Sparse Equidistribution of Unipotent Orbits in Finite-Volume Quotients of $\mathrm{PSL}(2,\mathbb{R})$

Dissertation

Presented in Partial Fulfillment of the Requirements for the Degree Doctor of Philosophy in the Graduate School of the Ohio State University

By

Cheng Zheng, B.S.

Graduate Program in Mathematics

The Ohio State University

2016

Dissertation Committee:

Nimish A. Shah, Advisor

Vitaly Bergelson

James Cogdell

Abstract

We consider the orbits $\{pu(n^{1+\gamma})|n \in \mathbb{N}\}$ in $\Gamma\backslash \operatorname{PSL}(2,\mathbb{R})$, where Γ is a non-uniform lattice in $\operatorname{PSL}(2,\mathbb{R})$ and $\{u(t)\}$ is the standard unipotent one-parameter subgroup in $\operatorname{PSL}(2,\mathbb{R})$. Under a Diophantine condition on the intial point p, we can prove that the trajectory $\{pu(n^{1+\gamma})|n \in \mathbb{N}\}$ is equidistributed in $\Gamma\backslash \operatorname{PSL}(2,\mathbb{R})$ for small $\gamma > 0$, which generalizes a result of Venkatesh [V10]. In Chapter 2, we will compute Hausdorff dimensions of subsets of non-Diophantine points in $\Gamma\backslash \operatorname{PSL}(2,\mathbb{R})$, using results of lattice counting problem. In Chapter 3 we will use the exponential mixing property of a semisimple flow to prove the effective equidistribution of horospherical orbits. In Chapter 4, we will give a definition of Diophantine points of type γ for $\gamma \geq 0$ in a homogeneous space $\Gamma\backslash G$ and compute the Hausdorff dimension of the subset of points which are not Diophantine of type γ when G is a semisimple Lie group of real rank one. As an application, we will deduce a Jarnik-Besicovitch Theorem on Diophantine approximation in Heisenberg groups.

To my parents

Acknowledgements

I would like to thank my advisor Professor Nimish Shah for his helps and encouragements during my PhD study. He is always patient to explain things that I need to know, which greatly broaden my view in mathematics, and many ideas in this thesis come from discussions with him. Also I would like to thank Davis Buenger for many discussions in our joint work [BZ16]. I would like to thank the Committee members Professor Vitaly Bergelson and Professor James Cogdell. And I would like to thank all the people in Mathematics Department for providing such a great academic environment.

I am also grateful to Nicolas de Saxce who suggested Lemma 1.3.4 in this thesis. I would like to thank Professor Andreas Strömbergsson and Samuel Edwards for many discussions in Chapter 3. I was told that they had results about the effective equidistribution of horocycle orbits in homogeneous spaces using number theoretic tools. In Chapter 3 our purpose is just to show how to use mixing property only to get such results. I thank the MSRI for hospitality during spring 2015.

Finally I would like to thank my parents and all of my friends for their support and help.

Vita

2010 . . . . B.S., Mathematics, Peking University

2010 to present . . . . Graduate Teaching Associate
Department of Mathematics
The Ohio State University

Fields of Study

Major Field: Mathematics

Table of Contents

Chapter 1

Sparse Equidistribution of Unipotent Orbits in $\Gamma\backslash\operatorname{PSL}(2,\mathbb{R})$

1.1 Introduction

The theory of equidistribution of unipotent flows on homogeneous spaces has been studied extensively over the past few decades. Furstenberg [F73] first proved that the unipotent flow on $\Gamma\backslash\operatorname{PSL}(2,\mathbb{R})$, where Γ is a uniform lattice, is uniquely ergodic. In [D78] Dani classified ergodic invariant measures for unipotent flows on finite volume homogeneous spaces of $\operatorname{PSL}(2,\mathbb{R})$, and using this result Dani and Smillie [DS84] proved that any non-periodic unipotent orbit is equidistributed on $\Gamma\backslash\operatorname{PSL}(2,\mathbb{R})$ for any lattice Γ. The proof of the Oppenheim Conjecture due to Margulis [M89] by proving a special case of Raghunathan's conjecture drew a lot of attention to this subject. Soon afterwords, Ratner published her seminal work [R90a, R90b, R91a] proving measure classification theorem for unipotent actions on homogeneous spaces as conjectured by Raghunathan and Dani [D81]. Using these results, Ratner [R91b] proved that any unipotent orbit in a finite volume homogeneous space is equidistributed in its orbit closure; see also Shah [Sh91] for the case of Rank-1 semisimple groups.

Ratner's work has led to many new extensions and number theoretic applications of ergodic theory of unipotent flows. One of these results, which is related to this paper, was the work by Shah [Sh94]. In that paper, Shah asked whether $\{pu(n^2)|n \in \mathbb{N}\}$ is equidistributed in a sub-homogeneous space of $\operatorname{PSL}(2,\mathbb{Z})\backslash\operatorname{PSL}(2,\mathbb{R})$, where

$u : \mathbb{R} \to \mathrm{PSL}(2, \mathbb{R})$ is the standard unipotent 1-parameter subgroup

$$u(t) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}.$$

In this direction, Venkatesh published a result about sparse equidistribution ([V10], Theorem 3.1). There he introduced a soft technique of calculations by using a discrepancy trick, and proved that if Γ is a cocompact lattice in $\mathrm{PSL}(2, \mathbb{R})$ and $\gamma > 0$ is a small number depending on the spectral gap of the Laplacian on $\Gamma\backslash \mathrm{PSL}(2, \mathbb{R})$, then for any point $p \in \Gamma\backslash \mathrm{PSL}(2, \mathbb{R})$ we have

$$\frac{1}{N}\sum_{n=0}^{N-1} f(pu(n^{1+\gamma})) \to \int_{\Gamma\backslash \mathrm{PSL}(2,\mathbb{R})} f d\mu.$$

In other words, in the case of $\Gamma\backslash \mathrm{PSL}(2, \mathbb{R})$ being compact, the equidistribtion holds for the sparse subset $\{n^{1+\gamma} | n \in \mathbb{N}\}$. It is worth noting that recently Tanis and Vishe [TV15] improve some results of Venkatesh [V10] and they obtain an absolute constant $\gamma > 0$ which does not depend on the spectral gap.

In this paper, we will consider the sparse subset $\{n^{1+\gamma} | n \in \mathbb{N}\}$ and orbits of $\{u(n^{1+\gamma}) | n \in \mathbb{N}\}$ in $\Gamma\backslash \mathrm{PSL}(2, \mathbb{R})$, where Γ is a non-uniform lattice. We want to prove a sparse equidistribution theorem similar to Shah's conjecture [Sh94] and the work of Venkatesh [V10] and that of Tanis and Vishe [TV15]. To deal with the complexity caused by initial points of unipotent orbits, we introduce a Diophantine condition for points in $\Gamma\backslash \mathrm{PSL}(2, \mathbb{R})$ as follows.

Let $G = \mathrm{PSL}(2, \mathbb{R})$ and we consider the Siegel sets $N_\Omega A_\alpha K$ where

$$N_\Omega = \left\{ \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \middle| t \text{ is in a bounded subset } \Omega \subset \mathbb{R} \right\}$$

$$A_\alpha = \left\{ \begin{pmatrix} s & 0 \\ 0 & s^{-1} \end{pmatrix} \Big| s \geq \alpha \right\}$$

and $K = SO(2)$. For the non-uniform lattice Γ, there exist $\sigma_j \in G$ and bounded intervals $\Omega_j \subset \mathbb{R}$ $(1 \leq j \leq k)$ with the following property ([GR70], [DS84])

1. For some $\alpha > 0$, $G = \bigcup_{j=1}^{k} \Gamma \sigma_j N_{\Omega_j} A_\alpha K$.

2. $\sigma_j^{-1} \Gamma \sigma_j \cap N$ is a cocompact lattice in N.

3. N_{Ω_j} is a fundamental domain of $\sigma_j^{-1} \Gamma \sigma_j \cap N \backslash N$.

We will fix σ_j $(1 \leq j \leq k)$ in such a way that in the upper half plane $\mathfrak{H}$, each σ_j corresponds to a cusp η_j, i.e. $\lim_{t\to\infty} \sigma_j \cdot it = \eta_j$, and $\eta_1, \eta_2, \ldots, \eta_k$ are the inequivalent cusps of $\Gamma\backslash\mathfrak{H}$. Let $\Gamma_j = \Gamma \cap \sigma_j N \sigma_j^{-1}$. Let π_j be the covering map

$$\pi_j : \Gamma_j \backslash G \to \Gamma \backslash G.$$

Now consider the usual action of G on $\mathbb{R}^2$ and let $e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. For each j, we can define a map

$$m_j : \Gamma_j \backslash G \to \mathbb{R}^2/\pm$$

by

$$m_j(q) = g^{-1} \sigma_j e_1$$

for $q = \Gamma_j g \in \Gamma_j \backslash G$, where $\mathbb{R}^2/\pm$ means that we identify every $v \in \mathbb{R}^2$ with its opposite $-v$. In this way, we obtain k maps m_j $(j = 1, 2, \ldots, k)$ whose images are all in $\mathbb{R}^2/\pm$. Using these notations, we can give the following definition of Diophantine condition of a point $p \in \Gamma\backslash G$.

Definition 1.1.1. Let $p \in \Gamma\backslash G$. We say that p is Diophantine of type $(\kappa_1, \kappa_2, \dots, \kappa_k)$ for some $\kappa_j > 0$ $(j = 1, 2, \dots, k)$ if for each j, there exist $\mu_j, \nu_j > 0$ such that for every point $\begin{pmatrix} a \\ b \end{pmatrix} \in m_j(\pi_j^{-1}(p))$, we have either $|b| \geq \mu_j$ or $|a|^{\kappa_j}|b| \geq \nu_j$.

Remark 1.1.1. This notion of Diophantine type on $p \in \Gamma\backslash G$ has been studied well in an equivalent form; it can be connected to the excursion rate of the geodesic orbit $\{g_t(p)\}_{t>0}$. We will prove this in section 1.3.

It is straightforward to verify that if $g \in AN$ then the Diophantine types of p and pg are the same; although the choices of $\mu_j, \nu_j > 0$ in the above definition may differ. The hausdorff dimension of the complement of the set of points of the Diophantine type $(\kappa_1, \kappa_2, \dots, \kappa_k)$ will be discussed in section 1.7. We will see that almost every point satisfies the Diophantine condition of type $(\kappa_1, \kappa_2, \dots, \kappa_k)$ when $\kappa_1, \kappa_2, \dots, \kappa_k > 1$. When $\min\{\kappa_1, \kappa_2, \dots, \kappa_k\} = 1$, the set of points of the Diophantine type $(\kappa_1, \kappa_2, \dots, \kappa_k)$ has zero Haar measure but has full Hausdorff dimension.

Now we state the main theorem in this paper.

