71-18,027

JOHNSON, Charles Edward, 1937-
A LOCAL THEORY OF GROUP EXTENSIONS.

The Ohio State University, Ph.D., 1970
Mathematics

A LOCAL THEORY OF GROUP EXTENSIONS

DISSERTATION

Presented in Partial Fulfillment of the Requirements for the Degree Doctor of Philosophy in the Graduate School of The Ohio State University

By

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The Ohio State University
1970

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ACKNOWLEDGEMENTS

I wish to thank the following two people: my adviser, Professor Hans Zassenhaus, and Marilyn Teeter Welles.

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FIELDS OF STUDY

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Studies in Group Theory: W. Kappe, Z. Janko, H. Wielandt

Studies in Representation Theory: H. Zassenhaus, W. Ledermann

Studies in Algebra: P. Roquette

Studies in Analysis: F. Caroll, B. Baishanshi, P. Reicheldorfer

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NOTATION

$A \subseteq B$. A is a subgroup of B.

$A \not\subseteq B$. A is not a subgroup of B.

$A < B$. A is a proper subgroup of B.

$A \lhd B$. A is a normal subgroup of B.

$a \in A$. a is an element of A.

$NK = \{ nk \mid n \in N, k \in K \}$.

$N \cdot K$. The subgroup NK when N is normalized by K.

$N \cap K$. The intersection of N and K.

$a^b = b^{-1}ab$.

$\langle x,y \rangle$. The group generated by x and y.

$|G|$. The order of the group G.

$|x|$. The order of the element x.

G'. The derived group of G.

$C_G(N)$. The centralizer in G of N.

$N_G(N)$. The normalizer in G of N.

$\Phi(G)$. The Frattini subgroup of G.

$Z(G)$. The center of G.

$[x,y] = x^{-1}y^{-1}xy$.

$[x,A] = \langle [x,a] \mid a \in A \rangle$.

Introduction

Given two finite groups N and F, there are many extensions of N by F, i.e. groups G with normal subgroup N and factor group $G/N = F$. The question as to what determines the extension was answered by Schrier (see e.g. Hall [5] theorem 15.1.1) in terms of automorphisms and factor sets, and two extensions were called equivalent if a change in coset representatives would give the same automorphisms and factor sets. As for any algebraic structure, the question as to what extent the local structure influences the global structure arises. In this respect, the question is asked if E_1 and E_2 are extensions of N by F, which subgroups of E_1 and E_2 must be equivalent in order to force E_1 and E_2 to be equivalent.

Many of the problems reduce to the question of splitting, i.e. does there exist a subgroup K of G such that $G = NK$ and $N \cap K = 1$? For that reason Chapter I deals with splitting theorems. The well-known theorems which are to be used later are given, and some new theorems are proved.

In Chapter II a general theory of local equivalence is developed. Subgroups U_1 and U_2 of E_1

and E_2 respectively are said to be equivalent if changing coset representatives of N in E_1 and E_2 gives the same automorphisms and factor sets when restricted to U_1 and U_2. A class of subgroups of G is called a local family if the class determines the extension, i.e. if when every member of the class in E_1 is equivalent to a member of the class of E_2 implies that E_1 and E_2 are equivalent. Several examples are given to show that certain families of subgroups are not local. For example, it is shown that the family of subgroups whose intersect with N is nilpotent is not a local family.

In Chapter III the concept of local equivalence is expanded to include all subgroups containing N and two extensions E_1 and E_2 are said to be C-equivalent, where C is a class of subgroups of F, if every subgroup U_1 of E_1 which contains N is equivalent to a subgroup U_2 of E_2. In the case where N is abelian, the theorem of Gaschutz saying that splitting of G over N is determined by the Sylow groups of G, is extended to give equivalence of two extensions if they are Sylow-equivalent. It is shown that if two extensions are cyclic-equivalent, then they have transversals which give the same automorphisms on N. A theorem is proven that if two extensions have such transversals, they have factor sets differing by an element in $C^2(F,Z(N))$, the second cocycles of F into $Z(N)$, i.e. an element of $B^2(F,Z(N))$. Several corollaries follow from this result, as does the main theorem, which is a generalization of Gaschutz Theorem to the non-abelian case. The theorem is that two extensions are equivalent if and only if they are Sylow-equivalent.

I. SPLITTING THEOREMS

A group G is said to split over a normal subgroup N if there is a subgroup K of G such that $G = NK$ and $N \cap K = 1$. K is called a complement of N in G. It is of great interest to know when G splits and much research has been done in this area. One of the basic results is

<u>Theorem 1.1</u> (Schur-Zassenhaus)

If N is a normal subgroup of G and $(|N|, |G:N|) = 1$ then N has a complement in G and all complements are conjugate.

A local-global theory for the splitting of groups has been sought for quite some time. One well-known result in this direction is the following theorem of Frobenius. K is called a normal p-complement if $K \lhd G$, $G = KP$, and $K \cap P = 1$ where P is some Sylow p-group of G. In this theorem then, the normal subgroup is sought, it is not assumed to exist.

<u>Theorem 1.2</u> (Frobenius)

G has a normal p-complement if and only if the normalizer of each p-subgroup does.

A more recent result in this same vein is

<u>Theorem 1.3</u> (Thompson)

Let p be an odd prime, P a Sylow p-subgroup, and $Z \subseteq Z(P)$. Let $J = \langle A \subseteq P \mid A$ is abelian and of maximal rank in $P\rangle$. If $C_G(Z)$ and $N_G(J)$ have normal p-complements so does G.

There are papers by Sah [13], Schoenwalder [15], Suzuki [19], Baer [1], and Roquette [12] in which complements of Hall subgroups are discussed. The first four of these papers discuss normal Hall complements. A paper by P. Hall [6] characterizes groups in which all subgroups are complements, one by Christensen [4] deals with groups in which every normal subgroup is complemented, and G. Higman [7] discusses complements of abelian normal subgroups. Theorems by Carter [3] and Schenkman [14] imply G splits over a certain term in a normal series. But basically these papers do not attack the problem from a local point of view. They do not assume the existence of a normal subgroup, but attempt to show the existence of a normal subgroup over which G splits. The work in this chapter does, and is more in the direction of

Theorem 1.4 (Gaschütz)

Let G be a finite group and N a normal abelian subgroup of G. Then G splits over N if and only if each Sylow group P of G splits over $P \cap N$.

As is well known, the theorem is not true if the normal subgroup is not abelian. An example which demonstrates this behavior is given in Scott [17]. This particular example will be used to demonstrate other behavior later as well so it is given now.

Example 1.1

Let $N = \langle a,b \mid a^4 = b^4 = 1,\ a^b = a^{-1},\ a^2 = b^2 \rangle$ be the group of quaternions, T be the central product of N with a cyclic group of order 4, $T = \langle N,x \mid x^2 = a^2,\ [N,x] = 1 \rangle$ and σ an automorphism of T of order three having the action $a^\sigma = b$, $b^\sigma = ab$, $x^\sigma = x$. Take G to be the split extension of T by $\langle\sigma\rangle$. It can be easily seen that although T splits over N, $T = N\langle ax\rangle$, $N \cap \langle ax\rangle =$ $= 1$, G does not. Since G/N is cyclic of order 6, any complement must contain an involution which is centralized by an element of order 3. Any complement of N in G would contain a complement of N in T and the complements of N in T are $\langle ax\rangle$, $\langle a^{-1}x\rangle$, $\langle bx\rangle$, $\langle b^{-1}x\rangle$, $\langle abx\rangle$, and $\langle (ab)^{-1}x\rangle$. These are all conjugate so it suffices to look at $\langle ax \rangle$. Since $x \in Z(G)$ then $C_G(ax) = C_G(a) = \langle a,x \rangle$ which has order 8. The involution ax is not centralized by any element of order 3 so cannot lie in a complement of N in G, nor can any of its conjugates, so there is no such complement.

Efforts to extend this result so as to not require that N be abelian have been notably unsuccessful, and only slight progress has been made in this direction. The following theorem places no constraints on the normal subgroup and is a beginning in the direction in which we want to proceed.

Theorem 1.5

Let G be a finite group and $N \lhd G$. Suppose every Sylow group P of G splits over $P \cap N$ and every maximal subgroup M of G splits over $M \cap N$. Then G splits over N.

