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BASES AND QUASI-REFLEXIVE SPACES

DISSERTATION

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CONTENTS

INTRODUCTION

In [2] Civin and Yood introduced the notion of a quasi-reflexive space. We say a Banach space X is quasi-reflexive of order n if the natural embedding of X into its second conjugate has codimension n (written ord $(X) \doteq n$). A space with this property had been introduced earlier by James [8]. In their paper Civin and Yood proved the following important theorem: If ord $(X) = n$ and if Y is a subspace of X then ord $(X) =$ $=$ ord (Y) + ord $\left(\frac{X}{Y}\right)$ where $\frac{X}{Y}$ is the quotient space of X with respect to Y. Thus, if ord $(X) = n$ then all subspaces of X are quasi-reflexive of order k where $k \leq n$. The main theorem of this work, Theorem 3.22, states that if ord $(X) = n$ then X contains subspaces of all orders less than n. The proof of Theorem 3.22 follows quickly once we show that if ord $(X) = n > 0$ then X contains a subspace Y such that ord $(Y) = 1$. In order to show this, we use a result of Pelczynski [9] which states that every non-reflexive space X (in particular assume ord $(X) = n > 0$) contains a non-reflexive subspace Y with a basis (so $n \geq$ ord $(Y) \geq 1$). We then apply basis theory techniques to Y.

Singer [10] introduces the notions of a basic sequence being k-shrinking or k-boundedly complete, which are generalizations of the notions of a basic sequence being shrinking or boundedly complete respectively (i.e. 0-shrinking corresponds to shrinking and 0-boundedly complete corresponds to boundedly complete). Singer shows that if Y has a basis $(x_i)_{i=1}^{\infty}$ then ord $(Y) = m$ if and only if $(x_i)_{i=1}^{\infty}$ is k-shrinking and q-boundedly complete, where $k + q = m$. This generalizes James' result [8] that Y is reflexive if and only if $(x_i)_{i=1}^{\infty}$ is shrinking and boundedly complete. We will show (Theorem 3.19) that if $(x_i)_{i=1}^{\infty}$ is k-shrinking and q-boundedly complete where $k + q > 0$ then there is a block basic sequence $(z_i)_{i=1}^{\infty}$ of $(x_i)_{i=1}^{\infty}$ such that $(z_i)_{i=1}^{\infty}$ is 1-shrinking and 0-boundedly complete or $(z_i)_{i=1}^{\infty}$ is 0-shrinking and 1-boundedly complete. This shows that ord $([z_i]_{i=1}^{\infty}) = 1$. Chapter 3 contains the proof of the main theorem, Theorem 3.22 in a manner as indicated above.

In Chapter 2 we give some standard information and definitions dealing mostly with basis and projection theory. There is nothing essentially new in this chapter. Lemma 2.10, Proposition 2.14 and Proposition 2.24 which are proved in Chapter 2 could not be found in the literature by the author, but are most likely known results to those who have worked with bases and projections.

In Chapter 4 we generalize the notions of k-shrinking and k-boundedly complete to include Markushevich bases. We give some examples of quasi-reflexive spaces, and finally conclude with some problems which may lead to further research in quasi-reflexive spaces.

NOTATION, DEFINITIONS AND WELL-KNOWN THEOREMS

This section includes standard information and notation which will be used throughout this work.

Notation 2.1. The letters X, Y, and H will always denote Banach spaces. The conjugate (dual) of X we write as X^* and the second conjugate as X^{**}.

Notation 2.2. There is a natural map Q from X into X^{**} defined by $(Q(x))f = f(x)$ where $x \in X$ and $f \in X^*$. We will write $Q(x)$ simply as x, and the distinction between x and $Q(x)$ will be obvious from the context. We call $Q(X)$ the natural embedding of X in X^{**}.

Notation 2.3. When we say A is a subspace of X we will always take A to be closed. If $X \supset Y$ and $f \in X^*$ then $f|Y \in Y^*$, where $(f|Y)(y) = f(y)$ for all $y \in Y$. We call $f|Y$ the restriction of f to Y.

Definition 2.4. By $y = \sum_{i=1}^{\infty} y_i$, where $y_i \in Y$ for all i, we mean $\sum_{i=1}^{n} y_i$ converges to y with n, in the norm (strong) topology of Y. We will write $\sum_{i=1}^{\infty} y_i$ simply as $\sum y_i$, and will include the limits of summation only for finite sums or where the index of summation is not clear.

<u>Notation 2.5</u>. We will write sequences $(a_i)_{i=1}^{\infty}$ as (a_i).

<u>Notation 2.6</u>. If $(x_i) \subset X$, we denote the smallest subspace of X containing (x_i) by $[x_i]$.

We now give some results from basis theory.

<u>Definition 2.7</u>. A sequence $(x_i) \subset X$ is a basis for X if for each $x \varepsilon X$ there is a unique sequence (a_i) such that $x = \sum a_i x_i$. If (x_i) is a basis for $[x_i]$ we say (x_i) is a basic sequence. If $p_1 < p_2 < \ldots$ and $z_n = \sum_{i=p_n+1}^{p_{n+1}} b_i x_i$, where $z_n \neq 0$ for all n, we say z_n is a block basic sequence of (x_i). (The fact that (z_i) is basic will follow from Lemma 2.9, given later.)

Pelczynski [9] has proved the following theorem.

<u>Theorem 2.8</u>. If X is not reflexive then X contains a basic sequence (x_i) such that $[x_i]$ is not reflexive.

The next lemma is an important characterization of basic sequences.

<u>Lemma 2.9</u>. $(x_i) \subset X$, where $x_i \neq 0$ for all i, is a basic sequence if and only if there is a K such that $K||\sum_{i=1}^{n+m} a_i x_i|| \geq ||\sum_{i=1}^{m} a_i x_i||$ for all positive integers n and m and for all choices of a_i. Also if (x_i) is basic then $2K||\sum a_i x_i|| \geq ||\sum_{i=n}^{m} a_i x_i||$ for $m \geq n$, [12, p. 211].

Lemma 2.10. Let (x_i) be a basic sequence and let $p_1 < p_2 \ldots$. If $\sum_{n=1}^{m} \left(\sum_{i=p_n+1}^{p_n+1} a_i x_i \right)$ converges in m then $\sum_{i=1}^{q} a_i x_i$ converges in q. Also if $\left\| \sum_{n=1}^{m} \left(\sum_{i=p_n+1}^{p_{n+1}} a_i x_i \right) \right\|$ is bounded in m then $\left\| \sum_{i=1}^{q} a_i x_i \right\|$ is bounded in q (i.e. bounded in blocks implies bounded).

Proof of lemma. Let $k \leq q$. Let $p_n < k \leq q < p_m$. We can write $\sum_{i=k}^{q} a_i x_i = \sum_{i=p_n}^{p_m} a_i x_i - \sum_{i=p_n}^{k-1} a_i x_i - \sum_{i=q+1}^{p_m} a_i x_i$. Let K be as in Lemma 2.9. The conclusion follows simply by noting that

$$\left\| \sum_{i=p_n}^{k-1} a_i x_i \right\| \leq 2K \left\| \sum_{i=p_n}^{p_m} a_i x_i \right\| \text{ and } \left\| \sum_{i=q+1}^{p_m} a_i x_i \right\| \leq 2K \left\| \sum_{i=p_n}^{p_m} a_i x_i \right\|.$$

Notation 2.11. Let (x_i) be basic and let $f_j \in [x_i]^*$ and $f_j(x_i) = \delta_{ij}$. We call (f_j) the biorthognal functionals of (x_i). Whenever we are given a basic sequence (x_i), and we write (f_j), we will take (f_j) to be the biorthognal functionals of (x_i).

Lemma 2.12. If (x_i) is a basis for X then (f_i) is basic, [12, p. 210].

Definition 2.13. For $(y_i) \subseteq Y^*$ we say (y_i) converges weak star to y_0 if $(y_i(x))$ approaches $y_0(x)$ for all $x \in Y$. If

$\sum_{i=1}^{n} y_i$ converges in n weak star to y_0 we write $\sum^* y_i = y_0$.

<u>Proposition 2.14</u>. If (x_i) is a basic sequence and $f \in [x_i]^*$ then $f = \sum^* f(x_i)f_i$ and $f \in [f_i]$ if and only if $\sum^* f(x_i)f_i = \sum f(x_i)f_i$. Also if we define a new norm $|||\ |||$ on $[x_i]^*$ by $|||\sum^* f(x_i)f_i||| = \sup ||\sum_{i=1}^{n} f(x_i)f_i||$ where the supremum is over all n, then the new norm is equivalent to the old norm.

<u>Proof of proposition</u>. The only non-trival part of the proposition is proving the equivalence of the norms. (The first part follows simply by evaluating $\sum_{i=1}^{n} f(x_i)f_i$ at $\sum a_i x_i$.) Since $\sum^* f(x_i)f_i = f$ for all $f \in [x_i]^*$ we have that $|||f||| \geq ||f||$. Let K be as in Lemma 2.9, and let $||\sum a_i x_i|| \leq 1$ so $||\sum_{i=1}^{m} a_i x_i|| \leq K$.

$$\left|\left(\sum_{i=1}^{m} f(x_i)f_i\right)\left(\sum a_i x_i\right)\right| = \left|\left(\sum_{i=1}^{m} f(x_i)f_i\right)\left(\sum_{i=1}^{m} a_i x_i\right)\right| = \left|(f)\left(\sum_{i=1}^{m} a_i x_i\right)\right|$$

$\leq K||\ f\ ||$. Therefore $K||f|| \geq ||\sum_{i=1}^{m} f(x_i)f_i||$ for all m which shows $K||f|| \geq |||f|||$. Thus the two norms are equivalent.