Theorem 1.1.1 (Main theorem)**.** *Let Γ be a non-uniform lattice in $\operatorname{PSL}(2,\mathbb{R})$ and k the number of inequivalent cusps of $\Gamma\backslash\operatorname{PSL}(2,\mathbb{R})$. Suppose that $p \in \Gamma\backslash\operatorname{PSL}(2,\mathbb{R})$ is Diophantine of type $(\kappa_1, \kappa_2, \dots, \kappa_k)$. Then there exists a constant $\gamma_0 > 0$ such that for any $0 < \gamma < \gamma_0$, we have*

$$\frac{1}{N}\sum_{n=0}^{N-1} f(pu(n^{1+\gamma})) \to \int_{\Gamma\backslash\operatorname{PSL}(2,\mathbb{R})} f\,d\mu.$$

Here the constant γ_0 depends on $\kappa_1, \kappa_2, \dots, \kappa_k$ and Γ, and f is any bounded continuous function on $\Gamma\backslash\operatorname{PSL}(2,\mathbb{R})$.

Remark 1.1.2. From the proof of the main theorem, we will see that the constant

$$\gamma_0 = \min\left\{ \frac{s^2}{(s+4)(\kappa_j+4)} \Big| j = 1, 2, \ldots, k \right\}.$$

Here s is defined as follows: if we let $\lambda > 0$ denote the smallest eigenvalue in the discrete spectrum of the Laplacian Δ on $\Gamma\backslash\mathfrak{H}$ then

$$s = \begin{cases} \frac{1-\sqrt{1-4\lambda}}{2}, & \text{if } 0 < \lambda < \frac{1}{4}; \\ \frac{1}{2}, & \text{otherwise.} \end{cases}$$

Now let Γ be a subgroup of finite index of $\mathrm{PSL}(2,\mathbb{Z})$. Then we have the following corollary of the main theorem, which will be explained in section 1.3.

Corollary 1.1.1. *Let* Γ *be a subgroup of finite index of* $\mathrm{PSL}(2,\mathbb{Z})$. *Let* $p = \Gamma g \in \Gamma\backslash \mathrm{PSL}(2,\mathbb{R})$ *with*

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

If $a/c \in \mathbb{R}$ *is a Diophantine number of type* ζ*; that is, there exists* $C > 0$ *such that for all* $m/n \in \mathbb{Q}$*, we have*

$$|n|^\zeta \left| n \cdot \left(\frac{a}{c}\right) - m \right| \geq C,$$

then the orbit $\{pu(n^{1+\gamma}) | n \in \mathbb{N}\}$ *is equidistributed in* $\Gamma\backslash \mathrm{PSL}(2,\mathbb{R})$ *for* $0 < \gamma < \gamma_0 := \frac{s^2}{(4+s)(\zeta+4)}$.

To prove the main theorem, we shall use the technique of Venkatesh in [V10] and Strömbergsson's result in [S13] about effective version of Dani and Smillie's result [DS84] on $\Gamma\backslash \mathrm{PSL}(2,\mathbb{R})$. In fact, an immediate consequence of the technique of [V10] and result of [S13] is obtained in the following theorem. Before stating the theorem, we need some notations. For $f \in C^k(\Gamma\backslash G)$ we let $\|f\|_{p,k}$ be the Sobolev L^p-norm involving all the Lie derivatives of order $\leq k$ of f. Note that $\|f\|_{\infty,0}$ is the supremum norm of f. We know that G acts on the upper half plane $\mathfrak{H}$ by the action

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az+b}{cz+d}$$

and we have the standard projection of $\Gamma\backslash G$ to the fundamental domain of Γ in $\mathfrak{H}$

$$\pi : \Gamma\backslash G \to \Gamma\backslash\mathfrak{H}$$

by sending Γg to $\Gamma g(i)$. We define the geodesic flow on $\Gamma\backslash G$ by

$$g_t(\Gamma g) = \Gamma g \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix}.$$

Fix, once for all, a point $p_0 \in \Gamma\backslash\mathfrak{H}$. For $p \in \Gamma\backslash G$ let

$$\mathrm{dist}(p) = d_{\mathfrak{H}}(p_0, \pi(p))$$

where $d_{\mathfrak{H}}(\cdot,\cdot)$ is the hyperbolic distance on $\Gamma\backslash\mathfrak{H}$.

Theorem 1.1.2 (Cf.[V10] Theorem 3.1). *Let $T > K > 2$ and $f \in C^\infty(\Gamma\backslash G)$ satisfying $\int_{\Gamma\backslash G} f d\mu = 0$ and $\|f\|_{\infty,4} < \infty$. Suppose that $q \in \Gamma\backslash G$ satisfies $r = r(q,T) = T \cdot e^{-dist(g_{\log T}(q))} \geq 1$. Then we have*

$$\left| \frac{1}{T/K} \sum_{\substack{j\in\mathbb{Z} \\ 0\leq Kj<T}} f(qu(Kj)) \right| \ll \frac{K^{\frac{1}{2}} \ln^{\frac{3}{2}}(r+2)}{r^{\frac{\beta}{2}}} \|f\|_{\infty,4}$$

for $\beta = \frac{s\kappa}{2(8+\kappa)}$. Here κ is the constant in the mixing property of the unipotent flow *(see Theorem 1.2.1 and Remark 1.2.1) and s is defined as in Remark 1.1.2.*

This theorem gives an estimate for the average of the unipotent action along an arithmetic progression with gap K, which is crucial in our proof of the main theorem. This was proved first in [V10] and later in [TV15], both in the case of $\Gamma\backslash G$ being compact.

The strategy of the paper is the following: note that the bound in Theorem 1.1.2 depends on the initial point, and hence when we combine the results with different arithmetic progressions and different initial points, the outcomes would get out of control. To overcome this difficulty, we need the Diophantine condition. With the help of this Diophantine condition along with the notion of $(C, \alpha; \rho, \epsilon_0)$-good functions, we will be able to control the rates of these effective results. In section 1.2, we list the concepts and theorems that we need in this paper. In section 1.3, we study the Diophantine condition and deduce Corollary 1.1.1 from the main theorem. In section 1.4, we will study dynamics of a special class of orbits in $\Gamma\backslash G$. The dynamical properties of these orbits will help us control the rates of the effective results in this paper. Since we are dealing with the noncompact case of $\Gamma\backslash G$, and also for the sake of completeness, we include the technique of [V10] and prove Theorem 1.1.2 in section 1.5. We will finish the proof of the main theorem in section 1.6. Further discussions will be included in section 1.7.

It may be interesting to explore the relation between the techniques used in this work and those developed in the work of Sarnak and Ubis [SU15], where they have described the limiting distribution of horocycles at primes.

1.2 Prerequisites

Throughout this note, if there exists an absolute constant $C > 0$ such that $f \leq Cg$, then we write $f \ll g$. If $f \ll g$ and $g \ll f$, then we use the notation $f \sim g$. We denote $G = \mathrm{PSL}(2, \mathbb{R})$ and Γ a non-uniform lattice in G. Let

$$N = \{u(t) | t \in \mathbb{R}\}, \quad A = \left\{ \begin{pmatrix} s & 0 \\ 0 & s^{-1} \end{pmatrix} \middle| s \in \mathbb{R}_+ \right\}.$$

For any element $a \in A$, we denote $\alpha(a) = s$.

One of the ingredients in our calculations is the effective version of the mixing property of unipotent flows in $\Gamma\backslash G$. The following effective version is proved by Kleinbock and Margulis [KM99].

Theorem 1.2.1 (Kleinbock and Margulis [KM99]). *There exists $\kappa > 0$ such that for any $f, g \in C^\infty(\Gamma\backslash G)$, we have*

$$\left| \int_{\Gamma\backslash G} f(xu(t))g(x)d\mu(x) - \int_{\Gamma\backslash G} f \int_{\Gamma\backslash G} g \right| \ll (1 + |t|)^{-\kappa} \|f\|_{\infty,1} \|g\|_{\infty,1}.$$

Here μ is the Haar measure on $\Gamma\backslash G$.

Remark 1.2.1. Note that when $G = \mathrm{PSL}(2, \mathbb{R})$, we can calculate κ explicitly. Indeed, let $\lambda > 0$ denote the smallest eigenvalue in the discrete spectrum of the Laplacian Δ on $\Gamma\backslash\mathfrak{H}$, then it follows from [V10] formula (9.7) and the technique of Lemma 2.3 in [R87] that $\kappa = 2s - \epsilon$ for any $\epsilon > 0$. Here s is defined as in Remark 1.1.2.

Another ingredient in the calculations is the effective version of Dani and Smillie's result [DS84] proved by Strömbergsson [S13].

Theorem 1.2.2 (Strömbergsson [S13]). *For all $p \in \Gamma\backslash G$, $T \geq 10$, and all $f \in C^4(\Gamma\backslash G)$ such that $\|f\|_{\infty,4} < \infty$ we have*

$$\left| \frac{1}{T} \int_0^T f(pu(t))dt - \int_{\Gamma\backslash G} f d\mu \right| \leq O(\|f\|_{\infty,4}) r^{-s} \ln^3(r+2)$$

provided that $r \geq 1$. Here $s > 0$ is a number depending on the spectrum of the Laplacian on $\Gamma\backslash\mathfrak{H}$ and $r = r(p,T) = T \cdot e^{-dist(g_{\log T}(p))}$. The implied constants depend only on Γ and p_0.

Remark 1.2.2. Here we can take s as in Remark 1.1.2, i.e., let $\lambda > 0$ be the smallest eigenvalue in the discrete spectrum of the Laplacian Δ on $\Gamma\backslash\mathfrak{H}$, and

$$s = \begin{cases} \frac{1-\sqrt{1-4\lambda}}{2}, & \text{if } 0 < \lambda < \frac{1}{4}; \\ \frac{1}{2}, & \text{otherwise.} \end{cases}$$

Readers may refer to [S13] for more details. We will prove a weaker version of this theorem in Chapter 3 using only mixing property, and Theorem 1.1.1 could be proved by using techniques only from dynamical systems for many Diophantine points (at least for a subset of full Haar measure).

1.3 The Diophantine Condition

First we deduce Corollary 1.1.1 from the main theorem.

Proof of Corollary 1.1.1. If Γ is a subgroup of finite index of $\mathrm{PSL}(2,\mathbb{Z})$, then we can pick $\sigma_j \in \mathrm{PSL}(2,\mathbb{Z})$ $(1 \leq j \leq k)$. Now let $p = \Gamma g \in \Gamma\backslash G$ with

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Note that for each m_j, we have

$$m_j(\pi_j^{-1}(p)) \subseteq g^{-1}\mathbb{Z}^2 \setminus \{0\} = \left\{ \begin{pmatrix} dm - bn \\ -cm + an \end{pmatrix} \middle| \begin{pmatrix} m \\ n \end{pmatrix} \in \mathbb{Z}^2 \setminus \{0\} \right\}.$$

If there exist constants $\zeta > 0, \mu, \nu > 0$ such that for any $(m, n) \in \mathbb{Z}^2 \setminus \{0\}$

$$|an - cm| \geq \mu \ \text{ or } \ |dm - bn|^\zeta |an - cm| \geq \nu, \tag{1.1}$$

then p is Diophantine of type $(\zeta, \ldots, \zeta)$ by the definition above. In particular, if $a/c \in \mathbb{R}$ is a Diophantine number of type ζ, i.e. there exists $C > 0$ such that for $m/n \in \mathbb{Q}$,

$$|n|^\zeta \left| n \cdot \frac{a}{c} - m \right| \geq C,$$

then condition (1.1) holds because when $|an - cm|$ is sufficiently small,

$$|dm - bn| = \frac{|cdm - bcn|}{c} = \frac{|cdm - (ad-1)n|}{c} = \frac{|d(cm - an) + n|}{c} \sim |n|.$$

Hence, Corollary 1.1.1 follows from the main theorem. □

In order to prove the main theorem, we have to analyze the map $m_j : \Gamma_j \backslash G \to \mathbb{R}^2/\pm$ for each j. The following lemma is well known. The reader may refer to [DS84]. We will denote B_d the ball of radius d around the origin in $\mathbb{R}^2$.