Proof:

If N lies in all maximal subgroups, N lies in their intersection, $\Phi(G)$. Since $\Phi(G)$ is well known to be nilpotent, N is nilpotent and $N' < N$. N/N' is an abelian normal subgroup of G/N' and each Sylow group PN'/N' splits over $(PN'/N') \cap (N/N')$ since $P = (P \cap N) \cdot K_p$ so $PN'/N' \cap N/N' = (P \cap N)N'/N'$. Applying the theorem of Gaschutz then N/N' has a complement in G/N'. Hence N is complemented in G contrary to N lying in $\Phi(G)$, so there must be at least one maximal subgroup M which does not contain N and $M = (M \cap N) \cdot K$, $(M \cap N) \cap K = 1$. Since M is maximal $G = NM$ and

$$|G| = \frac{|N|\ |M|}{|M \cap N|} = \frac{|N|\ |M \cap N|\ |K|}{|M \cap N|} = |N|\ |K|$$

so K is a complement of N in G.

Q.E.D.

In example 1.1 only one maximal subgroup fails to split over its intersection with N and that is a cyclic subgroup of order 12, $\langle x\sigma \rangle$. It is of course not necessary that all maximal subgroups split over their intersection with N, for if that were the case, applying the result to the maximal subgroups of the maximal subgroups, we could show that all subgroups split over their intersection with N.

Another result is that of the author's in [10] in which it is shown that

Theorem 1.6

Let G be a finite group and N a normal subgroup of G. If for each p, every p-subgroup P of G splits over $P \cap N$, then G splits over N.

Since this theorem places no restriction on N, it is of some interest then to know under what circumstances every subgroup of a group of prime power order will split over a normal subgroup. This is answered in the negative in the paper just refered to by giving a characterization of p-groups which do not split over a normal subgroup, while all proper subgroups do split over their intersection with the normal subgroup. Here we derive a sufficient condition for a prime power group and all its subgroups to split over their intersection with a normal subgroup.

Theorem 1.7

Let G be a finite p-group for some prime p and $N \triangleleft G$. If each subgroup P has the property that for each $A \subseteq \Phi(P)$, $A \triangleleft P$, every two generator subgroup S/A of P/A splits over $(S \cap N) \cdot A/A$, then G splits over N and all subgroups P of G split over $P \cap N$.

Proof:

Let G be a counterexample of minimal order. If P is a proper subgroup of g, then P satisfies the conditions of the theorem relative to the normal subgroup $P \cap N$. Therefore we may assume that every proper subgroup P of G splits over $P \cap N$.

If $N \nsubseteq \Phi(G)$ then N would have a supplement $S < G$ so that $G = NS$. Since S would be proper $S = (S \cap N) \cdot K$, $(S \cap N) \cap K = N \cap K = 1$ and K would be a complement of N in G. So it may be assumed that $N \subseteq \Phi(G)$.

Let M be a maximal subgroup of G. Then M contains

N and by assumption splits over N, so $N \not\subseteq \Phi(M)$. Let A be a normal subgroup of G maximal subject to the following requirements, $\Phi(M) \subseteq A \subseteq \Phi(G)$ and $N \not\subseteq A$. Then NA/A is minimal normal in G/A and if B/A is also minimal normal in G and $B/A \subseteq \Phi(G/A)$ then $BA \subseteq \Phi(G)$ and $BA \triangleleft G$ so that by the maximality of A, no other minimal normal subgroup of G can lie in $\Phi(G/A)$. Suppose B/A is a minimal normal subgroup of G/A and $B/A \not\subseteq \Phi(G/A)$. Since $B/A \not\subseteq \Phi(G/A)$, B/A is supplemented, and being of order p is therefore complemented. Since a minimal normal subgroup of a p-group lies in the center, both B/A and its complement are normal so that G/A is a direct product. Each minimal normal subgroup which does not lie in $\Phi(G/A)$ is a direct factor of G/A. The complement of the direct factor B/A will have as direct factors each minimal normal subgroup not lying in $\Phi(G/A)$. Thus G/A can be written $B_1/A \times B_2/A \times \ldots \times B_s/A \times L/A$ where L/A has a unique minimal normal subgroup NA/A. If L is a proper subgroup of G then L splits over N so L/A splits over NA/A and $L/A = NA/A \times K/A$ since $NA/A \subseteq Z(G/A)$. But then a minimal normal subgroup of K/A is also minimal normal in G/A so L/A would not have a unique minimal normal subgroup. Thus L must not be a proper subgroup of G and we may assume that G has a unique minimal normal subgroup.

Now $A \supseteq \Phi(M)$ so M/A is elementary abelian. Let $g \in G/A \setminus M/A$. The action of g on M/A can be considered as operating on a vector space, so a basis for M/A can be selected so that the action of g can be written as a matrix in Jordan form. Each Jordan block is normalized by g, and of course by other elements of M/A. Each block then contains a minimal normal subgroup, and there being only one, there can be only one block. In other words, the action of g is cyclic. Thus $G/A = \langle g, M \rangle /A$ is a two generator group and by assumption splits over NA/A and N has a supplement contradicting $N \subseteq \Phi(G)$.

Q.E.D.

We next observe that if a group G is nilpotent and each Sylow group P of G splits over $P \cap N$, $P = (P \cap N) \cdot K_p$, then the complements K_p, K_q commute so that $K = \langle K_p \rangle = \prod K_p$ is a complement of N in G. We extend this concept in the following result.

<u>Theorem 1.8</u>

Let G be a solvable group, $N \triangleleft G$ and G/N be nilpotent. Suppose every Sylow subgroup P of G splits over $P \cap N$, $P = (P \cap N)K_p$. If there is a chain of subgroups $N = N_0 \triangleright N_1 \triangleright N_2 \triangleright \cdots \triangleright N_s = 1$ which is stabilized by each K_p, then G splits over N and there is a complement K which contains all the K_p.

Proof:

By induction on s, the length of the chain. It may be assumed that each factor N_i/N_{i+1} is abelian, for if not intermediate steps can be inserted.

$s = 1$

N is abelian and $C_G(N) \supseteq N \cdot \langle K_p \mid p \mid |G| \rangle = G$ so $N \subseteq Z(G)$ and G is nilpotent, and the conclusion follows.

Assume now that the conclusion holds for groups with s<n. For each prime p the multiplier group $K(N,K_p) = \langle g^x g^{-1} \mid g \in N,\ x \in K_p \rangle$ is a normal subgroup of N by Satz 1 of Kaloujnine [11]. It is also normalized by K_p, since for $y \in K_p$, $y^{-1}g^x g^{-1} y = g^{xy} g^{-y} = g^{yy^{-1}xy} g^{-y} =$ $= (g^y)^{y^{-1}xy}(g^y)^{-1} \in K(N,K_p)$. Then $N'K(N,K_p) \triangleleft NK_p$ and $N/N'K(N,K_p)$ is the maximal abelian factor group of N which K_p stabilizes, so $N_1 \supseteq N'K(N,K_p)$.

$N/N'K(N,K_p) \subseteq Z(NK_p/N'K(N,K_p))$ so $N_1/N'K(N_1K_p) \triangleleft NK_p/N'K(N_1K_p)$ and $N_1 \triangleleft NK_p$. N_1 is normalized by $N\langle K_p \mid p \mid |G| \rangle = G$. G/N_1 is nilpotent since $N/N_1 \subseteq Z(G/N_1)$ and $\dfrac{G/N_1}{N/N_1} \cong G/N$ is nilpotent. Therefore there is a complement K/N_1 which contains each of the K_pN_1/N_1. We note that $N \cap K \subseteq N_1$.

Now consider the group K relative to the chain $N_1 \triangleright N_2 \triangleright \cdots \triangleright N_s = 1$. The K_p stablize the chain so by induction N_1 has a complement C which contains all the

K_p. $K = N_1 \cdot C$, and $N_1 \cap C = 1$. Since $G = N \cdot K = N \cdot N_1 C = NC$ and $N \cap C \subseteq N \cap K \subseteq N_1$, it follows that $N \cap C = N_1 \cap C = 1$, and C is also a complement for N in G.

Q.E.D.

In [11] Kaloujnine calls a group N K-nilpotent, if K acts on N to stabilize a chain of normal subgroups of N. N need not be nilpotent itself, as can be seen in the following example.

Example 1.2

Let $N = \langle a,b,c \mid a^2 = b^2 = c^3 = [a,b] = 1,\ a^c = b,\ b^c = ab \rangle$, and τ the automorphism of order 2 which maps $c^\tau = ca$, $a^\tau = a$, $b^\tau = b$. Let $G = N \cdot \langle x \rangle$ where $x^6 = 1$ and x acts as τ on N. G is not nilpotent, but satisfies the hypothesis of theorem 1.8 relative to the normal chain $1 \lhd \langle a,b \rangle \lhd N$, so G splits over N and $\langle x^2 \rangle \cdot \langle x^3 \rangle$ is a complement.

