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DUALITY RELATIONSHIPS FOR A NONLINEAR VERSION OF THE GENERALIZED NEYMAN-PEARSON PROBLEM

DISSERTATION

Presented in Partial Fulfillment of the Requirements for the Degree Doctor of Philosophy in the Graduate School of the Ohio State University

By

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* * * * * *

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GLOSSARY

E_n	Euclidean n-space
E_n^+	Non-negative orthant of E_n
X	Lebesgue measurable subset of E_{n-1}
$t(x)$	Real-valued, bounded, Lebesgue measurable function defined on X
$u(x)$	Real-valued, bounded, Lebesgue measurable function defined on X
W	$= \{f: t(x) \leq f(x) \leq u(x) \text{ for all } x \in X\}$
b	Column vector of real numbers with m components
$\overline{0}$	Null vector
$e(x,y)$	Function defined on $\{(x,y): x \in X,\ t(x) \leq y \leq u(x)\}$, strictly concave in y, partial derivative with respect to y exists, and Lebesgue integrable with respect to x.
$c_i(x,y)$	Function defined on $\{(x,y): x \in X,\ t(x) \leq y \leq u(x)\}$, convex in y, partial derivative with respect to y exists, and Lebesgue integrable with respect to x.
$c(x,y)$	Column vector $(c_1(x,y), c_2(x,y), \ldots, c_m(x,y))^T$
z	Row vector of non-negative real numbers with m components.
$h(x,y;z) =$	$e(x,y) - zc(x,y)$
$D_n e(x,y)$	Partial derivative with respect to y, (the n^{th} component of a vector in E_n)
$D_n c_i(x,y)$	Partial derivative with respect to y, (the n^{th} component of a vector in E_n)

$D_n c(x,y)$		Column vector $(D_n c_1(x,y), D_n c_2(x,y), \ldots, D_n c_m(x,y))^T$
$S_1(z)$	=	$\{x \in X : D_n h(x,y;z) < 0$ for every $y \in [t(x),u(x)]\}$
$S_2(z)$	=	$\{x \in X : D_n h(x,y;z) > 0$ for every $y \in [t(x),u(x)]\}$
$S_3(z)$	=	$\{x \in X : D_n h(x,y;z) = 0$ for exactly one $y \in [t(x),u(x)]\}$
$g(x;z)$		Unique point $y \in [t(x),u(x)]$ where $D_n h(x,y;z)=0$
$E(f)$	=	$\int_X e(x,f(x))dx$
$C_i(f)$	=	$\int_X c_i(x,f(x))dx$
$C(f)$		Column vector $(C_1(f), C_2(f), \ldots, C_m(f))^T$
$D(f)$	=	$C(f)-b$
$P(b)$	=	$\{f \in W : C(f) \le b\}$, (the set of feasible solutions to the dual problem)

CHAPTER I

INTRODUCTION

In linear programming problems a common reason for finding the dual problem is that the dual problem may be easier to solve. Then the solution to the primal problem can easily be determined since there is a direct relationship between the solutions of the primal and dual problems.

For most nonlinear programming problems the dual problem cannot be determined until the optimal solution to the primal problem is found. However, the dual to the nonlinear Neyman-Pearson problem which is the topic of this dissertation (both problems are defined in Chapter II) involves the solution to the Neyman-Pearson problem only in a general form, i.e., it is not required that a complete solution for the Neyman-Pearson problem be determined before the dual to the Neyman-Pearson problem can be stated.

The problem solved in Chapter V is an example of a Neyman-Pearson problem which can be solved in a more straightforward manner by attacking the dual of the Neyman-Pearson problem. Thus, the duality results of this dissertation have advantages similar to those of duality theory in linear programming. Further, duality provides additional theoretical insight and results.

In Chapter II, the Neyman-Pearson problem and its dual, the assumptions, and definitions are stated. The literature review in Chapter II is a discussion of related published work and the numerous areas of application in which the Neyman-Pearson problem occurs.

In Chapter III, the usual duality relationship is stated: any value for the objective function primal problem is always greater than or equal to any value of the objective function of the dual problem. Also, when the optimal solutions are determined the values of the objective functions are equal for these two problems. The necessary and sufficient conditions for solutions to the two problems are obtained, and are constructive in the sense that the solutions are implicit in the statement of these necessary and sufficient conditions.

Chapter IV states the duality results for a discrete version of the Neyman-Pearson problem and its dual. The reader may find the definitions of Chapter IV easier to understand; hence Chapter IV may provide insight into the results and proofs of Chapter III.

Chapter V is a numerical example which is solved in a step-by-step manner to point out the use of the results of Chapter III; it also may be useful to the reader when going through Chapter III. The numerical example is also presented to aid the reader when he is looking at the applications problems of Chapter VI.

Conclusions and recommendations based on the results of this dissertation as well as a summary of the results, are presented in Chapter VII.

CHAPTER II

DUALITY PROBLEMS AND LITERATURE REVIEW

2.1 Traditional Definitions and Dissertation Motivation

Before stating the problems analyzed in this dissertation it is worthwhile to point out some traditional terminology. Historically, the Neyman-Pearson problem has been referred to as the dual problem and the dual of the Neyman-Pearson problem is referred to as the primal problem. This will be continued throughout the dissertation.

Literature relative to this dissertation concerns the Neyman-Pearson problem and/or the Neyman-Pearson lemma. To avoid confusion as to the difference between the problem and the lemma we will consider the Neyman-Pearson problem as a particular optimization problem and we will consider the Neyman-Pearson Lemma as sufficient conditions for a solution to the Neyman-Pearson problem.

In 1969, Francis and Wright [18] published duality relationships for a linear functional version of the Neyman-Pearson problem. This dissertation was motivated by their paper and a paper by Wagner [50] which presented a nonlinear version of the Neyman-Pearson lemma.

2.2 Assumptions, Definitions, and Notation

Let X be a Lebesgue measurable subset of the Euclidean n-1 space E_{n-1}, and let t and u be real-valued, bounded, Lebesgue measurable functions such that $t(x) \leq u(x)$ for all $x \in X$. The set of real-valued, Lebesgue measurable functions on X and bounded by t and u will be denoted by W, so that $W = \{f : t(x) \leq f(x) \leq u(x) \text{ for all } x \in X\}$. We note that W is a convex set, for if $f_1, f_2 \in W$, $0 \leq a \leq 1$ and $f(x) = af_1(x) + (1-a) f_2(x)$ for all $x \in X$, then $f \in W$.

The definition of concavity and convexity of functionals will also be used during the analysis in this chapter. A real-valued functional C(f) is convex if $C(af_1 + (1-a)f_2) \leq aC(f_1) + (1-a)C(f_2)$, $0 < a < 1$, for all elements f_1, f_2 in a convex subset of a linear vector space. Also, a real-valued functional E(f) is concave if $E(af_1 + (1-a)f_2) \geq aE(f_1) + (1-a)E(f_2)$, $0 < a < 1$, for all elements f_1, f_2 in a convex subset of a linear vector space. Real-valued functionals are strictly convex or strictly concave if the inequalities of the previous definitions are strict inequalities when $f_1 \neq f_2$

The convex set W defined previously is a subset of the linear vector space of real-valued functions and will be the domain of the convex and concave functionals used in subsequent analysis.

We will also need real numbers $b_1, b_2, \ldots, b_m$ in the subsequent definition of the constraints for the dual problem. This set of numbers can be expressed as the column vector $b = (b_1, b_2, \ldots, b_m)^T$. To avoid confusion between the scalar zero and the zero (null) vector we will denote the zero vector by $\overline{0}$.

Let the real-valued functions $e(x,y)$ and $c_i(x,y)$, $i=1,\ldots,m$ be given and defined on the set $\{(x,y): x \in X, t(x) \leq y \leq u(x)\}$. For each $x \in X$, assume $e(x,y)$ is strictly concave in the n^{th} variable and $c_i(x,y)$ is convex in the n^{th} variable for $i=1,\ldots,m$. Assume the partial derivatives of $e(x,y)$ and $c_i(x,y)$, $i=1,\ldots,m$ with respect to the n^{th} variable exist. Represent these partial derivatives as $D_n e(x;\cdot)$ and $D_n c_i(x;\cdot)$, $i=1,\ldots,m$. Also, for any $f \in W$, assume $e(x,f(x))$ and $c_i(x,f(x))$, $i=1,\ldots,m$ are Lebesgue integrable with respect to x.

Vector notation will be less cumbersome in subsequent definitions and analysis. Therefore, it is appropriate to use the notation $c(x,y) = (c_1(x,y), c_2(x,y),\ldots,c_m(x,y))^T$ to define the column vector of functions with entries $c_i(x,y)$, $i=1,\ldots,m$. Also, denote the column vector of derivatives by $D_n c(x,y) = (D_n c_1(x,y), D_n c_2(x,y),\ldots, D_n c_m(x,y))^T$.

For ease in stating the primal and dual problems define $h(x,y;\, z) = e(x,y) - \sum_{i=1}^{m} z_i c_i(x,y) = e(x,y) - zc(x,y)$ where $z = (z_1,\ldots,z_m)$ is a row vector of non-negative real numbers. Since it assumed $e(x,y)$ is strictly concave in the n^{th} variable and $c_i(x,y)$ is convex in the n^{th} variable for $i=1,\ldots,m$, the function $h(x,y;\, z)$ is strictly concave in the n^{th} variable.

For every vector $z \in E_m^+$, the non-negative orthant of E_m, we define the following sets:

$S_1(z) = \{x \in X : D_n h(x,y;z) < 0$ for every $y \in [t(x), u(x)]\}$;

$S_2(z) = \{x \in X : D_n h(x,y;z) > 0$ for every $y \in [t(x), u(x)]\}$;

$S_3(z) = \{x \in X : D_n h(x,y;z) = 0$ for exactly one $y \in [t(x), u(x)]\}$;

Reference to Figures 1, 2 and 3 which follow may be useful during the subsequent discussion of these sets.

These sets can be interpreted in terms of strictly increasing and strictly decreasing functions of y. Fix x and z; define $\hat{h}(y) = h(x,y;z)$ and $\hat{h}'(y) = D_n h(x,y;z)$. If $\hat{h}(y)$ is strictly decreasing on $[t(x), u(x)]$ then $\hat{h}'(y) < 0$ on $[t(x), u(x)]$ which implies $x \in S_1(z)$. If $\hat{h}(y)$ is strictly increasing on $[t(x), u(x)]$ then $\hat{h}'(y) > 0$ on $[t(x), u(x)]$ which implies $x \in S_2(z)$. For each element $x \in S_3(z)$ the function $\hat{h}(y)$ is neither strictly increasing nor strictly decreasing since $\hat{h}'(y) = 0$ for one $y \in [t(x), u(x)]$.

In subsequent analysis we will need a definition of the point in $[t(x), u(x)]$ where $\hat{h}'(y) = 0$. Recall $\hat{h}(y)$ is differentiable on $[t(x), u(x)]$ which implies $\hat{h}(y)$ is continuous on $[t(x), u(x)]$. Thus, by the Weierstrass theorem stated in Appendix A, the max $\{\hat{h}(y); y \in [t(x), u(x)]\}$ exists. For those elements x in $S_3(z)$ define $g(x;z)$ as the point in $[t(x), u(x)]$ such that $\hat{h}(g(x;z)) = \max\{\hat{h}(y) : y \in [t(x), u(x)]\}$.

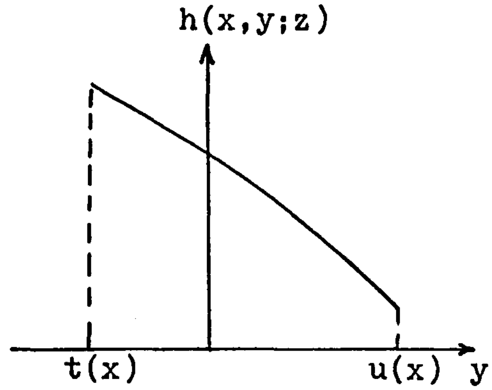
Now we will show $D_n h(x,y;z) = 0$ when $y = g(x;z)$, i.e., $\hat{h}'(y) = 0$ at the point where the maximum value of $\hat{h}(y)$ occurs for $x \in S_3(z)$.

Let $y_1 \in [t(x), g(x;z))$ and $y_2 \in (g(x;z), u(x)]$. Since $\hat{h}(y)$ is strictly concave, $\hat{h}'(y) > 0$ for all $y \in [t(x),g(x;z))$ which implies $\hat{h}'(y_1) > 0$. Also, $\hat{h}'(y) < 0$ for all $y \in (g(x;z),u(x)]$ which implies $\hat{h}'(y_2) < 0$. Proposition 1 in Appendix B guarantees that $\hat{h}'(\cdot)$ takes on every value between $\hat{h}'(y_1)$ and $\hat{h}'(y_2)$. This implies there is a point $y \in [y_1,y_2]$ where $\hat{h}'(y) = 0$. This point is unique and is $g(x;z)$ by the strict concavity of $\hat{h}(\cdot)$. Therefore, $D_n h(x,y;z) = 0$ when $y = g(x;z)$.

Note that if $g(x;z) = t(x)$ then $[t(x),g(x;z))$ is the empty set and $\hat{h}'(y) < 0$ for all $y \in (g(x;z),u(x)]$. Thus, the strict concavity of $\hat{h}(y)$ and $x \in S_3(z)$ implies there is only one $y \in [t(x),u(x))$ such that $\hat{h}'(y) = 0$ and this point must be $g(x;z) = t(x)$. Similarly, it can be shown that if $g(x;z) = u(x)$ then $\hat{h}'(y) = 0$ when $y = u(x)$.

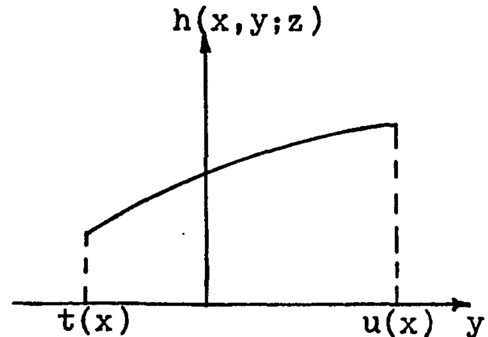
Based on these arguments we can conclude $y = g(x;z)$ is the point in $[t(x), u(x)]$ where $D_n h(x,y;z) = 0$.

The sets $S_1(z)$, $S_2(z)$, and $S_3(z)$ are described in the following sketches in the sense that the "typical" functions are shown for fixed x and z. For the purpose of reasonable sketches, it is assumed that $X \subseteq E_1$, $z \in E_1$, and $c(x,y)$ is a one component vector.



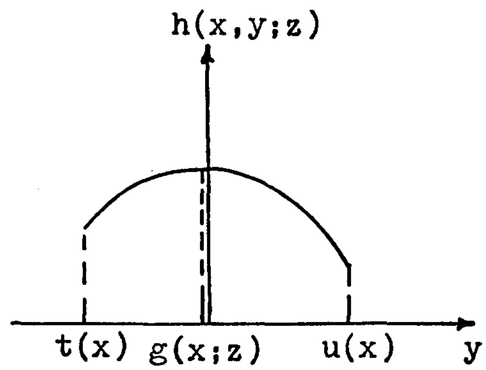
x $S_1(z)$

Figure 1



x $S_2(z)$

Figure 2



x $S_3(z)$

Figure 3

An important property of the sets $S_1(z)$, $S_2(z)$ and $S_3(z)$ is that $X = S_1(z) \cup S_2(z) \cup S_3(z)$ and $S_i(z) \cap S_j(z)$ is the empty set for all $i \neq j$. The proof of this property is given in Appendix C.

2.3 Statement of Primal and Dual Problems

The Fenchal Duality Theorem, several example problems in Luenberger [35], and the work of Francis and Wright, provided the motivation for the construction of the primal problem from the Neyman-Pearson problem. The primal problem and the dual problem which form the basis of the analysis will now be stated.

Primal Problem

$$\underset{z \in E_m^+}{\text{Minimize}}\ F(z) = \int_{S_1(z)} h(x,t(x);z)dx + \int_{S_2(z)} h(x,u(x);z)dx + \int_{S_3(z)} h(x,g(x;z);z)dx + b^T z \qquad (1)$$

Recall $g(x;z)$ is the value of $y \in [t(x),u(x)]$ such that $h(x;g(x;z);z) = \max\{h(x,y;z) : y \in [t(x),u(x)]\}$ for fixed x and z and $D_n h(x;y;z) = 0$ when $y = g(x;z)$. E_m^+ is the non-negative orthant of E_m.

Dual Problem (Nonlinear Neyman-Pearson Problem)

$$\text{Maximize } E(f) = \int_X e(x,f(x))dx \qquad (2)$$

subject to

$$C_i(f) = \int_X c_i(x,f(x))dx \leq b_i,\ i=1,\ldots,m \qquad (3)$$

and $t(x) \leq f(x) \leq u(x)$ for all $x \in X$ (4)

Equations (3) and (4) can be rewritten as

$$C(f) = \int_X c(x,f(x))dx \leq b$$

and $f \in W$, respectively, where $C(f)$ is defined as the column vector $C(f) = (C_1(f), C_2(f),\ldots,C_m(f))^T$.

For use in some of the subsequent properties it is appropriate to define the collection of functions satisfying the constraints of the dual problem in set notation as $P(b) = \{f \in W : C(f) \leq b\}$.

As in the paper by Francis and Wright and in [32], the following definitions will be used: a solution to the primal problem is any global minimum of $F(\cdot)$; a feasible solution to the dual problem is any function $f(\cdot)$ which satisfies the dual constraints (3) and (4); a solution to the dual problem is any feasible solution to the dual problem that maximizes $E(\cdot)$. These definitions imply $P(b)$ may be considered to be the set of all feasible solutions to the dual problem.

Now that we have defined the primal problem and the Neyman-Pearson problem, it is appropriate to discuss work related to the primal, dual, and the duality relationships.

2.4 Related Primal Problems

In 1963, Kuhn [29] discovered a dual problem to the location problem which is often referred to as the generalized Fermat problem or Steiner-Weber problem. Kuhn [29] also gives an interesting discussion of the history of the generalized Fermat problem; the problem can be stated as follows:

$$\text{Minimize} \sum_{i=1}^{p} w_k |x - x_k|$$

where the x_k for $k = 1,2,\ldots,p$ are given points in the plane. This problem is a special case of the discrete version of the primal problem.

Witzgall and Rockafellar [52] also discovered the dual to the generalized Fermat problem using W. Fenchal's theory of conjugate functions. Francis and Wright [18] give references to other related works on the location problem.

If $b = 0$ and z is a positive real number then we can write the linear functional version of the primal problem as follows:

$$\text{Minimize } F(z) = \int_0^z -t(x)(z-x)dx + \int_z^\infty u(x)(x-z)dx.$$

This problem is one form of the one-period stochastic inventory model where x is the demand, z is the inventory, $-t(x)$ is the storage cost, and $u(x)$ is the shortage cost. Also, this might be considered to be a continuous version of the Christmas tree problem or the newsboy problem.

Francis and Wright [18] indicate the location problem and the one-period stochastic inventory model appear to be the only examples of applications of the primal problem.

2.5 Related Dual Problems

The dual problem was formulated by Neyman and Pearson and then sufficient conditions for a solution (Neyman-Pearson lemma) were published in 1936 [38]. Neyman and Pearson were considering the testing of simple hypotheses in statistical problems. We can state the problems as follows: maximize the power of the test (probability of rejecting a hypothesis when it is false) subject to the constraint of a given Type I error (probability of rejecting a hypothesis when it is true). Dantzig proved the necessity of these conditions in 1939 as stated in [11]. Wald developed the necessary conditions independently and the results were published jointly with Dantzig in 1951 [12].

Chernoff and Scheffé extended the results of Dantzig and Wald for a more general version of the Neyman-Pearson problem in 1952 [10]. Virsan [48] obtained necessary conditions for a solution to a linear functional version of the previously stated Neyman-Pearson problem.

Zahl [55] presented necessary and sufficient conditions for a solution to the nonlinear version of the Neyman-Pearson problem (with one constraint) stated in Section 2.3. However, the assumptions are different from those stated in Section 2.2. He considered the problem as an allocation of resources problem but indicated that search problems and some game theory problems have the same form.

In 1969, Wagner [50] presented sufficient conditions (Neyman-Pearson lemma) for a solution to the nonlinear version of the Neyman-Pearson problem when the constraints hold as equalities for the solution. He solves five different example problems to indicate the simplicity of using the Neyman-Pearson lemma rather than Lagrangian multipliers or dynamic programming as the solution techniques. The example in Chapter V of this dissertation is a variation of one of Wagner's example problems.

Another example problem given by Wagner was based on the following problem by Black and Proschan [7]. A complex system is to be placed in the field and the only replacement parts available are those sent initially with the system. Maximum assurance of continued operation of the system is desired subject to the constraint that the optimum spare parts kit is limited by a fixed budget. In equation form, this problem is a discrete version of the Neyman-Pearson problem stated in Section 2.2. As Wagner stated, Black and Proschan achieve the same results but they do not state the results as the Neyman-Pearson Lemma.

Wagner gives an extensive literature review of those papers presenting linear functional versions of the Neyman-Pearson problem to solve the nonlinear functional version in addition to those discussed in the following paragraph. He also presents a review of those papers giving other solution techniques to the Neyman-Pearson problem.

Karlin [22], [24], and Rustagi [45] have transformed the nonlinear functional version of the Neyman-Pearson problem into a linear functional version to solve applications problems. Rustagi [45] indicates there are many statistical applications for the nonlinear functional version of the Neyman-Pearson problem. In Chapter VI, the technique developed in this dissertation is used to give sufficient conditions for a solution to the general problem presented and solved by Rustagi [45].