Lemma 1.3.1 ([DS84] Lemma 2.2)**.** *For each j with the maps $\pi_j : \Gamma_j \backslash G \to \Gamma \backslash G$ and $m_j : \Gamma_j \backslash G \to \mathbb{R}^2/\pm$, there exists a constant $d_j > 0$ such that for any $p \in \Gamma \backslash G$ there exists at most one point of $m_j(\pi_j^{-1}(p))$ which lies in B_{d_j}.*

Remark 1.3.1. We will fix these d_j's for $j = 1, 2, \ldots, k$ throughout this note.

Lemma 1.3.2. *If $p \in \Gamma\backslash G$ is Diophantine of type $(\kappa_1, \kappa_2, \ldots, \kappa_k)$, then the orbit $\{g_t(p)|t \geq 0\}$ is non-divergent.*

Proof. Suppose that $\{g_t(p)|t \geq 0\}$ is divergent. Let η_j be the cusp where $\{g_t(p)|t \geq 0\}$ diverges. By Lemma 11.29 in [EW10], we know that $\{g_t(p)|t \geq 0\}$ is divergent if and only if $\{pu(t)\}$ is periodic in $\Gamma\backslash G$. Combined with Lemma 2.1 in [DS84], this would imply that there is a point $\begin{pmatrix} x \\ y \end{pmatrix} \in m_j(\pi_j^{-1}(p))$ lying on the x-axis in $\mathbb{R}^2$, i.e. $y = 0$, which contradicts the Diophantine condition. Therefore, $\{g_t(p)|t \geq 0\}$ is non-divergent. □

Definition 1.3.1. For $p \in \Gamma\backslash G$, we define

$$\|p\|_j := \min \left\{ \left\| \begin{pmatrix} a \\ b \end{pmatrix} \right\| \,\middle|\, \begin{pmatrix} a \\ b \end{pmatrix} \in m_j(\pi_j^{-1}(p)) \right\}$$

where $\|\cdot\|$ denotes the standard Euclidean norm in $\mathbb{R}^2$. Moreover, we define

$$d(p) = \min\{\|p\|_j | j = 1, 2, \ldots, k\}.$$

Lemma 1.3.3. *For any $p \in \Gamma\backslash SL(2, \mathbb{R})$, we have*

$$e^{\mathrm{dist}(p)} \sim \frac{1}{d(p)^2}.$$

Proof. Recall that η_j $(1 \leq j \leq k)$ are the inequivalent cusps of $\Gamma\backslash\mathfrak{H}$. For each $1 \leq j \leq k$, we fix a small neighborhood C_j of η_j in $\Gamma\backslash G$ such that $C_1, C_2, \ldots, C_k$ are pairwise disjoint. Also we fix a point $q_j \in C_j$ for each $1 \leq j \leq k$. We observe that it suffices to prove the lemma for $p \in C_j$ $(j = 1, 2, \ldots, k)$ since the complement of $\bigcup C_j$ is compact. Let $p \in C_j$ for some $j \in \{1, 2, \ldots, k\}$. Let $\alpha_j > 0$ be such that

π_j maps $\sigma_j N_{\Omega_j} A_{\alpha_j} K$ isomorphically to C_j. Then we can pick a representative for p in $\sigma_j N_\Omega A_{\alpha_j} K$, say $\sigma_j n_p a_p k_p$, i.e. $p = \Gamma \sigma_j n_p a_p k_p = \pi_j(\Gamma_j \sigma_j n_p a_p k_p)$. By definition we know that

$$d(p) = \|p\|_j = \|k_p^{-1} a_p^{-1} e_1\| = \alpha(a_p)^{-1}.$$

On the other hand, in the fundamental domain of $\Gamma \backslash \mathfrak{H}$, the point corresponding to $p = \pi_j(\Gamma_j \sigma_j n_p a_p k_p) \in C_j$ is equal to

$$\Gamma \sigma_j n_p a_p k_p \cdot i = \Gamma \sigma_j (n_p a_p \cdot i) = \Gamma \sigma_j (n_p \cdot (\alpha(a_p)^2 i)).$$

Since σ_j is fixed and n_p is in the compact set N_{Ω_j} of N, we obtain

$$|d_{\mathfrak{H}}(\pi(q_j), \pi(p)) - \ln \alpha(a_p)^2| \leq C_j$$

for some constant $C_j > 0$. Since q_j is fixed, we have

$$|\operatorname{dist}(p) - d_{\mathfrak{H}}(\pi(q_j), \pi(p))| \leq C_j'$$

for some $C_j' > 0$. Therefore we get

$$|\operatorname{dist}(p) - \ln \alpha(a_p)^2| \leq C$$

for $C = \max\{C_1 + C_1', C_2 + C_2', \ldots, C_k + C_k'\}$ and hence

$$e^{\operatorname{dist}(p)} = e^{(\operatorname{dist}(p) - \ln \alpha(a_p)^2) + \ln \alpha(a_p)^2} \sim e^{\ln \alpha(a_p)^2} = \alpha(a_p)^2 = \frac{1}{d(p)^2}.$$

$\square$

Now we prove that the Diophantine condition on $p \in \Gamma \backslash G$ can be defined by the excursion rate of the geodesic orbit $\{g_t(p)\}_{t>0}$. We need some notations. As in the proof of Lemma 1.3.3, let $\eta_1, \eta_2, \dots, \eta_k$ be the inequivalent cusps of $\Gamma\backslash\mathfrak{H}$ and we choose the neighborhood C_j of η_j $(1 \le j \le k)$ in $\Gamma\backslash G$ such that $C_1, C_2, \dots, C_k$ are pairwise disjoint. For each $1 \le j \le k$, we define a function on $\Gamma\backslash G$ by

$$\operatorname{dist}^{(j)}(p) = \begin{cases} \operatorname{dist}(p), & \text{if } p \in C_j; \\ 0, & \text{otherwise.} \end{cases}$$

Lemma 1.3.4. *A point $p \in \Gamma\backslash G$ has Diophantine type $(\kappa_1, \kappa_2, \dots, \kappa_k)$ if and only if $\kappa_j \ge 1$ and*

$$\limsup_{t\to\infty} \left(\operatorname{dist}^{(j)}(g_t(p)) - \frac{\kappa_j - 1}{\kappa_j + 1} t \right) < \infty$$

for each $j \in \{1, 2, \dots, k\}$.

Proof. As in the proof of Lemma 1.3.3, we know that there exists $\alpha_j > 0$ such that π_j maps $\sigma_j N_{\Omega_j} A_{\alpha_j} K$ isomorphically to C_j. Also using the same argument in the proof of Lemma 1.3.3, we can get that for any $q \in C_j$,

$$e^{\operatorname{dist}^{(j)}(q)} \sim \frac{1}{\|q\|_j^2}. \tag{1.2}$$

If $p \in \Gamma\backslash G$ is Diophantine of type $(\kappa_1, \kappa_2, \dots, \kappa_k)$, then for each j, any $\begin{pmatrix} a \\ b \end{pmatrix} \in m_j(\pi_j^{-1}(p))$ satisfies

$$|b| \ge \mu_j \text{ or } |a|^{\kappa_j} |b| \ge \nu_j.$$

Since

$$m_j(\pi_j^{-1}(g_t(p))) = \begin{pmatrix} e^{-t/2} & 0 \\ 0 & e^{t/2} \end{pmatrix} m_j(\pi_j^{-1}(p)), \tag{1.3}$$

this implies that any $\begin{pmatrix} x \\ y \end{pmatrix} \in m_j(\pi_j^{-1}(g_t(p)))$ satisfies

$$|y| \geq e^{t/2}\mu_j \text{ or } |x|^{\kappa_j}|y| \geq \nu_j e^{(1-\kappa_j)t/2}. \tag{1.4}$$

Note that this holds for all $t > 0$.

By Lemma 1.3.2, we know that $\{g_t(p)|t \geq 0\}$ is nondivergent. So there exists a compact subset $S \subset \Gamma\backslash G$ such that $g_{t_i}(p)$ remains in S for infinitely many $t_i \to \infty$. By the compactness of S, we can find a constant $C_0 > 0$ such that for each t_i, there exists $\begin{pmatrix} x_i \\ y_i \end{pmatrix} \in m_j(\pi_j^{-1}(g_{t_i}(p)))$ satisfying

$$\left\| \begin{pmatrix} x_i \\ y_i \end{pmatrix} \right\| \leq C_0.$$

This implies via equation (1.4) that $\kappa_j \geq 1$. Now fix t and for any $\begin{pmatrix} x \\ y \end{pmatrix} \in m_j(\pi_j^{-1}(g_t(p)))$ we have

$$\|(x,y)\| \geq |y| \geq e^{t/2}\mu_j \text{ or } \|(x,y)\| \geq \max\{|x|,|y|\} \geq |x|^{\frac{\kappa_j}{\kappa_j+1}}|y|^{\frac{1}{\kappa_j+1}} \geq \nu_j^{\frac{1}{\kappa_j+1}} e^{\frac{1-\kappa_j}{1+\kappa_j}\frac{t}{2}}.$$

Therefore, $\|g_t(p)\|_j \geq e^{t/2}\mu_j$ or $\nu_j^{\frac{1}{\kappa_j+1}} e^{\frac{1-\kappa_j}{1+\kappa_j}t/2}$. By equation (1.2) and $\kappa_j \geq 1$, we get that

$$\limsup_{t\to\infty}\left(\mathrm{dist}^{(j)}(g_t(p)) - \frac{\kappa_j - 1}{\kappa_j + 1}t\right) < \infty.$$

Conversely, if the above inequality holds for each j with $\kappa_j \geq 1$, then by equation (1.2) there exists a constant $C > 0$ such that for all $t > 0$ we have

$$\|g_t(p)\|_j \geq Ce^{\frac{1-\kappa_j}{1+\kappa_j}\frac{t}{2}}.$$

This implies via equation (1.3) that for any $\begin{pmatrix} a \\ b \end{pmatrix} \in m_j(\pi_j^{-1}(p))$ we have

$$e^{-t}a^2 + e^t b^2 \geq Ce^{\frac{1-\kappa_j}{1+\kappa_j}t}. \tag{1.5}$$

By discreteness of $m_j(\pi_j^{-1}(p))$ in $\mathbb{R}^2$, there exists a constant $\mu_j > 0$ such that if $\begin{pmatrix} a \\ b \end{pmatrix} \in m_j(\pi_j^{-1}(p))$ satisfies $|b| < \mu_j$, then $|b| < |a|$. Now for such $\begin{pmatrix} a \\ b \end{pmatrix}$, we take $t > 0$ such that $e^{-t}a^2 = e^t b^2$. By equation (1.5), this implies that $|b| \geq \sqrt{\frac{C}{2}}e^{-\frac{\kappa_j}{1+\kappa_j}t}$ and hence

$$|a|^{\kappa_j}|b| = |e^t b|^{\kappa_j}|b| \geq \left(\sqrt{\frac{C}{2}}\right)^{\kappa_j+1}.$$

This implies that p is Diophantine of type $(\kappa_1, \kappa_2, \ldots, \kappa_k)$. $\square$

1.4 $(C, \alpha; \rho, \epsilon_0)$-Good Functions in Presence of Diophantine Condition

This section will be important in the proof of the main theorem. First, we need a modified version of the concept of (C, α)-good functions (see [KM98] for the definition of (C, α)-good functions).