We now consider quotient spaces.

Definition 2.15. Let $X \supset Y$. We write the quotient space of X with respect to Y as $\frac{X}{Y}$. The elements of $\frac{X}{Y}$ are the cosets $x + Y$ where $x \varepsilon X$ and $x + Y = \{w : w = x + y \text{ for } y \varepsilon Y\}$. We define $||x + Y|| = \inf ||x + y||$ where the infimum is taken over all $y \varepsilon Y$. With this norm $\frac{X}{Y}$ is a Banach space [4, p. 6].

Civin and Yood use the following three theorems in [2].

Lemma 2.16. If $H \subset \frac{X}{Y}$ then there is a closed subspace A of X such that $H = \frac{A}{Y}$.

Lemma 2.17. $\dfrac{\frac{X}{Y}}{\frac{H}{Y}}$ is isomorphic to $\frac{X}{H}$ where $X \supset H \supset Y$.

Lemma 2.18. If $\frac{X}{Y}$ is separable and Y is separable then X is separable.

We will need some theorems concerning direct sums and projections.

Definition 2.19. Let H_1 and H_2 be contained in X, and for each $x \varepsilon X$ there is a unique $h_1 \varepsilon H_1$ and $h_2 \varepsilon H_2$ such that $x = h_1 + h_2$. We say H_1 is a direct factor of X and write $X = H_1 \oplus H_2$. If we define an operator P from X to X by $P(h_1 + h_2) = h_1$ then $P^2 = P$, and we call P a projection of X onto H_1 along H_2. P is bounded in norm [5, p. 70]. Conversly if P is a bounded linear operator from X to X and if $P^2 = P$ then $X = P(X) \oplus (I-P)(X)$ where I is the identity map from X to X.

Notation 2.20. By $H \oplus x$ where $x \varepsilon X$, $H \subset X$ and $x \not\varepsilon H$ we mean the direct sum of H and the one dimensional subspace [x].

<u>Definition 2.21</u>. We say $(y_1, y_2 \ldots y_n)$ is independent of H if $\sum_{i=1}^{n} \alpha_i y_i \in H$ implies $\alpha_i = 0$ for $i = 1, 2, \ldots n$. Thus if $(y_1, y_2, \ldots y_n)$ is independent of H we have $[H, y_1, y_2, \ldots y_n] = H \oplus y_1 \oplus y_2 \oplus \ldots \oplus y_n$. Clearly if $(y_1, y_2, \ldots y_n)$ is independent of H then $(y_1, y_2 + \alpha_2 y_1 + h_2, y_3 + \alpha_3 y_1 + h_3, \ldots, y_n + \alpha_n y_1 + h_n)$ is independent of H where $h_2, h_3 \ldots h_n \in H$.

<u>Definition 2.22</u>. Let $X = H_1 \oplus H_2$. If the dimension of H_2 is n we say H_1 has codimension n in X. H_1 has codimension n in X if and only if any set of n + 1 elements in X is not independent of H_1, and if there exist a set of n elements $(y_1, y_2 \ldots y_n)$ which is independent of H_1, in which case $X = H_1 \oplus y_1 \oplus y_2 \oplus \ldots \oplus y_n$ [10].

<u>Lemma 2.23</u>. Let P be a projection from X onto H_1 along H_2. Thus $X = H_1 \oplus H_2$. Let $f \in H_1^*$ and let g be the extension of f to X defined by making $g(H_2) = 0$. Then $||g|| \leq ||f||\ ||P||$.

<u>Proof of lemma</u>. Since $||h_1 + h_2|| \leq 1$ implies $||h_1|| \leq ||P||$ for all $h_1 \in H_1$ and $h_2 \in H_2$, we have $||g|| = \sup|g(h_1 + h_2)| = $ $= \sup|g(h_1)| = \sup|f(h_1)| \leq ||f||\ ||P||$ where the supremum is taken over all $h_1 \in H_1$ and $h_2 \in H_2$ where $||h_1 + h_2|| \leq 1$.

<u>Proposition 2.24</u>. H has codimension n in X if and only if $H = \bigcap_{i=1}^{n} g_i^{-1}(0)$ where $(g_1, g_2, \ldots g_n)$ is a linearly independent set in X*.

Proof of proposition. Let $X = H \oplus y_1 \oplus y_2 \oplus \dots \oplus y_n$. Define g_j by $g_j(y_i) = \delta_{ij}$ and $g_j(H) = 0$ for $i,j = 1,2, \dots n$. By the preceding theorem $(g_1, g_2 \dots g_n) \subseteq X^*$. Clearly $H = \bigcap_{i=1}^{n} g_i^{-1}(0)$ and $(g_1, g_2, \dots g_n)$ is a linearly independent set.

Now assume $(g_1, g_2, \dots g_n)$ is a linearly independent set. There exist $y_i \in X$ such that $g_j(y_i) = \delta_{ij}$ [4, p. 6]. Define P by $P(x) = \sum_{i=1}^{n} g_i(x) y_i$ for all $x \in X$. P is a projection from X onto $[y_1, y_2, \dots y_n]$ and $(I-P)$ is a projection from X onto $\bigcap_{i=1}^{n} g_i^{-1}(0)$.

Theorem 2.25. Let H have codimension n in X. There is a projection P_η from X onto H such that $||P_\eta|| < 2^n + \eta$ for all $\eta > 0$ [6].

Proof of theorem. By proposition 2.24, we can find $(g_1, g_2, \dots g_n) \subset X^*$ such that $\bigcap_{i=1}^{n} g_i^{-1}(0) = H$ and $(g_1, g_2, \dots g_n)$ are linearly independent. Fix $\delta > 0$. Without loss of generality we can assume that $||g_1|| = 1$. Thus there is a $y_1 \in X$ such that $g_1(y_1) = 1$ and $||y_1|| < 1 + \delta$. Define the projection Q_1 from X onto $[y_1]$ by $Q_1(x) = g_1(x) y_1$ for all $x \in X$.
$||Q_1|| \leqq ||g_1|| \; ||y_1|| < 1 + \delta$. Let $P_1 = I_1 - Q_1$ where I_1 is the identity map from X to X. P_1 is a projection from X onto $g_1^{-1}(0)$ and $||P_1|| \leq 2 + \delta$. Since g_1 and g_2 are linearly independent, then $g_2 | g_1^{-1}(0) \neq 0$ [5, p. 421]. Let $g_2 | g_1^{-1}(0) = \overline{g_2}$. We can assume without loss of generality that $||\overline{g_2}|| = 1$. Thus there is

a $y_2 \in g_1^{-1}(0)$ such that $||y_2|| < 1 + \delta$ and $\overline{g_2}(y_2) = 1$. We define a projection Q_2 from $g_1^{-1}(0)$ to $[y_2]$ by $Q_2(x) = \overline{g_2}(x)y_2$ for all $x \in g_1^{-1}(0)$. Thus $||Q_2|| \leq 1 + \delta$. Therefore, for the projection $I_2 - Q_2$ from $g_1^{-1}(0)$ onto $\bigcap_{i=1}^{2} g_i^{-1}(0)$, where I_2 is the identity map from $g_1^{-1}(0)$ to $g_1^{-1}(0)$, we have $||I_2 - Q_2|| < 2 + \delta$. Let $P_2 = I_2 - Q_2$. Thus P_2P_1 is a projection from X onto $\bigcap_{i=1}^{2} g_i^{-1}(0)$ and $||P_2P_1|| < (2 + \delta)^2$. In the same manner we can construct P_k, a projection from $\bigcap_{i=1}^{k-1} g_i^{-1}(0)$ onto $\bigcap_{i=1}^{k} g_i^{-1}(0)$ for $k = 2,3, \ldots n$, such that $||P_k|| < 2 + \delta$. Thus if $P = P_nP_{n-1} \ldots P_1$ then P is a projection from X onto $\bigcap_{i=1}^{n} g_i^{-1}(0)$ and $||P|| < (2 + \delta)^n$. Since δ can be taken as small as desired, we have proved the theorem.

<u>Notation 2.26</u>. We say (a_{n_i}) is a proper subsequence of (a_i) if infinitely many integers are not contained in (n_i). If (m_i) is the subsequence of the integers obtained from those integers not contained in (n_i), then we say (a_{m_i}) is the complementary subsequence of (a_{n_i}). In the above i runs through the positive integers only.

SUBSPACES OF QUASI-REFLEXIVE SPACES

We define a quasi-reflexive space and then give known theorems which we will need on such spaces.

<u>Definition 3.1</u>. If the codimension (see Definition 2.22) of X in X^{**} (see Notation 2.1) under the natural embedding (see Notation 2.2) is finite we say X is quasi-reflexive. If the codimension is n we say X is quasi-reflexive of order n, and write this as $\text{ord}(X) = n$.