II. BASIC LOCAL THEORY AND EXAMPLES

Considerations arising from problems in extensions of number fields and their relation to their Galois groups lead us toward a particular concept of localization. In his thesis Sonn [18] studies the problem of determining the existence of a Galois extension L of an algebraic number field k with Galois group $G(L/k) = G$. Reduction theorems lead to the embedding problem of determining the existence of L given that L must also contain a Galois extension K of k, where $G(L/k)$ is a given extension of $N = G(L/K)$ by $G(K/k)$. In this chapter we look at this concept of localization and see several examples in which the conditions set forth are not sufficient to give a local-global theory of group extensions and a stronger notion of localization is required.

Results in extension theory of groups are given in two different notations; one as given in Marshall Hall [5], and another as in W. R. Scott [17]. Both notations are to be used so they are given here.

In Hall E is said to be an extension of a normal subgroup N by a group H if $N \triangleleft E$ and $E/N \cong H$. The basic result is

<u>Theorem 2.1</u> (Schrier)

Given a group E with a normal subgroup N and factor group $F = E/N$. If we choose coset representatives $\bar{u}$ where $\bar{u}N \rightarrow u \in F$, taking $\bar{1} = 1$, then automorphisms and a factor set are determined, satisfying

$$(a^u)^v = (u,v)^{-1}(a^{uv})(u,v); \; a, \; (u,v) \in N; \; u, \; v \in F$$
$$(uv,w)(u,v)^w = (u,vw)(v,w); \; (1,1) = 1.$$

Conversely, if for every $u \in F$ there is given an automorphism $a \rightarrow a^u$ of N, and if for these automorphisms and the factor set $(u,v) \in N$; $u, v \in F$, the above conditions hold, then elements $\bar{u}a$, $u \in F$, $a \in N$, with the product rule $\bar{u}a \, \bar{v}b = \overline{uv}(u,v)a^v b$ define a group E with normal subgroup N and $E/N \cong F$.

The extension is denoted $E[N,H,a^u,(u,v)]$ and equivalence of two extensions is given in the following:

Definition 2.1

Two extensions $E_1 = E[N,H,a^{u^1},(u,v)^1]$ and $E_2 = E[N,H,a^{u^2},(u,v)^2]$ are equivalent if the automorphisms and factor sets are related by

$$a^{u^2} = \alpha(u)^{-1}a^{u^1}\alpha(u)$$
$$(u,v)^2 = \alpha(u,v)^{-1}(u,v)^1\alpha(u)^v\alpha(v)$$

where $\alpha(u)$is a function of elements $u \in F$ with values in N and $\alpha(1) = 1$.

These same concepts are given by Scott in the following manner. An extension of a group N by a group F is an exact sequence of groups

$$1 \longrightarrow N \xrightarrow{i} E \xrightarrow{\varepsilon} F \longrightarrow 1.$$

Equivalence of extensions is given in

<u>Definition 2.2</u>

An extension

$$1 \longrightarrow N \xrightarrow{i_1} E_1 \xrightarrow{\varepsilon_1} F \longrightarrow 1$$

is equivalent to an extension

$$1 \longrightarrow N \xrightarrow{i_2} E_2 \xrightarrow{\varepsilon_2} F \longrightarrow 1$$

if and only if there is an isomorphism $\phi : E_1 \longrightarrow E_2$ such that the following diagram commutes and is exact:

$$\begin{tikzcd}
 & 1 \arrow[d] & 1 \arrow[d] & 1 \arrow[d] & \\
1 \arrow[r] & N \arrow[r, "i_1"] \arrow[d, "id"'] & E_1 \arrow[r, "\varepsilon_1"] \arrow[d, "\phi"'] & F \arrow[r] \arrow[d, "id"] & 1 \\
1 \arrow[r] & N \arrow[r, "i_2"] \arrow[d] & E_2 \arrow[r, "\varepsilon_2"] \arrow[d] & F \arrow[r] \arrow[d] & 1 \\
 & 1 & 1 & 1 &
\end{tikzcd}$$

<u>Lemma 2.1</u>

For each subgroup U of E and each natural epimorphism $\Theta : E \longrightarrow E/M$ we have the extension

$$1 \longrightarrow N(U,\Theta) \xrightarrow{i(U,\Theta)} \Theta U \xrightarrow{\varepsilon(U,\Theta)} F(U,\Theta) \longrightarrow 1$$

where $N(U,\Theta) = i^{-1}(U \cap iN)/i^{-1}(U \cap iN \cap \ker \Theta)$

$F(U,\Theta) = \varepsilon U/(\varepsilon U \cap \varepsilon \ker \Theta)$

$i(U,\Theta)(n/i^{-1}(U \cap iN \cap \ker \Theta)) = \Theta in, \; n \in i^{-1}(iN \cap U)$

and $\varepsilon(U,\Theta)(\Theta u) = \varepsilon u/(\varepsilon U \cap \varepsilon \ker \Theta) \qquad u \in U.$

Proof:

It is necessary only to show that the sequence is exact.

i) exactness at $i(U,\theta)$.

Let $\bar{n} \in i^{-1}(U \cap iN)/i^{-1}(U \cap iN \cap \ker \theta)$.
$i(U,\theta)\bar{n} = \theta in$ where n is a representative of $\bar{n}$ in $U \cap iN$, and since $\ker i = 1$ so $\ker i(U,\theta) = i^{-1}(\ker\theta \cap iN \cap U)$. If $\bar{n} \in \ker i(U,\theta)$ then $\bar{n} = 1$.

ii) exactness at $\varepsilon(U,\theta)$.

Let $u \in U$. $\varepsilon(U,\theta)\theta u = 1$ if and only if $\varepsilon u \in \varepsilon(U \cap \ker\theta)$, i.e. if $\varepsilon u \in \varepsilon \ker\theta$. That happens if $u \in \ker\theta$ or $u \in \ker\varepsilon$. Now if $u \in \ker\theta$ $\theta u = 1$, and if $u \in \ker\varepsilon = iN$ then $u \in iN \cap U$ so $u = in$, $n \in N$ and $\theta u \in \ker\varepsilon(U,\theta)$ if and only if u is in the image of $i(U,\theta)$.

Theorem 2.2

If E_1 and E_2 are equivalent extensions of N by F and θ_1 is the natural epimorphism of E_1 on E_1/M then there is an epimorphism θ_2 of E_2 such that for any subgroup U_1 of E_1 there is a subgroup U_2 of E_2 which satisfies

a) $N(U_1,\theta_1) = N(U_2,\theta_2)$

b) $F(U_1,\theta_1) = F(U_2,\theta_2)$

c) the extensions

$$1 \longrightarrow N(U_1, \Theta_1) \xrightarrow{i(U_1,\Theta_1)} \Theta_1 U_1 \xrightarrow{\varepsilon(U_1,\Theta_1)} F(U_1, \Theta_1) \longrightarrow 1$$

$$1 \longrightarrow N(U_2, \Theta_2) \xrightarrow{i(U_2,\Theta_2)} \Theta_2 U_2 \xrightarrow{\varepsilon(U_2,\Theta_2)} F(U_2, \Theta_2) \longrightarrow 1$$

are equivalent.

Proof:

Let ϕ be an isomorphism $\phi : E_1 \to E_2$ which satisfies the conditions in Definition 2.2. Let $\ker \Theta_1 = M_1$. Let $M_2 = \phi M_1$ be the kernel of the natural epimorphism $\Theta_2 : E_2 \to E_2/M_2$ and $U_2 = \phi U_1$.

a) Then $N(U_2, \Theta_2) = \dfrac{i_2^{-1}(U_2 \cap i_2 N)}{i_2^{-1}(U_2 \cap i_2 N \cap \ker \Theta_2)} =$

$= \dfrac{i_2^{-1}(\phi U_1 \cap \phi i_1 N)}{i_2^{-1}(\phi U_1 \cap \phi i_1 N \cap \phi \ker \Theta_1)}$ and since $i_2 = \phi i_1$,

$i_2^{-1} = i_1^{-1} \phi^{-1}$ it follows that $N(U_2, \Theta_2) =$

$$\frac{i_1^{-1}\phi^{-1}(\phi U_1 \cap \phi i_1 N)}{i_1^{-1}\phi^{-1}(\phi U_1 \cap \phi i_1 N \cap \phi \ker \Theta_1)} = \frac{i_1^{-1}(U_1 \cap i_1 N)}{i_1^{-1}(U_1 \cap i_1 N \cap \ker \Theta_1)} =$$

$N(U_1, \Theta_1)$.

b) $F(U_2, \Theta_2) = \varepsilon_2 U_2 / (\varepsilon_2 U_2 \cap \varepsilon_2 \ker \Theta_2) =$