Definition 1.4.1. A function $f(x)$ is said to be $(C, \alpha; \rho, \epsilon_0)$-good if for any $0 < \epsilon < \epsilon_0$ and any $I = (x_1, x_2) \subset [1, \infty)$ with $|f(x_1)| = \rho$, we have

$$m(\{x \in I \| f(x)| \leq \epsilon\}) \leq C \left(\frac{\epsilon}{\rho}\right)^{\alpha} m(I)$$

where m denotes the Lebesgue measure on $\mathbb{R}$.

Now we shall begin to study a special class of functions and prove that they are $(C, \alpha; \rho, \epsilon_0)$-good for some C, α, ρ and $\epsilon_0 > 0$. Note that we restrict these functions to the domain $[1, \infty)$.

Lemma 1.4.1. *Let $\kappa, \mu, \nu > 0$ and $0 < \gamma < \frac{1}{\kappa+4}$. Let $\begin{pmatrix} a \\ b \end{pmatrix} \in \mathbb{R}^2 \setminus \{0\}$ be such that*

$$|b| \geq \mu \quad or \quad |a|^{\kappa}|b| \geq \nu.$$

Then there exist $C, \epsilon_0 > 0$ such that

$$f(x) = (bx^{\frac{3}{4}+\gamma} - ax^{-\frac{1}{4}})^2 (x^{\frac{1}{4}-\frac{1}{\kappa+4}})^2 + (bx^{\frac{1}{4}})^2 (x^{\frac{1}{4}-\frac{1}{\kappa+4}})^2$$

is $(C, \frac{1}{2}; \rho, \epsilon_0)$-good on $[1, \infty)$, where ρ is any fixed constant $\leq f(1)$. Here the constants C, ϵ_0 depend only on ρ, κ, μ, ν and γ.

Proof. We observe that if $|b| \geq \mu$, then $f(x) \geq (bx^{\frac{1}{4}})^2 (x^{\frac{1}{4}-\frac{1}{\kappa+4}})^2 \geq b^2 \geq \mu^2$ and $f(x)$ is automatically $(C, \alpha; \rho, \epsilon_0)$-good for any C, α, ρ and $\epsilon_0 = \mu^2/2$. Therefore, in the following we assume that $|b| < \mu$ and hence $|a|^{\kappa}|b| \geq \nu$. We have two cases: $ab < 0$ and $ab > 0$.

Case 1: $ab < 0$. Our function $f(x)$ then becomes

$$f(x) = (|b|x^{\frac{3}{4}+\gamma} + |a|x^{-\frac{1}{4}})^2(x^{\frac{1}{4}-\frac{1}{\kappa+4}})^2 + (bx^{\frac{1}{4}})^2(x^{\frac{1}{4}-\frac{1}{\kappa+4}})^2.$$

We have

$$\begin{aligned} f(x) &\geq \left((|b|x^{\frac{3}{4}+\gamma} + |a|x^{-\frac{1}{4}})x^{\frac{1}{4}-\frac{1}{\kappa+4}}\right)^2 \\ &\geq \left(\max\{|b|x^{1+\gamma-\frac{1}{\kappa+4}}, |a|x^{-\frac{1}{\kappa+4}}\}\right)^2 \\ &\geq \left((|b|x^{1+\gamma-\frac{1}{\kappa+4}})^{\frac{1}{\kappa+1}}(|a|x^{-\frac{1}{\kappa+4}})^{\frac{\kappa}{\kappa+1}}\right)^2 \\ &\geq \left((|b||a|^{\kappa}x^{1+\gamma-\frac{\kappa+1}{\kappa+4}})^{\frac{1}{\kappa+1}}\right)^2 \\ &\geq \nu^{\frac{2}{\kappa+1}}. \end{aligned}$$

This implies that $f(x)$ is $(C_2, \alpha_2; \rho, \epsilon_0)$-good for any $C_2, \alpha_2 > 0$ and $\epsilon_0 = \frac{1}{2}\nu^{\frac{2}{\kappa+1}}$.

Case 2: $ab > 0$. Without loss of generality, we assume that $a > 0, b > 0$. Now let $I = (x_1, x_2) \subset [1, \infty)$ be an interval (x_1, x_2) where $f(x_1) = \rho$. Since $f(x_1) = \rho$, we know that

$$\text{either } (bx_1^{\frac{3}{4}+\gamma} - ax_1^{-\frac{1}{4}})^2(x_1^{\frac{1}{4}-\frac{1}{\kappa+4}})^2 \geq \frac{\rho}{2} \text{ or } (bx_1^{\frac{1}{4}})^2(x_1^{\frac{1}{4}-\frac{1}{\kappa+4}})^2 \geq \frac{\rho}{2}.$$

If $(bx_1^{\frac{1}{4}})^2(x_1^{\frac{1}{4}-\frac{1}{\kappa+4}})^2 \geq \rho/2$, then there is nothing to prove because $f(x) \geq \rho/2$ for all $x \geq x_1$.

Otherwise, we have

$$(bx_1^{\frac{3}{4}+\gamma} - ax_1^{-\frac{1}{4}})^2(x_1^{\frac{1}{4}-\frac{1}{\kappa+4}})^2 \geq \frac{\rho}{2}.$$

Note that

$$\{x \in I \| f(x)| \le \epsilon\} \subseteq \{x \in I | (bx^{\frac{3}{4}+\gamma} - ax^{-\frac{1}{4}})^2 (x^{\frac{1}{4}-\frac{1}{\kappa+4}})^2 \le \epsilon\}.$$

Therefore, to finish the proof of the lemma, it suffices to show that there exist $C, \epsilon_0 > 0$ depending only on $\rho, \kappa, \mu, \nu, \gamma$ such that for any $0 < \epsilon < \epsilon_0$ we have

$$\frac{1}{x_2 - x_1} m(\{x \in (x_1, x_2) \big| |g(x)| \le \sqrt{\epsilon}\}) \le C \left(\frac{\epsilon}{\rho}\right)^{\frac{1}{2}} \tag{1.6}$$

where $g(x) = (bx^{\frac{3}{4}+\gamma} - ax^{-\frac{1}{4}})x^{\frac{1}{4}-\frac{1}{\kappa+4}} = bx^{1+\gamma-\frac{1}{\kappa+4}} - ax^{-\frac{1}{\kappa+4}}$. Note that $g(x)$ is increasing and $|g(x_1)| \ge \sqrt{\rho/2}$. Without loss of generality, we may assume that $|g(x_1)| = \sqrt{\rho/2}$. If $g(x_1) = \sqrt{\rho/2}$, since $g(x)$ is increasing, the $(C, \alpha; \rho, \epsilon_0)$-good property automatically holds in this case with $\epsilon_0 = \frac{1}{2}\sqrt{\rho/2}$. Therefore we assume that $g(x_1) = -\sqrt{\rho/2}$. In this case, we will prove that the inequality (1.6) holds with $\epsilon_0 = \frac{1}{2}\sqrt{\rho/2}$.

Let $0 < \epsilon < \epsilon_0$. Since $g(x)$ is increasing with $g(x_1) = -\sqrt{\rho/2}$, if we fix x_1 and let x_2 vary as a variable, then the ratio

$$\frac{1}{x_2 - x_1} m(\{x \in (x_1, x_2) \big| |g(x)| \le \sqrt{\epsilon}\})$$

would attain its maximum when $g(x_2) = \sqrt{\epsilon}$. So we will assume that $g(x_2) = \sqrt{\epsilon}$. To compute this maximal ratio, let $z \in (x_1, x_2)$ such that $g(z) = -\sqrt{\epsilon}$, and then by the mean value theorem we obtain

$$\begin{aligned} & \frac{1}{x_2 - x_1} m(\{x \in (x_1, x_2) \big| |g(x)| \le \sqrt{\epsilon}\}) \\ = \quad & \frac{x_2 - z}{x_2 - x_1} = \frac{g'(\xi_2)}{g'(\xi_1)} \cdot \frac{g(x_2) - g(z)}{g(x_2) - g(x_1)} \end{aligned}$$

$$= \frac{g'(\xi_2)}{g'(\xi_1)} \cdot \frac{2\sqrt{\epsilon}}{\sqrt{\epsilon}+\sqrt{\rho/2}} \tag{1.7}$$

where ξ_1 is between x_2 and z, ξ_2 is between x_1 and x_2.

Let $x_3 \in [1, \infty)$ such that $g(x_3) = \sqrt{\rho/2}$. Then $(x_1, x_2) \subset (x_1, x_3)$. According to equation (1.7), to prove formula (1.6), it suffices to prove that for any $x, y \in (x_1, x_3)$ the ratio

$$\frac{g'(x)}{g'(y)}$$

is bounded above by constants depending only on ρ, κ, μ, ν and γ. Observe that

$$g'(x) = \left(1 + \gamma - \frac{1}{\kappa+4}\right) b x^{\gamma - \frac{1}{\kappa+4}} + \frac{a}{\kappa+4} x^{-\frac{\kappa+5}{\kappa+4}}$$

is decreasing since $\gamma < \frac{1}{\kappa+4}$. Therefore, to get an upper bound for $g'(x)/g'(y)$ $(x, y \in (x_1, x_3))$, we only need to estimate $g'(x_1)/g'(x_3)$. By the condition that $g(x_1) = -\sqrt{\rho/2}$ and $g(x_3) = \sqrt{\rho/2}$, we have

$$\begin{aligned}
\frac{g'(x_1)}{g'(x_3)} &= \frac{\left(1+\gamma-\frac{1}{\kappa+4}\right) b x_1^{\gamma-\frac{1}{\kappa+4}} + \frac{a}{\kappa+4} x_1^{-\frac{\kappa+5}{\kappa+4}}}{\left(1+\gamma-\frac{1}{\kappa+4}\right) b x_3^{\gamma-\frac{1}{\kappa+4}} + \frac{a}{\kappa+4} x_3^{-\frac{\kappa+5}{\kappa+4}}} \\
&= \frac{\left(1+\gamma-\frac{1}{\kappa+4}\right) (a x_1^{-\frac{1}{\kappa+4}} - \sqrt{\rho/2})/x_1 + \frac{a}{\kappa+4} x_1^{-\frac{\kappa+5}{\kappa+4}}}{\left(1+\gamma-\frac{1}{\kappa+4}\right) (a x_3^{-\frac{1}{\kappa+4}} + \sqrt{\rho/2})/x_3 + \frac{a}{\kappa+4} x_3^{-\frac{\kappa+5}{\kappa+4}}} \\
&\leq \frac{\left(1+\gamma-\frac{1}{\kappa+4}\right) a x_1^{-\frac{1}{\kappa+4}}/x_1 + \frac{a}{\kappa+4} x_1^{-\frac{\kappa+5}{\kappa+4}}}{\left(1+\gamma-\frac{1}{\kappa+4}\right) a x_3^{-\frac{1}{\kappa+4}}/x_3 + \frac{a}{\kappa+4} x_3^{-\frac{\kappa+5}{\kappa+4}}} = \left(\frac{x_3}{x_1}\right)^{\frac{\kappa+5}{\kappa+4}}.
\end{aligned}$$