Clearly $\text{ord}(X) = 0$ if and only if X is reflexive.

Civin and Yood [2] have proved the following two important theorems on quasi-reflexive spaces.

<u>Theorem 3.2</u>. If $\text{ord}(X) = n$ and if Y is a subspace (see Notation 2.3) of X then $\text{ord}(Y)$ is less than or equal to n and $\text{ord}(X) = \text{ord}(Y) + \text{ord}\left(\frac{X}{Y}\right)$ (see Definition 2.15).

<u>Theorem 3.3</u>. If $\text{ord}(X) = n$, then there is a reflexive subspace Y, of X such that $\frac{X}{Y}$ is separable.

These two theorems imply that if $\text{ord}(X) = n$ then there is a separable quotient space of X which is quasi-reflexive of order n.

Cuttle [3] has proved the following proposition.

<u>Proposition 3.4</u>. If $\text{ord}(X) = n$ then $\text{ord}(X^*) = n$.

I. Singer [10] introduces the notions of k-shrinking and k-boundedly complete for basic sequences (see Definition 2.7).

Theorems dealing with these two notions will lead directly to the main theorem of this work (Theorem 3.22).

<u>Definition 3.5</u>. The basic sequence (x_i) (see Notation 2.5) is k-shrinking if the codimension of $[f_i]$ (see Notation 2.6, Notation 2.11 and Lemma 2.12) in $[x_i]^*$ is k.

We note that (x_i) is 0-shrinking implies that $[f_i] = [x_i]^*$. We say (x_i) is shrinking [4, p. 70] if $||f|[x_n, x_{n+1}, \ldots]||$ (see Notation 2.3) approaches 0 in n for all f in $[x_i]^*$. (x_i) is 0-shrinking if and only if (x_i) is shrinking [4, p. 70].

<u>Definition 3.6</u>. To define k-boundedly complete we consider the set $B_{[x_i]}$ of sequences $(a_i x_i)$ where (x_i) is a basic sequence and $||\sum_{i=1}^{n} a_i x_i||$ is bounded in n. Let $C_{[x_i]}$ denote the subset of $B_{[x_i]}$ consisting of those sequences $(a_i x_i)$ such that $\sum_{i=1}^{n} a_i x_i$ converges in n. We note that $B_{[x_i]}$ may be considered a vector space in the natural way. If there are k sequences $(a_{ij} x_i)_{i=1}^{\infty}$ for j = 1,2, ... k which belong to $B_{[x_i]}$ such that $B_{[x_i]} = C_{[x_i]} \oplus (a_{i1} x_i) \oplus (a_{i2} x_i) \oplus \ldots \oplus (a_{ik} x_i)$ (see Definition 2.19 and Notation 2.20), then we say (x_i) is k-boundedly complete. Thus $C_{[x_i]}$ has codimension k in $B_{[x_i]}$ if and only if (x_i) is k-boundedly complete.

A basic sequence (x_i) is said to be boundedly complete

[4, p. 69] if whenever $\|\sum_{i=1}^{n} a_i x_i\|$ is bounded in n then $\sum_{i=1}^{n} a_i x_i$

converges in n.

Thus, (x_i) is 0-boundedly complete if and only if (x_i) is boundedly complete.

James [8] proved that if (x_i) is a basic sequence then $[x_i]$ is reflexive if and only if (x_i) is shrinking and boundedly complete. Singer [10] generalizes this result in the following theorem.

Theorem 3.7. Let (x_i) be a basic sequence. Ord $([x_i]) = n$ if and only if (x_i) is k_1-shrinking and k_2-boundedly complete where $k_1 + k_2 = n$.

Definition 3.8. Let (x_i) be basic in X. f in X* is said to be shrinking on (x_i) if $\|f|[x_n, x_{n+1}, \ldots]\|$ approaches 0 in n.

Thus (x_i) is shrinking if and only if $f \in [x_i]^*$ implies f is shrinking on (x_i).

Lemma 3.9. Let (x_i) be a basis for X and let $f \in X^*$. The following three statements are equivalent:

(1) f is not shrinking on (x_i).

(2) $f \notin [f_i]$.

(3) There exist $p_1 < p_2 < p_3 \ldots$ and $\eta > 0$ such that

$$\Big\| \sum_{i=p_n+1}^{p_{n+1}} a_i x_i \Big\| = 1 \text{ and } \Big| f\Big(\sum_{i=p_n+1}^{p_{n+1}} a_i x_i \Big) \Big| > \eta \text{ for } n = 1, 2, \ldots .$$

Proof of lemma. By Proposition 2.14 we can write $f = \sum^{*} b_i f_i$

where $b_i = f(x_i)$. If $f \in [f_i]$ then $\sum_{i=1}^{n} b_i f_i$ converges in n so

$|| \sum_{i=n}^{\infty} b_i f_i ||$ approaches 0 in n. But this means

$|| f | [x_n, x_{n+1}, \ldots .] ||$ approaches 0 in n, since

$\sum_{i=n}^{\infty} b_i f_i | [x_n, x_{n+1}, \ldots] = f | [x_n, x_{n+1}, \ldots]$. Thus (1) implies (2).

If $f \notin [f_i]$ then there exists a $\delta > 0$ and $p_1 < p_2 < \ldots$ such that $|| \sum_{i=p_n+1}^{p_{n+1}} b_i f_i || > \delta$. Thus there exist for each n, $\sum a_i^n x_i$ such that $|| \sum a_i^n x_i || = 1$ and

$\left| \left(\sum_{i=p_n+1}^{p_{n+1}} b_i f_i \right) \left(\sum a_i^n x_i \right) \right| > \delta$. But $|| \sum_{i=p_n+1}^{p_{n+1}} a_i^n x_i || < 2K$ where K is as in Lemma 2.9. This shows (2) implies (3) where $\eta = \frac{\delta}{2K}$.

Clearly (3) implies (1) completing the proof of the lemma.

We now proceed to prove several theorems concerning the notions of k-shrinking and k-boundedly complete. These theorems will enable us to obtain, from a quasi-reflexive space of order n, subspaces of lesser order.

Proposition 3.10. If (x_i) is basic and (z_i) is a block basic sequence of (x_i) (see Definition 2.7) then:

(1) If (x_i) is k-shrinking then (z_i) is $\bar{k}$-shrinking where $\bar{k} \leq k$.

(2) If (x_i) is k-boundedly complete then (z_i) is $\bar{k}$-boundedly complete where $\bar{k} \le k$.

<u>Proof of proposition</u>. Let $z_n = \sum_{i=p_n+1}^{p_{n+1}} b_i x_i$ and $z_n \neq 0$ for all n and for $p_1 < p_2 < \dots$.

Assume (x_i) is k-shrinking. Therefore we can write $[x_i]^* = [f_i] \oplus f_{01} \oplus f_{02} \oplus \dots \oplus f_{0k}$ (see Definition 3.5 and Definition 2.22). Let $(g_j) \in [z_i]^*$ be the biorthognal functionals of (z_i). We wish to show that the codimension of $[g_i]$ in $[z_i]^*$ is less than or equal to k. We deny this by assuming $(g_{01}, g_{02}, \dots g_{0k+1}) \subset [z_i]^*$ is independent of $[g_i]$. Let $\bar{g}_{0i}$ be a functional in $[x_i]^*$ such that $\bar{g}_{0i} | [z_i] = g_{0i}$ for $i = 1, 2, \dots k + 1$. Since $[f_i]$ has codimension k in $[x_i]^*$ there exist $\alpha_1, \alpha_2, \dots \alpha_{k+1}$, not all zero, such that $\sum_{i=1}^{k+1} \alpha_i \bar{g}_{0i} \in [f_i]$. Thus $\sum_{i=1}^{k+1} \alpha_i \bar{g}_{0i}$ is shrinking on (x_i) (see Lemma 3.9 and Definition 3.8). But this implies $\sum_{i=1}^{k+1} \alpha_i g_{0i}$ is shrinking on (z_i). Thus $\sum_{i=1}^{k+1} \alpha_i g_{0i} \in [g_i]$ where not all $(\alpha_1, \alpha_2, \dots \alpha_{k+1})$ are zero. This is a contradiction and thus (1) is proved.

Now assume (x_i) is k-boundedly complete. Assume $B_{[z_i]}$ (see definition 3.6) contains k + 1 elements $(a_{ij} z_i)_{i=1}^{\infty}$ for

$j = 1, 2, \ldots k+1$, which are independent of $C_{[z_i]}$ (see Definition 2.21). Let $(c_{ij}x_i)_{i=1}^{\infty}$ be the expansion of $(a_{ij}z_i)_{i=1}^{\infty}$ in terms of (x_i) (i.e. $c_{11} = a_{11}b_1, c_{21} = a_{11}b_2 \ldots, c_{p_1 1} =$ $= c_{11}b_{p_1}, c_{p_1+1 1} = a_{21}b_{p_1+1} \ldots$). By Lemma 2.10, $(c_{ij}x_i)_{i=1}^{\infty} \varepsilon B_{[x_i]}$ for $j = 1, 2, \ldots k+1$. Since $C_{[x_i]}$ has codimension k in $B_{[x_i]}$, there exist $\alpha_1, \alpha_2 \ldots \alpha_{k+1}$ such that $\sum_{j=1}^{k+1} \alpha_j(c_{ij}x_i) \varepsilon C_{[x_i]}$ where not all α_j are zero (see Definition 2.22). Thus, we have

$$\sum_{j=1}^{k+1} \alpha_j(a_{ij}z_i) \varepsilon C_{[z_i]}.$$

This contradicts the assumption that $(a_{ij}z_i)_{i=1}^{\infty}$ are independent of $C_{[z_i]}$. Therefore by Definition 2.22 the codimension of $C_{[z_i]}$ in $B_{[z_i]}$ is less than or equal to k. This proves (2).

