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if A is generated by a, then each $\alpha \in \mathrm{Mult}(A)$ is completely determined by the distributive law if we know $a\alpha a$. Thus Mult(A) is the cyclic group having the same order as A and generated by ρ if ρ is defined by $a\rho a = a$.

For another example of Mult(A), suppose that A is a finitely generated free group generated by a_i, $1 \leq i \leq t$. Then Mult(A) is a free group on t^3 generators. To see this note that each element $\alpha \in \mathrm{Mult}(A)$ is determined by knowing $a_i\alpha a_j$ for all i, j. Moreover, we may choose $a_i\alpha a_j$ arbitrarily and extend α to all of A by the distributive law. No difficulties arise from this construction since A is free. If now we define elements $(i,j,k) \in \mathrm{Mult}(A)$ for each i, j, k such that $1 \leq i, j, k \leq t$ by

$$a_n(i,j,k)a_m = 0 \quad \text{if } n \neq i \text{ or } m \neq j$$
$$a_i(i,j,k)a_j = a_k$$

each $\alpha \in \mathrm{Mult}(A)$ is a linear combination of the (i,j,k). Mult(A) is then the free group on the t^3 generators (i,j,k).

By making use of Theorem 1.1 and properties of the tensor product it is possible to determine Mult(A) for groups A that are direct sums of cyclic groups. The details of this may be found in [5].

According to 2) of Theorem 1.1 every $\alpha \in \mathrm{Mult}(A)$ corresponds to a subgroup K of A which is the kernel of a homomorphism of A into End(A). The following result gives necessary and sufficient conditions for a subgroup K of A

to determine an element $\alpha \in \mathrm{Mult}(A)$.

<u>Theorem 1.2</u>. Let K be a subgroup of a group A. Then K is the kernel of a homomorphism of A into End(A), i. e. K determines an element $\alpha \in \mathrm{Mult}(A)$, if and only if there is a subgroup $\Lambda = \{\varphi_a\}_{a \in A}$ of Hom(A/K,A) such that

1) Λ is a homomorphic image of A;

2) $\bigcap_x \mathrm{Ker}(\varphi_x) = K$.

<u>Proof</u>. Suppose K is the kernel of a homomorphism $\eta: A \longrightarrow \mathrm{End}(A)$. Then as in the proof of Theorem 1.1 $\eta \longrightarrow \alpha$, for $\alpha \in \mathrm{Mult}(A)$, if $a\eta = R_a^\alpha$ for each $a \in A$. Now for each $x \in A$ define $\varphi_x: A/K \longrightarrow A$ as follows:

$$(a + K)\varphi_x = xR_a^\alpha.$$

This is a mapping since $a_1 - a_2 \in K$ implies

$$R_{a_1-a_2}^\alpha = (a_1 - a_2)\eta = 0$$

or $R_{a_1}^\alpha = R_{a_2}^\alpha$ so that $xR_{a_1}^\alpha = xR_{a_2}^\alpha$ for all $x \in A$. Also

$$\begin{aligned}[(a + K) + (b + K)]\varphi_x &= [(a + b) + K]\varphi_x \\ &= xR_{a+b}^\alpha = x(R_a^\alpha + R_b^\alpha) \\ &= xR_a^\alpha + xR_b^\alpha \\ &= (a + K)\varphi_x + (b + K)\varphi_x.\end{aligned}$$

Thus each φ_x is a homomorphism of A/K into A. The set $\Lambda = \{\varphi_x\}_{x \in A}$ is a homomorphic image of A under the correspondence $x \longrightarrow \varphi_x$. For this is clearly a mapping and

$$\begin{aligned}(a + K)\varphi_{x+y} &= (x + y)R_a^\alpha = xR_a^\alpha + yR_a^\alpha \\ &= (a + K)\varphi_x + (a + K)\varphi_y \\ &= (a + K)(\varphi_x + \varphi_y)\end{aligned}$$

implies that $\varphi_{x+y} = \varphi_x + \varphi_y$. This proves 1).

To prove 2) suppose $a + K \in \bigcap_x \mathrm{Ker}(\varphi_x)$. For every $x \in A$

$$(a + K)\varphi_x = xR_a^\alpha = x(a\alpha) = 0.$$

It then follows that $R_a^\alpha = a\eta$ is the zero endomorphism and hence that $a \in K$, the kernel of η. Therefore $\bigcap_x \mathrm{Ker}(\varphi_x) = K$.

Conversely, assume the existence of the subgroup Λ of $\mathrm{Hom}(A/K,A)$ satisfying 1) and 2). For elements $a, x \in A$ define a product $x\alpha a \in A$ as follows: choose $\varphi_x \in \Lambda$ and if $a \in y + K$, set

$$x\alpha a = (y + K)\varphi_x.$$

α is obviously well-defined. For $x, y \in A$, $a \in y + K$, $z \in A$

$$\begin{aligned}(x + z)\alpha a &= (y + K)\varphi_{x+z} \\ &= (y + K)\varphi_x + (y + K)\varphi_z \\ &= x\alpha a + z\alpha a.\end{aligned}$$

If $a \in y + K$, $b \in w + K$, $x \in A$

$$\begin{aligned}x\alpha(a + b) &= [(y + w) + K]\varphi_x \\ &= [(y + K) + (w + K)]\varphi_x \\ &= (y + K)\varphi_x + (w + K)\varphi_x \\ &= x\alpha a + x\alpha b.\end{aligned}$$

We conclude that α is distributive with respect to + and that $\alpha \in \mathrm{Mult}(A)$. Also, K is the kernel of the homomorphism $\eta: A \longrightarrow \mathrm{End}(A)$ such that $a\eta = R_a^\alpha$. For if $a\eta = R_a^\alpha = 0$, we have by definition of

$$0 = xR_a^\alpha = x\alpha a = (y + K)\varphi_x \qquad \text{for all } x \in A,$$

where $a \in y + K$. But then $y + K$ is in the kernel of every φ_x. By condition 2) $y \in K$ and consequently $a \in K$. This shows that the kernel of η is contained in K and since the reverse inclusion follows from the definition of α, the assertion is proved.

If $\alpha \in \mathrm{Mult}(A)$ and $\eta \in \mathrm{End}(A)$, a product $\alpha\eta \in \mathrm{Mult}(A)$ may be defined:

$$x(\alpha\eta)y = (x\alpha y)\eta.$$

It is routine to verify that $\mathrm{Mult}(A)$ is a right module over its endomorphism ring $\mathrm{End}(A)$ with this definition of product.

We will have occasion to make use of the following results.

<u>Theorem</u> <u>1.3</u>. Let $A = B + C$ be a direct sum of the groups B and C and suppose that $\alpha \in \mathrm{Mult}(B)$. Then α may be extended to an element $\alpha^* \in \mathrm{Mult}(A)$.

<u>Proof</u>. If $\alpha \in \mathrm{Mult}(B)$ and $a_1 = b_1 + c_1$, $a_2 = b_2 + c_2$ for $b_1, b_2 \in B$, $c_1, c_2 \in C$, we may define α^* by

$$a_1\alpha^* a_2 = (b_1 + c_1)\alpha^*(b_2 + c_2) = b_1\alpha b_2.$$

α^* is obviously in $\mathrm{Mult}(A)$ and an extension of α. We call α^* the <u>trivial extension</u> of α to all of A.

<u>Theorem</u> <u>1.4</u>. If A is a divisible group and B a torsion group, then $A \otimes B = 0$.

<u>Proof</u>. Suppose $a \in A$ and $b \in B$ with b having order n. Since A is divisible there is an element $x \in A$ such that $nx = a$.

Then by the distributive property of the tensor product

$$a \otimes b = (nx) \otimes b = x \otimes (nb) = 0.$$

Hence $A \otimes B = 0$.

<u>Corollary 1</u>. If A is a torsion divisible group, then $\mathrm{Mult}(A) = 0$.

<u>Proof</u>. This follows from Theorem 1.1 since by the present theorem $A \otimes A = 0$.

<u>Corollary 2</u>. If $A = B + C$ is the direct sum of a divisible group B and a torsion group C, then any ring on A is the ring direct sum of rings on B and C.

<u>Proof</u>. If $b \in B$ and $c \in C$, we have as in the proof of the theorem $b \cdot c = 0$.

CHAPTER II

THE BEAUMONT CONDITIONS FOR p-GROUPS

The object of this chapter is to describe all possible rings on a given group A which has a basis or a quasi-basis. More precisely, we seek to determine the multiplication tables of the basal elements in much the same way as one does this for an algebra over a field. The first results of this type for groups are due to Beaumont [1]; later these were generalized by Rédei [8] to arbitrary groups with operators. The main result of this chapter is also a generalization of Beaumont's result in which all multiplications on an arbitrary torsion group are determined.

Suppose a group $A = \sum_{i\in\Lambda}\{e_i\}$ is given as the direct sum of the cyclic groups $\{e_i\}_{i\in\Lambda}$ and the orders $O(e_i) = n_i$ are finite. Then a multiplication α on A is completely determined if we know the values of $e_i\alpha e_j$ for all $i, j \in \Lambda$. Let these values $e_i\alpha e_j$ be given by

$$e_i\alpha e_j = \sum_k t_{ijk}e_k \tag{1}$$

where $i, j, k \in \Lambda$ and at most a finite number of the terms $t_{ijk}e_k$ are different from zero and t_{ijk} are integers. If $A(+,\alpha)$ is to be a ring, the distributive laws imply

$$n_i(e_i\alpha e_j) = 0$$

and

$$n_j(e_i\alpha e_j) = 0.$$

Consequently $(n_i, n_j, n_k)t_{ijk}e_k = 0$, where (n_i, n_j, n_k) is the greatest common divisor of the integers n_i, n_j, n_k. Hence

(2) $$t_{ijk} \equiv 0 \mod \frac{n_k}{(n_i, n_j, n_k)}.$$

If $A(+,\alpha)$ is to be an associative ring, we must have, for all $i, j, k \in \Lambda$, $(e_i \alpha e_j)\alpha e_k = e_i \alpha (e_j \alpha e_k)$ or

$$(\sum_r t_{ijr}e_r)\alpha e_k = e_i \alpha (\sum_r t_{jkr}e_r)$$

$$\sum_r t_{ijr}e_r \alpha e_k = \sum_r t_{jkr}e_i \alpha e_r$$

$$\sum_r t_{ijr}(\sum_s t_{rks}e_s) = \sum_r t_{jkr}(\sum_s t_{irs}e_s)$$

$$\sum_s (\sum_r t_{ijr}t_{rks})e_s = \sum_s (\sum_r t_{jkr}t_{irs})e_s.$$

Therefore, for each i, j, k, s

(3) $$\sum_r t_{ijr}t_{rks} \equiv \sum_r t_{ikr}t_{irs} \mod (n_s).$$

Since $(e_i \alpha e_j)\alpha e_k = \sum_s (\sum_r t_{ijr}t_{rks})e_s$, $A(+,\alpha)$ will not be a Lie ring unless

(4) $$\sum_r t_{ijr}t_{rks} + \sum_r t_{jkr}t_{ris} + \sum_r t_{kir}t_{rjs} \equiv 0 \mod (n_s).$$

The identity (4) is the condition imposed by the Jacobi identity: $(x\alpha y)\alpha z + (y\alpha z)\alpha x + (z\alpha x)\alpha y = 0$. But to be a Lie ring $A(+,\alpha)$ must also satisfy $x\alpha x = 0$ and $x\alpha y = -(y\alpha x)$. These two conditions lead to the following:

(5) $$t_{iik} \equiv 0 \mod (n_k)$$

(6) $$t_{ijk} \equiv -t_{jik} \mod (n_k).$$

One can also show that the above conditions are

sufficient to insure that $A(+,\alpha)$ satisfies the appropriate laws.