Now let $x_0 \in (x_1, x_3)$ such that $g(x_0) = 0$. ($x_0 = (a/b)^{\frac{1}{1+\gamma}}$ by solving the equation $g(x) = 0$). We set $x_1 = \delta_1 x_0$ and $x_3 = \delta_2 x_0$ for some δ_1, δ_2. Then δ_1, δ_2 satisfy the following equation

$$|b(x_0\delta)^{1+\gamma-\frac{1}{\kappa+4}} - a(x_0\delta)^{-\frac{1}{\kappa+4}}| = \sqrt{\frac{\rho}{2}}$$

since $|g(x_1)| = |g(x_3)| = \sqrt{\rho/2}$. By the fact that $bx_0^{1+\gamma-\frac{1}{\kappa+4}} = ax_0^{-\frac{1}{\kappa+4}}$ and $ba^\kappa \geq \nu$, this equation becomes

$$|ax_0^{-\frac{1}{\kappa+4}}\delta^{1+\gamma-\frac{1}{\kappa+4}} - ax_0^{-\frac{1}{\kappa+4}}\delta^{-\frac{1}{\kappa+4}}| = \sqrt{\frac{\rho}{2}}$$

$$\begin{aligned} |\delta^{1+\gamma-\frac{1}{\kappa+4}} - \delta^{-\frac{1}{\kappa+4}}| &= \sqrt{\frac{\rho}{2}}\frac{x_0^{\frac{1}{\kappa+4}}}{a} = \sqrt{\frac{\rho}{2}}\frac{(a^{\kappa+1}/ba^\kappa)^{\frac{1}{(1+\gamma)(\kappa+4)}}}{a} \\ &\leq \sqrt{\frac{\rho}{2}}\frac{1}{\nu^{\frac{1}{(1+\gamma)(\kappa+4)}}}\frac{a^{\frac{\kappa+1}{(1+\gamma)(\kappa+4)}}}{a}. \end{aligned}$$

Here $\frac{\kappa+1}{(1+\gamma)(\kappa+4)} < 1$. Since $b \leq \mu$ and hence $a \geq \sqrt[\kappa]{\nu/\mu}$, the above inequality becomes

$$|\delta^{1+\gamma-\frac{1}{\kappa+4}} - \delta^{-\frac{1}{\kappa+4}}| \leq \sqrt{\frac{\rho}{2}}\frac{1}{\nu^{\frac{1}{(1+\gamma)(\kappa+4)}}}\left(\sqrt[\kappa]{\frac{\nu}{\mu}}\right)^{\frac{\kappa+1}{(1+\gamma)(\kappa+4)}-1}$$

which holds for $\delta = \delta_1$ and δ_2. This shows that δ_1, δ_2 are bounded above and below by constants depending only on ρ, κ, μ, ν and γ. Therefore

$$\frac{g'(x_1)}{g'(x_3)} \leq \left(\frac{x_3}{x_1}\right)^{(\kappa+5)/(\kappa+4)} = \left(\frac{\delta_2}{\delta_1}\right)^{(\kappa+5)/(\kappa+4)}$$

is bounded above by a constant depending only on ρ, κ, μ, ν and γ. This completes the proof of the lemma.

$\square$

For the rest of this section, we turn to the dynamics on $\Gamma\backslash G$. For later use, we give the following definition.

Definition 1.4.2. For any $\delta > 0$ and any $j \in \{1, 2, \ldots, k\}$, we define the subset of

$\Gamma\backslash G$

$$S_{j,\delta} := \{q \in \Gamma\backslash G \big| \|q\|_j \leq \delta\}.$$

Moreover, we define

$$S_\delta := \bigcup_j S_{j,\delta} = \{q \in \Gamma\backslash G \big| d(q) \leq \delta\}.$$

Lemma 1.4.2. *Let $p \in \Gamma\backslash G$ be Diophantine of type $(\kappa_1, \kappa_2, \ldots, \kappa_k)$. We fix $j \in \{1, 2, \ldots, k\}$ and let $0 < \gamma < 1/(\kappa_j + 4)$. Then for sufficiently small $\epsilon > 0$ and $T \geq 1$, we have*

$$\frac{1}{T} m \left(\left\{ x \in [1, T] \middle| p \begin{pmatrix} x^{\frac{1}{4}} & x^{\frac{3}{4}+\gamma} \\ 0 & x^{-\frac{1}{4}} \end{pmatrix} \in S_{j, \epsilon x^{-\frac{1}{4}+\frac{1}{\kappa_j+4}}} \right\} \right) \leq C\epsilon$$

where C is a constant only depending on p and γ.

Proof. We will use the notations, the maps m_j and π_j in section 3. Then the image of $\pi_j^{-1}\left(p\begin{pmatrix} x^{\frac{1}{4}} & x^{\frac{3}{4}+\gamma} \\ 0 & x^{-\frac{1}{4}} \end{pmatrix}\right)$ under m_j is equal to

$$\begin{aligned} & \begin{pmatrix} x^{\frac{1}{4}} & x^{\frac{3}{4}+\gamma} \\ 0 & x^{-\frac{1}{4}} \end{pmatrix}^{-1} m_j(\pi_j^{-1}(p)) \\ = & \left\{ \begin{pmatrix} x^{-\frac{1}{4}} & -x^{\frac{3}{4}+\gamma} \\ 0 & x^{\frac{1}{4}} \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} \right\} = \left\{ \begin{pmatrix} ax^{-\frac{1}{4}} - bx^{\frac{3}{4}+\gamma} \\ bx^{\frac{1}{4}} \end{pmatrix} \right\} \end{aligned}$$

where $\begin{pmatrix} a \\ b \end{pmatrix}$ runs over all points in $m_j(\pi_j^{-1}(p))$. By definition, what we need to prove is equivalent to the following

$$m\left(\left\{x\in[1,T]\middle|\left\|p\begin{pmatrix}x^{\frac{1}{4}} & x^{\frac{3}{4}+\gamma}\\ 0 & x^{-\frac{1}{4}}\end{pmatrix}\right\|_j\leq \epsilon x^{-\frac{1}{4}+\frac{1}{\kappa_j+4}}\right\}\right)\leq C\epsilon T.$$

which is equivalent to the following

$$\begin{aligned} & m\left(\left\{x\in[1,T]\middle|\exists \text{ a point in } m_j\left(\pi_j^{-1}\left(p\begin{pmatrix}x^{\frac{1}{4}} & x^{\frac{3}{4}+\gamma}\\ 0 & x^{-\frac{1}{4}}\end{pmatrix}\right)\right)\right.\right. \\ & \qquad\left.\left.\text{with length} \leq \epsilon x^{-\frac{1}{4}+\frac{1}{\kappa_j+4}}\right\}\right) \\ \leq\; & C\epsilon T. \end{aligned}$$

Let $\rho=\min\left\{d\left(p\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}\right),d_1,d_2,\dots,d_k\right\}$, where d_j's are as in Remark 1.3.1. We denote by P the subset $m_j(\pi_j^{-1}(p))$. For $(a,b)\in P$, let $I^l_{(a,b)}(l=1,2,\dots)$ be all the maximal connected subintervals in $[1,T]$ such that for any $x\in I^l_{(a,b)}$ the point of $\pi_j^{-1}\left(p\begin{pmatrix}x^{\frac{1}{4}} & x^{\frac{3}{4}+\gamma}\\ 0 & x^{-\frac{1}{4}}\end{pmatrix}\right)$ corresponding to (a,b); that is

$$\begin{pmatrix}x^{\frac{1}{4}} & x^{\frac{3}{4}+\gamma}\\ 0 & x^{-\frac{1}{4}}\end{pmatrix}^{-1}\begin{pmatrix}a\\ b\end{pmatrix}=\begin{pmatrix}ax^{-\frac{1}{4}}-bx^{\frac{3}{4}+\gamma}\\ bx^{\frac{1}{4}}\end{pmatrix}$$

has norm $\leq \rho x^{-\frac{1}{4}+\frac{1}{\kappa_j+4}}$. Since $x\geq 1$, Lemma 1.3.1 implies that all the intervals

$$\{I^l_{(a,b)}|(a,b)\in P,\ l=1,2,\dots\}$$

are pairwise disjoint. Therefore, we have

$$
\begin{aligned}
& m\left(\left\{x\in[1,T]\middle|\exists\text{a point in } m_j\left(\pi_j^{-1}\left(p\begin{pmatrix} x^{\frac14} & x^{\frac34+\gamma}\\ 0 & x^{-\frac14}\end{pmatrix}\right)\right)\right.\right.\\
&\qquad\qquad \left.\left.\text{with length}\le \epsilon x^{-\frac14+\frac{1}{\kappa_j+4}}\right\}\right)\\
=\ & \sum_{(a,b)\in P}\sum_l m\left(x\in I^l_{(a,b)}\middle|\left\|\begin{pmatrix} ax^{-\frac14}-bx^{\frac34+\gamma}\\ bx^{\frac14}\end{pmatrix}\right\|\le \epsilon x^{-\frac14+\frac{1}{\kappa_j+4}}\right)
\end{aligned}
$$

Because of this, to prove the lemma, it suffices to prove the following

$$
m\left(\left\{x\in I^l_{(a,b)}\middle|\left\|\begin{pmatrix} ax^{-\frac14}-bx^{\frac34+\gamma}\\ bx^{\frac14}\end{pmatrix}\right\|\le \epsilon x^{-\frac14+\frac{1}{\kappa_j+4}}\right\}\right)\le C\epsilon m(I^l_{(a,b)}),
$$

for some C,ϵ_0 with $0<\epsilon<\epsilon_0$, or to prove that the function

$$
f(x)=(bx^{\frac34+\gamma}-ax^{-\frac14})^2(x^{\frac14-\frac{1}{\kappa_j+4}})^2+(bx^{\frac14})^2(x^{\frac14-\frac{1}{\kappa_j+4}})^2
$$

has $(C',1/2;\rho^2,\epsilon_0^2)$-good property for some $C'=C\rho$. This follows immediately from Lemma 1.4.1.

□

To conclude this section, we give the following proposition, which is crucial in our proof of the main theorem. It is the discrete version of Lemma 1.4.2.

