (10) we show that the primal constraint set is uniformly bounded. This is not a new result since our primal constraint set is the same as that used in later versions of the continuous linear problem. However, the proof of (10) does not appear explicitly in the literature. Proceeding, we then apply duality results for Sinha's problems to our discrete problems in (11) through (13).

In (14) through (17), we are able to utilize properties of our dual constraint set and algebraic conditions (2) in order to subsequently bound the optimal solutions to the discrete dual problem. This is done in (18) and (19) by extending the ideas of Levinson [25]. We thus show the existence of uniformly bounded optimal solutions to each discrete problem pair.

From these optimal discrete solutions we are then able to extract a convergent subsequence of functions. To do this, we review the pertinent background material in (20) through (24); then, in (25) we define step functions to which we apply Tyndall's "diagonal process". The convergent subsequence thus yields functions which we then show in (27) and (28) to be feasible by extending the arguments of Grinold [13]. These functions are then shown to be optimal with the aid of Theorem 2.3.

As in Chapter II, some of the definitions and results are essentially the same as found in the works mentioned above and are included primarily for completeness.

In particular, we shall make use of the same algebraic conditions and regularity conditions as were used in the linear case in the most recent paper by Tyndall [36].

1. Regularity Conditions:

(i) The functions $a_j(t)$, $B_{ij}(t)$ and $c_i(t)$ are continuous for almost every $t \in [0,T]$, $i = 1,\dots,M$, $j = 1,\dots,N$.

(ii) The functions $D_{jk}(t)$ are piecewise continuous for $t \in [0,T]$, $j,k = 1,\dots,N$.

2. Algebraic Conditions:

(i) $c_i(t) \geq 0$ for all $t \in [0,T]$

$K_{ij}(t) \geq 0$ for all $(s,t) \in [0,T] \times [0,T]$.

(ii) There exists $\delta > 0$ such that for each $i = 1,\ldots,M$, $j = 1,\ldots,N$, $t \in [0,T]$, either $B_{ij}(t) = 0$ or $B_{ij}(t) \geq \delta$.

(iii) For each $j = 1,\ldots,N$, $t \in [0,T]$, there exists an i_j (perhaps depending on t) such that $B_{i_j j}(t) \geq \delta$.

3. Remark: Condition (2-iii) insures that each column of $B(t)$ has a positive entry, while Condition (2-ii) bounds each non-zero entry of $B(t)$ away from zero. This is, of course, slightly stronger than the condition $B(t) \geq 0$; it is needed in the proof of Lemma 17.

Algebraic Conditions (2) were originally employed by Levinson [25]. Earlier Tyndall [34] had used the following constraint qualification for constant non-negative matrices B :

$$\{z\colon Bz \leq 0,\ z \geq 0\} = \{0\} \quad \text{for every} \quad t \in [0,T] .$$

Clearly this is equivalent to (2-iii) above when $B(t) \geq 0$.

The main result of this chapter is the proof of the following theorem.

4. Theorem (Duality): If the regularity conditions (1) and the algebraic conditions (2) are satisfied, then there exist

optimal solutions for P and D, feasible for all $t \in [0,T]$, and with equal objective function values.

The remainder of this chapter is devoted to the proof of Theorem 4. We begin by defining discrete variables and problems, only slightly different than those used by Tyndall [34].

5. Definition: For any $n = 1,2,\ldots,$ and $k = 1,2,\ldots,n,$ let $\Delta_n = \frac{T}{n}$, $t^{n,k} = k\Delta_n$, and $I^{n,k} = [t^{n,k-1}, t^{n,k})$.

Also let

$$a^{n,k} = a(k\Delta_n),$$
$$B^{n,k} = B(k\Delta_n),$$
$$c^{n,k} = c(k\Delta_n),$$
$$D^{n,k} = D(k\Delta_n), \text{ and}$$
$$K^{n,\ell,k} = \Delta_n\, K(\ell\Delta_n, k\Delta_n), \quad \ell = 1,2,\ldots,n.$$

When working with a fixed n, the superscript n will often be omitted.

6. Definition: Problem P^n. Find vectors $z^{n,1},\ldots,z^{n,n} \in E^N$ which maximize

$$\sum_{k=1}^{n} \left(a^{n,k}\right)^T z^{n,k} - \sum_{k=1}^{n} \left[\left(z^{n,k}\right)^T D^{n,k} z^{n,k}\right]^{1/2}$$

subject to

$$B^{n,k}z^{n,k} \leq c^{n,k} + \sum_{\ell=1}^{k-1} K^{n,\ell,k}z^{n,\ell}, \quad k = 1,\ldots,n$$

$$z^{n,k} \geq 0, \quad k = 1,\ldots,n.$$

Problem D^n: Find vectors $w^{n,1},\ldots,w^{n,n} \in E^M$ which minimize

$$\sum_{k=1}^{n} (c^{n,k})^\tau w^{n,k}$$

subject to

$$\left(B^{n,k}\right)^\tau w^{n,k} \geq a^{n,k} + \sum_{\ell=k+1}^{n} \left(K^{n,k,\ell}\right)^\tau w^{n,\ell} - D^{n,k}y^{n,k}$$

$$k = 1,2,\ldots,n$$

$$\left(y^{n,k}\right)^\tau D^{n,k}y^{n,k} \leq 1, \quad k = 1,\ldots,n$$

$$w^{n,k} \leq 0, \quad k = 1,2,\ldots,n.$$

We can shorten the statement of problems P^n and D^n with the following definition.

