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EQUIVALENT NORMS AND THE CHARACTERISTIC OF SUBSPACES IN THE CONJUGATE OF A NORMED LINEAR SPACE.

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and only if for every $y \in A$ and every scalar α with $|\alpha| \leq 1$, it is true that $\alpha y \in A$.

The following facts are easily checked. The closure of a balanced set in any linear topological space is also a balanced set. If A is a balanced subset of X, then

$$A^o = \{x' : x' \in X^*, |x'x| \leq 1 \quad \text{for all } x \in A\} .$$

Similarly, if A' is a balanced subset of X*, then

$$^oA' = \{x : x \in X, |x'x| \leq 1 \quad \text{for all } x' \in A'\} .$$

It is then immediate from these relations, the fact that S_1 and S_1' are balanced and from (0) and (1) that

(8) $$S_1^o = S_1' \text{ and } {}^oS_1' = S_1 .$$

<u>Definition 2</u>. A subset A of a linear space Y is <u>absorbing</u> if and only if for each $y \in Y$ there exists a positive real number r such that $|\beta| > r$ implies $y \in \beta A$ (see equations (3)).

An absorbing set contains the zero vector. Also it contains a nonzero multiple of every vector in the space since $y \in \beta A$ and $\beta \neq 0$ imply $(1/\beta)y \in A$.

<u>Definition 3</u>. Let A be a subset of the linear space Y. The <u>balanced and convex hull of</u> A is the intersection of all balanced and convex sets which contain A.

We note that the intersection of balanced sets is balanced and the intersection of convex sets is convex. Thus, the balanced and convex hull of A is the smallest convex and balanced set containing A.

Definition 4. Let A be a balanced, convex, and absorbing set in a linear space Y. For each $y \in Y$ let

$$A(y) = \{ r \ : \ r \text{ is a positive real and } y \in rA \} .$$

Because A is absorbing, A(y) is nonempty. For each $y \in Y$ define

$$p(y) = \inf_{r \in A(y)} r .$$

p is called the Minkowski functional of A (7, pp. 134-137).

If (α, ∞) and $[\alpha, \infty)$ represent the open and closed semi-infinite intervals to the right of the real number α, then it is clear from the fact that A is convex and absorbing that

$$(p(y), \infty) \subseteq A(y) \subseteq [p(y), \infty) .$$

Also it is easily checked that if A is the unit sphere of a norm $\|\cdot\|_0$ on X, then $\|\cdot\|_0$ is the Minkowski functional of A.

Definition 5. A norm $\|\cdot\|_0$ on X is equivalent to $\|\cdot\|$ on X if and only if $(X, \|\cdot\|_0)$ and $(X, \|\cdot\|)$ are equivalent topological spaces.

Definition 6. A subset A of X is a norm set for $(X, \|\cdot\|)$ if and only if A is the unit sphere of some norm on X equivalent to $\|\cdot\|$.

We would like to state here several theorems involving the above concepts.

Theorem 1. The norm $\|\cdot\|_o$ on X is equivalent to $\|\cdot\|$ on X if and only if there exist positive real numbers r and t such that

$$r\|x\|_o \leq \|x\| \leq t\|x\|_o$$

for all $x \in X$.

Proof. See (7, p. 87).

Theorem 2. Suppose $\|\cdot\|_o$ is a norm on X. For any nonnegative real number r, let

$$T_r = \{x : x \in X, \|x\|_o \leq r\}.$$

Then, if r_1 is a positive real number,

$$r_1\|x\|_o \leq \|x\| \qquad \text{for all } x \in X$$

if and only if

$$S_{r_1} \subseteq T_1.$$

If r_2 is a positive real number, then

$$\|x\| \leq r_2\|x\|_o \qquad \text{for all } x \in X$$

if and only if

$$T_1 \subseteq S_{r_2}.$$

Proof. The necessity of the condition follows easily. For the converse of the first assertion, suppose $S_{r_1} \subseteq T_1$. Then $x \neq 0$

implies $\frac{r_1 x}{\|x\|} \in S_{r_1}$ and hence also in T_1 . Thus we have $r_1 \left\| \frac{x}{\|x\|} \right\|_o \leq 1$. This shows that, for all $x \neq 0$,

$$r_1 \|x\|_o \leq \|x\| \quad .$$

Trivially $x = 0$ also satisfies this inequality. For the converse of the second assertion we apply the first assertion of the theorem and the fact that, if U is a sphere of radius r for some norm in X, then $r_2 U$ is the sphere of radius rr_2 for that norm.

Theorem 3. If $\|\cdot\|_o$ is a norm on X and T_1 is the unit sphere of $(X, \|\cdot\|_o)$, then $\|x\|_o = \|x\|$ for all $x \in X$ if and only if $S_1 = T_1$.

This theorem is simply stating that two norms on the space X are the same if and only if they have the same unit sphere. Its proof follows by using Theorem 2 with $r_1 = r_2 = 1$.

Theorem 4. Let Y be a locally convex linear topological space and let A be a nonempty, closed, convex subset of Y. If $y_o \notin A$, then there exist a continuous linear functional y' on Y and a real number α such that

$$R(y' y_o) > \alpha \geq R(y' y) \qquad \text{for all } y \in A.$$

(The importance of this theorem lies in the fact that the points in Y need not be closed sets. There are several standard theorems

[see (1, No. 1189, p. 71) and (3, p. 417)] in which y_o is replaced in the above inequality by an arbitrary element of a closed, compact, convex set B not intersecting A. Taylor shows (7, p. 151) that the hypothesis that B is closed can be eliminated. Although Theorem 4 is a special case of this result, we include a proof of the theorem since we can give a simpler proof in this special case.)

Proof. We will first suppose that the scalar field of Y is the real field. Since Y is a regular space (7, p. 126), there exist open nonintersecting sets W and W_1 with $y_o \in W$ and $A \subseteq W_1$. Since Y is locally convex, we may assume W is convex. Then there exist - see (1, No. 1189, p. 97) or (7, p. 139) - a nonzero continuous linear functional y' on Y and a real number α such that

$$y'y \geq \alpha \qquad \text{for all } y \in W$$

and

$$y'y \leq \alpha \qquad \text{for all } y \in A.$$

There is a balanced and convex neighborhood U of 0 such that $y_o + U \subseteq W$. Since y' is not the zero functional, there exists some $y_1 \in Y$ such that $y'y_1 \neq 0$. Because it is a neighborhood of zero, U is absorbing. Then there exists some real number $\beta \neq 0$ such that $y_2 = \beta y_1 \in U$. Since U is balanced, we may assume $y'y_2 = \beta\, y'y_1 < 0$. Then $y_o + y_2 \in y_o + U \subseteq W$ implies $y'(y_o + y_2) \geq \alpha$. But then $y'y_o + y'y_2 \geq \alpha$ and since $y'y_2 < 0$, $y'y_o > \alpha$. This completes the proof when the scalar field of Y is the real field.

If the scalar field of Y is the complex field, then we can also regard Y as a locally convex linear topological space Y_R over the real numbers. The proof above is valid for Y_R. But, the continuous linear functionals y_R' on Y_R can be characterized as the real parts of continuous linear functionals on Y in the following way: $y'y = y_R'(y) - i\, y_R'(iy)$ where y' is a functional on Y and $y_R'(y) = R(y'y)$. If y' is linear and continuous, so in y_R' and conversely. Thus there exist a continuous linear functional y' on Y and a real number α such that

$$R(y'y_o) > \alpha \geq R(y'y) \qquad \text{for all } y \in A.$$

<u>Corollary</u>. Let Y be a locally convex linear topological space and let A be a nonempty, closed, convex, and balanced subset of Y. If $y_o \notin A$, then there exist a continuous linear functional y' on Y and a real number α such that

$$|y'y_o| > \alpha \geq |y'y| \qquad \text{for all } y \in A.$$

Proof. Let the scalar field of Y be the complex numbers (the proof for the real case is similar). By Theorem 4, there exist a continuous linear functional y' on Y and a real number α such that

$$R(y'y_o) > \alpha \geq R(y'y) \qquad \text{for all } y \in A.$$

The inequality $|y'y_o| > \alpha$ is immediate since $|y'y_o| \geq R(y'y_o)$. Suppose $y \in A$ and φ is such that $0 \leq \varphi \leq 2\pi$ and $y'y = |y'y|\, e^{i\varphi}$. Because A is balanced, $e^{-i\varphi} y \in A$ and $\alpha \geq R(y'(e^{-i\varphi} y)) = R(e^{-i\varphi}\, y'y) = |y'y|$. Thus

$$|y'y| \leq \alpha \qquad \text{for all } y \in A.$$

The next theorem will concern the weak topology $\mathcal{T}^{\circ}$ on X* generated by a linear subspace M of X. A base at 0' for this topology is the family of all sets of the form

$$V'(A,\epsilon) = \{x' : |x'x| < \epsilon \quad \text{for all } x \in A\}$$

where A is a finite subset of M and ϵ is a positive real number. Since by definition, $x' \in M^+$ implies $x'x = 0$ if $x \in M$, it is true that $M^+ \subseteq V'(A,\epsilon)$ for all $A \subseteq M$ and $\epsilon > 0$. We note that when $M = X$, $\mathcal{T}^{\circ}$ is the weak* topology on X*.

<u>Theorem</u> <u>5</u>. For every $B' \subseteq X^*$ with $\mathcal{T}^{\circ}$ as above,

$$B' + M^+ \subseteq \overline{B'}^{\mathcal{T}^{\circ}} .$$

Proof. Let $x' = b' + m'$, where $b' \in B'$ and $m' \in M^+$. One base at x' is the collection of all sets $x' + V'(A,\epsilon)$ for A a finite subset of M and ϵ a positive real number. But $b' = x' + (-m')$ is in every set $x' + V'(A,\epsilon)$ since $-m' \in M^+ \subseteq V'(A,\epsilon)$ for all $A \subseteq M$ and $\epsilon > 0$. Every neighborhood of x' contains b' and thus $x' \in \overline{B'}^{\mathcal{T}^{\circ}}$.

<u>Corollary</u>. If M is a subspace of X with $M^+ \neq \{0'\}$ and $\mathcal{T}^{\circ}$ is the weak topology on X* generated by M, then the closure in the topology $\mathcal{T}^{\circ}$ of a nonempty subset of X* is unbounded. (A subset C' of X* is bounded if and only if there is a real number $r > 0$ such that $\|x'\| \leq r$ when $x' \in C'$.)