We wish to show that if (x_i) is basic and ord $([x_i]) = n > 0$ then we can find a block basic sequence (z_i) of (x_i) which will reduce the shrinking order and the boundedly complete order of (x_i) so that ord $([z_i]) = 1$ (in other words so that (z_i) is 1-shrinking and 0-boundedly complete or 0-shrinking and 1-boundedly complete (see Theorem 3.7)).

The previous theorem shows that neither order will be increased by taking a block basis. The next theorem deals with reducing the shrinking order. The theorem after that will show that we can reduce the boundedly complete order.

Theorem 3.11. If (x_i) is a basic sequence and if (x_i) is n-shrinking and if $k \leq n$ then there is a block basic sequence (z_i) of (x_i) such that (z_i) is k-shrinking.

To prove this theorem we first prove two lemmas.

Lemma 3.12. If (x_i) is basic and $f_{01}, f_{02}, \ldots f_{0n} \in [x_i]^*$ are independent of $[f_i]$ then $\|f_{0j} | [x_i]_{i=m}^{\infty} \cap \bigcap_{\substack{i=1 \\ i \neq j}}^{n} f_{0i}^{-1}(0)\| > \delta > 0$ for all m, $j = 1,2, \ldots n$ and some δ.

Proof of lemma. We will prove this lemma for $j = n$. This is obviously sufficient. Let $B_m = [x_i]_{i=m}^{\infty} \cap \bigcap_{i=1}^{n-1} f_{0i}^{-1}(0)$. We deny the conclusion of the lemma by assuming $\|f_{0n}|B_m\|$ approaches 0 in m. Let $\delta > 0$. There is an M such that $\|f_{0n}|B_M\| < \delta$. Let $f_{0i}|[x_i]_{i=M}^{\infty} = g_{0i}$ for $i = 1,2, \ldots n$. If

$\sum_{i=1}^{n} \alpha_i g_{0i} = 0$ then $\sum_{i=1}^{n} \alpha_i f_{0i} \in [f_i]_{i=1}^{M-1}$ $\Big($i.e. $\sum_{i=1}^{n} \alpha_i f_{0i} = \sum_{i=1}^{M-1} \beta_i f_i$ where $\beta_j = \left(\left(\sum_{i=1}^{n} \alpha_i f_{0i}\right) x_j\right)$$\Big)$. Thus $\alpha_1 = \alpha_2 = \ldots = \alpha_n = 0$ since f_{0i} is independent of $[f_i]$ (see Definition 2.22). Since $\sum_{i=1}^{n} \alpha_i g_{0i} = 0$ implies $\alpha_1 = \alpha_2 = \ldots \alpha_n = 0$ we have that $g_{01}, g_{02}, \ldots g_{0n}$ are linearly independent. By Proposition 2.24, $\bigcap_{i=1}^{n-1} g_{0i}^{-1}(0)$ has codimension $(n-1)$ in $[x_i]_{i=M}^{\infty}$. By Theorem 2.25, we can find $z_1, z_2, \ldots z_{n-1} \varepsilon [x_i]_{i=M}^{\infty}$ such that there is a projection P, from $[x_i]_{i=M}^{\infty}$ onto B_M along $[z_1, z_2, \ldots z_{n-1}]$ (see Definition 2.19) such that $||P|| \le 2^{n-1}+1$. Thus we have $[x_i]_{i=M}^{\infty} = B_M \oplus z_1 \oplus z_2 \oplus \ldots \oplus z_{n-1}$ and if $||b + z|| = 1$ then $||b|| \le 2^{n-1} + 1$ where $b \varepsilon B_M$ and $z \varepsilon [z_1, z_2, \ldots z_{n-1}]$. Since $g_{01}, g_{02}, \ldots g_{0n-1}$ vanish on B_M and are linearly independent on $[x_i]_{i=M}^{\infty}$, it follows that $g_{01}, g_{02}, \ldots g_{0n-1}$ are linearly independent on $[z_1, z_2, \ldots z_{n-1}]$. Therefore there exist $\alpha_1, \alpha_2, \ldots \alpha_{n-1}$ such that $g_{0n}(z_j) = \left(\sum_{i=1}^{n-1} \alpha_i g_{0i}\right) z_j$ for $j = 1,2, \ldots n-1$ (i.e. some linear combination of $g_{01}, g_{02}, \ldots g_{0n-1}$ must agree with g_{0n} on $[z_1, z_2, \ldots z_{n-1}]$). But $\left(g_{0n} - \sum_{i=1}^{n-1} \alpha_i g_{0i}\right) z_j = 0$ for $j = 1,2, \ldots n-1$ and $\left(g_{0n} - \sum_{i=1}^{n-1} \alpha_i g_{0i}\right) | B_M = g_{0n}$, since $g_{0i} \equiv 0$ on B_M for $i = 1,2, \ldots n-1$. Also $||g_{0n}|| = ||f_{0n} | B_M|| < \delta$. Therefore by Lemma 2.23, $\left|\left|\left(g_{0n} - \sum_{i=1}^{n-1} \alpha_i g_{0i}\right)\right|\right| < \delta$ and $||P|| < 2^{n-1} + 1$ imply

$$\left|\left|\left(g_{0n} - \sum_{i=1}^{n-1} \alpha_i g_{0i}\right)\right|\right| = \left|\left|\left(f_{0n} - \sum_{i=1}^{n-1} \alpha_i f_{0i}\right) | [x_i]_{i=M}^{\infty}\right|\right| < (2^{n-1} + 1)\delta.$$

The natural projection P_M from $[x_i]$ onto $[x_i]_{i=M}^{\infty}$

$\left(\text{i.e. } P_M\left(\sum_{i=1}^{\infty} a_i x_i\right) = \sum_{i=M}^{\infty} a_i x_i\right)$ has norm bounded in M by 2K where K is given in Lemma 2.9. Let $\beta_j = \left(f_{0n} - \sum_{i=1}^{n-1} \alpha_i f_{0i}\right)x_j$ for $j = 1,2, \ldots M-1$. Let $h_{0n} = f_{0n} - \sum_{i=1}^{n} a_i f_{0i} - \sum_{j=1}^{M-1} \beta_j f_j$. Therefore $h_{0n}|[x_i]_{i=1}^{M-1} = 0$ and $h_{0n}|[x_i]_{i=M}^{\infty} = g_{0n} - \sum_{i=1}^{n-1} \alpha_i g_{0i}$. Therefore

$||h_{0n}|[x_i]_{i=M}^{\infty}|| \leq (2^{n-1} + 1)\delta$. Therefore by Lemma 2.23, since $||P_M|| \leq 2K$, we have $||h_{0n}|| \leq 2K(2^{n-1} + 1)\delta$. Since δ can be taken arbitrarily small we conclude that

$f_{0n} \varepsilon [(f_i), f_{01}, f_{02}, \ldots f_{0n-1}]$. This is a contradiction and so the lemma is proved.

<u>Lemma 3.13</u>. Let (x_i) be a basic sequence in X and $||x_i|| > \delta > 0$ for some δ and for all i. If $\Sigma|f(x_i)|$ converges then f is shrinking on (x_i). Thus if (x_i) is a basic sequence and $|f(x_i)|$ approaches 0 in i then there is a subsequence (x_{n_i}) of (x_i) such that f is shrinking on (x_{n_i}).

<u>Proof of lemma</u>. Let $\Sigma|f(x_i)|$ converge. Let $f = \Sigma^*\alpha_i f_i$ (see Definition 2.13 and Proposition 2.14). Let $||\Sigma b_i x_i|| = 1$. Therefore, there is a K such that $||b_i x_i|| < 2K$ for all i. Thus

$$|b_i| < \frac{2K}{\delta} \cdot |\Big(\sum_{i=n}^{m} \alpha_i f_i\Big)\Big(\sum b_i x_i\Big)| = |f\Big(\sum_{i=n}^{m} b_i x_i\Big)| \leq \frac{2K}{\delta} \sum_{i=n}^{m} |f(x_i)|$$

for all $n \leq m$. Since $\sum_{i=n}^{m} |f(x_i)|$ approaches 0 uniformly in n and m, and uniformly for $\Sigma b_i x_i$ such that $||\Sigma b_i x_i|| = 1$ we conclude that $||\sum_{i=n}^{m} a_i f_i||$ approaches 0 uniformly in n and m so $\sum_{i=1}^{n} a_i f_i$ converges in n. Therefore $f \varepsilon [f_i]$ and thus by Lemma 3.9 f is shrinking on (x_i).

