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# DESIGN OF AN AUTOMOBILE CONTROLLER FOR OPTIMUM TRAFFIC RESPONSE TO STOCHASTIC DISTURBANCES

DISSERTATION

Presented in Partial Fulfillment of the Requirements for the Degree Doctor of Philosophy in the Graduate School of The Ohio State University

By

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## PUBLICATIONS

"An Adaptive Control System With Application to the Measurement of Human Operator Parameters," Proceedings of the International Congress on Human Factors in Electronics of the I. R. E., May, 1962.

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## FIELDS OF STUDY

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Studies in Physics. Professor Wave H. Shaffer

Studies in Servomechanisms. Professors Robert L. Cosgriff and F. Carlin Weimer

Studies in Communication Circuits. Professors C. Earl Warren and William C. Davis

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# CHAPTER I

## INTRODUCTION

Within the past decade there has been a major shift of interest in transportation research. Previously primary interest was in construction techniques, materials, and design. Every attempt was made to provide transport vehicles with an optimum passive environment through which they could pass with minimum delay and impedance. As traffic increased it became obvious that this was not adequate to meet the requirements of higher density traffic flow. These remarks are not restricted to the highway mode of transportation, although it is there that the problem has become most acute. With the increases in traffic, disturbances to the flow have become common.

Many of these disturbances have not been so much a result of the traffic environment but more a result of the manner in which the vehicles are controlled. Consequently the shift of interest has been from the environment to the dynamics of the traffic itself. Since the dynamic characteristics of vehicular traffic are determined by the mode of control of the traffic, increasing interest has been directed toward traffic control.[1, 2]

A considerable amount of research[3, 4, 5, 6] has been and is underway in an effort to gain an understanding of the nature of the present system of manual vehicular control. This task is made difficult by the quasi-random nature of the human controller. Simultaneously with this research, effort is being expended to improve the present system of vehicular control.

Electronic devices can be used in a number of ways to improve traffic flow. In general, the techniques fall into two broad categories. The first is the control of the overall traffic stream by external controllers. Examples of these are electronic control of traffic signals and electronic control of entrances and exits from freeways. On the other hand, electronic devices can improve control of individual vehicles within the traffic stream. This has been of particular interest to vehicle manufacturers and the federal government[8] for improving vehicle transportation, while retaining a maximum of freedom for the individual. This paper deals with the control of individual vehicles.

The Ohio Department of Highways and the U. S. Bureau of Public Roads have jointly sponsored a research project at The Ohio State University to study electronic devices as traffic aids. The investigation described in this paper is one phase of the study. Early in the study it was decided that it would be most fruitful to use electronic devices to improve control of individual vehicles in response to their local traffic situation. It was shown that by activating the road bed itself, i.e., to place vehicle detectors in the road, it is possible, with the aid of small electronic circuits connected to the detectors, to transmit information about the preceding vehicle to a following controlled vehicle.[9] The speed of the preceding vehicle and its distance from the following vehicle can be determined. With this as a basis, the study of methods for automatically controlling the following vehicle in response to its local situation was undertaken.[10]

Although it is not considered practical at the present time to proceed directly to automatic control, the development of such an automatic system demonstrates its feasibility and also indicates an upper limit on the ultimate capability of any man-machine controllers for vehicles. The study of the automatic longitudinal control system for individual vehicles has been quite instrumental in showing how restrictions on transport vehicles containing passengers determine the nature of the vehicle controller. Furthermore it has resulted in the design of the vehicle controller.[11]

The particular controller designed has nonlinear modes of control to accomplish such maneuvers as avoiding rear-end collisions. On the other hand, it has been determined that a linear mode controller is needed to provide stability in the traffic queue.[12]

The stability of linear mode controllers has been studied before.[13, 14] In these studies the linear mode controllers were mathematical models used to approximate manually controlled vehicles. Only the response of such controllers to idealized disturbances was studied.

In the development of automatic controllers for highway vehicles, the more realistic types of disturbances of the traffic system must be studied. Design considerations for the control systems of the vehicles depend on the response of a queue of controlled vehicles to the real world disturbances. These real world disturbances cannot be predicted with certainty for they are largely random in nature. Consequently they must be treated as stochastic processes. The random components of these disturbances can only be described by their statistical properties.[15] The response of a queue with linear mode controllers to these disturbances is also stochastic, and is also characterized by its statistical properties. The characterization of the disturbances and the resulting queue response by their statistical properties provides the basis for designing an optimum linear system.[16]

It is the philosophy of this paper that ideally a certain desired equilibrium condition should be maintained in steady state traffic flow. Random disturbances induced by external sources result in a random deviation of the queue from the equilibrium condition. This random deviation must be minimized by optimum design of linear controllers for the vehicles of the queue. In this manner maximum traffic flow and safety are achieved.

It is the purpose of this paper to develop the techniques of analysis and to show how the statistical properties of queue response measures are related to statistical measures of the disturbances. Then the general nature of the sources of disturbances are discussed, and the relationships between the statistical properties of the disturbances and those of their sources are derived. Finally, the techniques of analysis and descriptions of the disturbances are applied in the design of the optimum linear controller.

The second chapter introduces frequency response techniques which are required throughout this paper. The nature of the design problem is also introduced in this chapter. Measures of queue response to random disturbances introduced by the initial vehicle are developed in the third chapter. The nature of the response of a queue of vehicles to uncorrelated disturbances introduced by each vehicle is also considered in the third chapter. The fourth chapter treats the problem of random road-induced disturbances. Measures of queue response to road-induced disturbances are developed. In the fifth chapter the optimum design of the control system is developed. Conclusions and recommendations of areas of further study conclude this paper.

# CHAPTER II

# RESPONSE OF A QUEUE OF AUTOMATICALLY CONTROLLED VEHICLES TO SINUSOIDAL DISTURBANCES

## Introduction

It is the purpose of this chapter to introduce frequency response techniques and to use these techniques in the study of the dynamic characteristics of a queue of automatically controlled vehicles. Likewise, certain static characteristics of these controlled vehicles are introduced. This material is developed in the following order.

The definition of a linear system and the properties of linearity are expressed first. The properties of the mathematical representation of these systems leads to the definition of the gain function, and it is then shown that the response of a linear system is directly related to a sinusoidal disturbance by the gain function. System variables are defined for a queue of automatically controlled vehicles. Because of linearity of the system, it is possible to separate the queue's dynamic motion into a desired constant or equilibrium component and a disturbance from this equilibrium state due to disturbances externally induced into the queue. The queue's response is related to sinusoidal induced disturbances by the derived gain functions. These gain functions are necessary in later sections of this work in the study of random disturbances.

## Linear Time Invariant Systems

If the equations relating the response of a system to the input excitation are linear, the system is said to be linear. The equations characterizing the response y to input disturbance x are linear if x and y can be related by a linear combination of terms in x and y and their derivatives. This linear combination is simply a sum of the variables multiplied by coefficients independent of the dependent variables as expressed by Eq. (2-1).

$$a_n \frac{d^n}{dt^n} y(t) + a_{n-1} \frac{d^{n-1}}{dt^{n-1}} y(t) + \cdots + a_1 \frac{d}{dt} y(t) + a_0 y(t) = b_m \frac{d^m}{dt^m} x(t) + \cdots + b_1 \frac{d}{dt} x(t) + b_0 x(t) \tag{2-1}$$

For time invariant linear systems, to which attention is confined in this paper, the coefficients, the a's and b's, are constants. The symbol p is used to represent the operation of taking the derivative of the variable it precedes, such that

$$p\,y(t) = \frac{d}{dt}\,y(t),$$

$$p^2\,y(t) = \frac{d^2}{dt^2}\,y(t),$$

$$- - - - - - -$$

(2-2) $$p^n\,y(t) = \frac{d^n}{dt^n}\,y(t).$$

It is noted here that if A is a constant, then

(2-3) $$p^n A y(t) = A p^n y(t),$$

and also that

(2-4) $$p^n\left[y_1(t) + y_2(t)\right] = p^n y_1(t) + p^n y_2(t).$$

Substituting Eq. (2-2) into (2-1) yields

(2-5) $$\begin{aligned} &a_n p^n y(t) + a_{n-1} p^{n-1} y(t) + --- + a_1 p y(t) + a_0 y(t) \\ &= b_m p^m x(t) + --- + b_1 p x(t) + b_0 x(t). \end{aligned}$$

This equation is expressed in the form

(2-6) $$\begin{aligned} &(a_n p^n + a_{n-1} p^{n-1} + --- + a_1 p + a_0)\,y(t) \\ &= (b_m p^m + --- + b_1 p + b_0)\,x(t), \end{aligned}$$

or

$$D(p)\,y(t) = N(p)\,x(t),$$

and finally expressed by

$$y(t) = \frac{N(p)}{D(p)} x(t).$$

Here $N(p)/D(p)$ is the transfer function of the system characterized by Eq. (2-1).

By substituting Eqs. (2-3) and (2-4) into (2-5) two basic properties of linearity may be observed. First, if $y_1(t)$ is the response to the input $x_1(t)$, then the response to the input $Ax_1(t)$, where $A$ is a constant, is $Ay_1(t)$. Secondly, if $y_1(t)$ is the response to $x_1(t)$ and $y_2(t)$ the response to $x_2(t)$, then a response $y(t)$ to the input $x(t) = x_1(t) + x_2(t)$ is given by

$$y(t) = y_1(t) + y_2(t). \tag{2-7}$$

This property of linear systems is of prime inportance and is termed superposition. If the input signal disturbance to a linear system is equal to the sum of several components, then the response to this input is equal to the sum of the responses to each of the input components taken separately. The superposition principle will be used extensively throughout this paper to simplify considerations.

## The Gain Function

Let the input to the linear system be given by

$$x(t) = A e^{j\omega t}. \tag{2-8}$$

Because the linear combination of the response and its derivatives is equal to a linear combination of x(t) and its derivatives, the response is given by

$$y(t) = B e^{j(\omega t + \theta)}, \tag{2-9}$$

where B and $\Theta$ are constants. This is termed the steady state response and does not contain transient terms. It is seen from Eqs. (2-5) and (2-6) that

$$\begin{aligned}&\left[a_n(j\omega)^n + a_{n-1}(j\omega)^{n-1} + \cdots + a_1 j\omega + a_0\right] y(t) \\ &= \left[b_m(j\omega)^m + \cdots + b_1 j\omega + b_0\right] x(t).\end{aligned} \tag{2-10}$$

The ratio

$$\frac{\left[b_m(j\omega)^m + \cdots + b_1 j\omega + b_0\right]}{\left[a_n(j\omega)^n + a_{n-1}(j\omega)^{n-1} + \cdots + a_1 j\omega + a_0\right]}$$

is the "gain" function. It is noted that this gain function is found from the transfer function by setting $p = j\omega$, so the gain function is $\frac{N(j\omega)}{D(j\omega)}$. Also, if x(t) is given by Eq. (2-8), then

$$y(t) = \frac{N(p)}{D(p)} x(t) = \frac{N(j\omega)}{D(j\omega)} A e^{j\omega t}. \tag{2-11}$$

Since the gain is a complex number for any given frequency $\omega$, it may be written in polar form as

$$\frac{N(j\omega)}{D(j\omega)} = \left|\frac{N(j\omega)}{D(j\omega)}\right| e^{j\theta(\omega)}, \tag{2-12}$$

where $\Theta(\omega)$ is the argument of the gain function. Substituting Eq. (2-12) into (2-11)

$$y(t) = \left|\frac{N(j\omega)}{D(j\omega)}\right| A e^{j(\omega t+\theta)}. \tag{2-13}$$

Knowing the steady state solution for the exponential function, one can quickly define the response for $x(t)$ sinusoidal, namely,

$$x(t) = A \cos \omega t = \frac{A}{2}\left(e^{j\omega t} + e^{-j\omega t}\right). \tag{2-14}$$

By superposition and Eqs. (2-11) and (2-13), $y(t)$ is given by

$$y(t) = \left|\frac{N(j\omega)}{D(j\omega)}\right| \frac{A}{2} e^{j[\omega t+\theta(\omega)]} + \left|\frac{N(-j\omega)}{D(-j\omega)}\right| \frac{A}{2} e^{-j[\omega t-\theta(-\omega)]}.$$

But from theory of complex numbers it is known that

$$\left|\frac{N(-j\omega)}{D(-j\omega)}\right| = \left|\frac{N(j\omega)}{D(j\omega)}\right|$$

and $\Theta(-\omega) = -\Theta(\omega)$ . Thus

$$y(t) = \left|\frac{N(j\omega)}{D(j\omega)}\right| \frac{A}{2}\left[e^{j(\omega t+\theta)} + e^{-j(\omega t+\theta)}\right] = \left|\frac{N(j\omega)}{D(j\omega)}\right| A \cos(\omega t+\theta). \tag{2-15}$$

It is observed that for sinusoidal inputs, the response is also sinusoidal. Its amplitude is multiplied by the magnitude of the gain function and advanced or delayed in time by $\frac{\Theta(\omega)}{\omega}$ , depending on the sign of $\Theta(\omega)$ . Methods of analysis based upon periodic disturbances are termed frequency response methods, which are used throughout this work.

Problem Definition

Before proceeding it is necessary to define variables to be used in representing a queue of vehicles and to define the equilibrium state of the vehicles of the queue. In this development, the equilibrium state exists when the velocities of all vehicles are equal and constant. This equilibrium velocity is denoted by $v_{ss}$ . The vehicles are assumed to be traveling in the positive $x$ direction. The variables describing a queue of vehicles are shown in Figure 1.

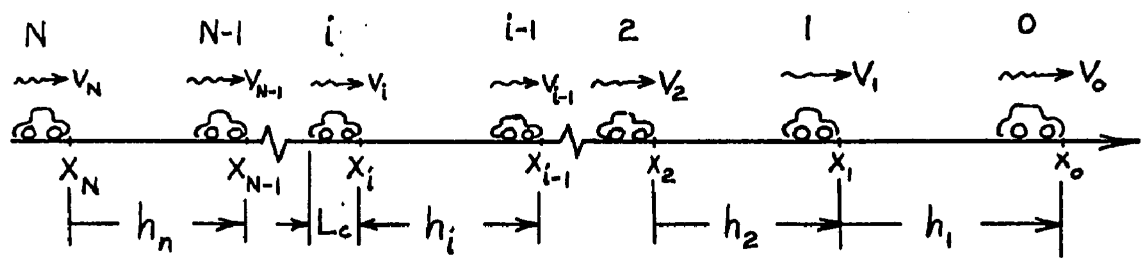


Fig. 1.--Traffic Queue Coordinates

The initial vehicle is indexed zero, with indices of the following vehicles progressing from 1 to N . The instantaneous position relative to an arbitrary reference on the road of the leading edge of the ith vehicle is $x_i$ as indicated previously. The headway $h_i$ of the ith vehicle is defined to be the position of the i-1st vehicle minus the position of the ith vehicle. The absolute velocity $v_i$ of the ith vehicle is

$$v_i = px_i . \tag{2-16}$$

The absolute acceleration of the ith vehicle is

$$(2\text{-}17) \qquad a_i = p^2 x_i .$$

The relative velocity of the ith vehicle is the derivative of its headway and is denoted by $v_{r_i}$ , where

$$(2\text{-}18) \qquad v_{r_i} = p h_i = p x_{i-1} - p x_i .$$

The length of each vehicle is assumed to be $L_c$ , as shown in Figure 1 for the ith vehicle.

The equilibrium condition is described explicitly as follows:

$$\begin{aligned} p^n x_i &= 0 ; & n \geqq 2, \quad i \geqq 0 \\ p x_i &= v_{ss} ; & i \geqq 0 \\ h_i &= h_s & i \geqq 1 \end{aligned}$$

Here $v_{ss}$ is a constant velocity and $h_s$ is a constant headway.

The first problem to be studied is the case where the velocity $v_0(t)$ of the initial vehicle is composed of two parts, the constant equilibrium component $v_{ss}$ and a sinusoidal component $\Delta v_0(t)$ given by $V_0 \cos \omega t$ , namely

$$(2\text{-}19) \qquad v_0(t) = v_{ss} + \Delta v_0(t).$$

Here it is noted that the superposition principle can be applied. $T(p)$ is defined to be the transfer function of the linear control system relating the velocity of the controlled vehicle to that of its predecessor for each vehicle in the queue. Then

$$V_1 = T(p)V_0 ,$$

$$V_2 = T(p)V_1 = [T(p)]^2 V_0 ,$$

and

(2-20) $$V_n = [T(p)]^n V_0 .$$

The velocity $v_n$ like $v_0$ is composed of two parts, a constant response $v_{ss}$ due to the constant component of $v_0$, and $\Delta v_n$, a response to the disturbance component $\Delta v_0(t)$. The superposition principle allows separate treatment of the two parts. The solution for the constant component of $v_n(t)$ with no disturbance has already been specified as the equilibrium state in which each vehicle has the same velocity $v_{ss}$. The first requirement of $T$ is determined from the equilibrium state requirement. Now by assumption, $T(p)$ is of the form

(2-21) $$T(p) = \frac{\sum_{i=0}^{m} b_i p^i}{\sum_{k=0}^{n} a_k p^k}$$

or

(2-22) $$\sum_{k=0}^{n} a_k p^k V_i(t) = \sum_{l=0}^{m} b_l p^l V_{i-1}(t) ,$$

where $i$ is the vehicle index. Substitution of $v_i = v_{i-1} = v_{ss}$ into Eq. (2-23) and remembering that $p^n v_{ss} = 0$, $n > 0$, reduces it to $v_{ss} a_0 = b_0 v_{ss}$ which indicates that $a_0 = b_0$. Alternatively, this solution can be represented by

(2-23) $$V_{ss} = T(0) V_{ss} .$$

Here the equilibrium state introduces the restriction on $T(p)$ that $T(0) = 1$. The solution for the position $x_n$ of the nth vehicle for $v_n(t) = v_{ss}$ is given by

$$x_n(t) = \int_0^t v_{ss}\,dt + x_n(0) = v_{ss}t + x_n(0). \tag{2-24}$$

If $T(p)$ relates the velocities of adjacent vehicles as indicated, the equilibrium velocities of adjacent vehicles will be equal if $T(0) = 1$. However, the separation of adjacent vehicles at equilibrium will be arbitrary and equal to $x_{n-1}(0) - x_n(0)$. Because this separation must be controlled, the headway at entry into the linear mode is controlled. This is a fundamental characteristic of the velocity controller.

Suppose now that $T_x(p)$ is defined by

$$x_i(t) = T(p)x_{i-1}(t). \tag{2-25}$$

Such a system properly adjusted is termed a headway controller in that $x_{n-1} - x_n$ at equilibrium condition is constant for a given $v_{ss}$. Because linear operators can be expressed by

$$px_i(t) = pT(p)x_{i-1}(t) = T(p)px_{i-1}(t), \tag{2-26}$$

and this reduces to the form given by Eq. (2-22). The justification here is that both sides of Eq. (2-25) can be differentiated with respect to time. At equilibrium $v_{ss} = T(p)\,v_{ss}$, so again $T(0) = 1$, and $a_0 = b_0$. Systems whose equations are given by Eqs. (2-22) and (2-25) although similar in form are quite different in their characteristics. For this reason elementary systems of both types will be considered briefly before proceeding further with the analysis of these systems.

Linear systems which will produce the desired equilibrium state in traffic fall into two basic categories. The first of these is the velocity controller which in simplest form is characterized by the equation,

$$p v_i = k(v_{i-1} - v_i). \tag{2-27}$$

This may be rewritten in the form

$$v_i = \frac{k}{p+k} v_{i-1}, \tag{2-28}$$

which is the same form as Eq. (2-22). Manipulation of this equation yields

$$p(x_{i-1} - x_i) = \frac{p}{p+k} v_{i-1} = p h_i .$$

If the velocity $v_{i-1} = v_{ss}$ and if control according to Eq. (2-27) is started at $t = t_0$, then the complete solution for $h$ is given by

$$h_i(t) = C e^{-k(t-t_0)} + h_f .$$

The first term is the transient term with $C$ dependent upon initial conditions. After a long period of time, $h_i(t)$ is given by

$$h_i(t) = h_f , \qquad (t - t_0) >> 1 .$$

Likewise, if $v_i(t)$ is given by

$$v_i(t) = A \cos \omega t + V_{ss} ,$$

the constant component of $h_i(t)$ will still be $h_f$ .

An alternative system intended for achieving the equilibrium state is the headway controller which in simplest form is characterized by the equation,

$$\tau p^2 x_i + p x_i = k_1(x_{i-1} - x_i - h_o) + k_2 p(x_{i-1} - x_i). \tag{2-29}$$

Here $h_0$ is a safety factor introduced so as to increase the equilibrium headway. Neglecting $h_0$, since it can be simply added to the value of $h$ determined with $h_0 = 0$, $x_i$ and $x_{i-1}$ are related by

$$x_i = \frac{k_1 + k_2 p}{\tau p^2 + (1 + k_2)p + k_1} x_{i-1},$$

which has the form of Eq. (2-26). It is shown in reference 2 that restrictions on $k_1$ and $k_2$ reduce the transfer function to $\frac{k_2}{\tau p + k_2}$. For $p x_{i-1} = v_{ss}$, the steady state headway is then given by

$$h_s = \frac{\tau V_{ss}}{k_2} + h_o.$$

Here $h_0$ is the preset reference headway set into each headway controller. It is further shown in reference 2 that the headway controller causes large accelerations if $| h_i(t_0) - h_s |$ is large. However, both these controllers are considered possible alternatives for control within a small linear mode of an automatic longitudinal control system which has nonlinear modes to accomplish control for large deviations from equilibrium. The nonlinear modes are so designed as to guarantee that $| h_i(t_0) - h_s |$ is small for both systems. In other words, the nonlinear modes force the system response toward equilibrium, so the proper initial spacing is approximately achieved for the velocity controller mode and accelerations are small with the headway controller mode.

### Queue Response to Sinusoidal Disturbance of the Initial Vehicle

Although the pure or almost pure sinusoidal disturbance is almost nonexistent in manually controlled traffic, it is conceivable that such motion could occur under automatic control. A somewhat cyclic disturbance has been observed in the response of a human driver following a lead vehicle of constant velocity in a simulated driving environment. At any rate if the assumption is made that such disturbances can exist in the lead vehicle velocity, then the problem is to determine where in the queue the sinusoidal headway response is greatest and to adjust the equilibrium spacing $h_s$ so that it is the minimum possible without the occurence of collisions within the queue. This will give a conservative estimate of the traffic flow in vehicles/hour for the given velocity $v_{ss}$.

The disturbance response of the first vehicle of the queue is given by

$$\Delta v_1(t) = T(j\omega)\Delta v_0(t),$$

where $\Delta v_0(t) = A \cos \omega t$ . If $T(j\omega)$ is given in polar form as $|T(j\omega)| e^{j\Theta(\omega)}$ , then from Eq. (2-15), $\Delta v_i$ is given by

$$\Delta v_1(t) = |T(j\omega)| A \cos(\omega t + \theta). \tag{2-30}$$

Now $\Delta v_2(t) = T(p)\ \Delta v_1(t) = T(p)^2\ \Delta v_1(t)$ . It may be seen by induction that

(2-31) $$\Delta V_i(t) = \left[T(p)\right]^i \Delta V_0(t),$$

which has been shown to reduce to

(2-32) $$\Delta V_i(t) = |T(j\omega)|^i A\cos(\omega t + i\theta).$$

An observation can be made here about asymptotic stability.[2,3] Local stability insures that disturbances associated with any single vehicle will not build up without limit as time increases. Asymptotic stability insures that disturbances transmitted from vehicle to vehicle by the control systems die out as they propagate back along the queue. For local stability it is sufficient that the poles of $T(j\omega)$ lie in the upper half plane, i.e., the roots of $D(j\omega)$ are of the form $j\omega = -\alpha + j\beta$ , where $\alpha > 0$ . For asymptotic stability, the Barbosa criteria states that if $\omega_0$ is a frequency at which a relative maximum of $|T(j\omega)|$ exists, then

(2-33) $$|T(j\omega_0)| \leqq 1 \quad .$$

If $|T(j\omega)| \leqq 1$ , then it is seen from Eq. (2-32) that

(2-34) $$\lim_{i\to\infty} \Delta V_i(t) = 0.$$

From Eq. (2-31)

(2-35) $$\Delta V_{i-1}(t) = \left[T(p)\right]^{i-1} \Delta V_0(t).$$

Subtracting Eq. (2-31) from Eq. (2-35),

(2-36) $$\Delta V_{r_i} = \Delta V_{i-1} - \Delta V_i = \left[1-T(p)\right]\left[T(p)\right]^{i-1} \Delta V_0 .$$

Since $\Delta V_{r_i}(t) = p\Delta h_i(t)$,

then

(2-37) $$\Delta h_i(t) = \left[\frac{1-T(p)}{p}\right]\left[T(p)\right]^{i-1}\Delta V_o(t),$$

and

(2-38) $$\Delta h_i(t) = \left|\frac{1-T(j\omega)}{j\omega}\right|\left|T(j\omega)\right|^{i-1} A\cos\left[\omega t+(i-1)\theta(\omega)+\phi(\omega)\right]$$

where $\Theta(\omega)$ is the argument of $T(j\omega)$ and $\phi(\omega)$ is the argument of $\frac{1-T(j\omega)}{j\omega}$.