Proof. As the reader can quickly check, M^+ is a linear subspace of X*. If $M^+ \neq \{0'\}$, then there is some $x' \in M^+$ with $x' \neq 0'$. Then the subspace generated by x' is in M^+, i. e., for all scalars α, $\alpha x' \in M^+$. Suppose B' is a nonempty subset of X^*. By Theorem 5,

$$B' + M^+ \subseteq \overline{B'}^{\mathcal{J}^\circ} .$$

If x_o' B', then $x_o' + \alpha x'$ B' for all scalars α.

$$\| x_o' + \alpha x' \| \geq \big| \|x_o'\| - \|\alpha x'\| \big| = \big| \|x_o'\| - |\alpha| \, \|x'\| \big| .$$

Since $\|x'\| \neq 0$ and α is any scalar, the quantity on the right is unbounded as a function of α. Therefore $\overline{B'}^{\mathcal{J}^\circ}$ is also unbounded.

We will later find it convenient to know just what conditions a set must satisfy to be a norm set (Def. 6) in a Banach space $(X, \|\cdot\|)$.

<u>Theorem</u> <u>6</u>. Let $(X, \|\cdot\|)$ be a Banach space. A subset U of X is a norm set for $(X, \|\cdot\|)$ if and only if U is balanced, convex, absorbing, and bounded and closed in $(X, \|\cdot\|)$. If U is a norm set, its Minkowski functional is the norm (equivalent to $\|\cdot\|$) for which U is the unit sphere.

Proof. It is immediate that any unit sphere of a norm equivalent to $\|\cdot\|$ must have the stated properties.

Conversely, suppose U is balanced, convex, and absorbing. Denote the Minkowski functional (Def. 4) of U by p . If $x, x_1 \in X$, and α is any scalar,

$$p(\alpha x) = |\alpha| p(x), \qquad p(x + x_1) \leq p(x) + p(x_1)$$

and

$$p(x) \geq 0 , \qquad p(0) = 0.$$

Since U is bounded, there is a positive number r such that $x \in U$ implies $\|x\| \leq r$. Then $U \subseteq S_r$. If $x \neq 0$, $\frac{(r+1)x}{\|x\|} \notin S_r$ and hence also not a member of U. Then $x \notin \frac{\|x\|}{r+1} U$. By Definition 4 and the remarks which follow it, this implies, for $x \neq 0$,

$$\frac{\|x\|}{r+1} \leq p(x) .$$

Since the last inequality is trivially true for $x = 0$,

(9) $$\|x\| \leq (r+1)p(x) \qquad \text{for all } x \in X.$$

It is clear from (9) that $p(x) = 0$ implies $x = 0$, which completes the proof that p is a norm.

Because U is absorbing,

$$\bigcup_{n=1}^{\infty} nU = X.$$

Since multiplication by a nonzero constant is a homeomorphism of X onto itself and U is closed, for each n the set nU is closed. By the Baire category theorem (at this point we need to know $(X, \|\cdot\|)$ is complete, i. e., a Banach space), some set nU has a nonempty interior, and **hence** $U = (1/n)nU$ must also have a nonempty interior. In particular, there exist an $x_o \in U$ and a real number t such that $t > 0$ and

$$S = \{x : \|x - x_o\| < t\} \subseteq U.$$

Let $x \in X$ with $\|x\| < t$. Then $x_o + x$ and $x_o - x$ are both in S. Also, since U is balanced, $-(x_o - x) = x - x_o$ is in U. Since U is convex,

$$x = (1/2)(x - x_o) + (1/2)(x_o + x)$$

must also be in U. Hence

$$\{x : \|x\| < t\} \subseteq U .$$

Since the closure of $\{x : \|x\| < t\}$ is S_t and U is closed,

(10) $$S_t \subseteq U.$$

Also since U is closed (7, p. 134),

(11) $$U = \{x : p(x) \leq 1\} .$$

From (10), (11), Theorem 2, and the fact p is a norm,

(12) $$t\, p(x) \leq \|x\| \qquad \text{for all } x \in X.$$

From (9) and (12), with Theorem 1, we conclude p and $\|\cdot\|$ are equivalent norms. Then, from (11), U is a norm set.

The following theorem is an immediate consequence of standard theorems (6, p. 194).

Theorem 7. Let X and Y be two normed linear spaces with norms $\|\cdot\|$ and $\|\cdot\|_0$ respectively. Let T be a linear mapping from X into Y and let

$$r = \inf_{x \neq 0} \frac{\|Tx\|_0}{\|x\|} .$$

Then $r \neq 0$ if and only if T has a continuous inverse on its range. If $r \neq 0$ and T^{-1} is the inverse of T on its range, then $\|T^{-1}\| = 1/r$.

Theorem 8. (Riesz's Lemma) Let M be a closed and proper subspace of X. Then for each r such that $0 < r < 1$ there exists an element $x_r \in X$ such that $\|x_r\| = 1$ and $\|x - x_r\| > r$ if $x \in M$.

A proof of this theorem is given in (7, p. 96).

EQUIVALENT NORMS AND THE CHARACTERISTIC OF SUBSPACES IN THE CONJUGATE OF A NORMED LINEAR SPACE

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CHAPTER II

EQUIVALENT NORMS IN THE CONJUGATE OR A NORMED LINEAR SPACE

In this chapter we would like to discuss how closely the usual norm in X^*, as defined by (0), is linked to the fact that $(X^*, \|\cdot\|)$ is the conjugate of $(X, \|\cdot\|)$. In particular, we want to ask this question: If $\|\cdot\|_1$ is a norm equivalent to $\|\cdot\|$ on X^*, then will there exist om X a norm $\|\cdot\|_2$ such that $(X^*, \|\cdot\|_1)$ is the conjugate of $(X, \|\cdot\|_2)$?

The starting point for the solution of this problem is simple. One consequence of the Hahn-Banach theorem tells us that there is only one way to define $\|x\|_2$ if we wish $(X^*, \|\cdot\|_1)$ to be the conjugate of $(X, \|\cdot\|_2)$. According to formula (1), we would have to define

$$\|x\|_2 = \sup_{\|x'\|_1 \leq 1} |x'x| \qquad \text{for all } x \in X.$$

Or, if we set

(13) $$U_1' = \{x' : \|x'\|_1 \leq 1\} ,$$

we have

(14) $$\|x\|_2 = \sup_{x' \in U_1'} |x'x| \qquad \text{for all } x \in X.$$

If this is a norm on X, it is the only norm $\|\cdot\|_0$ which would make $(X^*, \|\cdot\|_1)$ the conjugate of $(X, \|\cdot\|_0)$.

As an example, let Y be the space of all n-tuples of real numbers with the usual norm. Thus, if $y = (\alpha_1, \ldots, \alpha_n)$ is an element of Y, then $\|y\| = \sqrt{\alpha_1^2 + \ldots + \alpha_n^2}$. Every element y' of Y* can be made to correspond with one and only one n-tuple of real numbers $(\beta_1, \ldots, \beta_n)$ where, for $y = (\alpha_1, \ldots, \alpha_n)$,

$$y'y = \sum_{i=1}^{n} \alpha_i \beta_i \ .$$

When this correspondence, which is a linear isometry, is made, the norm of y' is given by the formula $\|y'\| = \sqrt{\beta_1^2 + \ldots + \beta_n^2}$.

It is easily checked that

$$\|y'\|_1 = \max_{1 \leq i \leq n} |\beta_i|$$

defines a norm on Y* which is equivalent to $\|\cdot\|$. (In fact, $\|y'\|_1 \leq \|y'\| \leq \sqrt{n}\ \|y'\|_1$.) A quick calculation shows that, if $\|y\|_2$ is defined as in formula (14), for an element of Y given by $y = (\alpha_1, \ldots, \alpha_n)$

$$\|y\|_2 = |\alpha_1| + \ldots + |\alpha_n| \ .$$

It is well known that $\|\cdot\|_2$ is a norm on Y equivalent to $\|\cdot\|$. Also, it can be easily checked that the conjugate of $(Y, \|\cdot\|_2)$ is Y* with norm $\|\cdot\|_1$.

In the example, what might be expected does in fact occur; $\|\cdot\|_2$ is a norm equivalent to $\|\cdot\|$ and the conjugate of $(Y, \|\cdot\|_2)$ is $(Y^*, \|\cdot\|_1)$. Now we would like to examine this situation in general.

Theorem 9. Let $\|\cdot\|_1$ be a norm on X^* and define U_1' and $\|\cdot\|_2$ by (13) and (14). In order that $\|\cdot\|_2$ be a norm on X it is sufficient that $\|\cdot\|_1$ generate a finer topology on X^* than $\|\cdot\|$ generates. If X is complete, this condition is also necessary. If $\|\cdot\|$ and $\|\cdot\|_1$ are equivalent on X^*, then $\|\cdot\|$ and $\|\cdot\|_2$ are equivalent on X.[1]

Proof. It is clear that $\|x\|_2 \geq 0$, and that $\|\alpha x\|_2 = |\alpha|\,\|x\|_2$ for all scalars α and all $x \in X$. Also, since

$$|x'(x_1 + x_2)| = |x'x_1 + x'x_2| \leq |x'x_1| + |x'x_2| ,$$

it is immediate that

$$\|x_1 + x_2\|_2 \leq \|x_1\|_2 + \|x_2\|_2$$

for all x_1 and x_2 in X. If $\|x\|_2 = 0$, then $x'x = 0$ for all $x' \in U_1'$. Since U_1' is the unit sphere of a norm on X^*, U_1' must be absorbing. Then $x_o' \in X^*$ implies there is some scalar $\beta \neq 0$ such that $\beta x_o' \in U_1'$ and hence $\beta x_o'x = 0$. Therefore $x_o'x = 0$ for all $x_o' \in X^*$. By a corollary of the Hahn-Banach theorem (3, p. 65), this implies $x = 0$. This finishes the verification that $\|\cdot\|_2$ is a norm when $\|x\|_2 < \infty$ for all $x \in X$.

If $\|\cdot\|_1$ generates a finer topology than $\|\cdot\|$ generates on X^*, then there is a positive real number r such that $\|x'\| \leq r\,\|x'\|_1$ for all x' in X^*. Then for any x in X,

$$\|x\|_2 = \sup_{x' \in U_1'} |x'x| \leq \sup_{x' \in U_1'} \|x'\|\,\|x\|$$

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[1]The author wishes to thank Professor Norman Levine who pointed out an error in the original statement of this theorem.

$$\leq \sup_{x' \in U_1'} r\|x'\|_1 \, \|x\| = r\|x\| < \infty .$$

Thus, in this case, $\|\cdot\|_2$ is a norm on X.