<u>Theorem 2.1</u>. Let $A = \sum_{i \in \Lambda} \{e_i\}$ be a group on which there is defined a binary composition α such that

$$e_i \alpha e_j = \sum_k t_{ijk} e_k$$

for integers $i, j, k \in \Lambda$. Then

i. (Beaumont [1]) $A(+,\alpha)$ is a ring if and only if condition (2) holds;

ii. (Beaumont [1]) $A(+,\alpha)$ is an associative ring if and only if conditions (2) and (3) hold;

iii. $A(+,\alpha)$ is a Lie ring if and only if conditions (2), (4), (6) and (7) hold.

Of course the statement of Theorem 2.1 could be extended to include other kinds of rings.

We wish now to consider multiplications on an arbitrary torsion group A. Here the problem may be simplified by resolving A into its p-components. The simplification arises from the fact that if $A(+,\alpha)$ is a ring, then $a\alpha b = 0$ when a, b are from different p-components of A. Hence $A(+,\alpha)$ is the ring direct sum of its p-components. We will therefore confine our attention to p-groups.

The notion of pure subgroup and basic subgroup are fundamental in the study of p-groups.

<u>Definition</u>. A subgroup B is <u>pure</u> in a group A if solvability of $nx = b \in B$ in A implies solvability in B (n is an integer).

<u>Definition</u>. (Kulikov [7]) If A is a p-group, then B is a <u>basic</u> subgroup of A if and only if

(a) B is a direct sum of cyclic groups;

(b) B is pure in A;

(c) A/B is divisible.

Kulikov [7] has proved the existence of basic subgroups for every p-group.

Let B be a basic subgroup of a p-group A. A/B is a p-group and by definition is divisible. It then follows by the structure theory for divisible groups that

$$A/B = \sum C_\mu, \quad C_\mu \cong \mathcal{C}(p^\infty).$$

Each C_μ is generated by cosets $\bar{c}_\mu^{(1)}, \bar{c}_\mu^{(2)}, \ldots, \bar{c}_\mu^{(n)}, \ldots$ satisfying the defining relations: $p\bar{c}_\mu^{(1)} = \bar{0}$, $p\bar{c}_\mu^{(2)} = \bar{c}_\mu^{(1)}$, $\ldots, p\bar{c}_\mu^{(n+1)} = \bar{c}_\mu^{(n)}, \ldots$ A fundamental result for pure subgroups is that if $\bar{x} \in A/B$, where B is pure in A, then there is an element $y \in \bar{x}$ such that $O(y) = O(\bar{x})$. Applying this result in the present case we choose from each $\bar{c}_\mu^{(n)}$ an element $c_\mu^{(n)}$ such that $p^n c_\mu^{(n)} = 0$. Because of the relation $p\bar{c}_\mu^{(n+1)} = \bar{c}_\mu^{(n)}$, $pc_\mu^{(n+1)} \in \bar{c}_\mu^{(n)}$ and $pc_\mu^{(n+1)} = c_\mu^{(n)} - b_\mu^{(n)}$ for

some $b_\mu^{(n)} \in B$ when $n > 1$. For $n = 0$, $pC_\mu^{(1)} = 0$. Suppose $a \in A$, $a + B \in A/B$. Then

$$a + B = a_1\bar{C}_{\mu_1}^{(n_1)} + a_2\bar{C}_{\mu_2}^{(n_2)} + \cdots + a_t\bar{C}_{\mu_t}^{(n_t)}$$

where the a_i are assumed not to be divisible by p. Having chosen the representatives for each component C_μ in the manner described above, we have

$$a + B = (a_1C_{\mu_1}^{(n_1)} + a_2C_{\mu_2}^{(n_2)} + \cdots + a_tC_{\mu_t}^{(n_t)}) + B.$$

Hence $a - (a_1C_{\mu_1}^{(n_1)} + a_2C_{\mu_2}^{(n_2)} + \cdots + a_tC_{\mu_t}^{(n_t)}) \in B$. Taking a basis $\{b_\lambda\}$ for B

$$a - (a_1C_{\mu_1}^{(n_1)} + \cdots + a_tC_{\mu_t}^{(n_t)}) = k_1b_{\lambda_1} + k_2b_{\lambda_2} + \cdots + k_vb_{\lambda_v}$$

or

$$a = k_1b_{\lambda_1} + k_2b_{\lambda_2} + \cdots + k_vb_{\lambda_v} + a_1C_{\mu_1}^{(n_1)} + \cdots + a_tC_{\mu_t}^{(n_t)}.$$

If such an expression were 0, it would follow that

$$a_1C_{\mu_1}^{(n_1)} + a_2C_{\mu_2}^{(n_2)} + \cdots + a_tC_{\mu_t}^{(n_t)} \in B$$

so that

$$a_1\bar{C}_{\mu_1}^{(n_1)} + a_2\bar{C}_{\mu_2}^{(n_2)} + \cdots + a_t\bar{C}_{\mu_t}^{(n_t)} = \bar{0}.$$

But A/B is a direct sum and hence

$$a_1\bar{C}_{\mu_i}^{(n_i)} = \bar{0} \text{ and } a_1C_{\mu_i}^{(n_i)} = 0.$$

Then

$$k_1b_{\lambda_1} + k_2b_{\lambda_2} + \cdots + k_vb_{\lambda_v} = 0$$

and since the b's are a basis for B, $k_1b_{\lambda_i} = 0$. We may

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then conclude that the elements $k_i b_{\lambda_i}$ and $a_i c_{\mu_i}^{(n_i)}$ are uniquely determined by the element $a \in A$. The set $\{\{b_\lambda\}, \{c_\mu^{(n)}\}\}$ is called a <u>quasi-basis</u> for A. This proves the following theorem of Fuchs [2].

<u>Theorem 2.2</u>. Let A be a p-group and B be a basic subgroup of A. If $\{\{b_\lambda\}, \{c_\mu^{(n)}\}\}$ is a quasi-basis for A, then each element $a \in A$ can be expressed as

$$a = k_1 b_{\lambda_1} + k_2 b_{\lambda_2} + \cdots + k_v b_{\lambda_v} + a_1 c_{\mu_1}^{(n_1)} + \cdots + a_t c_{\mu_t}^{(n_t)}$$

where the k_i and the a_j are integers with the a_j not divisible by p. Moreover, the terms $k_i b_{\lambda_i}$ and $a_j c_{\mu_j}^{(n_j)}$ are uniquely determined by the element a in A.

Let B, C be subgroups of the groups G and H respectively. In general it is not possible to embed $B \otimes C$ isomorphically in $G \otimes H$ in the natural way. However, if B, C are pure in G, H respectively, then this can be done. By definition basic subgroups are pure so that we may identify $B \otimes C$ with its isomorphic image in $G \otimes H$. Also by definition of pure subgroup G/B and H/C are both divisible. We may then solve the rquations

$$n(x + B) = g + B$$

$$m(y + C) = h + C$$

where n, m are non-zero integers. In particular, we may choose integers k and t such that $k \geq E(h)$, $t \geq E(b)$ and

$$p^k x + b = g$$
$$p^t y + c = h$$

for certain elements b and c in B and C respectively. Then

$$\begin{aligned} g\otimes h &= (p^k + b)\otimes h = (p^k x\otimes h + (b\otimes h) \\ &= x\otimes(p^k h) + b\otimes(p^t y + c) \\ &= b\otimes(p^t y) + (b\otimes c) = (p^t b)\otimes y + b\otimes c = b\otimes c. \end{aligned}$$

This shows that $G\otimes H \subseteq B\otimes C$ and proves the following.

<u>Lemma 2.1</u>. (Fuchs [4]). If B, C are basic subgroups of the p-groups G, H respectively, then $G\otimes H \cong B\otimes C$.

As a special case of this lemma, $B\otimes B \cong A\otimes A$ when B is a basic subgroup of A. Then since $\mathrm{Mult}(A) \cong \mathrm{Hom}(A\otimes A, A)$,

$$\mathrm{Mult}(A) \cong \mathrm{Hom}(B\otimes B, A).$$

<u>Theorem 2.3</u>. (Fuchs [5]). Let A be a p-group, B a basic subgroup of A and $\alpha \in \mathrm{Mult}(A)$. Then for any $x, y \in A$ there are elements $b, c \in B$ such that $x\alpha y = b\alpha c$. Moreover, any mapping of $B\otimes B$ into A may be extended to a multiplication on A.

<u>Proof</u>. Let $\alpha \in \mathrm{Mult}(A)$ and $x, y \in A$. Then there is a homomorphism η of $A\otimes A$ into A such that $x\alpha y = (x\otimes y)\eta$. But by the proof of Lemma 2.1 there are elements b, c in B such that $x\otimes y = b\otimes c$. Hence $x\alpha y = (x\otimes y)\eta = (b\otimes c)\eta = b\alpha c$. This proves the first part of the theorem. If a mapping of $B\otimes B$ into A is given, we may use the isomorphism $A\otimes A \cong B\otimes B$ to define $x\alpha y = b\alpha c$ if $x\otimes y$ corresponds to $b\otimes c$ for $x, y \in A$.