$\varepsilon_2 \phi U_1 / (\varepsilon_2 \phi U_1 \cap \varepsilon_2 \phi \ker \Theta_1) = \varepsilon_1 U_1 / (\varepsilon_1 U_1 \cap \varepsilon_1 \ker \Theta_1) =$

$F(U_1, \Theta_1)$.

c) It must be shown that the diagram following commutes and is exact, where $\phi(U_1, \Theta_1)$ is the map induced by ϕ and is given by $\phi(U_1, \Theta_1)\Theta_1 u = \Theta_2 \phi u$, $u \in U_1$.

$$\begin{tikzcd}
 & 1 \arrow[d] & & 1 \arrow[d] & & 1 \arrow[d] & \\
1 \arrow[r] & N(U_1,\Theta_1) \arrow[r, "i(U_1,\Theta_1)"] \arrow[d] & \Theta_1 U_1 \arrow[rr, "\varepsilon(U_1,\Theta_1)"] \arrow[d, "\phi(U_1,\Theta_1)"] & & F(U_1,\Theta_1) \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & N(U_2,\Theta_2) \arrow[r, "i(U_2,\Theta_2)"] \arrow[d] & \Theta_2 U_2 \arrow[rr, "\varepsilon(U_2,\Theta_2)"] \arrow[d] & & F(U_2,\Theta_2) \arrow[r] \arrow[d] & 1 \\
 & 1 & 1 & & 1 &
\end{tikzcd}$$

Exactness of the horizontal sequences is given by lemma 2.1, and $\Phi(U_1,\Theta_1)$ is exact since $\Theta_1 u \in \ker \Phi(U_1,\Theta_1)$ if and only if $u \in \ker \Theta_2\Phi$, i.e. if and only if $\Phi u \in \ker \Theta_2$, which happens if and only if $u \in \ker \Theta_1$ so that $\Theta_1 u = 1$.

To show that the left square is commutative, let $n \in i_1^{-1}(U_1 \cap i_1 N) = i_2^{-1}(U_2 \cap i_2 N)$.
$\Phi(U_1,\Theta_1)\ i_1(U_1,\Theta_1)(n/i_1^{-1}(U_1 \cap i_1 N \cap \ker \Theta_1)) =$
$\Phi(U_1,\Theta_1)\Theta_1 i_1 n = \Theta_2 \Phi i_1 n = \Theta_2 i_2 n =$
$i(U_2,\Theta_2)(n/i_2^{-1}(U_2 \cap i_2 N \cap \ker \Theta_2))$.

To show that the right square is commutative, let $u \in U_1$. $\varepsilon(U_1,\Theta_1)\Theta_1 u = \varepsilon_1 u/(\varepsilon_1 U_1 \cap \varepsilon_1 \ker \Theta_1) =$ $\varepsilon_2 \Phi u/(\varepsilon_2 \Phi U_1 \cap \varepsilon_2 \Phi \ker \Theta_1$, and $\varepsilon(U_2,\Theta_2)\Phi(U_1,\Theta_1)\Theta_1 u$ $= \varepsilon(U_2,\Theta_2)\Theta_2 \Phi u = \varepsilon_2 \Phi u/(\varepsilon_2 U_2 \cap \varepsilon_2 \ker \Theta_2)$. Since $\Phi U_1 = U_2$ the two are equal and the diagram commutes and is exact so that the two extensions are equivalent.

Q.E.D.

In view of Theorem 2.2 the question arises as to whether or not two extensions are equivalent if epimorphisms of certain subgroups are. In order to investigate this question more fully we make the following definitions.

Definition 2.3

Let E_1 and E_2 be two group extensions of N by F. Let θ_1 be a natural epimorphism of E_1 and θ_2 a natural epimorphism of E_2. The pair (U_1, θ_1) of a subgroup U_1 of E_1 and θ_1 and the pair (U_2, θ_2) of a subgroup U_2 of E_2 and θ_2 are said to be equivalent if

a) $N(U_1, \theta_1) = N(U_2, \theta_2)$

b) $F(U_1, \theta_1) = F(U_2, \theta_2)$

c) The extensions

$$1 \rightarrow N(U_1, \theta_1) \xrightarrow{i(U_1, \theta_1)} \theta_1 U_1 \xrightarrow{\varepsilon(U_1, \theta_1)} F(U_1, \theta_1) \rightarrow 1$$

$$1 \rightarrow N(U_2, \theta_2) \xrightarrow{i(U_2, \theta_2)} \theta_2 U_2 \xrightarrow{\varepsilon(U_2, \theta_2)} F(U_2, \theta_2) \rightarrow 1$$

are equivalent.

Definition 2.4

Let E_1 and E_2 be extensions of N by F. Let θ_1 be an epimorphism of E_1 and θ_2 an epimorphism of E_2. Let $\mathcal{L}(E_1)$ be a family of pairs (U_1, θ_1) for E_1 and $\mathcal{L}(E_2)$ be a family of pairs (U_2, θ_2) of E_2. The extensions E_1 and E_2 are said to be $\mathcal{L}$-equivalent if for each $(U_1, \theta_1) \in \mathcal{L}(E_1)$ there is a $(U_2, \theta_2) \in \mathcal{L}(E_2)$ to which it is equivalent, and conversely given a pair in $\mathcal{L}(E_2)$ there is a pair in $\mathcal{L}(E_1)$ to which it is equivalent.

If E_1 and E_2 are equivalent extensions of N by F, Θ_1 an epimorphism of E_1 and $\mathcal{L}(E_1)$ a family of pairs (U_1, Θ_1) of E_1, then let
$\mathcal{L}(E_2) = \{(\phi U_1, \phi\Theta_1\phi^{-1}) \mid (U_1, \Theta_1) \in \mathcal{L}(E_1)$
be a family for E_2. According to theorem 2.2 these families are equivalent so that E_1 and E_2 are $\mathcal{L}$-equivalent. In order to phrase the question of the converse, when does $\mathcal{L}$-equivalence imply equivalence, we make the following definition.

<u>Definition 2.5</u>

A family $\mathcal{L}(E)$ is said to be a local family if $\mathcal{L}$-equivalence implies equivalence.

We see immediately that there are local families. For example $\mathcal{L}(E) = \{(E,1)\}$ is a local family. The collection of all families can be ordered by set inclusion and then any family containing a local family is also local. Our problem then becomes one of determining non-trivial local families. Some examples show rather quickly that a local family probably will need to have (N, Θ) as one of its elements. In order to indicate this we will use the following notation.

Notation 2.1

Let $\mathcal{E}_1$ and $\mathcal{E}_2$ be group theoretic properties. Let E be an extension of N by F and Θ a natural epimorphism of E. Denote by $\mathcal{L}(\mathcal{E}_1, \mathcal{E}_2)$ the family of pairs (U, Θ) for which $N(U, \Theta)$ has property $\mathcal{E}_1$ and $F(U, \Theta)$ has property $\mathcal{E}_2$.

The next theorem shows that the family of subgroups whose intersections with N are abelian does not determine the extension. Let $\mathcal{A}$ be the property of being abelian and $\mathcal{G}$ the property of being a group.

Theorem 2.3

$\mathcal{L}(\mathcal{A}, \mathcal{G})$ is not a local family.

Proof: by example 2.1

Example 2.1

Let $N = \langle a,b \mid a^4 = b^4 = 1,\ a^2 = b^2,\ a^b = a^{-1} \rangle$ be the quaternion group of order 8 and $F = \langle c \mid c^3 = 1 \rangle$. The two extensions to be considered are

$$E_1 = \langle N,x \mid a^x = b,\ b^x = ab,\ x^3 = 1 \rangle \text{ and}$$
$$E_2 = \langle N,y \mid a^y = ab,\ b^y = a,\ y^3 = 1 \rangle .$$

For $1 \longrightarrow N \xrightarrow{i_1} E_1 \xrightarrow{\mathcal{E}_1} F \longrightarrow 1$

i_1 is the identity map and $\mathcal{E}_1(xn) = c$ for $n \in i_1 N$.

For $1 \longrightarrow N \xrightarrow{i_1} E_2 \xrightarrow{\mathcal{E}_1} F \longrightarrow 1$ i_2 is the identity map and $\varepsilon_2(yn) = c$ for $n \in i_2N$. Let θ_1 be the identity map.