Suppose $\|\cdot\|_2$ is a norm on X. Then $\|x\|_2 < \infty$ for all x in X. If X is complete, then by the principle of uniform boundedness, U_1' is a bounded set in $(X^*, \|\cdot\|)$. This implies there is some positive real number r such that $U_1' \subseteq S_r'$. Then by Theorem 2, $\|x'\| \leq r\|x'\|_1$ for all x' in X*. Hence $\|\cdot\|_1$ generates a finer topology on X* than $\|\cdot\|$.

If $\|\cdot\|_1$ and $\|\cdot\|$ are equivalent in X*, there exist (Theorem 1) positive real numbers s and t such that

$$s\|x'\|_1 \leq \|x'\| \leq t\|x'\|_1 \qquad \text{for all } x' \in X^*.$$

From Theorem 2, this is equivalent to

(15) $$S_s' \subseteq U_1' \subseteq S_t' .$$

For any nonnegative real number r, let T_r be the sphere of radius r for $(X, \|\cdot\|_2)$: $T_r = \{x : \|x\|_2 \leq r\}$. From equation (14) and the remarks after Definition 1, it is clear that

(16) $$T_1 = {}^oU_1'.$$

Then from (15)

(17) $${}^oS_s' \supseteq {}^oU_1' \supseteq {}^oS_t' .$$

But, ${}^oS_s' = {}^o(sS_1') = (1/s){}^oS_1' = (1/s)S_1 = S_{(1/s)}$, as can be easily checked. Relation (17) can be restated as

$$S_{(1/s)} \supseteq T_1 \supseteq S_{(1/t)}$$

or, in view of Theorem 2,

$$(1/s)\|x\|_2 \geq \|x\| \geq (1/t)\|x\|_2 .$$

Then by Theorem 1, $\|\cdot\|_2$ and $\|\cdot\|$ are equivalent norms for X.

Now that we know $\|\cdot\|_2$ is a norm on X, we turn to the problem of determining whether or not we obtain $(X^*, \|\cdot\|_1)$ by taking the conjugate of the space $(X, \|\cdot\|_2)$ in the sense that, for $\|\cdot\|_2$ as a norm in $(X^*, \|\cdot\|_2)$, we have

$$\|x'\|_2 = \|x'\|_1 \qquad \text{for all } x' \in X^*.$$

<u>Theorem</u> <u>10</u>. If $\|\cdot\|_1$ is equivalent to $\|\cdot\|$ in X*, then for all x' in X* $\|x'\|_2 \leq \|x'\|_1$.

Proof. From (16), $T_1 = {}^{\circ}U_1'$. Then $T_1^{\circ} = ({}^{\circ}U_1')^{\circ} \supseteq U_1'$. By (8), T_1° is the unit sphere of $(X^*, \|\cdot\|_2)$. Then the inclusion $T_1^{\circ} \supseteq U_1'$ implies (Theorem 2)

$$\|x'\|_2 \leq \|x'\|_1 \qquad \text{for all } x' \in X^*.$$

For each nonnegative real number r, let T_r' be the sphere of radius r for $(X^*, \|\cdot\|_2)$: $T_r' = \{x' : \|x'\|_2 \leq r\}$. From (8), $T_1^{\circ} = T_1'$.

We recall that $\overline{U_1'}$ stands for the closure of U_1' in the weak* topology.

<u>Theorem</u> <u>11</u>. If $\|\cdot\|_1$ is equivalent to $\|\cdot\|$ on X*, then $\overline{U_1'} = T_1'$.

Proof. T_1', as the unit sphere of the conjugate of $(X^*, \|\cdot\|_2)$, is weak* closed. From the proof of Theorem 10, we know that always

$$T_1' \geq U_1' .$$

Then since T_1' is weak* closed,

$$T_1' \geq \overline{U_1'}.$$

Suppose $x_o' \notin \overline{U_1'}$. By the corollary to Theorem 4 there exist an element $x \in X$ (in the weak* topology, continuous linear functionals f on X* are of the form $f(x') = x'x$ where x is some element of X) and a real number α such that

$$|x_o'x| > \alpha \geq |x'x| \quad \text{for all } x' \in \overline{U_1'}.$$

Then, since $U_1' \subseteq \overline{U_1'}$,

$$|x_o'x| > \alpha \geq \sup_{x' \in U_1'} |x'x| = \|x\|_2 .$$

Because $\|x_o'\|_2 \|x\|_2 \geq |x_o'x|$, the above inequality implies $\|x_o'\|_2 > 1$. Hence $x_o' \notin T_1'$. This completes the proof that $\overline{U_1'} = T_1'$.

Theorem 11 is also a direct consequence of a very general theorem of Bourbaki (1, No. 1229, p. 52).

Theorem 12. Let $\|\cdot\|_1$ be a norm on X*, equivalent to $\|\cdot\|$ on X*. If U_1' is defined by equation (13) and $\|\cdot\|_2$ by equation (14), and $(X^*, \|\cdot\|_2)$ is the conjugate of $(X, \|\cdot\|_2)$, then $\|x'\|_2 = \|x'\|_1$ for all $x' \in X^*$ if and only if U_1' is weak* closed.

Proof. From Theorem 3, $\|x'\|_2 = \|x'\|_1$ for all $x' \in X^*$ if and only if $U_1' = T_1'$. From Theorem 11, the assertion $U_1' = T_1'$ is equivalent to the statement that U_1' is weak* closed.

Theorem 12 is contained implicitly in the proof of a theorem by Klee (5, p. 37).

The last theorem tells us necessary and sufficient conditions that we retrieve the norm $\|\cdot\|_1$ by the process of passing back to a norm $\|\cdot\|_2$ on X through equation (14) and then returning to $\|\cdot\|_2$ on X* by taking the conjugate of $(X, \|\cdot\|_2)$. But Theorem 12, with Theorem 11, also raises a question. Since $\overline{U_1'} = T_1'$, we know U_1' and T_1' are, in a certain sense, "close to being equal." Are they perhaps always equal ? Are they always equal if X is some particular type of space ?

These are the questions we wish to investigate next. $\|\cdot\|_1$ will be a norm on X* equivalent to $\|\cdot\|$ if and only if U_1' is a norm set (Definition 6) for $\|\cdot\|$ in X*. With this in mind we may rephrase our question as follows. Is it true that every norm set in $(X^*, \|\cdot\|)$ is weak* closed ? If this is not always true, are there any classes of Banach spaces $(X, \|\cdot\|)$ for which it is true ?

We can answer the first question with an example, which we will include here in a theorem preceded by a sequence of lemmas.

For the following six lemmas let X be a nonreflexive Banach space and let J denote the canonical map from X into X**. Thus, for $x \in X$, $Jxx' = x'x$ for all $x' \in X^*$.

<u>Lemma 1</u>. ${}^{\circ}(JS_1) = S_1'$.

Proof. If $x \in S_1$, then $|Jxx'| = |x'x|$. Then $x' \in {}^{\circ}(JS_1)$ implies $|x'x| \leq 1$ for all $x \in S_1$ and hence $\|x'\| \leq 1$ or $x' \in S_1'$. Conversely, if $x' \in S_1'$, then for all $x \in S_1$, $|Jxx'| = |x'x| \leq \| x'\| \| x \| \leq 1$ and hence $x' \in {}^{\circ}(JS_1)$.

Let T'' be the balanced and convex hull (Def. 3) of the set $JS_1 \cup S_{1/2}''$.

<u>Lemma 2</u>. $T'' = \bigcup_{0 \leq r \leq 1}(rJS_1 + (1 - r)S_{1/2}'')$.

Proof. Denote the set on the right of the statement of the lemma by W''. It is clear that W'' contains $JS_1 \cup S_{1/2}''$ and that W'' is balanced since JS_1 and $S_{1/2}''$ are balanced. Also, clearly the elements of W'' are contained in any convex and balanced set which contains JS_1 and $S_{1/2}''$. If we can show W'' is itself convex, then we will have proved $T'' = W''$.

Let γ be a real number, $0 < \gamma < 1$, $x_1'' = r_1Js_1 + (1 - r_1)s_1''$ and $x_2'' = r_2Js_2 + (1 - r_2)s_2''$ where $s_i \in S_1$ and $s_i'' \in S_{1/2}''$ for $i = 1,2$.

Then $\gamma x_1'' + (1-\gamma)x_2'' = J(\gamma r_1 s_1 + (1-\gamma)r_2 s_2) + (\gamma(1-r_1)s_1'' + (1-\gamma)(1-r_2)s_2'')$. Let $r_3 = \gamma r_1 + (1-\gamma)r_2$. Then $0 \leq r_3 \leq 1$ and $1 - r_3 = \gamma(1-r_1) + (1-\gamma)(1-r_2)$. Define:

$$s_3 = (\gamma r_1/r_3)s_1 + ((1-\gamma)r_2/r_3)s_2$$

$$s_3'' = (\gamma(1-r_1)/(1-r_3))s_1'' + ((1-\gamma)(1-r_2)/(1-r_3))s_2''.$$

From the facts that S_1 is convex, $s_1 \in S_1$, $s_2 \in S_1$, and $(\gamma r_1/r_3) + ((1-\gamma)r_2/r_3) = 1$, we conclude $s_3 \in S_1$. Similarly we may conclude $s_3'' \in S_{1/2}''$. But it is immediate that

$$\gamma x_1'' + (1-\gamma)x_2'' = r_3 J s_3 + (1-r_3)s_3''$$

and therefore is an element of W'' since $0 \leq r_3 \leq 1$. Thus W'' is convex and therefore $T'' = W''$.

Let $d(x'',A'')$ denote the distance from the set A'' to the point x'' in the usual sense:

$$d(x'',A'') = \inf_{a'' \in A''} \|x'' - a''\| .$$

<u>Lemma</u> 3. If $x'' \in T''$, then $d(x'',JX) \leq 1/2$.

Proof. If $x'' \in T''$, then there exist a real number r with $0 \leq r \leq 1$, an $s \in S_1$ and an $s'' \in S_{1/2}''$ such that $x'' = rJs + (1-r)s''$.

$$d(x'',JX) \leq \|x'' - J(rs)\| = |1-r|\,\|s''\| \leq 1/2.$$

Let U_1'' be the closure in $(X^{**}, \|\cdot\|)$ of T''.

<u>Lemma</u> 4. If $x'' \in U_1''$, then $d(x'', JX) \leq 1/2$.

Proof. $d(x'', JX)$ is a continuous function of x'' in X^{**} with the $\|\cdot\|$ topology. Since U_1'' is the closure of T'' in this topology, the result follows from the continuity of $d(x'', JX)$ and Lemma 3.

<u>Lemma</u> 5. U_1'' is a norm set for $\|\cdot\|$ in X^{**} with ${}^{o}U_1'' = S_1'$.