Proposition 1.4.1. *Let $p\in\Gamma\backslash G$ be Diophantine of type $(\kappa_1,\kappa_2,\dots,\kappa_k)$. Let $0<\gamma<\min\{1/(\kappa_j+4):j=1,2,\dots,k\}$. Then there exists a constant $C_0>0$ depending* *only on p and γ such that for sufficiently small $\epsilon > 0$ and any $N \in \mathbb{N}$,*

$$\frac{1}{N}\left|\left\{n \in [1,N] \cap \mathbb{N} \middle| p \begin{pmatrix} n^{\frac{1}{4}} & n^{\frac{3}{4}+\gamma} \\ 0 & n^{-\frac{1}{4}} \end{pmatrix} \in S_{\theta(n,\epsilon)}\right\}\right| \leq C_0\epsilon$$

where

$$\theta(x,\epsilon) = \epsilon \min\{x^{-\frac{1}{4}+\frac{1}{\kappa_j+4}} | j = 1, 2, \dots, k\}. \tag{1.8}$$

Proof. By the definition of S_δ, it suffices to prove that there exists a constant $C_0 > 0$ depending only on p and γ such that for each j and any $\epsilon > 0$,

$$\frac{1}{N}\left|\left\{n \in [1,N] \cap \mathbb{N} \middle| p \begin{pmatrix} n^{\frac{1}{4}} & n^{\frac{3}{4}+\gamma} \\ 0 & n^{-\frac{1}{4}} \end{pmatrix} \in S_{j,\epsilon n^{-\frac{1}{4}+\frac{1}{\kappa_j+4}}}\right\}\right| \leq C_0\epsilon.$$

We compute that for any $\delta \in (-1, 1)$ and $n \geq 1$

$$\begin{aligned}
&\begin{pmatrix} n^{\frac{1}{4}} & n^{\frac{3}{4}+\gamma} \\ 0 & n^{-\frac{1}{4}} \end{pmatrix}^{-1} \begin{pmatrix} (n+\delta)^{\frac{1}{4}} & (n+\delta)^{\frac{3}{4}+\gamma} \\ 0 & (n+\delta)^{-\frac{1}{4}} \end{pmatrix} \\
=& \begin{pmatrix} n^{-\frac{1}{4}} & -n^{\frac{3}{4}+\gamma} \\ 0 & n^{\frac{1}{4}} \end{pmatrix} \begin{pmatrix} (n+\delta)^{\frac{1}{4}} & (n+\delta)^{\frac{3}{4}+\gamma} \\ 0 & (n+\delta)^{-\frac{1}{4}} \end{pmatrix} \\
=& \begin{pmatrix} (1+\delta/n)^{\frac{1}{4}} & n^{-\frac{1}{4}}(n+\delta)^{\frac{3}{4}+\gamma} - n^{\frac{3}{4}+\gamma}(n+\delta)^{-\frac{1}{4}} \\ 0 & (1+\delta/n)^{-\frac{1}{4}} \end{pmatrix} \\
=& \begin{pmatrix} (1+\delta/n)^{\frac{1}{4}} & ((n+\delta)^{1+\gamma} - n^{1+\gamma})(n(n+\delta))^{-\frac{1}{4}} \\ 0 & (1+\delta/n)^{-\frac{1}{4}} \end{pmatrix}
\end{aligned}$$

which lies in a compact neighborhood U of identity in $\mathrm{PSL}(2,\mathbb{R})$. Let

$$L = \max\{\|g\| | g \in U\}$$

where $\|g\|$ denotes the operator norm of g on $\mathbb{R}^2$. Then by the computations above, we know that

$$\begin{aligned}
&\frac{1}{N}\left|\left\{n \in [1,N]\cap\mathbb{N}\middle| p\begin{pmatrix} n^{\frac{1}{4}} & n^{\frac{3}{4}+\gamma} \\ 0 & n^{-\frac{1}{4}}\end{pmatrix} \in S_{j,\epsilon n^{-\frac{1}{4}+\frac{1}{\kappa_j+4}}}\right\}\right| \\
\leq\ &\frac{1}{N} m\left(\left\{x \in [1,N]\middle| p\begin{pmatrix} x^{\frac{1}{4}} & x^{\frac{3}{4}+\gamma} \\ 0 & x^{-\frac{1}{4}}\end{pmatrix} \in S_{j,L\epsilon x^{-\frac{1}{4}+\frac{1}{\kappa_j+4}}}\right\}\right).
\end{aligned}$$

Now the proposition follows immediately from Lemma 1.4.2.

□

1.5 Calculations

In this section, we shall apply the technique of Venkatesh to obtain some effective results about averaging over arithmetic progressions. It is very similar to [V10], where Venkatesh proved the sparse equidistribution theorem for Γ being cocompact. Since in our setting Γ is non-uniform, and for the sake of self-containedness, we include the details of the calculations in this section. We will follow the notations in [V10]. Throughout this section, we fix an arbitrary point $q \in \Gamma\backslash G$. For a character $\psi : \mathbb{R} \to S^1$, we define

$$\mu_{T,\psi}(f) = \frac{1}{T}\int_0^T \psi(t) f(qu(t))dt$$

for f on $\Gamma\backslash G$.

Lemma 1.5.1 (Cf. [V10, Lemma 3.1]). *There exists a constant $\beta > 0$ which only depends on Γ such that for any $f \in C^\infty(\Gamma\backslash G)$ satisfying $\|f\|_{\infty,4} < \infty$ and $\int_{\Gamma\backslash G} f d\mu = 0$, any character $\psi : \mathbb{R} \to S^1$, any $T \geq 1$ and any $q \in \Gamma\backslash G$ satisfying*

$$r = r(q,T) = T \cdot e^{-\operatorname{dist}(g_{\log T}(q))} \geq 1, \tag{1.9}$$

we have

$$|\mu_{T,\psi}(f)| \ll r^{-\beta} \ln^3(r+2)\|f\|_{\infty,4}$$

and the implicit constant is independent of ψ.

Proof. The proof is almost the same as that of [V10, Lemma 3.1] combined with [S13]. We define

$$\sigma_H(f)(x) = \frac{1}{H}\int_0^H \psi(s) f(xu(s))ds.$$

First it is easy to get that $|\mu_{T,\psi}(f) - \mu_{T,\psi}(\sigma_H(f))| \leq \frac{H}{T}\|f\|_{\infty,0} \leq \frac{H}{r}\|f\|_{\infty,0}$. Now we estimate $\mu_{T,\psi}(\sigma_H(f))$. By Cauchy-Shwartz inequality, we have

$$\begin{aligned}
|\mu_{T,\psi}(\sigma_H(f))| &\leq \frac{1}{T}\left(\int_0^T |\psi(t)|^2 dt\right)^{\frac{1}{2}} \left(\int_0^T |\sigma_H(f)(qu(t))|^2 dt\right)^{\frac{1}{2}} \\
&\leq \left(\frac{1}{T}\int_0^T |\sigma_H(f)(qu(t))|^2 dt\right)^{\frac{1}{2}} \\
&\leq \left(\frac{1}{H^2}\int_0^H \int_0^H \left|\frac{1}{T}\int_0^T \overline{f^y} f^z(qu(t))dt\right| dydz\right)^{\frac{1}{2}}.
\end{aligned}$$

Here f^y and f^z denote the right translation of f by $u(y)$ and the right translation of f by $u(z)$, respectively. Therefore, by Strombergsson's effective equidistribution Theorem 1.2.2, we have

$$\begin{aligned}
|\mu_{T,\psi}(\sigma_H(f))| \leq \Bigg(&\frac{1}{H^2}\int_0^H \int_0^H O(\|\overline{f^y} f^z\|_{\infty,4}) r^{-s} \ln^3(r+2) dydz \\
&+ \frac{1}{H^2}\int_0^H \int_0^H \left|(f^{y-z}, f)\right| dydz\Bigg)^{\frac{1}{2}}.
\end{aligned}$$

for some $s > 0$ depending only on Γ (the spectral gap) and r is as in (1.9) (see Theorem 1.2.2).

By mixing property of unipotent flows (Theorem 1.2.1), we know that

$$|(f^h, f)| \ll (1+|h|)^{-\kappa} \|f\|_{\infty,1}^2.$$

Also by product rule and chain rule in Calculus (see [V10] Lemma 2.2 for details), we know that

$$O(\|\overline{f^y} f^z\|_{\infty,4}) \ll O(\|\overline{f^y}\|_{\infty,4} \|f^z\|_{\infty,4}) \ll y^4 z^4 O(\|f\|_{\infty,4}^2).$$

Therefore, combining all the computations above, we obtain

$$\begin{aligned} |\mu_{T,\psi}(f)| &\leq |\mu_{T,\psi}(f) - \mu_{T,\psi}(\sigma_H(f))| + |\mu_{T,\psi}(\sigma_H(f))| \\ &\ll \frac{H}{r}\|f\|_\infty + (H^8 r^{-s} \ln^3(r+2) + H^{-\kappa})^{\frac{1}{2}} \|f\|_{\infty,4}. \end{aligned}$$

Let $H = r^{\frac{s}{8+\kappa}}$ and we get $\beta = s\kappa/2(8+\kappa)$. This completes the proof of the lemma.

□

We deduce Theorem 1.1.2 from Lemma 1.5.1. It will be crucial in the proof of the main theorem in section 1.6.

Proof of Theorem 1.1.2. The proof is almost the same as that of [V10, Theorem 3.1]. Let $\delta > 0$ and $g_\delta(x) = \max\{\delta^{-2}(\delta - |x|), 0\}$. Let

$$g(x) = \sum_{j \in \mathbb{Z}} g_\delta(x + Kj).$$

On the one hand, since $g(x)$ has most mass on the points $\{Kj | j \in \mathbb{Z}\}$, we know that

$$\left| \int_0^T g(t) f(qu(t)) dt - \sum_{\substack{j \in \mathbb{Z} \\ 0 \le Kj < T}} f(qu(Kj)) \right| \le 2\|f\|_\infty + \frac{T}{K} \delta \|f\|_{\infty,1}.$$

On the other hand, since $g(x)$ is periodic, we have the Fourier expansion

$$g(x) = \sum_{k \in \mathbb{Z}} a_k e^{2\pi i k x / K}.$$

A simple calculation shows that

$$\sum_{k \in \mathbb{Z}} |a_k| = |g(0)| = \frac{1}{\delta}.$$

By Lemma 1.5.1 with characters $\psi_k = e^{2\pi i k x / K}$, we have

$$\left| \int_0^T g(t) f(qu(t)) dt \right| \quad \le \quad \sum_{k \in \mathbb{Z}} |a_k| \left| \int_0^T e^{2\pi i k t / K} f(qu(t)) dt \right| \ll \frac{T \ln^3(r+2)}{\delta r^\beta} \|f\|_{\infty,4}.$$

Combining the calculations above, we have

$$\left| \frac{1}{T/K} \sum_{\substack{j \in \mathbb{Z} \\ 0 \le Kj < T}} f(qu(Kj)) \right| \ll \left(\frac{K}{T} + \delta + \frac{K \ln^3(r+2)}{\delta r^\beta} \right) \|f\|_{\infty,4}.$$

Note that $K < T$, $r \le T$ and $\beta < 1$. Let $\delta = \sqrt{\frac{K \ln^3(r+2)}{r^\beta}}$ and we complete the proof of the theorem.