If $U_1 \subsetneq N$, clearly the corresponding group $U_2 \subseteq E_2$ is such that $(U_1,1)$ is equivalent to $(U_2,1)$, so we need examine only subgroups $U_1 \not\subseteq N$. Therefore U_1 must contain elements of order 3. One of the elements of order 3 must be in the coset xN and one in x^2N. If $U_1 \cap N$ is to be abelian, it must be one of the groups $\langle a^2 \rangle$, $\langle a \rangle$, $\langle b \rangle$, or $\langle ab \rangle$. But $\langle a \rangle^{xn} = \langle b \rangle$ for all $n \in N$ so $U_1 \cap N \neq \langle a \rangle$. Similarly it cannot be $\langle b \rangle$ or $\langle ab \rangle$ and must be $\langle a^2 \rangle$. a^2 lies in the center of E_1 so U_1 is generated by commuting elements of order 2 and 3 and must therefore be a cyclic group of order 6. We may write $U_1 = \langle a^2, xn \rangle$ $n \in N$ such that $(nx)^3 = 1$.

Since E_1 and E_2 are isomorphic under the map which fixes N and takes $x \longrightarrow y^2$, the same conditions hold in E_2 so the only subgroups of E_2 whose intersection with N is abelian are $\langle a^2, y^2n \rangle$ where n is such that $(y^2n)^3 = 1$.

For $U_1 = \langle a^2, xn \rangle$ there is a one to one correspondence with $U_2 = \langle a^2, y^2n \rangle$. Then

a) $N(U_1,1) = \langle a^2 \rangle = N(U_2,1)$

b) $F(U_1,1) = F = F(U_2,1)$

c) the extensions are equivalent by the map

$\phi(U_1,1)(xn) = (y^2n)^{-1}$, $\phi(U_1,1)a^2 = a^2$.

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \langle a^2 \rangle & \xrightarrow{i(U_1,1)} & \langle a^2, xn \rangle & \xrightarrow{\varepsilon(U_1,1)} & F & \longrightarrow & 1 \\ & & \downarrow & & \downarrow \phi(U_1,1) & & \downarrow & & \\ 1 & \longrightarrow & \langle a^2 \rangle & \xrightarrow{i(U_2,1)} & \langle a^2, y^2n \rangle & \xrightarrow{\varepsilon(U_2,1)} & F & \longrightarrow & 1 \end{array}$$

the left square commutes because each map is the identity on a^2 and the right square commutes because $\mathcal{E}(U,1)xn =$ $= \mathcal{E}_1(xn) = c = \mathcal{E}_2(y^2n)^{-1} = \mathcal{E}(U_1,1)(y^2n)^{-1} =$ $= \mathcal{E}(U_1,1)\phi(U_1,1)(xn)$. Thus by definition 2.3 the pair $(U_1,1)$ and the pair $(U_2,1)$ are equivalent so the extensions E_1 and E_2 are $\mathcal{L}(\mathcal{A},\mathcal{G})$ - equivalent.

It must be shown now that the extensions are not themselves equivalent. If they were, by definition 2.1 the automorphisms would be related by

$$g^x = \alpha(c)^{-1} g^y \alpha(c)$$

where $\alpha(c) \in N$, since $\mathcal{E}_1(x) = \mathcal{E}_2(y) = c$. In other words, $g^x = g^{yn}$ for some $n \in N$, or $g^{y^2x} = g^{y^2yn} = g^n$, so that y^2x would be an inner automorphism of N. But the automorphism induced by y^2 is the same as that by x so $g^{y^2x} = g^{x^2}$ which is an automorphism of order 3 and cannot be inner, and E_1 and E_2 cannot be equivalent.

The next theorem shows that subgroups U which intersect N in groups of prime power order do not characterize the extension either. Let pp be the property of being a group of prime power order.

Theorem 2.4

$\mathcal{L}(pp, \mathcal{D})$ is not a local family.

Proof: by example 2.2

Example 2.2

Let $N = \langle a,b \mid a^{15} = b^2 = 1,\ a^b = a^{-1} \rangle$ be the dihedral group of order 30 and $F = \langle c \mid c^2 = 1 \rangle$. Let $E_1 = \langle N,x \mid a^x = a,\ b^x = b,\ x^2 = 1 \rangle$ and $E_2 = \langle N,y \mid a^x = a^4,\ b^x = b,\ y^2 = 1 \rangle$ where for $1 \longrightarrow N \xrightarrow{i_1} E_1 \xrightarrow{\varepsilon_1} F \longrightarrow 1$ i_1 is the identity map and $\varepsilon_1(xn) = c$ for $n \in N$ and for $1 \longrightarrow N \xrightarrow{i_2} E_2 \xrightarrow{\varepsilon_2} F \longrightarrow 1$ i_2 is the identity map and $\varepsilon_2(yn) = c$ for $n \in N$. E_2 can be described as a direct product of a group of order 3 by one of order 5, extended by an involution which inverts all elements, further extended by an involution which fixes elements of order 3 and inverts elements of order 5. Note that y acts on C_3 in the same manner as x, while the element by acts on C_5 in the same manner as x.

We first observe that the extensions are not equivalent, for the automorphism induced on N in E_1 by x is inner, while the automorphism induced on N in E_2 by y is not.

As was observed in the previous example we need only look at subgroups $U_1 \nsubseteq N$. $U_1 \cap N$, being of prime power order, is of order 2, 3 or 5. $|U_1 : U_1 \cap N| = 2$ so U_1 is of order 4, 6, or 10. Let us assume Θ is the identity.

<u>case $|U_1| = 4$</u>

U_1 is a Sylow 2-group of E_1 and is therefore a conjugate of $\langle b,x \rangle$ which is an elementary abelian group of order 4. U_2 would be a Sylow 2-group of E_2 and would be conjugate to $\langle b,y \rangle$, which is also an elementary abelian group of order 4. Then

a) $N(U_1,1) = \langle b \rangle = N(U_2,1)$

b) $F(U_1,1) = F = F(U_2,1)$

c) The two extensions U_1 and U_2 are equivalent under the map $\Phi(U_1,1)(b) = b$ and $\Phi(U_1,1)(x) = y$. Any conjugate of U_1 is also equivalent to U_2 by first conjugating and then applying $\Phi(U_1,1)$. Also conjugates of U_2 are equivalent to U_1.

<u>case $|U_1| = 6$</u>

$U_1 \cap N = \langle a^5 \rangle$ the only subgroup of order 3.

$|E_1 : C_{E_1}(a^5)| = 2$ and $C_{E_1}(a^5) = \langle a,x \rangle$ while

$|E_2 : C_{E_2}(a^5)| = 2$ and $C_{E_2}(a^5) = \langle a,y \rangle$.

If xn is any involution in $C_{E_1}(a^5)$ then let $U_1 = \langle a^5, xn \rangle$ and $U_2 = \langle a^5, ym \rangle$ where ym can be taken to be any involution in $C_{E_2}(a^5)$. If xn is any involution not in $C_{E_1}(a^5)$ then let $U_1 = \langle a^5, xn \rangle$ and $U_2 = \langle a^5, ym \rangle$ where ym can be taken to be any involution not in $C_{E_2}(a^5)$. Then in either event

a) $N(U_1,1) = \langle a^5 \rangle = N(U_2,1)$

b) $F(U_1,1) = F = F(U_2,1)$

c) U_1 and U_2 are equivalent extensions under the map $\Phi(U_1,1)(a^5) = a^5$ and $\Phi(U_1,1)(xn) = ym$

<u>case $|U_1| = 10$</u>

$U_1 \cap N = \langle a^3 \rangle$, the only subgroup of order 5. As in the previous case $|E_1 : C_{E_1}(a^3)| = 2$ and $C_{E_1}(a^3) = \langle a, x \rangle$ while $|E_2 : C_{E_2}(a^3)| = 2$ and $C_{E_2}(a^3) = \langle a, by \rangle$. Again as in the previous case $U_1 \cap N$ has only one automorphism of order 2 so any involution in E_1 not in $C_{E_1}(a^3)$ must induce that automorphism, and so must any involution in E_2 not in $C_{E_2}(a^3)$. The situation then is exactly as in the previous case and so if $U_1 \cap N = 5$ the pair $(U_1,1)$ and the pair $(U_2,1)$ are equivalent.