Karlin [22] made the comment that a great deal of ingenuity is required when using his linearizing technique. Wagner indicates the use of his sufficient conditions is more direct since a transformation of the Neyman-Pearson problem into a linear function version is not required. The author believes the solution technique developed in Chapter III is more straightforward than Wagner's since the necessary and sufficient conditions are constructive in the sense that the optimal solution to the Neyman-Pearson problem is given explicitly in these conditions. Wagner does not state how the solution is determined when the upper bound $u(x)$ or lower bound $t(x)$ is achieved by the optimal solution.

As indicated by Francis and Wright [18] and Wagner [50], the Neyman-Pearson problem occurs in allocation problems, search problems, control theory, information theory, and facility design.

2.6 Duality Relationships

As previously stated, Francis and Wright [18] presented duality relationships for a linear functional version of the Neyman-Pearson problem. The following discussion shows that the primal problem of this dissertation and the primal problem of the linear functional version are equivalent when the Neyman-Pearson problem given in Section 2.3 is stated as a linear functional version.

Let $e(x,y) = e(x)y$ so that $e(x,f(x)) = e(x)f(x)$ and let $c(x,y) = c(x)y$ so that $c(x,f(x)) = c(x)f(x)$. Then we have

$$h(x,y;z) = e(x)y - zc(x)y = y[e(x) - zc(x)]$$

which implies

$$D_n h(x,y;z) = e(x) - zc(x).$$

If $e(x) - zc(x) > 0$ for some $x \in X$, the function $h(x,y;z)$ is a strictly decreasing function of y which implies $x \in S_1(z)$. If $e(x) - zc(x) > 0$ for some $x \in X$ then $h(x,y;z)$ is a strictly increasing function which implies $x \in S_2(z)$. When $e(x) - zc(x) = 0$ for some $x \in X$ then $x \in (S_1(z) \cup S_2(z))^c$. Define $(S_1(z) \cup S_2(z))^c = S_3'(z)$. Note that $S_3'(z)$ does not equal $S_3(z)$ in a strict sense as $S_3(z)$ was defined for $h(\cdot)$ being strictly concave in y.

Thus, for the linear case

$$\int_{S_3'(z)} h(x,f(x);z)dx = \int_{S_3'(z)} f(x)[e(x) - zc(x)]dx$$

$$= \int_{S_3'(z)} f(x)D_n h(x,y;z)dx = 0$$

since $D_n h(x,y;z) = 0$ for all $x \in S_3'(z)$. Therefore,

$$\int_{S_3'(z)} h(x,f(x);z)dx = 0 \text{ for all } f \in P(b);$$

and, in particular, when we make the arbitrary choice $f(x) = t(x)$ a.e. on $S_3'(z)$. Now the primal problem can be written as

$$
\begin{aligned}
\underset{z \in E_m^+}{\text{Min}} \; F(z) &= \int_{S_1(z)} t(x)[e(x)-zc(x)]dx + \int_{S_2(z)} u(x)[e(x)-zc(x)]dx \\
&+ \int_{S_3'(z)} t(x)[e(x)-zc(x)]dx + b^T z \\
&= \int_{S_1(z)\cup S_3'(z)} t(x)[e(x)-zc(x)]dx + \int_{S_2(z)} u(x)[e(x)-zc(x)]dx + b^T z
\end{aligned}
$$

which is the primal problem developed by Francis and Wright. The construction of the dual problem given by Francis and Wright follows directly from the equations $e(x,f(x)) = e(x)f(x)$ and $c_i(x,f(x)) = c_i(x)f(x)$, $i=1,2,\ldots,m$.

Hsiang-Chun Yen [53] presents the dual to the nonlinear functional version of the Neyman-Pearson problem which he developed using the Lagrangain Multiplier technique (following tradition he also refers to the primal problem as the dual of the Neyman-Pearson Problem). His results require the sufficiency conditions developed by Wagner; a rather restrictive requirement since he requires the solution to the Neyman-Pearson problem before he can obtain the primal problem. He also restricts the domain of the solution to the Neyman-Pearson problem to an open interval on the real line. He does not indicate how the solution is determined for the Neyman-Pearson problem and does not indicate explicitly that the solution to the Neyman-Pearson problem is a function of the Lagrangain multipliers.

Hsiang-Chun Yen also presents necessary and sufficient conditions for a solution to the nonlinear Neyman-Pearson problem based on the restrictions mentioned above.

Necessary conditions are obtained via the Calculus of Variations.

The author's development of the primal problem is more general since Hsiang-Chun Yen required the use of Wagner's sufficiency conditions for the development of the primal problem while the author has developed the primal problem using concavity-convexity arguments and insight based on the Fenchel Duality Theorem and Lagrangian multipliers. In Chapter III, the author compares Wagner's conditions and the conditions established in this dissertation. The author also develops necessary and sufficient conditions for a solution to the primal problem; the primal problem may be easier to solve as is indicated in Chapter V.

2.7 Recent Developments

In July 1970, the author's advisor received correspondence from Wagner and Stone [51] giving a proof that Wagner's sufficient conditions [50] for a solution to the Neyman-Pearson problem are also necessary when the constraint $C(f) = b$ is satisfied and $C(f)$ is a linear functional. The proof is based on the result given by Dantzig and Wald [12].

Necessary conditions were also developed for a solution to the Neyman-Pearson problem stated in Section 2.2 when $e(x,y)$ is non-decreasing in y and $c(x,y)$ is strictly increasing in y. These assumptions are different than those given in Section 2.2. This proof is also based on the result by

Dantzig and Wald. This author's proof of the necessary conditions for a solution to the Neyman-Pearson problem is based on Karlin's extension of the Kuhn-Tucker theorem.

CHAPTER III

ANALYSIS

3.1 Introduction

The purpose of this chapter is to state and analyze the duality relationships of the primal and dual problems.

A proof of the necessary conditions for a solution to the linear version of the Neyman-Pearson problem [18] is also included since it provides an alternative approach based on a technique used by Slater in proving the Kuhn-Tucker Theorem. The technique is also used subsequently in deriving the necessary conditions for a solution to the non-linear version of the Neyman-Pearson problem.

A single example is developed in Chapter V to illustrate the results of this chapter; the reader may find the example useful to refer to when going through this chapter.

3.2 Properties of the Primal and Dual Problems

Result 1. - For any feasible solution $f(\cdot)$ to the dual problem, and any $z \in E_m^+$, $E(f) \le F(z)$.

Proof. - Let $F(\cdot)$ be any feasible solution to the dual problem and let $z \in E_m^+$. Then

$$E(f) = \int_X e(x,f(x))dx$$

$$\le \int_X e(x,f(x))dx + \sum_{i=1}^{m} z_i[b_i - \int_X c_i(x,f(x))dx]$$

$$= \int_X e(x,f(x))dx + b^T z - \int_X zc(x,f(x))dx$$

$$= \int_X [e(x,f(x)) - zc(x,f(x))]dx + b^T z$$

$$= \int_{S_1(z)} h(x,f(x);z)dx + \int_{S_2(z)} h(x,f(x);z)dx$$

$$+ \int_{S_3(z)} h(x,f(x);z)dx + b^T z$$

since $S_1(z)$, $S_2(z)$, and $S_3(z)$ are pairwise disjoint and their union is X. Now $h(x,f(x);z) \le h(x,t(x);z)$ on $S_1(z)$ since $D_n h(x,y;z) < 0$ for all $x \in S_1(z)$ and $t(x) \le f(x) \le u(x)$, as is illustrated in Figure 1. Also, $h(x,f(x);z) \le h(x,u(x);z)$ on $S_2(z)$ since $D_n h(x,y;z) > 0$ for all $x \in S_2(z)$ as is illustrated in Figure 2. For each $x \in S_3(z)$, we have $h(x,f(x);z) \le h(x,g(x;z);z)$ from the definition of $S_3(z)$. Looking at

Figure 3 we see the latter inequality is satisfied by the "typical" function $h(x,y;z)$.

Using the inequalities of the previous paragraph we have

$$\int_{S_1(z)} h(x,f(x);z)dx \leq \int_{S_1(z)} h(x,t(x);z)dx,$$

$$\int_{S_2(z)} h(x,f(x);z)dx \leq \int_{S_2(z)} h(x,u(x);z)dx,$$

$$\int_{S_3(z)} h(x,f(x);z)dx \leq \int_{S_3(z)} h(x,g(x;z);z)dx.$$

Adding the above inequalities gives

$$\begin{aligned} E(f) &\leq \int_{S_1(z)} h(x,t(x);z)dx + \int_{S_2(z)} h(x,u(x);z)dx \\ &\qquad + \int_{S_3(z)} h(x,g(x;z);z)dx + b^T z \\ &= F(z) \end{aligned}$$

which is the desired result.

Sufficient Conditions for Solutions to Primal and Dual Problem

Result 2. - If there exists a feasible solution $f(\cdot)$ to the dual problem, a $z \in E_m^+$ such that

$$\sum_{i=1}^{m} z_i \left[\int_X c_i(x,f(x)dx - b_i\right] = 0, \tag{5}$$

$$f(x) = t(x) \quad \text{a.e. on } S_1(z), \tag{6}$$

$$f(x) = u(x) \quad \text{a.e. on } S_2(z), \tag{7}$$

$$f(x) = g(x;z) \text{ a.e. on } S_3(z), \tag{8}$$

then z is a solution to the primal problem, $f(\cdot)$ is a solution to the dual problem, and $E(f) = F(z)$.

Proof. - It is clear that z is a solution to the primal problem and f(x) is a solution to the dual problem by Result 1 provided we can show $E(f) = F(z)$. Note that

$$\begin{aligned} E(f) &= \int_X e(x,f(x))dx \\ &= \int_X e(x,f(x))dx - \sum_{i=1}^{m} z_i\left[\int_X c_i(x,f(x))dx - b_i\right] \\ &= \int_X \left[e(x,f(x)) - zc(x,f(x))\right]dx + b^Tz \end{aligned}$$

by hypothesis (5) above. Therefore, we can write

$$\begin{aligned} E(f) &= \int_X h(x,f(x);z)dx + b^Tz \\ &= \int_{S_1(z)} h(x,f(x);z)dx + \int_{S_2(z)} h(x,f(x);z)dx + \int_{S_3(z)} h(x,f(x);z)dx + b^Tz \\ &= \int_{S_1(z)} h(x,t(x);z)dx + \int_{S_2(z)} h(x,u(x);z)dx + \int_{S_3(z)} h(x,g(x;z);z)dx + b^Tz \end{aligned} \tag{9}$$

by equations (6), (7) and (8) of the hypothesis. The right hand side of equation (9) is the objective function of the primal problem. Therefore, we have $E(f) = F(z)$.

Necessary Conditions for a Solution to the Dual Problem

While the techniques such as those given in Luenberger [35] could be used for establishing the necessary conditions for a solution to the dual problem, these general techniques are not required for the Neyman-Pearson problem stated in this dissertation. The author is indebted to his advisor for suggesting the use of a simpler and more straightforward approach based on Slater's and Karlin's proofs of the necessary conditions for the Kuhn-Tucker Theorem. For the purpose of completeness of this section, the Kuhn-Tucker Theorem is stated in Appendix D.

The proof of Property 1 to follow is essentially the same as one involving the Kuhn-Tucker conditions by Karlin [22], and is included in the interest of completeness.

The necessary conditions are developed via Properties 1, 2, and 3. Result 3 is stated as a summary of results for the necessary conditions for a solution to the dual problem.

For Properties 1, 2, and 3, it is assumed there is at least one $f \in P(b)$ such that $D(f) = C(f) - b < \overline{0}$.

Property 1. - If $f_0(\cdot)$ is a solution to the dual problem then there exists a $z_0 \in E_m^+$ such that

$$E(f) - z^0 D(f) \leq E(f_0) - z^0 D(f_0) \leq E(f_0) - zD(f_0) \quad (10)$$

for all $f \in W$ and all $z \in E_m^+$. Also, we have

$$z^0 D(f_0) = 0. \quad (11)$$

<u>Proof:</u> - Let $f_0(\cdot)$ be a solution to the dual problem. Define $A = \{(u,v) \in E_{m+1}: u \geq D(f),\ v \leq E(f)$ for at least one $f \in W\}$ and $B = \{(u,v) \in E_{m+1}: u < \overline{0},\ v > E(f_0)\}$. For the sets A and B, $u = (u_1, u_2, \ldots, u_m)^T \in E_m$ and $v \in E_1$. $D(f) = (D_1(f), D_2(f), \ldots, D_m(f))^T$ is an m-component column vector where $D_i(f) = C_i(f) - b_i$, $i=1,\ldots,m$.

A geometric illustration giving an example of the relationship between the set A and the set B is shown in Figure 4.

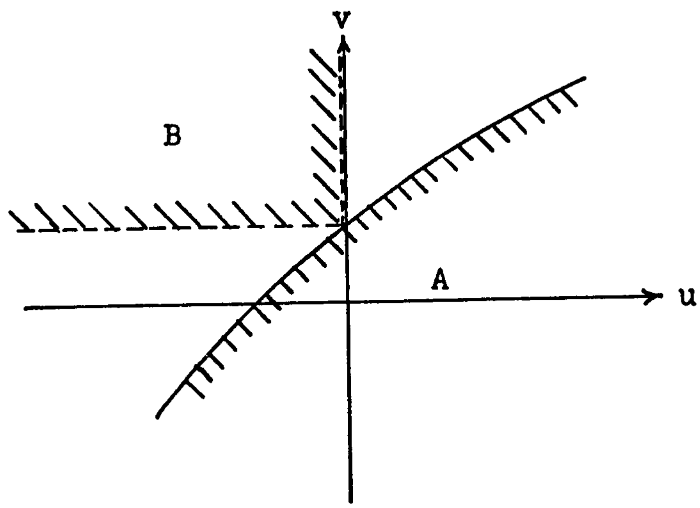


Figure 4

It is clear that B is a convex set and the set A is shown to be convex in Proposition 6 of Appendix E. Figure 4 shows geometrically that A and B are disjoint. This is shown analytically in the following proof by contradiction.

Assume $(u,v) \in A \cap B$. Then $(u,v) \in A$ which implies there exists an $f \in W$ such that $u \geq D(f)$ and $v \leq E(f)$. Also, $(u,v) \in B$ implies $u < \overline{0}$ and $v > E(f_0)$. Thus $D(f) \leq u < \overline{0}$ and $E(f_0) < v \leq E(f)$ which implies $f(\cdot)$ is a feasible solution to the dual and $f_0(\cdot)$ is not a solution to the dual problem. But the statement of this property indicates $f_0(\cdot)$ is a solution to the dual problem. Hence, A and B are disjoint.

Consequently, the hypotheses of the separating hyperplane theorem given in Appendix F are satisfied so there is a separating hyperplane $c\cdot(u,v) = d$, where d is a real number, with $c = (z^*, -r_0) \neq \overline{0}$ ($r_0 \in E_1$, $c \in E_{m+1}$) such that

$$-r_0\, v'' + z^*\, u'' \geq -r_0\, v' + z^*\, u', \tag{12}$$

for all $(u', v') \in B$ and all $(u'', v'') \in A$.

We have $z^* \geq \overline{0}$ for if at least one $z_i^* < 0$, we can make the i^{th} component of u' arbitrarily small for $(u', v') \in B$ which would imply the inequality (12) would be violated.

It is also true that $r_0 \geq 0$ since v' can be made arbitrarily large for $(u', v') \in B$ which implies $r_0 < 0$ would violate inequality (12).

Now we will show $r_0 > 0$. Choose $f \in W$ and let $x_1 = (\overline{0}, E(f_0))$ and $x_2 = (D(f), E(f))$ so that $x_2 \in A$. Since x_1 is a boundary point (defined in Appendix F) of B (12) still holds, by the separating hyperplane theorem. Substituting x_2 and x_1 in (12) yields

$$-r_0\, E(f) + z^* D(f) \geq -r_0\, E(f_0) + z^*\, \overline{0}$$

and we have

$$-r_0\, E(f) + z^*\, D(f) \geq -r_0\, E(f_0). \tag{13}$$

Consider two cases, $z^* = \overline{0}$; $z_i^* > 0$ for at least one i, where $i=1,\ldots,m$.

<u>Case 1.</u> - Let $z^* = \overline{0}$.

We know $r_0 \geq 0$ and $(z^*, -r_0) \neq \overline{0}$. Therefore $(\overline{0}, -r_0) \neq \overline{0}$ if and only if $r_0 > 0$.

<u>Case 2.</u> - Let $z_i^* > 0$ for at least one i.

If $r_0 = 0$, (13) implies $z^* D(f) \geq 0$ for all $f \in W$; but it is assumed there exists an $f \in W$, say f_1, such that

$D(f_1) < \overline{0}$ which implies $z_i^* D_i(f_1) < 0$ for all $z_i^* > 0$ so that $z^* D(f_1) < 0$. Thus, r_0 must be positive.

Combining both cases we conclude $r_0 > 0$ for $z^* \epsilon E_m^+$. Now dividing (13) by r_0 gives

$$-E(f) + z^0 D(f) \geq - E(f_0) \text{ for all } f \epsilon W \tag{14}$$

where

$$z^0 = (\frac{z_1^*}{r_0}, \frac{z_2^*}{r_0}, \ldots, \frac{z_m^*}{r_0}) \geq \overline{0}. \tag{15}$$

In particular, for f_0 we have

$$-E(f_0) + z^0 D(f_0) \geq - E(f_0)$$

which implies

$$z^0 D(f_0) \geq 0.$$

Since $f_0 \epsilon P(b)$ implies $D(f_0) \leq \overline{0}$ and $z^0 \geq \overline{0}$ we know

$$z^0 D(f_0) \leq 0$$

and thus conclude

$$z_0 D(f_0) = 0, \tag{16}$$

which proves (11).

From (14) and (16) we can now conclude

$$-E(f) + z^0 D(f) \geq - E(f_0) + z^0 D(f_0) \text{ for all } f \epsilon W.$$

Multiplying by (-1) in the previous inequality yields

$$E(f) - z^0 D(f) \leq E(f_0) - z^0 D(f_0) \tag{17}$$

Since $D(f_0) \leq \overline{0}$ we have $-z D(f_0) \geq 0$ for all $z \epsilon E_m^+$. Therefore,

$E(f_0) \leq E(f_0) - z D(f_0)$, for all $z \epsilon E_m^+$, and since $z^0 D(f_0) = 0$ we have

$$E(f_0) - z^0 D(f_0) \leq E(f_0) - zD(f_0), \text{ for all } z \epsilon E_m^+. \tag{18}$$

Combining (17) and (18) yields

$$E(f) - z^0 D(f) \leq E(f_0) - z^0 D(f_0) \leq E(f_0) - z\ D(f_0)$$

for all $z \in E_m^+$ and all $f \in W$ which is the desired result (10).

Property 2. - If $f_0(\cdot)$ is a solution to the dual problem then $z^0 \in E_m^+$, defined in (13) of the previous property, is a solution to the primal problem and $E(f_0) = F(z^0)$.

Proof. - Let

$$\begin{aligned} \bar{f}(x) &= t(x) && \text{a.e. on } S_1(z^0), \\ \bar{f}(x) &= u(x) && \text{a.e. on } S_2(z^0), \\ \bar{f}(x) &= g(x;z^0) && \text{a.e. on } S_3(z^0). \end{aligned}$$

Then we can see $\bar{f}(\cdot) \in W$. Also we can express $F(z^0)$ in the following manner:

$$\begin{aligned} F(z^0) &= \int_{S_1(z^0)} h(x,t(x);z^0)dx + \int_{S_2(z^0)} h(x,u(x);z^0)dx \\ &+ \int_{S_3(z^0)} h(x,g(x;z^0);z^0)dx + b^T z^0 \\ &= \int_{S_1(z^0)} e(x,t(x))dx + \int_{S_2(z^0)} e(x,u(x))dx \\ &+ \int_{S_3(z^0)} e(x,g(x;z^0))dx \\ &- z^0 \Big[\int_{S_1(z^0)} c(x,t(x))dx + \int_{S_2(z^0)} c(x,u(x))dx \\ &+ \int_{S_3(z^0)} c(x,g(x;z^0))dx - b\Big] \end{aligned}$$

from the definition of $h(x,y;z)$.

Using the above definition of $\bar{f}(\cdot)$ and the expression for $F(z^0)$ given above we can then write

$$F(z^0) = \int_X e(x,\bar{f}(x))dx - z^0[\int_X c(x,\bar{f}(x))dx - b]$$

$$= E(\bar{f}) - z^0 D(\bar{f}).$$

By (10) in Property 1 we have

$$F(z^0) = E(\bar{f}) - z^0D(\bar{f}) \leq E(f^0) - z^0D(f_0) = E(f_0). \quad (19)$$

From Result 1 we have

$$F(z) \geq E(f) \text{ for all } z \in E_m^+ \text{ and all } f \in P(b).$$

Since $f_0 \in P(b)$ and $z^0 \in E_m^+$ we have

$$F(z^0) \geq E(f_0). \quad (20)$$

Combining (19) and (20) yields

$$F(z^0) = E(f_0).$$

Hence, z^0 is a solution to the primal problem.