7. Definition: Problem $\bar{\bar{P}}^n$: Find a vector $Z^n \in E^{nN}$ which maximizes

$$F^n(Z^n) = (A^n)^\tau Z^n - \sum_{k=1}^{n} \left[(Z^n)^\tau \mathcal{D}_k^n Z^n\right]^{1/2}$$

subject to $\mathcal{B}^n Z^n \leq C^n$

$$Z^n \geq 0 .$$

Problem $\bar{\bar{D}}^n$: Find vector $W^n \in E^{nM}$ which minimizes

$$G^n(W^n) = (C^n)^\tau W^n$$

subject to
$$\left(\mathcal{B}^n\right)^{\tau} W^n \geq A^n - \sum_{k=1}^{n} \mathcal{S}_k^n Y^n$$
$$Y^n \mathcal{S}_k^n Y^n \leq 1 \qquad k = 1, \ldots, n$$
$$W^n \geq 0 \ .$$

8. Remark. It is easily seen that problems P^n and $\bar{P}^n$ are equivalent, as are problems D^n and $\bar{D}^n$, where (omitting the superscript n):

$$Z = \begin{bmatrix} z^1 \\ \vdots \\ z^n \end{bmatrix}, \quad A = \begin{bmatrix} a^1 \\ \vdots \\ a^n \end{bmatrix}, \quad Y = \begin{bmatrix} y^1 \\ \vdots \\ y^n \end{bmatrix} \text{ are vectors in } E^{nN},$$

$$W = \begin{bmatrix} w^1 \\ \vdots \\ w^n \end{bmatrix}, \quad C = \begin{bmatrix} c^1 \\ \vdots \\ c^n \end{bmatrix} \quad \text{are vectors in } E^{nM},$$

$$\mathcal{B} = \begin{bmatrix} B^1 & 0 & \cdots & 0 \\ -K^{1,2} & B^2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ -K^{1,n} & -K^{2,n} & \cdots & B^n \end{bmatrix} \quad \text{is an } nM \times nN \text{ matrix,}$$

and
$$\mathcal{S}_k = \begin{bmatrix} 0 & \cdots & 0 \\ \vdots & D^k & \vdots \\ 0 & \cdots & 0 \end{bmatrix} \quad \text{is a symmetric } nN \times nN \text{ matrix.}$$

We should also notice that $\bar{P}^n$ and $\bar{D}^n$ are equivalent to Sinha's problems.

In particular, each D^k is positive semidefinite by Definitions (1:1) and (5), so each Δ_k is positive semidefinite also.

An immediate yet important observation is the following:

9. Lemma: If algebraic conditions (2) hold for P and D, then the corresponding conditions hold for P^n and D^n.

We first devote our attention to establishing the existence of optimal solutions to P^n. Our Lemma (10) is a more compact version of a result originally proved by Tyndall.

10. Lemma: If algebraic conditions (2) hold, then P^n is feasible for each n. Further, the set of feasible solutions to P^n is bounded.

Proof: Let n be fixed, and omit the superscript n. By Lemma 9, $z^k = 0 \quad k = 1,\ldots,n$ is a feasible solution for P^n.

Now consider z^k, the k^{th} term of a feasible solution. Let $e^\tau = (1,1,\ldots,1)$, $e \in E^M$. Then, using (2), $e^\tau B \geq \delta e^\tau > 0$. From Definitions (1:1) and (5), c^k and $K^{k,\ell}$ are bounded, say $|c^k| \leq \gamma$ and $|K^{\ell,k}| \leq \beta\Delta$ for $\ell = 1,\ldots,n$. The constraints of P^n when multiplied by e^τ become

$$(i) \quad e^\tau B^k z^k \leq e^\tau c^k + \sum_{\ell=1}^{k-1} e^\tau K^{\ell,k} z^\ell \qquad k = 1,\ldots,n \, .$$

But $e^\tau B^k z^k \geq \delta e^\tau z^k = \delta |z^k|$. Also $e^\tau c^k = |c^k| \leq \gamma$, and $\sum_{\ell=1}^{k-1} e^{\tau}K^{\ell,k}z^{\ell} \leq \sum_{\ell=1}^{k-1} |e^{\tau}K^{k,\ell}||z^{\ell}|$ so that we have

$\delta|z^{k}| \leq \gamma + \sum_{\ell=1}^{k-1} |e^{\tau}K^{\ell,k}||z^{\ell}| \leq \gamma + \beta\Delta \sum_{\ell=1}^{k-1} |z^{\ell}|$, or

$|z^{k}| \leq \frac{\gamma}{\delta} + \frac{\beta\Delta}{\delta} \sum_{\ell=1}^{k-1} |z^{\ell}|$ for $k = 1,2,\ldots,n$. Using induction,

we can show $|z^{k}| \leq \frac{\gamma}{\delta}\left[1 + \frac{\beta\Delta}{\delta}\right]^{k-1}$. Clearly, this is true

for $k = 1,2$. Assume it is true for k. Then

$$|z^{k+1}| \leq \frac{\gamma}{\delta} + \frac{\beta\Delta}{\delta} \sum_{\ell=1}^{k} |z^{\ell}| \leq \frac{\gamma}{\delta} + \frac{\beta\Delta}{\delta} \sum_{\ell=1}^{k} \frac{\gamma}{\delta}\left[1 + \frac{\beta\Delta}{\delta}\right]^{\ell-1}$$

$$= \frac{\gamma}{\delta}\left\{1 + \frac{\beta\Delta}{\delta}\left[\frac{\left(1 + \frac{\beta\Delta}{\delta}\right)^{k} - 1}{1 + \frac{\beta\Delta}{\delta} - 1}\right]\right\} = \frac{\gamma}{\delta}\left(1 + \frac{\beta\Delta}{\delta}\right)^{k}.$$

Now $e^{x} \geq 1 + x$ implies

$$|z^{k}| \leq \frac{\gamma}{\delta}\left(1 + \frac{\beta\Delta}{\delta}\right)^{k} \leq \frac{\gamma}{\delta}\left(1 + \frac{\beta\Delta}{\delta}\right)^{n} \leq \frac{\gamma}{\delta} \exp\frac{\beta\Delta n}{\delta} = \frac{\gamma}{\delta} \exp\frac{\beta T}{\delta}.$$

Thus each $|z^{k}|$ is bounded independent of n.