As a specific example, it is interesting to consider the velocity controller. Its disturbance response is given by

(2-39) $$\Delta V_i(t) = \frac{k}{p+k}\Delta V_{i-1}(t)$$

and by

$$\Delta V_i(t) = \frac{k}{j\omega+k}\Delta V_{i-1}(t),$$

for

$$\Delta V_{i-1}(t) = Ae^{j\omega t}.$$

In this case

$$\left[T(j\omega)\right]^i = \left(\frac{\omega^2}{k^2}+1\right)^{-\frac{i}{2}} e^{-ji\tan^{-1}\frac{\omega}{k}},$$

and $$\frac{1-T(j\omega)}{j\omega} = \frac{1}{k}T(j\omega).$$

(2-40) $$\frac{\Delta h_i(t)}{\Delta V_o(t)} = \frac{1}{k}\left[T(j\omega)\right]^i = \frac{1}{k}\left(\frac{\omega^2}{k^2}+1\right)^{-\frac{i}{2}} e^{-ji\tan^{-1}\frac{\omega}{k}}$$

The headway variation due to a sinusoidal disturbance $\Delta v_0(t) = V_0 \cos \omega_0 t$ is

$$(2\text{-}41) \qquad \Delta h_i(t) = \frac{1}{K}\left(\frac{\omega_0^2}{K^2}+1\right)^{-\frac{i}{2}} V_0 \cos\left(\omega_0 t - i \tan^{-1}\frac{\omega_0}{K}\right)$$

It is obvious from Eq. (2-41) that the magnitude of the headway response to a given sinusoidal input is greatest when $i = 1$ for $\omega_0 \neq 0$ or when $\omega_0 = 0$. Then the amplitude of $\Delta h_1$ for $\omega_0 = 0$ is simply

$$(2\text{-}42) \qquad |\Delta h_1(t)| = \frac{V_0}{K}$$

However, it should be observed here that $\Delta v_0(t) = V_0 \cos \omega_0 t$ is not a very realistic representation of the extreme periodic velocity of the lead vehicle, since the lead vehicle's acceleration would then be

$$a_0(t) = -\omega_0 V_0 \sin \omega_0 t$$

and the peak value of $a_0$ would increase with $\omega_0$ without bound.

It is more realistic to assume that the initial vehicle is not automatically controlled. In this case the vehicle's actual velocity $\Delta v_0(t)$ is approximately related to a commanded velocity $\Delta v_c(t)$ (assumed linearly proportional to accelerator pedal displacement) by the equation

$$(2\text{-}43) \qquad (\tau_a p + 1)\Delta V_0(t) = \Delta V_c(t)$$

where $\tau_a$ is the major time constant of the automobile. The time constant $\tau_a$ is on the order of 20 seconds. In order to affix an envelope of response on $\Delta v_0(t)$ , let it be assumed that $\Delta v_c(t) = V \cos \omega t$ . On a periodic basis $V = 44$ ft/sec (30 mph) seems to be a reasonable upper limit.* Then the amplitude of $\Delta v_0(t)$ as a function of frequency is given by

$$(2\text{-}44) \qquad |\Delta v_0(t)| = \left| \frac{V}{\tau_a j\omega + 1} \right| = \frac{44}{\sqrt{1+400\,\omega^2}}$$

The initial vehicle's acceleration amplitude is then

$$|a_0(t)| = \left| \frac{j\omega V}{\tau_a j\omega + 1} \right| = \frac{44\,\omega}{\sqrt{1+400\,\omega^2}}$$

The acceleration amplitude approaches a limit as $\omega$ becomes large ($\omega > .5$) given by $|a_0(t)| = 2.2$ ft/sec$^2$ . This corresponds to approximately .07g , which is well within vehicle capability.

Using the above input function for $\Delta v_c(t)$ , the maximum headway variation amplitude is then given by

$$|\Delta h_1(t)| = \frac{1}{K}\left(\frac{\omega^2}{K^2} + 1\right)^{-\frac{1}{2}} \left| \frac{44}{20 j\omega + 1} \right| .$$

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*In this case the velocity of the controlled vehicle would vary from 60 to zero miles per hour with $v_{ss} = 30$ miles per hour.

Obviously $\Delta h_1$ is still maximized when $\omega \to 0$, in which case

$$(2\text{-}45) \qquad \lim_{\omega \to 0} \left| \Delta h_1(t) \right| = \frac{44}{k}$$

On the basis of avoiding collisions due to the above extreme disturbance using this linear system with $k = 0.644$, the equilibrium spacing should be one car length plus the maximum headway variation amplitude, or

$$(2\text{-}46) \qquad h_s = L_c + \frac{44}{K} \approx 20 + 70 = 90 \text{ ft.}$$

This separation of vehicles based upon extreme variations of the lead vehicle is considered excessive, and must be limited.* Two methods can be used other than increasing $k$. One system involves a modification of the structure of $T$, while the second involves a nonlinear controller which comes into play whenever extreme conditions such as considered here exist. This controller causes $h_s$ to increase, once extreme conditions such as considered here occur.

Propagation of disturbances also occurs in the queue, as is shown in the following manner. The general equation for the headway variation of the ith vehicle due to sinusoidal disturbance of the initial

---

*Note that if maximum and minimum speeds are specified as 60 and 40 miles per hour, the above value of $h_s$ would be 43 ft.

vehicle is repeated here for convenience and rewritten slightly.

$$\Delta h_i(t) = \left|\frac{1-T(j\omega)}{j\omega}\right| \left|T(j\omega)\right|^{i-1} V_o \cos\left[\omega t + (i-1)h_s \frac{\theta(\omega)}{h_s} + \phi(\omega)\right] \tag{2-47}$$

Note that since small perturbations from equilibrium are being considered $(i-1)\,h_s$ is approximately the distance from the initial vehicle to the ith vehicle. That is,

$$(i-1)h_s \approx x_o - x_i$$

Similarly,

$$(i-2)h_s \approx x_o - x_{i-1}$$

Now consider a particular point of the oscillation of $\Delta h_i(t)$ such as the peak, for which

$$\omega t_i + (x_o - x_i)\frac{\theta(\omega)}{h_s} + \phi(\omega) = 0 \tag{2-48}$$

where $t_i$ is the time at which the peak occurs. Similarly the headway oscillation of the $(i-1)$st vehicle will reach a peak at $t_{i-1}$, for which

$$\omega t_{i-1} + (x_o - x_{i-1})\frac{\theta(\omega)}{h_s} + \phi(\omega) = 0. \tag{2-49}$$

Subtracting Eq. (2-49) from Eq. (2-48) and solving for $\dfrac{x_{i-1} - x_i}{t_{i-1} - t_i}$ yields

$$\frac{x_{i-1} - x_i}{t_{i-1} - t_i} = \frac{-\omega h_s}{\theta(\omega)} \tag{2-50}$$

which is the velocity with which the peak in the headway oscillation appears to move with respect to the queue, or the propagation velocity $v_p$ . Since $\Theta(\omega)$ and $t_{i-1} - t_i$ are both negative, $v_p < 0$ , indicating that propagation occurs in the negative direction, or back away from the initial vehicle. If an observer were watching the queue from the side of the road and if $v_{ss} = v_p$ , then the observer would see the peak of the headway disturbance as each vehicle passed. For the specific example previously considered

$$\theta(\omega) = -\tan^{-1}\frac{\omega}{k}$$

and

$$v_p \approx -\frac{\omega h_s}{\tan^{-1}\frac{\omega}{k}} \tag{2-51}$$

indicating that propagation occurs back away from the initial vehicle. The propagation velocity is independent of i and is thus constant along the queue. It is readily seen that at very low frequencies ($\omega << k$)

$$v_p \approx -k h_s \tag{2-52}$$

Note that $\tan^{-1}\frac{\omega}{k}$ increases monotonically from 0 to $\frac{\Pi}{2}$ as $\omega$ increases from 0 to $\infty$ . The normalized propagation velocity $\frac{v_p}{k\,h_s}$ is plotted in Figure 2. Because of controller characteristics which attenuate the higher frequencies, the velocity $k\,h_s$ is often termed <u>the</u> propagation velocity.

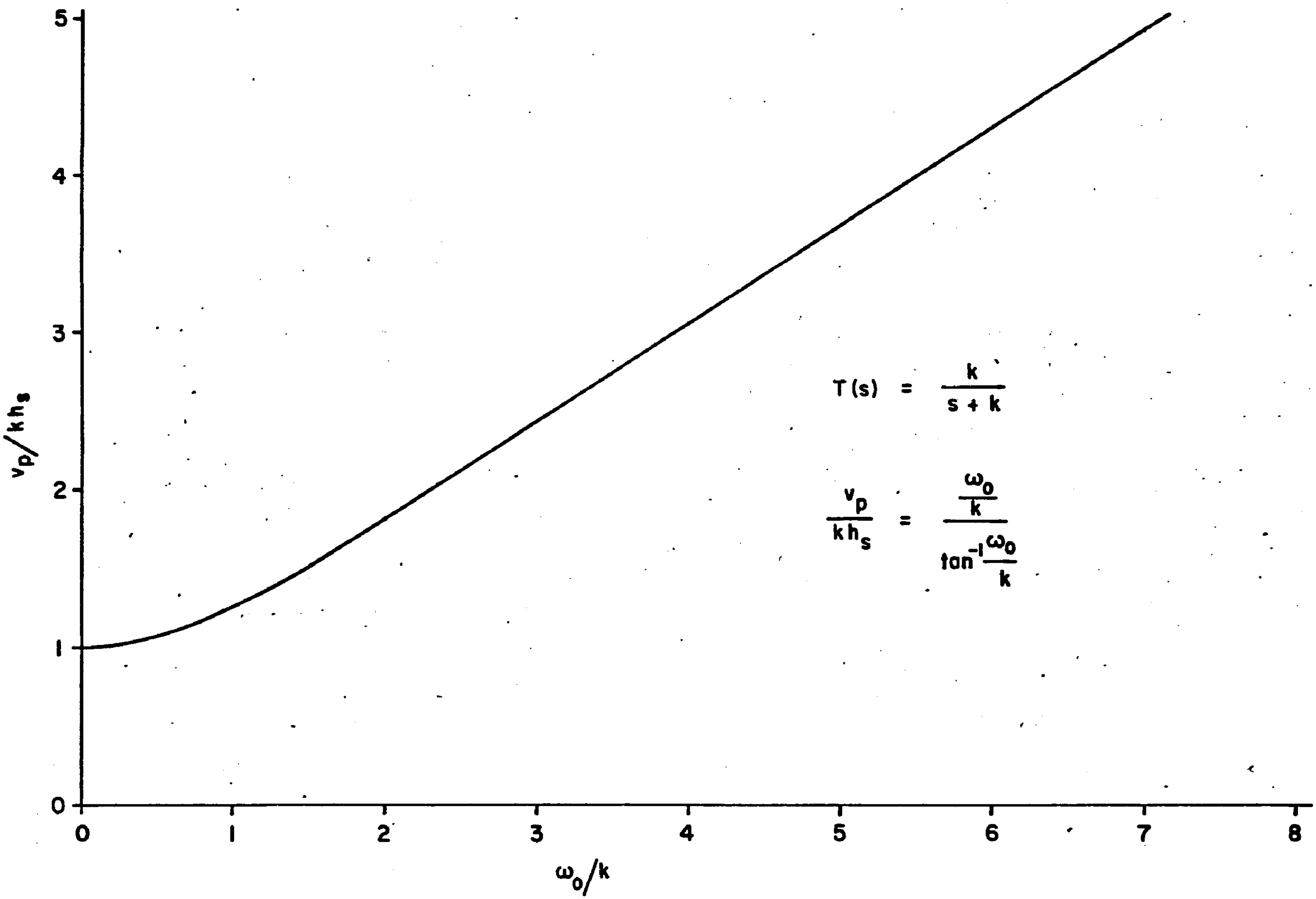


Fig. 2.--Propagation Velocity for Linear Mode System, T(s)

## Queue Response to Sinusoidal Road-Induced Disturbances

Up to this point the study has been concerned with queue response to sinusoidal disturbances $\Delta v_0(t)$ introduced by external excitation of only the initial vehicle. Now attention is turned to the possibility of a disturbance induced into each vehicle of the queue in sequence. Again attention is confined to sinusoidal disturbances. In particular, disturbances of a type which might be induced by a road with a small sinusoidal vertical profile are considered. In this case each vehicle is disturbed externally in the same manner as the preceding vehicle, but the disturbance function is delayed in time by the time spacing $\Upsilon$ . This is an approximation which holds only for small disturbances of each vehicle from equilibrium. Equilibrium is the same as previously defined. In this case the total disturbance of each vehicle is the sum of the disturbance induced by the road and the disturbances transmitted to it from each of the preceding vehicles.

It might occur to the reader at this point that it may not be entirely true that the road, under the assumptions above, will induce identical disturbances into each vehicle, even if the queue is controlled by identical control systems. This situation arises if the initial vehicle is not automatically controlled. First assume that it <u>is</u> automatically controlled in such a way that

(2-53) $$V_0(t) = T(p)\, V_{ss}$$

Now consider a disturbance induced by the road into the initial vehicle given by $\Delta v_0(t)$ . The road will induce a disturbance $\Delta v_0(t-\tau)$ in the first vehicle, $\Delta v_0(t-2\tau)$ in the second vehicle, etc. If $\Delta v_0(t)$ is given by

$$\Delta V_0(t-\tau) = A e^{j\omega t}$$

then the disturbance induced by the road in the first vehicle will be

(2-54) $$\Delta V_0(t-\tau) = A e^{j(\omega t-\tau)} = A e^{j\omega t} e^{-j\omega\tau} = e^{-j\omega\tau} \Delta V_0(t)$$

Similarly, the disturbance induced by the road in the second vehicle is

$$\Delta V_0(t-2\tau) = e^{-j2\omega\tau} \Delta V_0(t)$$

The disturbance induced in the ith vehicle is the sum of all such disturbances transmitted to it plus $e^{-j\omega i\tau}\Delta v_0(t)$ . Thus the total disturbance of the ith vehicle may be related by a gain function, yet to be determined, to $\Delta v_0(t)$ .

On the other hand if the initial vehicle is <u>not</u> automatically controlled, the road induced disturbance of the first vehicle will not be that of the initial vehicle delayed by $\tau$ seconds, but will be given by

$$\Delta V_1(t) = T(j\omega)\Delta V_0(t) + \Delta V_R(t) \neq T(j\omega)\Delta V_0(t) + e^{-j\omega\tau}\Delta V_0(t) \qquad (2\text{-}55)$$

Here $\Delta v_R(t)$ is the disturbance induced by the road in the first vehicle. It can be seen that the overall queue response will be the sum of two components. One is due to $\Delta v_0(t)$ externally induced in only the initial vehicle. This response was described in the first section of this chapter. The second component is due to the disturbance $\Delta v_1(t)$ being induced with appropriate delay into each vehicle of the queue. This problem is identical to the one for which Eq. (2-53) holds, but with the indices increased by one. Therefore in the remainder of this chapter it is assumed that Eq. (2-53) is true, and all vehicles are similarly controlled.

In order to determine where the largest sinusoidal headway oscillations occur in the queue, it is desirable to find a gain function for the headway disturbance of the ith vehicle of the queue as a function of the input disturbance, $\Delta v_0(t) = Ae^{j\omega t}$. Specifically it is desired to find $\Delta h_i(t)$, where

$$\Delta h_i(t) = H_i(j\omega)\Delta V_0(t) \qquad (2\text{-}56)$$

This is found in the following manner. The response of each vehicle is the sum of the response transmitted by the gain function from the preceding vehicle plus the response induced by the road. For the kth vehicle,

$$\Delta V_k(t) = T(j\omega)\Delta V_{k-1}(t) + e^{-jk\omega\tau}\Delta V_0(t) \tag{2-57}$$

For the 1st vehicle,

$$\Delta V_1(t) = T(j\omega)\Delta V_0(t) + e^{-j\omega\tau}\Delta V_0(t) \tag{2-58}$$

For the 2nd vehicle, applying Eq. (2-57),

$$\Delta V_2(t) = T(j\omega)^2\Delta V_0(t) + T(j\omega)e^{-j\omega\tau}\Delta V_0(t) + e^{-j2\omega\tau}\Delta V_0(t) \tag{2-59}$$

In general, then, for the ith vehicle,

$$\Delta V_i(t) = \left[T(j\omega)^i + T(j\omega)^{i-1}e^{-j\omega\tau} + \cdots + T(j\omega)e^{-j(i-1)\omega\tau} + e^{-ji\omega\tau}\right]\Delta V_0(t) \tag{2-60}$$

Substituting Eq. (2-60) into Eq. (2-57), where $k-1 = i$ ,

$$\Delta V_{i+1}(t) = \left[T(j\omega)^{i+1} + T(j\omega)^{i}e^{-j\omega\tau} + \cdots + T(j\omega)e^{-ji\omega\tau} + e^{-j(i+1)\omega\tau}\right]\Delta V_0(t)$$

Therefore, by induction Eq. (2-60) is seen to be valid for all $i$ . It is further noted that the right member of Eq. (2-60) may be written in closed form, given by

$$\Delta V_i(t) = e^{-ji\omega\tau}\left[\frac{1 - T(j\omega)^{i+1}e^{j(i+1)\omega\tau}}{1 - T(j\omega)e^{j\omega\tau}}\right]\Delta V_0(t) \tag{2-61}$$

The relative velocity disturbance of the ith vehicle is

$$(2\text{-}62) \qquad \Delta V_{r_i}(t) = \Delta V_{i-1}(t) - \Delta V_i(t)$$

After substitution of Eq. (   ) into Eq. (   ) and some algebraic manipulation

$$(2\text{-}63) \qquad \Delta V_{r_i}(t) = \frac{e^{j\omega\tau}\left[e^{-j\omega\tau}(1-e^{-j\omega\tau}) + \left(T(j\omega)-1\right)T(j\omega)^i\right]}{1-T(j\omega)e^{j\omega\tau}} \Delta V_0(t)$$

Now $\Delta v_{r_i}(t) = p\Delta\, h_i(t)$ . So,

$$(2\text{-}64) \qquad \Delta h_i(t) = \frac{\Delta V_{r_i}(t)}{j\omega}$$

where the constant initial condition on $\Delta h_i(t)$ is to be adjusted independently.

$$(2\text{-}65) \qquad \Delta h_i(t) = \frac{e^{j\omega\tau}\left[e^{-ji\omega\tau}(1-e^{-j\omega\tau}) + \left(T(j\omega)-1\right)T(j\omega)^i\right]}{j\omega\left[1-T(j\omega)e^{j\omega\tau}\right]} \Delta V_0(t)$$

Eq. (2-65) gives the required gain function.

In order to find the gain function, $\Delta v_0(t)$ was given by $\Delta v_0(t) = Ae^{j\omega t}$ , which is not a realistic input disturbance. Road-induced disturbances are external disturbances and must be accelerations or forces on the vehicles. Because of the feedback systems used to achieve equilibrium, $\Delta v_0(t)$ cannot be specified as an independent disturbance. Rather, $\Delta v_0(t)$ is the response of the feedback system to an independent disturbance $\Delta a_0(t)$ . From block diagrams of the controllers, it may be determined that for the velocity controller,

$$\Delta V_0(t) = \frac{1}{p+k}\Delta a_0(t), \tag{2-66}$$

and for the headway controller,

$$\Delta V_0(t) = \frac{\tau_a p}{\tau_a p^2 + (1+k_2)p + k_1}\Delta a_0(t) \tag{2-67}$$

In order to solve the problem then, $\Delta a_0(t)$ must be determined. This is done as follows.