Property 3. - If $f_0(\cdot)$ is a solution to the dual problem then there exists $z^0 \in E_m^+$ such that

$$f_0(x) = t(x) \quad \text{a.e. when } x \in S_1(z^0), \quad (21)$$

$$f_0(x) = u(x) \quad \text{a.e. when } x \in S_2(z^0), \quad (22)$$

$$f_0(x) = g(x;z^0) \quad \text{a.e. when } x \in S_3(z^0). \quad (23)$$

Proof. - Since it was shown that there exists a solution z^0 to the primal problem when $f_0(\cdot)$ is a solution to the dual problem and $z^0D(f_0) = 0$ we can write

$$E(f_0) = E(f_0) - z^0 D(f_0) = \int_X e(x,f_0(x))dx - z^0\left[\int_X c(x,f_0(x)) - b\right]$$

$$= \int_X h(x,f_0(x);z^0)dx + b^T z^0$$

$$= \int_{S_1(z^0)} h(x,f_0(x);z^0)dx + \int_{S_2(z^0)} h(x,f_0(x);z^0)dx$$

$$+ \int_{S_3(z^0)} h(x,f_0(x);z^0)dx + b^T z^0 \qquad (24)$$

Also,

$$F(z^0) = \int_{S_1(z^0)} h(x,t(x);z^0)dx + \int_{S_2(z^0)} h(x,u(x);z^0)dx$$

$$+ \int_{S_3(z^0)} h(x,g(x;z^0);z^0)dx + b^T z^0 \qquad (25)$$

In the previous property it was shown $F(z^0) = E(f_0)$ so equating (24) and (25) yields

$$\int_{S_1(z^0)} h(x,f_0(x);z^0)dx + \int_{S_2(z^0)} h(x,f_0(x);z^0)dx + \int_{S_3(z^0)} h(x,f_0(x);z^0)dx$$

$$= \int_{S_1(z^0)} h(x,t(x);z^0)dx + \int_{S_2(z^0)} h(x,u(x);z^0)dx + \int_{S_3(z^0)} h(x,g(x;z^0);z^0)dx.$$

The above equation can be rewritten as follows:

$$\int_{S_1(z^0)} [h(x,t(x);z^0) - h(x,f_0(x);z^0)]dx + \int_{S_2(z^0)} [h(x,u(x);z^0) - h(x,f_0(x);z^0)]dx$$

$$+ \int_{S_3(z^0)} [h(x,g(x;z^0);z^0) - h(x,f_0(x);z^0)]dx = 0. \qquad (26)$$

Since $h(x,y;z)$ is strictly decreasing in y for all $x \in S_1(z^0)$ this implies

$$h(x,t(x);z^0) - h(x,f_0(x);z^0) \geq 0. \qquad (27)$$

Also, $h(x,y;z)$ is strictly increasing in y for all $x \in S_2(z^0)$ and so

$$h(x,u(x);z^0) - h(x,f_0(x);z^0) \geq 0. \tag{28}$$

From the definition of $S_3(z^0)$ we also have

$$h(x,g(x;z^0);z^0) - h(x,f_0(x);z^0) \geq 0. \tag{29}$$

for all $x \in S_3(z^0)$. The sketches in Figures 1, 2 and 3 also illustrate geometrically that these inequalities must hold.

Therefore, each integrand in (26) is non-negative and the sum of non-negative terms is zero only when each term is zero, which yields

$$\int_{S_1(z^0)} [h(x,t(x);z^0) - h(x,f_0(x);z^0)]dx = 0, \tag{30}$$

$$\int_{S_2(z^0)} [h(x,u(x);z^0) - h(x,f_0(x);z^0)]dx = 0, \tag{31}$$

and

$$\int_{S_3(z^0)} [h(x,g(x;z^0);z^0) - h(x,f_0(x);z^0)]dx = 0. \tag{32}$$

Equations (27) and (30) imply

$$h(x,t(x);z^0) = h(x,f_0(x);z^0) \text{ a.e. on } S_1(z^0). \tag{33}$$

If $f_0(x) \neq t(x)$ a.e. on $S_1(z^0)$ then $f_0(x) > t(x)$ on a subset of $S_1(z^0)$ of positive measure, which from the definition of $S_1(z^0)$, ($h(\cdot)$ is strictly decreasing in the n^{th} variable) implies $h(x,f_0(x);z^0) < h(x,t(x);z^0)$ on a subset of $S_1(z^0)$ of positive measure contradicting (33). Thus, (21) is established.

Similarly, (28) and (31) imply

$$h(x,u(x);z^0) = h(x,f_0(x);z^0) \text{ a.e. on } S_2(z^0). \tag{34}$$

Assume $f_0(x) \neq u(x)$ a.e. on $S_2(z^0)$ then $f_0(x) < u(x)$ on a subset of $S_2(z^0)$ of positive measure. From the definition of $S_2(z^0)$, $h(\cdot)$ is strictly increasing in the n^{th} variable which implies $h(x,f_0(x);z) < h(x,u(x);z^0)$ on a subset of $S_2(z^0)$ of positive measure contradicting (34). Thus, (22) is established.

Equations (29) and (32) imply

$$h(x,g(x;z^0);z^0) = h(x,f_0(x),z^0) \text{ a.e. on } S_3(z^0). \quad (35)$$

If $f_0(x) \neq g(x;z^0)$ a.e. on $S_3(z^0)$ then $f_0(x) \neq g(x;z^0)$ on a subset of $S_3(z^0)$ of positive measure. From the definition of $S_3(z^0)$ and Figure 3, it can be seen that $h(\cdot)$ is a strictly increasing function of y when $y < g(x;z^0)$ and $h(\cdot)$ is a strictly decreasing function of y when $y > g(x;z^0)$. Thus, $h(x,f_0(x);z^0) < h(x,g(x;z^0);z^0)$ on a subset of $S_3(z^0)$ of positive measure which contradicts (35). Therefore, (23) is established.

Properties 1, 2, and 3 can be combined in the following result to give necessary conditions for a solution to the dual problem.

<u>Result 3.</u> - Suppose there is at least one $f \in P(b)$ such that $[\int_X c(u,f(x))dx - b] < \bar{0}$. If $f_0(\cdot)$ is a solution to the dual problem then there exists a $z^0 \in E_m{}^+$ such that

$$z^0[\int_X c(x,f(x))dx - b] = 0;$$

$$f_0(x) = t(x) \quad \text{a.e. on } S_1(z^0);$$

$f_0(x) = u(x)$ a.e. on $S_2(z^0)$;

$f_0(x) = g(x;z^0)$ a.e. on $S_3(z)$;

and $E(f_0 = F(z_0)$.

Results 2 and 3 can be combined in the following manner as necessary and sufficient conditions for a solution to the dual problem.

Result 4. - Suppose there is at least one $f \in P(b)$ such that $[\int_X c(x,f(x))dx - b] < \bar{0}$. Then the feasible solution $f_0(\cdot)$ is a solution to the dual problem if and only if there exists a $z^0 \in E_m^+$ such that

$z^0[\int_X c(x,f_0(x))dx - b] = 0$;

$f_0(x) = t(x)$ a.e. on $S_1(z^0)$;

$f_0(x) = u(x)$ a.e. on $S_2(z^0)$;

$f_0(x) = g(x;z^0)$ a.e. on $S_3(z^0)$.

Also, $E(f_0) = F(z^0)$.

Note that when Result 4 holds, Result 1 implies z^0 is a solution to the primal problem.

The assumption that there exists an $f \in P(b)$ such that $\int_X c(x,f(x))dx - b < 0$ is usually called a regularity assumption. When this regularity assumption is satisfied, i.e., there is a feasible solution for which all constraints are inactive, then the solutions to the primal and dual problems can be found from the necessary and sufficient conditions (Result 4) and the definitions of $S_1(z)$, $S_2(z)$ and $S_3(z)$. Also, the values for $E(f_0)$ and $F(z^0)$ are the same.

An example problem is given in Chapter V to illustrate the previous results, the use of the sets $S_1(z)$, $S_2(z)$,

$S_3(z)$, and the necessary and sufficient conditions for determining the solutions to the primal and dual problems.

3.3 Necessary Conditions for a Solution to the Linear Functional Version of the Neyman-Pearson Problem

Francis and Wright [18] give a proof of the necessary conditions for the linear functional version of the Neyman-Pearson problem based on a proof by Virsan [48]. However, the linear functional version is a special case of the non-linear functional version as shown in Section 2.6 which implies Result 3 holds. Since Slater's technique appears to be more straightforward, it is appropriate to present the proof for the linear version using his technique.

Although it is assumed that $e(x,\cdot)$ is strictly concave in the n^{th} variable for the non-linear functional version, the Kuhn-Tucker Theorem (Appendix D) requires $E(\cdot)$ to be only concave. It will be shown that $E(\cdot)$ and $C(\cdot)$ being linear functionals does not affect the hypotheses of the necessary conditions.

Recall from Section 2.2 that the definitions of $S_1(z)$ and $S_2(z)$ are identical with those for the nonlinear functionals. We also defined

$$\begin{aligned} S_3'(z) &= \{x \in X : D_n h(x,y;z) = e(x) - zc(x) = 0\} \\ &= (S_1(z) \cup S_2(z))^c. \end{aligned}$$

<u>Result 5.</u> - Assume there is a feasible solution $f(\cdot)$ to the dual problem such that $\int_X c(x)f(x)dx - b < \overline{0}$.

If $f_0(\cdot)$ is a solution to the dual problem, then there exists a $z^0 \in E_m^+$ for which

$$z^0\Big[\int_X c(x)f(x)dx - b\Big] = 0, \tag{36}$$

$$f_0(x) = t(x) \quad \text{a.e. on } S_1(z^0), \tag{37}$$

$$f_0(x) = u(x) \quad \text{a.e. on } S_2(z^0). \tag{38}$$

Proof: Equations (11) and (36) are the same when $c(x,f(x)) = c(x)f(x)$. Therefore, the proof of Property 1 proves (36) since the strict concavity assumption for the dual problem was not required for (11) to hold.

Now we can show z^0 (defined in (15)) is a solution to the primal problem when $f_0(\cdot)$ is a solution to the dual problem, i.e, we can prove Property 2 for the linear functional version.

Let

$$\bar{f}(x) = t(x) \text{ a.e. on } S_1(z^0),$$
$$\bar{f}(x) = u(x) \text{ a.e. on } S_2(z^0),$$
$$\bar{f}(x) = t(x) \text{ on } S_3'(z^0),$$

($\bar{f}(\cdot)$ may be chosen arbitrarily on $S_3'(z^0)$)

and then we can observe $\bar{f}(\cdot) \in W$. Recalling the linear version of the primal problem objective function we have

$$F(z_0) = \int_{S_1(z^0)\cup S_3'(z^0)} t(x)[e(x) - zc(x)]dx \int_{S_2(z^0)} u(x)[e(x) - zc(x)]dx + b^T z^0$$

$$= \int_{S_1(z^0)\cup S_3'(z^0)} e(x)t(x)dx + \int_{S_2(z^0)} e(x)u(x)dx$$

$$- z^0 \Big[\int_{S_1(z^0)\cup S_3'(z^0)} c(x)t(x)dx + \int_{S_2(z^0)} c(x)u(x)dx - b\Big]$$

Using the above definition of $\bar{f}(\cdot)$ and the expression for $F(z^0)$ given above we can write

$$F(z^0) = \int_X e(x)\bar{f}(x)dx - z^0[\int_X c(x)\bar{f}(x)dx - b]$$

$$= E(\bar{f}) - z^0 D(\bar{f}).$$

By (10) in Property 1 we have

$$F(z^0) = E(\bar{f}) - z^0 D(\bar{f}) \leq E(f_0) - z^0 D(f_0) = E(f_0). \quad (39)$$

From the work of Francis and Wright [18], we have $F(z) \geq E(f)$ for all $z \in E_m^+$ and all feasible solutions to the linear problem. Since $f_0(\cdot)$ is a feasible solution we have

$$F(z^0) \geq E(f_0). \quad (40)$$

Combining (39) and (40) yields $F(z^0) = E(f_0)$; hence z^0 is a solution to the dual problem.

Using the proof of Property 3 we conclude, without using the strict concavity assumption, that

$$h(x,t(x);z^0) = h(x,f_0(x);z^0) \quad \text{a.e. on } S_1(z^0) \quad (41)$$

and

$$h(x,u(x);z^0) = h(x,f_0(x);z^0) \quad \text{a.e. on } S_2(z^0). \quad (42)$$

Equation (41) is equivalent to

$$t(x)[e(x)-z^0c(x)] = f_0(x)[e(x)-z^0c(x)] \text{ a.e. on } S_1(z^0)$$

which implies (37) is satisfied since $e(x) - z^0c(x) < 0$.

Equation (42) is equivalent to

$$u(x)[e(x)-z^0c(x)] = f_0(x)[e(x)-z^0c(x)] \text{ a.e. on } S_2(z^0)$$

which implies (38) is satisfied since $e(x) - z^0c(x) > 0$.

3.4 Wagner's Sufficient Conditions for a Solution to the Dual Problem

In Wagner's paper [50] sufficient conditions were given for a slightly different problem than the dual problem of this dissertation. Since this dissertation is partially based on Wagner's paper, it is appropriate to compare Wagner's sufficiency conditions and the previously derived sufficient conditions. This comparison will be based on the following: concavity - convexity assumptions of this dissertation, one integral constraint, and Wagner's sufficiency conditions for the dual problem.

The following result is Wagner's sufficiency condition written in the notation of this dissertation.

Result 6. - Suppose $k(\cdot) \epsilon W$ has the following property: there exists a $z > 0$ such that for all $x \epsilon X \subseteq E_1$ (except on a set of measure zero)

$$D_2e(x,y) \leq zD_2c(x,y) \text{ whenever } k(x) < y < u(x) \qquad (43)$$

$$D_2e(x,y) \geq zD_2c(x,y) \text{ whenever } t(x) < y < k(x) \qquad (44)$$

Then

$$E(k) = \text{Max } \{E(f): f \epsilon W \text{ and } C(f) \leq C(k)\} \qquad (45)$$

$$C(k) = \text{Min } \{C(f): f \epsilon W \text{ and } E(f) \geq E(k)\}. \qquad (46)$$

Since it has been assumed in this dissertation that $h(x,y;z)$ is strictly concave in y, the author will prove the following result.

Result 7.- When $h(x,y;z) = e(x,y)-zc(x,y)$ is strictly concave in y then (43) and (44) become

$$D_2e(x,y) < zD_2c(x,y) \text{ whenever } k(x)<y<u(x) \qquad (47)$$

$$D_2e(x,y) > zD_2c(x,y) \text{ whenever } t(x)<y<k(x), \qquad (48)$$

respectively.

Proof. - Assume there is a point $y^* \in (k(x),u(x))$ such that $D_2h(x,y^*,z) = 0$. Since $h(\cdot)$ is strictly concave in y there is at least one $y \in (k(x),y^*)$ such that $D_2h(x,y;z) > 0$ which contradicts (43). Therefore, there is no point $y \in (k(x),u(x))$ such that $D_2h(x,y;z) = 0$. Hence, (43) holds as a strict inequality as is given in inequality (47).

Similarly, if $y^* \in (t(x),k(x))$ and $D_2h(x,y^*;z) = 0$ then there is at least one point $y \in (y^*,k(x))$ such that $D_2h(x,y;z) < 0$ which contradicts (44). Thus, (48) is a restatement of (44) when $h(\cdot)$ is strictly concave in y.

When a function $k(\cdot) \in W$ satisfies (43) and (44), (47) and (48) for $h(\cdot)$ strictly concave in y, Wagner defines the function $k(\cdot)$ to be "cost-effective". This problem is more general than the dual problem of this dissertation in the sense that there is no upper bound on $C(k)$ other than $C(k)$ being finite. If we bound $C(k)$, say $C(k) \leq b$, then (46) has no meaning as the problem becomes a constrained maximization problem, namely the dual problem of this dissertation. Wagner considers this situation in Remark 1 of [50] and states that if there is a $k^*(\cdot) \in W$ such that $k^*(\cdot)$ satisfies (43) and (44) and $C(k^*) = b$ then $k^*(\cdot)$ is a solution to the constrained maximization problem. In other words, $k^*(\cdot)$ is the most effective solution for a given cost restriction.

However, there is another case that must be considered. This is when the solution to the constrained maximization problem occurs with the constraints inactive. In other words, the maximum value of the objective function to the constrained problem is the same as the maximum value for the unconstrained problem. In fact, it will be shown in the proof of the following property that (47) and (48) are not enough information to guarantee that $k(\cdot)$ (defined by (47) and (48)) is a solution to the dual problem if $C(k)<b$.

Before stating Wagner's sufficiency conditions for the dual problem, it is worthwhile to note that $z > 0$ was required to prove (46). Since (46) is not involved in the dual problem of this dissertation it will be seen that the requirement can be relaxed to z being non-negative in Wagner's sufficiency conditions for the dual problem. However, $C(f) \leq b$ so we have $k(\cdot)$ restricted to the set $P(b)$.

<u>Property 4.</u> - Suppose $k(\cdot) \in P(b)$ has the following property: there exists a $z \geq 0$ such that for all $x \in X$ (except on a set of measure zero)

$$D_2e(x,y) < zD_2c(x,y) \text{ whenever } k(x) < y < u(x), \quad (49)$$

$$D_2e(x,y) > zD_2c(x,y) \text{ whenever } t(x) < y < k(x). \quad (50)$$

Then $E(k) = \text{Max}\left\{E(f)\colon f \in W \text{ and } C(f) \leq b\right\}$ (51)

if $C(k)=b$, i.e., $k(\cdot)$ is a solution to the dual problem.

<u>Proof.</u> - The proof follows the technique used by Wagner and proceeds in a fashion much the same as the original proof of the Neyman-Pearson Lemma.

For all $x \in X$ and $f(x) \in (k(x),u(x))$ we have

$$e(x,f(x)) - e(x,k(x)) = \int_{k(x)}^{f(x)} D_2e(x,y)dy$$

$$\leq z \int_{k(x)}^{f(x)} D_2c(x,y)dy \qquad (52)$$

by (49).

Also, when $f(x) \in (t(x),k(x))$ we have

$$e(x,k(x)) - e(x,f(x)) = \int_{f(x)}^{k(x)} D_2e(x,y)dy$$

$$\geq z \int_{f(x)}^{k(x)} D_2c(x,y)dy$$

by inequality (50). However, multiplying the inequality by -1 yields

$$e(x,f(x)) - e(x,k(x)) \leq -z \int_{f(x)}^{k(x)} D_2c(x,y)dy$$

$$\leq z \int_{k(x)}^{f(x)} D_2c(x,y)dy \qquad (53)$$

Since (52) and (53) are the same, we have

$$e(x,f(x)) - e(x,k(x)) \leq z[c(x,f(x)) - c(x,k(x))] \qquad (54)$$

for all $x \in X$ and $z \geq 0$. Integrating both sides of (54) yields

$$\int_X e(x,f(x))dx - \int_X e(x,k(x)dx \leq z\int_X c(x,f(x))dx - \int_X c(x,k(x))dx$$

and we have

$$E(f) - E(k) \leq z[C(f) - C(k)]. \qquad (55)$$

Since $C(k) = b$ and $C(f) \leq b$ we have

$$E(f) - E(k) \leq 0$$

which is the desired result.

However, when the constraint is inactive ($C(k)<b$) for a solution to the Neyman-Pearson problem, there is no guarantee

that $z[C(f)-C(k)] \leq 0$ since there is a possibility that $C(f) > C(k)$ for some feasible solution f. Therefore, the complementary slackness condition $z[C(k)-b] = 0$ is needed to "fix up" Wagner's sufficiency conditions for the dual problem.

Adding and subtracting bz on the right hand side of (55) gives

$$\begin{aligned} E(f) - E(k) &\leq z[C(f)-b + b - C(k)] \\ &\leq z[C(f)-b] - z[C(k)-b]. \end{aligned}$$

The term $z[C(f)-b]$ is nonpositive and the addition of the complementary slackness condition $z[C(k)-b] = 0$ yields $E(f) - E(k) \leq 0$ which implies $k(\cdot)$ is a solution to the dual problem.

Including the complementary slackness condition and deleting the requirement that $C(k)=b$ in Property 4 yields sufficient conditions for the solution to the dual problem, which are stated formally in the following result.

<u>Result 8.</u> - Suppose $k(\cdot) \in P(b)$ has the following property: there exists a $z \geq 0$ such that for all $x \in X$, (except on a set of measure zero)

$$D_2e(x,y) < zD_2c(x,y) \text{ whenever } k(x)<y<u(x), \tag{56}$$

$$D_2e(x,y) > zD_2c(x,y) \text{ whenever } t(x)<y<k(x), \tag{57}$$

$$z[C(k) - b] = 0. \tag{58}$$

Then $k(\cdot)$ is a solution to the dual problem.

Since Wagner's conditions have been changed to be sufficient for a solution to the dual problem it is appropriate to show Result 2 and Result 8 are equivalent. Since (5) and (58) are identical we can show any feasible solution that satisfies the properties of Result 2 also satisfies the properties of Result 8 by proving the following property.

<u>Property 5:</u> Let $k(\cdot)$ be in $P(b)$ and $z \geq 0$. If

$$k(x) = t(x) \quad \text{a.e. on } S_1(z), \tag{59}$$

$$k(x) = u(x) \quad \text{a.e. on } S_2(z), \tag{60}$$

$$k(x) = g(x;z) \text{ a.e. on } S_3(z) \tag{61}$$

then $k(\cdot)$ satisfies

$$D_2h(x,y;z) < 0 \qquad \text{when } y \in (k(x),u(x)), \tag{62}$$

$$D_2h(x,y;z) > 0 \qquad \text{when } y \in (t(x),k(x)), \tag{63}$$

for all $x \in X$ except on a set of measure zero.

<u>Proof:</u> Let x be any point in $S_1(z)$ such that $k(x) = t(x)$. Then $D_2h(x,y;z) < 0$ for all $y \in [t(x),u(x)]$ which implies $D_2h(x,y;z) < 0$ for $y \in (k(x),u(x))$ so (62) is satisfied. Since $k(x) = t(x)$ the interval $(t(x),k(x))$ is empty which implies (63) hold vacuously.

Let x be any point in $S_2(z)$ such that $k(x) = u(x)$. Then $D_2h(x,y;z) > 0$ for all $y \in [t(x),u(x)]$ which implies $D_2h(x,y;z) > 0$ for $y \in (t(x),k(x))$. Also, $(k(x),u(x))$ is the empty set so (62) and (63) are satisfied.

Now let x be any point in $S_3(z)$ such that $k(x) = g(x;z)$. Then, the strict concavity of $h(\cdot)$ with respect to y and the definition of $S_3(z)$ implies

$$D_2h(x,y;z) < 0 \text{ for all } y \in (g(x;z),u(x)]$$

$$D_2h(x,y;z) > 0 \text{ for all } y \in (t(x),g(x;z)].$$

Hence, (62) and (63) are satisfied.