11. <u>Lemma</u>: If the algebraic conditions (2) hold, then each P^{n} has an optimal solution.

Proof: The set of feasible solutions for each P^{n} is non-empty and bounded by Lemma 10, and closed. Since the objective function of each P^{n} is continuous, it attains a maximum over a closed bounded set.

12. Theorem (Bhatia, [7 , p. 605]): If there exists an optimal solution to $\overline{P}^n$, then there exists an optimal solution to $\overline{D}^n$. Further, the objective function values are equal.

13. Lemma: If algebraic conditions (2) hold, then there exists an optimal solution to each D^n.

Proof: This is a direct consequence of Lemma (11), Theorem (12), and Remark (8).

Notice that Lemma (10) insures uniform boundedness for the optimal solutions $\{z^k: k = 1,\ldots,n\}$ to P^n. We shall now show that the optimal solutions $\{w^k: k = 1,\ldots,n\}$ and $\{y^k: k = 1,\ldots,n\}$ to D^n are also uniformly bounded. We shall first factor D and transform variables.

14. Lemma: If $D(t)$ is a symmetric matrix whose entries are piecewise continuous functions, then there exists a matrix $Q(t)$, also piecewise continuous, such that $D(t) = Q^T(t)Q(t)$.

Proof: Denote the eigenvalues of D by λ_i with corresponding normalized eigenvector v_i. If R is the matrix whose i^{th} column is v_i, then R is orthogonal and R^TDR is the diagonal matrix with entries λ_i. Thus $D = Q^TQ$ where $Q_{ij} = \sqrt{\lambda_i}\, v_{ij}$ and v_{ij} is the j^{th}-component of v_i.

Now $P(\lambda) = 0$ where $P(\lambda)$ is the characteristic polynomial of D. Since λ is a continuous function of the coefficients of P [26, p.3] which themselves are continuous functions of the piecewise continuous entries of D , λ is a piecewise continuous function of t . Thus each minor of $D - \lambda I$ is a piecewise continuous function of t .

Now consider any subinterval such that no points of discontinuity of any minor occur within that subinterval. In each of these (finitely many) intervals we may invert the largest non-singular submatrix of $D - \lambda I$ to obtain eigenvectors continuous on that interval. Thus a piecewise Q exists.

15. Definition: In Problem D^n , let

$$D^{n,k} = \left(Q^{n,k}\right)^\tau Q^{n,k} \text{ , and}$$

$$x^{n,k} = Q^{n,k} y^{n,k} \text{ .}$$

Notice that we can rewrite the constraints of D^n as follows:

$$\left(B^k\right)^\tau w^k \geq a^k + \sum_{\ell=k+1}^{n} \left(K^{k,\ell}\right)^\tau w^\ell - \left(Q^k\right)^\tau x^k \text{ ,}$$

$$\left(x^k\right)^\tau x^k \leq 1, \quad w^k \geq 0, \; k = 1,\ldots,n \text{ .}$$

16. Lemma: For $k = 1,\ldots,n$, $(x^k)^\tau x^k \leq 1$ implies $\|(Q^k)^\tau x^k\|_\infty$ is bounded independent of n.

Proof: If $Q_{\cdot j}$ denotes the j^{th} column of Q and hence the j^{th} row of Q^τ, then

$$\|(Q^k)^\tau x^k\|_\infty^2 = \max_j |Q_{\cdot j}^k x^k|^2 \leq \max_j \|Q_{\cdot j}^k\|_2^2 \|x^k\|_2^2 \leq \max_j \|Q_{\cdot j}^k\|_2^2$$

$$= \max_j (Q_{\cdot j})^\tau Q_{\cdot j} = \max_j D_{jj}^k .$$

But D_{jj}^k is bounded by Definitions (1:1) and (3:5). Thus $\|(Q^k)^\tau x^k\|_\infty$ is bounded independent of n.

Since $c^k \geq 0$ and $w^k \geq 0$, to minimize $\sum_{k=1}^{n} c^k w^k$ we wish to make each w^k as small as possible maintaining feasibility. Following Levinson [25], we shall obtain a bound on w^k. We first prove an intermediate result.

17. Lemma: In Problem D^n let $|a^k| \leq \alpha$, $|K^{\ell,k}| \leq \beta\Delta_n$, and $\|(Q^k)^\tau x^k\|_\infty \leq \eta$ according to Definitions (5), (14), and (1:1) and Lemma (16). Define the scalar

$\rho^{n,k} = \left(\frac{\alpha + \eta}{\delta}\right)\left(1 + \frac{\beta\Delta_n}{\delta}\right)^{n-k}$. Then, omitting the superscript n,

$$\delta\rho^k \geq a_j^k + \sum_{\ell=k+1}^{n} \left(\sum_{i=1}^{M} K_{ij}^{k,\ell} \rho^\ell \right) - \sum_{m=1}^{N} Q_{jm}^k x_m^k \quad \text{for } k = 1,2,\ldots,n.$$

Proof: $\sum_{\ell=k+1}^{n} \rho^{\ell} = \sum_{\ell=0}^{n-k-1} \rho^{n-\ell} = \sum_{\ell=0}^{n-k-1} \left(\frac{\alpha+\eta}{\delta}\right)\left(1+\frac{\beta\Delta}{\delta}\right)^{\ell}$

$$= \left(\frac{\alpha+\eta}{\delta}\right)\left(\frac{\delta}{\beta\Delta}\right)\left[\left(1+\frac{\beta\Delta}{\delta}\right)^{n-k} - 1\right].$$