Sinusoidal road-induced disturbances may be caused by a road section with a sinusoidal vertical profile as shown in Figure 3. The elevation of any point along the road is given by

$$y(x) = Y\cos\omega_x x \tag{2-68}$$

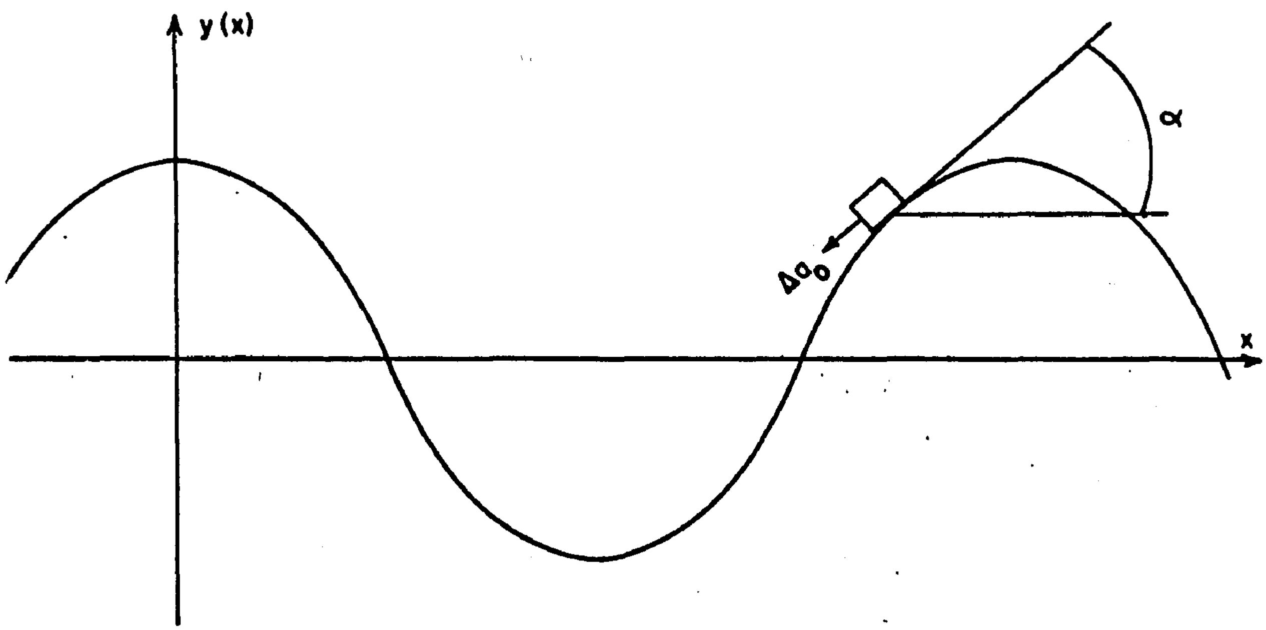


Fig. 3.--Road With Sinusoidal Vertical Profile

It is seen from Figure 3 that the gravitational force component tangential to the road is

(2-69) $$\Delta a_0 = -g \sin \alpha(x)$$

For small $\alpha$ , Eq. (    ) becomes

(2-70) $$\Delta a_0 \approx -g\alpha(x)$$

where

(2-71) $$\alpha(x) = \frac{dy(x)}{dx} = -Y\omega_x \sin \omega_x x$$

The vehicle horizontal velocity component is given to a first approximation by $v_{ss}$ , so that

(2-72) $$x \approx v_{ss} t$$

Substituting Eq. (2-72) into Eq. (2-71) yields

$$\alpha(t) = -Y\omega_x \sin \omega_x v_{ss} t$$

Letting $\omega_x v_{ss} = \omega$ ,

(2-73) $$\alpha(t) = -Y\omega_x \sin \omega t$$

Then $\Delta a_0(t)$ is approximately given by substituting Eq. (2-73) back into Eq. (2-70), so that

(2-74) $$\Delta a_0(t) = gY\omega_x \sin \omega t = gY\frac{\omega}{v_{ss}} \sin \omega t$$

It is noted that if Eq. (2-72 ) is substituted into Eq. (2-68), y(t) is given approximately by

$$y(t) = Y \cos \omega t \tag{2-75}$$

and

$$\Delta a_0(t) = -\frac{g}{V_{ss}} p y(t) \tag{2-76}$$

Substituting Eq. (  ) back into Eq. (  ) gives $\Delta v_0(t)$ for the velocity controller as

$$\Delta V_0(t) = -\frac{g}{V_{ss}} \frac{p}{p+k} y(t) \tag{2-77}$$

Substituting Eq. (  ) back into (  ) gives $\Delta v_0(t)$ for the headway controller as

$$\Delta V_0(t) = \frac{-g \tau_a p^2}{V_{ss}\left[\tau_a p^2 + (1+k_2)p + k_1\right]} y(t) \tag{2-78}$$

Thus the corresponding gain function relating $\Delta v_0(t)$ to y(t) for the two controllers are

$$\frac{\Delta V_0(t)}{y(t)} = \frac{-g}{V_{ss}} \frac{j\omega}{j\omega + k} \quad \text{for } y(t) = Ae^{j\omega t} \tag{2-79}$$

and

$$\frac{\Delta V_0(t)}{y(t)} = \frac{g \tau_a \omega^2}{V_{ss}\left[-\tau_a \omega^2 + (1+k_2)j\omega + k_1\right]}, \tag{2-80}$$

respectively. Substituting Eqs. (2-79) and (2-80) back into Eq. (2-65) gives the headway response for the road-induced disturbance, $y(t) = Y\cos\omega t$, as

$$(2\text{-}81)\quad \Delta h_i(t) = \left| \frac{e^{j\omega\tau}\left[e^{-j\omega\tau}(1-e^{-j\omega\tau}) + (T(j\omega)-1)\,T(j\omega)^i\right]}{\left[1-T(j\omega)e^{j\omega\tau}\right](j\omega+K)} \right| \frac{gY}{V_{ss}} \cos(\omega t+\phi)$$

and

$$(2\text{-}82)\quad \Delta h_i(t) = \left| \frac{j\omega e^{j\omega\tau}\left[e^{-j\omega\tau}(1-e^{-j\omega\tau}) + (T(j\omega)-1)T(j\omega)^i\right]}{\left[1-T(j\omega)e^{j\omega\tau}\right]\left[-\tau_a\omega^2+(1+K_2)j\omega + K_1\right]} \right| \frac{\tau_a gY}{V_{ss}} \cos(\omega t+\psi)$$

respectively for the velocity and headway controllers. Here $\phi$ is the argument of the complex magnitude function in Eq. (2-81), and $\psi$ is the argument of the complex magnitude function in Eq. (2-82).

The magnitude of $\Delta h_i(t)$ appears to be indeterminate if $\omega = 0$, since $T(0) = 1$. This can be treated by L'Hospital's rule or by a series expansion of the terms which approach zero as $\omega \rightarrow 0$. In the present situation the latter course is the more efficient and will be used. First $T(j\omega)$ near $\omega = 0$ can be approximated by

$$\frac{j\omega a_1 + a_0}{j\omega b_1 + a_0} \underset{\omega\to 0}{\approx} 1 + j\omega\left(\frac{a_1}{a_0} - \frac{b_1}{a_0}\right)$$

and

$$e^{j\omega\tau} \approx 1 + j\omega\tau$$

As a consequence the terms in the numerator and denominator of the equations which approach zero as $\omega \rightarrow 0$ become

$$1 - e^{-j\omega\gamma} \longrightarrow j\omega\gamma$$

$$T(j\omega) - 1 \longrightarrow j\omega\left(\frac{a_1}{a_0} - \frac{b_1}{a_0}\right)$$

and

$$1 - T(j\omega)\, e^{j\omega\gamma} \longrightarrow -j\omega\left(\gamma + \frac{a_1}{a_0} - \frac{b_1}{a_0}\right)$$

The terms causing difficulty are the bracketed terms in numerator and denominator. Substituting these series into the bracketed terms one has

$$\frac{\left[e^{-ji\omega\gamma}(j\omega\gamma) + \left(j\omega\left(\frac{a_1}{a_0} - \frac{b_1}{a_0}\right) T(j\omega)^{i}\right]}{-j\omega\left(\gamma + \frac{a_1}{a_0} - \frac{b_1}{a_0}\right)}$$

The two terms in the numerator of the above expression approach unity as $\omega \rightarrow 0$ causing the above ratio to approach $-1$ as $\omega \rightarrow 0$. As a consequence $\Delta h_i(t)$ for the velocity controller approaches

$$\Delta h_i(t) \longrightarrow \frac{-gY}{k V_{ss}}$$

and $\Delta h_i(t)$ for the headway controller approaches zero. These results are valid only if

$$\gamma + \frac{a_1}{a_0} - \frac{b_1}{a_0} \neq 0.$$

Conclusion

It is remarked in closing that the sinusoidal analysis and the gain function may be applied to evaluation of the queue response to more general periodic input disturbances. The periodic input may be expanded in a Fourier series, a sum of sinusoids. Because superposition applies, the terms of the series may be treated as separate inputs. The response to each of the inputs is related to the input sinusoid by the magnitude and phase of the gain function as in the previous sinusoidal response analysis. The periodic queue response is then the sum of the sinusoidal response terms. For example, a periodic disturbance of the initial vehicle only may be expanded in the series

$$\Delta V_o(t) = \sum_{n=1}^{\infty} c_n \cos(n\omega_o t + \alpha_n) \tag{2-83}$$

where $\omega_0 = \frac{2\pi}{T_0}$, $T_0$ is the period of the disturbance, and $\alpha_n$ is the relative phase of the nth component. The headway response of the ith queue vehicle $\Delta h_i(t)$ is then given by

$$\Delta h_i(t) = \sum_{n=1}^{\infty} c_n \left|H_i(jn\omega_o)\right| \cos\left[n\omega_o t + \alpha_n + \phi_i(n\omega_o)\right], \tag{2-84}$$

where $H_i(jn\omega_0)$ is the gain function,

$$(2\text{-}85) \qquad H_i(jn\omega_0) = \left[\frac{1 - T(jn\omega_0)}{jn\omega_0}\right]\left[T(jn\omega_0)\right]^{i-1},$$

and $\phi_i(n\omega_0)$ is the argument or phase of $H_i(jn\omega_0)$ . The maximum magnitude of $\Delta h_i(t)$ may then be determined, and the static headway $h_s$ is made larger than maximum $\Delta h_i$ . This indicates a type of bound on the flow possible with the automatically controlled queue. The periodic road-induced disturbances can be similarly treated.

However, the sinusoidal and periodic disturbances are arbitrary and idealized disturbances, not likely to be encountered in traffic. The disturbances to be encountered in traffic are not predictable, but are random in nature. Queue response to these more realistic random disturbances is discussed at length in the remaining chapters of this paper. Throughout the discussion, the same gain factors derived from sinusoidal analysis will be extensively employed.

## CHAPTER III

## RESPONSE OF THE AUTOMATICALLY CONTROLLED QUEUE TO INDEPENDENT STATIONARY RANDOM DISTURBANCES

### Introduction

In this chapter the response of a queue of automatically controlled vehicles to stationary random disturbances is studied. Typical sources of such random disturbances are considered, as well as their occurrence in the queue. Characterization of the disturbances and the applicability of their treatment as stationary random functions is discussed. The power which must be supplied by the vehicles is related to the disturbances. The method is then shown for determining the mean square values of certain queue response functions from knowledge of autocorrelations or power spectra of the disturbance sources. Finally, the engineering problem is discussed of determining the equilibrium spacing to avoid collisions when each vehicle is a disturbance source.

### Sources of Random Disturbances of Automatic Queues

In the study of the automated highway, it has been considered advisable that the automatic system be compatible with the present manually controlled vehicles. In the automatic system the response of each vehicle to the motion of the preceding vehicle is controlled by an automatic longitudinal control system. A compatible controller must also detect and respond to manually driven vehicles in the traffic system. In this case it is expected that occasional manually controlled vehicles will be found in long queues of automatically controlled vehicles and that the manually controlled vehicles may either break the queue or simply accumulate a following queue of automatic vehicles. In these situations the human driver introduces random disturbances from the equilibrium condition of the queue. The random disturbances are transmitted upstream by the automatic controllers. The determination of measures of the random queue response to the disturbances may be treated by considering the manual vehicle to be the initial vehicle of an automatic queue and to be the sole source of disturbance of the queue. This is one situation to be analyzed in this chapter.

Another situation which may be treated in the same manner is that of a queue which is entirely automatic, but in which random disturbances are generated internally in one of the vehicles. Again the random disturbance is transmitted upstream by the automatic controllers, and the source vehicle may be considered to be the initial vehicle of the queue for the purpose of analyzing the queue response to the disturbance.

It is readily conceivable that at any point in an automatic queue there may be a random disturbance response to more than one disturbance source downstream. Since attention is confined to the linear mode operation of the automatic system, superposition again applies, and the total disturbance may be found by summing the responses at each point of the queue determined separately by treating each source vehicle as an initial vehicle of an automatic queue. If the sources of random disturbances are statistically independent, then the response to each of the sources will be uncorrelated, and the measures of queue response will be the sum of the measures of response to each independent disturbance.

Finally, it is to be expected in the realistic automatic traffic queue that there will be random disturbance sources in each vehicle of the queue due to quantization error in the measurement of velocity and headway by each automatic controller. A first approximation to the determination of the response to these disturbances can be made by considering these sources to be independent, although they may actually be correlated. The validity of such an approximation should be the subject of an advanced study, which is beyond the scope of this paper. Correlated random disturbances of the queue due to road-induced disturbances are considered in the next chapter.

Stationary Random Disturbances

Random functions of time are distinguished from the deterministic functions of the previous chapter by the fact that it is not possible to predict their value at any given instant with certainty. Consequently, they are not characterized by specifying them as known functions of time. Instead, random functions are characterized by their time averages. Only the mean and the autocorrelation of the random functions are needed for engineering purposes of this paper.

In general the mean $\overline{y(t)}$ of a random function of time $y(t)$ is defined to be

(3-1) $$\overline{y(t)} = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} y(t)\, dt .$$

The functions of interest are accelerations, velocities, headways, etc. The above definition of the mean of a variable is not applicable to the real traffic system because it is not possible to observe the variable for an infinite period of time. The question then arises, if it is only possible to average a variable over a finite time period, then what is the period to be. The variable associated with an automated queue of vehicles is ofinterest here, so the average of interest is to be taken while the vehicle is on a finite section of automatic highway. In this case the measured mean is given by

$$(3\text{-}2) \qquad \overline{y(t)} = \frac{1}{2T}\int_{-T}^{T} y(t)\,dt \;,$$

where the vehicle enters the automatic section at $t = -T$ and leaves at $t = T$.

Queues of vehicles form because the members of the queue desire a higher mean velocity than the initial vehicle of the queue. If passing cannot occur, the queue members are forced to remain behind the initial vehicle and avoid collision with the preceding vehicle. This is true of either manually controlled traffic or traffic in which each vehicle is controlled individually by an automatic controller. Since the queue response to initial vehicle disturbances is of interest here, the period $2T$ over which the response is averaged occurs after the formation of the queue and may be shorter than the time for the initial vehicle to traverse the automatic section. On this basis, the initial vehicle determines the average velocity of each vehicle of the queue, and this is considered the equilibrium speed. For queues of automatically controlled vehicles the equilibrium spacing need not be a function of the equilibrium speed, but must be large enough to eliminate collisions due to disturbances from equilibrium. Therefore, it is sufficient from here on to consider only disturbances from equilibrium.

The autocorrelation of a random function $\Delta y(t)$ is denoted $\phi_{\Delta y}(t_1 \,,\, \tau)$ and is given by

$$\phi_{\Delta y}(t_1, \tau) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} \Delta y(t-t_1) \Delta y(t-t_1+\tau)\, dt. \tag{3-3}$$

This function again cannot be obtained on real road sections. Furthermore its usefulness is severely limited by its dependence on $t_1$. A function which can be obtained on the real road section is given by

$$\phi_{\Delta y}^{T}(t_1, \tau) = \frac{1}{2T} \int_{-T}^{T} \Delta y(t-t_1) \Delta y(t-t_1+\tau)\, dt. \tag{3-4}$$

The question then arises as to how large T must be in the function $\phi_{\Delta y}^{T}(t_1, \tau)$ in order that it reasonably approximate the autocorrelation and be independent of $t_1$. This function, which can be obtained in reality, is a measure of the disturbance. The answer to the above question depends on the disturbance source.

The first disturbance source considered is electrical noise generated in an automatic controller's components. Such noise is in general dependent on the temperature (such as transistor noise), the material and design of the component (such as potentiometer transducer noise) or on the equilibrium speed of the vehicle (such as tachometer noise). The temperature and equilibrium speed are assumed constant, so that the electrical noise can be considered stationary. This is generally a relatively high frequency noise source, and its autocorrelation can be expected to approach zero for $\tau > 1$ sec. However, it is generated in a feedback control loop which is expected to have a major time constant of several seconds at most. The resulting control system output disturbance, that of the vehicle, at any time instant can be expected to be independent of its value at instants more than several seconds earlier. In other words, if $\phi_{\Delta y}(\tau)$ is the autocorrelation of the vehicle response to electrical noise in its controller, $\phi_{\Delta y}(\tau)$ will be essentially zero for $\tau$ greater than several seconds. In this case, $\phi^T_{\Delta y}(\tau)$ is a reasonable approximation to $\phi_{\Delta y}(\tau)$ if T is only a few minutes. Also, $\phi^T_{\Delta y}(\tau)$ is stationary and independent of the time of measurement.

The quantization or instrumentation noise in the input to the feedback controller is dependent primarily on the spacing of detectors in the highway and on the equilibrium speed for the automatic system presently considered. According to the present concept of the automatic system, the spacing of the detectors will be constant. Since the equilibrium speed is also assumed constant, it is expected that this type of disturbance source is also stationary. The feedback controller filters this noise in the same fashion as the internal electrical noise, so that again a T of a few minutes will be sufficient for a stationary measurement of the autocorrelation of the resulting vehicle disturbance.

The final source of disturbances to be considered is the human driver. He is also the most nonstationary. His characteristics tend to change with mood, fatigue, environment, etc. During the past few years the human driver response in the car-following situation has been studied experimentally here at The Ohio State University. Specifically, measurements have been taken of the driver's response while he is attempting to follow a preceding vehicle of constant velocity. This situation is quite similar to that of the driver as a member of a long traffic queue. These measurements have been taken in a simulated traffic environment and on the highway. In the simulated environment it was found that measurements of the mean square value of the driver's disturbance showed reasonable consistency for a given subject from day to day and for several different measurements during a day if the measurements were taken over 10 to 12 minute periods. In the road tests it has been found that ten minute recordings of data reduce with reasonable consistency also. Thus it seems that if $T$ is greater than 15 minutes, the measurements of $\phi^T_{\Delta y}(t_1, \tau)$ are relatively independent of $t_1$. Furthermore, autocorrelations have been calculated for velocity disturbance in car following, and they are found to be essentially zero for $\tau$ greater than 100 seconds. Thus it is expected that $\phi^T_{\Delta y}(\tau)$ for $T > 15$ minutes will reasonably approximate $\phi_{\Delta y}(\tau)$.

From the above considerations it is estimated that on sections of automatic highway where the above types of disturbances can exist, averages of duration 15 minutes or more will appear stationary. Of course, the validity of this estimate requires the ruling out of extremes of mood changes, learning, environmental changes, etc. Also, the same individual disturbance sources are assumed for all time on the sections. Different sources will generate different stationary disturbances. The exploration of the characteristics of the individual sources is beyond the scope of this paper. Here representative characteristics only will be assumed.

Power Dissipated By a Velocity Disturbance

It is of some interest to attach some physical significance to the autocorrelation of the velocity disturbance of a vehicle. In general the vehicle is quite nonlinear. Also, it normally has vertical translation motion and pitching motion as well as the longitudinal translation which is of interest here. The vertical and pitching motion are normally road-induced and need not be considered here. The friction force on the vehicle is a nonlinear function of velocity. It can be expected to be similar to aerodynamic drag, proportional to $v^n$, where $n > 1$. In this case it will be similar to the characteristic shown in Figure 4. Here it is seen that $F(v)$ may be expanded in a Taylor

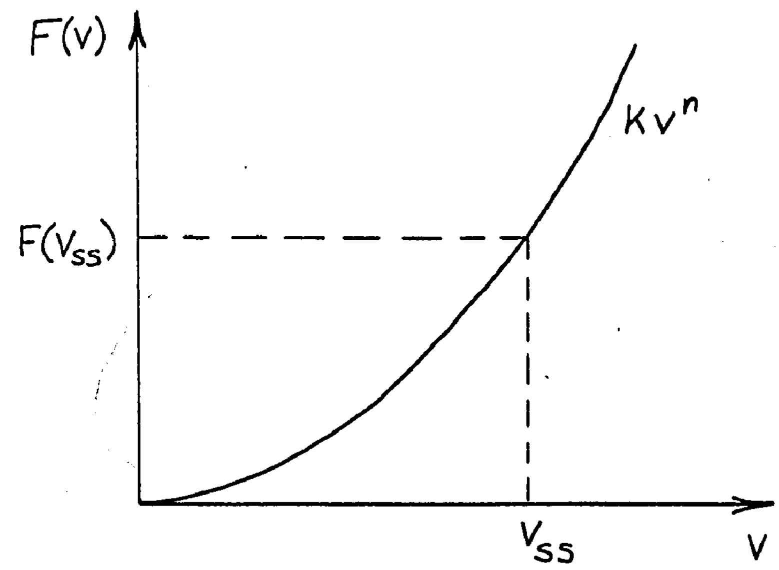


Fig. 4.--Nonlinear Friction Force Characteristic

series about $v_{ss}$ , given by

$$F(v) = F(V_{ss}) + \frac{d}{dv}F(v)\Big|_{v=V_{ss}}(v-V_{ss}) + \frac{1}{2}\frac{d^2}{dv^2}F(v)\Big|_{v=V_{ss}}(v-V_{ss})^2. \tag{3-5}$$

For small disturbances from equilibrium, the disturbance velocity $\Delta v = v - v_{ss}$ produces a disturbance drag force $\Delta F = F(v) - F(v_{ss})$. The relationship between them is linear and is given by the first two terms of the Taylor series,

$$\Delta F = \frac{dF(v)}{dv}\Big|_{v=V_{ss}}\Delta V = F_v \Delta V \tag{3-6}$$

Finally, the changes in potential energy are not important in this discussion.

The infinitesimal change in energy which must be supplied by the engine due to inertia and friction forces resulting from disturbances of velocity is

$$dW = M \frac{d\Delta v}{dt} dx + F_v \Delta v \, dx , \tag{3-7}$$

where $x = v_{ss} t + \Delta x$ . The required engine power is then given by

$$P = \frac{dW}{dt} = M \Delta V \frac{d}{dt} \Delta V + F_v (\Delta V)^2 + M V_{ss} \frac{d}{dt} \Delta V + F_v V_{ss} \Delta V .$$

The average power is

$$\overline{P} = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} P \, dt = \lim_{T \to \infty} \frac{M}{2T} \int_{-T}^{T} \Delta V \frac{d}{dt} \Delta V \, dt + \lim_{T \to \infty} \frac{M V_{ss}}{2T} \int_{-T}^{T} \frac{d}{dt} \Delta V \, dt + \lim_{T \to \infty} \frac{F_v}{2T} \int_{-T}^{T} (\Delta V)^2 \, dt + \lim_{T \to \infty} \frac{F_v V_{ss}}{2T} \int_{-T}^{T} \Delta V \, dt . \tag{3-8}$$

The first integral of the right hand member is zero because $\Delta v(t)$ is uncorrelated with its derivative. The second integral is zero because the disturbance $\frac{d\Delta v(t)}{dt}$ has zero mean. ($v(t) = v_{ss} + \Delta v(t)$ , where $v_{ss}$ is the mean of $v(t)$.) The third integral is recognized as the autocorrelation of $\Delta v$ with zero argument and with the constant multiplier $F_v$ .

$$F_v \phi_{\Delta v}(0) = \overline{P} \tag{3-9}$$

The fourth term is zero because $\Delta v$ has zero mean.

Therefore the autocorrelation of the velocity disturbance with zero argument is the average power dissipated due to the disturbance $\Delta v(t)$ . This power must be supplied by the engine in addition to the power required to maintain $v_{ss}$ , given by $\overline{P}_{v_{ss}} = F_v V_{ss}^2$ .

Linear System Response to Stationary
Random Input Disturbances

Frequency response techniques are used to determine the response of a linear system to random input disturbances in a manner analogous to the sinusoidal analysis. The power spectrum $\Phi_{\Delta y}(\omega)$ of a random disturbance $\Delta y(t)$ is defined to be the Fourier integral of its autocorrelation, given by

$$\Phi_{\Delta y}(\omega) = \int_{-\infty}^{+\infty} \phi_{\Delta y}(\tau)\, e^{-j\omega\tau} d\tau . \tag{3-10}$$

It is shown in various references[15, 16] that the spectrum of the response of a linear system $\Delta y(t)$ to a random input disturbance $\Delta x(t)$ is related to the spectrum of the input disturbance $\Phi_{\Delta x}(\omega)$ by the equation

$$\Phi_{\Delta y}(\omega) = G(j\omega) G(-j\omega) \Phi_{\Delta x}(\omega) , \tag{3-11}$$

where $G(j\omega)$ is the gain function of the linear system, defined in the sinusoidal analysis.

The autocorrelation of the linear system response is the inverse transform of $\Phi_{\Delta y}(\omega)$, given by

$$\phi_{\Delta y}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \Phi_{\Delta y}(\omega)\, e^{j\omega\tau} d\omega. \tag{3-12}$$

The average power associated with the response or its mean square value is

$$\phi_{\Delta y}(0) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \Phi_{\Delta y}(\omega)\, d\omega. \tag{3-13}$$

Thus if the spectrum or autocorrelation of the input disturbance is known and the gain function of the linear system is determined, then the mean square value of the system response may be determined.

### Measures of Queue Response to Random Disturbances of the Initial Vehicle

There are several measures of queue response to random disturbances of the initial vehicle, which may be of considerable use for characterizing the response in the engineering of the automatic traffic system. These are listed below:

1. The average power dissipated by the nth vehicle of the queue, denoted $\overline{P}_{\Delta n}$ .

2. The total average power dissipated in a queue of n vehicles, denoted $\Sigma \overline{p}_{\Delta n}$ .

3. The mean square headway disturbance of the nth vehicle of the queue, denoted $\sigma_{\Delta h_n}{}^2$ .

4. The mean square relative velocity disturbance of the nth queue vehicle, denoted $\sigma_{\Delta v_{r_n}}{}^2$ .

5. The mean square absolute acceleration disturbance of the nth vehicle of the queue, denoted $\sigma_{\Delta a_n}{}^2$ .

Each of these measures will now be related to the spectrum $\Phi_{\Delta v_0}(\omega)$ of the velocity disturbance of the initial vehicle, $\Delta v_0(t)$ . The procedure is simply to determine the pertinent gain from sinusoidal analysis for each measure and then to apply Eqs. (3-11) and (3-13).