All possible $x \in X$ except possibly a set of measure zero has been checked. Hence, Property 5 is proven and we can conclude solutions satisfying the conditions of Result 2 will also satisfy Result 8.

Suppose there are feasible solutions which satisfy the properties of Result 8 but do not satisfy those of Result 2. Then there is a solution which satisfies the properties of Result 8 but not those of Result 2. This is not possible if there is a feasible solution, say f_1, to the dual problem of this dissertation such that $C(f_1)-b<0$ for then the properties of Result 2 are also necessary for a solution to dual problem.

Therefore, the author concludes Wagner's sufficiency conditions for a solution to the dual problem and Result 2 are equivalent when the regularity assumption $C(f) < b$ is satisfied for some feasible solution to the dual problem.

3. Alternate Necessary and Sufficient Conditions for Solutions to the Primal and Dual Problems

As in the paper by Francis and Wright [18], there are alternate necessary and sufficient conditions for solutions to the primal and dual problems. The sufficient conditions developed by the author are given in this section.

Sufficient Conditions for Solutions to the Primal and Dual Problem

Result 9. - Suppose there exists a $z^0 \in E_m^+$ such that

$$\int_{S_1(z^0)} c_i(x,t(x))dx + \int_{S_2(z^0)} c_i(x,u(x))dx + \int_{S_3(z^0)} c_i(x,g(x;z_0))dx \le b_i, i=1,\dots,m \tag{64}$$

and

$$\sum_{i=1}^{m} z_i^0 \Big[\int_{S_1(z^0)} c_i(x,t(x))dx + \int_{S_2(z^0)} c_i(x,u(x))dx + \int_{S_3(z^0)} c_i(x,g(x;z_0))dx - b_i\Big] = 0. \quad (65)$$

Define the function $f_0(\cdot)$ on X as follows:

$$f_0(x) = t(x), \quad \text{a.e. on } S_1(z^0);$$

$$f_0(x) = u(x), \quad \text{a.e. on } S_2(z^0);$$

$$f_0(x) = g(x;z_0), \text{ a.e. on } S_3(z^0).$$

Then z^0 is a solution to the primal problem, $f_0(\cdot)$ is a solution to the dual problem, and $E(f_0) = F(z^0)$.

<u>Proof.</u> - First it must be shown that $f_0(\cdot)$ is a feasible solution to the dual problem. By the definition of $f_0(\cdot)$, $t(x) \leq f_0(x) \leq u(x)$ for all $x \in X$. By the definition of $f_0(\cdot)$ and (64) we have

$$\int_X c_i(x,f_0(x))dx = \int_{S_1(z^0)} c_i(x,t(x))dx + \int_{S_2(z^0)} c_i(x,u(x))dx + \int_{S_3(z^0)} c_i(x,g(x;z^0))dx$$

$$\leq b_i, \; i=1,\dots,m$$

Therefore, $f_0(\cdot)$ is a feasible solution.

Now if we show $E(f_0) = F(z^0)$ the conclusion will follow from Result 1. By (64) and the definition of $f_0(\cdot)$ we can write

$$E(f_0) = \int_X e(x,f_0(x))dx$$

$$= \int_X e(x,f_0(x))dx - \sum_{i=1}^{m} z_i^0 [\int_{S_1(z^0)} c_i(x,t(x))dx + \int_{S_2(z^0)} c_i(x,u(x))dx$$

$$+ \int_{S_3(z^0)} c_i(x,g(x;z^0))dx - b_i]$$

$$= \int_{S_1(z^0)} e(x,t(x) - z^0 c(x,f(x))dx + \int_{S_2(z^0)} [e(x,u(x)) - z^0 c(x,u(x))]dx$$

$$+ \int_{S_3(z^0)} [e(x,g(x;z_0)) - z^0 c(x,g(x;z^0)]dx + b^T z^0$$

$$= \int_{S_1(z^0)} h(x,t(x);z^0)dx + \int_{S_2(z^0)} h(x,u(x);z^0)dx + \int_{S_3(z^0)} h(x,g(x;z^0);z^0)dx$$

$$+ b^T z^0$$

$$= F(z^0)$$

which is the desired result.

Necessary Conditions for a Solution to the Primal Problem

To prove the sufficient conditions in Result 9 are necessary, we need to know the function $F(z)$ is convex. By redefining $F(z)$ in an equivalent form the convexity can be shown quite easily. The redefinition of $F(z)$ is based on the global and local optimization theory as presented in Luenberger's text [35].

<u>Property 6.</u> - The objective function in the primal problem is $F(z) = \sup_{f \in W} [E(f) - zD(f)]$.

<u>Proof.</u> - In Result 1 the proof of $E(f) \leq F(z)$ also showed

$E(f) - zD(f) \leq F(z)$ for all $f \in W$ and $z \in E_m^+$. Therefore,

$$\sup_{f \in W} [E(f) - zD(f)] \leq F(z). \tag{66}$$

If we let

$$\bar{f}(x) = t(x) \quad \text{a.e. on } S_1(z),$$

$$\bar{f}(x) = u(x) \quad \text{a.e. on } S_2(z),$$

$$\bar{f}(x) = g(x;z) \quad \text{a.e. on } S_3(z),$$

then $\bar{f}(\cdot) \in W$ and

$$F(z) = \int_{S_1(z)} h(x,t(x);z)dx + \int_{S_2(z)} h(x,u(x);z)dx + \int_{S_3(z)} h(x,g(x;z);z)dx$$

$$+ b^T z$$

$$= \int_X h(x,\bar{f}(x);z)dx + b^T z$$

$$= \int_X e(x,\bar{f}(x))dx - z[\int_X c(x,\bar{f}(x))dx - b]$$

$$= E(\bar{f}) - zD(\bar{f}) \leq \sup_{f \in W} [E(f) - zD(f)] \tag{67}$$

Combining (66) and (67) yields the desired equality.

With this property we can now prove the convexity of the objective function $F(z)$.

<u>Property 7.</u> - $F(z) = \sup_{f \in W} [E(f) - zD(f)]$ is a convex function of z on E_m^+.

<u>Proof</u>. - Let $0 \leq a \leq 1$ and z_1, $z_2 \in E_m^+$ then

$$F(az_1 + (1-a)z_2) = \sup_{f \in W}[E(f) - (az_1 + (1-a)z_2)D(f)]$$
$$= \sup_{f \in W}[aE(f)+(1-a)E(f) - (az_1+(1-a)z_2)D(f)]$$
$$= \sup_{f \in W}\{a[E(f)-z_1D(f)] + (1-a)[E(f)-z_2D(f)]\}$$
$$\leq a \sup_{f \in W}[E(f)-z_1D(f)] + (1-a)\sup_{f \in W}[E(f)-z_2D(f)]$$
$$= a\,F(z_1) + (1-a)\,F(z_2)$$

which is the desired result.

Assume $F(z)$ is a differentiable function and denote the partial derivative with respect to z_i as $D_iF(z)$ for $i=1,\ldots,m$. Also, the gradient of $F(z)$ will be denoted by $\nabla F(z)$ where $\nabla F(z) = (D_1F(z), D_2F(z),\ldots,D_mF(z))^T$. The Kuhn-Tucker conditions [31] state that $z\nabla F(z) = 0$ is a necessary condition for z to be a solution to the primal problem.

Therefore, we only have to show the form of $D_iF(z)$, $i=1,\ldots,m$ to state the necessary conditions for a solution to the primal problem. To develop the expression for $D_iF(z)$, $i=1,\ldots,m$ we need the expression $F(z_0) + F'(z_0)(z-z_0)$ where $F'(z_0)(z-z_0)$ is the inner product of $F'(z_0)$ and $(z-z_0)$. The function $F'(z_0)$ we define as

$$F'(z_0) = b - \int_{S_1(z_0)} c(x,t(x))dx - \int_{S_2(z_0)} c(x,u(x))dx - \int_{S_3(z_0)} c(x,g(x;z_0))dx \quad (68)$$

where z_0 is an arbitrary point in E_m^+.

Writing out $F(z_0) + F'(z_0)(z-z_0)$ in terms of $h(\cdot)$, z, and b we have

$$
\begin{aligned}
&F(z_0) + F'(z_0)(z-z_0) \\
&\quad = b^T z_0 + \int_{S_1(z_0)} h(x,t(x);z_0)dx + \int_{S_2(z_0)} h(x,u(x);z_0)dx \\
&\quad + \int_{S_3(z_0)} h(x,g(x;z_0);z_0)dx + b^T z - b^T z_0 \\
&\quad - z\int_{S_1(z_0)} c(x,t(x))dx - z\int_{S_2(z_0)} c(x,u(x))dx - z\int_{S_3(z_0)} c(x,g(x;z_0))dx \\
&\quad + z_0\int_{S_1(z_0)} c(x,t(x))dx + z_0\int_{S_2(z_0)} c(x,u(x))dx + z_0\int_{S_3(z_0)} c(x,g(x;z_0))dx \\
&\quad = b^T z + \int_{S_1(z_0)} e(x,t(x))dx + \int_{S_2(z_0)} e(x,u(x))dx + \int_{S_3(z_0)} e(x,g(x;z_0))dx \\
&\quad - z\int_{S_1(z_0)} c(x,t(x))dx - z\int_{S_2(z_0)} c(x,u(x))dx - z\int_{S_3(z_0)} c(x,g(x;z_0))dx \\
&\quad = b^T z + \int_{S_1(z_0)} [e(x,t(x)) - zc(x,t(x))]dx \\
&\quad + \int_{S_2(z_0)} [e(x,u(x)) - zc(x,u(x))]dx \\
&\quad + \int_{S_3(z_0)} e(x,g(x;z_0)) - zc(x,g(x;z_0))dx
\end{aligned}
$$

Summarizing in terms of $h(\cdot)$ yields

$$
\begin{aligned}
&F(z_0) + F'(z_0)(z-z_0) \\
&= b^T z + \int_{S_1(z_0)} h(x,t(x);z)dx + \int_{S_2(z_0)} h(x,u(x);z)dx \\
&+ \int_{S_3(z_0)} h(x,g(x;z_0);z)dx
\end{aligned}
$$

Add and subtract $\int_{S_1(z_0)} h(x,u(x);z)dx + \int_{S_3(z_0)} h(x,u(x);z)dx$ from the preceeding expression and we have

$$F(z_0) + F'(z_0)(z-z_0) = b^T z + \int_{S_1(z_0)} [h(x,t(x);z) - h(x,u(x);z)]dx$$

$$+ \int_{S_3(z_0)} [h(x,g(x;z_0);z) - h(x,u(x);z)]dx$$

$$+ \int_X h(x,u(x);z)dx \qquad (69)$$

Now we can also change the form of $F(z)$ by adding and subtracting $\int_{S_1(z)} h(x,u(x);z)dx + \int_{S_3(z)} h(x,u(x);z)dx$, to give

$$F(z) = \int_{S_1(z)} [h(x,t(x);z)-h(x,u(x);z)]dx$$

$$+ \int_{S_3(z)} [h(x,g(x;z);z)-h(x,u(x);z)]dx$$

$$+ \int_X h(x,u(x);z)\,dx + b^T z. \qquad (70)$$

We are now in a position to prove the form of $\nabla F(z)$.

<u>Result 10.</u> - The gradient of $F(\cdot)$ is

$$\nabla F(z) = b - \int_{S_1(z)} c(x,t(x))dx - \int_{S_2(z)} c(x,u(x))dx - \int_{S_3(z)} c(x,g(x;z))dx \qquad (71)$$

<u>Proof.</u> - Rockafellar [43] defines a subgradient, for a finite, convex function $F(\cdot)$, at a point z_0 as a point in E_m, say $F'(z_0)$, such that

$$F(z) \geq F(z_0) + F'(z_0)(z-z_0) \text{ for all } z \in E_m.$$

Rockafellar also states the result that the subgradient (here defined in (68)) is the usual gradient for the function $F(\cdot)$ when it is differentiable. Thus, we only have to show $F(z)-F(z_0)-F'(z_0)(z-z_0) \geq 0$ for all $z \in E_m$ to prove (71) holds.

Define $H(z) = F(z) - F(z_0) - F'(z_0)(z-z_0)$: then we have to show $H(z) \geq 0$ for all $z \in E_m$.

From (69) and (70) we have

$$
\begin{aligned}
H(z) = & \int_{S_1(z)} [h(x,t(x);z)-h(x,u(x);z)]dx \\
& - \int_{S_1(z_0)} [h(x,t(x);z)-h(x,u(x);z)]dx \\
& + \int_{S_3(z)} [h(x,g(x;z);z) - h(x,u(x);z)]dx \\
& - \int_{S_3(z_0)} [h(x,g(x;z_0);z)- h(x,u(x);z)]dx
\end{aligned}
\tag{72}
$$

Since $S_i(z) \cap S_i(z_0)$, $S_i(z) \cap S_i(z_0)^c$, and $S_i(z)^c \cap S_i(z_0)$ are disjoint for $i=1,3$ we can rewrite (72) in the following manner:

$$
\begin{aligned}
H(z) \\
= & \int_{S_1(z) \cap S_1(z_0)^c} [h(x,t(x);z)-h(x,u(x);z)]dx + \int_{S_1(z) \cap S_1(z_0)} [h(x,t(x);z)-h(x,u(x);z)]dx \\
& - \int_{S_1(z) \cap S_1(z_0)} [h(x,t(x);z)-h(x,u(x),z)]dx - \int_{S_1(z)^c \cap S_1(z_0)} [h(x,t(x);z)-h(x,u(x);z)]dx \\
& + \int_{S_3(z) \cap S_3(z_0)^c} [h(x,g(x;z);z)-h(x,u(x);z)]dx + \int_{S_3(z) \cap S_3(z_0)} [h(x,g(x;z);z)-h(x,u(x);z)]dx \\
& - \int_{S_3(z) \cap S_3(z_0)} [h(x,g(x;z_0);z)-h(x,u(x);z)]dx - \int_{S_3(z)^c \cap S_3(z_0)} [h(x,g(x;z_0);z)-h(x,u(x);z)]dx
\end{aligned}
$$

This equation reduces to:

$$H(z)$$

$$= \int_{S_1(z)\cap S_1(z_0)^c}[h(x,t(x);z)-h(x,u(x);z)]dx - \int_{S_1(z)^c\cap S_1(z_0)}[h(x,t(x);z)-h(x,u(x);z)]dx$$

$$+\int_{S_3(z)\cap S_3(z_0)^c}[h(x,g(x;z);z)-h(x,u(x);z)]dx + \int_{S_3(z)\cap S_3(z_0)}[h(x,g(x;z);z)-h(x,g(x;z_0);z)]dx$$

$$- \int_{S_3(z)^c\cap S_3(z_0)}[h(x,g(x;z_0);z) - h(x,u(x);z)]dx \tag{73}$$

Since $S_1(z)$, $S_2(z)$, and $S_3(z)$ are disjoint for each $z \in E_m$ this implies

$$\begin{aligned} S_1(z_0)^c &= S_2(z_0) \cup S_3(z_0) \\ S_1(z)^c &= S_2(z) \cup S_3(z) \\ S_3(z_0)^c &= S_1(z_0) \cup S_2(z_0) \\ S_3(z) &= S_3(z) \cup S_2(z). \end{aligned} \tag{74}$$

Note that every integral in (73) involving $S_i(\cdot)^c$ can be decomposed into two integrals using equalities in (74). Thus, a direct computation establishes that $H(z) = \sum_{i=1}^{7} H_i(z)$ where

$$H_1(z) = \int_{S_1(z)\cap S_2(z_0)}[h(x,t(x);z) - h(x,u(x);z)]dx,$$

$$H_2(z) = -\int_{S_2(z)\cap S_1(z_0)}[h(x,t(x);z) - h(x,u(x);z)]dx,$$

$$H_3(z) = \int_{S_3(z)\cap S_2(z_0)}[h(x,g(x;z);z) - h(x,u(x);z)dx,$$

$$H_4(z) = \int_{S_3(z)\cap S_3(z_0)}[h(x,g(x;z);z) - h(x,g(x;z_0);z)]dx,$$

$$H_5(z) = \int_{S_1(z)\cap S_3(z_0)}[h(x,t(x);z)-h(x,u(x);z)]dx$$

$$- \int_{S_1(z)\cap S_3(z_0)}[h(x,g(x;z_0);z) - h(x,u(x);z)]dx$$

$$= \int_{S_1(z)\cap S_3(z_0)}[h(x,t(x);z) - h(x,g(x;z_0);z)]dx,$$

$$H_6(z) = \int_{S_3(z)\cap S_1(z_0)}[h(x,g(x;z);z) - h(x,u(x);z)]dx$$

$$- \int_{S_3(z)\cap S_1(z_0)}[h(x,t(x);z) - h(x,u(x);z)]dx$$

$$= \int_{S_3(z)\cap S_1(z_0)}[h(x,g(x;z);z) - h(x,t(x);z)]dx,$$

and

$$H_7(z) = -\int_{S_2(z)\cap S_3(z_0)}[h(x,g(x;z_0);z) - h(x,u(x);z)]dx$$

$$= \int_{S_2(z)\cap S_3(z_0)}[h(x,u(x);z) - h(x,g(x;z_0);z)]dx.$$

To show $H(z) \geq 0$ we only have to show $H_i(z) \geq 0$, $i=1, \ldots, 7$.

From the definition of $S_1(z)$ and the typical sketch in Figure 1 it can be seen that $h(\cdot)$ is a decreasing function of the n^{th} variable for $x \in S_1(z)$. Therefore, $h(x,t(x);z) - h(x,u(x);z) \geq 0$ for all $x \in S_1(z)$ and, in particular, for $x \in S_1(z)\cap S_2(z_0)$ which implies $H_1(z) \geq 0$. Also, $h(x,t(x);z) - h(x,g(x;z_0);z) \geq 0$ for all $x \in S_1(z)\cap S_3(z_0)$ which implies $H_5(z) \geq 0$.

The definition of $S_2(z)$ and Figure 2 shows $h(\cdot)$ is an increasing function of the n^{th} variable for $x \in S_2(z)$. Therefore, $h(x,u(x);z) - h(x,t(x);z) \geq 0$ for $x \in S_2(z)\cap S_1(z_0)$

which implies $H_2(z) \geq 0$. Also, $h(x,u(x);z) - h(x,g(x;z_0);z) \geq 0$ for $x \in S_2(z) \cap S_3(z_0)$ which implies $H_7(z) \geq 0$.

From Figure 3 and the definition that $h(x,g(x;z);z) = \sup \{h(x,y;z) : y \in [t(x),u(x)]\}$ for all $x \in S_3(z)$ we have

$$h(x,g(x;z);z)-h(x,u(x);z) \geq 0 \text{ for } x \in S_3(z) \cap S_2(z_0), \tag{75}$$

$$h(x,g(x;z);z)-h(x,g(x;z_0);z) \geq \text{for } x \in S_3(z) \cap S_3(z_0), \tag{76}$$

and

$$h(x,g(x;z);z)-h(x,t(x);z) \geq 0 \text{ for } x \in S_3(z) \cap S_1(z_0). \tag{77}$$

The inequalities (75), (76) and (77) imply $H_3(z) \geq 0$, $H_4(z) \geq 0$, and $H_6(z) \geq 0$, respectively.

Therefore, $H(z) \geq 0$ which means $F'(z_0)$ is the subgradient of F evaluated at z_0 and (71) holds.

In Result 10 the form of $\nabla F(z)$ was determined so we can state the necessary conditions for a solution to the primal problem when F(z) is differentiable with respect to $z_i, i=1,\dots,m$.

<u>Result 11.</u> - If z^0 is a solution to the primal problem then

$$\nabla F(z^0) = [b - \int_{S_1(z^0)} c(x,t(x))dx - \int_{S_2(z^0)} c(x,u(x))dx - \int_{S_3(z^0)} c(x,g(x;z^0))dx] \geq \bar{0}$$

and

$$z^0 \quad F(z^0) = 0.$$

<u>Proof.</u> - Using the Kuhn-Tucker Conditions [31] for differentiable functions it is clear that $\nabla F(z^0) \geq \bar{0}$ and $z^0 \nabla F(z^0) = 0$ which is the stated result.

Combining Result 9 and Result 11 yields necessary and sufficient conditions for a solution to the primal problem.

Result 12. - The point $z^0 \epsilon E_m^+$ is a solution to the primal problem if, and only if

$$\nabla F(z^0) \geq \overline{0} \tag{78}$$

$$z^0 \; \nabla F(z^0) = 0. \tag{79}$$

This result then gives the interesting result that finding a solution z^0 satisfying (78) and (79) implies (from the proof of Result 9) that we can define

$$\begin{aligned} f_0(x) &= t(x) \quad \text{a.e. on } S_1(z^0), \\ f_0(x) &= u(x) \quad \text{a.e. on } S_2(z^0), \\ f_0(x) &= g(x;z) \ \text{a.e. on } S_3(z^0). \end{aligned} \tag{80}$$

and then $f_0(\cdot)$ is a solution to the dual problem. Therefore, we have the following result relating the primal and dual problems.

Result 13. - If z^0 is a solution to the primal problem then $f_0(\cdot)$, defined by (80), is a solution to the dual problem and $E(f_0) = F(z^0)$.

From the necessary and sufficient conditions for a solution to the dual problem given in Result 4 and the necessary and sufficient conditions for a solution to the primal problem as stated in Result 12, it can be seen that we can determine the solutions to the primal or the dual problem using either set of conditions. This can be summarized as follows.

<u>Result 14.</u> - The primal problem has a solution $z \in E_m^+$ if, and only if, the dual problem has a solution $f(\cdot)$, and $E(f) = F(z)$.

This result generates the question as to when a solution does exist for the primal problem or dual problem so that we know a solution exists for each problem. Therefore, the following section considers the existence of a solution to the Neyman-Pearson (Dual) Problem.