Now $a_j^k + \sum_{\ell=k+1}^{n}\left(\sum_{i=1}^{M} K_{ij}^{k\ell}\,\rho^{\ell}\right) - \sum_{m=1}^{N} Q_{jm}^{k} x_m^k \le \alpha + \beta\Delta \sum_{\ell=k+1}^{n} \rho^{\ell} + \eta$

$$= \alpha + \eta + \beta\Delta\left(\frac{\alpha+\eta}{\beta\Delta}\right)\left[\left(1+\frac{\beta\Delta}{\delta}\right)^{n-k} - 1\right]$$

$$= (\alpha+\eta)\left(1+\frac{\beta\Delta}{\delta}\right)^{n-k} = \delta\rho^{k} \quad \text{as desired.}$$

18. Lemma: If $w^1,\ldots,w^n$ are feasible for D^n, and if algebraic conditions (2) hold, then there exists vectors $v^1,\ldots,v^n$ feasible for D^n such that

$$0 \le v^k \le w^k \quad \text{and} \quad 0 \le v_i^k \le \rho^k, \quad k = 1,\ldots,n; \quad i = 1,\ldots,M,$$

where ρ^k is defined in Lemma (17).

Proof: The proof is essentially the same as Levinson's Lemma 3.1 [25, p. 78] with obvious modifications using our Lemma (17). A proof is included in Appendix B.

19. Lemma: If algebraic conditions (2) hold, then there exist uniformly bounded optimal solutions for D^n.

Proof: Lemma (13) insures existence. According to Lemma (18), each $v_i^{n,k}$ is bounded. Since $c^{n,k} \ge 0$, $\sum_{k=1}^{n} c^{n,k} v^{n,k} \le \sum_{k=1}^{n} c^{n,k} w^{n,k}$. Thus, for fixed n, if there exists an optimal solution, there exists a bounded optimal solution. Further,

$$v_1^{n,k} \leq \rho^{n,k} = \frac{\alpha + \eta}{\delta}\left[1 + \frac{\beta\Delta_n}{\delta}\right]^{n-k} \leq \frac{\alpha + \eta}{\delta}\left[1 + \frac{\beta\Delta_n}{\delta}\right]^n$$

so $$v_1^{n,k} \leq \frac{\alpha + \eta}{\delta}\exp\left(\frac{\beta\Delta_n}{\delta}\right)n = \frac{\alpha + \eta}{\delta}\exp\frac{\beta T}{\delta} ,$$

independent of n.

Now that we have shown the existence of uniformly bounded optimal solutions to the discrete problems P^n and D^n, we shall follow Grinold's arguments [13] to show that they converge to feasible solutions to P and D. We then show that these solutions are in fact optimal for P and D. We begin with statements of some standard theorems needed subsequently.

20. Theorem (Lebesgue Convergence Theorem [30, p. 76]):
Let g be integrable over S and let $\{g_n\}_{n=1}^{\infty}$ be a sequence of measurable functions such that $|g_n| \leq g$ on S and $g_n(x) \to g(x)$ a.e. on S. Then

$$\int_S g_n(x)dx \to \int_S g(x)dx .$$

21. Definition (Weak Star Convergence):
Let $g_n \in L_\infty[0,T]$ for $n = 1,2,\ldots$. Then $g_n \to g$ in a weak star sense if

$$\int_0^T f(t)g_n(t)dt \to \int_0^T f(t)g(t)dt \quad \text{for every } f \in L_1[0,T] .$$

Also, a vector-valued function converges in a weak star sense if each component function converges weak star.

22. Lemma (Tyndall [34, p. 653]): Let $g_n \in L_\infty[0,T]$ for $n = 1,2,\ldots.$ Then there exists a function $g \in L_\infty[0,T]$ and a subsequence $\{n_k\}$ such that $g_{n_k} \to g$ (weak star).

23. Lemma (Tyndall [34, p. 653]): Let $z^n \in L_\infty^N[0,T]$ $n = 1,2,\ldots$. Then there exists a vector-valued function $z \in L_\infty^N[0,T]$ and a subsequence $\{n_k\}$ such that

$$\int_{t_1}^{t_2} z^{n_k}(t)dt \to \int_{t_1}^{t_2} z(t)dt \quad \text{for} \quad 0 \le t_1 < t_2 \le T \ .$$

24. Lemma: Let

$$C_P' = \Big\{ z \in L_\infty^N[0,T] : \int_{t_1}^{t_2} B(t)z(t)dt \le \int_{t_1}^{t_2} c(t)dt + \int_{t_1}^{t_2}\int_0^t K(s,t)z(s)dsdt,$$

$$z(t) \ge 0, \quad \text{for every} \quad t_1,\ t_2 \quad \text{where} \quad 0 \le t_1 < t_2 \le T \Big\} .$$

$$C_D' = \Big\{ w \in L_\infty^M[0,T] : \int_{t_1}^{t_2} B^\tau(t)w(t)dt \ge \int_{t_1}^{t_2} a(t)dt$$

$$+ \int_{t_1}^{t_2}\int_t^T K^\tau(t,s)w(s)ds\,dt - \int_{t_1}^{t_2} D(t)y(t)dt \ ,$$

$$\int_{t_1}^{t_2} y^\tau(t)D(t)y(t)dt \le t_2 - t_1,\ y \in L_\infty^N[0,T] , \quad w(t) \ge 0 \ ,$$

$$\text{for every} \quad t_1,\ t_2 \quad \text{where} \quad 0 \le t_1 \le t_2 \le T \Big\} .$$

Then (i) $z \in C_P'$ if, and only if, z is feasible a.e. for P;

(ii) $w \in C_D'$ if, and only if, w is feasible a.e. for D.