The average power dissipated by the nth vehicle is simply the product of the constant multiplier $F_v$ and the mean square absolute velocity $\sigma_{\Delta v_n}{}^2$ of the nth vehicle. The gain function relating absolute velocity of the nth vehicle to that of the initial vehicle was shown in Chapter II to be simply $T(j\omega)^n$ . Therefore by Eq. (3-11),

$$\Phi_{\Delta v_n}(\omega) = |T(j\omega)|^{2n} \Phi_{\Delta v_0}(\omega). \tag{3-14}$$

Then from Eq. (3-13) the mean square velocity is

$$\sigma_{\Delta v_n}{}^2 = \frac{1}{2\pi} \int_{-\infty}^{+\infty} |T(j\omega)|^{2n} \Phi_{\Delta v_0}(\omega)\, d\omega,$$

and the average power of the nth vehicle disturbance is

(3-15) $$\overline{P}_{\Delta n} = \frac{F_v}{2\pi}\int_{-\infty}^{+\infty} |T(j\omega)|^{2n}\, \Phi_{\Delta V_0}(\omega)\, d\omega.$$

The total average power dissipated in the queue of n vehicles is

$$\Sigma \overline{P}_{\Delta n} = \frac{F_v}{2\pi}\int_{-\infty}^{+\infty} \sum_{i=0}^{n-1} |T(j\omega)|^{2i}\, \Phi_{\Delta V_0}(\omega)\, d\omega,$$

which can be written in closed form as

(3-16) $$\Sigma \overline{P}_{\Delta n} = \frac{F_v}{2\pi}\int_{-\infty}^{+\infty} \frac{1-|T(j\omega)|^{2n}}{1-|T(j\omega)|^{2}}\, \Phi_{\Delta V_0}(\omega)\, d\omega.$$

The relative velocity of the nth vehicle,

(3-17) $$\Delta V_{r_n} = \Delta V_{n-1} - \Delta V_n ,$$

is related to $\Delta v_0$ by the gain function, $\left[1-T(j\omega)\right]\left[T(j\omega)\right]^{n-1}$ .
The spectrum of the relative velocity disturbance of the nth vehicle is then

(3-18) $$\Phi_{\Delta V_{r_n}}(\omega) = |1-T(j\omega)|^{2}\, |T(j\omega)|^{2(n-1)}\, \Phi_{\Delta V_0}(\omega),$$

and the mean square relative velocity disturbance is

$$(3\text{-}19) \quad \sigma_{\Delta V_{r_n}}^{2} = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \left|1-T(j\omega)\right|^{2} \left|T(j\omega)\right|^{2(n-1)} \Phi_{\Delta V_o}(\omega)\, d\omega .$$

The relative velocity is the derivative of the headway, so that

$$(3\text{-}20) \quad \Delta V_{r_n}(t) = p\Delta h_n .$$

Then the headway gain function is given by

$$\left[\frac{1-T(j\omega)}{j\omega}\right]\left[T(j\omega)\right]^{n-1} .$$

It is necessary to determine if the apparent pole at $\omega = 0$ actually exists. The gain function $T(j\omega)$ for low frequencies is approximately given by

$$T(j\omega) \approx \frac{b_o + b_1 j\omega}{b_o + a_1 j\omega} ,$$

since $T(0) = 1$ . The first two terms of the expansion of $T(j\omega)$ are

$$T(j\omega) \approx 1 + j\,\frac{(b_1 - a_1)}{b_o}\,\omega .$$

Then $$\lim_{\omega \to 0} \left[\frac{1-T(j\omega)}{j\omega}\right] = \frac{b_1 - a_1}{b_o} ,$$

and it is seen that the pole does not exist at $\omega = 0$ . The spectrum of the headway response $\Phi_{\Delta h_n}(\omega)$ of the nth vehicle is

$$(3\text{-}21) \qquad \Phi_{\Delta h_n}(\omega) = \left|\frac{1-T(j\omega)}{j\omega}\right|^2 \left|T(j\omega)\right|^{2(n-1)} \Phi_{\Delta V_0}(\omega).$$

The mean square headway disturbance of the nth vehicle is then

$$(3\text{-}22) \qquad \sigma_{\Delta h_n}^2 = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \left|\frac{1-T(j\omega)}{j\omega}\right|^2 \left|T(j\omega)\right|^{2(n-1)} \Phi_{\Delta V_0}(\omega)\, d\omega.$$

Finally, the acceleration of the nth vehicle is the derivative of its velocity, and

$$\Delta a_n(t) = p\Delta V_n(t).$$

The gain relating $\Delta a_n(t)$ to $\Delta v_0)t)$ is then given by

$$j\omega\left[T(j\omega)\right]^n.$$

The spectrum of the acceleration $\Phi_{\Delta a_n}(\omega)$ of the nth vehicle is

$$(3\text{-}23) \qquad \Phi_{\Delta a_n}(\omega) = \omega^2 \left|T(j\omega)\right|^{2n} \Phi_{\Delta V_0}(\omega).$$

The mean square value of the nth vehicle's acceleration is

$$(3\text{-}24) \qquad \sigma_{\Delta a_n}^2 = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \omega^2 \left|T(j\omega)\right|^{2n} \Phi_{\Delta V_0}(\omega)\, d\omega.$$

It is thus seen that each of the above measures is of the form,

$$\sigma^2 = \frac{1}{2\pi}\int_{-\infty}^{+\infty} f\left[T(j\omega)\right] \Phi_{\Delta V_0}(\omega)\, d\omega .$$

The above measures are summarized in Table 1.

TABLE 1

| MEASURE | SYMBOL | f(T) |
|---|---|---|
| Power dissipated by nth vehicle due to disturbance | $\bar{P}_{\Delta n}$ | $F_v \lvert T \rvert^{2n}$ |
| Total power dissipated in a queue of n vehicles due to the disturbance | $\sum \bar{P}_{\Delta n}$ | $F_v \left( \frac{1 - \lvert T \rvert^{2n}}{1 - \lvert T \rvert^{2}} \right)$ |
| Headway variance of the nth vehicle | $\sigma^2_{\Delta h_n}$ | $\frac{\lvert 1-T \rvert^2 \lvert T \rvert^{2(n-1)}}{\omega^2}$ |
| Acceleration variance of the nth vehicle | $\sigma^2_{\Delta a_n}$ | $\omega^2 \lvert T \rvert^{2n}$ |
| Relative velocity variance of the nth vehicle | $\sigma^2_{\Delta V_{r_n}}$ | $\lvert 1-T \rvert^2 \lvert T \rvert^{2(n-1)}$ |

Queue Response to Multiple Independent Disturbance Sources

The engineering design problem of allowing large enough equilibrium headway to avoid collisions in the queue due to random disturbances is illustrated by considering the headway disturbance of the nth vehicle. The disturbances generated by each preceding vehicle are assumed independent. In this case the mean square headway disturbance of the nth vehicle is given by

$$(3\text{-}25)\qquad \sigma_{\Delta h_n}^{2} = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\sum_{k=0}^{n-1}\left|\frac{1-T(j\omega)}{j\omega}\right|^2\left|T(j\omega)\right|^{2(n-k-1)}\Phi_{\Delta v_k}(\omega)\,d\omega.$$

where $\Phi_{\Delta v_k}(\omega)$ is the spectrum of the velocity disturbance of the kth vehicle due to its internal disturbance source. Eq. (3-25) may be written,

$$(3\text{-}26)\qquad \sigma_{\Delta h_n}^{2} = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\left|\frac{1-T(j\omega)}{j\omega}\right|^2\sum_{K=0}^{n-1}\left|T(j\omega)\right|^{2K}\Phi_{\Delta v_{n-K-1}}(\omega)\,d\omega.$$

The most severe problem here is that of disturbances generated by each vehicle in very long queues. Suppose for the sake of discussion that the spectrum of the disturbance of each vehicle due to its internal source is the same and is $\Phi_{\Delta v_i}(\omega)$ . Then Eq. (3-26) becomes

$$(3\text{-}27)\qquad \sigma_{\Delta h_n}^{2} = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\left|\frac{1-T(j\omega)}{j\omega}\right|^2\Phi_{\Delta v_i}(\omega)\sum_{K=0}^{n-1}\left|T(j\omega)\right|^{2K}d\omega.$$

Recalling that $T(0) = 1$ to meet the equilibrium requirements and that $|T(j\omega)| \leq 1$ to meet the asymptotic stability requirement, and noting that $|T(j\omega)|$ can be made to be less than unity for $\omega \neq 0$ , one can observe that the series in the integrand converges for $n \to \infty$ if $\omega \neq 0$ . However, the series may converge to a very large number for small $\omega \neq 0$ . The series will still be a large number if $n$ is large but finite. In order to avoid a building up without limit of disturbance along the queue as the length of the queue becomes large, it is necessary that the $\Phi_{\Delta v_i}(\omega)$ in Eq. (3-27) be zero at $\omega = 0$ and that it be small for small $\omega$ . This will only be possible if $\Phi_{\Delta v_i}(\omega)$ has zeros at $\omega = 0$ . It will now be shown that by proper system design $\Phi_{\Delta v_i}(\omega) \to 0$ as $\omega \to 0$ .

Consideration of the sources themselves shows first of all that any constant bias of the disturbance sources must be removed to maintain the equilibrium condition in the traffic queue. The remaining disturbance from the source is bounded. Therefore the autocorrelation of the source disturbance is bounded. Generally the value of the disturbance of the source at any time is independent of its value a few seconds or more earlier at most (excluding road-induced disturbances, which eventually become independent). Therefore the autocorrelations approach zero for large $\tau$ . Consequently, the power spectra of these source disturbances are bounded. This means at least that the power spectra have no poles at $\omega = 0$ . The spectra of quantization noise in the measurement of $\Delta v_{i-1}$ and tachometer noise, which both depend on the equilibrium speed of the vehicles, may have zeros at $\omega = 0$ . However, there is no obvious basis for assuming that the spectra of the other source disturbances are not finite and non-zero at $\omega = 0$ . These could cause $\sigma^2_{\Delta h_n}$ to increase without limit eventually if the length of the queue is unlimited. and if necessary zeros are not introduced by the gain function relating $\Delta v_i$ to the source disturbances. This gain is characteristic of the type of controller used. Several of these are now considered.

The velocity of each of the vehicles can only be changed by a force on the vehicle. The sources of disturbance considered result in forces on the automatically controlled vehicle through its controller. Consider first the block diagram of the ideal velocity controller shown in Fig. 5

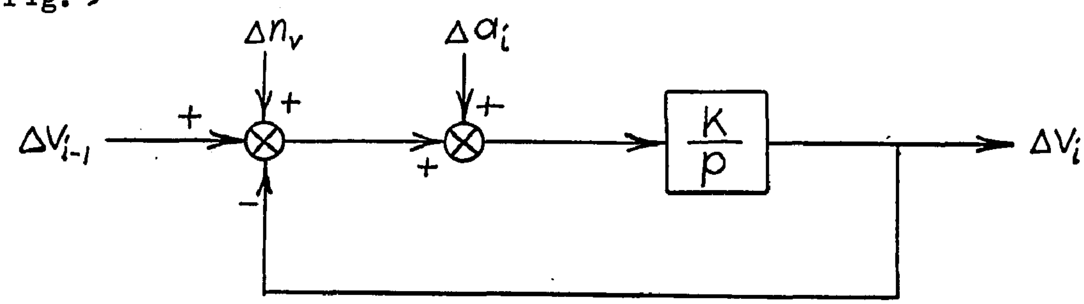


Fig. 5 --Ideal Velocity Controller

The disturbance sources are $\Delta n_v$ , the noise and error in the measurement of $\Delta v_{i-1}$ and $\Delta v_i$ , and $\Delta a_i$ , which includes external accelerations on the vehicle as well as noise generated in the servomechanism components of the vehicle. From this diagram it can be seen that the gain relating the measurement noise, component noise, and accelerations to the velocity disturbance of the vehicle is simply $T(j\omega)$ . Then

(3-28) $$\Phi_{\Delta v_i}(\omega) = |T(j\omega)|^2 \, \Phi_{\Delta n_v}(\omega) \quad \text{or} \quad |T(j\omega)|^2 \Phi_{\Delta a_i}(\omega).$$

Since $T(j\omega)$ will not provide $\Phi_{\Delta v_i}(\omega)$ with zeros at $\omega = 0$, and since apparently $\Phi_{\Delta n_v}(\omega)$ and $\Phi_{\Delta a_i}(\omega)$ don't have the zeros themselves, it is anticipated that headway disturbances in long queues with this simple controller would build up without bound.

An alternative linear velocity controller using acceleration feedback on the controlled vehicle was studied by English and Lim.[17, 18] A simplified block diagram for this controller is shown in Figure 6

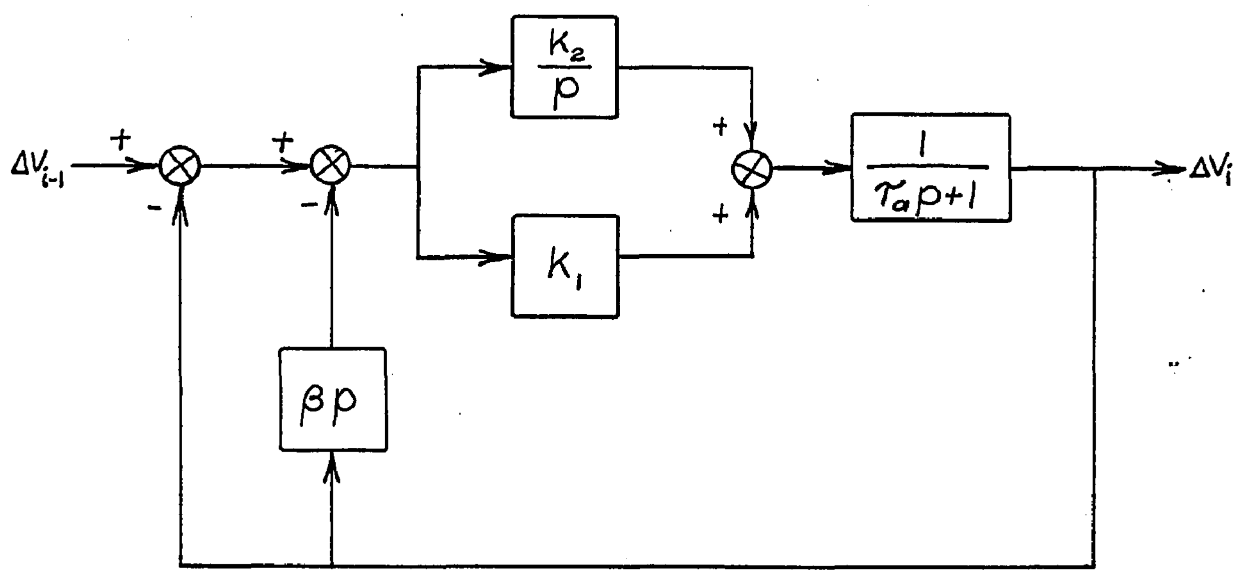


Fig. 6 --Velocity Controller With Acceleration Feedback

The response $\Delta v_i$ to $\Delta v_{i-1}$ for this controller is given by

$$\text{(3-29)} \qquad \left[(\tau_a + K_1\beta)p^2 + (1 + K_1 + K_2\beta)p + K_2\right]\Delta v_i = (K_1 p + K_2)\Delta v_{i-1} .$$

The block in Fig. 6 containing the transfer function $\dfrac{1}{\tau_a p + 1}$ is an ideal linearization of the automobile response $\Delta v_i$ to its accelerator and brake pedal displacements. These characteristics have been measured on the road in a representative sedan. It was found that both the speed vs. accelerator displacement and the speed vs. brake pedal displacement could each be approximated with a single time constant filter. Although the time constants were somewhat different for acceleration and for braking, they can be reasonably approximated by the same for both. The measured value of $\tau_a$ was approximately 20 seconds. In an actual vehicle there is a certain amount of dead zone in the brake pedal. It is assumed that this can be removed and a smooth transition from acceleration to braking can be achieved in a real servo controller. The forcing function on the linearized vehicle which can be controlled is then the throttle position and brake pressure, and the above transfer function reasonably relates the vehicle velocity to this forcing function.

Practical implementation of this controller takes the form shown in Fig. 7 .

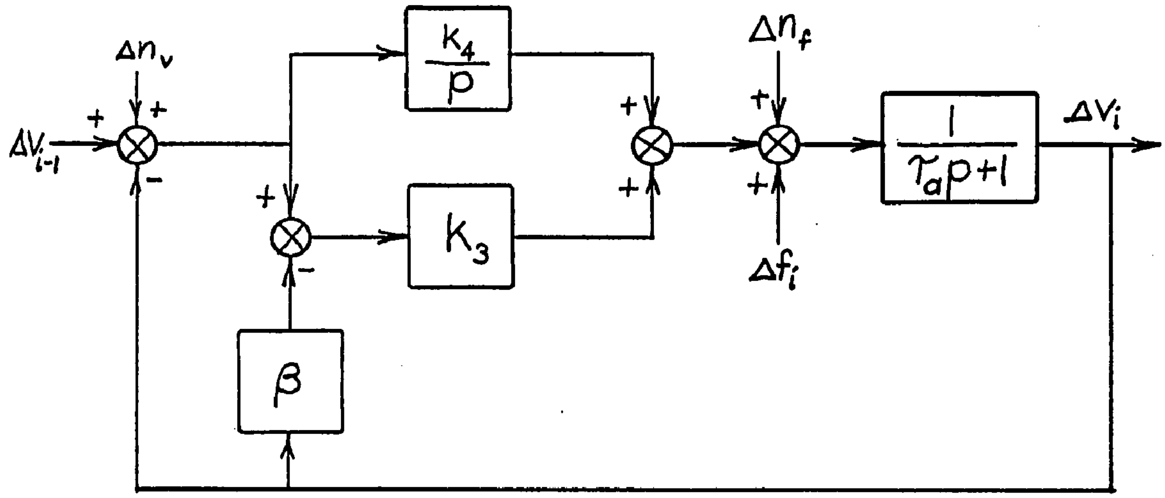


Fig. 7.--A Practical Velocity Controller

The response $\Delta v_i$ to $\Delta v_{i-1}$ of the practical controller is given by

$$(3\text{-}30) \qquad \left[\tau_a p^2 + (1+K_3+K_3\beta)p + K_4\right]\Delta V_i = \left(K_3 p + K_4\right)\Delta V_{i-1} .$$

The three sources of noise shown in Fig. 7 are denoted $\Delta n_v$ for noise associated with the measurement of the relative velocity $(v_{i-1} - v_i)$, $\Delta n_f$ associated with component noise of the servo-mechanism, and $\Delta f_i$ representing wind force and road-induced gravitational forces. Some discussion of these sources is now in order.

Road-induced forces are not intended to be considered here, but it will be shown in the next chapter that their spectra contain zeros at $\omega = 0$. The wind force on the vehicle is certainly a continuous bounded function. Also, its value at any time is considered to be independent of its value at a considerably earlier time. Consequently it must have a bounded spectrum.*

As has been discussed before, component noise is assumed to have finite spectra at $\omega = 0$.

The measurement noise requires consideration of the measurement technique. It is assumed that $v_{i-1}$ and $v_i$ are measured by identical electronic sampling and clamping circuits actuated by pulses received in the controlled vehicle. Detectors are imbedded in the road at constant spacing, which by techniques already available can provide the controlled vehicle with a pulse each time the preceding vehicle crosses a detector. Similarly if provision is made, pulses may be received in the controlled vehicle as it crosses detectors. This results in two pulse trains received in the controlled vehicle as shown in Fig. 8. Here the first order approximation is made that both vehicles are at nearly the same constant speed $v_{ss}$. This results in a constant sampling period $T_s$, the time required for each vehicle to travel from one detector to the next. Although the pulse trains are displaced by $\tau_s$, $nT_s + \tau_s$ is the time spacing between vehicles. The small integer $n$ may be on the order of 1 to 5.

*The details of this argument are presented in the discussion of the spectrum of road elevation in the first section of the following chapter.

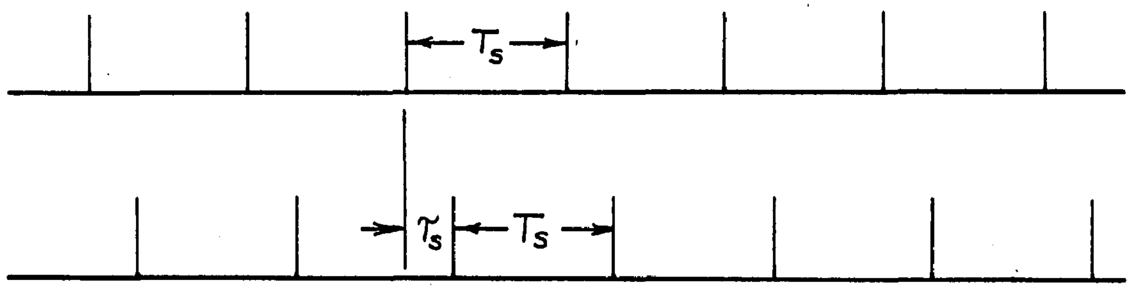


Fig. 8.--Pulse Trains Received In The Controlled Vehicle

In one velocity measurement technique the time T between pulse is registered each time a new pulse arrives. The time T is the sum of three terms given by

$$T = \frac{l_b}{V_{ss}} + \frac{\Delta l_b}{V_{ss}} + \Delta t_v .$$

The first term is the time that would be registered if the speed of the vehicle, whose velocity is being measured, was the constant $v_{ss}$ and there was no error in the placement of the detectors. The second term is a time increment added due to error in the position of the detectors. The third term is the time increment added due to the velocity disturbance $\Delta v$ from $v_{ss}$. The approximate velocity, computed electronically, is

$$V = \frac{l_b}{T} = \frac{V_{ss}}{1 + \frac{\Delta l_b}{l_b} + \frac{\Delta t_v V_{ss}}{l_b}} .$$

The first two terms of the expansion of the right member for small errors give the approximation,

$$v \approx v_{ss}\left(1 - \frac{\Delta l_b}{l_b} - \frac{v_{ss}\Delta t_v}{l_b}\right) = v_{ss}\left(1 - \frac{\Delta l_b}{l_b} - \frac{\Delta t_v}{T_s}\right).$$

The samples of lead vehicle velocity $v_{i-1}$ obtained at instants $t = kT_s$

are $$v_{i-1}(kT_s) = v_{ss} + \Delta d_{i-1}(kT_s) + \Delta v_{i-1}(kT_s),$$

where $\Delta d_{i-1}(kT_s) = -v_{ss}\frac{\Delta b}{b}$ evaluated at $kT_s$ and

$$\Delta v_{i-1}(kT_s) = -\frac{v_{ss}\Delta t_v}{l_b}$$

evaluated at $kT_s$ . The corresponding values of the controlled vehicle velocity $v_i$ are obtained at the time instants $t = kT_s + \tau_s$ , and are given by

$$v_i(kT_s + \tau_s) = v_{ss} + \Delta d_i(kT_s + \tau_s) + \Delta v_i(kT_s + \tau_s). \tag{3-31}$$

These sampled values are held until the next samples are taken. A difference amplifier follows the sampling and holding devices. The output of the difference amplifier is the input of the vehicle servomechanism. It is noted that $v_{ss}$ will not appear here, and will not be considered. The difference amplifier output is given by

$$\Delta v_r = \frac{(1 - e^{-T_s p})}{p}\left\{(\Delta v_{i-1} + \Delta d_{i-1})^* - e^{-\tau_s p}\left[e^{\tau_s p}(\Delta v_i + \Delta d_i)\right]^*\right\}. \tag{3-32}$$

Here the asterisk indicates that the variables are sampled at the time intervals $t = kT_s$. The error $\Delta d_i$ appears in both vehicles, delayed in time by the time spacing $nT_s + \Upsilon_s$ in the controlled vehicle. Thus

$$\Delta d_{i-1}(kT_s - nT_s - \Upsilon_s) = \Delta d_i(kT_s). \tag{3-33}$$

Substituting Eq. (3-33) and $\Delta v_i = T(p)\Delta v_{i-1}$ into Eq. (3-32), one obtains

$$\Delta V_r = \frac{(1-e^{-T_s p})}{p}\left\{\begin{matrix} \Delta V_{i-1}^{*} - e^{-\Upsilon_s p}\left[e^{\Upsilon_s p}T(p)\Delta V_{i-1}\right]^{*} \\ + \Delta d_{i-1}^{*} - e^{-\Upsilon_s p}\left[e^{-nT_s p}\Delta d_{i-1}\right]^{*} \end{matrix}\right\}. \tag{3-34}$$