We have shown then that E_1 and E_2 are $\mathcal{L}(pp, \mathfrak{J})$-equivalent but not equivalent. Note that if Θ_1 is a natural epimorphism and ker $\Theta_1 \subsetneq N$ then ker$\Theta_1 = \langle a^3 \rangle$ or $\langle a^5 \rangle$. Take Θ_2 to be the natural epimorphism $E_2 \twoheadrightarrow E_2/M$ and kerΘ_2 = kerΘ_1. Then in either event the pair (E_1, Θ_1) is equivalent to the pair (E_2, Θ_2) since

a) $N(E_1,\Theta_1) = i_1^{-1}(i,N/i_1^{-1}(i_1N \cap \ker\Theta_1) = N/\ker\Theta_1$ and $N(E_2,\Theta_2) = i_2^{-1}(i_2N)/i_2^{-1}(i_2N \cap \ker\Theta_2) = N/\ker\Theta_1$

b) $F(E_1,\Theta_1) = \mathcal{E}_1E_1/(\mathcal{E}_1E_1 \cap \mathcal{E}_1\ker\Theta_1) = \mathcal{E}_1E_1 = F$ and $E(E_2,\Theta_2) = \mathcal{E}_2E_2/(\mathcal{E}_2E_2 \cap \mathcal{E}_2\ker\Theta_2) = \mathcal{E}_2E_2 = F$.

c) the extensions E_1/M and E_2/M are equivalent under the map $\Phi(E_1,\Theta_1)\, nM = nM$ for $n \in N$ and $\Phi(E_1,\Theta_1)xM = yM$ if $M = \langle a^3 \rangle$ and $\Phi(E_1,\Theta_1)xM = byM$ if $M = \langle a^5 \rangle$.

Any subgroup U_1 of E_1 has a subgroup U_2 of E_2 so that (U_1,Θ_1) is equivalent to (U_2,Θ_2). We have shown then, that for any epimorphism Θ_1 of E_1 with $\ker\Theta_1 \subseteq N$, $\mathcal{L}(pp,\mathfrak{Y})$ is not a local family.

The essential point of this example is that the automorphism of any coset representative of N in E_2 is like one inner automorphism on a Sylow 5-group and like another inner automorphism on a Sylow 3-group. Since the example is dependent upon the existence of inner automorphisms one might hope that if N is abelian then $\mathcal{L}(pp,\mathfrak{Y})$ would be a local family. That this is not the case is shown in the next theorem.

<u>Theorem 2.5</u>

Let E be an extension of an abelian group N by a group F. Then $\mathcal{L}(pp,\mathfrak{Y})$ is not a local family for E.

<u>Proof</u>: by example 2.3

Example 2.3

Let $N = \langle a \mid a^{21} = 1 \rangle$ and $F = \langle c \mid c^3 = 1 \rangle$.
Let $E_1 = \langle N, x \mid a^x = a^4,\ x^3 = a^7 \rangle$ and
$E_2 = \langle N, y \mid a^y = a^{16},\ y^3 = a^7 \rangle$ where for
$1 \longrightarrow N \xrightarrow{i_1} E_1 \xrightarrow{\mathcal{E}_1} F \longrightarrow 1$ i_1 is the identity
and $\mathcal{E}_1(xn) = c$ for $n \in N$, and for
$1 \longrightarrow N \xrightarrow{i_2} E_1 \xrightarrow{\mathcal{E}_2} F \longrightarrow 1$ i_2 is the identity
and $\mathcal{E}_2(yn) = c$ for $n \in N$. Let Θ_1 be the identity
map. We want to show that E_1 and E_2 are $\mathcal{L}(pp, \mathcal{Y})$-
equivalent but that E_1 and E_2 are not equivalent.

-Any coset representative of xN in E, say xa^i,
has order 9 since $(xa^i)^3 = x^3(a^i)^{1+x^2+x} = x^3(a^i)^{1+16+4} =$
$= x^3 = a^7$ which has order 3. Thus if U_1 is a subgroup
of E_1 not contained in N, and if $U_1 \cap N$ is a group of
prime power order, then the prime is 3. The same
situation obtains in E_2, since
$(ya^i)^3 = y^3(a^i)^{1+x^2+x} = y^3(a^i)^{1+4+16} = y^3 = a^7$ is of
order 3. Thus the family $\mathcal{L}(pp, \mathcal{Y})$ for $(E_1, 1)$ consists only of the pairs $(\langle a^3 \rangle, 1), (\langle a^7 \rangle, 1)$ and
$(P_1, 1)$ where P_1 is a Sylow 3-group of E_1. The family
$\mathcal{L}(pp, \mathcal{Y})$ for $(E_2, 1)$ consists of $(\langle a^3 \rangle, 1)$,
$(\langle a^7 \rangle, 1)$, and $(P_2, 1)$ where P_2 is a Sylow 3-group of
E_2. In the first two pairs, the subgroup lies in N so
the pair for E_1 and E_2 are equivalent. $(P_1, 1)$ and
$(P_2, 1)$ are equivalent since

a) $N(P_1,1) = \langle a^7 \rangle = N(P_2,1)$

b) $F(P_1,1) = F = F(P_2,1)$, and

c) under the map $\Phi(P_1,1)(a^7) = a^7$

$\Phi(P_1,1)(xa^i) = ya$ the extensions

$$\begin{tikzcd}
1 \arrow[r] & \langle a^7 \rangle \arrow[r, "i_1(P_1,1)"] \arrow[d] & \langle xa^i \rangle \arrow[r, "\varepsilon_1(P_1,1)"] \arrow[d, "\Phi(P_1,1)"] & F \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & \langle a^7 \rangle \arrow[r, "i_2(P_2,1)"] & \langle ya^i \rangle \arrow[r, "\varepsilon_2(P_2,1)"] & F \arrow[r] & 1
\end{tikzcd}$$

are equivalent. Thus the families are equivalent and the extensions are not.

The last result considering families where $(N,1)$ is not in the family is in showing that even all subgroups U where $U \cap N$ is nilpotent do not characterize the extension. Let $\mathcal{N}$ be the property of being nilpotent.

<u>Theorem 2.6</u>

$\mathcal{L}(\mathcal{N}, \mathcal{U})$ is not a local family.

<u>Proof:</u> by example 2.4

<u>Example 2.4</u>

Let A be an elementary abelian group of order 64. There is an automorphism of order 63 which is transitive on the non-identity elements of A. Denote the seventh power of this automorphism by σ. Thus σ has order 9. We now look at a Sylow 3-group of the automorphism group of A, i.e at Syl(3) of GL(6,2), which has order $3^4 = 81$. Let P be a Sylow 3-group containing σ. We want an element τ of P such that $\tau^3 = \sigma^3$. P is not cyclic since 81 > 64. If $x \in C_p(\sigma)$ and $x^3 = 1$, then let $\tau = x\sigma$ and $\tau^3 = x^3\sigma^3 = \sigma^3$. If no such x exists then $C_p(\sigma)$ is cyclic. If $C_p(\sigma) > \langle\sigma\rangle$ then α with $\alpha^3 = \sigma$ would be a fixed point free automorphism of order 27 of a group of order 64 which does not occur. So $\langle\sigma\rangle$ is self-centralizing. Then $N_p(\langle\sigma\rangle)$ is of order 3^3 and there is an x in $N_p(\langle\sigma\rangle)$ such that $x^3 = 1$. The only automorphisms of order 3 of a cyclic group of order 9 are $\sigma \to \sigma^4$ and its square $\sigma \to \sigma^7$. In either event $(x\sigma)^3 = x^3\sigma^{1+x^2+x} =$ $= \sigma^{1+4+7} = \sigma^3$ so we take $\tau = x\sigma$.

Let $E_1 = \langle A, \sigma \mid \sigma^9 = 1$ and $E_2 =$ $= \langle A, \tau \mid \tau^9 = 1\rangle$. If we let $N = \langle A, \sigma^3\rangle$ and identify $\sigma^3 = \tau^3$, then both E_1 and E_2 are extensions of N by $F = \langle c \mid c^3 = 1\rangle$. Although these groups are isomorphic we want to show that the extensions are not equivalent.

To be equivalent the automorphisms σ and τ must be related by $a^{\sigma} = a^{n\tau}$ where $n \in N$. Then $a^{\sigma} = n^{nx\sigma}$ where, $x \notin \langle\sigma\rangle$ is an automorphism of order 3, and so $a^{nx} = a$, $a^n = a^{x^{-1}}$ and the only innder automorphism of N of order 3 is σ^3.