<u>Lemma 2.2</u>. Let A be a p-group and let B be a basic subgroup of A. If $\alpha \in \mathrm{Mult}(A)$ and α is associative on B, then α is associative on all of A.

<u>Proof</u>. The proof of this is similar to the proof of Lemma 2.1. Here, if a, b, c are elements of A, we may choose k as the maximum of the integers E(a), E(b), E(c) and find elements d, e, f $\in$ B such that

$$p^k x + d = a$$
$$p^k y + e = b$$
$$p^k z + f = c$$

where x, y, z are certain elements of A. It then follows that

$$(a\alpha b)\alpha c = d\alpha e\alpha f = a\alpha(b\alpha c).$$

Now if A is a p-group and $B = \sum_i \{e_i\}$ is a basic subgroup of A, we may determine the multiplications on A by determining the values $e_i \alpha e_j$. Note that the values $e_i \alpha e_j$ may not be in B. Choose a quasi-basis $\{\{e_i\}, \{c_\mu^{(n)}\}\}$ for A and suppose the values $e_i \alpha e_j$ are given as follows:

$$(9) \qquad e_i \alpha e_j = \sum t_{ij\epsilon} e_\epsilon + \sum f_{ij\delta} c_{\mu_\delta}^{(n_\delta)}$$

where $t_{ij\epsilon}$, $f_{ij\delta}$ are integers and $p \nmid f_{ij\delta}$.

Just as in the previous case the integers $t_{ij\epsilon}$ must satisfy the condition (2). We also must require

$$(10) \qquad E(c_{\mu_\delta}^{(n_\delta)}) \leq E(e_i) \text{ and } E(c_{\mu_\delta}^{(n_\delta)}) \leq E(e_j).$$

Condition (10) results from the fact that $p^{E(e_i)}(e_i \alpha e_j) = 0$ and hence $p^{E(e_i)} f_{ij\delta} C_{\mu_\delta}^{(n_\delta)} = 0$ so that $p^{E(C_{\mu_\delta}^{(n_\delta)})} \mid p^{E(e_i)} f_{ij\delta}$. But since $p \nmid f_{ij\delta}$, $p^{E(C_{\mu_\delta}^{(n_\delta)})} \mid p^{E(e_i)}$. The other part of (10) is obtained similarly.

In order to obtain the conditions for associativity we compute $(e_i \alpha e_j) \alpha e_k$:

$$(e_i \alpha e_j) \alpha e_k = (\sum_s t_{ijs} e_s + \sum_r f_{ijr} C_{\mu_r}^{(n_r)}) \alpha e_k$$

$$= \sum_s t_{ijs} e_s \alpha e_k + \sum_r f_{ijr} C_{\mu_r}^{(n_r)} \alpha e_k$$

$$= \sum_s t_{ijs} (\sum_m t_{skm} e_m + \sum_w f_{skw} C_{\mu_w}^{(n_w)}) + \sum_r f_{ijr} C_{\mu_r}^{(n_r)} \alpha e_k. \qquad (1)$$

By Theorem 2.3 we may choose $b_r = \sum_\tau v_\tau^{(r)} e_\tau \in B$ for each r such that

$$(*) \qquad C_{\mu_r}^{(n_r)} \alpha e_k = b_r \alpha e_k = (\sum_\tau v_\tau^{(r)} e_\tau) \alpha e_k.$$

Using the relation given by (*) in (1)

$$(e_i \alpha e_j) \alpha e_k = \sum_m (\sum_s t_{ijs} t_{skm}) e_m + \sum_w (\sum_s t_{ijs} f_{skw}) C_{\mu_w}^{(n_w)}$$

$$+ \sum_r f_{ijr} (\sum_\tau v_\tau^{(r)} e_\tau \alpha e_k)$$

$$= \sum_m (\sum_s t_{ijs} t_{skm}) e_m + \sum_w (\sum_s t_{ijs} f_{skw}) C_{\mu_w}^{(n_w)}$$

$$+ \sum_r f_{ijr} (\sum_\tau v_\tau^{(r)} [\sum_s t_{\tau ks} e_s + \sum_\sigma f_{\tau k\sigma} C_{\mu_\sigma}^{(n_\sigma)}])$$

$$= \sum_m (\sum_s t_{ijs} t_{skm}) e_m + \sum_w (\sum_s t_{ijs} f_{skw}) C_{\mu_w}^{(n_w)}$$

$$+ \sum_s [\sum_\tau (\sum_r v_\tau^{(r)} t_{\tau ks} f_{ijr})] e_s + \sum_\sigma [\sum_\tau (\sum_r v_\tau^{(r)} f_{ijr} f_{\tau k\sigma})] C_{\mu_\sigma}^{(n_\sigma)}$$

and finally

$$(e_i \alpha e_j)\alpha e_k = \sum_m \Big[\sum_s t_{ijs} t_{skm} + \sum_\tau \big(\sum_r v_\tau^{(r)} t_{\tau km} f_{ijr}\big)\Big] e_m$$
$$+ \sum_w \Big[\sum_s t_{ijs} f_{skw} + \sum_\tau \big(\sum_r v_\tau^{(r)} f_{ijr} f_{\tau kw}\big)\Big] C_{\mu_w}^{(n_w)}.$$

In like manner we find

$$e_i \alpha (e_j \alpha e_k) = \sum_m \Big[\sum_s t_{jks} t_{ism} + \sum_\tau \big(\sum_r v_\tau^{(r)} f_{jkr} t_{i\tau m}\big)\Big] e_m$$
$$+ \sum_w \Big[\sum_s t_{jks} f_{isw} + \sum_\tau \big(\sum_r \tilde{v}_\tau^{(r)} f_{jkr} f_{i\tau w}\big)\Big] C_{\mu_w}^{(n_w)}$$

where

$$(**) \qquad e_i \alpha C_{\mu_r}^{(n_r)} = e_i \alpha \sum_\tau \tilde{v}_\tau^{(r)} e_\tau.$$

By using Lemma 2.2 these conditions can also be shown to be sufficient so that we conclude that a necessary and sufficient condition for α to be associative is that the following hold:

$$(11) \qquad \sum_s t_{ijs} t_{skm} + \sum_\tau \big(\sum_r v_\tau^{(r)} t_{\tau km} f_{ijr}\big) \equiv$$
$$\sum_s t_{jks} t_{ism} + \sum_\tau \big(\sum_r \tilde{v}_\tau^{(r)} t_{jkr} t_{i\tau m}\big) \qquad \mathrm{mod}(p^{E(e_m)})$$

$$(12) \qquad \sum_s t_{ijs} f_{skw} + \sum_\tau \big(\sum_r v_\tau^{(r)} f_{ijr} f_{\tau kw}\big) \equiv$$
$$\sum_s t_{jks} f_{isw} + \sum_\tau \big(\sum_r \tilde{v}_\tau^{(r)} f_{jkr} f_{i\tau w}\big) \qquad \mathrm{mod}(p^{E(C_{\mu_w}^{(n_w)})}).$$

<u>Theorem 2.4</u>. Let A be a p-group and let $B = \sum_i \{e_i\}$ be a basic subgroup of A. If $\{\{e_i\}, \{C_\mu^{(n)}\}\}$ is a quasi-basis for A and α is a binary composition on A given by (9), then

(a) $A(+,\alpha)$ is a ring if and only if (2) and (10) hold;

(b) $A(+,\alpha)$ is an associative ring if and only if

(2), (10), (11) and (12) hold.

It is also possible to obtain necessary and sufficient conditions for $A(+,\alpha)$ to be a Lie ring.

CHAPTER III

ASSOCIATIVE-CLOSED GROUPS

1. <u>Introduction</u>. Let P be a ring property and define

$$P(A) = \{\alpha \mid \alpha \in \mathrm{Mult}(A),\ A(+,\alpha) \text{ has property } P\}.$$

In [5] Fuchs asks for a characterization of those groups A such that P(A) is a subgroup of Mult(A). In this chapter we solve this problem for groups with a basis or quasi-basis when P is the associative property $\mathcal{A}$. If P(A) is a subgroup of Mult(A), we shall say that A is <u>P-closed</u>.

<u>Lemma 2.1</u>. If $\alpha, \beta \in \mathcal{A}(A)$, then $\alpha \circledast \beta \in \mathcal{A}(A)$ if and only if for all x, y, z $\in$ A

$$(x\alpha y)\beta z + (x\beta y)\alpha z = x\alpha(y\beta z) + x\beta(y\alpha z).$$

<u>Proof</u>. Suppose $\alpha \circledast \beta \in \mathcal{A}(A)$ for $\alpha, \beta \in \mathcal{A}(A)$. Then for all x, y, z $\in$ A

$$[x(\alpha \circledast \beta)y](\alpha \circledast \beta)z = x(\alpha \circledast \beta)[y(\alpha \circledast \beta)z].$$

Using the definition of $\alpha \circledast \beta$ we obtain

$$(x\alpha y + x\beta y)\alpha z + (x\alpha y + x\beta y)\beta z = x\alpha(y\alpha z + y\beta z) + x\beta(y\alpha z + y\beta z).$$

Since both α and β are distributive over +, the above equation becomes

$$(x\alpha y)\alpha z + (x\alpha y)\beta z + (x\beta y)\alpha z + (x\beta y)\beta z =$$
$$x\alpha(y\alpha z) + x\alpha(y\beta z) + x\beta(y\alpha z) + x\beta(y\beta z).$$

Now since $\alpha, \beta \in \mathcal{A}(A)$, this last identity is equivalent to the identity of our lemma. This identity is also clearly

sufficient to ensure that $\alpha \oplus \beta \in \mathcal{A}(A)$.

<u>Definition</u>. If $\alpha \in \mathcal{A}(A)$ and $\varphi_1, \varphi_2 \in \mathrm{End}(A)$, the mapping of $A \times A$ into A is defined by

$$x(\varphi_1 \alpha \varphi_2)y = (x\varphi_1)\alpha(y\varphi_2)$$

for all $x, y \in A$.