Proof. The inclusion

$$JS_1 \cup S_{1/2}'' \subseteq S_1''$$

is immediate. Since S_1'' is balanced and convex,

$$JS_1 \cup S_{1/2}'' \subseteq T'' \subseteq S_1''.$$

Then because S_1'' is a closed set in $(X^{**}, \|\cdot\|)$,

(18) $$JS_1 \cup S_{1/2}'' \subseteq U_1'' \subseteq S_1''.$$

This in turn implies

$$S_{1/2}'' \subseteq U_1'' \subseteq S_1''$$

which shows U_1'' is bounded and absorbing. Since it is also balanced, convex, and closed in $(X^{**}, \|\cdot\|)$, U_1'' is a norm set for $\|\cdot\|$ in X^{**} (Theorem 6).

Relation (18) also implies

$$JS_1 \subseteq U_1'' \subseteq S_1''.$$

Then

$${}^{o}(JS_1) \supseteq {}^{o}U_1'' \supseteq {}^{o}S_1''.$$

Now, from (8) and Lemma 1,

$$S_1' \supseteq {}^{o}U_1'' \supseteq S_1'$$

or

$$S_1' = {}^{o}U_1''.$$

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Let $\|\cdot\|_1$ denote the Minkowski functional of U_1''. From Theorem 6, $\|\cdot\|_1$ is a norm for X^{**} equivalent to $\|\cdot\|$ and with unit sphere U_1'' by (11). If we define a norm $\|\cdot\|_2$ in X^* by the analogue of (14), then the unit sphere of $\|\cdot\|_2$ is ${}^oU_1''$, as in (16). But since ${}^oU_1'' = S_1'$, Theorem 3 implies $\|x'\|_2 = \|x'\|$ for all $x' \in X^*$. Then the conjugate of $(X^*, \|\cdot\|_2)$ is $(X^{**}, \|\cdot\|)$.

<u>Lemma</u> <u>6</u>. $U_1'' \neq S_1''$.

Proof. JX is a closed proper subspace of X^{**} since X is not reflexive. By Riesz's lemma (Theorem 8), there exists in S_1'' some x'' such that $d(x'', JX) > 1/2$. Then from Lemma 4, $x'' \notin U_1''$.

<u>Theorem</u> <u>13</u>. Let $(X, \|\cdot\|)$ be a nonreflexive Banach space. There exists in $(X^{**}, \|\cdot\|)$ a norm set which is not weak* closed.

Proof. U_1'' is a norm set for which $U_1'' \neq S_1''$ (Lemma 6). By Theorem 6, U_1'' is the unit sphere of $(X^{**}, \|\cdot\|_1)$. But from the remarks after Lemma 5, the conjugate of $(X^*, \|\cdot\|_2)$ is $(X^{**}, \|\cdot\|)$ which has unit sphere S_1''. Then Lemma 6, Theorem 3, and Theorem 12 imply U_1'' is not weak* closed.

This theorem answers the first of the questions asked in the discussion after Theorem 12. Also, it will help to give a partial answer to the second one.

<u>Theorem</u> <u>14</u>. If $(X, \|\cdot\|)$ is reflexive, every norm set for $\|\cdot\|$ in X^* is weak* closed.

Proof. A reflexive normed linear space must be complete. Hence $(X, \|\cdot\|)$ is a Banach space. Since a norm set is closed in $(X^*, \|\cdot\|)$ and convex, it must be weakly closed in X^* (<u>7</u>, p. 153), i.e., closed in the weak topology on X^* generated by X^{**}. However, if X is reflexive, the weak and weak* topologies on X^* are identical. Then every norm set is weak* closed.

<u>Theorem</u> <u>15</u>. Let X be the conjugate of some Banach space Y. Then every norm set in X^* is weak* closed if and only if X is reflexive.

Proof. This follows immediately from Theorem 14, Theorem 13, and the fact that $X = Y^*$ is reflexive if and only if Y is reflexive.

This answers the second question raised in the discussion after Theorem 12, but not in a complete manner. It would be more satisfying to have a theorem similar to Theorem 15 in which the hypothesis that X is the conjugate of some Banach space Y did not appear.

For example, let c_o, l^1 and l^∞ refer to the spaces of sequences converging to zero, absolutely convergent series, and bounded sequences respectively, with their usual norms (<u>7</u>, p. 194 and p. 201). c_o^* can be identified with l^1 and $(l^1)^*$ with l^∞ (both identifications are by means of linear isometries). Since c_o is nonreflexive, Theorems 12 and 13 show it is possible to define norms in l^∞ so that l^∞ with its new topology is not the conjugate of l^1 with any norm topology although it is linearly homeomorphic under the identity mapping to l^∞ with its usual topology. However, the question of whether or not this can be done in l^1 as the conjugate of c_o is untouched by the above theorems since c_o is not a conjugate space (8).

The example in Theorem 13 has an interesting feature. The elements of $(X^{**}, \|\cdot\|_1)$ are continuous linear functions on $(X^*, \|\cdot\|)$. Since $\|\cdot\|_1$ and $\|\cdot\|$ are equivalent norms on X^{**}, the identity map from $(X^{**}, \|\cdot\|_1)$ onto $(X^{**}, \|\cdot\|)$ is a linear homeomorphism. Nevertheless, Theorems 13 and 12 and the remarks after equation (14) show that $(X^{**}, \|\cdot\|_1)$ is not the conjugate of any Banach space.

CHAPTER III

THE CHARACTERISTIC OF A SUBSPACE OF THE CONJUGATE OF A NORMED LINEAR SPACE

Dixmier has defined the "characteristic" of a subspace of X* and used the concept to characterize those Banach spaces which are linearly homeomorphic to a conjugate space (2). Taylor has also used the characteristic to give a variation of the principle of uniform boundedness (7, pp. 201-208). In Dixmier's article several formulas for the characteristic are developed. Here we will give new derivations of these formulas by an application of the results of the previous chapters and by interpreting various quantities as the norms of suitable mappings. These interpretations will usually be based on showing two norm topologies are equivalent and identifying the unit spheres in the topologies. The various formulas will appear as consequences of the study of the topologies.

Dixmier's proofs of his formulas are *ad hoc* proofs in the sense that they verify the equality of two numbers by "proving the inequality holds both ways," but do not reveal the underlying structure to which the equalities can be related.

In this chapter, $(X, \|\cdot\|)$ will still denote a normed linear space. V' will denote a linear subspace of $(X^*, \|\cdot\|)$. $\mathcal{J}$ and $\mathcal{J}_1$ will signify the weak topologies generated by V' on X and X^{**} respectively. As before, a bar "$\overline{}$" over the symbol for a set will denote the weak* closure of that set.

Definition 7. The characteristic of the subspace V' of X^* is

$$r_1 = \sup \left\{ r : S_r' \subseteq \overline{V' \cap S_1'} \right\} . \tag{19}$$

Throughout this chapter let

$$P = \left\{ r : S_r' \subseteq \overline{V' \cap S_1'} \right\} . \tag{20}$$

Because S_1' is weak* closed, $V' \cap S_1' \subseteq S_1'$ implies $\overline{V' \cap S_1'} \subseteq S_1'$ so that 1 is an upper bound for P. Also, P is non-empty since 0 in an element of P. This shows r_1 is well defined and $0 \leq r_1 \leq 1$.

There are three other quantities which enter into Dixmier's discussion of the characteristic.

$$r_2 = \inf_{x \neq 0} \ \sup_{x' \in V' \cap S_1'} \frac{|x'x|}{\|x\|} . \tag{21}$$

$$r_3 = \sup_{x \in \overline{S_1}^{\mathcal{J}}} \|x\| . \tag{22}$$

$$r_4 = \inf_{x \neq 0,\ z'' \in V'^{+}} \frac{\|Jx + z''\|}{\|x\|} . \tag{23}$$

J represents the canonical mapping from X into X^{**} and, as above,
$V'^{+} = \left\{ x'' : x'' \in X^{**},\ x''x' = 0 \quad \text{for all } x' \in V' \right\}$.

With the usual interpretation in case any of these quantities are zero or infinite, Dixmier has shown

(24) $$r_1 = r_2 = (1/r_3) = r_4 .$$

These equations, with Definition 7, give four formulas for the characteristic of V'.

We will also consider three other quantities:

(25) $$r_5 = \inf_{x'' \neq 0''} \sup_{x' \in \overline{V' \cap S_1'}} \frac{|x''x'|}{\|x'\|} ,$$

(26) $$r_6 = \sup_{x'' \in \overline{JS_1}^{\mathcal{J}_2}} \|x''\| ,$$

and

(27) $$r_7 = \sup_{x'' \in \overline{S_1''}^{\mathcal{J}_1}} \|x''\| .$$

Here $\mathcal{J}_2$ is the topology induced on JX by $\mathcal{J}_1$. We will find

(28) $$r_1 = r_5 = (1/r_6) .$$

However, while we might expect r_3 and r_7 to be equal, this will not in general be true unless X is reflexive.

Consider the set $\overline{V' \cap S_1'}$. Since the weak* topology is less fine than the norm topology of X*, $\overline{V' \cap S_1'}$ is norm closed. It is also convex and balanced since $V' \cap S_1'$ has these properties. We saw above that $\overline{V' \cap S_1'} \subseteq S_1'$. Thus V' S_1' is a balanced and convex set bounded and closed in $(X^*, \|\cdot\|)$.

We note that $r_1 \in P$ (formulas (19) and (20)). This is immediate if $r_1 = 0$. If $r_1 > 0$, it is clear from (19) that

$$\{x' : \|x'\| < r_1\} \subseteq \overline{V' \cap S_1'} .$$

Since $\overline{V' \cap S_1'}$ is closed in the $\|\cdot\|$ topology and S_{r_1}' is the closure of

$\{ x' : \|x'\| < r_1 \}$ in that topology, it follows that $S_{r_1}' \subseteq \overline{V' \cap S_1'}$ or $r_1 \in P$. Thus we may write

$$r_1 = \max P.$$

Suppose $r_1 > 0$. $\overline{V' \cap S_1'}$ is then absorbing since it contains the sphere S_{r_1}'. Since $\overline{V' \cap S_1'}$ is always balanced, convex, bounded and closed in $(X^*, \|\cdot\|)$, $r_1 > 0$ implies $\overline{V' \cap S_1'}$ is a norm set for $\|\cdot\|$ in X^* (Theorem 6). Denote the Minkowski functional of $\overline{V' \cap S_1'}$ by $\|\cdot\|_1$. From Theorem 6, $\|\cdot\|_1$ is a norm equivalent to $\|\cdot\|$ on X^* and hence the identity mapping from $(X^*, \|\cdot\|)$ onto $(X^*, \|\cdot\|_1)$ will be a linear homeomorphism. Denote this mapping by j_{01}.