Proof: The proof of (i) is the same as that for a similar result of Tyndall [34, p. 652]. The proof of (ii) is similar and makes use of the fact that, if $F(x) = \int_a^x f(t)dt$, $F'(x) = f(x)$ a.e.. A proof appears in Appendix B.

In order to progress from the discrete problems P^n and D^n to the continuous problems P and D, we now define step functions using Definitions (5) and Lemmas (11), (13) and (15).

25. <u>Definition</u>: For $n = 1,2,\ldots,$ and $k = 1,\ldots,n$, let $\{z^{n,k}\}$ and $\{w^{n,k}\}$ be the optimal solutions to P^n and D^n respectively. For $t \in I^{n,k}$ let

$$a^n(t) = a^{n,k}$$
$$c^n(t) = c^{n,k}$$
$$B^n(t) = B^{n,k}$$
$$Q^n(t) = Q^{n,k}$$
$$w^n(t) = w^{n,k}$$
$$z^n(t) = z^{n,k}$$
$$x^n(t) = x^{n,k} .$$

For $(s,t) \in I^{n,\ell} \times I^{n,k}$, let

$$K^n(s,t) = \frac{1}{\Delta_n} K^{n,\ell,k}$$

Now that we have shown the existence of uniformly bounded optimal solutions to the discrete problems P^n and D^n, we can use Tyndall's "diagonal process" to extract a common subsequence $\{n_j\}$ such that there exist bounded measurable functions z, w, and x to which the subsequences of our step functions, that is $\left\{z^{n_j}(t)\right\}$, $\left\{w^{n_j}(t)\right\}$ and $\left\{x^{n_j}(t)\right\}$, converge in a weak star sense.

26. Lemma: If condition (2) holds, then there exist functions $z,x \in L_\infty^N[0,T]$ and $w \in L_\infty^M[0,T]$ and a subsequence $\{n_j\}$ such that, using Definition (25), $z^{n_j}(t) \to z(t)$, $x^{n_j}(t) \to x(t)$ and $w^{n_j}(t) \to w(t)$, all in a weak star sense on $[0,T]$.

Proof: We simply extend Tyndall's argument [34, p. 653] to the N components of $x^n(t)$ as well, using Lemmas (10), (16), (19), and (23).

Hereafter, we shall denote the convergent common subsequences of functions as simply $z^n(t)$, $x^n(t)$ and $w^n(t)$.

We next show that z and w are feasible a.e. for P and D. Grinold has shown feasibility for z using slightly different notation, and we need only slightly modify his argument to show that w is feasible also.

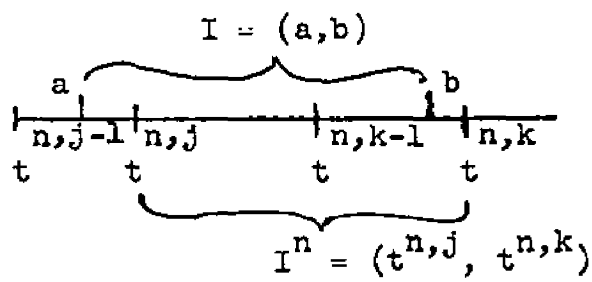
27. Lemma: If Regularity Conditions (1) hold, then, in Lemma (26), z is feasible for P a.e. on $[0,T]$.

Proof: Grinold [13, App. C].

28. Lemma: Given Definitions (15) and (25) and Lemma (26), there exists a $y(t)$ such that $x(t) = Q(t)y(t)$.

Proof: As in the proof of Lemma (14), consider any nondegenerate subinterval such that all minors of Q are continuous in it. [The continuity of each minor insures that we can further subdivide that interval into finitely many non-degenerate subintervals such that some submatrix of order r is non-singular but all submatrices of order $r + 1$ are singular]. Let $I = (a,b)$ denote any one of these (finitely-many) subintervals with this latest property, and consider $Q(t)$ on I . That is, the rank of $Q(t)$ on I is r . Since $Q_{ij}(t) = \sqrt{\lambda_i(t)}\, v_{ij}(t)$ and the eigenvectors v_i are linearly independent, $\lambda_i = 0$ on I for some $N - r$ of the indices i . Thus the corresponding $N - r$ rows of Q are zero on I .

For n sufficiently large, we have the following:



From the definition of $Q^n(t)$, the same $N - r$ rows of $Q^n(t)$ and hence of $x^n(t)$ are zero on I^n .

Thus the corresponding $N - r$ entries of $x(t)$ must be zero on I since the lengths of $[t^{n,j-1},a]$ and $[b,t^{n,k}]$ become arbitrarily small for large n. Finally, since $Q(t)$ has rank r on I, the conclusion follows.

29. Lemma: If Regularity Conditions (1) hold, then, in Lemma (26) w is feasible for D a.e. on $[0,T]$.

Proof: Following Grinold [13, App. C] for $t \in [0,T)$, let

$$h^n(t) = \left[B^n(t) - B(t)\right]^\tau w^n(t),$$

$$H^n(t) = \int_{t+\Delta_n}^{T} \left[K^n(t,s) - K(t,s)\right]^\tau w^n(s)ds \text{ , and}$$

$$q^n(t) = \left[Q^n(t) - Q(t)\right] x^n(t)$$

where $D(t) = Q^\tau(t)Q(t)$ and Q is piecewise continuous on $[0,T]$. Also let $h^n(T) = H^n(T) = q^n(T) = 0$. Following Grinold's arguments, it is easy to see that $h^n(t)$, $H^n(t)$ and $q^n(t)$ are bounded in norm from Definition (1:1) and the fact that $x^n \in L_\infty^N[0,T]$ and $w^n \in L_\infty^M[0,T]$. Further, Regularity Conditions (1) insure that $h^n(t) \to 0$, $H^n(t) \to 0$, and $q^n(t) \to 0$ a.e. on $[0,T]$. Now the feasibility of $w^{n,k}$ for Problem D^n implies