The gain function relating $\Delta v_r = \Delta v_{i-1} - \Delta v_i$ to $\Delta v_{i-1}$ may be found if the gain function of a sampled variable is first considered. In general if a response $y$ is related to an input $x$ by a relation $y = g(p)x$, the transform of $y$ is given by $Y(j\omega) = G(j\omega)\ X(j\omega)$, where $Y(j\omega)$ is the Fourier transform of $y(t)$ and $X(j\omega)$ is the Fourier transform of $X(t)$. The effect of sampling $y$ is to make $\overset{*}{Y}(j\omega)$, the transform of the sampled variable $y^*$, a periodic function of $\omega$ with period $\omega_0 = \frac{2\pi}{T_s}$. On the nth period

$Y^*(j\omega \pm jn\frac{2\pi}{T_s}) = \frac{Y(j\omega)}{T_s}$. If $z(t) = h(p)\ y^*(t)$, then the transform of $z(t)$ is

$$Z(j\omega) = H(j\omega)Y^{*}(j\omega).$$

For frequencies in the range $-\frac{\pi}{T_s} < \omega < \frac{\pi}{T_s}$,

$$Z(j\omega) = \frac{H(j\omega)Y(j\omega)}{T_s} = \frac{H(j\omega)G(j\omega)X(j\omega)}{T_s} .$$

In this range one can consider $z(t)$ to be related to $x(t)$ by the gain function $\frac{H(j\omega)\,G(j\omega)}{T_s}$. Applying this discussion to Eq. (3-34), the nature of the gain functions as $\omega \to 0$ relating $\Delta v_r$ to $\Delta v_{i-1}$ and to $\Delta d_{i-1}$ may be determined. The gain function relating $\Delta v_r$ to $\Delta v_{i-1}$ is

$$\frac{1-e^{-T_s j\omega}}{T_s j\omega}\left[1-T(j\omega)\right]. \qquad \left(-\frac{\pi}{T_s} < \omega < \frac{\pi}{T_s}\right)$$

Also, the gain function relating $\Delta v_r$ to $\Delta d_{i-1}$ is

$$\frac{1-e^{-T_s j\omega}}{T_s j\omega}\left[1-e^{-(nT_s+\tau_s)j\omega}\right]. \qquad \left(-\frac{\pi}{T_s} < \omega < \frac{\pi}{T_s}\right)$$

It is seen immediately that

(3-35) $$\lim_{\omega \to 0}\left\{\frac{(1-e^{-T_s j\omega})}{T_s j\omega}\left[1-e^{-(nT_s+\tau_s)j\omega}\right]\right\} = 0 .$$

At very low frequency the approximation may be made that

$$T(j\omega) \approx 1 - \frac{(1+K_3\beta)j\omega}{K_4} .$$

Then it can be seen that

$$\text{(3-36)} \qquad \lim_{\omega \to 0} \left\{ \frac{(1-e^{-T_s j\omega})}{T_s j\omega} \left[ 1 - T(j\omega) \right] \right\} = 0.$$

Thus if the spectrum of the random spacing error of the detectors converted to a time function by $x = v_{ss}t$ is denoted $\Phi_{\Delta d}(\omega)$, then the spectrum of the resulting noise in the relative velocity signal to the servos is

$$\text{(3-37)} \qquad \Phi_{n_{v_d}}(\omega) = \left| \frac{(1-e^{-T_s j\omega})}{T_s j\omega} \left[ 1 - e^{-(nT_s + \tau_s) j\omega} \right] \right|^2 \Phi_{\Delta d}(\omega).$$

Also the component of this noise spectrum due to sampling and holding error is

$$\text{(3-38)} \qquad \Phi_{n_{v_s}}(\omega) = \left| \frac{(1-e^{-T_s j\omega})}{T_s j\omega} \left[ 1 - \left(1 - \frac{1+K_{3\beta}}{K_4} j\omega\right) \right] \right|^2 \Phi_{\Delta v_{i-1}}(\omega).$$

It is reasonable to assume that $\Phi_{\Delta v_{i-1}}(\omega)$ and $\Phi_{\Delta d}(\omega)$ are bounded and are thus finite at the origin $(\omega = 0)$. Then $\Phi_{n_{v_d}}(\omega)$ and $\Phi_{n_{v_s}}(\omega)$ have zeros at $\omega = 0$.

Now the gain functions relating $\Delta v_i$ to the various disturbances are found from Fig. 7. The gain relating $\Delta v_i$ to $\Delta n_v$ is the same as that relating $\Delta v_i$ to $\Delta v_{i-1}$, readily determined from Eq. (3-31) to be

$$\frac{K_3 j\omega + K_4}{\tau_a (j\omega)^2 + (1 + K_3 + K_{3\beta}) j\omega + K_4} .$$

which has no zeros at $\omega = 0$ . However, because $\Phi_{n_{v_d}}(\omega)$ and $\Phi_{n_{v_s}}(\omega)$ do have the zeros, the resulting $\Phi_{\Delta v_i}(\omega)$ will have zeros at $\omega = 0$ . The gain function relating $\Delta v_i$ to $\Delta n_f$ is from Fig. 7 found to be

$$\frac{j\omega}{\tau_a (j\omega)^2 + (1 + K_3 + K_{3\beta}) j\omega + K_4} .$$

Since it was shown above that all other disturbance sources could be referred to $\Delta n_f$ with spectra which are finite at $\omega = 0$ , the resulting $\Phi_{\Delta v_i}(\omega)$ will have zeros at $\omega = 0$ . Thus the $\Phi_{\Delta v_i}(\omega)$ has zeros at the origin for all expected sources of random disturbance, and it is anticipated that $\sigma_{\Delta h_n}{}^2$ will remain finite for unlimited n or queue lengths with this type of velocity controller.

The response of the queue with the headway controller is now considered. Its simplified block diagram is shown in Fig. 9. In Fig. 9 the velocity measurement noise is denoted $\Delta n_v$ and the headway measurement noise, $\Delta n_h$ . Component and external force noise is designated by $\Delta f_i$ . The gain relating $\Delta x_i$ to $\Delta x_{i-1}$ is found from Fig. 9 to be $\dfrac{k_2 j\omega + k_1}{\tau_a (j\omega)^2 + (1 + k_2) j\omega + k_1}$ .

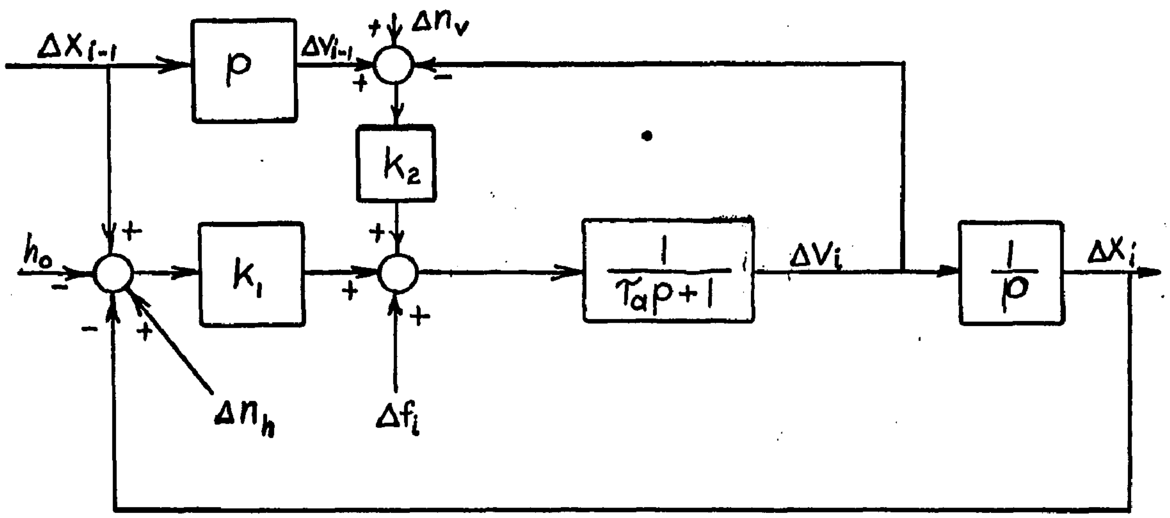


Fig. 9.--Simplified Headway Controller

The same gain function relates $\Delta v_i$ to $\Delta v_{i-1}$ . The gain relating $\Delta v_i$ to $\Delta f_i$ is $\dfrac{j\omega}{\tau_a(j\omega)^2 + (1 + k_2)j\omega + k_1}$ , and $\Phi_{\Delta v_i}(\omega)$ is

is related to $\Phi_{\Delta f_i}(\omega)$ by

$$\Phi_{\Delta v_i}(\omega) = \frac{\omega^2}{\left|\tau_a(j\omega)^2 + (1+k_2)j\omega + k_1\right|^2}\,\Phi_{\Delta f_i}(\omega). \tag{3-39}$$

The gain relating $\Delta v_i$ to the velocity measurement noise $\Delta n_v$ is

$$\frac{k_2 j\omega}{\tau_a(j\omega)^2 + (1+k_2)j\omega + k_1}.$$

The component of $\Phi_{\Delta v_i}(\omega)$ due to velocity measurement noise is

(3-40) $$\Phi_{\Delta v_i}(\omega) = \frac{k_2^2\,\omega^2}{\left|\tau_a(j\omega)^2 + (1+k_2)j\omega + K_1\right|^2}\,\Phi_{\Delta n_v}(\omega).$$

Finally, the gain relating $\Delta v_i$ to headway measurement noise $\Delta n_h$ is

$$\frac{K_1 j\omega}{\tau_a(j\omega)^2 + (1+k_2)j\omega + K_1}.$$

The component of $\Phi_{\Delta v_i}(\omega)$ due to headway measurement noise is

(3-41) $$\Phi_{\Delta v_i}(\omega) = \frac{K_1^2\,\omega^2}{\left|\tau_a(j\omega)^2 + (1+k_2)j\omega + k_1\right|^2}\,\Phi_{\Delta n_h}(\omega).$$

The velocity disturbances of the vehicle with a headway controller have a spectrum with the required zeros at $\omega = 0$, as can be seen in Eqs.(3-39), (3-40), and (3-41). Furthermore it is noticed in passing, from the above gain functions, that constant biases in the noise will not be a problem with the headway controller. A small constant error in headway can be tolerated. It will not exist in the relative velocity. If the headway controller is used, $\sigma^2_{\Delta h_n}$ of Eq. (3-28) will remain finite for unlimited queues. Its magnitude will depend on the parameters of $T(j\omega)$, which are $k_1$, $k_2$, and $\tau_a$.

It is not known whether the spectrum $\Phi_{\Delta v_i}(\omega)$ of human driver response has zeros at $\omega = 0$. However, it is expected that the number of manually driven vehicles in the queue will be limited.

Furthermore it is anticipated that the human driver will break the queue and generate independent disturbances as the controller of the initial vehicle of the following queue. Therefore consideration is not given to compounding of manually driven vehicle response.

## CHAPTER IV

## QUEUE RESPONSE TO ROAD-INDUCED DISTURBANCES

### Introduction

In this chapter the nature of road-induced accelerations on the automatically controlled vehicles is discussed. Restrictions on the spectra of the road-induced accelerations are related to the source of the disturbances. The behavior of the queue of automatically controlled vehicles in response to road-induced disturbances is considered, and measures of the queue response are derived. Finally the problem of building up of disturbance in long queues is discussed.

### Source of Road-Induced Disturbances

In considering the diagram of a vehicle travelling over a road with a random vertical profile as shown in Fig. 10, it is seen that the gravitational force on a vehicle $\Delta f_i$ is

$$\Delta f_i = -Mg\sin\alpha, \tag{4-1}$$

and for small $\alpha$

$$\Delta f_i \approx -Mg\alpha. \tag{4-2}$$

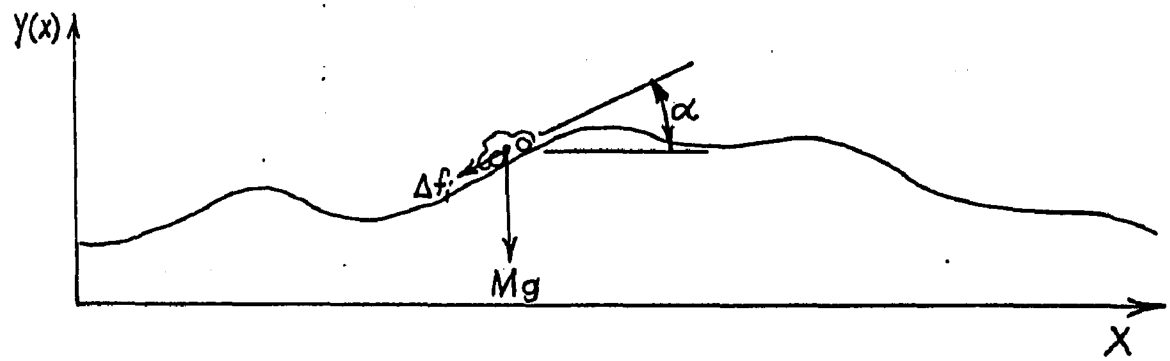


Fig. 10--Road-Induced Gravitational Force on Vehicle

The grade, $\alpha$ , at any point x is given by

(4-3) $$\alpha = \frac{dy(x)}{dx} .$$

If small disturbances (small hills) are considered, a first approximation of the position of the vehicle is given by

(4-4) $$x = v_{ss}t + x_o ,$$

where $v_{ss}$ is the mean speed of the vehicle on the road section. Substituting $x = v_{ss}t$ for the variable of $y(x)$ , $y(x)$ becomes a function of time, and

(4-5) $$\alpha = \frac{dy(t)}{dt}\frac{dt}{dx} = \frac{1}{v_{ss}}\frac{dy(t)}{dt} .$$

Then

(4-6) $$\Delta f_i \approx -\frac{gM}{v_{ss}}\frac{dy(t)}{dt} = -\frac{Mg}{v_{ss}}py .$$

The road configuration and the mean speed over the road section thus permit the tangential force on the vehicle (which is approximately in the horizontal direction of x because of the assumption of small grades) to be written as the derivative of the elevation of the vehicle, a function of time.

The mean value of $y(t)$ is denoted $\overline{y(t)}$, and

(4-7) $$y(t) = \overline{y(t)} + \Delta y(t) .$$

The force disturbance on the vehicle is given by

(4-8) $$\Delta f_i(t) = -\frac{Mg}{V_{ss}} p \Delta y(t) .$$

A number of observations can be made about $\Delta y(t)$. First of all, since vehicles are constrained to the surface of the earth, $\Delta y(t)$ is bounded. Consequently it has a finite average power,

(4-9) $$\lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} [\Delta y(t)]^2 dt \leqq \Delta y(t)^2_{max} .$$

Then $\phi_{\Delta y}(0) \leqq \Delta y(t)^2_{max}$ and is finite. Since

$$\left| \phi_{\Delta y}(\tau) \right| \leqq \phi_{\Delta y}(0), \quad \phi_{\Delta y}(\tau) = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} \Delta y(t)\Delta y(t+\tau) dt$$

is bounded for all $\tau$.

Also it is noted that $y(t)$ must be continuous for all $t$ from the fact that usable road surfaces are continuous. If $y(t)$ is continuous, $y(t+\tau)$ is a continuous function of $\tau$ also. Then the product $y(t)\,y(t+\tau)$ is continuous for all $t$ and $\tau$. Then $\phi_{\Delta y}(\tau)$ is continuous over all $\tau$ [19]. Furthermore $\left| \phi_{\Delta y}(\tau) \right|$ is continuous for all $\tau$.

Finally the observation is made that in general $\Delta y(t + \tau)$ is independent of $\Delta y(t)$ for large values of $\tau$. This is due to the generally random nature of terrain. Since $x = v_{ss}t$, the distance between positions of the vehicle at times $t$ and $t + \tau$ is $v_{ss}\tau$. It is considered that the elevations of the road at any two points widely separated (large $\tau$) are independent. The autocorrelation of $\Delta y$, $\phi_{\Delta y}(\tau)$, is a measure of the correlation of vertical displacements of pairs of points separated by distance $v_{ss}\tau$ all along the road. If the elevations of the pairs are independent, then they will be uncorrelated, and $\phi_{\Delta y}(\tau)$ will be zero. Then

$$\lim_{\tau \to \infty} \phi_{\Delta y}(\tau) = 0.$$

Since $|\phi_{\Delta y}(\tau)|$ is bounded for all $\tau$ by the finite $\phi_{\Delta y}(0)$, and since $|\phi_{\Delta y}(\tau)|$ is continuous for all $\tau$ and

$$\lim_{\tau \to \infty} \phi_{\Delta y}(\tau) = 0,$$

then

$$\int_{-\infty}^{+\infty} |\phi_{\Delta y}(\tau)| \, d\tau$$

is finite. Furthermore, since $\phi_{\Delta y}(\tau)$ is an even function of $\tau$,

$$\int_{-\infty}^{+\infty} \phi_{\Delta y}(\tau) d\tau = 2\int_{0}^{\infty} \phi_{\Delta y}(\tau) \, d\tau \leqq 2\int_{0}^{\infty} |\phi_{\Delta y}(\tau)| \, d\tau. \tag{4-10}$$

Now the spectrum of $\Delta y(t)$ is defined by

$$\Phi_{\Delta y}(\omega) = \int_{-\infty}^{+\infty} \phi_{\Delta y}(\tau) e^{-j\omega\tau} d\tau. \tag{4-11}$$

Because of the evenness of $\phi_{\Delta y}(\tau)$,

$$(4\text{-}12) \qquad \Phi_{\Delta y}(\omega) = 2\int_0^{\infty} \phi_{\Delta y}(\tau)\cos\omega\tau \, d\tau \leqq 2\int_0^{\infty} |\phi_{\Delta y}(\tau)| \, d\tau .$$

Consequently $\phi_{\Delta y}(\omega)$ is bounded by

$$2\int_0^{\infty} |\phi_{\Delta y}(\tau)| \, d\tau$$

for all $\omega$. Thus $\Phi_{\Delta y}(\omega)$ has no poles on the real $\omega$ axis and no poles at $\omega = 0$.

Since $\Delta y(t)$ has finite average power given by

$$(4\text{-}13) \qquad \overline{\Delta y(t)^2} = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \Phi_{\Delta y}(\omega) \, d\omega ,$$

$\Phi_{\Delta y}(\omega)$ must have more poles than zeros. Thus the simplest form that $\Phi_{\Delta y}(\omega)$ can have is

$$\Phi_{\Delta y}(\omega) = \frac{K}{\omega^2 + \alpha^2} ,$$

where K and $\alpha$ are finite constants.

Similar statements can be made about $\Delta f_i(t)$, since it is observed that $\Delta f_i(t)$ is always bounded by Mg ; and by assumption of small grades, its bound will be considerably less than that. Then its average power is finite. Since $\frac{d\,y(x)}{dx}$ is continuous for all roads, it follows that $\Delta f_i(t)$ is also continuous. Finally, $\Delta f_i(t+\tau)$ is again independent of $\Delta f_i(t)$ for large $\tau$. Then all the preceding discussion may be applied to the $\phi_{\Delta f_i}(\tau)$ and $\Phi_{\Delta f_i}(\omega)$ functions.

But in addition,

(4-14) $$\Phi_{\Delta f_i}(\omega) = \omega^2\, \Phi_{\Delta y}(\omega).$$

Thus $\Phi_{\Delta f_i}(\omega)$ has zeros at the origin because $\Phi_{\Delta y}(0)$ is finite. Also, since $\Delta f_i(t)$ has finite average power, by virtue of its boundedness, the integral given by

$$\overline{[\Delta f_i(t)]^2} = \phi_{\Delta f_i}(0) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \Phi_{\Delta f_i}(\omega)\, d\omega$$

must be finite. This in turn means that $\Phi_{\Delta f_i}(\omega)$ must have more poles than zeros. Since $\Phi_{\Delta f_i}(\omega)$ must have a pair of zeros at $\omega = 0$, it must have two pairs of poles - not on the real $\omega$ axis because of the boundedness of $\Phi_{\Delta f_i}(\omega)$ as of $\Phi_{\Delta y}(\omega)$. Since $\Phi_{\Delta f_i}(\omega)$ must have the same poles as $\Phi_{\Delta y}(\omega)$, the simplest rational function which may be assumed for $\Phi_{\Delta y}(\omega)$ is

(4-15) $$\Phi_{\Delta y}(\omega) = \frac{K}{(\omega^2+a^2)(\omega^2+b^2)},$$

where K, a, and b are constants. Then

(4-16) $$\Phi_{\Delta f_i}(\omega) = \frac{K\omega^2}{(\omega^2+a^2)(\omega^2+b^2)}.$$

Response of the Finite Queue to Road-Induced Disturbances

The road-induced disturbances described in the previous section cause disturbances in the velocity of each vehicle of the queue. The response of each vehicle due to road-induced disturbances is by the superposition principle a summation of disturbances induced by the road in preceding vehicles and transmitted to the particular vehicle of interest by the automatic controller. Here the problem differs from that of the last chapter in that the individual disturbance terms are not independent, but are correlated with the other terms. It is not possible to find the measures of queue response by simply summing the measures due to individual terms. It will be seen that a gain function can be found relating the total response of the queue to the road elevation $\Delta y(t)$. Then as before the spectrum of the response variable is related to the spectrum of the input $\Phi_{\Delta y}(\omega)$ by the gain function, and finally the mean square value of the response variable is the integral of its spectrum. It is convenient for development of the gain function in terms of a general controller gain function $T(j\omega)$ to deal with disturbances of velocity of each vehicle $\Delta v_i$. Finally although other disturbances are normally present, superposition permits the separate treatment of road-induced disturbances as if there were no other disturbances.