For $r > 0$,

$$S_r' \subseteq \overline{V' \cap S_1'}$$

if and only if

$$\frac{rx'}{\|x'\|} \in \overline{V' \cap S_1'} \qquad \text{for all } x' \neq 0'.$$

Since $\overline{V' \cap S_1'}$ is the unit sphere for $\|\cdot\|_1$ (Theorem 6), this is equivalent to

$$\frac{r\,\|x'\|_1}{\|x'\|} \leq 1 \qquad \text{for all } x' \neq 0',$$

or

$$\|x'\|_1 \leq (1/r)\|x'\| \qquad \text{for all } x' \in X^*.$$

Then for $r_1 > 0$,

$$(29) \quad \{ r : r > 0,\ \|x'\|_1 \leq (1/r)\|x'\| \ \text{ for all } x' \in X^* \}$$
$$= \{ r : r > 0,\ S_r' \subseteq \overline{V' \cap S_1'} \} .$$

If we set $Q = \{r : r > 0, S_r' \subseteq \overline{V' \cap S_1'}\}$, then, for $r_1 > 0$, clearly $r_1 = \sup Q$ and

$$(1/r_1) = (1/ \sup_{r \in Q} r) = \inf_{r \in Q} (1/r).$$

But from the other form of the set Q in (29) we know

$$\inf_{r \in Q} (1/r) = \| j_{01} \|$$

and hence

$$(1/r_1) = \| j_{01} \| .$$

We collect these results as a theorem.

<u>Theorem</u> <u>16</u>. Let $(X, \| \cdot \|)$ be a normed linear space, V' a linear subspace of X^*, and let r_1 be defined by (19). When $r_1 > 0$, $\overline{V' \cap S_1'}$ is a norm set for $\| \cdot \|$ in X^*. If $\| \cdot \|_1$ is the Minkowski functional of $\overline{V' \cap S_1'}$ and j_{01} is the identity mapping from $(X^*, \| \cdot \|)$ onto $(X^*, \| \cdot \|_1)$, then

$$(1/r_1) = \| j_{01} \| .$$

Next we attempt to find an analogous theorem involving r_2. For each $x \in X$, let

(30) $$\| x \|_2 = \sup_{x' \in V' \cap S_1'} | x'x |$$

Although the notation so suggests, we do not assert $\| \cdot \|_2$ is always a norm on X.

We note $\|x\|_2 \leq \sup_{x' \in \overline{V' \cap S_1'}} |x'x|$. But if $x_0' \in \overline{V' \cap S_1'}$ and $x \in X$, then there exists, for each integer n, an element $x_n' \in V' \cap S_1'$ such that $|x_0'x - x_n'x| < 1/n$. Therefore,

$$|x_0'x| = \lim_{n \to \infty} |x_n'x| \leq \sup_{x' \in V' \cap S_1'} |x'x| = \|x\|_2$$

and

$$\sup_{x_0' \in \overline{V' \cap S_1'}} |x_0'x| \leq \|x\|_2 \ .$$

Hence

(31) $$\|x\|_2 = \sup_{x' \in \overline{V' \cap S_1'}} |x'x| \ .$$

The identity (31) shows that, when $r_1 > 0$, $\|\cdot\|_2$ is just the norm generated in X by the norm $\|\cdot\|_1$ in X* as in Chapter II (formula (14)). Since $\overline{V' \cap S_1'}$, the unit sphere for $\|\cdot\|_1$, is weak* closed, it follows from Theorem 12 that (X*, $\|\cdot\|_1$) is the conjugate of (X, $\|\cdot\|_2$).

This result is proved in a different manner in (4, p. 370) by using a theorem of Helly (3, p. 86) about the solvability of a finite system of equations $x_i'x = c_i$ where x_i' and c_i are given.

Let i_{20} be the identity map from (X, $\|\cdot\|_2$) onto (X, $\|\cdot\|$) (still assuming $r_1 > 0$). Since $\|\cdot\|_2$ and $\|\cdot\|$ are equivalent in X (Theorem 9) in this case, i_{20} will be a linear homeomorphism. From (21) and (30),

(32) $$r_2 = \inf_{x \neq 0} \sup_{x' \in V' \cap S_1'} \frac{|x'x|}{\|x\|} = \inf_{x \neq 0} \frac{\|x\|_2}{\|x\|}$$

Thus $\frac{1}{r_2} = \frac{1}{\inf_{x \neq 0} \frac{\|x\|_2}{\|x\|}} = \sup_{x \neq 0} \frac{\|x\|}{\|x\|_2} = \|i_{20}\|$. This proves the next theorem.

<u>Theorem</u> <u>17</u>. Let $(X, \|\cdot\|)$ be a normed linear space, V' a linear subspace of X^*, and let r_1 and r_2 be defined by (19) and (21). When $r_1 > 0$, $\|\cdot\|_2$ is a norm equivalent to $\|\cdot\|$ in X. If i_{20} is the identity mapping from $(X, \|\cdot\|_2)$ onto $(X, \|\cdot\|)$, then $\|i_{20}\| = 1/r_2$.

According to Theorems 16 and 17 we have this situation:

$$(X, \|\cdot\|_2) \xrightarrow{i_{20}} (X, \|\cdot\|)$$

$$(X^*, \|\cdot\|_1) \xleftarrow{j_{01}} (X^*, \|\cdot\|).$$

Both i_{20} and j_{01} are identity mappings. This makes it clear that j_{01} is the conjugate of i_{20}, and hence that

$$\|j_{01}\| = \|i_{20}\| .$$

This equation, with Theorems 16 and 17, shows that for $r_1 > 0$, $r_1 = r_2$.

If $r_1 = 0$, $\overline{V' \cap S_1'}$ is not absorbing. (This is true since any norm closed, absorbing, balanced, and convex set in X^* must contain some sphere of nonzero radius about $0'$. This was shown in the proof of Theorem 6.) Then there is some $x_0' \in X^*$ such that $rx_0' \notin \overline{V' \cap S_1'}$ for any real positive number r. By the corollary to Theorem 4, for

INTRODUCTION

Let X be a normed linear space (over the real or complex numbers) with norm $\|\cdot\|$, and let X* be its conjugate space. The usual norm on the conjugate space is defined by

$$(0) \qquad \|x'\| = \sup_{x \in X,\ \|x\| \leq 1} |x'x| \qquad \text{for all } x' \in X^*.$$

If two equivalent norms are defined on X, the norms defined on X* from the norms on X as in (0) are also equivalent.

One consequence of the Hahn-Banach theorem shows that we can "reverse" formula (0) to obtain the norm in X from the norm it generates in X*:

$$(1) \qquad \|x\| = \sup_{x' \in X^*,\ \|x'\| \leq 1} |x'x| \qquad \text{for all } x \in X.$$

This formula suggests an interesting possibility. As (1) is stated, the norm $\|\cdot\|$ in X* must have arisen as in (0) from a norm in X. However, given any norm $\|\cdot\|_1$ in X*, in analogy to (1), we could define a function (which may or may not be a norm) in X as follows:

$$(2) \qquad \|x\|_2 = \sup_{x' \in X^*,\ \|x'\|_1 \leq 1} |x'x| \qquad \text{for all } x \in X.$$

Some questions which arise are these: Is $\|\cdot\|_2$ a norm on X? If it is a norm, and $\|\cdot\|_1$ is equivalent to $\|\cdot\|$ on X*, will $\|\cdot\|_2$ be equivalent to $\|\cdot\|$ on X? In view of formulas (0), (1), and (2), if each positive real number r there exists a real number α_r and an element $x_r \in X$ such that

$$|rx_o'x_r| > \alpha_r \geq \sup_{x' \in V' \cap S_1'} |x'x_r|$$

and

$$|r|\,\|x_o'\| \geq \frac{|r|\,|x_o'x_r|}{\|x_r\|} > \frac{\alpha_r}{\|x_r\|} \geq \sup_{x' \in V' \cap S_1'} \frac{|x'x_r|}{\|x_r\|} = \frac{\|x_r\|_2}{\|x_r\|} .$$

From (21)

$$r_2 = \inf_{x \neq 0} \frac{\|x\|_2}{\|x\|} \leq \inf_{r > 0} \frac{\|x_r\|_2}{\|x_r\|} \leq \inf_{r > 0} |r|\,\|x_o'\| = 0.$$

Thus, whether or not $r_1 > 0$, we have

(33) $$r_1 = r_2.$$

Next we consider the set $\overline{S_1}^{\mathcal{J}}$ which enters into the definition of r_3.

<u>Theorem</u> <u>18</u>. $\overline{S_1}^{\mathcal{J}} = {}^{o}(V' \cap S_1')$.

Proof. From the remarks following Definition 1, we know

$${}^{o}(V' \cap S_1') = \{x : |x'x| \leq 1 \quad \text{for all } x' \in V' \cap S_1'\} .$$

or

$${}^{o}(V' \cap S_1') = \bigcap_{x' \in V' \cap S_1'} \{x : |x'x| \leq 1\} .$$

Each $x' \in V'$ is continuous in the $\mathcal{J}$ topology on X. Therefore, for each $x' \in V'$, $\{x : |x'x| \leq 1\}$ is a closed set in the topology $\mathcal{J}$. Then ${}^{o}(V' \cap S_1')$ must also be closed in the topology $\mathcal{J}$. The inclusion $V' \cap S_1' \subseteq S_1'$ implies ${}^{\circ}(V' \cap S_1') \supseteq {}^{\circ}S_1'$ and, from (8), ${}^{\circ}(V' \cap S_1') \supseteq S_1$. Since ${}^{\circ}(V' \cap S_1')$ is closed in the topology $\mathfrak{J}$, the last inclusion implies

$$ {}^{\circ}(V' \cap S_1') \supseteq \overline{S_1}^{\mathfrak{J}} . $$

Suppose $x_0 \notin \overline{S_1}^{\mathfrak{J}}$. By the corollary to Theorem 4, there exist a real number β and an $x_0' \in V'$ (the continuous functionals on X with topology $\mathfrak{J}$ are just the elements of V') such that

$$ |x_0'x_0| > \beta \geq |x_0'x| \qquad \text{for all } x \in \overline{S_1}^{\mathfrak{J}}. $$

We may assume that $\|x_0'\| = 1$ since we could divide by $\|x_0'\|$ in the above inequality. Thus $x_0' \in V' \cap S_1'$ and $|x_0'x_0| > \beta \geq \sup_{x \in \overline{S_1}^{\mathfrak{J}}} |x_0'x| \geq \sup_{x \in S_1} |x_0'x| = \|x_0'\| = 1$. This shows $|x_0'x_0| > 1$ and hence $x_0' \notin {}^{\circ}(V' \cap S_1')$, which completes the proof of Theorem 18.