<u>Proof of theorem</u>. By hypothesis there exist $f_{01}, f_{02}, \ldots f_{0n} \varepsilon [x_i]^*$ such that $f_{01}, f_{02}, \ldots f_{0n}$ are independent of $[f_i]$ and $[x_i]^* = [f_i] \oplus f_{01} \oplus f_{02} \oplus \ldots \oplus f_{0n}$. We construct a block basic sequence (y_j) (see Definition 2.7) of (x_i) with the following properties:

(1) $\frac{1}{2} < ||y_j|| < \frac{3}{2}$ for all j.

(2) $|f_{0i}(y_{nq+i})| > \delta > 0$ for $i = 1,2, \ldots n$, $q = 1,2, \ldots$ and some δ.

(3) $|f_{0j}(y_{nq+i})| < \frac{1}{2^q}$ for $i = 1,2, \ldots n$, $i \neq j$, and $q = 1,2, \ldots$.

We will show how y_1 and y_2 are constructed from which the construction of the rest of the sequence (y_j) will be obvious. By Lemma 3.12, letting M = 1 there, we can find $\Sigma a_i x_i$ and $\delta > 0$ such that $||\Sigma a_i x_i|| = 1$, $|f_{0j}(\Sigma a_i x_i)| = 0$ for $j = 2,3, \ldots n$ and

$|f_{01}(\Sigma a_i x_i)| > \delta$. There is an integer N_1 such that if

$y_1 = \sum_{i=1}^{N_1} a_i x_i$ then y_1 satisfies (1), (2), and (3). Letting $j = 2$

and $M = N_1 + 1$ in Lemma 3.12 we see that there is $\sum_{i=N_1+1}^{\infty} a_i x_i$ such

that $\|\sum_{i=N_1+1}^{\infty} a_i x_i\| = 1$, $|f_{02}(\sum_{i=N_1+1}^{\infty} a_i x_i)| > \delta$ and

$|f_{0j}(\sum_{i=N_1+1}^{\infty} a_i x_i)| < \frac{1}{2}$ for $j = 1,3,4, \ldots n$. Therefore there is

an N_2 such that if $y_2 = \sum_{i=N_1+1}^{N_2} a_i x_i$ then y_2 satisfies (1), (2),

and (3). We now assume (y_j) is a block basic sequence of (x_i) with properties (1), (2), and (3). Let $1 \leq k \leq n$. We consider a subsequence (z_j) of (y_j) consisting of the elements of the form $(y_{nq+i})_{q=0}^{\infty}$ where $i = 1,2, \ldots k$ (i.e. we take the first k elements of (y_j), drop the next n-k elements, take the next k elements and so on). We will show that (z_j) is k-shrinking. Let $g_j \in [z_j]^*$, $g_j(z_i) = \delta_{ij}$ and $f_{0i}|[z_j] = g_{0i}$ for $i = 1,2, \ldots n$. For each $g \in [z_i]^*$, there is an $f \in [x_i]^*$ such that $f|[z_i] = g$. If $f \in [f_i]$ then f is shrinking on $[x_i]$ (see Lemma 3.9) and thus f is shrinking on $[z_i]$ so $f|[z_i] \in [g_j]$. By Lemma 3.13, since

$\sum_{q=1}^{\infty} |f_{0j}(y_{nq+i})| < \sum_{q=1}^{\infty} \frac{1}{2^q}$ for $j = k + 1, k + 2, \ldots n$ and for

$i = 1,2, \ldots k$, we have $f_{0k+1}, f_{0k+2}, \ldots f_{0n}$ are shrinking on (z_i) so $g_{0k+1}, g_{0k+2}, \ldots, g_{0n} \in [g_j]$. If we can show that $g_{01}, g_{02}, \ldots g_{0k}$ are independent of $[g_j]$ we will have proved the theorem for $1 \leq k \leq n$ (see Definition 2.21) since then $[z_i]^* = [g_i] \oplus g_{01} \oplus g_{02} \oplus \ldots \oplus g_{0k}$. Let

$$\sum_{i=1}^{k} \alpha_i g_{0i} = g \in [g_j]. \text{ If } \alpha_1 \neq 0 \text{ then } \alpha_1 g_{01} = -\sum_{i=2}^{k} \alpha_i g_{0i} + g.$$

Consider the subsequence $(y_1, y_{n+1}, y_{2n+1} \ldots)$ of (z_i). g is shrinking on this subsequence and, by Lemma 3.13 and by (3) $g_{02}, g_{03} \ldots g_{0k}$ are also shrinking on it

$$\left(\text{i.e. } \sum_{i=1}^{\infty} |g_{0j}(y_{ni+1})| = \sum_{i=1}^{\infty} |f_{0j}(y_{ni+1})| < \infty \text{ for } j = 2,3, \ldots k\right).$$

But since $|g_{01}(y_{ni+1})| > \delta > 0$ by (2), and since $||y_{ni+1}|| < \frac{3}{2}$ by (1), we have that g_{01} is not shrinking on $(y_{ni+1})_{i=1}^{\infty}$. Since the sum of shrinking functionals is shrinking then $\alpha_1 = 0$. In the same way $\alpha_2 = \alpha_3 = \ldots \alpha_k = 0$. By Definition 2.21, $g_{01}, g_{02}, \ldots g_{0k}$ are independent of $[z_i]$. Therefore $[z_i]$ is k-shrinking. To complete the proof we have only to consider the case $k = 0$. But $\bigcap_{i=1}^{n} f_{0i}^{-1}(0) \cap [x_i]_{i=1}^{n+1}$ contains a non-zero element z_1, since the codimension of $\bigcap_{i=1}^{n} f_{0_i}^{-1}(0) \cap [x_i]_{i=1}^{n+1}$ in $[x_i]_{i=1}^{n+1}$ is at most n (see Proposition 2.24). In a similar way we can find $z_j \varepsilon \bigcap_{i=1}^{n} f_{0i}^{-1}(0) \cap [x_i]_{i=(j-1)n+j-1}^{jn+j}$ where $z_j \neq 0$. Clearly as in the above proof $[z_i]$ is 0-shrinking since $f_{0i}[z_i] \equiv 0$ for $i = 1,2, \ldots n$.

Corollary 3.14. If (x_i) is a basis for X and if ord (X) = n and (x_i) is boundedly complete, then there is a block basic sequence (z_i) of (x_i) such that ord $([z_i]) = k$ for $k = 0,1,2, \ldots n$.

Proof of corollary. By the preceding theorem, we can find a block basic sequence (z_i) of (x_i) such that (z_i) is k-shrinking, since by Theorem 3.7 (x_i) is n-shrinking. By Proposition 3.10, (z_i) is 0-boundedly complete. Thus by Theorem 3.7 ord $([z_i]) = k$.

The above corollary is the strongest result we have in the sense that it not only yields subspaces of all orders less than the order of the space, but it gives these subspaces a basis which is a block basic sequence of the original space. In trying to reduce the boundedly complete order we can not show that we get all orders, less than the original order by using block bases. The next theorem does show however, that if the boundedly complete order is positive we can always reduce that order to 1.

Theorem 3.15. Let (x_i) be a basic sequence and let (x_i) be k-boundedly complete. Then we conclude the following:

(1) if $k \geq 2$ and $||x_i|| > \delta > 0$ for all i and some δ and if $(a_i x_i) \varepsilon B_{[x_i]}$ (see Definition 3.6), where a_i does not approach 0 in i, then there is a subsequence (x_{n_i}) of (x_i) such that (x_{n_i}) is (k-1) - boundedly complete.

(2) if $k \geq 1$ then there is a block basic sequence (z_i) of (x_i) such that (z_i) is 1-boundedly complete.

(3) if $k = 1$ then there is a block basic sequence (z_i) of (x_i) such that $||z_i|| > \delta > 0$ for some δ and for all i, and (z_i) is 1-boundedly complete, where $||\sum_{i=1}^{n} z_i||$ is bounded in n. Also if (z_i) is 1-boundedly complete, $||z_i|| > \delta > 0$ for some δ and for all i, and if $||\sum_{i=1}^{n} z_i||$ is bounded in n then (z_{n_i}) is 0-boundedly complete where (z_{n_i}) is any proper subsequence (see Notation 2.26) of (z_i). (We note that in this case $B_{[z_i]} = C_{[z_i]} \oplus (z_i)$).