If $\alpha \in \mathrm{Mult}(A)$, $a \in A$ we have the right and left multiplication endomorphisms R_a^α and L_a^α determined by a and α. We write α_a for $R_a^\alpha \alpha I$, I the identity endomorphism on A. Here we have for $x, y \in A$

$$x\alpha_a y = x\alpha a\alpha y$$

when α is associative.

<u>Lemma 2.2</u>. Let $\alpha, \beta \in \mathcal{A}(A)$ and $\varphi_1, \varphi_2 \in \mathrm{End}(A)$. Then

(a) $\varphi_1 \alpha \varphi_2 \in \mathrm{Mult}(A)$;

(b) $\alpha_a \in \mathcal{A}(A)$ for each $a \in A$;

(c) $\{\alpha_a \mid a \in A\}$ is a subgroup of $\mathrm{Mult}(A)$ of associative multiplications and is a homomorphic image of A;

(d) if the ring $A(+,\alpha)$ contains no element $c \neq 0$ such that $x\alpha c\alpha y = 0$ for all $x, y \in A$, then $\{\alpha_a\}$ is isomorphic to A.

<u>Proof</u>. (a) For $x, y, z \in A$

$$\begin{aligned}(x + y)(\varphi_1 \alpha \varphi_2)z &= [(x + y)\varphi_1]\alpha(z\varphi_2) \\ &= [(x\varphi_1) + (y\varphi_1)]\alpha(z\varphi_2) \\ &= (x\varphi_1)\alpha(z\varphi_2) + (y\varphi_1)\alpha(z\varphi_2).\end{aligned}$$

The other distributive law is similarly obtained so that $\varphi_1 \alpha \varphi_2 \in \mathrm{Mult}(A)$.

(b) We have by definition of α_a

$$(x\alpha_a y)\alpha_a z = (x\alpha a\alpha y)\alpha a\alpha z = x\alpha a\alpha(y\alpha a\alpha z)$$
$$= x\alpha a\alpha(y\alpha_a z) = x\alpha_a(y\alpha_a z).$$

(c) Let $\alpha_a, \alpha_b \in \{\alpha_a\}$. Then

$$x(\alpha_a \oplus \alpha_b)y = x\alpha_a y + x\alpha_b y$$
$$= x\alpha a\alpha y + x\alpha b\alpha y$$
$$= x\alpha(a + b)\alpha y = x\alpha_{(a+b)}y.$$

Hence $\{\alpha_a\}$ is a subgroup of associative multiplications and the mapping $a \rightarrow \alpha_a$ is a homomorphism of A onto $\{\alpha_a\}$.

(d) We show that the homomorphism $a \rightarrow \alpha_a$ is 1-1 under the given hypothesis. Suppose $\alpha_a = \alpha_b$. Then for all $x, y \in A$

$$x\alpha_a y = x\alpha_b y$$

and

$$x\alpha a\alpha y = x\alpha b\alpha y.$$

This implies that $x\alpha(a - b)\alpha y = 0$ and it follows that $a = b$.

2. <u>Groups with a basis</u>. Assume the group A has a basis and is given as

$$A = \sum_{i \in \Lambda} \{e_i\}$$

a direct sum of the cyclic groups $\{e_i\}$. A multiplication is completely determined here if we know the products $e_i \alpha e_j$.

<u>Definition</u>. If the group $A = \sum_{i \in \Lambda} \{e_i\}$, a binary composition ω is called an <u>orthogonal</u> multiplication on A if it is defined as follows:

$$e_i \omega e_j = \delta_{ij} t_i e_i$$

for all $i, j \in \Lambda$ and δ_{ij} is the Kronecker delta, t_i is an integer.

<u>Lemma 2.3.</u> If $\mathfrak{A}$ is the set of all orthogonal multiplications on a group A, then $\mathfrak{A} \subseteq \mathcal{A}(A)$ and $\mathfrak{A}$ is a subgroup of Mult(A).

<u>Proof.</u> That $\mathfrak{A} \subseteq \mathcal{A}(A)$ is clear from the definition of elements in $\mathfrak{A}$. To prove that $\mathfrak{A}$ is a group, let $\omega, \beta \in \mathfrak{A}$. Then

$$\begin{aligned} e_i(\omega \oplus \beta)e_j &= e_i\omega e_j + e_i\beta e_j \\ &= \delta_{ij}t_i e_i + \delta_{ij}t_i' e_i \\ &= \delta_{ij}(t_i + t_i')e_i \end{aligned}$$

and $\omega \oplus \beta \in \mathfrak{A}$. Also

$$\delta_{ij}(-t_i)e_i = -(e_i\omega e_j) = e_i(\theta\omega)e_j$$

so that $\theta\omega \in \mathfrak{A}$.

<u>Lemma 2.4.</u> If ρ is the orthogonal multiplication such that

$$e_i\rho e_j = \delta_{ij}e_i$$

for all $i, j \in \Lambda$, then $A \cong \{\rho_a | a \in A\}$.

<u>Proof.</u> If A is finitely, then $A(+,\rho)$ has the unit $e = \sum e_i$ and the isomorphism $A \cong \{\rho_a\}$ follows at once from Lemma 2.2 part (d). It is not necessary to assume this finiteness, however. Let $f: A \to \rho_a$ be defined as above: $f(a) = \rho_a$. Let $a, b \in A$ and $a = \sum k_i e_i$, $b = \sum k_i' e_i$. Suppose that $\rho_a = \rho_b$. Then for each e_i

$$\begin{aligned} e_i\rho_a e_i &= e_i\rho_b e_i \\ e_i\rho a\rho e_i &= e_i\rho b\rho e_i. \end{aligned}$$

By the definition of ρ this reduces to $k_i e_i = k_i' e_i$ for each i. Hence a = b and f is 1-1.

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It follows from Lemma 2.3 that if A is a cyclic group, $\mathcal{A}(A)$ is a subgroup of Mult(A). For in this case $\mathcal{A}(A)$ is the same as Mult(A). When the group A has more than one generator we have the following.

<u>Theorem 3.1</u>. Let $A = \sum_{i \in \Lambda} \{e_i\}$ be a torsion group of rank $r > 1$ and let n_i be the order of e_i. Then $\mathcal{A}(A)$ is a subgroup of Mult(A) if and only if $(n_i, n_j) = 1$ for every i, j for which $i \neq j$.

<u>Proof</u>. Suppose the condition $(n_i, n_j) = 1$ holds and α is in Mult(A) and is given by

$$e_i \alpha e_j = \sum_k t_{ijk} e_k.$$

There exist integers t, s such that $1 = tn_i + sn_j$ and

$$\begin{aligned} e_i \alpha e_j &= 1 \cdot (e_i \alpha e_j) \\ &= (tn_i + sn_j) \cdot (e_i \alpha e_j) \\ &= (tn_i) \cdot (e_i \alpha e_j) + (sn_j) \cdot (e_i \alpha e_j) \\ &= (tn_i e_i) \alpha e_j + e_i \alpha (sn_j e_j) \\ &= 0 \alpha e_j + e_i \alpha 0 = 0 + 0 = 0. \end{aligned}$$

This shows immediately that any ring on A is the ring direct sum of rings on the cyclic summands of A. Since the integers t_{ijk} must satisfy the Beaumont condition (1)

$$t_{iik} \equiv 0 \quad \mathrm{mod} \frac{n_k}{(n_i, n_k)}$$

or

$$t_{iik} \equiv 0 \quad \mathrm{mod}(n_k)$$

when $i \neq k$, α is an orthogonal multiplication. Hence $\mathrm{Mult}(A) \subseteq \mathfrak{A}$. But since we always have $\mathfrak{A} \subseteq \mathcal{A}(A) \subseteq \mathrm{Mult}(A)$, for any group A, we conclude

$$\mathfrak{A} = \mathcal{A}(A) = \mathrm{Mult}(A).$$

Conversely, assume that for some $i \neq j$, we have $(n_i, n_j) = d > 1$. Now define $t_{iij} = n_j/d$ and thus a multiplication α by

$$e_i \alpha e_i = t_{iij} e_j$$
$$e_k \alpha e_m = 0 \quad \text{if } k \neq i \text{ or } m \neq i.$$

This multiplication α satisfies the Beaumont condition (1) since $t_{kmr} = 0$ for $k \neq i$ or $m \neq i$ or $r \neq j$ and

$$t_{iij} = n_j/d \equiv 0 \qquad \mathrm{mod} \frac{n_j}{(n_i, n_j)}$$

by definition of t_{iij}. This is also clearly associative. Now we have $\alpha, \rho \in \mathcal{A}(A)$ where $\rho \in \mathfrak{A}$ such that

$$e_i \rho e_j = \delta_{ij} e_i.$$

If $\mathcal{A}(A)$ is a group, we may apply Lemma 2.1 for $e_i, e_j \in A$:

$$(e_i \rho e_i)\alpha e_j + (e_i \alpha e_i)\rho e_j = e_i \rho(e_i \alpha e_j) + e_i \alpha(e_i \beta e_j).$$

By definition of α and ρ this gives

$$e_i \alpha e_j + (t_{iij} e_j)\rho e_j = e_i \rho 0 + e_i \alpha 0 = 0.$$

Hence

$$t_{iij} e_j = 0.$$

But this contradicts the order of e_j since by definition of t_{iij} we have $0 < t_{iij} < n_j$. This completes the proof.

<u>Corollary</u>. If A is a torsion associative-closed group with

a basis, then any ring $A(+,\alpha)$ is both commutative and associative.

<u>Theorem</u> 3.2. If A is a group with a basis and is not a torsion group, then A is associative-closed if and only if A is the infinite cyclic group.

<u>Proof</u>. If A is the infinite cyclic group generated by e, then for any choice of integer t

$$e\alpha e = te$$

defines a multiplication in $\mathcal{A}(A)$ and these are the only possibilities. Hence $\mathcal{A}(A)$ is a group.