From (30) it is clear that $\|x\|_2 \leq 1$ if and only if $|x'x| \leq 1$ for $x' \in V' \cap S_1'$. This implies

$$ {}^{\circ}(V' \cap S_1') = \{ x : \|x\|_2 \leq 1 \} $$

and then from Theorem 18,

$$ \overline{S_1}^{\mathfrak{J}} = \{ x : \|x\|_2 \leq 1 \} . $$

When $r_1 \neq 0$, $\|\cdot\|_2$ is a norm on X and $\overline{S_1}^{\mathfrak{J}}$ is the unit sphere of $(X, \|\cdot\|_2)$. By (22), (33) and Theorem 17, if $r_2 \neq 0$,

$$r_3 = \sup_{x \in \overline{S_1}^{\,\mathfrak{J}}} \|x\| = \sup_{\|x\|_2 \leq 1} \|x\| = \|i_{20}\| .$$

This verifies the following theorem.

<u>Theorem 19</u>. Let $(X, \|\cdot\|)$ be a normed linear space, V' a linear subspace of X^*, and let r_2 and r_3 be defined by (21) and (22). Let $r_2 > 0$ and let i_{20} be the identity mapping from $(X, \|\cdot\|_2)$ onto $(X, \|\cdot\|)$. Then $r_3 = \|i_{20}\|$.

It is a consequence of Theorems 19 and 17 and equation (33) that $r_3 = 1/r_2$ when $r_2 \neq 0$. To verify that $r_3 = \infty$ when $r_2 = 0$ we first show $\overline{V' \cap S_1'} = ({}^{\circ}(\overline{V' \cap S_1'}))^{\circ}$. From (31), ${}^{\circ}(\overline{V' \cap S_1'}) = \{x : \|x\|_2 \leq 1\}$. Also we know $({}^{\circ}(\overline{V' \cap S_1'}))^{\circ} \supseteq \overline{V' \cap S_1'}$. If $x_o' \notin \overline{V' \cap S_1'}$, then there is an $x_o \in X$ and a real number α such that

$$|x_o' x_o| > \alpha \geq |x' x_o|$$

for all $x' \in \overline{V' \cap S_1'}$. If we set $x_1 = x_o/\alpha$, then

$$|x_o' x_1| > 1 \geq |x' x_1| \qquad \text{for all } x' \in \overline{V' \cap S_1'}.$$

Then $x_1 \in {}^{\circ}(\overline{V' \cap S_1'})$ and hence $|x_o' x_1| > 1$ implies $x_o' \notin ({}^{\circ}(\overline{V' \cap S_1'}))^{\circ}$. Thus $\overline{V' \cap S_1'} = ({}^{\circ}(\overline{V' \cap S_1'}))^{\circ}$.

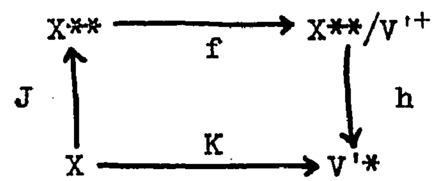
Now, if $r_3 < \infty$, then ${}^{\circ}(\overline{V' \cap S_1'}) = \overline{S_1}^{\,\mathfrak{J}} \subseteq S_{r_3}$ and $({}^{\circ}(\overline{V' \cap S_1'}))^{\circ} \supseteq (S_{r_3})^{\circ} = S'_{(1/r_3)}$. From (19), $r_1 > 0$. By (33), $r_2 > 0$. The contrapositive of what we have proven would be: $r_2 = 0$ implies $r_3 = \infty$. Then in all cases

(34) $$r_2 = 1/r_3.$$

The fourth quantity discussed by Dixmier is

(34) $$r_4 = \inf_{x \neq 0,\ z'' \in V'^{+}} \frac{\| Jx + z'' \|}{\| x \|} .$$

To relate this quantity to the others we will use several standard mappings as suggested in the following figure.



The coset relative to V'^{+} containing x'' will be indicated by $[x'']$. These cosets are the elements of X^{**}/V'^{+}. We note V'^{+} is a closed subspace. X^{**}/V'^{+} is given the quotient topology, which can be defined by a norm $\|\cdot\|$ given by

$$\| [x''] \| = \inf_{z'' \in V'^{+}} \| x'' + z'' \| , \qquad [x''] \in X^{**}/V'^{+}.$$

The restriction of $x'' \in X^{**}$ to the subspace V' of X^{*} will be denoted by $x''|V'$. f and h are the canonical mappings defined by

$$f(x'') = [x''] \qquad \text{for all } x'' \in X^{**},$$

$$h([x'']) = x''|V' \qquad \text{for all } [x''] \in X^{**}/V'^{+}.$$

f is linear and continuous from X^{**} onto X^{**}/V'^{+} while h is a linear isometry of X^{**}/V'^{+} onto V'^{*}. J is the canonical mapping from X into X^{**}. We define K to be the composition of J, f and h : $K = hfJ$. Then for $x \in X$ and $x' \in V'$,

(35) $$(Kx)x' = (hfJx)x' = (Jx|V')x' = Jxx' = x'x.$$

V', as a subspace of X^{*}, can itself be considered a normed linear space if we use as norm the function $\|\cdot\|$, the norm on X^{*}, restricted to V'. Denote this restricted norm on V' by $\|\cdot\|$ and also denote the norm in V'* as the conjugate of $(V', \|\cdot\|)$ by $\|\cdot\|$.

With these notational conventions,

$$r_4 = \inf_{x\neq 0,\ z''\in V'^+} \frac{\|Jx+z''\|}{\|x\|} = \inf_{x\neq 0} \frac{1}{\|x\|} \inf_{z''\in V'^+} \|Jx+z''\|$$

$$= \inf_{x\neq 0} \frac{1}{\|x\|} \|[Jx]\| = \inf_{x\neq 0} \frac{\|fJx\|}{\|x\|}$$

$$= \inf_{x\neq 0} \frac{\|hfJx\|}{\|x\|} \quad \text{(since h is an isometry).}$$

Thus

(36) $$r_4 = \inf_{x \neq 0} \frac{\|Kx\|}{\|x\|} .$$

Theorem 20. Let $(X, \|\cdot\|)$ be a normed linear space, V' a linear subspace of X*, let K be the mapping from X into V'* defined by (35), and let r_4 be defined by (23). Then K has a continuous inverse on its range if and only if $r_4 \neq 0$. If $r_4 \neq 0$, and $\overset{-1}{K}$ represents the inverse of K on its range, then

$$\|\overset{-1}{K}\| = 1/r_4 .$$

Proof. The theorem follows directly from Theorem 7 and equation (36).

For each x, $Kx \in V'^*$. Therefore,

$$\|Kx\| = \sup_{\|x'\|\leq 1,\ x'\in V'} |(Kx)x'| = \sup_{x'\in V'\cap S_1'} |(Kx)x'| .$$

Then by (35),

$$\|Kx\| = \sup_{x' \in V' \cap S_1'} |x'x| . \tag{37}$$

From (36), (37), and (21),

$$r_4 = \inf_{x \neq 0} \frac{\|Kx\|}{\|x\|} = \inf_{x \neq 0} \frac{1}{\|x\|} \sup_{x' \in V' \cap S_1'} |x'x| = r_2 .$$

Thus

$$r_2 = r_4 . \tag{38}$$

<u>Theorem 21</u>. Let $(X, \|\cdot\|)$ be a normed linear space, V' a linear subspace of X^* and let r_1, r_2, r_3, and r_4 be defined by (19), (21), (22), and (23). Then $r_1 = r_2 = (1/r_3) = r_4$.

Proof. Equations (33), (34), and (38) show the theorem is true.

Theorem 21 verifies the formulas of Dixmier while Theorems 16, 17, and 20 give geometric interpretations of the quantities involved (in the nondegenerate cases). To prove $r_1 = r_2$ when $r_1 \neq 0$, we noted that $j_{01} = i_{20}^*$, the conjugate of i_{20}. This suggests a method of finding a new formula for the characteristic in the following way.

If $r_1 > 0$, then $\|\cdot\|_1$ is equivalent to $\|\cdot\|$ in X^*. Taking conjugates, we know $\|\cdot\|_1$ and $\|\cdot\|$ are equivalent norms in X^{**}. Then j_{01}^*, the conjugate on j_{01}, will be the identity mapping from $(X^{**}, \|\cdot\|_1)$ onto $(X^{**}, \|\cdot\|)$. Hence

$$\frac{1}{\|j_{01}^*\|} = \frac{1}{\sup\limits_{x''\neq 0''} \frac{\|x''\|}{\|x''\|_1}} = \inf_{x''\neq 0''} \frac{\|x''\|_1}{\|x''\|} =$$

$$\inf_{x''\neq 0''} \frac{1}{\|x''\|} \sup_{x'\in \overline{V'\cap S_1'}} |x''x'| .$$

The last equality holds since $\overline{V'\cap S_1'}$ is the unit sphere of $(X^*, \|\cdot\|_1)$. Thus, when $r_1 > 0$, by (25)

$$\frac{1}{\|j_{01}^*\|} = \inf_{x''\neq 0''} \sup_{x'\in \overline{V'\cap S_1'}} \frac{|x''x'|}{\|x''\|} = r_5 .$$

<u>Theorem</u> <u>22</u>. Let X be a normed linear space and V' a linear subspace of X*. With r_5 defined by (25), if $r_1 > 0$, then $\|j_{01}^*\| = (1/r_5)$ where j_{01}^* is the identity mapping from $(X^{**}, \|\cdot\|_1)$ onto $(X^{**}, \|\cdot\|)$ and hence the conjugate of the identity mapping j_{01} from $(X^*, \|\cdot\|)$ onto $(X^*, \|\cdot\|_1)$.