3.6 Existence of a Solution to the Dual Problem

To prove the existence of a solution we will show the set of feasible solutions is compact when the domain T for each of the feasible solutions is a finite measurable subset of E_{n-1}. The domain T must be a set of finite measure since it is required that $\int_T dx$ be finite in Properties 9 and 10. Previous results required only a measurable set (defined as X). This restriction is not too severe since this just guarantees $\int_T c(x,f(x))dx$ is finite for all functions f in the set of feasible solutions. For example, any closed, bounded interval in E_1 is a set of finite measure.

It will be assumed that $|D_n c_i(x,y)| \leq M$ for $i=1,\ldots,m$ and $|D_n e(x,y)| \leq N$, for all $x \in T$, i.e., the partial derivatives of $c_i(x,\cdot)$ and $e(x,\cdot)$ not only exist with respect to y but are bounded.

It will be shown that $E(\cdot)$ is an upper semicontinuous functional and the Weierstrass theorem, given in Appendix A, guarantees a solution exists to the dual problem.

The proof of existence is rather lengthy so the proof of the existence theorem is divided into several properties and then summarized after the proof is completed.

Property 8. - The set $\overline{W} = \{f : t(x) \leq f(x) \leq u(x), x \in T\} = \prod_{x \in T} [t(x)u(x)]$ is a compact set.

Proof. - The definition of $\overline{W} = \prod_{x \in T} [t(x), u(x)]$ is discussed in Appendix G.

Since $t(\cdot)$ and $u(\cdot)$ are bounded, real-valued functions, the intervals $[t(x), u(x)]$ are compact for each $x \in T$. The Tychonoff Theorem, as given in Kelley [25] and other topology texts, states the Cartesian product of compact sets is compact. Therefore, $\overline{W} = \prod_{x \in T} [t(x), u(x)]$ is a compact set.

Property 9. - The set of feasible solutions $\overline{P}(b) = \{f \in \overline{W} : \int_T c_i(x, f(x)dx \leq b_i,\ i=1,\ldots,m\}$ is compact.

Proof. - To show that the set $\overline{P}(b)$ is compact we only have to show $\overline{P}(b)$ is a closed subset of the compact set $\overline{W}$. Clearly, $\overline{P}(b) \subseteq \overline{W}$ so we only have to show that $\overline{P}(b)$ is closed.

Let $\{f_\delta\}_{\delta \in \Delta}$ be a convergent net (definition of net in Appendix H) in $\overline{P}(b)$. Then $f_\delta \rightarrow f_0$ where $f_0 \in \overline{W}$ since $\overline{W}$ is compact. To show closure we must prove $f_0 \in \overline{P}(b)$.

Since $\{f_\delta\}_{\delta \in \Delta}$ is a convergent net in $\overline{W}$ we can choose an arbitrary $\epsilon > 0$ so that there exists a $\delta_0 \in \Delta$ such that $\delta \geq \delta_0$ implies

$$| f_\delta(x) - f_0(x) | \leq \frac{\epsilon}{M}, \text{ for each } x \in T.$$

Since $f_\delta \in \overline{P}(b)$ for all $\delta \in \Delta$ we have

$$\int_T c_i(x,f_\delta(x))dx \leq b_i,\ i=1,\ldots,m,\ \delta \in \Delta \quad ,$$

which implies

$$\int_T [c_i(x,f_\delta(x))-c_i(x,f_0(x)) + c_i(x,f_0(x))]\,dx \leq b_i, i=1,\ldots,m$$

and this is equivalent to

$$\int_T c_i(x,f_0(x))dx \leq b_i + \int_T [c_i(x,f_0(x))-c_i(x,f_\delta(x))]dx, i=1,\ldots,m$$

$$= b_i + \int_T [\int_{f_\delta(x)}^{f_0(x)} D_n c(x,y)dy]dx, i=1,\ldots,m.$$

Since it is assumed that $|D_n c_i(x,y)| \leq M$ for all $x \in T$ and $i=1,\ldots,m$, it follows that

$$\int_T c_i(x,f_0(x))dx \leq b_i + \int_T M(f_0(x)-f_\delta(x))dx, i=1,\ldots,m$$

$$\leq b_i + \int_T M(\tfrac{\epsilon}{M})dx,\ i=1,\ldots,m \text{ and } \delta \geq \delta_0$$

$$= b_i + \int_T \epsilon\ dx,\ i=1,\ldots,m \text{ and } \delta \geq \delta_0.$$

The positive number ϵ can be made arbitrarily small which implies

$$\int_T c_i(x,f_0(x))dx \leq b_i,\ i=1,\ldots,m.$$

Now f_0 satisfies the integral constraints for f_0 to be an element of $\overline{P}(b)$, but the question of whether or not $\int_T c_i(x,f_0(x))dx > -\infty$, $i=1,\ldots,m$ must also be considered since it is assumed that $c_i(x,\cdot)$ is Lesbegue integrable with respect to x.

For $\delta \geq \delta_0$ we have $|f_\delta(x)-f_0(x)| \leq \frac{\epsilon}{M}$ for each $x \in T$. Therefore,

$$\left|\int_T c_i(x,f_\delta(x)) - \int_T c_i(x,f_0(x))dx\right|$$
$$\leq \int_T \left|c_i(x,f_\delta(x)) - c_i(x,f_0(x))\right| dx, \quad i=1,\ldots,m$$
$$= \int_T \left|\int_{f_0(x)}^{f_\delta(x)} D_n c_i(x,y)dy\right| dx \quad , \quad i=1,\ldots,m$$
$$\leq \int_T M\left|f_\delta(x) - f_0(x)\right| dx \quad , \quad i=1,\ldots,m$$
$$\leq \int_T \epsilon \, dx \quad , \quad i=1,\ldots,m$$

which implies $\int_T c_i(x,f_0(x))dx > -\infty$, $i=1,\ldots,m$ since these integrals differ by no more than $\int_T \epsilon\, dx$ from $\int_T c_i(x,f_\delta(x))dx$, $i=1,\ldots,m$ and $\delta \geq \delta_0$. Thus,

$$-\infty < \int_T c_i(x,f_0(x))dx \leq b_i, \quad i=1,\ldots,m$$

which implies $f_0 \in \overline{P}(b)$; hence $\overline{P}(b)$ is closed.

<u>Property 10.</u> - The concave functional $\overline{E}(\cdot) = \int_T e(x,f(x))dx$ is upper semi-continuous on the compact set $\overline{P}(b)$.

<u>Proof.</u> - One of the forms of the definition of upper semi-continuity is as follows: $\overline{E}(\cdot)$ is upper semi-continuous on $\overline{P}(b)$ if $Q(r) = \{ f \in \overline{P}(b) : \overline{E}(f) \geq r,\ r \text{ is a real number} \}$ is a closed set for every real number r.

To show the closure of Q(r), let $\{f_\delta\}_{\delta \in \Delta}$ be a convergent net in Q(r); then we can choose $\epsilon > 0$ so that there exists a $\delta \geq \delta_0$ which implies

$$|f_\delta(x) - f_0(x)| \leq \epsilon \text{ for each } x \in T.$$

Since $f_\delta \in Q(r)$ for all $\delta \in \Delta$ this implies

$$\overline{E}(f_\delta) \geq r \text{ for all } \delta \in \Delta .$$

Therefore, $\overline{E}(f_\delta) - \overline{E}(\delta_0) + \overline{E}(f_0) \geq r$ which implies

$$\overline{E}(f_0) \geq r + \overline{E}(f_0) - E(f_\delta).$$

Proceeding in the same manner as the proof in Property 9 we have

$$\overline{E}(f_0) = \int_T e(x, f_0(x))dx \geq r - \int_T \epsilon N \, dx$$

where $N \geq |D_n e(x,y)|$ for all $x \in T$ since we have assumed that $D_n e(x,y)$ is bounded. Since ϵ is arbitrary, $\overline{E}(f_0) \geq r$ which implies $f_0 \in Q(r)$; hence $Q(r)$ is closed.

Now that we have shown that $\overline{E}(\cdot)$ is upper semi-continuous on $\overline{P}(b)$ we are in a position to formally state the property for existence of a solution to the dual problem.

Result 15.- The maximum value of $\overline{E}(\cdot)$ exists for some f $\overline{P}(b)$.

Proof. - Since $\overline{P}(b)$ is compact and $\overline{E}(\cdot)$ is upper semi-continuous for all $f \in \overline{P}(b)$ the Weierstrass theorem stated in Appendix A guarantees the existence of a solution.

Thus, we can conclude that when the domain X (of the set of feasible solutions P(b)) has finite measure, a solution to the Neyman-Pearson problem exists. Of course, $e(x,\cdot)$ and $c(x,\cdot)$ must have bounded derivatives with respect to the n^{th} variable and also $e(x,f(x))$ and $c(x,f(x))$ are Lebesgue integrable with respect to x for all $f \in W$.

Result 4 gives necessary and sufficient conditions for a solution to the Neyman-Pearson problem while Result 12 gives necessary and sufficient conditions for a solution to the primal problem. Either result can be used to develop solutions to both the primal and dual problems. The technique used will depend on the particular problem, as shown in the example in Chapter V.

A complete summary of results developed by the author is stated in Chapter VII.

CHAPTER IV

DISCRETE VERSION OF ANALYSIS

4.1 Introduction

This chapter includes the statement of the primal and dual problems for a discrete version of the Neyman-Pearson Problem. All the properties are stated but only the existence theorem is proved since the proofs of Chapter III follow in a direct manner with the integral symbol replaced by the summation symbol. The proof of the existence is given since it is more straightforward for the discrete version.

4.2 Assumptions, Definitions, and Notation

Only those definitions of Chapter III that are changed due to the consideration of the discrete version will be stated.

Let t and u be real-valued functions (or vectors) with the domain $J = \{j: j = 1,2,\ldots,n\}$, such that $t_j \leq u_j$ for all $j \in J$. The set of real-valued functions defined on J and bounded by t and u will be denoted by $W=\{f: t_j \leq f_j \leq u_j$, for all $j \in J\}$ which is clearly a convex set.

Let the real-valued functions $e(j,y)$ and $c_i(j,y)$,

$i=1,\ldots,m$ be given and defined on the set $\{(j,y): j \in J,$ $t_j \le y \le u_j\}$. For each $j \in J$, assume $e(j,y)$ is strictly concave in y and $c_i(j,y)$ is convex in y for $i=1,\ldots,m$. Assume $e(j,y)$ and $c_i(j,y)$, $i=1,\ldots,m$ are continuous with respect to y.

As in Chapter III we will define a function h as follows:

$$h(j,y;z) = e(j,y) - \sum_{i=1}^{m} z_i c_i(j,y) = e(j,y;z) - zc(j,y)$$

where $z \in E_m^+$.

For every vector $z \in E_m^+$, we define the following sets:

$$S_1(z) = \{j \in J : h(j,y';z) > h(j,y'';z), t_j \le y' < y'' \le u_j\};$$

$$S_2(z) = \{j \in J : h(j,y';z) < h(j,y'';z), t_j \le y' < y'' \le u_j\};$$

and

$$S_3(z) = \{j \in J : h(j,g(j;z);z) > h(j,y;z), y \in [t_j,u_j],$$
$$y \ne g(j;z), \text{ and } h(j,y;z) \text{ is not strictly monotonic in } y\}.$$

The point $g(j;z) \in [t_j;u_j]$ is defined in an analogous manner to $g(x;z)$ of Chapter III. The following sketches are similar to Figure 1, 2, and 3 but it is not required that $h(\cdot)$ be differentiable everywhere in y for the discrete version.

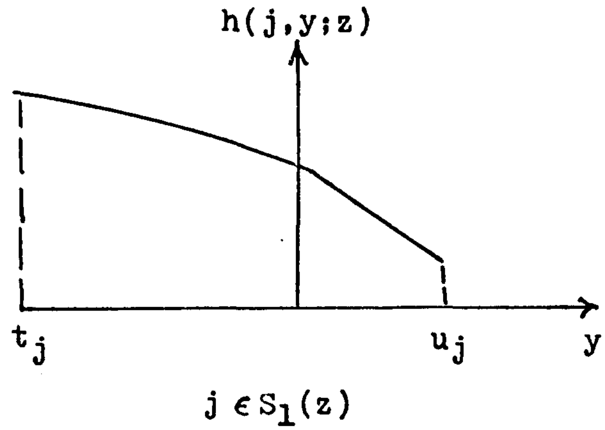


Figure 5

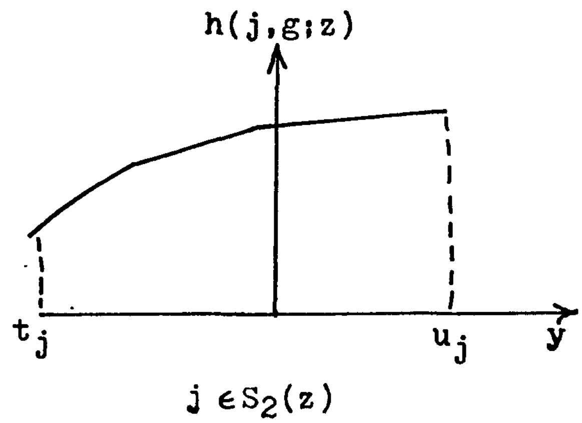


Figure 6

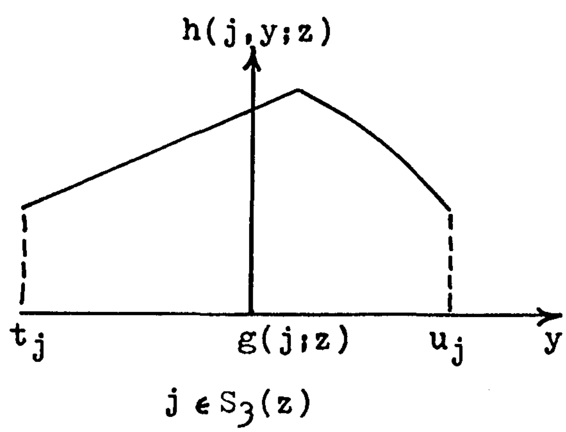


Figure 7

4.3 Statement of Primal and Dual Problems

Based on the work of Luenberger, Francis and Wright, Wagner, and the development of F(z) in Chapter III the following primal problem may be stated.

Primal Problem

$$\underset{z \in E_m^+}{\text{Minimize}}\ F(z) = \sum_{j \in S_1(z)} h(j,t_j;z) + \sum_{j \in S_2(z)} h(e_j,u_j;z) + \sum_{j \in S_3(z)} h(j,g(j;z);z) + b^T z$$

Dual Problem (Discrete Neyman-Pearson Problem)

$$\text{Maximize } E(f) = \sum_{j=1}^{n} e(j,f_j)$$

$$\text{subject to: } C_i(f) = \sum_{j=1}^{n} c_i(j,f_j) \leq b_i, i=1,\ldots,m$$

$$\text{or } C(f) = \sum_{j=1}^{n} c(j,f_j) \leq b$$

and $t_j \leq f_j \leq u_j$ for all $j \in J$

or $f \in W$.

As in Chapter III we define the set of feasible solutions $P(b) = \{f \in W : C(f) \leq b\}$.

If the non-linear discrete version of the Neyman-Pearson Problem is changed to a linear discrete version, which is then equivalent to a discrete version of the problem in [18], the dual problem becomes

$$\text{Maximize } E(f) = \sum_{j=1}^{n} e_j f_j$$

$$\text{subject to: } \sum_{j=1}^{n} c_{ij} f_j \leq b_i,\ i=1,\ldots,m$$

and $t_j \leq f_j \leq u_j$.

which is the bounded variable problem of linear programming.

If we rewrite $h(\cdot)$ for the linear discrete version we have

$$h(j,y;z) = e_j y - \sum_{i=1}^{m} z_i c_{ij} y$$

$$= y[e_j - \sum_{i=1}^{m} z_i c_{ij}].$$

Thus, $j \in S_1(z)$ when $e_j - \sum_{i=1}^{m} z_j c_{ij} < 0$,

$j \in S_2(z)$ when $e_j - \sum_{i=1}^{m} z_j c_{ij} > 0$,

$j \in S_3(z)$ when $e_j - \sum_{i=1}^{m} z_j c_{ij} = 0$

so the primal problem stated above becomes

$$\underset{z \in E_m^+}{\text{Minimize}}\ F(z) = \sum_{j \in S_1(z) \cup S_3(z)} t_j[e_j - zc_j] + \sum_{j \in S_2(z)} u_j[e_j - zc_j] + b^T z$$

Yoshimura [54] has shown that the latter problem is equivalent to the standard dual of the bounded variable linear programming problem.

Sections 4.4 and 4.5 are statements, without elaboration of properties relating the primal and dual problems. These properties follow directly from their counterparts in Sections 3.2 and 3.5 of Chapter III.

4.4 Properties of the Primal and Dual Problem

Result 16. - For any feasible solution f to the dual problem, and any $z \in E_m^+$, $E(f) \leq F(z)$.

Sufficient Conditions for Solutions to Primal and Dual Problems

Result 17. - If there exists a feasible solution f to the dual problem, a $z \in E_m^+$ such that

$$\sum_{i=1}^{m} z_i [\sum_{j=1}^{n} c_i(j,f_j) - b_i] = 0,$$

and

$$f_j = t_j \qquad \text{on } S_1(z),$$

$$f_j = u_j \qquad \text{on } S_2(z),$$

$$f_j = g(j;z) \qquad \text{on } S_3(z),$$

then z is a solution to the primal problem, f is a solution to the dual problem, and $E(f) = F(z)$.

Necessary Conditions for a Solution to the Dual Problem

Result 18. - If there is at least one $f \in P(b)$ such that $[\sum_{j=1}^{n} c(j,f_j) - b] < \overline{0}$ and if $\overline{f}$ is a solution to the dual problem, then there exists a $\overline{z} \in E_m^+$ such that

$$\sum_{i=1}^{m} \overline{z}_i [\sum_{j=1}^{n} c_i(j,f_j) - b_i] = 0,$$

$$\overline{f}_j = t_j \qquad \text{on } S_1(\overline{z}),$$

$$\overline{f}_j = u_j \qquad \text{on } S_2(\overline{z}),$$

$$\overline{f}_j = g(j;\overline{z}) \text{ on } S_3(\overline{z}).$$

Also, $E(\overline{f}) = F(\overline{z})$.

As in Chapter III the necessary and sufficient conditions can be combined into the following result.

Result 19. - If there is at least one $f \in P(b)$ such that $\sum_{j=1}^{n} c(j,f_j) - b < \bar{0}$ then the feasible solution $\bar{f}$ is a solution to the dual problem if and only if there exists a $\bar{z} \in E_m^+$ such that

$$\sum_{i=1}^{m} \bar{z}_i[\sum_{j=1}^{n} c_i(j,f_j) - b_i] = 0,$$

$$\bar{f}_j = t_j \quad \text{on } S_1(\bar{z}),$$

$$\bar{f}_j = u_j \quad \text{on } S_2(\bar{z}),$$

$$\bar{f}_j = g(j;\bar{z}) \text{ on } S_3(\bar{z}).$$

Also, $E(\bar{f}) = F(\bar{z})$.

Note that $\bar{z}$ is a solution to the primal by Result 16.

4.5 Alternate Necessary and Sufficient Conditions for Solutions to the Primal and Dual Problems

Sufficient Conditions for Solutions to Primal and Dual Problems

Result 20. - Suppose there exists a $\bar{z} \in E_m^+$ such that

$$\sum_{j \in S_1(\bar{z})} c_i(j,t_j) + \sum_{j \in S_2(\bar{z})} c_i(j;u_j) + \sum_{j \in S_3(\bar{z})} c_i(j,g(j;\bar{z})) \leq b_i, i=1,\ldots,m$$

and

$$\sum_{i=1}^{m} \bar{z}_i[\sum_{j \in S_1(\bar{z})} c_i(j,t_j) + \sum_{j \in S_2(\bar{z})} c_i(j,u_j) + \sum_{j \in S_3(\bar{z})} c_i(j,g(j;\bar{z})) - b_i] = 0.$$

Define the function $\bar{f}$ on J as follows:

$$\bar{f}_j = t_j \quad \text{on } S_1(\bar{z}),$$

$$\bar{f}_j = u_j \quad \text{on } S_2(\bar{z}),$$

$$\bar{f}_j = g(j;\bar{z}) \quad \text{on } S_3(\bar{z}).$$

Then $\bar{z}$ is a solution to the primal problem, $\bar{f}$ is a solution to the dual problem, and $E(\bar{f}) = F(\bar{z})$.

Necessary Conditions for a Solution to the Discrete Primal Problem

As in the continuous version we can rewrite the function $F(z)$ as follows:

$$F(z) = \sup_{f \in W} [E(f) - zD(f)].$$

Then it can be shown that $F(z)$ is a convex function in the same manner as the proof of Property 7.

Assume $F(z)$ is differentiable with respect to z_i, $i=1,\ldots,m$ as was done in the continuous version in Chapter III. With $F(z)$ being a convex, differentiable functional the development of the form of the gradient follows directly from Result 10 and we can immediately state the necessary conditions for a solution to the discrete primal problem.

Result 21. - If $\bar{z}$ is a solution to the primal problem then

$$\nabla F(\bar{z}) = b - \sum_{j \in S_1(z)} c(j,t_j) - \sum_{j \in S_2(z)} c(j,u_j) - \sum_{j \in S_3(z)} c(j,g(j;z)) \geq \bar{0}$$

and

$$\bar{z} \; \nabla F(\bar{z}) = 0.$$

Results 12, 13, and 14 of Section 3.5 are so similar to those for the discrete version, they are not repeated in this chapter. The proof of the existence of a solution to the discrete dual problem is simpler than the proof for the integrable version. Because there are some differences in the proofs of the existence theorems for the discrete and integrable versions, the proof for the discrete dual problem is given in the following section.