$$\text{(i)} \quad \left(B^n(t)\right)^\tau w^n(t) \geq a^n(t) + \int_{t+\Delta}^{T} \left(K^n(t,s)\right)^\tau w^n(s)ds$$
$$- \left(Q^n(t)\right)^\tau x^n(t) \text{ ,}$$

using the fact that, from Definition (23),

$$\sum_{\ell=k+1}^{n} \left(K^{n,k,\ell}\right)^{\tau} w^{n,k} = \int_{t^{n,k}}^{T} \left[K^{n}(t,s)\right]^{\tau} w^{n}(s)ds, \quad k = 1,\ldots,n-1 .$$

Rewriting (i) using the functions defined earlier, we have

$$B^{\tau}(t)w^{n}(t) + h^{n}(t) \geq a^{n}(t) + \int_{t+\Delta}^{T} K^{\tau}(t,s)w^{n}(s)ds + H^{n}(t)$$
$$- \left(Q(t)\right)^{\tau} x^{n}(t) - q^{n}(t) .$$

Integrating the above inequality over $[t_1,t_2]$ and then taking the limit as $n \to \infty$, we have

$$\text{(ii)} \quad \int_{t_1}^{t_2} B^{\tau}(t)w(t)dt \geq \int_{t_1}^{t_2} a(t)dt + \int_{t_1}^{t_2}\int_{t}^{T} K^{\tau}(t,s)w(s)dsdt$$
$$- \int_{t_1}^{t_2} Q^{\tau}(t)x(t)dt$$

where we have used Theorem (20) and the fact that $w^{n}(t) \to w(t)$ and $x^{n}(t) \to x(t)$ weak star by Lemma (26). Since we also have $\left[x^{n}(t)\right]^{\tau} x^{n}(t) \leq 1$, thus

$$\text{(iii)} \quad \int_{t_1}^{t_2} \left[x^{n}(t)\right]^{\tau} x^{n}(t)\, dt \to \int_{t_1}^{t_2} x^{\tau}(t)x(t)dt \leq t_2 - t_1 .$$

Thus Lemmas (24) and (28) imply that $w(t)$ is feasible a.e. on $[0,T]$ as $w(t) \geq 0$.

30. <u>Lemma</u>: If Regularity Condition (1) hold, then, in Lemma (26), $f(z) = g(w)$.

Proof: The proof is again similar to that of Tyndall [34], [36] or Grinold [13]. From Theorem (12) and Remark (8) we have $F^n(Z^n) = G^n(W^n)$. Thus

$$\Delta_n \sum_{k=1}^{n} \left(a^{n,k}\right)^\tau z^{n,k} - \Delta_n \sum_{k=1}^{n} \left[\left(z^{n,k}\right)^\tau D^{n,k} z^{n,k}\right]^{1/2}$$

$$= \int_0^T \left[a^n(t)\right]^\tau z^n(t)dt - \int_0^T \left[\left(z^n(t)\right)^\tau D^n(t) z^n(t)\right]^{1/2} dt$$

$$= \Delta_n \sum_{k=1}^{n} \left(c^{n,k}\right)^\tau w^{n,k} = \int_0^T \left[c^n(t)\right]^\tau w^n(t)dt.$$

Therefore, in a manner similar to that in the proof of Lemma (29), weak star convergence and Regularity Conditions (1) imply

$$\int_0^T \left[a^n(t)\right]^\tau z^n(t)dt \to \int_0^T a^\tau(t) z(t)dt,$$

$$\int^T \left[\left(z^n(t)\right)^\tau D^n(t) z^n(t)\right]^{1/2} dt \to \int^T \left[z^\tau(t) D(t) z(t)\right]^{1/2} dt, \text{ and}$$

$$\int_0^T \left[\left(z^n(t)\right)^\tau D^n(t) z^n(t)\right]^{1/2} dt \to \int_0^T \left[z^\tau(t) D(t) z(t)\right]^{1/2} dt, \text{ and}$$

$$\int_0^T \left[c^n(t)\right]^\tau w^n(t)dt \to \int_0^T c^\tau(t) w(t)dt.$$ Thus $f(z) = g(w)$.

The fact that z and w are feasible for P and D only a.e. on $[0, T]$ is no problem.

31. Lemma: Let z and w be as in Lemma (30). Then there exists $\bar{z}$ and $\bar{w}$ feasible on $[0,T]$ for P and D respectively with $f(\bar{z}) = g(\bar{w})$.

Proof: Let S_P be the set of measure zero where z is not feasible for P. Define

$$\bar{z}(t) = \begin{cases} z(t) & \text{if } t \notin S_P \\ 0 & \text{if } t \in S_P \end{cases}$$

Then $\bar{z}$ is feasible on $[0,T]$ and $f(z) = f(\bar{z})$.

Similarly let S_D be the set of measure zero where w is not feasible for D. Let $\alpha \geq |a(t)|$ and $\beta \geq |K(s,t)|$. Define

$$\bar{w}(t) = \begin{cases} w(t) & \text{if } t \notin S_D \\ U(t) & \text{if } t \in S_D \end{cases} \quad \text{and} \quad \bar{y}(t) = \begin{cases} y(t) & \text{if } t \notin S_D \\ 0 & \text{if } t \in S_D \end{cases}$$

where $U(t)$ is the $M \times 1$ vector with each entry equal to $u(t) = \frac{\alpha}{\delta} \exp \frac{\beta(T-t)}{\delta}$. Then, as in Levinson [25], $U(t)$ is feasible for D since $\delta u(t) = \alpha + \beta \int_t^T u(s)ds$ and $\beta^\tau(t)\, w(t) \geq \delta u(t) e \geq a(t) + \int_t^T K^\tau(t,s)\, w(s)ds$. Clearly we have $f(\bar{z}) = f(z) = g(w) = g(\bar{w})$ as desired.