Next we want to see that E_1 and E_2 with epimorphism $\Theta = 1$ are $\mathcal{L}(\mathcal{N}, \mathcal{G})$-equivalent. For $U_1 \notin N$, U_1 contains an element of order 9 which is transitive on the elements of order 2 so if U_1 contains any element of order 2, $U_1 = E_1$. Thus $\mathcal{L}(\mathcal{N}, \mathcal{G}) = \{ (\langle \sigma \rangle, 1)$ and $(S_1, 1)$ where S_1 is a nilpotent subgroup of $N \}$ for E_1 and $\{ (\langle \tau \rangle, 1)$ and $(S_2, 1) \}$ for E_2. The pair $(S_1, 1)$ and $(S_2, 1)$ are clearly equivalent for any nilpotent $S \subseteq N$ and

a) $N(\langle \sigma \rangle, 1) = \langle \sigma^3 \rangle = N(\langle \tau \rangle, 1)$

b) $F(\langle \sigma \rangle, 1) = F = F(\langle \tau \rangle, 1)$

c) $\Phi(\langle \sigma \rangle, 1)\ (\sigma) = \tau$ is an equivalence

of the extensions

$$1 \longrightarrow \langle \sigma^3 \rangle \xrightarrow{i_1} \langle \sigma \rangle \xrightarrow{\varepsilon_1} F \longrightarrow 1 \quad \text{and}$$

$$1 \longrightarrow \langle \tau^3 \rangle \xrightarrow{i_2} \langle \tau \rangle \xrightarrow{\varepsilon_2} F \longrightarrow 1.$$

III. MAIN RESULTS

One sees from the examples and results of Chapter II that as long as the family $\mathcal{L}(\mathcal{E}_1, \mathcal{E}_2)$ is such that subgroups having property $\mathcal{E}_1$ are all properly contained in N, we are likely to not have a local family. The primary difficulty seems to be that although two extensions may have automorphisms which when restricted to various subgroups have the same action, the two extensions do not both have an automorphism which gives the same action on all of N. Another difficulty is that there simply were not enough subgroups U, with $U \cap N$ having property $\mathcal{E}_1$ and $U/U \cap N$ having property $\mathcal{E}_2$. In order to get a local family then it seems almost necessary to include N in property $\mathcal{E}_1$. This gives a set of automorphisms to begin with as well as putting each element of N in some U. Since N will be assumed to have property $\mathcal{E}_1$, the family $\mathcal{L}(\mathcal{E}_1, \mathcal{E}_2)$ will depend only on the property $\mathcal{E}_2$, so we make the following definition.

<u>Definition 3.1</u>

Let N and F be groups and

$$1 \longrightarrow N \xrightarrow{i_1} E_1 \xrightarrow{\varepsilon_1} F \longrightarrow 1$$

$1 \longrightarrow N \xrightarrow{i_2} E_2 \xrightarrow{\varepsilon_2} F \longrightarrow 1$ be two extensions of N by F. Let C be a class of subgroups of F. The two extensions are said to be C-equivalent if for each subgroup D of F which is in C, the extensions of N by D in E_1 and in E_2 are equivalent. In other words E_1 and E_2 are $\mathcal{L}(\mathcal{G}, C)$ equivalent.

Using this definition we can extend theorem 1.4, the theorem of Gáschütz which says that if N is an abelian normal subgroup of G, the splitting of G is determined by the splitting of the Sylow groups of F, to give equivalence of the extensions if the Sylow extensions are equivalent. This theorem, then, shows that $\mathcal{L}(\mathcal{G}, \mathcal{S})$ is a local family if N is abelian.

Theorem 3.1

Let N be a finite abelian group and F a finite group. Let E_1 and E_2 be two extensions of N by F. Then E_1 and E_2 are equivalent if and only if they are Sylow equivalent.

Proof:

If E_1 and E_2 are equivalent, let $\Theta = 1$ and $U_1 = NP_1$ where P is a Sylow group of E_1. Then Theorem 2.2 tells us that E_1 and E_2 are Sylow-equivalent. Now for the converse.

Since N is abelian, the factor sets form a group and both extensions can be multiplied by the inverse of the factor set (u,v) of E_1, so we may assume that E_1 is a splitting extension of N by F.

Let D be a Sylow p-group of F and D_1 and D_2 the extensions of N by D in E_1 and E_2 respectively. Since E_1 splits over N, D_1 splits over N and the Sylow p-group P_1 of E_1 does too. Since D_1 and D_2 are equivalent by assumption, D_2 must split over N and the Sylow p-group P_2 of E_2 splits over $P_2 \cap N$. This is true for each prime p, so an application of Gaschütz' theorem gives that E_1 splits over N.

Since both extensions E_1 and E_2 split we may assume the factor sets $(u,v) = 1$. Thus the two extensions are equivalent if the automorphisms are related by $a^{u_2} = \alpha(u)^{-1} a^{u_1} \alpha(u)$ and since N is abelian it must be shown that $a^{u_2} = a^{u_1}$, where $\varepsilon_2(u_2) = \varepsilon_1(u_1) = u$ and $a \in N$. If u is an element of p-power order then let D be a Sylow p-group of F containing u. The pre-images D_1 and D_2 of D in E_1 and E_2 are equivalent by assumption so $a^{u_2} = a^{u_1}$ for all $a \in N$. Any element $\bar{u} \in E_1$ is the product of elements of prime power order, say $\bar{u} = \bar{u}_1 \bar{u}_2 \dots \bar{u}_k$. If $\varepsilon_1(\bar{u}_i) = u_i \in F$, choose $\bar{\bar{u}}_i$ in E_2 so that $\varepsilon_2(\bar{\bar{u}}_i) = u_i$ and let $u = \bar{\bar{u}}_1 \bar{\bar{u}}_2 \dots \bar{\bar{u}}_k$. Since $a^{\bar{u}_i} = a^{\bar{\bar{u}}_i}$ for each i, $a^{\bar{u}} = a^{\bar{\bar{u}}}$ for each $a \in N$, $\bar{u} \in E_1$, $\bar{\bar{u}} \in E_2$ such that $\varepsilon_1(u) = \varepsilon_2(u)$. Q.E.D.

The next result uses some of the standard notation of cohomology theory, as it relates to extension theory of groups. Let A be an abelian group which is acted on by a group G.

$C^2(G,A) = \left\{ f \mid f:G\times G \longrightarrow A,\ f(g_ig_j,g_k)f(g_i,g_j)^{g_k} = f(g_i,g_{jk})f(g_j,g_k) \right\}$. These are the factor sets of extensions of A by G, the two-cocycles of G to A.

$B^2(G,A) = \left\{ f \in C^2(G,A) \mid f(g_i,g_j) = \alpha(g_ig_j)\alpha(g_i)^{-g_j}\alpha(g_j),\quad \alpha: G \longrightarrow A \right\}$. These are the principal factor sets, the coboundaries of G to A.

$H^2(G,A) = \dfrac{C^2(G,A)}{B^2(G,A)}$ is the second cohomology group of G to A and is the group of extensions of A by G. Different elements of $H^2(G,A)$ correspond to inequivalent extensions of A by G.

Theorem 3.2

Let E_1 and E_2 be extensions of N by F and suppose there are transversals $\{x_i\}$ and $\{y_i\}$ for E_1 and E_2 such that $g^{x_i} = g^{y_i}$ for all $g \in N$. Let $\{c_{i,j}\}$ be the factor set determined by $x_ix_j = x_kc_{i,j}$ in E_1 and $\{d_{i,j}\}$ be the factor set determined by the y_i in E_2. Then

i) $c_{i,j}d_{i,j}^{-1} = z_{i,j} \in C^2(F,Z(N))$

ii) E_1 and E_2 are equivalent if and only if

$z_{i,j} \in B^2(F,Z(N))$

Proof:

i) Since $g^{x_i x_j} = g^{x_k c_{i,j}}$ and also $g^{x_i x_j} =$

$= g^{y_i y_j} = g^{y_k d_{i,j}}$, we have $(g^{x_k})^{c_{i,j}} = (g^{y_k})^{d_{i,j}}$, and thus the inner automorphisms induced by $c_{i,j}$ and $d_{i,j}$ are the same. So the automorphism induced by

$c_{i,j}d_{i,j}^{-i}$ is the identity on N and

$c_{i,j}d_{i,j}^{-1} = z_{i,j} \in Z(N)$.