Now suppose that A is a torsion-free group of rank $r > 1$ and choose any two generators e_1, e_2 of A. Define multiplications α, β as follows

$$e_1\alpha e_1 = e_2$$
$$e_i\alpha e_j = 0 \quad \text{if } i \neq 1 \text{ or } j \neq 1$$
$$e_2\beta e_2 = e_1$$
$$e_i\beta e_j = 0 \quad \text{if } i \neq 2 \text{ or } j \neq 2.$$

It is clear that $\alpha, \beta \in \mathcal{A}(A)$. If $\mathcal{A}(A)$ is a group, then $\alpha \oplus \beta \in \mathcal{A}(A)$. By Lemma 1.1, if $\alpha \oplus \beta \in \mathcal{A}(A)$, then

$$(e_1\alpha e_1)\beta e_2 + (e_1\beta e_1)\alpha e_2 = e_1\alpha(e_1\beta e_2) + e_1\beta(e_1\alpha e_2)$$

that is

$$e_2\beta e_2 + 0\alpha e_2 = e_1\alpha 0 + e_1\beta 0 = 0.$$

This implies $e_2\alpha e_2 = 0$ which is a contradiction.

If A is a mixed group with a basis, the torsion-free

part of A, say T, cannot be of rank greater than 1 if A is to be an associative-closed group; for if the rank $r > 1$, $\mathcal{A}(T)$ is not a group and there exist elements $\alpha, \beta \in \mathcal{A}(T)$ such that $\alpha \oplus \beta \notin \mathcal{A}(T)$. Now since T is a direct summand of A, we may apply Theorem 1.3 and have $\alpha^*, \beta^* \in \mathcal{A}(A)$ and $(\alpha \oplus \beta)^* = \alpha^* \oplus \beta^* \notin \mathcal{A}(A)$. Thus we assume that A has exactly one generator of infinite order. By the same reasoning, the torsion part of A must satisfy the conditions of Theorem 3.1. Let a be the generator of infinite order and define multiplications α, β as follows:

$$a\alpha e_i = e_i\alpha a = e_i\alpha e_j = 0 \text{ and } a\alpha a = a \quad \text{for all } i, j$$

$$a\beta e_i = e_i\beta a = e_i \text{ and } a\beta a = a, \; e_i\beta e_j = \delta_{ij}e_i \text{ for all } i, j.$$

If A is to be associative-closed, the identity of Lemma 1.1 must be satisfied. In particular the following must hold:

$$(e_i\alpha a)\beta a + (e_i\beta a)\alpha a = e_i\alpha(a\beta a) + e_i\beta(a\alpha a).$$

But by the definition of these multiplications this implies that $e_i = 0$ and this is a contradiction.

3. <u>Some results for groups without a basis</u>. In general the problem we are considering is more difficult for groups without a basis. Some specific cases, however, can be handled. The following theorems contain results for these specific cases.

<u>Theorem 3.3</u>. If A is a torsion-free group of rank 1, then A is an associative-closed group.

<u>Proof</u>. Any two elements of A are dependent since A has rank 1 and by a theorem of R. Baer A is isomorphic to a subgroup of the rationals. Hence we may choose a fixed non-zero element $a \in A$ and write every element $b \in A$ as $b = ra$ for some rational r. Then any multiplication is commutative since $xy = (ra)(sa) = (rs)(aa) = yx$. From this it follows that any multiplication on A is associative since $a(aa) = (aa)a$. Hence A is associative-closed.

<u>Theorem 3.4</u>. If A is a torsion-free divisible group, then A is associative-closed if and only if the rank of A is 1.

<u>Proof</u>. The condition is sufficient by Theorem 3.3. To prove that it is necessary, let r be the rank of A and assume that $r > 1$. In this case, since A is torsion-free and divisible, A is a vector space over the rationals and any multiplication on A gives us a linear algebra over the rationals. A multiplication is then completely determined by products of the basis elements. Let $(e_i)_{i \in \Lambda}$ be a basis and since $r > 1$ we may choose elements $e_1, e_2 \in (e_i)$ and define multiplications as follows:

$$e_1 \alpha e_1 = e_2$$
$$e_i \alpha e_j = 0 \quad \text{if } i \neq 1 \text{ or } j \neq 1$$
$$e_2 \beta e_2 = e_1$$
$$e_i \beta e_j = 0 \quad \text{if } i \neq 2 \text{ or } j \neq 2.$$

As in the proof of Theorem 3.2, $\alpha, \beta \in \mathcal{A}(A)$ and $\mathcal{A}(A)$ can-

not be a group since Lemma 1.1 is violated.

It is interesting to note that when the group A is torsion-free and of rank 1, then every non-trivial multiplication α on A endows A with the structure of an integral domain. For suppose that there are elements a, $b \in A$ and $\alpha \in \mathrm{Mult}(A)$ such that $a\alpha b \neq 0$. Then if x, $y \in A$ are not zero, there are rationals r, s such that $x = ra$, $y = sb$ so that

$$\begin{aligned} x\alpha y &= (ra)\alpha(sb) \\ &= rs(a\alpha b) \neq 0 \end{aligned}$$

since A is torsion-free. Now $A(+,\alpha)$ is associative by the proof of Theorem 3.4 and commutative as well. Rédei and Szele in [9] have determined necessary and sufficient conditions for a torsion-free group A of rank 1 to have a non-trivial multiplication (also see Fuchs [5]).

<u>Theorem 3.5.</u> Let A be a divisible group. Then A is associative-closed if and only if

$$A = R + \sum_j \mathcal{C}(p_j^\infty)$$

where each p_j is prime and R is isomorphic to the rationals.

<u>Proof.</u> Since A is divisible, $A = \sum R + \sum \mathcal{C}(p_j^\infty)$. But by Theorem 3.4 and Theorem 1.3 $\mathcal{A}(A)$ will not be a group unless $\sum R$ contains only one summand. The condition is therefore necessary. To prove the converse we show that every multiplication on

$$A = R + \sum_i \mathcal{C}(p_i^{\infty})$$

is associative. But by Corollary 2 of Theorem 1.4 any multiplication on A is a direct sum of multiplications on R and $\sum \mathcal{C}(p_i^{\infty})$. Then Theorem 3.4 and Corollary 1 of Theorem 1.4 imply that every multiplication on A is associative.

<u>Corollary</u>. Let A be a group such that $A = B + C$ where B is a divisible group and C is any group. Then $\mathcal{A}(A)$ is not a group if the torsion-free rank of B is greater than 1.

<u>Proof</u>. If the torsion-free rank of B is greater than 1, B is not associative-closed by the theorem and hence A is not by Theorem 1.3.

In Chapter II all rings on torsion groups were described. We now seek a determination of necessary and sufficient conditions for a torsion group A to be associative-closed. As before, it is sufficient to consider the p-components of A. Let A_p be a p-component of A and let $B = \sum\{e_i\}$ be a basic subgroup of A_p. Now suppose that the rank r of B is greater than 1 and write $B = B_1 + B_2$ where $B_1 = \{e_1\} + \{e_2\}$ is a direct summand of B and is pure in B. Therefore B_1 is pure in A_p and being finite is a direct summand of A_p:

$$A_p = B_1 + A_p'.$$

Any multiplication on B_1 can be extended in the trivial way to a multiplication on all of A_p by Theorem 1.3. But by

Theorem 3.1 $\mathcal{A}(B_1)$ is not a group since $(n_1, n_2) \neq 1$, both $\{e_1\}$ and $\{e_2\}$ being p-groups. Hence $\mathcal{A}(A_p)$ is not a group. Therefore if $\mathcal{A}(A_p)$ is a group, $r = 1$. This requires the basic subgroup B, pure in A_p, to be finite and since finite pure subgroups are direct summands

$$A_p = B + D$$

where D is divisible. D is divisible since $A_p/B \cong \sum \mathcal{C}(p^\infty)$. With A_p being the direct sum of a torsion group and a divisible group the proof of Theorem 3.5 shows that the only multiplications on A_p are the trivial extensions of multiplications on B and all of these are associative. This proves the following result.

<u>Theorem 3.6</u>. Let A be a torsion group. Then $\mathcal{A}(A)$ is a group if and only if A has the following form:

$$A = \sum_i \mathcal{C}(p_i^{k_i}) + \sum_t D_t$$

where the primes p_i are such that $p_i \neq p_j$ if $i \neq j$ and for each t

$$D_t \cong \sum \mathcal{C}(p^\infty).$$

In [5] Fuchs has determined the additive structure of associative, commutative rings with descending chain condition on left ideals; the determination of the additive structure of semi-simple rings is also obtained in [5]. We state these results in the theorem below.

<u>Theorem</u> 3.7 (Fuchs [5]). Let A be a group. Then

(a) It is possible to define a commutative and associative ring on A having descending chain condition on left ideals if and only if

$$A = \sum_{i} R + \sum_{i=1}^{n} \mathcal{C}(p_i^{\infty}) + \sum_{\eta} \mathcal{C}(p_j^{k_j})$$

where i and η are arbitrary cardinals and m is a fixed integer such that $p_j^{k_j} | m$, n is finite.

(b) It is possible to define an associative semi-simple ring on A if and only if

$$A = \sum_{i} R + \sum_{\eta_1} \mathcal{C}(p_1) + \cdots + \sum_{\eta_s} \mathcal{C}(p_s)$$

where $p_1, \ldots, p_s$ are different primes and i, $\eta_1, \ldots, \eta_s$ are arbitrary cardinals.

<u>Theorem</u> 3.8. Let $A(+,\alpha)$ be an associative, commutative ring with descending chain condition on left ideals. Then A is an associative-closed group if and only if A has the following additive structure:

$$A = R + \sum_{i=1}^{n} \mathcal{C}(p_i^{\infty}) + \sum_{j} \mathcal{C}(q_j^{k_j})$$

where the p_i and q_j are primes and the q_j are distinct.

<u>Proof</u>. By Theorem 3.7 we know that A = B + C + D where B, C, D are the respective components of A given by that theorem. By our previous results B can contain only one copy of the rationals and D must satisfy the conditions of

Theorem 3.1. This shows that our conditions are necessary. Now assume the conditions are satisfied. Since B + C is divisible, D is torsion, any ring on (B + C) + D is a ring direct sum of rings on the two components. But as we have seen before, $\mathcal{A}(B + C) = \mathrm{Mult}(B + C)$ and $\mathcal{A}(D) = \mathrm{Mult}(D)$. It then follows that $\mathcal{A}(A) = \mathrm{Mult}(A)$ and this completes the proof.

We may use Theorem 3.7 and an argument entirely similar to that used in proving Theorem 3.8 to obtain the following.