This completes the proof of Theorem (4) since we have shown the existence of equilibium solutions which are therefore optimal by Theorem (2:9).

CHAPTER IV

CONCLUDING REMARKS

Chapter I mentions several works related to this thesis. While our work directly generalizes results of Sinha [33], Tyndall [34], [35], [36] and Levinson [25], some other possible extensions are rather obvious.

One possibility is to attempt to use Grinold's Algebraic and Boundedness Conditions [14, p. 86, 88] which include our Condition (3:2) as a special case. There are several immediate difficulties with this approach. One is that our discrete non-linear forerunner does not possess the symmetry of the discrete linear programming problem stated in the Introduction. Another difficulty is that feasibility alone does not imply existence of optimal solutions to our discrete non-linear problem whereas it does imply existence for the discrete linear problem. Finally, in Grinold's proof [14, Lemma 3.12] the linear structure of the objective function is used directly to sequentially quarantee an optimal solution.

Another possible extension is to use the latest results of Mond [28] which generalize Sinha's problem [33] to convex polyhedral cones in complex space. The primary difficulty here apparently is finding suitable hypotheses to insure boundedness of the constraint sets in the discrete problems corresponding to our P^n and D^n.

In an effort to obtain more symmetry in the Primal and Dual Problems, it is conceivable that one may be able to generalize Sinha's problem by including additional semidefinite forms both in the dual objective function and in the primal constraint set. Of course, this is pure speculation.

One other formulation which has been investigated unsuccessfully by this author is the following:

Primal: $$\max_{z \in C_P} \bar{f}(z)$$

Dual: $$\min_{w \in \bar{C}_D} g(w)$$

Here g and C_P are as in Definition (1:1) while

$$\bar{f}(z) = \int_0^T a^\tau(t)z(t)dt - \left[\int_0^T\int_0^T z^\tau(s)D(s,t)z(t)dsdt\right]^{1/2}, \text{ and}$$

$$\int_0^T a^\tau(t)z(t)dt - \left[\int_0^T\int_0^T z^\tau(s)D(s,t)z(t)dsdt\right]^{1/2},$$

$$\bar{C}_D = \Big\{w \in L_\infty[0,T]:\ w(t) \geq 0,\ \int_0\int_0 y'(s)D(s,t)y(t)dsdt \leq 1, \text{ and}$$

$$B^\tau(t)w(t) \geq a(t) + \int_t^T K^\tau(t,s)w(s)ds - \int_0^T D(s,t)y(s)ds\Big\}$$

This problem is appealing in several ways. In particular, it is easy to show that $\bar{f}(z) \leq g(w)$ for $z \in C_P$ and $w \in \bar{C}_D$. Also this problem directly generalizes those of Tyndall and Sinha.

Further, if one defines discrete versions of this problem similar to Definition (3:6), these discrete problems are equivalent to Sinha's problems. However, if our Conditions (1) and (2) are assumed, there apparently is a problem in uniformly bounding the analogue to our $x^{n,k}$ and then passing from the discrete to the continuous case.

APPENDIX A

In this appendix we indicate changes necessary to prove the results stated for the problems of Definition (1:1) rather than those proved in the text for the case $r = 1$.

In Theorem (2:3) replace the constraint $y^{\tau}(t)D(t)y(t) \leq 1$ with $y_p^{\tau}(t)D_p(t)y_p(t) \leq 1$, $p = 1, \ldots, r$, and all other terms involving only $D(t)y(t)$ or $y^{\tau}(t)D(t)$ with $\sum_{p=1}^{r} D_p(t)y_p(t)$ and $\sum_{p=1}^{r} y_p^{\tau}(t)D_p(t)$ respectively. Also in equations (ii) and (iii) the term $[z^{\tau}(t)D(t)z(t)]^{1/2}$ is changed to $\sum_{p=1}^{r} [z^{\tau}(t)D_p(t)z(t)]^{1/2}$. These same changes are also needed in Definition (2:6-ii), Remark (2.7) and Theorem (2:8), while Definition (2:6-iii) is modified to read as follows:

$$z^{\tau}(t)D_p(t)y_p(t) = [z^{\tau}(t)D_p(t)z(t)]^{1/2} \quad p = 1, \ldots, r .$$

Regularity Condition (3:1-ii) is extended to apply to each entry of each D_p , $p = 1, \ldots, r$, as is Definition (3:5). Next the objective function of P^n in Definition (3:6) is changed to

$$\sum_{k=1}^{n} (a^{n,k})^{\tau} z^{n,k} - \sum_{p=1}^{r} \sum_{k=1}^{n} [(z^{n,k})^{\tau} D_p^{n,k} z^{n,k}]^{1/2}$$

and that of $\bar{P}^n$ in Definition (3:7) becomes

$$(A^n)^\tau Z^n - \sum_{p=1}^{r} \sum_{k=1}^{n} [(Z^n)^\tau \mathcal{D}^n_{k,p} Z^n]^{1/2} .$$

Similarly the dual constraints of Definitions (3:6) and (3:7) are changed in the manner prescribed above for Chapter 2, additional superscripts notwithstanding.