From Theorems 16 and 22 and the identity $\|j_{01}^*\| = \|j_{01}\|$, we obtain

$$r_1 = r_5 \qquad \text{if } r_1 > 0.$$

To verify $r_5 = 0$ when $r_1 = 0$, we note that

$$r_5 = \inf_{x''\neq 0''} \sup_{x'\in \overline{V'\cap S_1'}} \frac{|x''x'|}{\|x''\|} \leq \inf_{x''\neq 0'',\ x''\in JX} \sup_{x'\in \overline{V'\cap S_1'}} \frac{|x''x'|}{\|x''\|} .$$

But, if $x'' \in JX$, then there is some x such that $x'' = Jx$, $x''x' = x'x$, and $\|x''\| = \|x\|$. Then

$$r_5 \leq \inf_{x\neq 0} \sup_{x'\in \overline{V'\cap S_1'}} \frac{|x'x|}{\|x\|} = \inf_{x\neq 0} \frac{\|x\|_2}{\|x\|} = r_2 .$$

Thus, $r_1 = 0$ implies $r_2 = 0$ which in turn implies $r_5 \leq 0$ from the above inequality. Since r_5 is nonnegative, we have shown $r_5 = 0$ if $r_1 = 0$. In all cases,

(39) $$r_1 = r_5 .$$

It is easily checked that J is a homeomorphism of X with topology $\mathcal{J}$ onto JX with topology $\mathcal{J}_2$ (the topology on JX induced by $\mathcal{J}_1$). Then

(40) $$J(\overline{S_1}^{\mathcal{J}}) = \overline{JS_1}^{\mathcal{J}_2} .$$

From this and the fact J is an isometry, it follows immediately that

$$\sup_{x'' \in \overline{JS_1}^{\mathcal{J}_2}} \| x'' \| = \sup_{x'' \in J(\overline{S_1}^{\mathcal{J}})} \| x'' \| = \sup_{x \in \overline{S_1}^{\mathcal{J}}} \| x \| .$$

In view of (26) and (22), we have

(41) $$r_6 = r_3 .$$

<u>Theorem</u> <u>23</u>. If $(X, \|\cdot\|)$ is a normed linear space and V' is a linear subspace of X*, then if r_1 to r_6 are given by (19), (21), (22), (23), (25), and (26), we have

$$r_1 = r_2 = (1/r_3) = r_4 = r_5 = (1/r_6) .$$

Proof. This theorem is a consequence of Theorem 21, equation (39), and equation (41).

Let $r_1 > 0$ and consider $(X^{**}, \|\cdot\|_1)$, the conjugate of $(X^*, \|\cdot\|_1)$. In this case, $\|\cdot\|_1$ restricted to JX will be a norm on JX. (We denote this norm on JX by $\|\cdot\|_1$.) Let T_1'' be the unit sphere of $(X^{**}, \|\cdot\|_1)$. Since we are assuming $r_1 > 0$, $\|\cdot\|_2$ is a norm on X and $(X^{**}, \|\cdot\|_1)$ is the second conjugate of $(X, \|\cdot\|_2)$, which has unit sphere $\overline{S_1}^{\mathcal{J}}$ (Theorem

18 and equation (31)). Then J is an isometry from $(X, \|\cdot\|_2)$ into $(X^{**}, \|\cdot\|_1)$, and we have

$$\overline{JS_1}^{\mathcal{J}} = T_1{}'' \cap JX.$$

From (40), then

$$\overline{JS_1}^{\mathcal{J}_2} = T_1{}'' \cap JX$$

and $\overline{JS_1}^{\mathcal{J}_2}$ is the unit sphere of $(JX, \|\cdot\|_1)$. Let k_{10} be the identity mapping from $(JX, \|\cdot\|_1)$ into $(X^{**}, \|\cdot\|)$. Then

$$\|k_{10}\| = \sup_{x'' \in \overline{JS_1}^{\mathcal{J}_2}} \|x''\| = r_6 .$$

<u>Theorem 24</u>. Let $(X, \|\cdot\|)$ be a normed linear space and let V' be a linear subspace of X^*. Let r_6 be defined by (26). If $r_1 > 0$, denote the restriction of $\|\cdot\|_1$ to JX in $(X^{**}, \|\cdot\|_1)$ also by $\|\cdot\|_1$ and let k_{10} be the identity mapping from $(JX, \|\cdot\|_1)$ into $(X^{**}, \|\cdot\|)$. Then for $r_1 > 0$, $\|k_{10}\| = r_6$.

The similarity of the definitions of the topologies $\mathcal{J}$ and $\mathcal{J}_1$, both generated by V' on X and X^{**} respectively, suggests examining

$$r_7 = \sup_{x'' \in \overline{S_1{}''}^{\mathcal{J}_1}} \|x''\|$$

as a possible means of obtaining still another formula for the characteristic. We might expect $r_3 = r_7$, which would yield a new formula. However, it is not always true that $r_3 = r_7$ as the following example shows.

Let X be the conjugate of a nonreflexive Banach space Y and let J_1 be the canonical mapping from Y into $Y^{**} = X^*$. Let $V' = J_1 Y$. V' is a proper closed subspace of X* and hence $V'^{+} \neq \{0''\}$ (7, p. 186). In this case, the corollary to Theorem 5 implies $\overline{S_1''}^{\mathfrak{J}_1}$ is unbounded:

$$r_7 = \sup_{x'' \in \overline{S_1''}^{\mathfrak{J}_1}} \| x'' \| = \infty .$$

However, (7, p. 228)

$$\begin{aligned} V' \cap S_1' &= J_1 Y \cap \{ y'' : \| y'' \| \leq 1 \} \\ &= J_1 \{ y : \| y \| \leq 1 \} \end{aligned}$$

is weak* dense in $\{y'' : \| y'' \| \leq 1\} = S_1'$. From this fact and (19), $r_1 = 1$. By Theorem 21, then $r_3 = 1$. Thus $r_3 \neq r_7$ in this case.

The corollary to Theorem 5, as used above, makes it clear that $r_7 = \infty$ whenever $V'^{+} \neq \{0''\}$. Then the formula $r_3 = r_7$ must fail whenever $V'^{+} \neq \{0''\}$ and $r_1 \neq 0$.

However, we can give two situations in which r_3 and r_7 are equal. To do this we first note again that J is a linear homeomorphism of X with topology $\mathfrak{J}$ onto JX with topology $\mathfrak{J}_2$ induced on JX by $\mathfrak{J}_1$.

<u>Theorem</u> <u>25</u>. Let $(X, \| \cdot \|)$ be a normed linear space and let V' be a linear subspace of X*. Let r_3 and r_7 be defined by (22) and (27) respectively. Then $r_3 \leq r_7$. In particular, $r_3 = \infty$ implies $r_7 = \infty$ and $r_7 = 1$ implies $r_3 = 1$.

Proof. $JS_1 = JX \cap S_1''$ since J is an isometry of $(X, \| \cdot \|)$ into $(X^{**}, \| \cdot \|)$. From (40),

$\|\cdot\|_2$ is a norm on X and we take the conjugate of X with the norm $\|\cdot\|_2$, will we obtain X* with the norm $\|\cdot\|_1$?

We will find necessary and sufficient conditions that we do come back to $\|\cdot\|_1$ by this process. We will also discuss spaces in which this may or may not be true for all possible norms $\|\cdot\|_1$ equivalent to $\|\cdot\|$ on X*.

As an application of the results obtained on the problems mentioned above, we will derive one of the formulas for the "characteristic" of a subspace of X* as introduced by Dixmier. Then we will prove and amplify from a geometric viewpoint his other formulas for the characteristic by relating the quantities involved to the norms of various mappings. Several new formulas will be introduced, also.

A word about notation. x will be a generic symbol for elements of X, x' for elements of X*, and x" for elements of X** = (X*)*. In general, subsets of X will have no prime (A, V, etc.), subsets of X* will carry one prime (A', V', etc.), and subsets of X** will carry two primes (A", V", etc.). Lower case Greek letters will be scalars, either real or complex numbers, while the letter r will be a generic symbol for nonnegative real numbers. 0, 0', and 0" will represent the zero vectors of X, X*, and X** respectively. $R(\alpha)$, where α is any scalar, will denote the real part of α.

$$J(\overline{S_1}^{\mathcal{J}}) = \overline{JS_1}^{\mathcal{J}_2} = JX \cap \overline{JS_1}^{\mathcal{J}_1} = JX \cap \overline{(JX \cap S_1'')}^{\mathcal{J}_1} \subseteq \overline{S_1''}^{\mathcal{J}_1}.$$

Again using the isometry property of J, we see that the above inclusion implies the result stated in the theorem.

<u>Theorem</u> <u>26</u>. Let $(X, \|\cdot\|)$ be a normed linear space and let V' be a linear subspace of X*. Let r_3 and r_7 be defined by (22) and (27) respectively. If $(X, \|\cdot\|)$ is reflexive, then $r_3 = r_7$.

Proof. If X is reflexive, JX = X** and $JS_1 = S_1''$. Then J is a linear homeomorphism of X with topology $\mathcal{J}$ onto X** with topology $\mathcal{J}_1$. Therefore,

$$J(\overline{S_1}^{\mathcal{J}}) = \overline{JS_1}^{\mathcal{J}_1} = \overline{S_1''}^{\mathcal{J}_1}$$

and since J is an isometry,

$$\sup_{x \in \overline{S_1}^{\mathcal{J}}} \|x\| = \sup_{x'' \in \overline{S_1''}^{\mathcal{J}_1}} \|x''\|$$

or

$$r_3 = r_7 .$$

SUMMARY

In the conjugate X* of a normed linear space $(X, \|\cdot\|)$, a norm can be defined by the formula

$$(42) \qquad \|x'\| = \sup_{\|x\| \leq 1} |x'x| \quad \text{for all } x' \in X^*.$$

It is a consequence of the Hahn-Banach theorem that then

$$(43) \qquad \|x\| = \sup_{\|x'\| \leq 1} |x'x| \quad \text{for all } x \in X.$$

If $\|\cdot\|_0$ is a norm in X equivalent to $\|\cdot\|$, the two norms defined in X* by $\|\cdot\|_0$ and $\|\cdot\|$ through the process indicated in (42) will also be equivalent.

These facts suggest the following problem. If $\|\cdot\|_1$ is a norm on X* and, paralleling (43), we define

$$(44) \qquad \|x\|_2 = \sup_{\|x'\|_1 \leq 1} |x'x| \quad \text{for all } x \in X,$$

is this new function $\|\cdot\|_2$ a norm on X? $\|\cdot\|_2$ is indeed a norm on X for any norm $\|\cdot\|_1$ which generates a finer topology on X* than $\|\cdot\|$ does. Further, if $\|\cdot\|_1$ is equivalent to $\|\cdot\|$, as defined by (42), on X*, then $\|\cdot\|_2$ is equivalent to $\|\cdot\|$ on X.