4.6 Existence of a Solution to the Dual (Discrete) Problem

As in Section 3.6 we will show the set of feasible solutions is compact. Instead of showing semi-continuity of E(•) we will show E(•) is continuous on the set of feasible solutions and then appeal to the Weierstrass theorem for justifying the existence of a solution to the dual problem. The proof of the existence theorem is divided into several properties.

Property 11. - The set $W = \{f : t_j \leq f_j \leq u_j, j \in J\} = \prod_{j=1}^{n} [t_j, u_j]$ is a compact set.

Proof. - The equivalence of the sets follows from the more general case discussed in Appendix H. It is a well-known result that a closed, bounded subset of E_n is compact. Therefore, W is compact. Alternatively, W is a cross-product of closed, bounded intervals which is well known to be compact.

Property 12. - The set of feasible solutions

$$P(b) = \bigcap_{i=1}^{m} \left\{ f \in W : \sum_{j=1}^{n} c_i(j, f_j) \leq b_i \right\} \text{ is compact.}$$

Proof. - A well-known result in real analysis is that a finite sum of continuous functions is continuous. Since

$c_i(j,y)$ is continuous in y for each j, the functional $C_i(f) = \sum_{j=1}^{n} c_i(j,f_j)$ is a continuous function of f.

The interval $(-\infty,b_i]$ is closed and $C_i(\cdot)$ is a continuous function so the inverse image (definition in Appendix I) $C_i^{-1}((-\infty,b_i]) = \{f \in W: C_i(f) \leq b_i\}$ is a closed set based on the Proposition 7 stated in Appendix I. Since the intersection of closed sets is closed we have P(b) being a closed subset of W which implies P(b) is compact.

Property 13. - The functional $E(\cdot)$ is continuous for all $f \in P(b)$.

Proof. - As stated in Property 12 the finite sum of continuous functions is continuous which implies $E(f) = \sum_{j=1}^{n} e(j,f_j)$ is continuous since it was assumed $e(j,y)$ is continuous in y for all $j \in J$.

The existence property can now be stated.

Result 22. - The maximum value of E(f) exists for some $f \in P(b)$.

Proof. - The Weierstrass theorem in Appendix A guarantees a solution exists since P(b) is compact and $E(\cdot)$ is continuous.

Since these results follow directly from the results in Chapter III, we can use these results and the definitions of

$S_1(z)$, $S_2(z)$, and $S_3(z)$ for the discrete version to determine solutions to the Neyman-Pearson problem and its dual. The technique is similar to that for the continuous version.

CHAPTER V

NUMERICAL EXAMPLE

5.1 Introduction

In Chapter III and Chapter IV we stated the primal and dual problems for continuous and discrete versions of the Neyman-Pearson problem. Since necessary and sufficient conditions were stated and proved in two different forms for the continuous version it is appropriate to show the solution technique, based on these conditions, for a sample numerical problem. The problem chosen to show the solution technique is a variation of Example 2 in Wagner's paper[50] and a statistics problem discussed by Rustagi [45]. This problem fits the dual problem formulation of Chapter III and satisfies the strict concavity and the convexity assumptions.

The solution technique is similar, but not identical to Wagner's technique since his sufficiency conditions were revised for the dual problem by adding the complementary slackness condition ($z(C(f) - b) = 0$) as shown in Property 4 and Result 8.

5.2 Outline of Solution Technique

The following steps are used in determining the primal problem and finding solutions to the primal and dual problems:

(1) Check the dual problem to determine whether a solution exists (when X has finite measure, Properties 8, 9, 10 and Result 15 are satisfied);

(2) Determine the partial derivative of $h(x,y;z) = e(x,y) - zc(x,y)$ with respect to y;

(3) Determine the sets $S_1(z)$, $S_2(z)$, and $S_3(z)$ (Recall $S_i(z)$, $i=1,2,3$ is based on $D_2h(x,y;z)$);

(4) Write out the primal problem;

(5) Show a feasible solution to the dual problem exists when the integral constraint is inactive which is required for the conditions to be necessary (as in the following example this can be usually done by inspection);

(6) Use the necessary and sufficient conditions to determine the solutions to the primal and dual problems.

5.3 Example Problem

The general problem may be specialized to the example problem as follows

$$\text{Maximize } E(f) = \int_0^1 - (f(x) - x)^2 dx \tag{81}$$

$$\text{subject to: } C(f) = \int_0^1 f(x)dx \leq \frac{1}{8} \tag{82}$$

$$\text{and } 0 \leq f(x) \leq 1 \text{ for } 0 \leq x \leq 1 \tag{83}$$

The set $X = [0,1]$ which has measure 1. Also, $D_2e(x,y)$ and $D_2c(x,y)$ are bounded for all $x \in [0,1]$ and the functions $e(x,f(x))$ and $c(x,f(x))$ are Lebesgue integrable for all $f \in W = \{f: t(x) \leq f(x) \leq u(x), 0 \leq x \leq 1\}$. Thus, Properties 8, 9, 10 and Result 15 are satisfied which implies a solution to this numerical example of the Neyman-Pearson problem exists.

Note that the results of the previous paragraph were determined by inspection. This should usually be the case.

From (81) and (82) we have

$$h(x,y;z) = e(x,y) - zc(x,y)$$
$$= -(y-x)^2 - zy \text{ where } z \geq 0. \qquad (84)$$

Taking the partial derivative of equation (84) with respect to y yields

$$D_2h(x,y;z) = -2(y-x) - z. \qquad (85)$$

The sets $S_1(z)$, $S_2(z)$, $S_3(z)$ are determined by investigating the values of $D_2h(x,y;z)$ for all $y \in [0,1]$ when x and z are fixed. For this example problem
$W = \{f: 0 \leq f(x) \leq 1,\ 0 \leq x \leq 1\}$ so we have

$$S_3(z) = \{x \in [0,1]: D_2h(x,y;z) = 0 \text{ for only one } y \in [0,1]\}.$$

Therefore, setting (85) equal to zero we can solve for y in terms of x and z which yields

$$y = x - \frac{z}{2} \qquad (86)$$

so $S_3(z) = \{x \in [0,1]:\ y = x - \frac{z}{2} \text{ for exactly one } y \in [0,1]\}$.

For any particular z, equation (86) is a straight line which is sketched below.

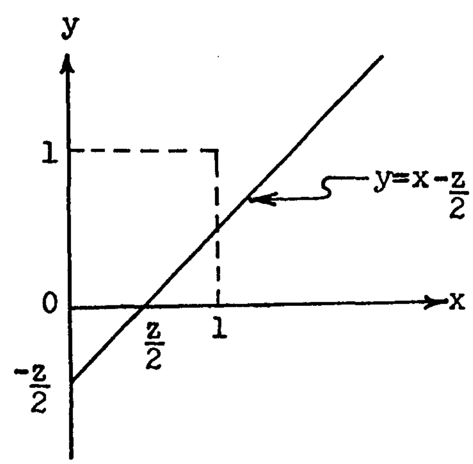


Figure 8

Since $D_2h(x,y;z) = 0$ only when $y = x - \frac{z}{2}$ we see from Figure 8 that for each $x \in [\frac{z}{2},1]$ (when $0 \leq z \leq 2$) there is only one $y \in [0,1]$ such that $D_2h(x,y;z) = 0$. Thus, $S_3(z) = [\frac{z}{2},1]$ for $0 \leq z \leq 2$ and $S_3(z)$ is the empty set for $z > 2$.

From the notation of Chapter III we defined $g(x;z)$ as the number such that $D_2h(x,y;z) = 0$ when $y = g(x;z)$. Since $D_2h(x,y;z) = 0$ when $y = x - \frac{z}{2}$ we have $g(x;z) = x - \frac{z}{2}$.

For this example the definition of $S_1(z)$ becomes

$$S_1(z) = \{x \in [0,1]: D_2h(x,y;z) = -2(y-x)-z < 0 \text{ for all } y \in [0,1]\}$$
$$= \{x \in [0,1]: y > x - \frac{z}{2} \text{ for all } y \in [0,1]\}.$$

The following sketch demonstrates clearly that $S_1(z) = [0, \frac{z}{2})$ when $0 \leq z \leq 2$.

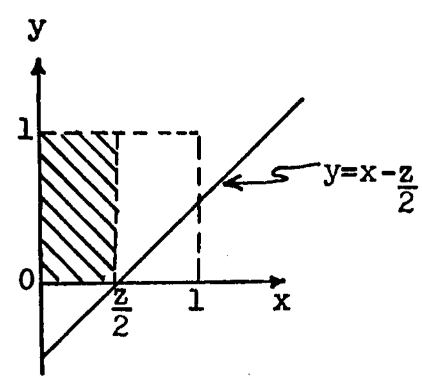


Figure 9

From the sketch we see that for any $0 < x < \frac{z}{2}$ we have $y > x - \frac{z}{2}$ for all $y \in [0,1]$. However, for $x > \frac{z}{2}$ it is not true that $y > x - \frac{z}{2}$ for all $y \in [0,1]$. Note that when $z > 2$ we have $y > x - \frac{z}{2}$ for all $y \in [0,1]$ and each $x \in [0,1]$. Therefore, $S_1(z) = [0,1]$ for all $z > 2$.

Since the domain of $f(x)$ is $[0,1]$ we have $S_1(z) \cup S_2(z) \cup S_3(z) = [0,1]$. However, since $S_i(z), i=1,2,3$ are pairwise disjoint and we have just shown that $S_1(z) \cup S_3(z) = [0,\frac{z}{2}) \cup [\frac{z}{2},1] = [0,1]$ for $0 \le z \le 2$, and $S_1(z) = [0,1]$ for $z > 2$ this implies $S_2(z)$ is the empty set for all $z \ge 0$.

Primal Problem

Recall the statement of the primal problem is

$$\underset{z \ge 0}{\text{Minimize}}\ F(z) = \int_{S_1(z)} h(x,t(x);z)dx + \int_{S_2(z)} h(x,u(x);z)dx$$
$$+ \int_{S_3(z)} h(x,g(x;z);z)dx + bz \tag{87}$$

when there is only one integral constraint for the dual problem.

From (83) we see that $t(x) = 0$ for $x \in [0,1]$ and $u(x) = 1$ for $x \in [0,1]$. Also, $S_2(z)$ is the empty set for this example problem. Therefore, we can rewrite (87) as follows:

$$\underset{z \ge 0}{\text{Minimize}}\ F(z) = \int_{S_1(z)} h(x,0;z)dx + \int_{S_3(z)} h(x,x-\tfrac{z}{2};z)dx + \frac{1}{8}z \tag{88}$$

where $g(x;z) = x - \frac{z}{2}$ as shown previously.

Since we have $S_1(z) = [0,\frac{z}{2})$ and $S_3(z) = [\frac{z}{2},1]$ for $z \in [0,2]$, $S_1(z) = [0,1]$ for $z > 2$, and $h(x,y;z) = e(x,y) - zc(x,y)$ we can write the objective function of (88) as follows:

$$F(z) = \begin{cases} \int_0^{\frac{z}{2}}[-(0-x)^2-z0]dx + \int_{\frac{z}{2}}^1[-(x-\frac{z}{2}-x)^2-z(x-\frac{z}{2})]dx + \frac{z}{8}, & 0 \le z \le 2 \\ \int_0^1[-(0-x)^2 - z0]dx + \frac{1}{8}z, & z > 2 \end{cases}$$

$$= \begin{cases} -\frac{z^3}{24} + \frac{z^2}{4} - \frac{3}{8}z, & 0 \le z \le 2 \\ -\frac{1}{3} + \frac{1}{8}z, & z > 2 \end{cases} \tag{89}$$

Clearly a feasible solution exists for the dual problem when the constraint (82) is inactive, for if we choose $f(x) = 0$ for all $x \in [0,1]$ then $\int_0^1 0\,dx = 0 < \frac{1}{8}$.

From the necessary and sufficient conditions we must have the solution $f_0(\cdot)$ defined as follows:

$$f_0(x) = 0 \quad \text{a.e. for } x \in S_1(z_0) \tag{90}$$

$$f_0(x) = x - \frac{z_0}{2} \text{ a.e. for } x \in S_3(z_0) \tag{91}$$

where z_0 is the solution to the primal problem. Also,

$$z_0 \left[\int_{S_1(z_0)} f_0(x)dx + \int_{S_3(z_0)} f_0(x)dx - \frac{1}{8}\right] = 0. \tag{92}$$

Therefore, either $z_0 = 0$ or the dual constraint holds as an equality.

If $z_0 = 0$, then the constraint becomes

$$\int_0^{\frac{0}{2}} 0\,dx + \int_{\frac{0}{2}}^1 (x-\tfrac{0}{2})dx = \frac{1}{2} > \frac{1}{8}.$$

Since the constraint (82) is not satisfied this implies $z_0 \neq 0$.

Now $\int_0^1 f_0(x)dx = \frac{1}{8}$. From (90) and (91) we have

$$C(f_0) = \int_0^{\frac{z_0}{2}} 0\, dx + \int_{\frac{z_0}{2}}^{1} (x - \frac{z_0}{2})dx = \frac{1}{8} \text{ for } 0 \le z \le 2 \quad (93)$$

and

$$C(f_0) = \int_0^1 0\, dx \neq \frac{1}{8} \text{ for } z > 2. \quad (94)$$

Equation (94) is incorrect which implies z_0 cannot be greater than 2. Therefore, from (93) we have

$$C(f_0) = (\frac{x^2}{2} - \frac{z_0 x}{2}) \Big|_{\frac{z_0}{2}}^{1} = \frac{1}{8} . \quad (95)$$

Solving (95) for z_0 yields

$$(z_0 - 3)(z_0 - 1) = 0.$$

Therefore, $z_0 = 1$ since (93) is the equality for $0 \le z \le 2$ and (94) indicates the equality of the integral constraint (82) does not hold for $z > 2$.

Since $z_0 = 1$ is the solution to the primal problem we have the following solution to the dual problem:

$$f_0(x) = 0 \quad \text{a.e. for } x \in S_1(1) = [0,\tfrac{1}{2}), \quad (96)$$

$$f_0(x) = x - \tfrac{1}{2} \text{ a.e. for } x \in S_3(1) = [\tfrac{1}{2},1] \quad (97)$$

which is illustrated in the following sketch.

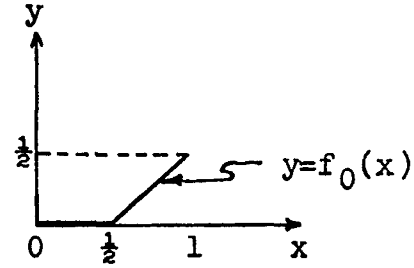


Figure 10

We can now show $E(f_0) = F(z_0)$ to verify further that z_0 and $f_0(\cdot)$ are solutions to the primal and dual problems, respectively.

Substituting $z_0 = 1$ into (89) yields $F(1) = -\frac{1}{6}$. Writing $E(f)$ in terms of $S_1(z)$ and $S_3(z)$ yields

$$E(f) = \int_0^{\frac{z}{2}} -(f(x)-x)^2 dx + \int_{\frac{z}{2}}^{1} -(f(x)-x)^2 dx \text{ for } 0 \leq z \leq 2. \tag{98}$$

Thus, (96) and (97) reduces (98) to

$$E(f_0) = \int_0^{\frac{1}{2}} - x^2 dx + \int_{\frac{1}{2}}^{1} - \frac{1}{4} dx = -\frac{1}{6} .$$

Hence, $E(f_0) = F(z_0)$.

5.4 Example Problem Solution Using Alternate Conditions

The solutions to the primal and dual problems can also be determined by using the alternate necessary and sufficient conditions which were developed with the emphasis on finding the solution to the primal problem. For this particular example, the alternate conditions are more straightforward than the conditions used in the previous section.

Recall the necessary and sufficient conditions for a solution to the primal problem are as follows:

$$\nabla F(z_0) \geq 0, \tag{99}$$

$$z_0 \nabla F(z_0) = 0 \tag{100}$$

if and only if z_0 is a solution to the primal problem. Since $z \in E_1$ the derivative of $F(z)$ and the gradient $\nabla F(z)$ are identical. Therefore, from (89) we have

$$\nabla F(z) = \begin{cases} -\frac{z^2}{8} + \frac{z}{2} - \frac{3}{8}, \text{ if } 0 \le z \le 2 \\ \frac{1}{8}, \text{ if } z > 2 \end{cases} \tag{101}$$

which can also be verified using Result 10.

Also,

$$z \nabla F(z) = \begin{cases} z(-\frac{z^2}{8} + \frac{z}{2} - \frac{3}{8}) = 0, (0 \le z \le 2) \\ \frac{1}{8} z \ne 0 \quad \text{for } z > 2. \end{cases} \tag{102}$$

From (102) we can see that $z_0 \in [0,2]$. If $z_0 = 0$ then from (101) we have $\nabla F(z) = -\frac{3}{8}$ when $z_0 = 0$ which contradicts (99). Therefore, $z_0 \in (0,2]$ which implies

$$\nabla F(z_0) = -\frac{z_0^2}{8} + \frac{z_0}{2} - \frac{3}{8} = 0 \text{ for } 0 < z_0 \le 2. \tag{103}$$

Solving (103) yields $z_0 = 1$ or $z_0 = 3$ but 3 does not satisfy the requirement that $0 < z_0 \le 2$. Therefore, $z_0 = 1$ as was determined using the other set of necessary and sufficient conditions. From Result 13 we have the solution to the dual problem:

$$f_0(x) = 0 \quad \text{a.e. on } S_1(1) = [0,\tfrac{1}{2})$$

$$f_0(x) = x - \tfrac{1}{2} \text{ a.e. on } S_3(1) = [\tfrac{1}{2},1].$$

Note that the solution to the dual problem can be determined by inspection of the primal problem when $F(\cdot)$ is written in terms of z_0, i.e.,

$$F(z_0) = \int_{S_1(z_0)} h(x,t(x);z_0)dx + \int_{S_2(z_0)} h(x,u(x);z_0)dx + \int_{S_3(z_0)} h(x,g(x;z_0);z_0)dx + b^T z_0$$

Recall that the solutions are the same as those found in Section 5.3 and shown in Figure 7.

As a comparison of the methods we have just used it seems likely that when the integrals of the objective function $F(z)$ (of the primal problem) are easy to evaluate and $F(z)$ is differentiable, the more direct method would be the use of the alternative set of conditions. In other words, solving $z_0[\nabla F(z_0)] = 0$ might be easier than solving

$$z_0[\int_X c(x, f_0(x))dx - b] = 0$$

CHAPTER VI

TWO APPLICATIONS PROBLEMS

6.1 Introduction

As indicated in the literature review there are a number of applications problems which may be formulated as nonlinear Neyman-Pearson problems. One example is the oil drilling problem which has been solved by Karlin [22], Luenberger [35], and Wagner [50].

Karlin solved this nonlinear functional Neyman-Pearson problem by choosing a transformation which converted the problem into a linear functional problem. Luenberger used a Lagrangian multiplier method which is somewhat similar to the technique used by Wagner.

Although Wagner's solution technique and the author's solution technique are similar and the solution to the oil drilling problem has been published several times, it is appropriate to solve the problem as a further illustration of results derived in this dissertation, and because some analysis is required to determine the primal problem (dual to the oil drilling problem). In subsequent sections, the primal problem is determined and an economic interpretation of the primal problem is discussed.

The other applications problem is stated and sufficient conditions for a solution are given in Section 6.5. This problem has been formulated and solved by Rustagi [45] using the technique of transforming the nonlinear functional version of the Neyman-Pearson problem into a linear functional version.

6.2 Problem Statement for Oil Drilling Problem

Assume there is a known amount of oil, say b barrels, in an oil reserve. Then we want to maximize the total discounted profit $E(f)$, by using an optimum extraction rate, $f_0(t)$, for removing the oil from the reserve. This can be formulated as a Neyman-Pearson problem as follows:

$$\text{Maximize } E(f) = \int_0^p e[f(t)]\, v(t)dt \tag{104}$$

$$\text{subject to: } C(f) = \int_0^p f(t)dt \le b, \tag{105}$$

$$0 \le f(t) \le M \text{ for all } t \in [0,p] \tag{106}$$

where p is chosen so that $b<\int_0^p Mdt<\infty$. (With p being finite we satisfy the assumption that $c(t,f(t)) = f(t)$ is Lebesgue integrable with respect to t for all $0 \le f(t) \le M$.)

The extraction rate $f(\cdot)$ has units of barrels/day. Clearly, the upper limit M is realistic since the oil cannot be removed instantaneously from the oil reserve. The function $e[\cdot]$ is the profit rate (dollars/day) in terms of the extraction rate $f(\cdot)$. It is assumed that $e[y]$ is a strictly increasing, strictly concave function of y with a continuous derivative with respect to y and $e[0] = 0$. Actually $e[0] = 0$ may be an economic real requirement since there can be no profit if oil is not being extracted from the oil field.

The function $v(\cdot)$ is the discount factor (dimensionless) and is assumed to be a continuous, strictly decreasing, positive, Lebesgue integrable function for all $t \in [0,p]$.