In Remark (3:8), for $p = 1 , \ldots, r$ let

$$Y_p = \begin{bmatrix} y_p^1 \\ \cdot \\ \cdot \\ \cdot \\ y_p^n \end{bmatrix} \qquad \text{and} \qquad \mathcal{D}_{k,p} \begin{bmatrix} 0 \ldots\ldots 0 \\ \cdot \qquad \cdot \\ \cdot \quad D_p^k \quad \cdot \\ \cdot \qquad \cdot \\ \cdot \qquad \cdot \\ 0 \ldots\ldots 0 \end{bmatrix}$$

where the position of D_p^k is determined by k , not p . That is, D_p^k is the k^{th} $N \times N$ diagonal submatrix. The equivalence of the modified P^n and $\bar{P}^n$ as well as D^n and $\bar{D}^n$ are then seen by rewriting the double sums over k and p as a single sum over α, $\alpha = 1, \ldots, nr$ where $\alpha = (p - 1) n + k$, $\mathcal{D}_{k,p} = \bar{\mathcal{D}}_\alpha$, and $\bar{Y}_a = Y_p$, $\alpha = (p - 1) k + 1, \ldots, pk$.

In (3:15), $Q_p^{n,k}$ and $x_p^{n,k}$ are defined for each $p = 1, \ldots, r$ and the constraints rewritten accordingly. In (3:16) each $(Q_p^k)^\tau x_p^k$, $p = 1 , \ldots, r$ is bounded in L_∞ norm , say by η . Then, in (3:17), the scalar $\rho^{n,k} = \left(\frac{\alpha + \eta r}{\delta}\right) \left(1 + \frac{\beta\Delta}{\delta}\right)^{n-k}$ will have the desired property, again replacing the terms $Q^\tau x$ by $\sum_{p=1}^{r} Q_p^\tau x_p$.

Of course in Lemma 3:19 we must again replace η with ηr .

In Definition (3:25) we replace $Q^n(t)$ and $x^n(t)$ by $Q_p^n(t)$ and $x_p^n(t)$ $p = 1, \ldots, r$ respectively. The diagonal process is extended to all $x_p^n(t)$ $p = 1, \ldots, r$ in (3:26) and Lemma (3:28) holds for each $p = 1, \ldots, r$. The necessary changes for the remainder of Chapter 3 are obvious - - for example, defining $q_p^n(t)$ appropriately for each $p = 1, \ldots, r$, replacing the term

$Q^T(t)\, x(t)$ by $\sum_{p=1}^{r} Q_p^T(t)\, x_p(t)$, and modifying constraints and

objective functions as before.

APPENDIX B

In this appendix we present proofs for Lemmas (3:18) and (3:24).

3:18 Lemma: If $w^1, \ldots, w^n$ are feasible for D^n, and if Algebraic Conditions (3:2) hold, then there exists vectors $v^1, \ldots, v^n$ feasible for D^n such that $0 \leq v^k \leq w^k$ and $0 \leq v_i^k \leq \rho^k$, $k = 1, \ldots, n$, $i = 1, \ldots, M$, where ρ^k is defined by Lemma 3:17.

Proof: Consider any k, $k = 1, \ldots, n$ and omit the superscript k where possible. Let

$I = \{i : w_i \leq \rho\}$ and $I' = \{i : w_i > \rho\}$. Let

$v_i = \begin{cases} w_i & \text{if } i \in I \\ \rho & \text{if } i \in I' \end{cases}$. If $I' = \emptyset$ we are done; if $I' \neq \emptyset$

we have

$$\sum_{i=1}^{M} B_{ij} v_i \geq \sum_{i \in I'} B_{ij} v_i = \sum_{i \in I'} B_{ij} \rho \geq \delta \rho \geq$$

$$a_j + \sum_{\ell=k+1}^{n} \sum_{i=1}^{M} K_{ij}^{\ell} \rho^{\ell} - \sum_{m=1}^{N} Q_{jm} x_m \geq$$

$$a_j + \sum_{\ell=k+1}^{n} \sum_{i=1}^{M} K_{ij}^{\ell} v_i^{\ell} - \sum_{m=1}^{N} Q_{jm} x_m \quad \text{by Lemma (3:17).}$$

Thus v^k is feasible.

3:24-(ii) <u>Lemma</u>: Let

$$C_D^1 = \{w \in L_\infty^M[0,T] : \int_{t_1}^{t_2} B^\tau(t)w(t)dt \geq \int_{t_1}^{t_2} a(t)dt +$$

$$\int_{t_1}^{t_2}\int_t^T K\,(t,\ s)\ w(s)ds\ dt - \int_{t_1}^{t_2} D(t)y(t)\,dt\ ,$$

$$\int_{t_1}^{t_2} y^\tau(t)\,D(t)\,y(t)dt \leq t_2 - t_1\ ,\quad y \in L_\infty^N[0,t],\quad w(t) \geq 0,$$

$$\text{for every}\ \ t_1,\ t_2\ \ \text{where}\ \ 0 \leq t_1 < t_2 \leq T\}\ .$$

Then $w \in C_D^1$ if, and only if, w is feasible a.e. for D .

Proof: If w is feasible a.e. for D, clearly $w \in C_D^1$. If $w \in C_D^1$, consider t_1 fixed, $0 \leq t_1 < t_2 \leq T$. We have

$$\frac{1}{t_2 - t_1}\int_{t_1}^{t_2} B^\tau(t)w(t)dt \geq \frac{1}{t_2 - t_1}\Big[\int_{t_1}^{t_2} a(t)dt +$$

$$\int_{t_1}^{t_2}\int_t^T K^\tau(t,s.)\ w(s)ds\ dt - \int_{t_1}^{t_2} D(t)y(t)dt\Big],\ \text{and}$$

$$\frac{1}{t_2 - t_1}\int_{t_1}^{t_2} y^\tau(t)D(t)y(t)dt \leq 1.$$

Taking the limit as $t_2 \to t_1$ yields the desired result (since t_1 is arbitrary) using the fact that $D_x \int_a^x f(t)dt = f(x)$ [30, p. 88] .

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