In the queue of vehicles where the velocity of each vehicle is related to the preceding vehicle by the gain $T(j\omega)$ , the total velocity disturbance of the nth vehicle is given by the relation,

$$\Delta V_n = \Delta V_{yn} + \sum_{K=0}^{n-1} T(p)^{n-K} \Delta V_{yK} , \tag{4-17}$$

where $\Delta v_{y_k}$ is the velocity disturbance induced by the road in the kth vehicle and $\Delta v_{y_n}$ the same for the nth vehicle. The same equilibrium conditions as before are assumed, namely that the vehicles have the same equilibrium speed $v_{ss}$ and that the time spacing between vehicles is constant $\gamma$ seconds. If the road induces a disturbance $\Delta v_0(t)$ in the initial vehicle of the queue, the disturbance induced by the road in the kth vehicle is given by

$$\Delta V_{yK} = \Delta V_0 (t - k\gamma) \tag{4-18}$$

or

$$\Delta V_{yK} = e^{-k\gamma p} \Delta V_0 . \tag{4-19}$$

Substituting Eq. (4-19) into Eq. (4-17),

$$\Delta V_n = \sum_{K=0}^{n-1} T(p)^{n-K} e^{-k\gamma p} \Delta V_0 + e^{-n\gamma p} \Delta V_0 . \tag{4-20}$$

Factoring out $e^{-n\gamma p}$ and combining terms, Eq. (4-20) becomes

$$\Delta V_n = e^{-n\gamma p} \sum_{K=0}^{n} T(p)^K e^{K\gamma p} \Delta V_0 . \tag{4-21}$$

This can be written in closed form as

$$\Delta V_n = e^{-n\gamma p} \left[ \frac{1 - T(p)^{n+1} e^{(n+1)\gamma p}}{1 - T(p) e^{\gamma p}} \right] \Delta V_0 . \tag{4-22}$$

The gain function relating the velocity disturbance of the nth vehicle to that induced in each vehicle by the road is

$$e^{-n\gamma j\omega}\left[\frac{1-T(j\omega)^{n+1}e^{(n+1)\gamma j\omega}}{1-T(j\omega)e^{\gamma j\omega}}\right].$$

The spectrum of the velocity disturbance of the nth vehicle is then

(4-23) $$\Phi_{\Delta V_n}(\omega) = \left|\frac{1-T(j\omega)^{n+1}e^{j(n+1)\gamma\omega}}{1-T(j\omega)e^{j\gamma\omega}}\right|^2 \Phi_{\Delta V_0}(\omega).$$

The average power of the nth vehicle is given in Chapter III as

(4-24) $$\overline{P}_{\Delta n} = \frac{F_v}{2\pi}\int_{-\infty}^{+\infty}\Phi_{\Delta V_n}(\omega)d\omega = \frac{F_v}{2\pi}\int_{-\infty}^{+\infty}\left|\frac{1-T(j\omega)^{n+1}e^{j(n+1)\gamma\omega}}{1-T(j\omega)e^{j\gamma\omega}}\right|^2\Phi_{\Delta V_0}(\omega)d\omega.$$

The acceleration of the nth vehicle is simply

$$\Delta a_n = p\Delta V_n,$$

so the gain function relating $\Delta a_n$ to $\Delta v_n$ is simply $j\omega$, and the spectrum of the nth vehicle's acceleration disturbance is

(4-25) $$\Phi_{\Delta a_n}(\omega) = \omega^2\Phi_{\Delta V_n}(\omega)$$

The mean square value of the random acceleration of the nth vehicle due to the road-induced disturbance is

(4-26) $$\sigma_{\Delta a_n}^2 = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\omega^2\left|\frac{1-T(j\omega)^{n+1}e^{j(n+1)\gamma\omega}}{1-T(j\omega)e^{j\gamma\omega}}\right|^2\Phi_{\Delta V_0}(\omega)d\omega.$$

The relative velocity of the nth vehicle, given by

$$\Delta V_{r_n} = \Delta V_{n-1} - \Delta V_n ,$$

is found from Eq. (4-21) to be related to

$$(4\text{-}27) \qquad \Delta V_{r_n} = e^{-nrp}\left[(e^{rp}-1)\sum_{k=0}^{n-1} T(p)^k e^{krp} - T(p)^n e^{nrp}\right]\Delta V_0 .$$

The corresponding gain function (with p = jω) is written in closed form,

$$(4\text{-}28) \qquad e^{-jnr\omega}\left\{(e^{jr\omega}-1)\left[\frac{1-T(j\omega)^n e^{jnr\omega}}{1-T(j\omega)e^{jr\omega}}\right] - T(j\omega)^n e^{jnr\omega}\right\}.$$

The spectrum of the random relative velocity response of the nth vehicle to the road induced velocity disturbance is given by

$$(4\text{-}29) \qquad \Phi_{\Delta V_{r_n}}(\omega) = \left|(e^{jr\omega}-1)\left[\frac{1-T(j\omega)^n e^{jnr\omega}}{1-T(j\omega)e^{jr\omega}}\right] - T(j\omega)^n e^{jnr\omega}\right|^2 \Phi_{\Delta V_0}(\omega).$$

The mean square value of the random relative velocity response of the nth vehicle to the road-induced disturbance is

$$(4\text{-}30) \qquad \sigma^2_{\Delta V_{r_n}} = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\left|(e^{jr\omega}-1)\left[\frac{1-T(j\omega)^n e^{jnr\omega}}{1-T(j\omega)e^{jr\omega}}\right] - T(j\omega)^n e^{jnr\omega}\right|^2 \Phi_{\Delta V_0}(\omega)\,d\omega.$$

The relative velocity of the nth vehicle is the time derivative of the headway of the nth vehicle, or

$$\Delta V_{r_n} = p\,\Delta h_n .$$

Then the gain function relating the headway to the relative velocity is simply $\frac{1}{j\omega}$ . Then the spectrum of the random variations in headway of the nth vehicle response to the road-induced disturbance is

$$(4\text{-}31) \qquad \Phi_{\Delta h_n}(\omega) = \frac{1}{\omega^2}\left|\left(e^{j\gamma\omega}-1\right)\left[\frac{1-T(j\omega)^n e^{jn\gamma\omega}}{1-T(j\omega)e^{j\gamma\omega}}\right]-T(j\omega)^n e^{jn\gamma\omega}\right|^2 \Phi_{\Delta v_0}(\omega).$$

The mean square value of the headway variation of the nth vehicle due to the road-induced velocity disturbance $\Delta v_0$ is

$$(4\text{-}32) \qquad \sigma_{\Delta h_n}^2 = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\left|\left(e^{j\gamma\omega}-1\right)\left[\frac{1-T(j\omega)^n e^{jn\gamma\omega}}{1-T(j\omega)e^{j\gamma\omega}}\right]-T(j\omega)^n e^{jn\gamma\omega}\right|^2 \Phi_{\Delta v_0}(\omega)\, d\omega.$$

Immediately it is seen that difficulties are encountered with the integrand as $\omega \to 0$ . Apparently the integrand becomes infinite as $\omega \to 0$ (has poles at the origin). If this is true, the integral will not converge and it can be expected that the magnitude of the headway disturbance of the nth vehicle will build up without limit and collision will occur in the queue. Considering first the nature of the gain function as $\omega \to 0$ , the apparently indeterminate form

$$\left[\frac{1-T(j\omega)^n e^{jn\gamma\omega}}{1-T(j\omega)e^{j\gamma\omega}}\right]$$

is best understood from its equivalent series,

$$\sum_{K=0}^{n-1} T(j\omega)^k e^{jk\gamma\omega}.$$

The limit of the series as $\omega \to 0$ is (recall that $T(0) = 1$)

$$\lim_{\omega \to 0} \sum_{k=0}^{n-1} T(j\omega)^k e^{jk\tau\omega} = n \, .$$

Then

$$\lim_{\omega \to 0} \left| \left(e^{j\tau\omega} - 1\right) \left[ \frac{1 - T(j\omega)^n e^{jn\tau\omega}}{1 - T(j\omega) e^{j\tau\omega}} \right] - T(j\omega)^n e^{jn\tau\omega} \right|^2 = 1 \, .$$

It can now be seen that the necessary condition that $\lim_{n\to\infty} \sigma^2_{\Delta h_n}$ be finite is that $\Phi_{\Delta v_0}(\omega)$ have zeros at the origin. Whether $\Phi_{\Delta v_0}(\omega)$ has zeros at the origin depends on how the road induces the disturbance in the initial vehicle's velocity, and since all vehicles are assumed identical it depends on how the road induces disturbance in any vehicle's velocity. This in turn depends on the type controller used for each vehicle. Several controllers were described in Chapter III. From the block diagram of the practical velocity controller of that chapter, the velocity disturbance of the vehicle to external acceleration is found to be given by

(4-33) $$\left[\tau_a p^2 + (k_3 + k_3\beta + 1)p + K_4\right] \Delta v_i = (\tau_a p + 1)\Delta f_i \, .$$

The road-induced force on the vehicle is given in terms of the elevation of the vehicle in Eq. (4-8). Then the gain function for this controller, which relates the velocity disturbance to the vehicle elevation is

(4-34) $$\frac{-Mg\, j\omega(\tau_a j\omega + 1)}{V_{ss}\left[\tau_a (j\omega)^2 + (1 + K_3 + K_3\beta) j\omega + K_4\right]} \, .$$

The spectrum of $\Phi_{\Delta v_0}(\omega)$ is given by

$$\Phi_{\Delta v_0}(\omega) = \frac{M^2 g^2 \omega^2 \left|(\tau_a j\omega + 1)\right|^2}{V_{ss}^2 \left| \tau_a (j\omega)^2 + (1 + K_3 + k_3\beta) j\omega + K_4 \right|^2} \Phi_{\Delta y}(\omega). \tag{4-35}$$

It was shown in the first section (see Eq. (4-15) ) that $\Phi_{\Delta y}(0)$ is finite, so $\Phi_{\Delta v_0}(\omega)$ does have the necessary zeros. Similarly it can be shown for each of the controllers considered that the spectrum of velocity response $\Phi_{\Delta v_0}(\omega)$ to road-induced disturbance has the necessary zeros at $\omega = 0$ .

The measures derived for queue response to road-induced disturbances are all of the form

$$\sigma_n^2 = \frac{1}{2\pi} \int_{-\infty}^{+\infty} g_n\left[T(j\omega)\right] \Phi_{\Delta v_0}(\omega)\, d\omega. \tag{4-36}$$

The results are summarized on the basis of Eq. (4-36) in Table 2 below.

TABLE 2

| MEASURE | SYMBOL | $g_n\left[T(j\omega)\right]$ |
|---|---|---|
| Power dissipated by the nth vehicle of the queue due to the road-induced disturbance. | $\overline{P}_{\Delta n}$ | $F\left\|\frac{1-T(j\omega)^{n+1}e^{j(n+1)\gamma\omega}}{1-T(j\omega)e^{j\gamma\omega}}\right\|^2$ |
| Mean square acceleration of the nth vehicle due to the road-induced disturbance. | $\sigma_{\Delta a_n}^2$ | $\omega^2\left\|\frac{1-T(j\omega)^{n+1}e^{j(n+1)\gamma\omega}}{1-T(j\omega)e^{j\gamma\omega}}\right\|^2$ |
| Mean square relative velocity of the nth vehicle due to the road-induced disturbance. | $\sigma_{\Delta v_{r_n}}^2$ | $\left\|\left(e^{j\gamma\omega}-1\right)\left[\frac{1-T(j\omega)^n e^{jn\gamma\omega}}{1-T(j\omega)e^{j\gamma\omega}}\right]-T(j\omega)^n e^{jn\gamma\omega}\right\|^2$ |
| Mean square headway of the nth vehicle due to the road-induced disturbance. | $\sigma_{\Delta h_n}^2$ | $\frac{1}{\omega^2}\left\|\left(e^{j\gamma\omega}-1\right)\left[\frac{1-T(j\omega)^n e^{jn\gamma\omega}}{1-T(j\omega)e^{j\gamma\omega}}\right]-T(j\omega)^n e^{jn\gamma\omega}\right\|^2$ |

Response of Very Long Queues
to Road-Induced Disturbances

It is interesting to determine the nature of $\sigma_{\Delta h_n}{}^2$ as a function of n for n very large but finite. The limit of the integrand in Eq. (4-32) as $\omega \longrightarrow 0$ has been determined to be

$$\Phi_{\Delta V_0}(\omega)/\omega^2\Big|_{\omega=0} .$$

Suppose $\omega$ is small but nonzero. Then

$$T(j\omega)^n e^{jn\gamma\omega} \approx 0 \qquad \text{for } n \text{ large,}$$

$$T(j\omega) \approx \frac{1 + b_1 j\omega}{1 + a_1 j\omega} \approx 1 + (b_1 - a_1) j\omega ,$$

$$e^{j\gamma\omega} \approx 1 + j\gamma\omega ,$$

$$T(j\omega) e^{j\gamma\omega} \approx 1 + (\gamma + b_1 - a_1) j\omega ,$$

and

$$\frac{e^{j\gamma\omega} - 1}{1 - T(j\omega) e^{j\gamma\omega}} \approx \frac{\gamma}{\gamma + b - a_1} \qquad (\gamma \neq a_1 - b_1) .$$

For very low frequencies then the integrand becomes

$$\left( \frac{\gamma}{\gamma + b_1 - a_1} \right)^2 \frac{\Phi_{\Delta V_0}(\omega)}{\omega^2} .$$

This integrand is independent of $n$ . Consequently it is anticipated that in very long queues, road-induced disturbances will cause finite $\sigma_{\Delta h_n}{}^2$ which builds up to a constant level with increasing $n$ . The value to which it builds is inversely related to $\gamma - (a_1 - b_1)$ . As $\gamma - (a_1 - b_1)$ tends toward zero, $\sigma_{\Delta h_n}{}^2$ builds up to extremely high values. Either $h_s$ would have to be extremely large or collisions would be expected. For the normal road condition and $\gamma \neq a_1 - b_1$ , the constant level at which $\sigma_{\Delta h_n}{}^2$ stabilizes in very long queues is related to the spectrum of $\Delta y(t)$ and the type of controller used. The necessary $h_s$ is then related to $\sigma_{\Delta h_n}{}^2$ by considering the probability density function associated with $\Delta y(t)$ .

## CHAPTER V

## DESIGN OF VEHICLE CONTROLLERS FOR OPTIMUM QUEUE RESPONSE

### Introduction

The disturbances of the automatically controlled traffic queue may be separated into two groups based upon the manner in which the disturbances are induced in the queue. The first group includes large random disturbances, which have their source in a particular vehicle of the queue. These disturbances are propagated from vehicle to vehicle back along the queue. The nature of the queue response to these disturbances is determined entirely by the closed loop characteristics of the vehicle controllers. It is apparent then that the response of the queue to these disturbances may be optimized by proper design of the closed loop gain function $T(j\omega)$ of the controller. This optimization problem is discussed first in this chapter. It includes consideration of criteria and problem formulation.

The second group includes disturbances which are usually of smaller magnitude, but which are induced into each vehicle of the queue. These include noise sources inherent in the vehicle controllers and road-induced disturbances. These disturbances are related to their resulting velocity disturbances of the vehicles by gain functions which are functions of the controller's components. It will be seen that their effect on queue response may be reduced by servo design techniques, while the optimum system closed loop function $T(j\omega)$ is maintained. This is treated after the optimization problem.

### Optimization of Queue Response to Disturbances of Only the Initial Vehicle

The queue of automatically controlled vehicles may be represented by a block diagram, as shown in Fig. 11 For the situation considered here, where the only external disturbance induced in the

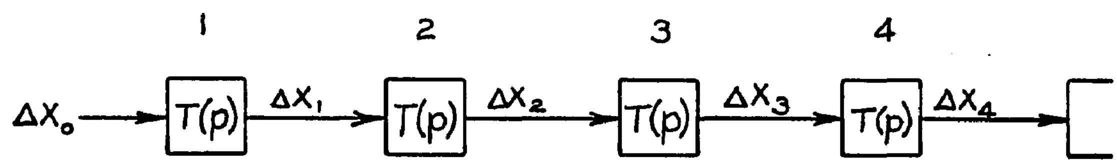


Fig. 11 --Block Diagram of a Queue of Automatically Controlled Vehicles

system is induced in the lead vehicle, the headway response of the ith vehicle to such a disturbance was found in the Chapter II to be given by the relation,

$$\Delta h_i = \Delta X_{i-1} - \Delta X_i = \left[1 - T(p)\right] T(p)^{i-1} \Delta X_o . \tag{5-1}$$

It is obvious that the headway disturbance for the entire queue would be minimum (zero) if $T(p) = 1$, corresponding to a rigid connection between vehicles. But here the disturbance is transmitted without reduction to the entire queue. In particular it is seen that $\Delta v_i = p\,\Delta x_i = p\,\Delta x_0$ for all $i$. Each vehicle of the queue dissipates as much power as the initial vehicle. This is rather an undesirable requirement on the queue vehicles. The velocity disturbance of the ith vehicle $\Delta v_i$, and consequently its power, can be reduced to zero by making $T(p) = 0$ corresponding to no coupling between vehicles. Of course the controller does not function for any purpose here and the headway disturbance between the first two vehicles will be $\Delta x_0$, the maximum headway disturbance possible for a stable queue with the single input disturbance. The above discussion suggests that the most desirable system allows some headway disturbance, but reduces disturbances as they are transmitted through the queue.

The mean-square value of headway disturbance of the ith vehicle was given in Chapter III as

$$\sigma_{\Delta h_i}^2 = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \left|1-T(j\omega)\right|^2 \left|T(j\omega)\right|^{2(i-1)} \Phi_{\Delta x_0}(\omega)\,d\omega\,. \tag{5-2}$$

It is recalled that the equilibrium condition required that $T(0) = 1$, and the asymptotic stability condition requires that $|T(j\omega)|^2 \leqq 1$ for all $\omega$. It is observed that for any physically realizable system,

$$\lim_{\omega \to \infty} T(j\omega) = 0\ .$$

In view of these restrictions of $T(j\omega)$, it is apparent from Eq. (5-2) that $\sigma_{\Delta h_i}^2$ for any given $\Phi_{\Delta x_0}(\omega)$ will be maximum for $i = 1$.

The average power dissipated by the ith queue vehicle is given by

$$\text{(5-3)} \qquad \overline{P}_{\Delta i} = \frac{F_v}{2\pi}\int_{-\infty}^{+\infty} \left|T(j\omega)\right|^{2i} \omega^2 \Phi_{\Delta x_o}(\omega)\, d\omega \, .$$

Again, in view of the above restrictions on $T(j\omega)$, it is apparent from Eq. (5-3) that $P_{\Delta i} < P_{\Delta 1}$ for all $i > 1$. Consequently, attention is focused on the optimization of the response of the first vehicle, since the maximum headway and velocity disturbances occur for this vehicle.

In general it is desirable to minimize the relative motion of the vehicles, the headway and relative velocity in particular. Reducing the headway disturbances reduces the equilibrium spacing $h_s$ required for collision avoidance, and thus the traffic flow is increased. Reducing the relative velocity increases safety. However, these reductions are obtained at the cost of larger individual motion, such as velocity and acceleration disturbances, than would occur if larger relative motion were allowed. Increase in velocity disturbance results in increased power dissipated by the vehicle, which is reflected in the cost of operation of the vehicle. Increase in acceleration disturbance results in increased discomfort of the passengers and increased wear in the vehicle. Tolerable limits on measures of the response motion, such as limits on average power and limits on the mean square acceleration, can be determined independently. Systems which produce measures less than the limits are acceptable. The design problem then reduces to the determination of the $T(j\omega)$ which minimizes the relative motion while simultaneously allowing no more than the limit of individual motion.

As an example it will be quite instructive to consider the very simple problem of minimization of the mean square headway disturbance $\sigma_{\Delta h_1}^2$, subject to the limit $\sigma_{\Delta v_1}^2 \leqq c_v$. In order to simplify the demonstration of techniques, the power density spectrum $\Phi_{\Delta v_0}(\omega)$ of the velocity disturbance $\Delta v_0$ of the initial vehicle is selected to be

$$\Phi_{\Delta v_0}(\omega) = \frac{\omega^2 N^2}{\omega^2 + a^2}$$

without regard to real physical considerations. The velocity disturbance spectrum is related to the position disturbance

(from $x_0(0) + v_{ss}t$) by*

$$\Phi_{\Delta v_0}(\omega) = \omega^2 \Phi_{\Delta x_0}(\omega), \tag{5-4}$$

so

$$\Phi_{\Delta x_0}(\omega) = \frac{N^2}{\omega^2 + a^2} \tag{5-5}$$

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*Recall that $\Delta v_0(t) = p\,\Delta x_0(t)$, so the gain function relating $\Delta v_0$ to $\Delta x_0$ is simply $j\omega$. Then $\Phi_{\Delta v_0}(\omega) = |j\omega|^2\,\Phi_{\Delta x_0}(\omega)$, which yields Eq. (5-4) directly.

A block diagram of this problem is shown in Fig. 12

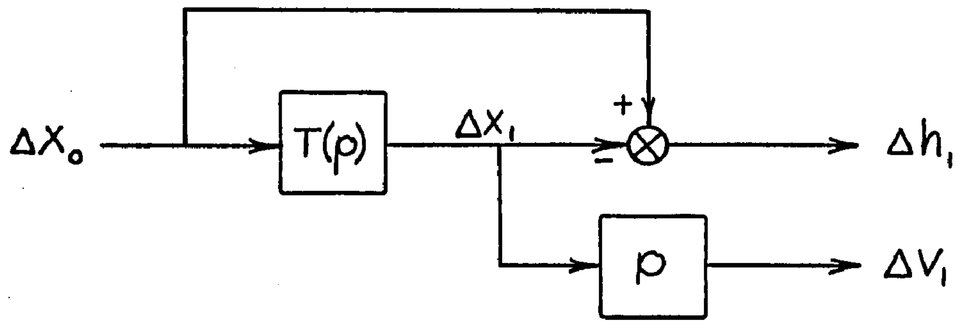


Fig. 12 --Block Diagram of Simple Headway-Velocity Optimization Problem

Here it is desired to minimize the integral $\sigma_{\Delta h_1}^2$ , subject to the limit (constraint) that $\sigma_{\Delta v_1}^2 = c_v$ . This is a calculus of variations problem with fixed end conditions and an integral constraint. This problem is solved by forming the integral,

$$(5\text{-}6) \qquad I = \sigma_{\Delta h_1}^2 + \lambda_v^2 \sigma_{\Delta V_1}^2 \; ,$$

where $\lambda_v^2$ is a Lagrange multiplier. The integral is then minimized by variation of $T(j\omega)$ , which minimizes the integral is determined as a function of $\omega$ and $\lambda_v$ , $T_0(j\omega, \lambda_v)$ . The optimum gain function is substituted back into the constraint, $\sigma_{\Delta v_1}^2 = c_v$ . Evaluation of the integral,

$$(5\text{-}7) \qquad \sigma_{\Delta V_1}^2 = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \left| T_0(j\omega, \lambda_v) \right|^2 \Phi_{\Delta V_0}(\omega)\, d\omega \; ,$$

yields an equation relating $\lambda_v$ to the parameters of $\Phi_{\Delta v_0}(\omega)$ and to $c_v$ . Finally, $T_0(j\omega)$ may be substituted back into the integral $\sigma_{\Delta h_1}^2$ , and the integral may be evaluated to determine the minimum mean square headway for the given spectrum $\Phi_{\Delta x_0}(\omega)$ and the given constraint.

There are two alternatives to the minimization of I by variation of the system gain function $T(j\omega)$ . One is to consider the integral

(5-8) $$I = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\left[\left|1-T(j\omega)\right|^2 + \lambda_v^2\,\omega^2\left|T(j\omega)\right|^2\right]\Phi_{\Delta x_0}(\omega)d\omega.$$

found by substituting the integrals for $\sigma_{\Delta h_1}^2$ and $\sigma_{\Delta v_1}^2$ into Eq. (5-6). Here the variation may be performed by letting $T(j\omega) = T_0(j\omega) + \epsilon\,\delta T(j\omega)$ , where $T_0(j\omega)$ is the optimum and $\epsilon\,\delta T(j\omega)$ is the change in the function from optimum. Substituting the modified expression for $T(j\omega)$ into Eq. (5-8), one obtains the integral as a function of $\epsilon$ , $I(\epsilon)$ . Then setting $\left.\frac{dI(\epsilon)}{d\epsilon}\right|_{\epsilon=0} = 0$ and examining the resulting integrands with the objective of meeting this requirement, one also obtains the necessary requirements on the poles and zeros of $T_0(j\omega)$ and thus the function itself.