<u>Proof of theorem</u>. Assume the hypothesis in (1). Since a_i does not approach 0 in i, $(a_i x_i) \not\in C_{[x_i]}$. Thus we can write $B_{[x_i]} = C_{[x_i]} \oplus (a_{i1}x_i) \oplus (a_{i2}x_i) \oplus (a_{i3}x_i) \oplus \ldots \oplus (a_{ik}x_i)$ where $(a_i x_i) = (a_{i1}x_i)$. Since $||\sum_{i=1}^{n} a_{ij}x_i||$ is bounded in n for $j = 1,2, \ldots k$, and since $||x_i|| > \delta$ for all i we have $|a_{ij}| < M$ for some M, $i = 1,2, \ldots,$ and $j = 1,2, \ldots k$ (see Lemma 2.9). Thus there is a subsequence $(a_{n_i 1})$ of (a_{i1}) such that $a_{n_i 1}$ approaches $a_1 \neq 0$ in i. Since $|a_{ij}| < M$ for $i = 1,2, \ldots,$ and $j = 1,2, \ldots k$, then we can assume without loss of generality

that $(a_{n_i j})$ approaches a_j in i for $j = 2,3, \ldots k$. By Definition 2.21 we can write

$$B_{[x_i]} = C_{[x_i]} \oplus (a_{i1}x_i) \oplus (a_{i2}x_i - \frac{a_2}{a_1} a_{i1}x_i)$$

$$\oplus (a_{i3}x_i - \frac{a_3}{a_1} a_{i1}x_i) \oplus \ldots \oplus (a_{ik}x_i - \frac{a_k}{a_1} a_{ik}x_i).$$

But $a_{n_i j} - \frac{a_j}{a_1} a_{n_i j}$ approaches 0 in i. Therefore we can write

$$B_{[x_i]} = C_{[x_i]} \oplus (a_{i1}x_i) \oplus (a_{i2}x_i) \oplus \ldots \oplus (a_{ik}x_i),$$

and assume without loss of generality that $a_{n_i j}$ approaches 0 in i for $j = 2,3, \ldots n$. Also without loss of generality we can assume that $|a_{n_i j}| < \frac{1}{2^i}$ for all i and $j = 2,3, \ldots k$. Therefore $\sum_{i=1}^{m} a_{n_i j} x_{n_i}$ converges in m for $j = 2,3, \ldots k$. Thus by Definition 2.21 we can assume without loss of generality that we can write $B_{[x_i]} = C_{[x_i]} \oplus (a_{1i}x_i) \oplus (a_{2i}x_i) \oplus \ldots (a_{ki}x_i)$ where $a_{n_i j} = 0$ for all i and $j = 2,3, \ldots k$, since $(a_{n_i j}x_{n_i}) \in C_{[x_i]}$ for $j = 2,3, \ldots k$. Let m_i be the complementary sequence (see Notation 2.26) of n_i. We will show that (x_{m_i}) is (k-1)-boundedly complete. First we note that $(a_{m_i 2}x_{m_i}), (a_{m_i 3}x_{m_i}), \ldots (a_{m_i k}x_{m_i})$ is independent of $C_{[x_{m_i}]}$ for if $\sum_{j=2}^{k} \alpha_j (a_{m_i j}x_{m_i}) \in C_{[x_{m_i}]}$ then

$\sum_{j=2}^{k} \alpha_j(a_{m_i j} x_{m_i}) \varepsilon\, C_{[x_i]}$ so $\alpha_2 = \alpha_3 = \dots = \alpha_k = 0$. Thus (x_{m_i}) is at least (k-1)-boundedly complete. Also by Proposition 3.10 (x_{m_i}) is less than or equal to k-boundedly complete. Assume (x_{m_i}) is k-boundedly complete. Then there are k sequences $(b_{m_i j} x_{m_i}) \varepsilon\, B_{[x_{m_i}]}$ for $j = 1,2, \dots k$ which are independent of $C_{[x_{m_i}]}$. But this implies the sequences are independent of $C_{[x_i]}$. Therefore $B_{[x_i]} = C_{[x_i]} \oplus (b_{m_i 1} x_{m_i}) \oplus (b_{m_i 2} x_{m_i}) \oplus \dots \oplus (b_{m_i k} x_{m_i})$. Thus there exist $\alpha_1, \alpha_2 \dots \alpha_k$ and $(c_i x_i) \varepsilon\, C_{[x_i]}$ such that $(a_{i1} x_i) = (c_i x_i) + \alpha_1(b_{m_i 1} x_{m_i}) + \alpha_2(b_{m_i 2} x_{m_i}) \dots + \alpha_k(b_{m_i k} x_{m_i})$. It follows that $a_{1n_i} = c_{n_i}$ and $a_{1m_i} = c_{m_i} + \alpha_1 b_{m_i 1} + \alpha_2 b_{m_i 2}$, for all i. But $a_{n_i 1}$ approaches $a_1 \neq 0$ in i and c_{n_i} approaches 0 in i since $(c_i x_i) \varepsilon\, C_{[x_i]}$. This is a contradiction. Thus (x_{m_i}) is not k-boundedly complete, so (x_{m_i}) is (k-1)-boundedly complete. This completes the proof of (1).

To prove (2) we show that there is a block basic sequence (z_i) of (x_i) such that (z_i) is strictly less than k-boundedly complete and greater than or equal to 1-boundedly complete. If we can prove this then (2) will be proved, since if (z_i) is a block basic sequence of (x_i) and if (y_i) is a block basic sequence of (z_i), then (y_i) is a block basic sequence of (x_i).

Let $(a_i x_i) \in B_{[x_i]}$ and $(a_i x_i) \notin C_{[x_i]}$. There exist $p_1 < p_2 < \dots$ and $\delta > 0$ such that if $z_n = \sum_{i=p_n+1}^{p_{n+1}} a_i x_i$ then $||z_n|| > \delta$ for $n = 1, 2, \dots$. $||\sum_{n=1}^{m} z_n||$ is bounded in m since $(a_i x_i) \in B_{[x_i]}$, and $\sum_{n=1}^{m} z_n = \sum_{i=p_1+1}^{p_{m+1}} a_i x_i$. Therefore (z_i) is at least 1-boundedly complete since $(z_i) \in B_{[z_i]}$ and $(z_i) \notin C_{[z_i]}$. Also by Proposition 3.10 (z_i) is less than or equal to k-boundedly complete. If (z_i) is strictly less than k-boundedly complete we are done. If not we can use (1) to get (z_{n_i}) (k-1)-boundedly complete for $k \geq 2$. (2) is now proved. Also, the construction of (z_i) is sufficient to prove the first statement in (3).

Now assume (z_i) is 1-boundedly complete and $||z_i|| > \delta > 0$ for some δ and all i. Also assume $||\sum_{i=1}^{n} z_i||$ is bounded in n. Let (z_{n_i}) be a proper subsequence of (z_i) and (z_{m_i}) the complementary sequence of (z_{n_i}). Assume (z_{n_i}) is 1-boundedly complete. Let $(a_{n_i} z_{n_i}) \in B_{[z_{n_i}]}$ but $(a_{n_i} z_{n_i}) \notin C_{[z_{n_i}]}$. Thus $B_{[z_i]} = C_{[z_i]} \oplus (a_{n_i} z_{n_i})$. Therefore there is an α and $(c_i z_i) \in C_{[z_i]}$ such that $(z_i) = \alpha(a_{n_i} z_{n_i}) + (c_i z_i)$, since

$(z_i) \in B_{[z_i]}$. But c_i approaches 0 in i and by the above equation $c_{m_i} = 1$. This is a contradiction. Therefore (z_{n_i}) is 0-boundedly complete. This proves (3) and completes the proof of the theorem.

Corollary 3.16. Let (x_i) be a basic sequence and let it be k-boundedly complete where $k \geq 1$. Then there is a block basic sequence (z_i) of (x_i) such that (z_i) is 1-boundedly complete,

$$||z_i|| > \delta > 0 \text{ for some } \delta \text{ and all } i, \; ||\sum_{i=1}^{n} z_i|| \text{ is bounded in } n,$$

and (z_{n_i}) is 0-boundedly complete where (z_{n_i}) is a proper subsequence of (z_i).

This corollary follows directly from (2) and (3) of the preceding theorem.

Corollary 3.17. Let (x_i) be a shrinking basic sequence. If ord $([x_i]) = n > 0$ then there is a block basic sequence (z_i) of (x_i) such that ord $([z_i]) = 1$ and ord $([z_{n_i}]) = 0$ if (z_{n_i}) is a proper subsequence of (z_i).

Proof of corollary. By Theorem 3.7 (x_i) is n-boundedly complete. We get (z_i) from the preceding corollary, and note that z_i is 0-shrinking by Proposition 3.10. Using Theorem 3.7 again, we complete the proof of the corollary.

We have been able to reduce the shrinking order of a non-shrinking basis to 1 (Theorem 3.11), and to reduce the boundedly complete order of a non-boundedly complete basis to 1

(Theorem 3.15). The next theorem deals with a basic sequence (x_i) which is 1-shrinking and 1-boundedly complete. It is the last result we need in order to show that if ord $([x_i]) = n > 0$ then ord $([z_i]) = 1$, for (z_i) which is some block basic sequence of (x_i). This last result follows the next theorem.

Theorem 3.18. Let (x_i) be basic. If (x_i) is 1-shrinking and 1-boundedly complete, there is a block basic sequence (z_i) of (x_i) such that ord $([z_i]) = 1$ (i.e. (z_i) is 1-shrinking and 0-boundedly complete or (z_i) is 0-shrinking and 1-boundedly complete).

Proof of theorem. Let (z_i) be a block basic sequence as described in Corollary 3.16. If (z_i) is 0-shrinking we are done. If not then (z_i) is 1-shrinking so there is an $f \in [z_i]^*$ which is not shrinking on $[z_i]$. Therefore by Lemma 3.9 there exist $p_1 < q_1 < p_2 < q_2 \ldots$ such that $\|f|[z_i]_{i=p_n}^{q_n}\| > \delta > 0$ for some δ. Therefore the proper subsequence $z_{p_1}, z_{p_1+1} \ldots z_{q_1}, z_{p_2}, z_{p_2+1} \ldots z_{q_2}, z_{p_3} \ldots$ of (z_i) is 1-shrinking. By Corollary 3.16 this proper subsequence is 0-boundedly complete. This proves the theorem.