6.3 Problem Solution and Statement of the Primal

The function $h(t,y;z) = e[y]v(t) - zy$ so we can differentiate $h(\cdot)$ with respect to y and then determine $S_1(z)$, $S_2(z)$, and $S_3(z)$. Denote the derivative of $e[\cdot]$ with respect to y as $e_y[\cdot]$. Then we have

$$D_2h(t,y;z) = e_y[y]v(t) - z. \tag{107}$$

Setting (107) equal to zero and solving for $e_y[y]$ yields

$$e_y[y] = \frac{z}{v(t)} .$$

Since $e[\cdot]$ is strictly concave the derived function $e_y[\cdot]$ is a strictly decreasing function of y. It is a well-known result that if a function is strictly decreasing and continuous then the inverse function exists and is a strictly decreasing, continuous function. Therefore,

$$y = e_y^{-1}\left(\frac{z}{v(t)}\right) \text{ when } D_2h(t,y;z) = 0$$

and we have

$$\begin{aligned} S_3(z) &= \{t \in [0,p]: D_2h(t,y;z) = 0 \text{ for exactly one } y \in [0,M]\} \\ &= \{t \in [0,p]: y = e_y^{-1}\left(\tfrac{z}{v(t)}\right) \text{ for } y \in [0,M]\}. \end{aligned}$$

From the definition of $S_1(z)$ and $S_2(z)$ we have

$$S_1(z) = \{t \in [0,p] : e_y[y]v(t) - z < 0 \text{ for all } y \in [0,M]\}$$

and

$$S_2(z) = \{ t \in [0,p] : e_y[y]v(t) - z > 0 \text{ for all } y \in [0,M] .$$

Now,

$$e_y[y]v(t) - z < 0 \text{ for all } y \in [0,M]$$

or

$$e_y[y] < \frac{z}{v(t)} \text{ for all } y \in [0,M]. \tag{108}$$

As previously stated $e_y[\cdot]$ being strictly decreasing implies $e_y^{-1}(\cdot)$ is strictly decreasing. Therefore, inequality (108) is equivalent to

$$y > e_y^{-1}\left(\frac{z}{v(t)}\right) \text{ for all } y \in [0,M].$$

Thus,

$$S_1(z) = \{t \in [0,p] : y > e_y^{-1}\left(\frac{z}{v(t)}\right) \text{ for all } y \in [0,M]\}.$$

Similarly, we can rewrite $S_2(z)$ since

$$e_y[y]v(t) - z > 0 \text{ for all } y \in [0,M]$$

if and only if

$$y < e_y^{-1}\left(\frac{z}{v(t)}\right) \text{ for all } y \in [0,M] \text{ which implies}$$

which implies

$$S_2(z) = \{t \in [0,p] : y < e_y^{-1}\left(\frac{z}{v(t)}\right) \text{ for all } y \in [0,M]\}.$$

If we knew $e[\cdot]$ and $v(\cdot)$ specifically we could proceed with the statement of the primal problem and/or solving the Neyman-Pearson problem. However, without knowing $e[\cdot]$ and $v(\cdot)$ specifically we can sketch the possible typical functions for $e_y^{-1}(\cdot)$. The five sketches which follow represent all possible cases that can occur for the function $g(t;z) \equiv e_y^{-1}(\frac{z}{v(t)})$

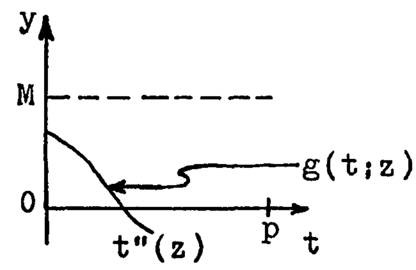


Figure 11

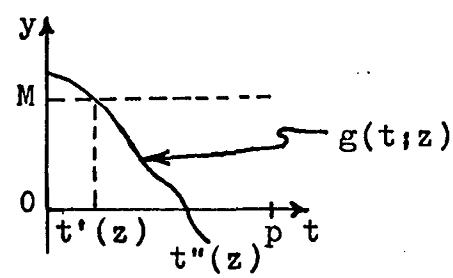


Figure 12

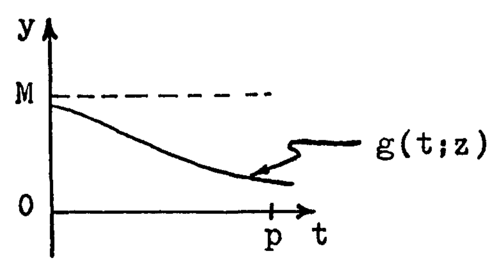


Figure 13

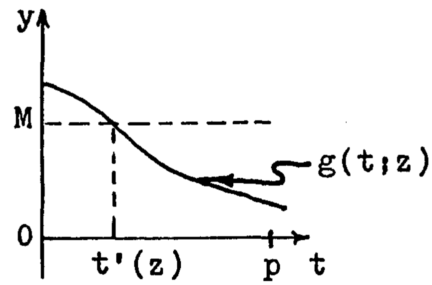


Figure 14

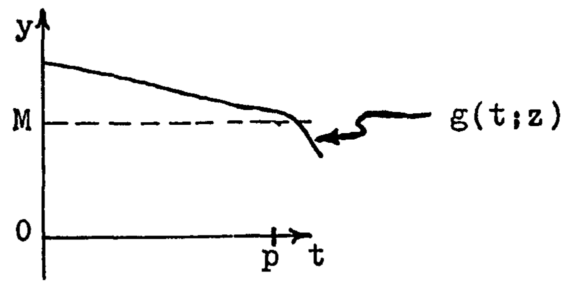


Figure 15

Note that the curve in each sketch above is actually one of a family of curves. There is a different curve for each value of $z \geq 0$.

It can be easily shown that $D_2h(x,y;z) > 0$ for the points (t,y) below the curve in each sketch; $D_2h(x,y;z) < 0$ for each point (t,y) above the curve in each sketch. Therefore, using sketches (Figures 11 through 15) we can also determine the sets $S_1(z)$, $S_2(z)$, and $S_3(z)$ geometrically.

Looking at Figure 15 it can be seen that $S_2(z) = [0,p]$. From the necessary and sufficient conditions for a solution to the dual problem if a solution exists then we have

$$f_0(t) = M \text{ a.e. on } [0,p].$$

However,

$$\int_0^p M dt > b$$

which implies $f_0(t) = M$ is not a feasible solution to the dual problem. Thus, we know that Figure 15 is not a representative sketch of a solution to the dual problem.

Since

$$g(t;z) = e_y^{-1}\left(\frac{z}{v(t)}\right)$$

is an extraction rate when $0 \leq y(t;z) \leq M$, Figures 11 and 12 are more representative of what we would expect (from a physical standpoint) for a specific $e[\cdot]$ and $v(\cdot)$. Thus, we will assume Figure 12 is representative of $g(t;z)$, in that there exist points $t'(z)$ and $t''(z)$ in $[0,p]$ such that $g(t'(z);z) = M$ and $g(t''(z);z) = 0$; then we will continue the analysis of the oil drilling problem.

The function $v(\cdot)$ is strictly decreasing and continuous which implies $v^{-1}(\cdot)$ exists and $v^{-1}(\cdot)$ is continuous. We can solve for $t'(z)$ and $t''(z)$ as follows:

$$g(t;z) = e_y^{-1}\left(\frac{z}{v(t)}\right) = M \text{ when } t = t'(z)$$

which implies

$$e_y[M] = \frac{z}{v(t)} \text{ when } t = t'(z)$$

and then

$$v(t'(z)) = \frac{z}{e_y[M]}.$$

Solving for $t'(z)$ yields

$$t'(z) = v^{-1}\left(\frac{z}{e_y[M]}\right).$$

Similarly, when $t = t''(z)$ we have

$$g(t;z) = e_y^{-1}\left(\frac{z}{v(t)}\right) = 0 \text{ when } t = t''(z)$$

and solving for $t''(z)$ yields

$$t''(z) = v^{-1}\left(\frac{z}{e_y[0]}\right).$$

From the previous definitions of $S_1(z)$, $S_2(z)$, and $S_3(z)$, and Figure 12 we have

$$S_1(z) = \left(t''(z), p\right] = \left(v^{-1}\left(\frac{z}{e_y[0]}\right), p\right],$$

$$S_2(z) = \left[0, t'(z)\right) = \left[0, v^{-1}\left(\frac{z}{e_y[M]}\right)\right),$$

$$S_3(z) = [t'(z), t''(z)] = \left[v^{-1}\left(\frac{z}{e_y[M]}\right),\ v^{-1}\left(\frac{z}{e_y[0]}\right)\right].$$

Using these definitions of $S_i(z)$, $i=1,2,3$ and recalling the statement of the primal problem in Chapter III we have the following:

$$\underset{z \geq 0}{\text{Minimize}}\ F(z) = \int_{v^{-1}\left(\frac{z}{e_y[0]}\right)}^{p} \{e[0]v(t) - 0z\}\,dt + \int_0^{v^{-1}\left(\frac{z}{e_y[M]}\right)} \{e[M]v(t) - Mz\}\,dt$$

$$+ \int_{v^{-1}\left(\frac{z}{e_y[M]}\right)}^{v^{-1}\left(\frac{z}{e_y[0]}\right)} \left\{e\left[e_y^{-1}\left(\frac{z}{v(t)}\right)\right]v(t) - z e_y^{-1}\left(\frac{z}{v(t)}\right)\right\} dt + bz$$

Since it is assumed that $e[0] = 0$ we can reduce the primal problem to

$$\underset{z \geq 0}{\text{Minimize}}\ F(z) = \int_0^{v^{-1}\left(\frac{z}{e_y[M]}\right)} \{e[M]v(t) - Mz\}\,dt + \int_{v^{-1}\left(\frac{z}{e_y[M]}\right)}^{v^{-1}\left(\frac{z}{e_y[0]}\right)} \left\{e\left[e_y^{-1}\left(\frac{z}{v(t)}\right)\right]v(t) - z e_y^{-1}\left(\frac{z}{v(t)}\right)\right\}dt + bz \quad (109)$$

From the necessary and sufficient conditions for a solution to the dual problem we know $f_0(\cdot)$ is a solution if and only if there exists a $z_0 \geq 0$ such that

$$f_0(t) = 0 \text{ a.e. on } S_1(z_0) = \left(v^{-1}\left(\frac{z_0}{e_y[0]}\right), p\right], \quad (110)$$

$$f_0(t) = M \text{ a.e. on } S_2(z_0) = \left[0, v^{-1}\left(\frac{z_0}{e_y[M]}\right)\right), \quad (111)$$

$$f_0(t) = e_y^{-1}\left(\frac{z_0}{v(t)}\right) \text{ a.e. on } S_3(z_0) = \left[v^{-1}\left(\frac{z_0}{e_y[M]}\right), v^{-1}\left(\frac{z_0}{e_y[0]}\right)\right], \quad (112)$$

and

$$z_0\left[\int_0^p f_0(t)dt - b\right] = 0. \quad (113)$$

If $z_0 = 0$ then $h(x,y;0) = e[y]v(t)$. Since $e[\cdot]$ is strictly increasing ($e_y[\cdot]>0$) and $v(\cdot)$ is positive we have

$$D_2h(x,y;0) = e_y[y]v(t) > 0$$

which implies $S_2(0) = [0,p]$. However,

$$\int_0^p M dt > b$$

which contradicts (105). Therefore, z_0 is a positive number. To satisfy the complementary slackness condition (113) we must have

$$\int_0^p f_0(t)dt = \int_0^{v^{-1}\left(\frac{z_0}{e_y[M]}\right)} M\,dt + \int_{v^{-1}\left(\frac{z_0}{e_y[M]}\right)}^{v^{-1}\left(\frac{z_0}{e_y[0]}\right)} e_y^{-1}\left(\frac{z}{v(t)}\right)dt = b. \tag{114}$$

Luenberger considers this problem when $S_2(z)$ is the empty set (there was no upper bound M on the feasible solutions) and stated

$$\int_0^{t(z_0)} e_y^{-1}\left(\frac{z_0}{v(t)}\right)dt = b$$

was continuous so that $t_1(z_0)$ could be determined; hence z_0 could be determined. It is conjectured here that

$$\int_0^{v^{-1}\left(\frac{z_0}{e_y[M]}\right)} M\,dt + \int_{v^{-1}\left(\frac{z_0}{e_y[M]}\right)}^{v^{-1}\left(\frac{z_0}{e_y[0]}\right)} e_y^{-1}\left(\frac{z_0}{v(t)}\right)dt$$

is continuous. Thus we can solve (114) for z_0, given specific functions $e[\cdot]$ and $v(\cdot)$.

After determining z_0 we verify that $t'(z_0)$ and $t''(z_0)$ are in $[0,p]$. If $t'(z_0)$ and $t''(z_0)$ are in $[0,p]$ then we can state $f_0(\cdot)$ (defined by (110), (111) and (112)) for specific subsets of $[0,p]$. If $t'(z_0)$ and/or $t''(z_0)$ are not in $[0,p]$, then the analysis must be redone considering one of the other representative sketches (Figure 11, 13, or 14).

Note that it might be easier to attack the primal problem by differentiation of the objective function rather than solving (114), as was done in the example given in Chapter V.

6.4 Interpretation of the Primal Problem

In nonlinear programming problems it is often difficult to give an interpretation to dual problems (the primal problem in this instance). However, we can gain some insight by checking the units of the terms in the primal problem. Consider the integral

$$\int_0^{t'(z)} e[M]v(t) - zM \, dt$$

which is the first term of (109), the primal problem. The function $e[\cdot]$ has units of dollars/day and $v(\cdot)$ is dimensionless. Therefore, zM must have the same units. Since M has units of barrels/day, the units for z must be dollars/barrel. Therefore, the integral has units of dollars and the objective function $F(z)$ must also have units of dollars for the primal problem to be dimensionally correct.

Luenberger states there is an economic significance to the Lagrangian multiplier for the oil drilling problem. He states that the Lagrangian multiplier is the derivative of the maximum total discounted profit with respect to the number of barrels of oil available in the oil field. It is conjectured here that z_0, the solution to the primal problem, has the same meaning since the development of the primal problem is similar to the Lagrangian equation technique.

6.5 Direct Solution of the Problem Solved by Rustagi [45]

Rustagi [45] indicates there are many statistical applications involving the minimization of a convex functional. He solves the following problem by using the technique (attributed to Karlin) of transforming the nonlinear functional into a linear functional version of the Neyman-Pearson problem and then showing the solution to the linear functional is also the solution to the nonlinear functional problem.

Problem. - Let $P = \{(x,y): -a \leq x \leq a,\ 0 \leq y \leq 1\}$ where "a" is a specified real number. Let $q(\cdot)$ be a function defined on the closed, bounded set P in E_2 such that $q(\cdot)$ is bounded and continuous on P. Also, $q(x,\cdot)$ is strictly convex and twice differentiable with respect to y.

$$\text{Minimize } Q(f) = \int_{-a}^{a} q(x,f(x))dx \tag{115}$$

subject to

$$\int_{-a}^{a} x\, d[f(x)] = \mu_1, \tag{116}$$

$$\int_{-a}^{a} x^2 d[f(x)] = \mu_2, \tag{117}$$

$$0 \leq f(x) \leq 1 \text{ for all } x \in [-a,a], \tag{118}$$

$$f(x) = 0,\ x < -a,\ f(x) = 1,\ x > a.$$

This problem is concerned with finding the cdf (cumulative distribution function) $f(\cdot)$ defined on $[-a,a]$ which minimizes the convex functional $Q(f)$. The real numbers μ_1 and μ_2 in (116) and (117) are the known (given) values of the first and second moments, respectively, of the cdf that solves the minimization problem.

Rustagi integrated (116) by parts to yield

$$\int_{-a}^{a} f(x)dx = a - \mu_1 \tag{119}$$

and integrated (117) by parts to yield

$$\int_{-a}^{a} xf(x)dx = \frac{a^2 - \mu_2}{2} \tag{120}$$

before attempting to solve this problem.

We can convert the objective function to a concave function and write (119) and (120) as inequalities to put the problem in the form of the dual problem. Let $e(x,y) = -q(x,y)$. Then $e(x,\cdot)$ is strictly concave and twice differentiable in y so we can restate the problem equivalently in the form of the dual problem as follows:

$$\text{Maximize } E(f) = \int_{-a}^{a} e(x,f(x))dx \tag{121}$$

subject to

$$\begin{aligned} C_1(f) &= \int_{-a}^{a} f(x)dx \le a - \mu_1, \\ C_1'(f) &= -\int_{-a}^{a} f(x)dx \le -(a-\mu_1), \\ C_2(f) &= \int_{-a}^{a} xf(x)dx \le \frac{a^2 - \mu_2}{2}, \\ C_2'(f) &= -\int_{-a}^{a} xf(x)dx \le -\frac{(a^2-\mu_2)}{2}, \end{aligned} \tag{122}$$

$$t(x) \le f(x) \le u(x), \text{ for } x \in [-a,a] \tag{123}$$

where

$$u(x) = \begin{cases} 1, & -a < x \leq a \\ 0, & x = -a, \end{cases}$$

and

$$t(x) = \begin{cases} 0, & -a \leq x < a \\ 1, & x = a. \end{cases}$$

Using the sufficiency conditions of this dissertation we will be able to solve this problem directly. Since there is no feasible solution $f(\cdot)$ such that the inequalities (122) are strictly satisfied, the regularity assumption required for the sufficient conditions to be necessary is not satisfied. However, it is conjectured that the regularity assumption is not required when the integral constraints are linear functionals.

Rustagi assumed $q(x,\cdot)$ is twice differentiable with respect to y so that $D_2q(x,\cdot)$ is continuous. The method of this dissertation requires only that $D_2e(x,\cdot)$ be continuous so this is the assumption we will use.

<u>Solution.</u> - Since $e(x,y)$ is not a specific function of x and y we will not be able to determine the solution explicitly. However, we can state the solution in general terms as was done by Rustagi.

For this problem we have

$$h(x,y;z) = e(x,y) - z_1 y - z_2(-y) - z_3 xy - z_4(-xy).$$

The partial derivative of $h(x,y;z)$ with respect to y is

$$D_2 h(x,y;z) = D_2 e(x,y) - (z_1-z_2) - (z_3-z_4)x.$$

Then we have

$$S_1(z) = \{x \in [-a,a]: D_2 e(x,y) - (z_1-z_2) - (z_3-z_4)x < 0 \text{ for all } y \in [t(x),u(x)]\}$$

$$S_2(z) = \{x \in [-a,a]: D_2 e(x,y) - (z_1-z_2) - (z_3-z_4)x > 0 \text{ for all } y \in [t(x),u(x)]\}$$

and

$$S_3(z) = \{x \in [-a,a]: D_2 e(x,y) - (z_1-z_2) - (z_3-z_4)x > 0 \text{ for exactly one } y \in [t(x),u(x)]\}.$$

Since $e(x,y)$ is strictly concave and $D_2\,e(x,y)$ is strictly decreasing and continuous; this implies the inverse of $D_2 e(x,\cdot)$ exists with respect to the variable y. Thus, given a specific function $e(x,y)$ we could state $S_1(z)$, $S_2(z)$, and $S_3(z)$ explicitly.

As in Chapter II let $g(x;z)$ be the value of y such that $D_2 h(x,y;z) = 0$; then we can state the solution to the problem in general terms. The sufficient conditions for a solution to the dual problem yields

$f_0(x)=t(x)$ a.e. for $x \in S_1(z_0)$,

$f_0(x)=u(x)$ a.e. for $x \in S_2(z_0)$,

$f_0(x) = g(x;z_0)$ a.e. for $x \in S_3(z_0)$,

where $S_1(z_0) \cup S_2(z_0) \cup S_3(z_0) = [-a,a]$, $f_0(\cdot)$ is a solution to the dual problem, and $z_0 \in E_4^+$ is the solution to the primal problem.

Note that when $f_0(\cdot)$ maximizes $E(f)$, the solution $f_0(\cdot)$ also minimizes $Q(f)$. Therefore, this is the same result determined by Rustagi except that he does not explicitly state the disjoint domains $S_1(z_0)$, $S_2(z_0)$, and $S_3(z_0)$ for the solution $f_0(\cdot)$.

As a matter of interest and for completeness we can immediately state the primal problem, which might be easier to solve for z_0 depending on the specific function given for $e(x,y)$. Recalling the general statement of the primal problem we have the following primal problem for this example:

$$\begin{aligned}\underset{z \in E_4^+}{\text{Minimize}}\ F(z) = & \int_{S_1(z)} e(x,0)dx + \int_{S_2(z)} [e(x,1)-(z_1-z_2)-(z_3-z_4)x]dx \\ & + \int_{S_3(z)} h(x,g(x;z);z)dx + (z_1-z_2)(a-\mu_1) \\ & + (z_3-z_4)\left(\frac{a^2-\mu_2}{2}\right).\end{aligned}$$

CHAPTER VII

CONCLUSIONS AND RECOMMENDATIONS

7.1 Summary and Conclusions

As previously indicated, the Neyman-Pearson problem occurs in the following areas: statistics, search theory, information theory, facility design, allocation problems, and some game theory problems. Since the linear functional version of the Neyman-Pearson problem is a special case of the nonlinear functional version, the results developed in Chapter III have expanded the class of Neyman-Pearson problems which can be solved in a straightforward manner.

In linear programming the dual problem is useful since it is often easier to solve the dual problem than it is to solve the primal problem. Although, the primal problem (dual to the Neyman-Pearson problem) involves the solution to the dual problem it was illustrated in the example problem of Chapter V that the primal problem could be solved more directly. This may not always be true, but the primal problem will often give insight which will prove useful when determining the solution to the Neyman-Pearson problem.

The author believes the solution technique based on the results in Chapter III is straightforward and easy to use. The use of the necessary and sufficient conditions to determine solutions to the primal and dual problems requires only a knowledge of elementary calculus and the properties of concave and convex functions. This is certainly an easier technique than using Calculus of Variations and requires less ingenuity than linearizing a nonlinear functional before determining a solution.

The following section gives a summary of the pertinent results developed in Chapter III.