The second alternative is to minimize I by variation of the impulse response of the system characterized by $T(p)$ . Denote this impulse response by $\eta(t)$ and the impulse response of the optimum system $(T_0(p))$ by $\eta_0(t)$ . From Fig. 12 it may be seen that

(5-9) $$\Delta h_1(t) = \Delta x_0(t) - \int_0^{\infty}\eta(\nu)\Delta x_0(t-\nu)d\nu.$$

The integral is the solution $\Delta x_1(t)$ of the differential equation represented by $\Delta x_1 = T(p)\, \Delta x_0$. Since $\Delta v_1 = \frac{d}{dt} x_1$,

$$\Delta V_1(t) = \int_0^\infty \eta(\nu) \frac{d}{dt} \Delta X_o(t-\nu)\, d\nu . \tag{5-10}$$

The mean square headway $\sigma^2_{\Delta h_1}$ is then given by

$$\sigma^2_{\Delta h_1} = \overline{\left\{ \Delta X_o(t) - \int_0^\infty \eta(\nu) \Delta X_o(t-\nu)\, d\nu \right\}^2} , \tag{5-11}$$

and $\sigma^2_{\Delta v_1}$ by

$$\sigma^2_{\Delta v_1} = \overline{\left\{ \int_0^\infty \eta(\nu) \frac{d}{dt} \Delta X_o(t-\nu)\, d\nu \right\}^2} . \tag{5-12}$$

The variation of $\eta(t)$ is given by

$$\eta(t) = \eta_o(t) + \epsilon \delta \eta(t) . \tag{5-13}$$

Substituting Eqs. (5-11), (5-12), and (5-13) into Eq. (5-6) yields $I(\epsilon)$,

$$\begin{aligned} I(\epsilon) = {} & \overline{\left\{ \Delta X_o(t) - \int_0^\infty \left[ \eta_o(\nu) + \epsilon \delta \eta(\nu) \right] \Delta X_o(t-\nu)\, d\nu \right\}^2} \\ & + \lambda_v^2 \overline{\left\{ \int_0^\infty \left[ \eta_o(\nu) + \epsilon \delta \eta(\nu) \right] \frac{d}{dt} \Delta X_o(t-\nu)\, d\nu \right\}^2} . \end{aligned} \tag{5-14}$$

Setting $\left. \frac{dI(\epsilon)}{d\epsilon} \right|_{\epsilon=0} = 0$ and reversing the order of averaging and integration on $\nu$, yields

$$\begin{aligned} \int_0^\infty \delta\eta(\alpha) \Bigg\{ & \overline{\Delta X_o(t) \Delta X_o(t-\alpha)} - \int_0^\infty \eta_o(\nu) \overline{\Delta X_o(t-\nu) \Delta X_o(t-\alpha)}\, d\nu \\ & - \lambda_v^2 \int_0^\infty \eta_o(\nu) \overline{\frac{d}{dt} \Delta X_o(t-\nu) \frac{d}{dt} \Delta X_o(t-\alpha)}\, d\nu \Bigg\} d\alpha = 0. \end{aligned} \tag{5-15}$$

Since $\delta\eta(\alpha)$ is an arbitrary function, its coefficient must be zero for $\alpha \geqq 0$, i.e.,

$$\overline{\Delta X_o(t)\,\Delta X_o(t-\alpha)} - \int_0^\infty 7_o(\nu)\,\overline{\Delta X_o(t-\nu)\Delta X_o(t-\alpha)}\,d\nu$$

(5-16)

$$-\lambda_v^2 \int_0^\infty 7_o(\nu)\,\overline{\frac{d}{dt}\Delta X_o(t-\nu)\frac{d}{dt}\Delta X_o(t-\alpha)}\,d\nu \quad = \quad 0$$

on the interval $\alpha \geqq 0$. Noting that the averages are autocorrelations and denoting $\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} x(t_1)x(t_2)\,dt$ by $\phi_x(t_1-t_2)$, Eq. (5-16) becomes

$$\phi_{\Delta X_o}(\alpha) = \int_0^\infty 7_o(\nu)\left[\phi_{\Delta X_o}(\alpha-\nu) + \lambda_v^2\,\phi_{\Delta V_o}(\alpha-\nu)\right]d\nu, \quad \alpha \geqq 0. \tag{5-17}$$

This equation is recognized as the well-known Wiener-Hopf equation, which is well covered in the literature.[20, 21] The solution of this equation will be outlined but not proved here.* At this point the definitions and nomenclature consistent with the literature are adopted. Using the definitions,

$$\phi_{ii}(\alpha-\nu) = \phi_{\Delta X_o}(\alpha-\nu) + \lambda_v^2\,\phi_{\Delta V_o}(\alpha-\nu) \tag{5-18}$$

and

$$\phi_{id}(\alpha) = \phi_{\Delta X_o}(\alpha), \tag{5-19}$$

Eq. (5-17) is reduced to

$$\phi_{id}(\alpha) = \int_0^\infty 7_o(\nu)\,\phi_{ii}(\alpha-\nu)\,d\nu, \quad \alpha \geqq 0. \tag{5-20}$$

This equation is customarily solved by spectral factorization of the two-sided Laplace transforms of the autocorrelation functions $\phi_{id}$ and $\phi_{ii}$, given by

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*For details of the solution the reader is referred to any of several good servomechanisms texts. The nomenclature defined in this section is aligned with that of discussions in "Automatic Feedback Control System Synthesis," by J. G. Truxal.

$$\Phi_{id}(s) = \int_{-\infty}^{+\infty} \phi_{id}(\alpha)\, e^{-\alpha s} d\alpha \tag{5-21}$$

and

$$\Phi_{ii}(s) = \int_{-\infty}^{+\infty} \phi_{ii}(\alpha)\, e^{-\alpha s} d\alpha \,. \tag{5-22}$$

Here $s$ is the complex frequency variable, $s = \sigma + j\omega$. For the autocorrelations considered here, the spectra exist for $\sigma = 0$. Consequently the power density spectra, $\Phi_{id}(\omega)$ and $\Phi_{ii}(\omega)$, also exist. They are found from $\Phi_{id}(s)$ and $\Phi_{ii}(s)$ by letting $s = j\omega$. The optimum system gain function found will be $T_0(s, \lambda_v)$, and

$$T_o(j\omega, \lambda_v) = T_o(s, \lambda_v)\Big|_{s=j\omega} \,. \tag{5-23}$$

The solution of the Wiener-Hopf equation is given by

$$T_o(s, \lambda_v) = \frac{\left[\Phi_{id}(s)/\Phi_{ii}^-(s)\right]_+}{\Phi_{ii}^+(s)} \,. \tag{5-24}$$

Here $\Phi_{ii}(s)$ is a rational function of $s$ which can be factored into two rational functions $\Phi_{ii}^-(s)$ and $\Phi_{ii}^+(s)$. The function $\Phi_{ii}^-(s)$ has poles and zeros only in the right half $(\sigma > 0)$ of the complex $s$ plane (abbreviated RHP hereafter), and $\Phi_{ii}^+(s)$ has poles and zeros only in the left half $(\sigma < 0)$ plane (LHP). The function $\left[\Phi_{id}(s)/\Phi_{ii}^-(s)\right]_+$ is the partial fraction expansion of $\Phi_{id}(s)/\Phi_{ii}^-(s)$ with the terms for the RHP poles eliminated. As mentioned earlier, the solution is completed by substituting $T_0(j\omega, \lambda_v)$ back into the constraint and solving for $\lambda_v$ .

The optimum system gain function $T_0(j\omega)$ will now be determined for the simple problem being considered. Taking the Fourier transform of Eq. (5-19), substituting in Eq. (5-5), and letting $s = j\omega$ , yields

$$\Phi_{id}(s) = \Phi_{\Delta x_0}(s) = \frac{N^2}{a^2 - s^2}$$

Taking the Fourier transform of Eq. (5-18), substituting in Eqs. (5-4) and (5-5), and letting $s = j\omega$ yields

$$\Phi_{ii}(s) = \left(1 - \lambda_v^2 s^2\right)\Phi_{\Delta x_0}(s) = \frac{(1-\lambda_v^2 s^2)N^2}{a^2 - s^2} .$$

The RHP and LHP factors of $\Phi_{ii}(s)$ are respectively

$$\Phi_{ii}^-(s) = \frac{(1-\lambda_v s)N}{a - s}$$

and

$$\Phi_{ii}^+(s) = \frac{(1+\lambda_v s)N}{a + s} .$$

Then

$$\frac{\Phi_{id}(s)}{\Phi_{ii}^-(s)} = \frac{N}{(a+s)(1-\lambda_v s)} .$$

The partial fraction expansion of $\Phi_{id}(s)/\Phi_{ii}^-(s)$ on the LHP poles only is given by

$$\left[\Phi_{id}(s)/\Phi_{ii}^-(s)\right]_+ = \frac{N}{(1+a\lambda_v)(s+a)} .$$

Finally, $T_0(s, \lambda_v)$ is given by

$$T_o(s,\lambda_v) = \frac{\left[\Phi_{id}(s)/\Phi_{ii}^-(s)\right]_+}{\Phi_{ii}^+(s)} = \frac{1}{(1+a\lambda_v)(1+\lambda_v s)}$$

The constraint equation,

$$\sigma_{\Delta V_1}^2 = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\left|T_o(j\omega,\lambda_v)\right|^2 \Phi_{\Delta V_o}(\omega)\,d\omega = c_v$$

upon substitution of $s$ for $j\omega$ becomes

$$\frac{1}{2\pi j}\int_{-j\infty}^{+j\infty} T_o(s,\lambda_v)\,T_o(-s,\lambda_v)\,\Phi_{\Delta V_o}(s)\,ds = c_v$$

or

$$\frac{1}{2\pi j}\int_{-j\infty}^{+j\infty}\frac{1}{(1+a\lambda_v)^2(1-\lambda_v s)(1+\lambda_v s)}\cdot\frac{-N^2 s^2}{(a-s)(a+s)}\,ds = c_v .$$

The integral is readily integrated by residue theory, yielding

$$(1+a\lambda_v)^2 c_v = \frac{N^2}{2}\frac{(a-1)}{(\lambda_v^2 a^2-1)} ,$$

from which one obtains the equation,

$$\lambda_v^4 + \frac{2}{a}\lambda_v^3 - \frac{2}{a^3}\lambda_v - \frac{1}{a^4} + \frac{N^2}{2c_v}\left(\frac{1}{a^4}-\frac{1}{a^3}\right) = 0.$$

When this equation is solved for $\lambda_v$ and the solution is substituted into $T_0(s, \lambda_v)$, the optimum design of the linear controller is accomplished.

In the above example it is observed that a particular difficulty exists with the optimum system function $T_0(s, \lambda_v)$. Note that

$$T_o(0, \lambda_v) = \frac{1}{(1 + a\lambda_v)} \neq 1 .$$

This system function will not meet the equilibrium requirement that $T(0) = 1$. This is due to the nature of the assumed $\Phi_{\Delta v_0}(\omega)$ in the vicinity of $\omega = 0$. An artifice may be introduced which will cause the optimization procedure to yield an optimum system gain function such that $T_0(0, \lambda_v) = 1$. The technique is to add a small constant to the spectrum of $\Phi_{\Delta v_0}(\omega)$, such that $\Phi_{\Delta v_0}(\omega)$ becomes

(5-25) $$\Phi_{\Delta v_o}(\omega) = \frac{\omega^2 N^2}{\omega^2 + a^2} + \Delta^2 ,$$

and $\Phi_{\Delta x_0}(\omega)$ becomes

(5-26) $$\Phi_{\Delta x_o}(\omega) = \frac{N^2}{\omega^2 + a^2} + \frac{\Delta^2}{\omega^2} = \frac{(N^2+\Delta^2)\omega^2 + a^2\Delta^2}{\omega^2(\omega^2+a^2)} .$$

The optimization method can not be applied to spectra of this kind. The poles at $\omega = 0$ are first replaced by poles at $\omega = \pm j\epsilon$, such that

(5-27) $$\Phi_{\Delta x_o}(\omega) = \frac{(N^2+\Delta^2)\omega^2 + a^2\Delta^2}{(\omega^2 + \epsilon^2)(\omega^2 + a^2)} ,$$

and $\epsilon$ is allowed to approach zero. The optimization is then performed, and finally $T_0(s, \lambda_v)$ is given by

(5-28) $$T_o(s, \lambda_v) = \lim_{\epsilon \to 0} T_o(s, \epsilon, \lambda_v) .$$

The optimum system function is also dependent on $\Delta$, which may be made arbitrarily small. The optimum function $T_0(s, \lambda_v)$ or $T_0(j\omega, \lambda_v)$, will first be found and then the effect of the change in spectrum of $\Phi_{\Delta v_0}(\omega)$ will be discussed.

The spectrum given in Eq. (5-27) ay be factored just as $\Phi_{ii}(s)$ is, such that

$$\Phi_{\Delta x_0}(s) = \Phi^-_{\Delta x_0}(s)\Phi^+_{\Delta x_0}(s) = \frac{\left[a\Delta - (N^2+\Delta^2)^{\frac{1}{2}} s\right]}{(\epsilon - s)(a-s)} \cdot \frac{\left[a\Delta + (N^2+\Delta^2)^{\frac{1}{2}} s\right]}{(\epsilon + s)(a+s)} .$$

Since $\Phi_{id}(s) = \Phi_{\Delta x_0}(s)$ and $\Phi^-_{ii}(s) = (1-\lambda_v s)\,\Phi^-_{\Delta x_0}(s)$,

$$\frac{\Phi_{id}(s)}{\Phi^-_{ii}(s)} = \frac{\Phi_{\Delta x_0}(s)}{(1-\lambda_v s)} = \frac{\left[a\Delta + (N^2+\Delta^2)^{\frac{1}{2}} s\right]}{(\epsilon+s)(a+s)(1-\lambda_v s)} .$$

$$\left[\frac{\Phi_{id}(s)}{\Phi^-_{ii}(s)}\right]_+ = \frac{(1+a\lambda_v)\left[a\Delta - (N^2+\Delta^2)^{\frac{1}{2}}\epsilon\right](a+s) - (1+\epsilon\lambda_v)\left[a\Delta - (N^2+\Delta^2)^{\frac{1}{2}} a\right](\epsilon+s)}{(a-\epsilon)(1+\epsilon\lambda_v)(1+a\lambda_v)(\epsilon+s)(a+s)}$$

$$T_0(s,\epsilon,\lambda_v) = \frac{\left[\Phi_{id}/\Phi^-_{ii}\right]_+}{\Phi^+_{ii}} = \frac{(1+a\lambda_v)\left[a\Delta - (N^2+\Delta^2)^{\frac{1}{2}}\epsilon\right](a+s) - (1+\epsilon\lambda_v)\left[a\Delta - (N^2+\Delta^2)^{\frac{1}{2}} a\right](\epsilon+s)}{(a-\epsilon)(1+\epsilon\lambda_v)(1+a\lambda_v)(1+\lambda_v s)\left[a\Delta + (N^2+\Delta^2)^{\frac{1}{2}} s\right]}$$

Letting $\epsilon = 0$ and simplifying yields

$$T_0(s,\lambda_v) = \frac{\dfrac{\left[\dfrac{(N^2+\Delta^2)^{\frac{1}{2}}}{a\Delta} + \lambda_v\right]}{(1+a\lambda_v)} s + 1}{\left[\dfrac{(N^2+\Delta^2)^{\frac{1}{2}}}{a\Delta} s + 1\right](\lambda_v s + 1)} . \tag{5-29}$$

As $\Delta$ becomes very small,

$$T_0(s,\lambda_v) \approx \frac{\dfrac{N}{a\Delta(1+a\lambda_v)} s + 1}{\left(\dfrac{N}{a\Delta} s + 1\right)(\lambda_v s + 1)} . \tag{5-30}$$

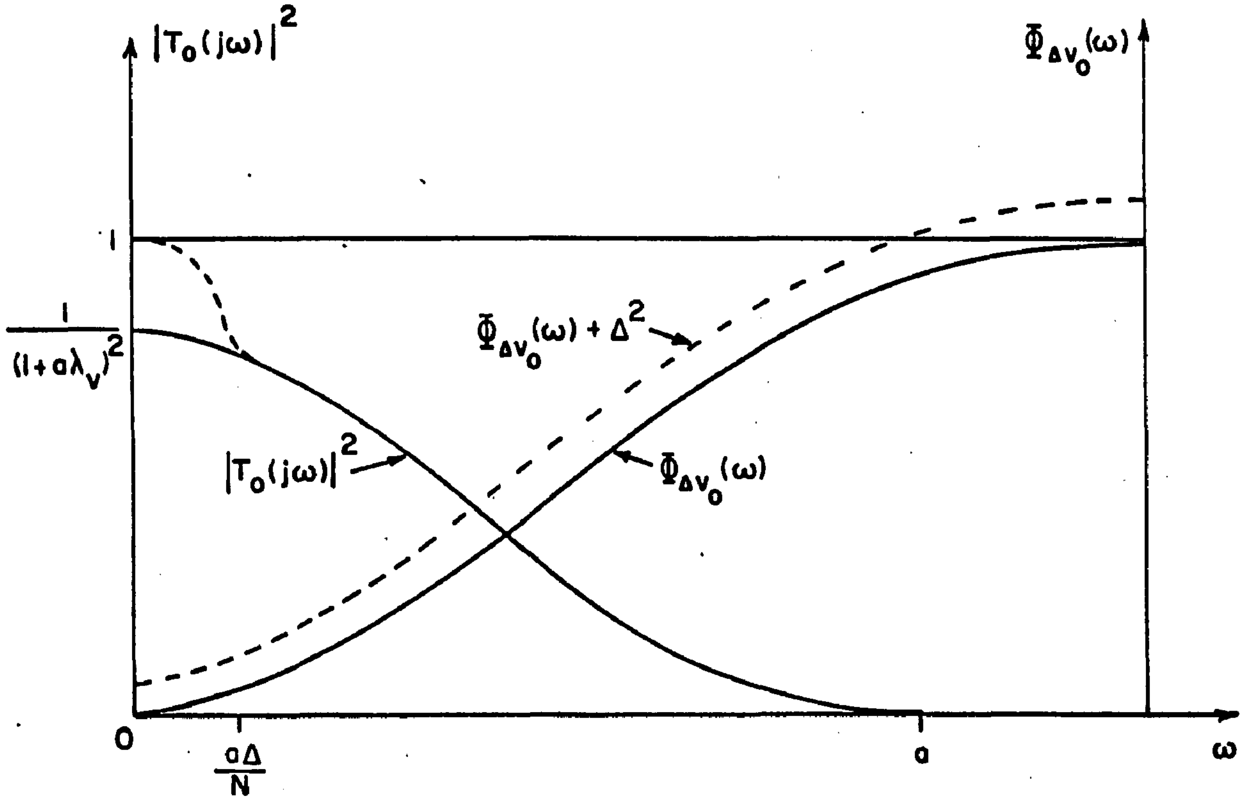


Fig. 13 --Modified Velocity Disturbance Spectrum and Weighting Function

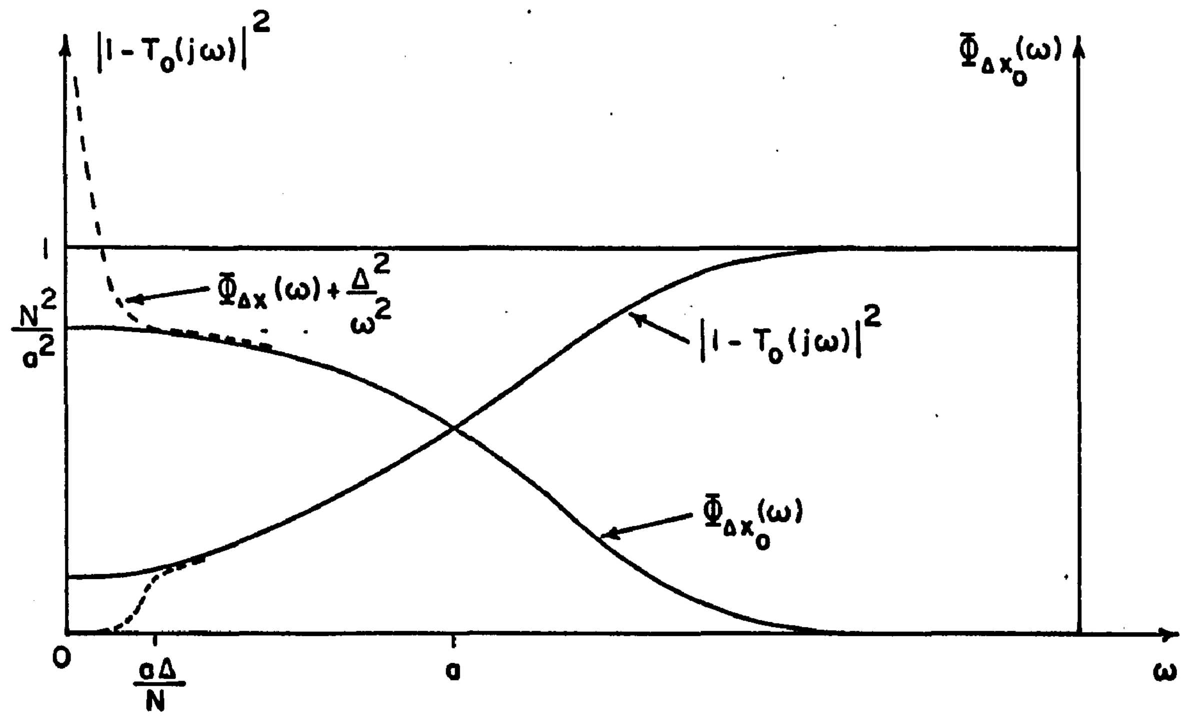


Fig. 14 --Modified Position Disturbance Spectrum and Weighting Function

From Eq. (5-29) it can be seen that $T(0) = 1$ . Furthermore it may be seen from Eq. (5-30) that since $a\lambda_v > 0$ , the small $\Delta$ has effectively caused a lag type filter in tandem with the filter $\frac{1}{\lambda_v s + 1}$. The ratio of the time constants of the lag type filter is independent of N and $\Delta$ and each is inversely proportional to $\Delta$ . The high frequency gain of the lag type filter is $\frac{1}{1 + a\lambda_v}$ the same gain as $T_0(0, \lambda_v)$ found before adding $\Delta^2$ to $\Phi_{\Delta v_0}(\omega)$ . The effect of $\Delta^2$ on the problem may be seen by considering Figs. 13 and 14 The dashed curves show the modified spectra and the modified $|T_0(j\omega)|^2$ and $|1 - T_0(j\omega)|^2$ due to the $\Delta^2$ term. It may be shown that for $\Delta << \frac{N}{a\lambda_v}$ and $T_0(s, \lambda_v)$ given by Eq. (5-29)

$$\text{(5-31)} \qquad \lim_{\omega \to 0} \frac{\Delta^2}{\omega^2} \left| 1 - T_0(j\omega, \lambda_v) \right|^2 = \frac{N^2 \lambda_v^2}{(1 + a\lambda_v)^2} .$$

Then the effect of the pole is to add an increment to the integral $\sigma_{\Delta h_1}^2$ found by using $T_0(j\omega, \lambda_v) = \frac{1}{(1 + a\lambda_v)(\lambda_v s + 1)}$ so that an estimate for the mean square headway using Eq. (5-29) is

$$\text{(5-32)} \qquad \sigma_{\Delta h_1}^2 = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{d\omega}{(1 + a\lambda_v)^2 (\lambda_v^2 \omega^2 + 1)} + \frac{1}{\pi} \left( \frac{a\Delta}{N} \right) \frac{N^2 \lambda_v^2}{(1 + a\lambda_v)^2} .$$

Also it may be seen from Fig. 13 that $\sigma_{\Delta v_1}^2$ may be estimated by

$$\sigma_{\Delta v_1}^2 = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{\omega^2 N^2}{(1+a\lambda_v)^2(\lambda_v^2\omega^2+1)(\omega^2+a^2)}\, d\omega \; +$$

(5-33)

$$\frac{1}{\pi}\left(\frac{a\Delta}{N}\right)\frac{(2a\lambda_v + a^2\lambda_v^2)}{(1+a\lambda_v)^2}\Delta^2 + \frac{\Delta^2}{2\pi}\int_{-\infty}^{+\infty}\frac{d\omega}{(1+a\lambda_v)^2(\lambda_v^2\omega^2+1)}$$

It is seen from Eqs. (5-32) and (5-33) that the modification in response from true optimum may be made arbitrarily small by making $\Delta^2$ very small, and yet the requirement $T(0) = 1$ may be met.

Any optimization technique, like methods of analysis, does not treat all aspects of the real problem. As a consequence it is generally necessary to modify the theoretical or optimum design. Generally, the modifications will not seriously impair system performance.