<u>Theorem 3.9</u>. Let $A(+,\alpha)$ be a ring which is semi-simple. Then A is an associative-closed group if and only if A has additive structure as follows:

$$A = R + \mathcal{C}(p_1) + \mathcal{C}(p_2) + \cdots + \mathcal{C}(p_t)$$

for distinct primes $p_1, p_2, \ldots, p_t$.

4. <u>A relation between commutative endomorphisms and associative-closure</u>. For a group A assume that its endomorphism ring End(A) is commutative. Then for any $\alpha \in \mathrm{Mult}(A)$, $a \in A$, $b \in A$, the endomorphisms L_a^α, R_b^α, are commutative. If $c \in A$,

$$a\alpha(c\alpha b) = a\alpha(cR_b^\alpha) = (cR_b^\alpha)L_a^\alpha = c(R_b^\alpha L_a^\alpha) = c(L_a^\alpha R_b^\alpha) = (cL_a^\alpha)R_b^\alpha$$
$$= (a\alpha c)\alpha b.$$

Hence if End(A) is commutative, every multiplication on A is associative. As a consequence we see that commutativity of End(A) implies associative-closure of A. The determination

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of those Abelian groups which have commutative endomorphism ring is not yet complete. This question has, however, been resolved by Szele and Szendrei in [13] for torsion groups. The result is that a torsion group A has commutative endomorphism ring if and only if

$$A = \mathcal{C}(p_1^{k_1}) + \mathcal{C}(p_2^{k_2}) + \cdots + \mathcal{C}(p_t^{k_t}) + \cdots$$

where the p_i are different primes and $0 \leq k_i \leq \infty$. This, when compared with Theorem 3.6, shows at once that associative-closure does not imply commutative endomorphisms. For example,

$$\mathcal{C}(2) + \mathcal{C}(2^{\infty})$$

is associative-closed by Theorem 3.6 but does not have commutative endomorphism ring by the Szele-Szendrei result.

CHAPTER IV

LIE-CLOSED GROUPS

In order to study the set $\mathcal{M}(A)$ of Lie multiplications we need a condition for closure with respect to $\bullet$ just as we did for $\mathcal{M}(A)$. This is the content of the following lemma. For convenience of notation we introduce the <u>symmetric Jacobian</u> $J_{\alpha,\beta}$:

$$J_{\alpha,\beta}(x, y, z) = (x\alpha y)\beta z + (x\beta y)\alpha z + (y\alpha z)\beta x + (y\beta z)\alpha x + (z\alpha x)\beta y + (z\beta x)\alpha y.$$

<u>Lemma 4.1</u>. If $\alpha, \beta \in \mathcal{M}(A)$, then $\alpha \bullet \beta \in \mathcal{M}(A)$ if and only if

$$J_{\alpha,\beta}(x, y, z) = 0$$

for all $x, y, z \in A$.

<u>Proof</u>. For all $x, y, z \in A$ and $\alpha, \beta \in \mathcal{M}(A)$ we have

$$[x(\alpha\bullet\beta)y](\alpha\bullet\beta)z = (x\alpha y)\alpha z + (x\alpha y)\beta z + (x\beta y)\alpha z + (x\beta y)\beta z,$$

$$[y(\alpha\bullet\beta)z](\alpha\bullet\beta)x = (y\alpha z)\alpha x + (y\alpha z)\beta x + (y\beta z)\alpha x + (y\beta z)\beta x,$$

$$[z(\alpha\bullet\beta)x](\alpha\bullet\beta)y = (z\alpha x)\alpha y + (z\alpha x)\beta y + (z\beta x)\alpha y + (z\beta x)\beta y.$$

Since $\alpha, \beta \in \mathcal{M}(A)$

$$(x\alpha y)\alpha z + (y\alpha z)\alpha x + (z\alpha x)\alpha y = 0$$

and

$$(x\beta y)\beta z + (y\beta z)\beta x + (z\beta x)\beta y = 0.$$

Clearly then, from the definition of the symmetric Jacobian, our condition is both necessary and sufficient.

When the group A has a basis, we say that α is a <u>semi-Lie</u> multiplication if $e_i\alpha e_i = 0$ and $e_i\alpha e_j = -(e_j\alpha e_i)$ for all i, j.

<u>Lemma 4.2</u>. The set $\bar{\mathcal{M}}(A)$ of semi-Lie multiplications is a group with respect to $\oplus$.

<u>Proof</u>. This follows from the definition.

<u>Lemma 4.3</u>. If A is a group with a basis having $r \leq 2$, then $\mathcal{M}(A) = \bar{\mathcal{M}}(A)$.

<u>Proof</u>. If $r = 1$, the result follows immediately since both $\mathcal{M}(A)$ and $\bar{\mathcal{M}}(A)$ contain only the trivial multiplication. Now suppose that $r = 2$. Since it is obvious that every Lie multiplication is a semi-Lie multiplication we only have to show that $\bar{\mathcal{M}}(A) \leq \mathcal{M}(A)$. In order to prove that $\bar{\mathcal{M}}(A) \leq \mathcal{M}(A)$ we only have to show that the Jacobi identity holds since a semi-Lie multiplication α is anti-commutative by definition. But if e_1, e_2 are the generators of A we have, for example,

$$(e_1\alpha e_2)\alpha e_1 + (e_2\alpha e_1)\alpha e_1 + (e_1\alpha e_1)\alpha e_2$$
$$= (e_1\alpha e_2)\alpha e_1 - (e_1\alpha e_2)\alpha e_1 + 0\alpha e_2 = 0$$

and in general

$$(e_i\alpha e_j)\alpha e_k + (e_j\alpha e_k)\alpha e_i + (e_k\alpha e_i)\alpha e_j = 0$$

follows in a similar manner for $i, j, k = 1$ or 2.

<u>Theorem 4.1</u>. If $A = \sum_i \{e_i\}$ is a torsion group such that the orders n_i satisfy the condition that for all pairwise distinct i, j, k, $(n_i, n_j, n_k) = 1$, then $\mathcal{M}(A) = \bar{\mathcal{M}}(A)$.

<u>Proof</u>. The case where the rank r of $A \leq 2$ is proved by Lemma 4.3. As in the proof of Lemma 4.3 we only have to show that the Jacobi identity is satisfied for every semi-

Lie multiplication and for this it is sufficient to show that $(e_i \alpha e_j)\alpha e_k + (e_j \alpha e_k)\alpha e_i + (e_k \alpha e_i)\alpha e_j = 0$ for pairwise distinct i, j, k. But for pairwise distinct i, j, k our hypothesis is that $(n_i, n_j, n_k) = 1$. Then there are integers r, s, t such that $rn_i + sn_j + tn_k = 1$ and hence

$$\begin{aligned}(e_i \alpha e_j)\alpha e_k &= 1\cdot(e_i \alpha e_j)\alpha e_k \\ &= (rn_i + sn_j + tn_k)\cdot(e_i \alpha e_j)\alpha e_k \\ &= (rn_i)\cdot(e_i \alpha e_j)\alpha e_k + (sn_j)\cdot(e_i \alpha e_j)\alpha e_k + (tn_k)\cdot(e_i \alpha e_j)\alpha e_k \\ &= 0.\end{aligned}$$

Hence $\mathcal{A}(A) = \bar{\mathcal{A}}(A)$ and the theorem is proved.

<u>Theorem 4.2</u>. Let $A = \sum_i \{e_i\}$ be a torsion group and let $O(e_i) = n_i$. Then $\mathcal{A}(A)$ is a group if and only if $(n_i, n_j, n_k) = 1$ for all pairwise distinct i, j, k.

<u>Proof</u>. That this condition is sufficient follows from Theorem 4.1 because $\bar{\mathcal{A}}(A)$ is a group for all A. To show that the condition is necessary suppose that for some set of pairwise distinct i, j, k we have $(n_i, n_j, n_k) > 1$. We consider the subgroup

$$A' = \mathcal{C}(n_i) + \mathcal{C}(n_j) + \mathcal{C}(n_k)$$

of A. Since A' is a direct summand of A, it will follow, as we have seen before, that $\mathcal{A}(A)$ is not a group if $\mathcal{A}(A')$ is not. We now consider the representation of n_i, n_j, n_k as products of primes. Let

$$n_i = p_1^{s_1} p_2^{s_2} \cdot \cdot \cdot p_u^{s_u}$$

$$n_j = q_1^{w_1} q_2^{w_2} \cdot \cdot \cdot q_v^{w_v}$$

$$n_k = r_1^{z_1} r_2^{z_2} \cdot \cdot \cdot r_x^{z_x}.$$

Then $(n_i, n_j, n_k) > 1$ implies $p_\eta = q_\mu = r_\sigma$ for some integers η, μ, σ such that $1 \leq \eta \leq u$, $1 \leq \mu \leq v$, $1 \leq \sigma \leq x$. Now resolve A' into the direct sum of its cyclic p-groups. A' then has a subgroup $B = \mathcal{C}(p^{s_i}) + \mathcal{C}(p^{w_j}) + \mathcal{C}(p^{z_k})$, where $p = p_\eta = q_\mu = r_\sigma$, which is a direct summand. Again by reasoning as before $\mathcal{A}(A')$ is not a group if $\mathcal{A}(B)$ is not. Consequently, $\mathcal{A}(A)$ is not a group if $\mathcal{A}(B)$ is not. It is therefore sufficient to show that $\mathcal{A}(B)$ is not a group when $(n_i, n_j, n_k) > 1$. Let the summands of B be generated by e_1, e_2, e_3 respectively and assume, without loss of generality, that $s_i \leq w_j \leq z_k$. Define

$$d_\delta = \frac{O(e_\delta)}{(O(e_1), O(e_2), O(e_3))} = \frac{O(e_\delta)}{p^{s_i}} \qquad \delta = 1, 2, 3.$$