To see that $z_{i,j} \in C^2(F,Z(N))$, first observe that F is an automorphism group on Z(N). For $g \in Z(N)$, $g^{x_i x_j} = g^{x_k c_{i,j}} = g^{x_k c_{i,j}} = g^{x_k}$. Taking the cocycle condition for $d_{i,j}$

$$d_{ij,k}d_{i,j}^{y_k} = d_{i,jk}d_{j,k}$$

we invert and get

$$d_{i,j}^{-y_k}d_{ij,k}^{-1} = d_{j,k}^{-1}d_{i,jk}^{-1}$$

and then multiply on the left by

$$c_{ij,k}c_{i,j}^{x_k} = c_{i,jk}c_{j,k} \quad ,$$

getting

$$c_{ij,k}c_{i,j}^{x_k}\, d_{i,j}^{-y_k}\, d_{ij,k}^{-1} = c_{i,jk}c_{j,k}d_{j,k}\, d_{j,k}^{-1}\, d_{i,jk}^{-1} \quad .$$

Since the automorphism $g \longrightarrow g^{y_k}$ is the same as $g \longrightarrow g^{x_k}$, we may write

$$c_{i,j}^{x_k}\, d_{i,j}^{-y_k} = c_{i,j}^{x_k}\, d_{i,j}^{-x_k} = (c_{i,j}\, d_{i,j}^{-1})^{x_k} \quad ,$$

and since $c_{i,j}\, d_{i,j}^{-1}$ and $c_{j,k}\, d_{j,k}^{-1}$ lie in $Z(N)$ our equation becomes

$$c_{ij,k}d_{ij,k}^{-1}(c_{i,j}\, d_{i,j}^{-1})^{x_k} = c_{i,jk}\, d_{i,jk}^{-1}\, (c_{j,k}\, d_{j,k}^{-1}),$$

which is

$$z_{ij,k}\, z_{i,j}^{x_k} \;=\; z_{i,jk}\, z_{j,k} \;.$$

ii) Assume that E_1 and E_2 are equivalent. By definition 2.1, there is a function α from F to N satisfying

$$a^x \;=\; \alpha(y)^{-1}a^y\;\alpha(y)$$

$$c_{i,j} = \alpha(y_i,y_j)^{-1}\, d_{i,j}\,\alpha(y_i)^{y_j}\alpha(y_j).$$

Since $a^y = a^x$, $\alpha(x)$ must be in $Z(N)$. Then

$$c_{i,j}\, d_{i,j}^{-1} = z_{i,j} = \alpha(y_i,y_j)^{-1}\alpha(y_i)^{y_i}\alpha(y_j),$$

which says that $z_{i,j}$ is a coboundary of F into $Z(N)$.

Conversely, suppose that $z_{i,j}$ is in $B^2(F,Z(N))$, so that there is a function α from F to $Z(N)$ such that $c_{i,j}\, d_{i,j}^{-1} = z_{i,j} = \alpha(y_i,y_j)^{-1}\alpha(y_i)^{y_j}\alpha(y_j)$. Then

$c_{i,j} = \alpha(y_i,y_j)^{-1} d_{i,j}\, \alpha(y_i)^{y_j} \alpha(y_i)$, and since $\alpha(y) \in Z(N)$, $\alpha(y)^{-1} a^y \alpha(y) = a^y = a^x$, and the extensions are equivalent. Q.E.D.

Lemma 3.1

If E_1 and E_2 are cyclic-equivalent extensions of N by F, then there are transversals $\{x_i\}$ for E_1 and $\{y_i\}$ for E_2 such that $g^{x_i} = g^{y_i}$ for all $g \in N$.

Proof:

Let $\{x_i\}$ be a transversal of N in E_1. Automorphisms $g \longrightarrow g^{x_i}$ and a factor set $\{c_{i,j}\}$ are determined satisfying $x_i x_j = x_k c_{i,j}$.

The subgroup $D_1 = \langle x_i, N\rangle$ is equivalent to some subgroup D_2 of E_2 by assumption, so there is an isomorphism $\Phi_i : D_1 \longrightarrow D_2$. Let $y_i = \Phi_i(x_i)$. $\{y_i\}$ is a transversal of N in E_2 since $\mathcal{E}_2(y_i) = \mathcal{E}_2(\Phi(x_i)) = \mathcal{E}_1(x_i)$ covers F.

If $g \in N$, $\Phi_i(g^{x_i}) = g^{x_i}$ since Φ_i must be the identity on N. But also $\Phi_i(x_i^{-1} g x_i) = \Phi_i(x_i^{-1}) \Phi_i(g) \Phi_i(x_i) = \Phi_i(x_i)^{-1} \Phi_i(g)\ \Phi(x_i) = y^{-1} g y_i = g^{y_i}$. Q.E.D.

Corollary 3.1

If E_1 and E_2 are extensions of N by F which are cyclic equivalent, and $H^2(F,Z(N) = 1$, then E_1 is equivalent to E_2.

Corollary 3.2

If E_1 and E_2 are extensions of N by F which are cyclic equivalent, and $Z(N) = 1$, then E_1 is equivalent to E_2.

Corollary 3.3

If E_1 and E_2 are extensions of N by F which are cyclic equivalent, and $(|F| , |Z(N)|) = 1$, then E_1 is equivalent to E_2.

Proof:

By Corollary 3-1-7 of Weiss 22 , if $(|F|, |Z(N)|) =$ $= 1$, then $H^2(F,Z(N)) = 1$. Q.E.D.

An example of groups which are cyclic-equivalent but not equivalent is:

Example 3.1

Let $N = \langle c \mid c^3 = 1 \rangle$, $F = \langle a,b \mid a^3 = b^3 =$ $= a,b = 1 \rangle$, and E_1 the non-abelian group of order 27 and exponent 3. Let E_2 have trivial factor set so that E_2 is the elementary abelian group of order 27. These groups are clearly not equivalent, but all proper subgroups are elementary abelian and are equivalent.

In the next theorem we see that efforts to generalize the theorem of Gaschutz, (theorem 1.4) to the case where N is non-abelian do not lead in the proper direction. The proper formulization for the theorem is that of theorem 3.1, and the splitting which is described by theorem 1.4 is due to the cohomology which is available when N is abelian.

Theorem 3.3

Let E_1 and E_2 be extensions of N by F. E_1 and E_2 are equivalent if and only if they are Sylow-equivalent.

Proof:

Let $D \subsetneq F$ be cyclic and $\varepsilon_1^{-1}D = D_1$, $\varepsilon_2^{-1}D = D_2$. If P is a p-Sylow group of F, then $\varepsilon_1^{-1}P = P_1$ is equivalent to $\varepsilon_2^{-1}P = P_2$ by assumption so let P be such that $P \cap D$ is a p-Sylow group of D. Then theorem 2.2 gives that $P_1 \cap D_1$ is equivalent to $P_2 \cap D_2$.

Let $D_1 = \langle x, N \rangle$ and $\{x_i\}$ be a transversal of N in D_1. For each x_i such that $|\varepsilon_1 x_i|$ is a prime power there is a $y_i \in D_2$ such that $\langle x_i, N \rangle$ and $\langle y_i, N \rangle$ are equivalent, and so $g^{x_i} = g^{y_i}$ for all $g \in N$. The set of y_i's is completed to a transversal of N in D_2 in the following manner.

Let yN be a coset of N in D_2. Let $\varepsilon_2 y = \varepsilon_1 x_k$

where $x_k \in \{ x_i \}$. There is a representation of $\mathcal{E}_2 y = \mathcal{E}_2 y_{i_1} \; \mathcal{E}_2 y_{i_2} \cdots \mathcal{E}_2 y_{i_r}$ where $\mathcal{E}_2 y_{i_j}$ has order a power of p_{i_j}, with $p_{i_j} \neq p_{i_k}$. Since $\mathcal{E}_2 y_i = \mathcal{E}_1 x_i$ we have $\mathcal{E}_2 y_{i_1} \; \mathcal{E}_2 y_{i_2} \cdots \mathcal{E}_2 y_{i_r} =$ $\mathcal{E}_1 x_{i_1} \; \mathcal{E}_1 x_{i_2} \cdots \mathcal{E}_1 x_{i_r} = \mathcal{E}_1 (x_{i_1} \, x_{i_2} \cdots x_{i_r}) =$ $\mathcal{E}_1 x_k$ and so $x_{i_1} \, x_{i_2} \cdots x_{i_r} u = x_k$ where $u \in N$.

Choose the coset representative for yN to be $y_k = y_{i_1} y_{i_2} \cdots y_{i_r} u$ and then

$$g^{y_k} = g^{y_{i_1} y_{i_2} \cdots y_{i_r} u} = g^{x_{i_1} x_{i_2} \cdots x_{i_r} u} = g^{x_k}.$$

Thus for each coset we choose a representative y_k so that for each $g \in N$, $g^{x_k} = g^{x_k}$. We may then apply theorem 3.2.

Consider the extension G of the abelian group Z(N) by F which is given by the automorphisms $g \longrightarrow g^{x_i}$, and the factor set $\{ z_{i,j} \}$, where $z_{i,j}$ is given by theorem 3.2. Restricting arguments i and j to a p-Sylow group P of F, $z_{i,j} \in B^2(P, Z(N))$ since P_1 and P_2 are equivalent. Thus every p-Sylow group of G splits over $P \cap Z(N)$. Theorem 1.4 now implies that G splits over Z(N). Since G splits over Z(N) it follows that $z_{i,j} \in B^2(F, Z(N))$ and by theorem 3.2, E_1 is equivalent to E_2. Q.E.D.

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