$\square$

1.6 Proof of the Main Theorem

Proof of the main theorem. By a standard approximation argument, we may assume that $f \in C^\infty(\Gamma\backslash G)$ with $\|f\|_{\infty,4} < \infty$ and

$$\int_{\Gamma\backslash G} f d\mu = 0.$$

We want to find $\gamma_0 > 0$ depending on $\kappa_1, \ldots, \kappa_k$ such that for any $0 < \gamma < \gamma_0$, the main theorem holds. Note that by Taylor expansion, for any $M \in \mathbb{N}$ and $k \in \mathbb{N}$,

$$(M+k)^{1+\gamma} = M^{1+\gamma} + (1+\gamma)M^\gamma k + O(M^{\gamma-1}k^2).$$

Therefore, if M is sufficiently large and $\gamma < 1/2$, then the sequence

$$\left\{(M+k)^{1+\gamma}\middle|\ 0 \leq k \leq \frac{1}{1+\gamma}M^{\frac{1}{2}-\gamma}(k \in \mathbb{N})\right\}$$

is approximately equal to the arithmetic progression

$$\left\{M^{1+\gamma} + (1+\gamma)M^\gamma k\middle|\ 0 \leq k \leq \frac{1}{1+\gamma}M^{\frac{1}{2}-\gamma}(k \in \mathbb{N})\right\}$$

since

$$O(M^{\gamma-1}k^2) \leq O(M^{\gamma-1}(M^{\frac{1}{2}-\gamma})^2) = O(M^{-\gamma}) \to 0$$

as $M \to \infty$.

By Proposition 1.4.1, we know that for any $\epsilon > 0$ and any $N > 0$,

$$\frac{1}{N}\left|\left\{n \in [1, N] \middle| p\begin{pmatrix} n^{\frac{1}{4}} & n^{\frac{3}{4}+\gamma} \\ 0 & n^{-\frac{1}{4}} \end{pmatrix} \in S_{\theta(n,\epsilon)}\right\}\right| \leq C_0\epsilon,$$

where $\theta(n) = \epsilon \min\{n^{-\frac{1}{4}+\frac{1}{\kappa_j+4}} | j = 1, 2, \ldots, k\}$. Set

$$B = \left\{n \in \mathbb{N} \middle| p\begin{pmatrix} n^{\frac{1}{4}} & n^{\frac{3}{4}+\gamma} \\ 0 & n^{-\frac{1}{4}} \end{pmatrix} \in S^c_{\theta(n,\epsilon)}\right\}.$$

We proceed as follows. Fix $\epsilon > 0$. We pick the first element $M_1 \in \mathbb{N}$ which lies in B. Then we take

$$P_1 = \left\{M_1 + k \middle| 0 \leq k \leq \frac{1}{1+\gamma} M_1^{\frac{1}{2}-\gamma}(k \in \mathbb{N})\right\}.$$

Next we pick the first element $M_2 \in \mathbb{N}$ which appears after P_1 and lies in B, and we take

$$P_2 = \left\{M_2 + k \middle| 0 \leq k \leq \frac{1}{1+\gamma} M_2^{\frac{1}{2}-\gamma}(k \in \mathbb{N})\right\}.$$

Then we pick the first element $M_3 \in \mathbb{N}$ which appears after P_2 and lies in B, and so on. In this manner, we get pieces $P_1, P_2, \ldots$ in $\mathbb{N}$ and by our choices of $M_1, M_2, \ldots$, we know that

$$B \subset P_1 \cup P_2 \cup \ldots$$

and hence for any $N > 0$

$$\frac{1}{N} |[1, N] \setminus (P_1 \cup P_2 \cup \ldots)| \leq C_0\epsilon. \tag{1.10}$$

Now we consider each of the pieces P_i. From the discussion above, we know that

$\{n^{1+\gamma} | n \in P_i\}$ is approximated by the arithmetic progression

$$\tilde{P}_i = \left\{ M_i^{1+\gamma} + (1+\gamma) M_i^{\gamma} k \,\middle|\, 0 \le k \le \frac{1}{1+\gamma} M_i^{\frac{1}{2}-\gamma} (k \in \mathbb{N}) \right\}.$$

We would like to apply Theorem 1.1.2 with $T = M_i^{1/2}$, $K = (1+\gamma)M_i^{\gamma}$ and $q = q_i := pu(M_i^{1+\gamma})$ for sufficiently large i. So first we have to check that $r_i := r(q_i, M_i^{\frac{1}{2}}) \ge 1$ for sufficiently large i. We compute that

$$g_{(\log M_i)/2}(q_i) = p \begin{pmatrix} 1 & M_i^{1+\gamma} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} M_i^{\frac{1}{4}} & 0 \\ 0 & M_i^{-\frac{1}{4}} \end{pmatrix} = p \begin{pmatrix} M_i^{\frac{1}{4}} & M_i^{\frac{3}{4}+\gamma} \\ 0 & M_i^{-\frac{1}{4}} \end{pmatrix} \in S^c_{\theta(M_i,\epsilon)},$$

by our choice of $M_i \in B$. By definition 1.4.2 and equation (1.8) , we have

$$d(g_{(\log M_i)/2}(q_i)) \ge \theta(M_i, \epsilon) = \epsilon \min\{M_i^{-\frac{1}{4}+\frac{1}{\kappa_j+4}} | j = 1, 2, \ldots, k\}.$$

By Lemma 1.3.3, $e^{-\operatorname{dist}(q)} \sim d(q)^2$. Hence

$$r_i = M_i^{1/2} e^{-\operatorname{dist}(g_{(\log M_i)/2}(q_i))} \gg \epsilon^2 \min\{M_i^{2/(\kappa_j+4)} | j = 1, 2, \ldots, k\}. \tag{1.11}$$

This implies that $r_i \to \infty$ as $i \to \infty$, since $M_i \to \infty$ by our choices of M_i's.

By Theorem 1.1.2 with $T = M_i^{1/2}$, $K = (1+\gamma)M_i^{\gamma}$, $q_i = pu(M_i^{1+\gamma})$ and $r_i = r(M_i^{\frac{1}{2}}, q_i)$, we have

$$\begin{aligned} \left| \frac{1}{|\tilde{P}_i|} \sum_{n \in \tilde{P}_i} f(pu(n)) \right| &= \left| \frac{1}{\lfloor M_i^{1/2}/(1+\gamma)M_i^{\gamma} \rfloor} \sum_{0 \le (1+\gamma)M_i^{\gamma} k < M_i^{\frac{1}{2}}} f(q_i u((1+\gamma)M_i^{\gamma} k)) \right| \\ &\ll \frac{((1+\gamma)M_i^{\gamma})^{\frac{1}{2}} \ln^{\frac{3}{2}}(r_i + 2)}{r_i^{\frac{\beta}{2}}} \|f\|_{\infty,4}. \end{aligned} \tag{1.12}$$

Since $M_i \to \infty$, according to inequalities (1.11) and (1.12), as long as

$$\gamma < \min\{2\beta/(\kappa_j + 4) | j = 1, 2, \ldots, k\},$$

we have

$$\left| \frac{1}{|\tilde{P}_i|} \sum_{n \in \tilde{P}_i} f(pu(n)) \right| \to 0$$

and hence by the fact that $\{n^{1+\gamma} | n \in P_i\}$ is approximated by $\tilde{P}_i$, i.e., for $0 \leq k \leq \frac{1}{1+\gamma} M_i^{\frac{1}{2}-\gamma}$,

$$|f((M_i + k)^{1+\gamma}) - f(M_i^{1+\gamma} + (1+\gamma)M_i^{\gamma} k)| \ll M_i^{-\gamma} \|f\|_{\infty,1}$$

and $\|f\|_{\infty,1} < \infty$ we obtain

$$\left| \frac{1}{|P_i|} \sum_{n \in P_i} f(pu(n^{1+\gamma})) \right| \to 0 \tag{1.13}$$

as $i \to \infty$. By formula (1.10), the proportion in $[1, N]$ which is not covered by P_i's is small relative to N. Also observe that for the P_i's which intersect $[1, N]$, their lengths are small relative to N. Therefore, by (1.13) we have

$$\begin{aligned}
& \limsup_{N\to\infty} \left| \frac{1}{N} \sum_{n=0}^{N-1} f(pu(n^{1+\gamma})) \right| \\
\leq \; & \limsup_{N\to\infty} \left| \frac{1}{N} \sum_{n\in[1,N]\setminus(\bigcup P_i)} f(pu(n^{1+\gamma})) \right| + \limsup_{N\to\infty} \left| \frac{1}{N} \sum_{n\in[1,N]\cap(\bigcup P_i)} f(pu(n^{1+\gamma})) \right| \\
\leq \; & \limsup_{N\to\infty} \left| \frac{1}{N} \sum_{n\in[1,N]\setminus(\bigcup P_i)} f(pu(n^{1+\gamma})) \right| + \limsup_{N\to\infty} \left| \frac{1}{N} \sum_{[1,N]\cap P_i \neq \emptyset} \sum_{n\in P_i} f(pu(n^{1+\gamma})) \right| \\
\leq \; & C_0 \epsilon \|f\|_{\infty,0} + 0 = C_0 \epsilon \|f\|_{\infty,0}.
\end{aligned}$$

Let $\epsilon \to 0$ and we complete the proof of the main theorem with $\gamma_0 = \min\{2\beta/(\kappa_j + 4) | j = 1, 2, \ldots, k\}$.

$\square$

1.7 Further Discussions

In the introduction, we have defined Diophantine points in $\Gamma\backslash \mathrm{PSL}(2, \mathbb{R})$. Let

$$S_{\kappa_1,\kappa_2,\ldots,\kappa_k} = \{p \in \Gamma\backslash \mathrm{PSL}(2, \mathbb{R}) | p \text{ is Diophantine of type } (\kappa_1, \kappa_2, \ldots, \kappa_k)\}.$$

Then we can calculate the Hausdorff dimension of the complement of $S_{\kappa_1,\kappa_2,\ldots,\kappa_k}$. In fact, we have the following

Theorem 1.7.1. *We have*

$$\dim_H S^c_{\kappa_1,\kappa_2,\ldots,\kappa_k} = 2 + \frac{2}{\min\{\kappa_j + 1 | 1 \le j \le k\}}.$$

If $\min\{\kappa_1, \kappa_2, \ldots, \kappa_k\} = 1$, *then* $S_{\kappa_1,\kappa_2,\ldots,\kappa_k}$ *has zero Lebesgue measure but has full Hausdorff dimension.*

Remark 1.7.1. Note that the Diophantine type remains constant on any weak unstable leaf of $\{g_t\}_{t>0}$. Therefore the set of non Diophantine points on any strong stable leaf has zero Hausdorff dimension. We will give a different proof of this theorem in Chapter 2.

Proof. For each cusp η_j $(1 \le j \le k)$, we define $S_{j,\kappa}$ to be the subset of points $p \in \Gamma\backslash \mathrm{PSL}(2, \mathbb{R})$ satisfying the condition that there exist $\mu, \nu > 0$ such that for every point $\begin{pmatrix} a \\ b \end{pmatrix} \in m_j(\pi_j^{-1}(p))$, either $|b| \ge \mu$ or $|a|^\kappa |b| \ge \nu$. Here μ and ν depend on p.