Two multiplications α, β are now defined as follows:

$$e_1 \alpha e_2 = -e_2 \alpha e_1 = d_3 e_3$$

$$e_1 \alpha e_3 = -e_3 \alpha e_1 = d_2 e_2$$

$$e_2 \alpha e_3 = -e_3 \alpha e_2 = d_1 e_1$$

$$e_f \alpha e_f = 0 \text{ for } f = 1, 2, 3$$

$$e_1 \beta e_2 = [O(e_1)/(O(e_1), O(e_2))] e_1 = e_1$$

$$e_f \beta e_f = 0 \text{ for } f = 1, 2, 3$$

$$e_f \beta e_g = 0 \text{ for } f = 3 \text{ or } g = 3.$$

The α, β were defined in such a way that the Beaumont conditions for distributivity are satisfied. A simple calculation also shows that $\alpha, \beta \in \mathcal{A}(B)$. By Lemma 4.1 $\alpha \oplus \beta \in \mathcal{A}(B)$ if and only if $J_{\alpha,\beta}(a, b, c) = 0$ for all $a, b, c \in B$. In particular $\mathcal{A}(B)$ is not a group unless $J_{\alpha,\beta}(e_1, e_2, e_3) = 0$. But

$$\begin{aligned} J_{\alpha,\beta}(e_1, e_2, e_3) &= (e_1\alpha e_2)\beta e_3 + (e_1\beta e_2)\alpha e_3 + (e_2\alpha e_3)\beta e_1 \\ &\quad + (e_2\beta e_3)\alpha e_1 + (e_3\alpha e_1)\beta e_2 + (e_3\beta e_1)\alpha e_2 \\ &= d_3e_3\beta e_3 + e_1\alpha e_3 + d_1e_1\beta e_1 + 0\alpha e_1 \\ &\quad + d_2e_2\beta e_2 + 0\alpha e_2 \\ &= 0 + d_2e_2 + 0 + 0 + 0 + 0 = d_2e_2. \end{aligned}$$

$d_2e_2 = 0$ if and only if $p^{w_j} | d_2$ or if and only if $p^{w_j} | p^{w_j - s_i}$. From the fact that $s_1 \geq 1$ it follows that $J_{\alpha,\beta}(e_1, e_2, e_3)$ is not zero so that $\mathcal{A}(B)$ is not a group. This completes the proof of Theorem 4.2.

<u>Theorem 4.3</u>. If $A = \sum_i \{e_i\}$ is a torsion-free group, then $\mathcal{A}(A)$ is a group if and only if the rank of $A < 3$.

<u>Proof</u>. That the condition is sufficient follows from Lemma 4.3. To show that this condition is necessary, assume that the rank $r \geq 3$. Then we may choose generators e_1, e_2, e_3 from the set (e_i) and define two Lie multiplications α, β as follows:

$$e_1\alpha e_2 = -e_2\alpha e_1 = e_3$$
$$e_1\alpha e_3 = -e_3\alpha e_1 = e_2$$

$$e_2\alpha e_3 = -e_3\alpha e_2 = e_1$$
$$e_i\alpha e_i = 0 \text{ for all } i$$
$$e_i\alpha e_j = 0 \text{ for } i, j \neq 1, 2, 3$$
$$e_1\beta e_2 = -e_2\beta e_1 = e_1$$
$$e_i\beta e_j = 0 \text{ for } i, j \neq 1, 2$$
$$e_i\beta e_i = 0 \text{ for all } i.$$

It is easily verified that $\alpha, \beta \in \mathcal{A}(A)$. Then applying Lemma 4.1:

$$\begin{aligned} J_{\alpha,\beta}(e_1, e_2, e_3) &= (e_1\alpha e_2)\beta e_3 + (e_1\beta e_2)\alpha e_3 + (e_2\alpha e_3)\beta e_1 \\ &\quad + (e_2\beta e_3)\alpha e_1 + (e_3\alpha e_1)\beta e_2 + (e_3\beta e_1)\alpha e_2 \\ &= e_3\beta e_3 + e_1\alpha e_3 + e_1\beta e_1 + 0\alpha e_1 + (-e_2)\beta e_2 + 0\alpha e_2 \\ &= 0 + e_2 + 0 + 0 + 0 + 0 = e_2. \end{aligned}$$

But since $e_2 \neq 0$ the proof is complete.

<u>Theorem 4.4</u>. Let A be a torsion-free divisible group of rank r. Then A is a Lie-closed group if and only if $r \leq 2$.

<u>Proof</u>. In this case A is a vector space over the rationals and any multiplication may be described in terms of a basis. If (a_i) is a basis for A and $r \leq 2$, we may show as in the proof of Lemma 4.3 that A is Lie-closed. If $r \geq 3$, we may choose a_1, a_2, a_3 from the set (a_i) and show, as in the proof of Theorem 4.3, that A is not Lie-closed and consequently that the condition is necessary. Q. E. D.

<u>Corollary</u>. If A is a divisible group of torsion-free rank r_0, then A is Lie-closed if and only if $r_0 \leq 2$.

<u>Proof</u>. Since A is divisible

$$A = \sum_i R_i + \sum_j \mathcal{C}(p_j^{\infty})$$

where each R_i is isomorphic to the rationals and the p_j are primes. By the present theorem $r_0 \leq 2$ if $\mathcal{A}(R_i)$ is to be a group. Hence the condition is necessary for $\mathcal{A}(A)$ to be a group. Conversely, the condition is sufficient by the present theorem and the corollary of Theorem 1.4.

<u>Theorem 4.5</u>. Let $A = T + F$ be a mixed group with a basis, where T is torsion and F is free. Then A is a Lie-closed group if and only if T and F are associative-closed.

<u>Proof</u>. A is not Lie-closed if either T or F is not Lie-closed. Hence assume that T satisfies the conditions of Theorem 4.2 and F satisfies the conditions of Theorem 4.3. Suppose F has rank 2: $F = \{a\} + \{b\}$. Then if $T = \sum_i \{e_i\}$, we define multiplications $\alpha, \beta \in \mathrm{Mult}(A)$ as follows:

$$e_i \alpha e_j = 0 = e_j \alpha e_i \text{ for all } i, j$$

$$e_j \alpha a = a \alpha e_j = 0 \text{ for all } j$$

$$a \alpha a = 0 = b \alpha b$$

$$e_j \alpha b = b \alpha e_j = 0 \text{ for all } j$$

$$a \alpha b = -b \alpha a = b$$

$$a \beta e_1 = -e_1 \beta a = e_1$$

$$b \beta e_1 = -e_1 \beta b = e_1$$

$$a \beta b = 0 = b \beta a$$

$$a \beta a = 0 = b \beta b$$

$$e_i\beta e_j = 0 = e_j\beta e_i \text{ for all } i, j$$

$$a\beta e_i = e_i\beta a = b\beta e_i = e_i\beta b = 0 \text{ if } i \neq 1.$$

A few simple calculations show that $\alpha, \beta \in \mathcal{N}(A)$. Using the symmetric Jacobian we have

$$\begin{aligned} J_{\alpha,\beta}(a, b, e_1) &= (a\alpha b)\beta e_1 + (a\beta b)\alpha e_1 + (b\alpha e_1)\beta a + (b\beta e_1)\alpha a \\ &\quad + (e_1\alpha a)\beta b + (e_1\beta a)\alpha b \\ &= b\beta e_1 + 0\alpha e_1 + 0\beta a + e_1\alpha a + 0\beta b + (-e_1)\alpha b \\ &= e_1 + 0 + 0 + 0 + 0 + 0 \neq 0. \end{aligned}$$

Thus F cannot have rank 2 and is therefore associative-closed. If T has rank 1, $\mathcal{N}(T)$ is a group and there is nothing to prove. Hence assume T has rank ≥ 2 and let e_1, e_2 be two generators of T such that $d = (n_1, n_2) > 1$. Define $\alpha, \beta \in \mathrm{Mult}(A)$:

$$a\alpha e_1 = -e_1\alpha a = e_1$$

$$a\alpha e_i = e_i\alpha a = 0 \text{ for } i \neq 1$$

$$a\alpha a = e_i\alpha e_i = 0 \text{ for all } i$$

$$e_1\alpha e_2 = -e_2\alpha e_1 = (n_1/d)\cdot e_1$$

$$e_i\alpha e_j = e_j\alpha e_i = 0 \text{ if } i \neq 1 \text{ or } 2 \text{ or } j \neq 1 \text{ or } 2$$

$$a\beta a = 0$$

$$e_i\beta e_i = 0 \text{ for all } i$$

$$a\beta e_2 = -e_2\beta a = e_2$$

$$a\beta e_i = e_i\beta a = 0 \text{ if } i \neq 2$$

$$e_i\beta e_j = e_j\beta e_i = 0 \text{ for all } i, j.$$

Again $\alpha, \beta \in \mathcal{N}(A)$ and

$$\begin{aligned} J_{\alpha,\beta}(e_1, e_2, a) &= (e_1\alpha e_2)\beta a + (e_1\beta e_2)\alpha a + (e_2\alpha a)\beta e_1 \\ &+ (e_2\beta a)\alpha e_1 + (a\alpha e_1)\beta e_2 + (a\beta e_1)\alpha e_2 \\ &= (n_1/d)\cdot e_1\beta a + 0\alpha a + 0\beta e_1 + (-e_2)\alpha e_1 \\ &+ e_1\beta e_2 + 0\alpha e_2 \\ &= 0 + 0 + 0 + (n_1/d)\cdot e_1 + 0 + 0. \end{aligned}$$

Since $d > 1$, $J_{\alpha,\beta}(e_1, e_2, a) \neq 0$. It follows that $\mathcal{A}(A)$ is not a group unless $(n_i, n_j) = 1$ for all i, j such that $i \neq j$, i. e., unless T is associative-closed. It is thus necessary that

$$A = T + \{a\}$$

where T is associative-closed. But this means that $\mathcal{A}(T) = 0$, $\mathcal{A}(\{a\}) = 0$ and the only possible non-trivial products for semi-Lie multiplications are

$$a\alpha e_i = -e_i\alpha a = \sum_k t_{aik}e_k$$

for integers t_{aik}. The Jacobi identity clearly holds for any such multiplication so that the semi-Lie multiplications are also Lie. This shows that our conditions are necessary and sufficient.

<u>Theorem</u> <u>4.6</u>. Let A be a torsion group. Then A is Lie-closed if and only if

$$A = \sum_{i\in\Lambda}\mathcal{C}(p_i^{k_i}) + \sum_{j\in M}\mathcal{C}(q_j^{\infty})$$

where the p_i and q_j are primes and for any $i\in\Lambda$ there is at most one $i'\in\Lambda$ such that $p_i = p_{i'}$.