Theorem 3.19. Let (x_i) be a basic sequence and let ord $([x_i]) = n > 0$. We have the following:

(1) There is a block basic sequence (z_i) of (x_i) such that ord $([z_i]) = 1$.

(2) If in (1) (z_i) is shrinking then we can obtain

a basic sequence (y_i) from (z_i) in such a way that $[y_i] = [z_i]$ and (y_i) is boundedly complete.

(3) If in (1) (z_i) is boundedly complete then we can obtain a basic sequence (y_i) from (z_i) such that $[y_i] = [z_i]$ and (y_i) is shrinking.

Proof of theorem. Let (x_i) be k-shrinking and q-boundedly complete. If $k = 0$ or $q = 0$ we have (1) by Corollary 3.14 or Corollary 3.16. If not then by Theorem 3.11 we can find a block basic sequence (y_{1i}) of (x_i) such that (y_{1i}) is 1-shrinking. If (y_{1i}) is 0-boundedly complete we are done. If not then by Theorem 3.15 we can find a block basic sequence (y_{2i}) of (y_{1i}) such that (y_{2i}) is 1-boundedly complete. If (y_{2i}) is 0-shrinking we are done. If not (y_{2i}) is 1-shrinking and 1-boundedly complete, so by Theorem 3.18 there is a block basic sequence (y_{3i}) of (y_{2i}) such that ord $([y_{3i}]) = \underline{1}$. This proves (1).

If we assume the hypothesis in (2) then (z_i) is 1-boundedly complete. By Corollary 3.16 there is a block basic sequence (y_i) of (z_i) such that $||y_i|| > \delta > 0$ for some δ and all i, $||\sum_{i=1}^{n} y_i||$ is bounded in n. Singer [11, p. 354] calls such basic sequences type P sequences and proves that if $w_n = \sum_{i=1}^{n} y_i$ then $[w_i] = [y_i]$ and (w_i) is a basic sequence. Singer also shows that there is an $f \in [w_i]^*$ such that $f(w_i) = 1$. But since $||w_i||$ is bounded in i, we have that f is not shrinking on (w_i).

Thus (w_i) is not shrinking. Since $||\sum_{i=1}^{n} y_i||$ is bounded in n we have that (y_i) is 1-boundedly complete and ord $([y_i]) = 1$. Thus, ord $([w_i])$ = ord $([y_i]) = 1$. Since (w_i) is not shrinking then (w_i) must be 1-shrinking and 0-boundedly complete. This proves (1).

Now let (z_i) be 1-shrinking and boundedly complete. Let $f \varepsilon [z_i]^*$ and f not shrinking on (z_i). By Lemma 3.9 there are $p_1 < p_2 < \ldots$, (a_i), $\delta > 0$ and M such that if $y_n = ||\sum_{i=p_n+1}^{p_{n+1}} a_i z_i||$ then $M > ||y_i|| > \delta$ for all i and $|f(y_i)| > \eta$ for all i and some η. Let $g_j \varepsilon [y_i]^*$ and $g_j(y_i) = \delta_{ij}$. Since $\Sigma^* g_i = f|[y_i]$ we have that $||\sum_{i=1}^{n} g_i||$ is bounded in n. For (y_i) with the above properties Singer [11, 356] uses the term P^*. He shows that if $w_1 = y_1$, $w_2 = y_1 - y_2$, $w_3 = y_2 - y_3$, $w_4 = y_3 - y_4$, ... then (w_i) is basic and $[w_i] = [y_i]$. Also $||\sum_{i=1}^{n} w_i|| = ||2y_1 - y_n||$ which is bounded in n, and $||w_i||$ is bounded away from 0 by Lemma 2.9. Thus $(w_i) \varepsilon B_{[w_i]}$ and $(w_i) \not\varepsilon C_{[w_i]}$. Thus (w_i) is not boundedly complete. Also ord $([w_i])$ = ord $([z_i]) = 1$, so (w_i) is 1-boundedly complete and 0-shrinking. This proves the theorem.

<u>Corollary 3.20</u>. If ord $(X) = n > 0$ then X contains a

subspace Y which is separable, has a basis and ord (Y) = 1.

Proof of corollary. By Theorem 2.8 X contains a basic sequence (x_i) so that $[x_i]$ is not reflexive. By Theorem 3.2 $n \geq$ ord $([x_i]) \geq 1$. By Theorem 3.19 there is a block basic sequence (z_i) of (x_i) such that ord $([z_i]) = 1$. The proof is completed by noting that a space with a basis is separable.

The following corollary was proved in [7] in a different manner.

Corollary 3.21. If ord (X) = n and $X \supset H$ then H contains an infinite dimensional subspace which is reflexive.

Proof of corollary. If ord (H) = 0 then we are done. If ord (H) = $k > 0$ then by the preceding theorem there is a subspace Y of H such that $Y = [x_i]$ where (x_i) is basic and ord $([x_i]) = 1$. If (x_i) is 1-shrinking we apply Theorem 3.11 and see that the corollary is proved. If (x_i) is 1-boundedly complete then we apply Corollary 3.16 and see that the theorem is proved.

We now conclude this chapter with the main theorem of this work.

Theorem 3.22. If ord (X) = n and $k \leq n$ then there is a subspace of A_k of X such that A_k is separable and ord $(A_k) = k$.

Proof of theorem. The case $k = 0$ is treated in the preceding theorem. The case $k = 1$ is treated in Corollary 3.20.

We will now show we can find A_2 from which the construction of $A_3, A_4 \dots A_n$ will be obvious. By Theorem 3.2 ord $(\frac{X}{A_1}) = n-1$. By Corollary 3.20 there is a separable subspace W of $\frac{X}{A_1}$ such that ord $(W) = 1$. By Lemma 2.16 there is an A_2 in X such that $W = \frac{A_2}{A_1}$. Let $Y = \frac{X}{A_1}$. Again by Theorem 3.2 ord $(\frac{Y}{W}) = n-2$. By Lemma 2.17 $\frac{Y}{W}$ is isomorphic to $\frac{X}{A_2}$. Thus ord $(\frac{X}{A_2}) = n-2$ and so ord $A_2 = 2$. Also since W is separable and A_1 is separable we have that A_2 is separable by Lemma 2.18. This completes the proof of the theorem.

We note here that the case $k = n$ in the above theorem was proved by Singer in [10].

GENERALIZATIONS, EXAMPLES AND UNSOLVED PROBLEMS

We can ask which of the results in Chapter III that deal with bases can be generalized to Markushevich bases, which we will now define [1].

<u>Definition 4.1</u>. We say (x_i) is a Markushevich basis for X if $[x_i] = X$, there exist biorthognal functionals (f_i) of (x_i) and (f_i) is total on X (i.e. if $x \varepsilon X$ and $f_i(x) = 0$ for all i then $x = 0$). Clearly a basis is a Markushevich basis.

A natural generalization of k-shrinking for Markushevich bases would be as follows:

<u>Definition 4.2</u>. We say the Markushevich basis (x_i) is k-shrinking if the codimension of $[f_i]$ in $[x_i]^*$ is k.

We will now give some background to justify a definition of k-boundedly complete for Markushevich bases.

<u>Proposition 4.3</u>. If (x_i) is a Markushevich basis for X then (f_i) is a Markushevich basis for $[f_i]$.

<u>Proof of proposition</u>. For $x \varepsilon X$ we define $\varphi(x)$ to be a linear functional on $[f_i]$ by $(\varphi(x))f = f(x)$ where $f \varepsilon [f_i]$. Clearly $||\varphi(x)|| \leq ||x||$ so $\varphi(x) \varepsilon [f_i]^*$. Also $(\varphi(x_i))$ are the biorthognal functionals of (f_i). Thus we need only show that $(\varphi(x_i))$ is total on $[f_i]$. Let $f \varepsilon [f_i]$ and assume $(\varphi(x_i))f = 0$. Thus $f(x_i) = 0$ for all i so f is 0 on X so $f = 0$.

We have the following theorem [10].

Proposition 4.4. Let (x_i) be a basis for X. (x_i) is k-boundedly complete if and only if (f_i) is k-shrinking, and (x_i) is k-shrinking if and only if (f_i) is k-boundedly complete.

We can now define k-boundedly complete for a Markushevich basis.

Definition 4.5. A Markushevich basis (x_i) is k-boundedly complete if (f_i) is k-shrinking.

By Proposition 4.4 this definition corresponds to Definition 3.6 in the case where the Markushevich basis (x_i) is a basis.

We can generalize Theorem 3.7 to include Markushevich bases with the help of the following lemma.

Lemma 4.6. Let $H \subseteq X^*$ and H total over X. Let φ map X into H^* by $(\varphi(x))(f) = f(x)$ for $x \in X$ and $f \in H$. Then ord $(X) = n$ if and only if the codimension, k_1, of H in X^* is finite and the codimension, k_2, of $\varphi(X)$ in H^* is finite and $k_1 + k_2 = n$.