When $\|\cdot\|_1$ is equivalent to $\|\cdot\|$ on X*, if we take the conjugate of X with the norm $\|\cdot\|_2$, we obtain a norm $\|\cdot\|_2$ on X* with the property $\|x'\|_2 \leq \|x'\|_1$ for all $x' \in X^*$. The norms $\|\cdot\|_2$ and $\|\cdot\|_1$ are actu-

ally the same on X* if and only if $\{x' : \|x'\|_1 \leq 1\}$ is weak* closed in X*. That is, if (44) is used to define from $\|\cdot\|_1$ in X* a norm in X, the conjugate of X with the resulting norm $\|\cdot\|_2$ is X* with the norm $\|\cdot\|_1$ if and only if the unit sphere in the $\|\cdot\|_1$ topology of X* is weak* closed.

An example demonstrates that, if X is the conjugate of some non-reflexive Banach space, then it is possible to define a norm equivalent to the original norm $\|\cdot\|$, but with its unit sphere not closed in the weak* topology. For such a norm $\|\cdot\|_1$, the process described in the previous paragraph would not lead back to $\|\cdot\|_1$.

These problems can be raised to a different plane by considering, instead of one norm equivalent to $\|\cdot\|$, all possible norms in X* equivalent to $\|\cdot\|$. If X is a reflexive space with norm $\|\cdot\|$, then the unit sphere of any norm in X* equivalent to $\|\cdot\|$ is weak* closed. Thus, in view of the example mentioned in the previous paragraph, in the class of spaces which are the conjugate of a Banach space, those which are reflexive are characterized by the property "the unit sphere of any norm equivalent to $\|\cdot\|$ in X* is weak* closed."

The last mentioned result is not quite satisfying since it applies only to spaces which are themselves already conjugate spaces. If this restriction could be removed, a new characterization of reflexivity might result.

More steps can be placed in this procedure of going from X* to X (by (44)) and returning by taking the conjugate. We might start in X** and move successively to norms in X* and X by following the process in (44) twice, and then taking the conjugate twice. If we start with a norm $\|\cdot\|_1$ in X** equivalent to $\|\cdot\|$, we would arrive back at X** with some norm equivalent to $\|\cdot\|_1$. Would we actually retrieve $\|\cdot\|_1$ in this way? From the results listed above and some elementary considerations, this will be true if and only if (a) the unit sphere in X** for $\|\cdot\|_1$ is weak* closed in X** (i.e., in the weak topology on X** generated by X*) and (b) the set $\{x' : |x''x'| \leq 1 \text{ for all } x'' \text{ with } \|x''\|_1 \leq 1\}$ is also weak* closed in X* (i.e., in the weak topology on X* generated by X). When (b) will be satisfied seems obscure. An illuminating equivalent restatement of (b) would be very helpful.

The second part of this paper discusses the "characteristic" of a linear subspace V' of the conjugate X* of a normed linear space $(X, \|\cdot\|)$. This concept was introduced by Dixmier (2). Seven quantities enter here:

$$r_1 = \sup \{ r : S_r' \subseteq \overline{V' \cap S_1'} \},$$

$$r_2 = \inf_{x \neq 0} \sup_{x' \in V' \cap S_1'} \frac{|x'x|}{\|x\|},$$

$$r_3 = \sup_{x \in \overline{S_1}^{\mathfrak{J}}} \|x\|,$$

$$r_4 = \inf_{x \neq 0,\ z'' \in V'^{+}} \frac{\|Jx + z''\|}{\|x\|},$$

$$r_5 = \inf_{x'' \neq 0''} \sup_{x' \in \overline{V' \cap S_1'}} \frac{|x''x'|}{\|x''\|},$$

$$r_6 = \sup_{x'' \in \overline{JS_1}^{\mathfrak{J}_2}} \|x''\| ,$$

and

$$r_7 = \sup_{x'' \in \overline{S_1''}^{\mathfrak{J}_1}} \|x''\| .$$

In these formulas J is the canonical mapping from X into X**, V'^{+} is the annihilator of V' in X**, $\mathfrak{J}$ is the weak topology on X generated by V' and $\mathfrak{J}_1$ is the weak topology on X** generated by V'. $\mathfrak{J}_2$ is the topology on JX induced by $\mathfrak{J}_1$.

By definition, r_1 is the characteristic of V'. The following equations hold:

$$r_1 = r_2 = (1/r_3) = r_4 = r_5 = (1/r_6).$$

Each of the quantities r_1 to r_6 has a "geometric" meaning. When $r_1 > 0$, $\overline{V' \cap S_1'}$ is absorbing. If $\|\cdot\|_1$ denotes its Minkowski functional, then $\|\cdot\|_1$ is equivalent to $\|\cdot\|$ on X*. If $\|x\|_2 = \sup_{x' \in V' \cap S_1'} |x'x|$, then, when $r_1 > 0$, $\|\cdot\|_2$ is a norm on X and $(X^*, \|\cdot\|_1)$ is the conjugate of $(X, \|\cdot\|_2)$. If j_{01} is the identity mapping from $(X^*, \|\cdot\|)$ onto $(X^*, \|\cdot\|_1)$ and i_{20} is the identity mapping from $(X, \|\cdot\|_2)$ onto $(X, \|\cdot\|)$, then $\|j_{01}\| = (1/r_1)$ and $\|i_{20}\| = (1/r_2) = r_3$.

If we define a mapping K from X into V'* by $(Kx)x' = x'x$ for all $x' \in V'$ and $x \in X$, then K has a continuous inverse on its range if and only if $r_4 \neq 0$. If $r_4 \neq 0$ and K^{-1} is the inverse of K on its range, then $\|K^{-1}\| = (1/r_4)$.

When $r_1 > 0$, $1/r_5$ is the norm of the identity mapping from $(X^{**}, \|\cdot\|_1)$, the conjugate of $(X^*, \|\cdot\|_1)$, onto $(X^{**}, \|\cdot\|)$, the conjugate of $(X^*, \|\cdot\|)$. Also in this case, r_6 is the norm of the identity mapping from $(JX, \|\cdot\|_1)$ into $(X^{**}, \|\cdot\|)$.

The quantity r_7 is not usually equal to r_3, although they might have been expected to be equal because of the similarity of their definitions. The connection between r_3 and r_7 could be more extensively studied.

These interpretations connecting the various formulas for the characteristic with mappings may simplify present (see (2) and (7, pp. 201-208)) or future applications of the characteristic.

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An underlined number enclosed in parentheses will refer to the item in the bibliography which has the same number. Usually page numbers for the reference will be included in the parentheses.

CHAPTER I

PRELIMINARIES

Throughout this paper $(X, \|\cdot\|)$ will denote a normed linear space, over the real or complex field, with norm $\|\cdot\|$. X^* will denote the set of continuous linear functions x' mapping $(X, \|\cdot\|)$ into its scalar field. The usual norm defined on X^* as in formula (0) will also be denoted by $\|\cdot\|$. Thus $(X^*, \|\cdot\|)$ is then the conjugate of $(X, \|\cdot\|)$ and $(X^{**}, \|\cdot\|)$ is similarly the conjugate of $(X^*, \|\cdot\|)$.

Let $A \subseteq X$, $B \subseteq X$, $x_o \in X$, and let β be an element of the scalar field of X. Then we define

$$\beta A = \{ x : x = \beta x_1 \text{ for some } x_1 \in A \} .$$

$$(3) \qquad x_o + A = \{ x : x = x_o + x_1 \text{ for some } x_1 \in A \} .$$

$$A + B = \{ x : x = x_1 + x_2 \text{ where } x_1 \in A \text{ and } x_2 \in B \} .$$

Similar definitions are used in X^* and X^{**}.

Let r be a nonnegative real number. We define

$$S_r = \{ x : x \in X, \|x\| \leq r \} .$$

$$S_r' = \{ x' : x' \in X^*, \|x'\| \leq r \} .$$

$$S_r'' = \{ x'' : x'' \in X^{**}, \|x''\| \leq r \} .$$

These sets will be called the "sphere of radius r" in X, X*, and X** respectively. When $r = 1$, each of these sets will be called the "unit sphere" of its space.

When it is necessary to refer to a particular norm topology, we will precede the word "topology" by the symbol for the norm, as "the $\|\cdot\|$ topology." The letter "$\mathcal{J}$" will be a generic symbol for a topology. However, we will refer to the weak topology on X* generated by X (or on X** by X*) as the "weak* topology" (7, pp. 151-154). The weak topology on X generated by X* (or on X* generated by X**) will be called the "weak topology."

A base at O' in the weak* topology on X* can be formed as follows. Let $A \subseteq X$, let ϵ be a positive real number and define

$$U'(A, \epsilon) = \{x' : |x'x| < \epsilon \quad \text{for all } x \in A\}.$$

The collection of all sets $U'(A, \epsilon)$ for finite subsets A of X and all positive ϵ is a base at O' for the weak* topology on X*. If $A' \subseteq X^*$ and ϵ is a positive real number, let

$$U(A', \epsilon) = \{x : |x'x| < \epsilon \quad \text{for all } x' \in A'\}.$$

The collection of all such sets, where A' is a finite subset of X* and ϵ is a positive real number, forms a base at O for the weak topology on X.

For the closure operation we will use the following conventions. A bar " — " above the symbol for a set will denote the closure of that set in the weak* topology. A bar "—" followed by a symbol "$\mathcal{J}$" for a topology, as "$-\mathcal{J}$", will indicate the closure in the topology $\mathcal{J}$.

In the four definitions contained in this paragraph, and in relations (4) to (7), let $A \subseteq X$ and $A' \subseteq X^*$. We define the <u>annihilator of A in X*</u> as

$$A^+ = \{x' : x' \in X^*,\ x'x = 0 \text{ for all } x \in A\}.$$

The <u>annihilator of A' in X</u> is

$$^+A' = \{x : x \in X,\ x'x = 0 \text{ for all } x' \in A'\}.$$

The <u>polar of A in X*</u> is

$$A^\circ = \{x' : x' \in X^*,\ R(x'x) \leq 1 \text{ for all } x \in A\}.$$

The <u>polar of A' in X</u> is

$$^\circ A' = \{x : x \in X,\ R(x'x) \leq 1 \text{ for all } x' \in A'\}.$$

The following relations are easily checked.

(4) $A_1 \subseteq A$ implies $A_1^+ \supseteq A^+$ and $A_1^\circ \supseteq A^\circ$.

(5) $A_1' \subseteq A'$ implies $^+A_1' \supseteq {}^+A'$ and $^\circ A_1' \supseteq {}^\circ A'$.

(6) $^+(A^+) \supseteq A$ and $^\circ(A^\circ) \supseteq A$.

(7) $(^+A')^+ \supseteq A'$ and $(^\circ A')^\circ \supseteq A'$.

At many points in this paper we will need the following definitions.

<u>Definition 1</u>. A subset A of a linear space Y is <u>balanced</u> if