When more complex input disturbance spectra are considered there are relationships which must exist between the spectra and the assigned constraints if the optimum system $T_0(s)$ is to be realistic. First, however, some consideration should be given to the realism of the assumed spectra. Consider the spectrum

$$\Phi_{\Delta x_0}(\omega) = \Phi_{\Delta x_0}^{+}(\omega)\, \Phi_{\Delta x_0}^{-}(\omega) \quad ,$$

where

(5-34)
$$\Phi_{\Delta x_0}^{+}(\omega) = \frac{A(j\omega+z_1)(j\omega+z_2)\cdots\cdots(j\omega+z_m)}{(j\omega+p_1)(j\omega+p_2)\cdots\cdots(j\omega+p_n)} = \frac{A\,N_m(j\omega)}{D_n(j\omega)}\,.$$

Since

$$\Phi_{\Delta v_0}(\omega) = \omega^2 \Phi_{\Delta x_0}(\omega) \tag{5-35}$$

and

$$\Phi_{\Delta a_0}(\omega) = \omega^4 \Phi_{\Delta x_0}(\omega), \tag{5-36}$$

and since it is known that $\Delta a_0$ is a bounded function and consequently has a finite mean square value

$$\sigma_{\Delta a_0}^2 = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{+T} [\Delta a_0(t)]^2 dt = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \Phi_{\Delta a_0}(\omega) d\omega.$$

it can be seen that $n > m + 2$ . Boundedness is also a property of higher order derivatives, which implies even more poles of $\Phi_{\Delta x_0}(\omega)$ , but usually only the lower frequency poles need be considered so that the constraints assigned do exist. In other words, the statement that $\sigma_{\Delta a_0}^2 = C_a$ implies that $n \geqq m + 3$ . Since the solution of optimization problems becomes extremely complex as $n$ increases, usually $n$ is made equal to $m + 3$ . This value of $n$ is required if attenuation constraints are specified, e.g.,

$$\sigma_{\Delta a_1}^2 \leqq \rho\, \sigma_{\Delta a_0}^2 , \qquad \text{where } \rho < 1 .$$

On the other hand, if $T_0(s)$ can be made to have more poles than zeros and if the constraints are given in terms of $\sigma_{\Delta a_1}^2$ and not $\sigma_{\Delta a_0}^2$ , then the problem may be greatly simplified by neglecting the higher frequency poles of $\Phi_{\Delta a_0}(\omega)$ and making $n = m + 2$ . The existence of $\sigma_{\Delta a_1}^2$ , is then due to $T_0(s)$ . This approximate optimization will be particularly accurate if the higher frequency poles neglected are widely separated from the dominant poles considered.

When additional constraints are placed on higher derivatives, the integral I corresponding to that of Eq. (5-6) becomes

$$I = \sigma_{\Delta h_1}^2 + \lambda_1 \sigma_{\Delta x_1^1}^2 + \lambda_2 \sigma_{\Delta x_1^2}^2 + \cdots + \lambda_q \sigma_{\Delta x_1^q}^2 \tag{5-37}$$

where

$$\Delta x_1^q = \frac{d^q \Delta x_1(t)}{dt^q}$$

In this case, the $q$ $\lambda$'s are found by solving simultaneously the $q$ constraint equations,

$$\sigma_{\Delta x_1^i}^2 = \frac{1}{2\pi}\int_{-\infty}^{+\infty} T_0(j\omega, \lambda_1, \lambda_2, \cdots, \lambda_q)\,\omega^{2i}\,\Phi_{\Delta x_0}(\omega)\,d\omega = C_i;\quad i = 1, 2, \cdots, q. \tag{5-38}$$

For small $q$ the integrals are tabulated,[22] but are nonlinear algebraic functions of the $\lambda$'s and the poles and zeros of $\Phi_{\Delta x_0}(\omega)$ . Also, the complexity of the functions increases very rapidly as $q$ and the number of poles of $\Phi_{\Delta x_0}(\omega)$ increase. Therefore it is very desirable to consider the minimum number of poles of $\Phi_{\Delta x_0}(\omega)$ possible and use the minimum number of constraints necessary.

When the integral I of Eq. (5-37) is varied as in Eqs. (5-13) to (5-16), the resulting Wiener-Hopf equation becomes

$$\phi_{\Delta x_0}(\alpha) = \int_0^{\infty} T_0(\nu)\left[\phi_{\Delta x_0}(\alpha-\nu) + \lambda_1 \phi_{\Delta x_0^1}(\alpha-\nu) + \cdots + \lambda_q \phi_{\Delta x_0^q}(\alpha-\nu)\right] d\nu, \quad \alpha \geq 0. \tag{5-39}$$

Then $\Phi_{ii}(\omega)$ becomes

$$\Phi_{ii}(\omega) = \left(1+\lambda_1\omega^2+\lambda_2\omega^4 + \cdots + \lambda_q\omega^{2q}\right)\Phi_{\Delta X_o}(\omega), \tag{5-40}$$

and

$$\Phi_{ii}(s) = \left(1-\lambda_1 s^2+\lambda_2 s^4 - \cdots + \lambda_q s^{2q}\right)\Phi_{\Delta X_o}(s). \tag{5-41}$$

It is possible to set

$$\left(1-\lambda_1 s^2+\lambda_2 s^4 - \cdots + \lambda_q s^{2q}\right) = \left(1-\frac{s^2}{r_1^2}\right)\left(1-\frac{s^2}{r_2^2}\right)\cdots\left(1-\frac{s^2}{r_q^2}\right), \tag{5-42}$$

and by multiplying out the right member and equating coefficients of like powers of $s$, the relations between the $\lambda$'s and $r$'s are obtained. Thus $\Phi_{ii}(s)$ may be factored as before into LHP and RHP factors, such that

$$\Phi_{ii}(s) = \left(1+\frac{s}{r_1}\right)\left(1+\frac{s}{r_2}\right)\cdots\left(1+\frac{s}{r_q}\right)\Phi^+_{\Delta X_o}(s)\left(1-\frac{s}{r_1}\right)\left(1-\frac{s}{r_2}\right)\cdots\left(1-\frac{s}{r_q}\right)\Phi^-_{\Delta X_o}(s).$$

Let

$$\Phi_{ii}(s) = Q_q^+(s)\Phi^+_{\Delta X_o}(s)Q_q^-(s)\Phi^-_{\Delta X_o}(s). \tag{5-43}$$

Then

$$\frac{\Phi_{id}(s)}{\Phi^-_{ii}(s)} = \frac{\Phi^+_{\Delta X_o}(s)}{Q_q^-(s)} = \frac{A N_m(s)}{D_n(s)Q_q^-(s)}. \tag{5-44}$$

Partial fraction expansion of this function on only the $n$ poles of $D_n(s)$ and recombination yields a function of the form

$$\left[\frac{\Phi_{id}(s)}{\Phi^-_{ii}(s)}\right]_+ = \frac{B M_{n-1}(s)}{D_n(s)}, \tag{5-45}$$

where the numerator polynomial $M_{n-1}(s)$ is of degree $n-1$ in $s$, as indicated by its subscript. Then

$$(5\text{-}46) \qquad T_0(s) = \frac{\left[\Phi_{id}(s)/\Phi_{ii}^{-}(s)\right]_+}{\Phi_{ii}^{+}(s)} = \frac{B\,M_{n-1}(s)}{Q_q^{+}(s)A\,N_m(s)} \,.$$

Here the numerator is of degree $n-1$ and the denominator of degree $m+q$. It is then seen that for $T_0(s)$ to be physically realizable,

$$(5\text{-}47) \qquad q + m \geqq n \,.$$

For instance of a $\Phi^{+}_{\Delta x_0}(\omega)$ is specified with two poles and no zeros, then a constraint must be assigned to $\sigma_{\Delta a_1}^2$, or the resulting $T_0(s)$ will not be realizable. Note that if a constraint is assigned on $\Delta a_1$, but not on $\Delta v_1$, such that $\lambda_2 = \lambda_a \neq 0$ but $\lambda_1 = \lambda_v = 0$, then $Q_q^{+}(s)$ will have complex poles. This may cause $|T_0(j\omega)|$ to be greater than unity for some $\omega$. A system characterized by such a $T_0(j\omega)$ will not meet the asymptotic stability requirements. The situation here will present the system designer with two alternatives. The first is that if limits are determined on $\sigma_{\Delta v_1}^2$ and $\sigma_{\Delta a_1}^2$, then the constants $c_v$ and $c_a$ of the constraints, $\sigma_{\Delta v_1}^2 = c_v$ and $\sigma_{\Delta a_1}^2 = c_a$ may be adjusted. The other is to accept $T_0(j\omega)$ and approximate it with a tractical system for which $|T(j\omega)| \leq 1$ and to accept the corresponding increase in $\sigma_{\Delta h_1}^2$.

It is mentioned in closing that there are other reasonable optimization problems in which it is desired to minimize $\sigma_{\Delta h_1}^2 + \mu\,\sigma_{\Delta v_r}^2$ or to put constraints on $\sigma_{\Delta v_r}^2$. The relative velocity $\Delta v_r$ is given by $\Delta v_r = \Delta v_0 - \Delta v_1$ in this case. In these problems the integral I takes the form

$$(5\text{-}48)\qquad I = \sigma_{\Delta h_1}^2 + \mu\,\sigma_{\Delta v_r}^2 + \lambda_1 \sigma_{\Delta v_1}^2 + \lambda_2 \sigma_{\Delta a_1}^2 + \cdots \quad .$$

A similar form of Wiener-Hopf equation is found. Now

$$(5\text{-}49)\qquad \Phi_{id}(\omega) = \left(1+\mu\omega^2\right)\Phi_{\Delta x_0}(\omega)\,.$$

$$(5\text{-}50)\qquad \Phi_{ii}(\omega) = \left[1 + (\mu+\lambda_1)\omega^2 + \lambda_2\omega^4 + \cdots\right]\Phi_{\Delta x_0}(\omega)$$

From Eq. (5-49) it is seen that effectively $\Phi_{\Delta x_0}(\omega)$ has an increase of two zeros, so that m is increased by one. This in turn, by Eq. (5-47), reduces the number of constraints by one, which are needed to insure a physically realizable $T_0(s)$ .

If the problem is of the first kind, minimization of $\sigma_{\Delta h_1}^2$ plus a weighted $\sigma_{\Delta v_r}^2$ , then $\mu$ is a known constant and not a constraint. The constraint on $\Delta v_1$ , $\sigma_{\Delta v_1}^2 = c_v$ , may be removed and the number of nonlinear equations and $\lambda$'s reduced by one.

### Optimization of Queue Response to Disturbances of Each Vehicle

Road induced disturbances and internal noise sources in the controllers of each vehicle can be referred to the force input to the automobile. Consider the general feedback configuration for the linear mode of the automatic longitudinal control system of each vehicle, shown in Fig. 15.

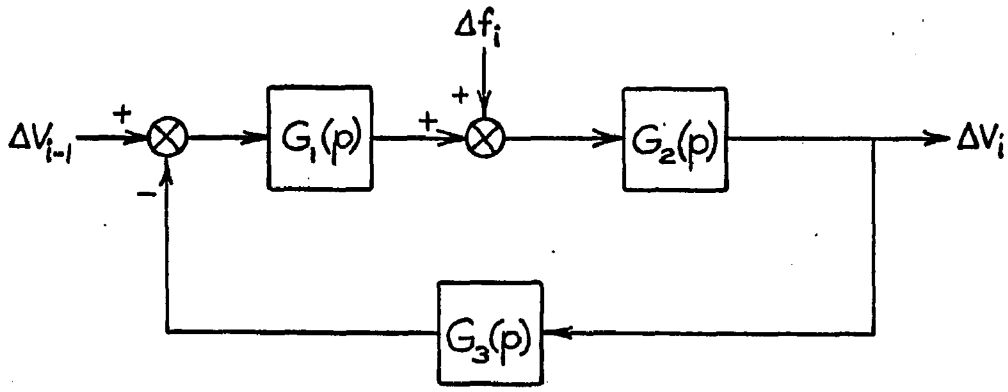


Fig. 15 --General Feedback Controller

It can be seen from Fig. 15 that the gain function relating $\Delta v_1$ to $\Delta v_{i-1}$ is

$$T(j\omega) = \frac{G_1(j\omega)G_2(j\omega)}{1 + G_1(j\omega)G_2(j\omega)G_3(j\omega)} \tag{5-51}$$

The gain function relating $\Delta v_i$ to the disturbance forces $\Delta f_i$ is

$$\frac{G_2(j\omega)}{1 + G_1(j\omega)G_2(j\omega)G_3(j\omega)} \tag{5-52}$$

Now it is assumed that $T(j\omega)$ is a physically realizable gain function with the property that $T(0) = 1$. The open loop gain of the feedback controller must have an integration so that $T(0) = 1$. The gain $G_2(j\omega)$, which relates the velocity of the vehicle to the applied force cannot have the pole at $\omega = 0$ because of the friction of the vehicle. This gain in simplified form is $\frac{1}{\tau_a j\omega + 1}$. Also, $G_3(j\omega)$ cannot have the pole, as this would make $T(0) = 0$. Therefore $G_1(j\omega)$ possesses the pole. It may be seen from the gain function Eq. (5-52) that at frequencies up to those for which $|G_2(j\omega)| \to 0$, if the gain $G_1(j\omega)\, G_3(j\omega) >> 1$, then the magnitude of the gain function (5-52) will be very small. This will greatly reduce the system response, and consequently the queue response, to road-induced disturbances and uncorrelated disturbances from other sources.

It can be observed from Eq. (5-51) that in order for $T(0) = 1$, it will be necessary for

(5-53) $$G_3(0) = 1 ,$$

since $G_1(0) \longrightarrow \infty$ and $\lim_{\omega \to 0} T(j\omega) = \dfrac{1}{G_3(0)}$. Furthermore, for frequencies for which

$$G_1(j\omega)\, G_2(j\omega)\, G_3(j\omega) >> 1 ,$$

(5-54) $$T(j\omega) \approx \frac{1}{G_3(j\omega)} .$$

Then the optimum system gain function $T_0(j\omega)$ may be achieved by setting

(5-55) $$G_3(j\omega) = \frac{1}{T_0(j\omega)}$$

for low frequencies. The bandwidth of $G_1(j\omega)\, G_2(j\omega)$ will provide the higher frequency cutoff of $T_0(j\omega)$.

For instance, consider that $T_0(j\omega)$ is given by

(5-56) $$T_0(j\omega) = \frac{\dfrac{N}{a\Delta(1 + a\lambda_v)} j\omega + 1}{\left(\dfrac{N}{a\Delta} j\omega + 1\right)\left(\lambda_v j\omega + 1\right)}$$

as found from the optimization of the two vehicle response to disturbances of the lead vehicle. Then an asymptotic Bode plot for the feedback controller is shown in Figure 16. In this plot it is assumed that

$G_1(j\omega) = \dfrac{K(\tau_a j\omega + 1)}{j\omega}$ for the plot of log $\left| G_1(j\omega)\, G_2(j\omega) \right|$.

The value of $K$ to obtain $T_0(j\omega)$ for the closed loop function is found

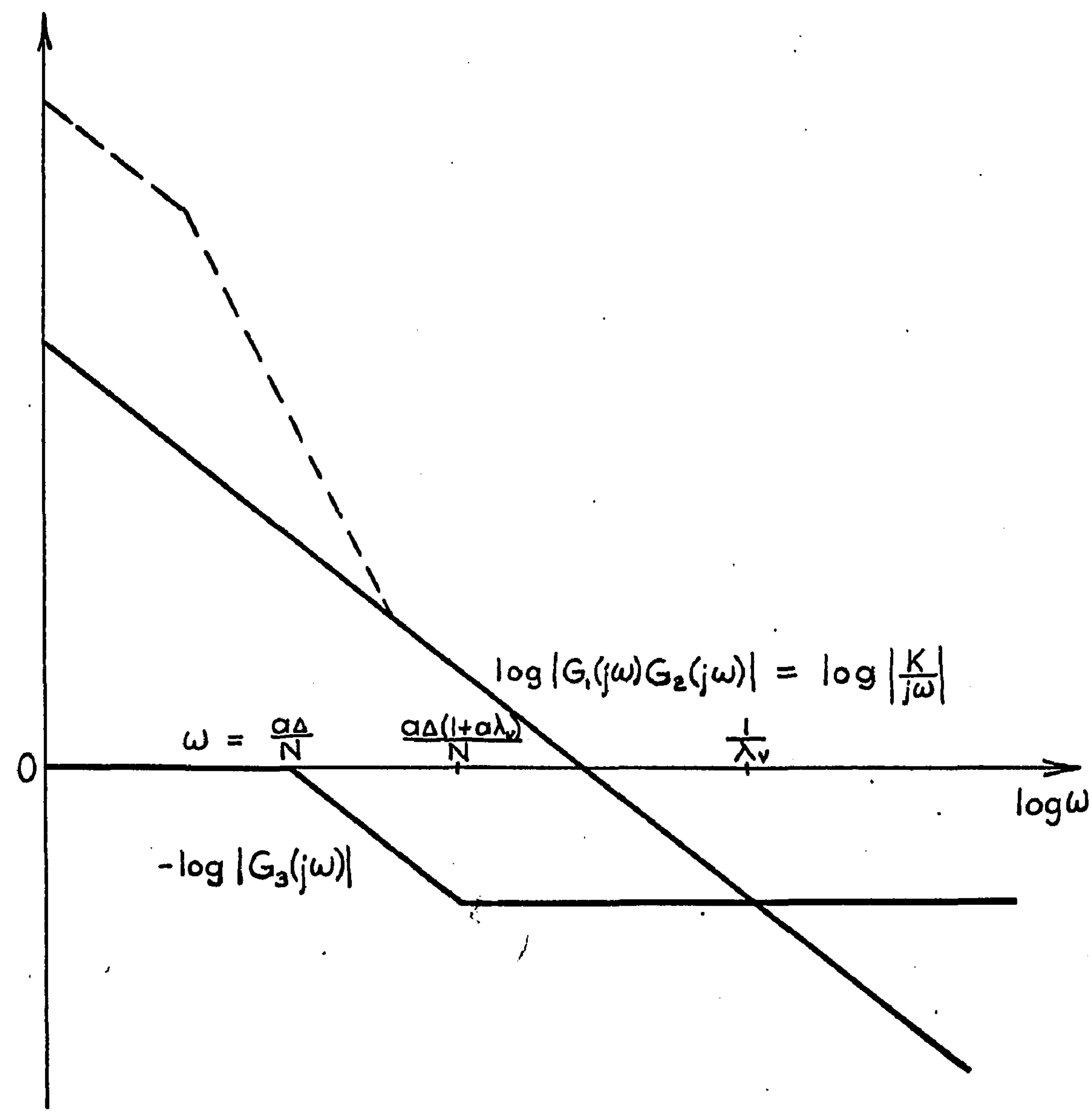


Fig. 16 --Asymptotic Gain Magnitude Plot

as follows. Noting that $\log \left| K/j\omega \right| = \log K - \log \omega$ , the cross-over point (where the loop gain is 1) is given by

$$\log K - \log \frac{1}{\lambda_v} = -\log(1 + a\lambda_v). \tag{5-57}$$

Thus

$$K = \frac{1}{\lambda_v(1 + a\lambda_v)} \tag{5-58}$$

For frequencies $\omega < \frac{1}{\lambda_v}$ , $\log \left| T_0(j\omega) \right| \approx -\log \left| G_3(j\omega) \right|$ .

For frequencies $\omega > \frac{1}{\lambda_v}$ , $\log \left| T_0(j\omega) \right| \approx \log \left| \frac{1}{\lambda_v(1 + a\lambda_v)\, j\omega} \right|$ .

The optimum gain function $T_0(j\omega)$ is thus achieved. Furthermore, it is noted that $K$ can be increased if a lag lead network is added to $G_1(j\omega)$ as indicated by the dashed line modification of the $\left| G_1(j\omega)\, G_2(j\omega) \right|$ plot. This will further reduce the system response to lower frequency components of force disturbances. This would be especially useful for reducing road-induced disturbances because of their very low frequency spectra.

CHAPTER VI

# CONCLUSIONS

In this paper several aspects of the response of queues of automatically controlled vehicles to various types of disturbances have been studied. Although the sinusoidal disturbance is quite unrealistic in the realm of traffic dynamics, it serves to define the various gain functions associated with the automatic queue.

Gain functions have been developed which relate headway, relative velocity, velocity and acceleration of each vehicle of the queue to the initial vehicle velocity and to velocity disturbance induced in each vehicle by the road. Based on these gain functions and the spectra associated with the random velocity disturbances, the mean square values of the resulting disturbances in the headway, relative velocity, velocity, and acceleration of each vehicle have been determined.

The traffic engineering problem is that of minimizing the equilibrium spacing, $h_s$, yet keeping it large enough to avoid collisions where the maximum headway disturbance occurs in the queue. For disturbances of the initial vehicle only, the maximum headway disturbance will occur between the initial vehicle and the first following vehicle. On the other hand, when disturbances are induced in each vehicle, the resulting maximum disturbance occurs well back in the queue. In very long queues the disturbance can become very large unless the gain functions relating the nth vehicle disturbance to the disturbance induced in each vehicle have zeros at $\omega = 0$ or the spectra of the disturbance sources themselves contain the zeros. Investigation showed that for all anticipated sources of disturbances in a practical controller, either the gain relating velocity to the source or the spectra of the sources do contain the zeros.

The main problem studied is that of determining the particular controller which yields optimum queue response to disturbances. It was shown that minimization of headway and relative velocity disturbances is accompanied by increases in the mean square velocity and acceleration of each vehicle of the queue. Mean square velocity and acceleration are related to power and discomfort, respectively, and must therefore be kept below tolerable limits. These limits are independent of the controller to be used. For disturbance sources in only the initial vehicle of the queue, the largest disturbances occur between the initial vehicle and the first following vehicle. Therefore the optimization is that of minimizing the mean square headway of the first vehicle while keeping its mean square velocity below the tolerable limit. An example of optimization of queue response was the minimization of the integral,

$$I = \sigma_{\Delta h_1}^2 + \lambda_v^2 \sigma_{\Delta v_1}^2 = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \left|1 - T(j\omega)\right|^2 \frac{\Phi_{\Delta v_0}(\omega)}{\omega^2} d\omega + \frac{\lambda_v^2}{2\pi}\int_{-\infty}^{+\infty} \left|T(j\omega)\right|^2 \Phi_{\Delta v_0}(\omega) d\omega,$$

by proper choice of $T(j\omega)$, while keeping $\sigma_{\Delta v_1}^2 = c_v$. Here $\lambda_v^2$ is a Lagrange multiplier and $c_v$ is a constant limit determined independently. Minimization of the integral $I$ by variation of $T(j\omega)$ results in the Wiener-Hopf equation. The solution of this equation, $T_0(j\omega, \lambda_v)$, is the optimum controller. The Lagrange multiplier is determined from the equation,

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty} \left|T_0(j\omega, \lambda_v)\right|^2 \Phi_{\Delta v_0}(\omega) d\omega = c_v .$$

The assumption that

$$\Phi_{\Delta v_0}(\omega) = \frac{\omega^2 N^2}{\omega^2 + a^2} + \Delta^2$$

where $\Delta^2$ is a small constant, results in an optimum $T(j\omega)$ given by

$$T_0(j\omega, \lambda_v) = \frac{\frac{N}{a\Delta(1 + a\lambda_v)} j\omega + 1}{\left(\frac{N}{a\Delta} j\omega + 1\right)\left(\lambda_v j\omega + 1\right)}$$

The application of the optimization technique to similar problems is discussed.

Optimization of the queue response to disturbances induced in each vehicle is accomplished by designing the controller so that the effects of such disturbances are minimized in each vehicle.

The optimization depends on the power density spectrum of the input disturbances. At the present time there is little information available on such spectra. Research is needed to investigate the spectra of the human driver's response and spectra of other sources of disturbance of the initial vehicle. Also, the integral measures of queue response to disturbances induced in each vehicle depend on the spectra of these disturbances. There may be enough information available to determine at least the spectra of road-induced disturbances on a regional basis. This should be investigated.

The optimization further depends on the tolerable limits of individual vehicular motion. Indices of comfort, wear, cost of operation, etc. should be determined in terms of mean square velocity, acceleration, etc., of each vehicle. This is still another important area for future study.

Finally, it is noted that results are obtained for unilateral systems, i.e., systems in which control of the motion of the vehicles depends only on the preceding vehicles. More specifically, the motion of each vehicle is controlled according to the motion of its immediate predecessor only. It may very very valuable in future systems to weight the motion of several vehicles ahead and behind into the control criteria for each vehicle. This may be a very profitable area of future study.

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