Proof of lemma. Assume ord $(X) = n$. Since H is total over X and $(\varphi(X))$ is total over H we have that k_1 and k_2 are finite and $k_1, k_2 \leq n$ [10]. Let $X^* = H \oplus f_{01} \oplus \dots \oplus f_{0k_1}$. Let $x_{01}, x_{02}, \dots x_{0k_1} \in X^{**}$ and $x_{0i}(H) = 0$ and $x_{0i}(f_{0j}) = \delta_{ij}$ for $i, j = 1,2, \dots k_1$. Let $H^* = \varphi(X) \oplus y_{01} \oplus \dots \oplus y_{0k_2}$. Let $\bar{y}_{0i} \in X^{**}$ and $\bar{y}_{0i}|H = y_{0i}$ and $\bar{y}_{0i}(f_{0j}) = 0$, for $i,j = 1,2, \dots k_2$. We will show

$(x_{01}, x_{02}, \dots x_{0k_1}, \bar{y}_{01}, \bar{y}_{02}, \dots \bar{y}_{0k_2})$ is independent of X. Assume

$\sum_{i=1}^{k_1} \alpha_i x_{0i} + \sum_{i=1}^{k_2} \beta_i \bar{y}_{0i} = x \in X$. But then $x|H = \sum_{i=1}^{k_2} \beta_i y_{0i} \in \varphi(X)$.

Since $(y_{01}, y_{02}, \dots y_{0k_2})$ is independent of $\varphi(X)$ we have $\beta_i = 0$ for $i = 1,2, \dots k_2$. Since $\sum_{i=1}^{k_1} \alpha_i x_{0i} = 0$ on H and H is total then $x = 0$. Thus $\alpha_i = 0$ for $i = 1,2, \dots k_1$ since $(x_{01}, x_{02}, \dots x_{0k_1})$ are linearly independent. Thus $(x_{01}, x_{02}, \dots x_{0k_1}, \bar{y}_{01}, \bar{y}_{02}, \dots \bar{y}_{0k_2})$ is independent of X. Thus $X^{**} \supseteq X \oplus x_{01} \oplus x_{02} \oplus \dots \oplus x_{0k_1} \oplus \bar{y}_{01} \oplus \bar{y}_{02} \oplus \dots \oplus \bar{y}_{0k_2}$. Let $F \in X^{**}$. Let $F|H = \varphi(x) + \sum_{i=1}^{k_2} \beta_i y_{0i}$. $F - \left(x + \sum_{i=1}^{k_2} \beta_i y_{0i}\right) = 0$ on H so there exist $\alpha_1, \alpha_2, \dots \alpha_{k_1}$ such that $F - \left(x + \sum_{i=1}^{k_2} \beta_i y_{0i}\right) = \sum_{i=1}^{k_1} \alpha_i x_{0i}$. Thus $F \in X \oplus x_{01} \oplus \dots x_{0k_1} \oplus \bar{y}_{01} \oplus \dots \oplus \bar{y}_{0k_2}$ and $X^{**} = X \oplus x_{01} \oplus x_{02} \oplus \dots \oplus x_{0k_1} \oplus \bar{y}_{01} \oplus \bar{y}_{02} \oplus \dots \oplus \bar{y}_{0k_2}$. Therefore $k_1 + k_2 = n$.

The proof of the converse is the same.

<u>Theorem 4.7</u>. Let (x_i) be a Markushevich basis for X. Ord (X) = n if and only if (x_i) is k_1-shrinking and k_2-boundedly complete where $k_1 + k_2 = n$.

Proof of theorem. By the previous theorem, since $[f_i]$ is total on X we need only show that (x_i) is k_2-boundedly complete if and only if the codimension of $\varphi(X)$ in $[f_i]^*$ is k_2, where φ is as defined in the previous theorem. (x_i) is k_2-boundedly complete if and only if $[\varphi(x_i)]$ has codimension k_2 in $[f_i]^*$. But $[\varphi(x_i)] = \varphi[X]$ and this completes the proof.

The following lemma gives a set of sufficient conditions for a basic sequence to be 1-boundedly complete. Thus if (x_i) is a basic shrinking sequence and satisfies the conditions of the lemma, then ord $([x_i]) = 1$.

Lemma 4.8. Let (x_i) be a basic sequence, and let $||x_i|| > \delta > 0$ for some δ and all i. (x_i) is 1-boundedly complete if the following conditions are satisfied:

(1) $||\sum_{i=1}^{n} x_i|| < M$ for all n and some M.

(2) If $||\sum_{i=1}^{n} a_i x_i||$ is bounded in n then (a_i) is a Cauchy sequence.

(3) If $||\sum_{i=1}^{n} a_i x_i||$ is bounded in n and if the sequence (a_i) approaches 0 then $\sum_{i=1}^{n} a_i x_i$ converges in n.

Proof of lemma. By (1) $(x_i) \in B_{[x_i]}$ and since $||x_i|| > \delta > 0$, $(x_i) \notin C_{[x_i]}$. We will show that

$B_{[x_i]} = C_{[x_i]} \oplus (x_i)$ and this will complete the proof. Let $(a_i x_i)$ belong to $B_{[x_i]}$. By (2) there exists an a such that (a_i) approaches a. By (3) $((a_i - a)x_i) \in C_{[x_i]}$ and $(a_i x_i) = ((a_i - a)x_i) + a(x_i)$. Thus $B_{[x_i]} = C_{[x_i]} \oplus (x_i)$.

Corollary 4.8. If (x_i) is basic, 0-shrinking and satisfies conditions (1), (2) and (3) of the above lemma then ord $([x_i]) = 1$.

The space which started the study of quasi-reflexive spaces is the James space denoted by J [8].

Definition 4.10. Let J consist of those sequences (a_i), which approach 0 and for which sup $\left(\sum_{i=1}^{n} (a_{q_i} - a_{p_i})^2\right)$ is finite where the supremum is taken over all sequences such that $p_1 < q_1 < p_2 < q_2 < p_3 \dots$. The norm of (a_i) is the supremum given above. If x_i denotes the element of J which has 1 in the i^{th} place and 0 elsewhere then (x_i) is a basis for J. R. C. James shows that (x_i) is shrinking and satisfies conditions (1), (2) and (3) of Lemma 4.8. Thus ord (J) = 1. Also if (f_i) are the biorthognal functionals of (x_i) then $J^* = [f_i]$ since (x_i) is shrinking and by Theorem 4.4 (f_i) is boundedly complete and 1-shrinking.

We remark here that if J_p consists of those sequences (a_i) as defined above except with the norm

$$||(a_i)|| = \sup \left(\sum_{i=1}^{n} |a_{q_i} - a_{p_i}|^p\right)^{\frac{1}{p}}$$ then J_p is a Banach space and

ord $(J_p) = 1$. This last statement is proved exactly the same way R. C. James proved ord $(J) = 1$ [8].

Since (x_i) is a basis of type P (see the proof of part (2) of Theorem 3.19) of J_p then (y_i) is basic, boundedly complete, and $(y_i) = J_p$ where $y_n = \sum_{i=1}^{n} x_i$. We now compute the norm of $\sum a_i y_i$. $\sum a_i y_i = \left(\sum_{j=1}^{\infty} a_j\right)x_1 + \left(\sum_{j=2}^{\infty} a_j\right)x_2 + \ldots + \left(\sum_{j=n}^{\infty} a_j\right)x_n + \ldots$.

Thus $||\sum a_i y_i|| = \sup \left(\sum_{j=1}^{n} |a_{p_j} + a_{p_j+1} + \ldots a_{q_j-1}|^p\right)^{\frac{1}{p}}$ where the supremum is taken over all sequences $p_1 < q_1 < p_2 < q_2 < p_3 \ldots$.

If $X = H \oplus Y$ then $\frac{X}{H}$ is isomorphic to Y. Thus by Theorem 3.2, ord $(X) = n$ if and only if ord (H) + ord $(Y) = n$. Since we have a quasi-reflexive space of order 1 we can get quasi-reflexive spaces of all orders simply by taking direct sums.

We conclude this work by listing some unsolved problems which are suggested by the results found here.

A) If (x_i) is a k-boundedly complete basis, can we find block bases of (x_i) which are boundedly complete of all orders less than k, as we can do in the k-shrinking case?

B) If (x_i) is basic and ord $[x_i] = n$ can we find a block basic sequence (z_i) of (x_i) such that ord $[z_i] = k$ for each $k < n$?

A negative answer to A would imply a negative answer to B, whereas a positive answer to A would go a long way in providing a positive answer to B. A positive answer to B would go far in solving the following problem.

C) If (x_i) is basic and m-shrinking and n-boundedly complete, can we find a block basic sequence (z_i) of (x_i) such that z_i is k-shrinking and q-boundedly complete for any k and q less than or equal to m and n respectively?

Theorem 3.11 answers C when $n = 0$.

D) Can we prove any of the theorems in Chapter 3 dealing with block bases in the more general setting of block Markushevich bases?

E) Being able to find quasi-reflexive subspaces of all orders less than the order of the space suggests the question as to whether we can project onto any of these subspaces. In particular do all quasi-reflexive spaces of order n contain a quasi-reflexive subspace of order n, which is the direct sum of n quasi-reflexive spaces of order 1? A positive answer to this question would make the proof of Theorem 3.22 trival. The author has not even been able to construct a quasi-reflexive space of order $n \geq 2$ in which there is not an obvious projection onto subspaces of order 1.

F) Is there a quasi-reflexive space of order n in which there is not a projection onto a separable quasi-reflexive space of order n?

G) What does the norm in $J_p{}^*$ look like?

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