7.2 Duality Relationships for the Nonlinear Neyman-Pearson Problem

Given:

set

X measurable subset of E_{n-1}

functions

$e(x,y)$ from $X \times E_1$ into E_1 (x in E_{n-1}, y in E_1)

$c_i(x,y)$ from $X \times E_1$ into E_1 for $i=1,\ldots,m$

$t(x)$, $u(x)$ from E_{n-1} into E_1

vectors

$z = (z_1,\ldots,z_m)$, $b = (b_1,\ldots,b_m)^T$ (b_i known constants)

Assumptions

$t(x) \leq u(x)$ for all x in X; $t(x)$, $u(x)$ are bounded and Lebesgue measurable

$e(x,y)$ is strictly concave in y for all x in X

$c_i(x,y)$ is convex in y for all x in X, $i=1,\ldots,m$

$D_n e(x,y) = \partial e(x,y)/\partial y$ exists and is bounded for all $y \in [t(x),u(x)]$ and almost all $x \in X$

$D_n c_i(x,y) = \partial c_i(x,y)/\partial y$ exists and is bounded for all $y \in [t(x),u(x)]$ and almost all $x \in X$, $i=1,\ldots,m$

Definitions

E_m^+: nonnegative orthant of E_m

$h(x,y;z) = e(x,y) - \sum_{i=1}^{m} z_i c_i(x,y)$ where

x is in X, y is in $[t(x),u(x)]$, z is in E_m^+

$S_1(z) = \{x \text{ in } X: D_n h(x,y;z) < 0 \text{ for all } y \text{ in } [t(x),u(x)]\}$

$S_2(z) = \{x \text{ in } X: D_n h(x,y;z) > 0 \text{ for all } y \text{ in } [t(x),u(x)]\}$

$S_3(z) = \{x \text{ in } X: D_n h(x,y;z) = 0 \text{ for exactly one } y \text{ in } [t(x),u(x)]\}$

$g(x;z)$: the unique point in $[t(x),u(x)]$ for which $h(x,g(x;z);z) = \max\{h(x,y;z) : y \text{ in } [t(x),u(x)]\}$

Primal Problem (P)

$$\min_{z \text{ in } E_m^+} F(z) = \int_{S_1(z)} h(x,t(x);z)dx + \int_{S_2(z)} h(x,u(x);z)dx + \int_{S_3(z)} h(x,g(x;z);z)dx + b^T z$$

Dual Problem (D)

$$\max E(f) = \int_X e(x,f(x))dx$$

subject to

$$\int_X c_i(x,f(x))dx \le b_i;\ i=1,\ldots,m$$

$$t(x) \le f(x) \le u(x), \text{ all } x \text{ in } X.$$

Definitions

feasible solution to P: any z in E_m^+

feasible solution to D: any function $f(\cdot)$ for which the integrals of D exist and the constraints are all satisfied

solution to P: any point in E_m^+ which minimizes $F(z)$

solution to D: any feasible solution to D which maximizes $E(f)$.

Properties of the Problems

Result 1. - For any feasible solution f to D. and any feasible solution z to P, $E(f) \leq F(z)$.

Result 2. - Suppose there exists a feasible solution f to D. and a feasible solution z to P such that

$$f(x) = t(x) \text{ a.e. on } S_1(z), \tag{124}$$

$$f(x) = u(x) \text{ a.e. on } S_2(z), \tag{125}$$

$$f(x) = g(x;z) \text{ a.e. on } S_3(z), \tag{126}$$

$$\sum_{i=1}^{m} z_i\Big[\int_X c_i(x,f(x))dx - b_i\Big] = 0. \tag{127}$$

Then z is a solution to P, $f(\cdot)$ is a solution to D, and $E(f) = F(z)$.

Regularity Assumption - There exists at least one feasible solution $f(\cdot)$ to D such that

$$\int_X c_i(x,f(x))dx < b_i \text{ for } i=1,\ldots,m.$$

Result 3. - Suppose the regularity assumption is satisfied.

If $f(\cdot)$ is a solution to D. then there exists a solution z to P such that $f(\cdot)$ and z satisfy (124), (125), (126), and (127), and $E(f) = F(z)$.

Result 4. - Results 2 and 3 combined give necessary and sufficient conditions for a solution to D.

Result 9. - Suppose there exists a feasible z to P such that

$$\int_{S_1(z)} c_i(x,t(x))dx + \int_{S_2(z)} c_i(x,u(x))dx + \int_{S_3(z)} c_i(x,g(x;z))dx \leq b_i, \; i=1,\ldots,m$$

and

$$\sum_{i=1}^{m} z_i\Big[\int_{S_1(z)} c_i(x,t(x))dx + \int_{S_2(z)} c(x,u(x))dx + \int_{S_3(z)} c_i(x,g(x;z))dx - b_i\Big] = 0.$$

Define the function $f(\cdot)$ on X as follows:

$$f(x) = t(x) \text{ a.e. on } S_1(z)$$

$$f(x) = u(x) \text{ a.e. on } S_2(z)$$

$$f(x) = g(x;z) \text{ a.e. on } S_3(z).$$

Then z is a solution to P, $f(\cdot)$ is a solution to D, and $E(f) = F(z)$.

Assumption. - The primal function $F(\cdot)$ is differentiable.

Result 10. - For $i=1,\ldots,m$

$$\frac{\partial F(z)}{\partial z_i} = b_i - \int_{S_1(z)} c_i(x,t(x))dx - \int_{S_2(z)} c_i(x,u(x))dx - \int_{S_3(z)} c_i(x,g(x;z))dx$$

Result 11. - If z is a solution to P, then z satisfies

$$\frac{\partial F(z)}{\partial z_i} \geq 0, \; i=1,\ldots,m$$

and

$$\sum_{i=1}^{m} z_i \frac{\partial F(z)}{\partial z_i} = 0.$$

Thus, if P has a solution z, then (by Results 9 and 10) D has a solution $f(\cdot)$, and $E(f) = F(z)$.

Result 12. - Results 9 and 11 combined give necessary and sufficient conditions for a solution to P.

Result 14. - The primal problem has a solution z if and only if the dual problem has a solution $f(\cdot)$, and $E(f) = F(z)$.

Result 15. - If the region of integration in the dual problem is restricted to a subset of finite positive measure, T of E_{n-1}, and the domain of $t(\cdot)$, $u(\cdot)$, and $f(\cdot)$ is also restricted to T, then there exists a solution to the dual problem.

7.3 Recommendations for Future Research

There are several areas where further work can be done to extend the results developed in this dissertation. The functions $e(x,y)$ and $c(x,y)$ were assumed to be strictly concave and convex in y, respectively, in Chapter II. The author conjectures that the sufficient conditions for a solution to the dual problem are easily satisfied when the assumption that $e(x,y)$ is strictly concave in y is weakened to concavity in y. However, one must be careful when redefining $S_1(z)$, $S_2(z)$, and $S_3(z)$. Considerable effort will probably be required to prove necessary conditions for a solution to the dual problem since it is conceivable that the solution to the dual problem may not be a unique function when $e(x,y)$ is only convex.

As indicated in Chapter II some work has been recently accomplished by Wagner and Stone [51] when the assumptions on $e(x,\cdot)$ and $c(,\cdot)$ have been relaxed in a slightly different manner.

Another possible area of study is that of proving or disproving the author's conjecture that the regularity assumption ($C(f) < b$) is not required for proving the necessary conditions for a solution to the dual problem when $C(f)$ is a linear functional. Luenberger [35] refers to a similar problem in the homework assignment section of Chapter 8 of his text.

A problem of theoretical interest might be the consideration of the following version of the Neyman-Pearson problem:

$$\text{Maximize } E(f) = \int_X e(x,f(x))dx$$

subject to

$$C(f) = \int_X c(x,f(x))dx \leq b(p),$$

$$t(x) \leq f(x) \leq u(x) \text{ for } x \quad X$$

and

$$0 \leq p \leq P$$

where p might be considered to have units of time in an applications problem and $b(p)$ is a continuous function of time, i.e., the upper bound on $C(f)$ might change continuously with respect to time. It would be interesting to determine the dual to this version of the Neyman-Pearson problem and then investigate the duality relationships.

Some work has been done by Stone [46] in this area when considering a generalized version of the oil drilling problem. The problem was written in the context of a search problem and $C(f)$ was a linear functional.

Another interesting problem is that of weakening the assumption that the objective function $F(\cdot)$ of the primal problem is differentiable. For example, in linear programming the objective function of the primal problem is not always differentiable.

From the previously stated areas of research there should be some interesting results still to be discovered. It is also likely that when these areas are investigated other problems related to those of this dissertation will be discovered.

APPENDIXES

APPENDIX A

<u>Weierstrass Theorem.</u> - An upper semicontinuous functional on a compact subset $\overline{P}(b)$ of a normed linear space L achieves a maximum on $\overline{P}(b)$.

<u>NOTE</u>. - The space L of all bounded real-valued functions on E_n is a linear vector space since $f_1, f_2 \in L$ implies $af_1(x) + bf_2(x)$ is a bounded real-valued function for all $x \in E_n$, where a and b are scalars.

The Weierstrass theorem given here is a general version stated in [35]. Many sources in real analysis give various statements of this theorem. For example, a closed, bounded subset of E_1 is compact so a common version of the Weierstrass theorem is as follows: If a function is continuous on a closed, bounded interval then the extreme values of the function are achieved on the interval.

APPENDIX B

The following well-known result in real analysis, which can be found in [19], is written in the notation of this dissertation.

<u>Proposition 1.</u> - Let $\hat{h}(\cdot)$ be a real-valued, differentiable function defined on $[y_1, y_2]$. Then $\hat{h}'(\cdot)$ takes on every value between $\hat{h}'(y_1)$ and $\hat{h}'(y_2)$.

APPENDIX C

<u>Proposition 2.</u> - Let $g(\cdot)$ be a strictly concave, differentiable function on the interval $[a,b]$ and let $y^* \in [a,b]$ maximize $g(\cdot)$.

If $g'(y^*)>0$ then $y^* = b$ and $g'(y)>0$ for all $y \in [a,b]$. (128)

If $g'(y^*)<0$ then $y^* = a$ and $g'(y)<0$ for all $y \in [a,b]$. (129)

If $g'(y^*)=0$ and $y^*=b$ then $g'(y) > 0$ for all $y \in [a,b)$. (130)

If $g'(y^*)=0$ and $y^* = a$ then $g'(y)<0$ for all $y \in (a,b]$. (131)

If $g'(y^*)$ 0 and $a<y^*<b$ then $g'(y)>0$ for all $y \in [a,y^*)$ and $g'(y)<0$ for all $y \in (y^*,b]$. (132)

<u>Proof.</u> - Since y^* maximizes $g(\cdot)$ we have $g(y^*) \geq g(y)$ for $y \in [a,b]$. By the strict concavity of $g(\cdot)$ we have $g'(y) > 0$ for all $y \in [a,y^*)$ and $g'(y) < 0$ for all $y \in (y^*,b]$.

Assume $g'(y^*) > 0$ and $(y^*,b]$ contains at least one point. Then $g'(y) < 0$ for $y \in (y^*,b]$ and $g'(y^*) > 0$ implies there is a point $y_0 \in (y^*,b]$ such that $g (y_0) = 0$ by Proposition 1 in Appendix B. Elementary calculus tells us that y_0 must be the point that maximizes $g(\cdot)$ but this contradicts the hypothesis that y^*, which is a unique maximum, maximizes $g(\cdot)$. Thus, $(y^*,b]$ must be the empty set, i.e., $y^* = b$. Since $g'(y) > 0$ for all $y \in [a,y^*)$ and $g (y^*) > 0$ we have $g (y) > 0$ for all $y \in [a,b]$ and (128) holds.

If $g'(y^*) < 0$ and $[a,y^*)$ contains at least one point, then $g'(y) > 0$ for $y \in [a,y^*)$ and $g'(y^*) < 0$ implies there is a point $y_0 \in [a,y^*)$ such that $g'(y_0) = 0$ by Proposition 1. This implies y_0 is the point which maximizes $g(\cdot)$ but this contradicts the hypothesis that y^*, which is a unique maximum, maximizes $g(\cdot)$. Thus, $y^* = a$ and we have $g'(y) < 0$ for all $y \in [a,b]$ and (129) holds.

Statements (130), (131), and (132) follow directly from the strict concavity and differentiability of $g(\cdot)$. Clearly, (128)-(132) are exclusive and exhaustive since $g(\cdot)$ is strictly concave.

<u>Proposition 3.</u> - $X = S_1(z) \cup S_2(z) \cup S_3(z)$ and the sets $S_i(z)$, $i=1,2,3$ are pairwise disjoint.

<u>Proof.</u> - To prove $X = S_1(z) \cup S_2(z) \cup S_3(z)$ we only have to show $X \subseteq S_1(z) \cup S_2(z) \cup S_3(z)$ since $S_1(z)$, $S_2(z)$, and $S_3(z)$ are all subsets of X which implies $S_1(z) \cup S_2(z) \cup S_3(z) \subseteq X$.

Recall $S_i(z)$ for $i=1,2,3$ is defined by the partial derivative $D_n h(x,y;z)$. As in the text we will define $\hat{h}'(y) = D_n h(x,y;z)$ and then for an arbitrary $x \in X$ we will show x is in $S_1(z)$ or $S_2(z)$ or $S_3(z)$ depending on the values of $\hat{h}'(y)$.

Let $x \in X$ and let y^* maximize $\hat{h}(y)$ on $[t(x),u(x)]$. If $\hat{h}'(y^*) > 0$ then (128) implies $x \in S_2(z)$. If $\hat{h}'(y^*) < 0$ then (129) implies $x \in S_1(z)$. If $\hat{h}'(y^*) = 0$ then (130), (131), and (132) imply $x \in S_3(z)$. Hence $X \subseteq S_1(z) \cup S_2(z) \cup S_3(z)$.

Clearly, the sets are pairwise disjoint since there is no way that statements (128) through (132) can occur simultaneously for a specific x.

APPENDIX D

Kuhn-Tucker Theorem - Assume there exists an f P(b) such that $C(f) - b < 0$. If $E(\cdot)$ is concave and $C(\cdot)$ is convex then $f_0(\cdot)$ is a solution to the dual problem if and only if a vector z_0 exists such that:

$$f_0 \epsilon W, \; z_0 \epsilon E_m^+$$

and

$$E(f_0) - z[C(f_0)-b] \geq E(f_0) - z_0[C(f_0)-b] \geq E(f) - z_0[C(f)-b]$$

for all $f \epsilon W$ and $z \epsilon E_m^+$.

The requirement that a feasible solution exists when the constraints are inactive is a regularity assumption due to Slater and Karlin.

APPENDIX E

<u>Proposition 4.</u> - Assume $c(x,\cdot)$ is convex in the n^{th} variable then $D(f) = C(f) - b$ is convex. In other words, $D_i(f)$ is a convex functional for $i=1,\ldots,m$.

<u>Proof.</u> - We must show $D(af_1 + (1-a)f_2) \leq aD(f_1) + (1-a)D(f_2)$, $0 < a < 1$, for all elements f_1, f_2 in the convex set W. Let $f = af_1 + (1-a)f_2$, $0 \leq a \leq 1$. Then

$$D(f) = \int_X c(x,f(x))dx - b$$

$$= \int_X [c(x,af_1(x) + (1-a)f_2(x))]dx - b$$

$$\leq \int_X ac(x,f_1(x))dx + \int_X (1-a)c(x,f_2(x))dx - (ab + (1-a)b)$$

since $c(x,\cdot)$ is convex in the n^{th} variable. Therefore,

$$D(f) \leq a\int_X c(x,f_1(x))dx - ab + (1-a) \int_X c(x,f_2(x))dx - (1-a)b$$

$$= a[\int_X c(x,f_1(x))-b] + (1-a)[\int_X c(x,f_2(x))dx-b]$$

$$= aD(f_1) + (1-a)\ D(f_2),\ 0 < a < 1$$

which is the desired result.

<u>Proposition 5.</u> - Assume $e(x,\cdot)$ is concave in the n^{th} variable then $E(\cdot)$ is a concave functional.

<u>Proof.</u> - Since $e(x,\cdot)$ is concave in the n^{th} variable,

$-e(x,\cdot)$ is convex in the n^{th} variable. From Proposition 4 we know $-E(\cdot)$ is convex. Hence $E(\cdot)$ is a concave functional.

<u>Proposition 6.</u> - The set $A = \{(u,v) \in E_{m+1} : u \geq D(f)$, $v \leq E(f)$ for at least one $f \in W\}$ is convex.

<u>Proof.</u> - The vector $u = (u_1, u_2, \ldots, u_m)T$ is in E_m and v is in E_1.

Let (u',v') and (u'',v'') be elements of A. Then we want to show $(u,v) = a(u',v') + (1-a)(u'',v'')$, for $0 \leq a \leq 1$, is in A.

Since $(u',v') \in A$ we have $u' \geq D(f')$ and $v' \leq E(f')$ for some $f' \in W$. Also, $(u'',v'') \in A$ implies $u'' \geq D(f'')$ and $v'' \leq E(f'')$ for some $f'' \in W$. Therefore,

$$u = au' + (1-a)u'' \geq aD(f') + (1-a)\,D(f'') \geq D(af' + (1-a)f'')$$

since $D(\cdot)$ is convex as shown in Proposition 4 of this appendix. Also,

$$v = av' + (1-a)v'' \leq aE(f') + (1-a)\,E(f'') \leq E(af' + (1-a)\,f''),\ 0 < a < 1$$

so $v \leq E(af' + (1-a)f'')$ for $0 \leq a \leq 1$ since $E(\cdot)$ is concave as shown in Proposition 5 of this appendix. The set W is convex which implies $[af' + (1-a)f'']$ W. Therefore, $(u,v) \in A$ which is the desired result.

APPENDIX F

Definition. - A point g is a boundary point of the set G if every neighborhood of g contains at least one point in the set G and contains at least one point not in the set G.

Definition. - A point $g \in G$, a subset of a normed linear space, is an interior point if there exists a neighborhood of g which contains only points of G.

Separating Hyperplane Theorem. - If G and K are two convex sets with no interior points in common, then there exists a hyperplane that separates G and K. In other words, there exists a non-zero vector c and a scalar d such that $cg \geq d$ for all $g \in G$ and $ck \leq d$ for all $k \in K$.

The above theorem can be found in [22], [35] and other texts.

An important point in this theorem is that the inequalities are still satisfied for boundary points of G and K since it only is required that the interiors of G and K be disjoint.

APPENDIX G

The following definition can be found in [25] and other topology texts.

Definition. - The set $\overline{W} = \{f : t(x) \leq f(x) \leq u(x),\ x \in T\}$ can also be written as $\overline{W} = \pi_{x \in T} [t(x), u(x)]$ where π is the symbol for the Cartesian Product.

To gain insight into this definition, consider the following example where the index of the Cartesian Product is finite. Let $X = \overset{2}{\underset{i=1}{\pi}} X_i$ where $X_i = [0,1]$ for i=1,2.

This is just a subset of the vector space E_2, namely the unit square as shown in the following sketch.

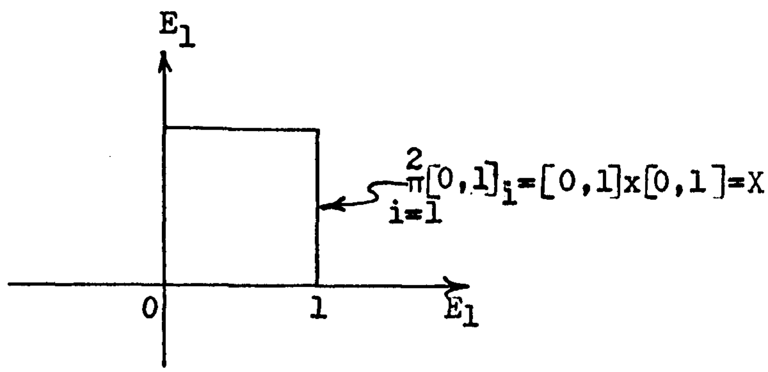


Figure 16

If x is a vector in X then $x = (x_1, x_2)$ with $x_1 \in X_1$ and $x_2 \in X_2$.

Now consider the set of all real-valued functions defined on T. This set is a vector space since the sum of real-valued functions is a real-valued function and a scalar multiple of a real-valued function is a real-valued function.

Therefore, $\overline{W}$ is a subset of the vector space of real-valued functions. Since $f \in \overline{W}$ is a vector we can consider $f(x)$ as the x^{th} component of the vector f.

The set X is a subset of a finite dimensional vector space E_2, while $\overline{W}$ is a subset of real-valued functions on T. Therefore, we could also compare X and $\overline{W}$ in the sense that $x \in X$ is a point in the plane E_2 and is denoted by (x_1, x_2) while $f \in \overline{W}$ is a point in the infinite dimensional space of real-valued functions with the x^{th} coordinate of f is denoted by $f(x)$.

APPENDIX H

Definition. - The set Δ is a directed set if $a \leq b$ and $b \leq c$ implies $a \leq c$ for a, b, c $\epsilon \Delta$, and if a,b $\epsilon \Delta$ then there is a c such that $a \leq c$ and $b \leq c$. In other words "$\leq$" is an ordering of the directed set Δ and Δ is unbounded.

An example of a directed set is the set of natural numbers where the symbol $\leq$ is the usual ordering of the natural numbers. Another example is the set of all subsets of a given set, say X. In other words, $C \subseteq D$ if and only if $D \leq C$. Note that $C \subseteq D$ and $D \subseteq E$ implies $C \subseteq E$ which is the same as $D \leq C$ and $E \leq D$ implies $E \leq C$.

The following definition is for the particular problem involved in proving the existence of a solution to the dual problem.

Definition. - Let Δ be a directed set and $f_\delta(\cdot)$ be a map from Δ into the reals for each $x \epsilon T$, a set of positive measure. Then $\{f_\delta\}_{\delta \epsilon \Delta}$ is called a net.

For example, if Δ is the set of natural numbers then $\{f_\delta\}_{\delta \epsilon \Delta}$ is a sequence. The reason for using nets rather than sequences is that a sequence converging to a point in a set is a necessary but not sufficient condition for closure of the set for some topological spaces. However, a net converging to an arbitrary point is necessary and sufficient for the closure of a set. In other words, by using nets we are not restricted to index sets having a countable number of elements.

A complete explanation of nets can be found in [36] and other topology texts.

APPENDIX I

<u>Definition.</u> - Let A be subset of the range Y of the mapping f from X into Y. The set $\{x \in X : f(x) \in A\}$ is called the inverse image of A. The usual notation for this set is $f^{-1}[A]$.

The following result can be found in many topology texts such as [36].

<u>Proposition 7.</u> - Let f be a mapping from X into Y. The function f is continuous if and only if the inverse image of each closed subset of Y is closed in X.

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