INTRODUCTION

Let $A(+)$ be an Abelian group and consider the set $\mathrm{Mult}(A)$ of all binary compositions on $A(+)$ which are distributive with respect to $+$. This is the set of all α such that $A(+,\alpha)$ is a ring, not necessarily associative. In recent years the set $\mathrm{Mult}(A)$ has been the object of some investigation. The research relating to $\mathrm{Mult}(A)$ may be divided into two classes of problems:

1) for a given group $A(+)$, to determine $\mathrm{Mult}(A)$;

2) for a given ring property P, to determine those groups $A(+)$ such that there exists a non-trivial $\alpha \in \mathrm{Mult}(A)$ and $A(+,\alpha)$ has property P.

$\mathrm{Mult}(A)$ is not the null set for any $A(+)$ since one may define the trivial ring on any Abelian group. If $\mathrm{Mult}(A)$ consists of only the trivial multiplication, we say that $A(+)$ is a <u>nil</u> group. T. Szele in [11] has determined the nil torsion groups. In fact Szele has proved that <u>a torsion group A is nil if and only if A is divisible</u> and <u>a mixed group is not nil</u>. The general problem of torsion-free nil groups has not been settled. However, in [10] R. Ree and R. J. Wisner have determined a special class of nil torsion-free groups.

A <u>quasi-nil</u> group is an Abelian group $A(+)$ such that

<u>Proof</u>. We write A as the direct sum of its p-components A_p. As has been observed before, any ring on A is the ring direct sum of rings on the A_p. Hence we investigate the A_p. Let B be a basic subgroup of A_p of rank r and suppose that $r \geq 3$. Then B has a subgroup of the form

$$C = \{e_1\} + \{e_2\} + \{e_3\}$$

which is a direct summand of A_p since finite pure subgroups are direct summands. Then $\mathcal{A}(C)$ is not a group since the conditions of Theorem 4.2 do not hold. Consequently $\mathcal{A}(A_p)$ is not a group. We conclude that any basic subgroup B of A_p must have rank $r < 3$. Then

$$A_p = B + D$$

where the rank of $B < 3$ and D is divisible. This leads to the condition on the primes p_1 and thus to the form of A stated in the theorem. Conversely, any multiplications on a group of this form must be nil on the divisible part and a product of an element from the non-divisible component with an element from the divisible component must be zero by Corollary 2 of Theorem 1.4. Thus products can be non-trivial only on the non-divisible component of A. But the non-divisible part of A is a Lie-closed group from which it follows that A is Lie-closed.

CHAPTER V

RINGS FREELY GENERATED BY GROUPS

Let $A(+)$ be a group and suppose that $R(A, +, \cdot)$ is a ring containing $A(+)$. $R(A, +, \cdot)$ is called the <u>freest ring generated by $A(+)$</u> if and only if the following condition is satisfied:

> If $A(+)$ is mapped into $T(+)$ of a ring $T(+,\cdot)$ by a homomorphism f, then there is a homomorphism f^* of $R(A, +, \cdot)$ into $T(+,\cdot)$ and f^* is an extension of f.

The existence of $R(A, +, \cdot)$ for every group $A(+)$ may be established via the free non-associative ring on the set A. For if $R(+,\cdot)$ is the free non-associative ring on the set A and N is the ideal of $R(+,\cdot)$ generated by the defining relations of $A(+)$, then $R/N \cong R(A, +, \cdot)$. Taking the free associative ring $W(+,\cdot)$ on the set A, W/N is the <u>freest associative ring generated by $A(+)$</u>, $W(A, +, \cdot)$. Both $R(A, +, \cdot)$ and $W(A, +, \cdot)$ have the interesting property that they are generalizations of the tensor product $A \otimes A$. This is a consequence of the fact that the subgroup R_2 of $R(A, +, \cdot)$ or $W(A, +, \cdot)$ of second degree terms is isomorphic to $A \otimes A$. We therefore have $\mathrm{Mult}(A)$ as the group of homomorphisms of second degree terms into first degree terms.

It is possible, of course, to derive a normal form for

the elements of $R(A, +, \cdot)$ but this will not be done here.

Suppose η is a homomorphism of $R_2 \leq R(A, +, \cdot)$ into the set A of first degree terms of $R(A, +, \cdot)$. Then η defines a ring $A(+,\alpha)$ by $a\alpha b = (ab)\eta$. Take f to be the identity mapping of $A(+)$ into the additive part of $A(+,\alpha)$. Then there is a homomorphic extension f^* of f mapping the ring $R(A, +, \cdot)$ into $A(+,\alpha)$. For elements $ab \in R_2$ we have

$$\begin{aligned}(ab)f^* &= (af^*)\alpha(bf^*)\\ &= (af)\alpha(bf) = a\alpha b = (ab)\eta.\end{aligned}$$

We thus conclude that <u>every homomorphism of R_2 into $A(+)$ can be extended to a homomorphism mapping $R(A, +, \cdot)$ into $A(+)$</u>.

In some special cases of $A(+)$ it has been possible to obtain some structure theory for $R(A, +, \cdot)$. This is omitted here, however, since our purpose in introducing $R(A, +, \cdot)$ is to increase our understanding of Mult(A) and nothing of significance has yet been obtained.

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AUTOBIOGRAPHY

I, Flournoy Lane Hardy, was born in Columbia, South Carolina, September 10, 1928. I received my secondary-school education in the public schools of East Point, Georgia, and my undergraduate training at Oglethorpe University, which granted me the Bachelor of Arts degree in 1955. From Emory University, I received the Master of Arts degree in 1956. In 1956 I was appointed to the position of Instructor of Mathematics at Emory University and became Assistant Professor of Mathematics at that University in 1960.

only a finite number of non-isomorphic rings on A(+) are definable. Quasi-nil groups are considered by L. Fuchs in [3] and [5].

Problem 1 may be solved easily in special cases of A(+) (see Chapter I). A description of Mult(A) when A is a direct sum of cyclic groups may be found in L. Fuchs [5]. In [1] R. A. Beaumont gives necessary and sufficient conditions for a binary composition α such that $\alpha \in$ Mult(A) when A is a direct sum of cyclic groups. This result has been generalized by L. Redei in [8] to groups with operators. In Chapter II a further generalization is obtained to include arbitrary torsion groups.

Problem 2 has been consider by L. Fuchs and T. Szele in [5], [6] and [12]. Results here include a) a determination of those groups on which it is possible to define a ring with descending chain condition on left ideals; b) a determination of those groups on which a semi-simple ring may be defined; c) a determination of those groups on which a regular ring may be defined.

An operation for Mult(A) may be defined in such a way that Mult(A) is an Abelian group (see Chapter I). In general a ring property P does not determine a subgroup of Mult(A). This dissertation is mainly concerned with determining those groups A(+) such that the associative

and Lie ring properties determine a subgroup of Mult(A). These two problems are solved for torsion groups in Chapters III and IV. These chapters contain the main results of the dissertation.

An area for future study is indicated in Chapter V where the <u>freest ring generated by an Abelian group A(+)</u> is defined. There is some indication that this concept will prove to be useful in the study of Mult(A).

CHAPTER I

PRELIMINARIES

Throughout the dissertation <u>group</u> will mean Abelian group and these will be denoted by upper case Latin letters while elements of groups will be denoted by lower case letters. $\mathcal{C}(m)$ means the cyclic group of order m, m a natural number; $\mathcal{C}(\infty)$ is the infinite cyclic group; $\mathcal{C}(p^\infty)$ denotes the quasi-cyclic group for the prime p. If x is an element of a p-group A, E(x) is the <u>exponent</u> of x, i. e., $p^{E(x)}$ is the order of x. When the term <u>direct sum</u> or the symbol Σ is used we mean the <u>restricted</u> direct sum. When A, B are groups Hom(A,B) is the notation for the group of homomorphisms of A into B and End(A) is the set of endomorphisms of A into itself.

If A(+) is a group, a mapping of A×A into A, i. e. a binary composition on A, is <u>distributive</u> with respect to + if both the right and left distributive laws hold. Define Mult(A) to be the set of all binary compositions on A that are distributive with respect to +. Hence α <u>is in Mult(A) if and only if A(+,α) is a ring</u>.

For elements x, y $\in$ A and $\alpha \in$ Mult(A) denote by $x\alpha y$ the image of (x, y) under α.

A binary composition $\oplus$ is defined on Mult(A) as follows:

$$x(\alpha \oplus \beta)y = x\alpha y + x\beta y, \qquad x, y \in A, \quad \alpha, \beta \in \text{Mult}(A).$$

With this definition of Θ Mult(A) is easily seen to be a group in which the trivial multiplication on A(+) is the identity element and $\Theta\alpha$ is the inverse of α where

$$x(\Theta\alpha)y = -(x\alpha y).$$

<u>Theorem</u> 1.1 (Fuchs [5]).

1) $$\mathrm{Mult}(A) \cong \mathrm{Hom}(A\otimes A, A)$$

2) $$\mathrm{Mult}(A) \cong \mathrm{Hom}(A, \mathrm{End}(A)).$$

<u>Proof</u>. To prove 1) let $\alpha \in \mathrm{Mult}(A)$ and consider the correspondence $(x, y) \longrightarrow x\alpha y$. This is a bilinear function of $A\times A$ into A and by a characterizing property of the tensor product there is a homomorphism η of $A\otimes A$ into A such that

$$(x\otimes y)\eta = x\alpha y.$$

Then the correspondence $\eta \rightarrow \alpha$ gives the desired isomorphism.

For the proof of 2) define for a fixed element $a \in A$ and $\alpha \in \mathrm{Mult}(A)$ the right multiplication R_a^α: $x \longrightarrow x\alpha a$. This is an endomorphism of A for each $a \in A$ and $\alpha \in \mathrm{Mult}(A)$. Then the correspondence $a \rightarrow R_a^\alpha$ is a homomorphism η of A into End(A). The isomorphism of 2) is now obtained by the mapping $\alpha \rightarrow \eta$.

The isomorphisms expressed in Theorem 1.1 can be used to advantage in some cases to determine Mult(A). For example, if A is cyclic, $A\otimes A = A$ and hence

$$\mathrm{Mult}(A) \cong \mathrm{Hom}(A,A) \cong \mathrm{End}(A) \cong A.$$

This is also easily obtained by direct computation. For