Then by definition, we have

$$S_{\kappa_1,\kappa_2,\ldots,\kappa_k} = S_{1,\kappa_1} \cap S_{2,\kappa_2} \cap \cdots \cap S_{k,\kappa_k}$$

and hence

$$S^c_{\kappa_1,\kappa_2,\ldots,\kappa_k} = S^c_{1,\kappa_1} \cup S^c_{2,\kappa_2} \cup \cdots \cup S^c_{k,\kappa_k}.$$

Let $\kappa_0 = \min\{\kappa_1, \kappa_2, \ldots, \kappa_k\}$. Note that by Lemma 1.3.4,

$$S_{\kappa_0,\kappa_0,\ldots,\kappa_0} \subset S_{\kappa_1,\kappa_2,\ldots,\kappa_k}.$$

Therefore we get

$$\bigcap_{j=1}^{k} S^c_{j,\kappa_1} \cup \bigcap_{j=1}^{k} S^c_{2,\kappa_2} \cup \cdots \cup \bigcap_{j=1}^{k} S^c_{k,\kappa_k} \subset S^c_{\kappa_1,\kappa_2,\ldots,\kappa_k} \subset S^c_{\kappa_0,\kappa_0,\ldots,\kappa_0}.$$

By Theorem 2 and Theorem 3 in [MP93], for any $\kappa \geq 1$ we have

$$\dim_H S^c_{\kappa,\kappa,\ldots,\kappa} = 2 + \frac{2}{\kappa+1} \text{ and } \dim_H \bigcap_{j=1}^{k} S^c_{j,\kappa} = 2 + \frac{2}{\kappa+1}.$$

This implies that

$$\begin{aligned} \dim_H S^c_{\kappa_1,\kappa_2,\ldots,\kappa_k} &= \dim_H S^c_{\kappa_0,\kappa_0,\ldots,\kappa_0} = \max\left\{ \dim_H \bigcap_{j=1}^{k} S^c_{j,\kappa_i} \middle| 1 \leq i \leq k \right\} \\ &= 2 + \frac{2}{\kappa_0+1} = 2 + \frac{2}{\min\{\kappa_j + 1 | 1 \leq j \leq k\}}. \end{aligned}$$

For the second statement, if $\min\{\kappa_1, \kappa_2, \ldots, \kappa_k\} = 1$, then by Lemma 1.3.4 and the ergodicity of the geodesic flow on $\Gamma\backslash \mathrm{PSL}(2,\mathbb{R})$, we know that $S_{\kappa_1,\kappa_2,\ldots,\kappa_k}$ has zero

Haar measure. Since

$$S_{1,1,\ldots,1} \subset S_{\kappa_1,\kappa_2,\ldots,\kappa_k}$$

and by Theorem 1.1 in [KM96] $S_{1,1,\ldots,1}$ has full Hausdorff dimension, this implies that $S_{\kappa_1,\kappa_2,\ldots,\kappa_k}$ has full Hausdorff dimension. $\square$

Finally, using the same argument as in section 4, we can actually prove that if p is Diophantine of type $(\kappa_1, \kappa_2, \ldots, \kappa_k)$ with all $\kappa_j < 3$ and $0 \leq \gamma < 1/4$, then for any $\epsilon > 0$, there exists a compact subset $K_\epsilon \subset \Gamma \backslash \operatorname{PSL}(2, \mathbb{R})$ such that for all $T \geq 0$,

$$\frac{1}{T} \left\{ x \in [1, T] \middle| p \begin{pmatrix} x^{\frac{1}{4}} & x^{\frac{3}{4}+\gamma} \\ 0 & x^{-\frac{1}{4}} \end{pmatrix} \in K_\epsilon \right\} \geq 1 - \epsilon.$$

Then using the arguments of [DS84] and [Sh94, Proposition 4.1], we get

Theorem 1.7.2. *If p is Diophantine of type $(\kappa_1, \kappa_2, \ldots, \kappa_k)$ with all $\kappa_j < 3$ and $0 \leq \gamma < 1/4$, then the trajectory*

$$\left\{ p \begin{pmatrix} x^{\frac{1}{4}} & x^{\frac{3}{4}+\gamma} \\ 0 & x^{-\frac{1}{4}} \end{pmatrix} \middle| x \geq 1 \right\}$$

is equidistributed in $\Gamma \backslash \operatorname{PSL}(2, \mathbb{R})$.

Chapter 2

Hausdorff Dimension of Diophantine Points in $\Gamma\backslash\operatorname{PSL}(2,\mathbb{R})$

2.1 Introduction and Preliminaries

Here we will give a a different proof of Theorem 1.7.1 using results of lattice counting problem, that is,

Theorem 2.1.1.

$$\dim_H S^c_{\kappa_1,\kappa_2,\ldots,\kappa_k} = 2 + \frac{2}{\min\{\kappa_j + 1 | 1 \leq j \leq k\}}.$$

To prove Theorem 2.1.1 we need some preliminaries. Readers may refer to [KM96]. Let X be a Riemannian manifold, m a volume form and E a compact subset of X. We will denote the diameter of a set E by $\operatorname{diam}(E)$. A countable collection $\mathcal{A}$ of compact subsets of E is said to be tree-like if $\mathcal{A}$ is the union of finite subcollections $\mathcal{A}_j$ such that

1. $\mathcal{A}_0 = \{E\}$.
2. For any j and $A, B \in \mathcal{A}_j$, either $A = B$ or $A \cap B = \emptyset$.
3. For any j and $B \in \mathcal{A}_{j+1}$, there exists $A \in \mathcal{A}_j$ such that $B \subset A$.
4. $d_j(\mathcal{A}) := \sup_{A\in\mathcal{A}_j} \operatorname{diam}(A) \to 0$ as $j \to \infty$.

We write $\mathbf{A}_j = \bigcup_{A\in\mathcal{A}_j} A$ and define $\mathbf{A}_\infty = \bigcap_{j\in\mathbb{N}} \mathbf{A}_j$. Moreover, we define

$$\Delta_j(\mathcal{A}) = \inf_{B\in\mathcal{A}_j} \frac{m(\mathbf{A}_{j+1}\cap B)}{m(B)}.$$

The following theorem gives a way to estimate the Hausdorff dimension of $\mathbf{A}_\infty$.

Theorem 2.1.2 ([M87], [U91] or [KM96])**.** *Let (X, m) be a Riemannian manifold. Assume that there exist constants $D > 0$ and $k > 0$ such that*

$$m(B(x,r)) \leq Dr^k$$

for any $x \in X$. Then for any tree-like collection $\mathcal{A}$ of subsets of E

$$\dim_H(\mathbf{A}_\infty) \geq k - \limsup_{j\to\infty} \frac{\sum_{i=0}^{j}\log(\frac{1}{\Delta_i(\mathcal{A})})}{\log(\frac{1}{d_{j+1}(\mathcal{A})})}$$

2.2 Some Properties of Lattices Points in $\mathbb{R}^2$

In this section, we will show some lemmas which will be used in the proof of Theorem 2.1.1. For each cusp η_j $(1 \leq j \leq k)$, we define $S_{j,\kappa}$ to be the subset of points $p \in \Gamma\backslash \mathrm{PSL}(2,\mathbb{R})$ satisfying the condition that there exist $\mu, \nu > 0$ such that for every point $\begin{pmatrix} a \\ b \end{pmatrix} \in m_j(\pi_j^{-1}(p))$, either $|b| \geq \mu$ or $|a|^\kappa|b| \geq \nu$. Here μ and ν depend on p. Then by definition, we have

$$S^{\mathrm{c}}_{\kappa_1,\kappa_2,\ldots,\kappa_k} = S^{\mathrm{c}}_{1,\kappa_1} \cup S^{\mathrm{c}}_{2,\kappa_2} \cup \cdots \cup S^{\mathrm{c}}_{k,\kappa_k}$$

and hence

$$\dim_H S^{\mathrm{c}}_{\kappa_1,\kappa_2,\ldots,\kappa_k} = \max\{\dim_H S^{\mathrm{c}}_{j,\kappa_j} | 1 \leq j \leq k\}.$$

Therefore, to prove Theorem 2.1.1, it suffices to prove

$$\dim_H S^{c}_{j,\kappa_j} = 2 + \frac{2}{\kappa_j + 1}.$$

In the rest of this part, we will consider $S^{c}_{j,\kappa}$ for a fixed cusp η_j. Without loss of generality, we may assume that $\sigma_j = e$, $\eta_j = i\infty$ and that

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in \Gamma.$$

Since $\Gamma \cap N \neq \{e\}$, this implies that Γe_1 is a discrete subset in $\mathbb{R}^2$. The following lemmas concern some properties of lattice points in $\Gamma e_1 \subset \mathbb{R}^2$.

Lemma 2.2.1. *There exists a constant $C > 0$ such that for any $(\alpha, \beta) \in \Gamma e_1$ we have $|\beta| \geq C$ or $\beta = 0$.*

Proof. We know that Γe_1 is discrete in $\mathbb{R}^2$. So there is a constant $C > 0$ such that for any point $(\alpha, \beta) \in \Gamma e_1$ we have

$$\|(\alpha, \beta)\| \geq 2C$$

where $\|\cdot\|$ is the standard Euclidean norm. Suppose that there exists $(\alpha_0, \beta_0) \in \Gamma e_1$ with $0 < |\beta_0| < C$. Then there exists an integer $n \in \mathbb{Z}$ such that

$$|\alpha_0 + n\beta_0| < \beta_0.$$

Since $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in \Gamma$, we have

$$\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha_0 \\ \beta_0 \end{pmatrix} = \begin{pmatrix} \alpha_0 + n\beta_0 \\ \beta_0 \end{pmatrix} \in \Gamma e_1$$

and

$$\|(\alpha_0 + n\beta_0, \beta_0)\| \leq \sqrt{2}C < 2C$$

which contradicts the definition of C. This completes the proof of the lemma. $\square$

Lemma 2.2.2. *There exists a constant $C > 0$ such that for any two distinct points (α_1, β_1) and (α_2, β_2) in Γe_1 we have*

$$|\alpha_1\beta_2 - \alpha_2\beta_1| \geq C.$$

Proof. Now let $\gamma_1, \gamma_2 \in \Gamma$ be such that

$$\gamma_1 = \begin{pmatrix} \alpha_1 & * \\ \beta_1 & * \end{pmatrix} \quad \gamma_2 = \begin{pmatrix} \alpha_2 & * \\ \beta_2 & * \end{pmatrix}.$$

Then we have

$$\gamma_1^{-1}\gamma_2 e_1 = \begin{pmatrix} * \\ \alpha_1\beta_2 - \alpha_2\beta_1 \end{pmatrix}.$$

Note that (α_1, β_1) and (α_2, β_2) are distinct and hence $\alpha_1\beta_2 - \alpha_2\beta_1 \neq 0$. By Lemma 2.2.1, we conclude that

$$|\alpha_1\beta_2 - \alpha_2\beta_1| \geq C$$

for some $C > 0$. $\square$

Remark 2.2.1. We will fix this constant C for later use. Note that by the definition of C, for any point $(\alpha,\beta)\in\Gamma e_1$ we have $\|(\alpha,\beta)\|\geq 2C$.

Definition 2.2.1. For $l>0$ and $0\leq\theta_1\leq\theta_2<2\pi$, we define the subset of $\mathbb{R}^2$

$$S(l,\theta_1,\theta_2):=\{(x,y)\in\mathbb{R}^2|l\leq r\leq 2l,\theta_1<\theta<\theta_2\}$$

where (r,θ) are the polar coordinates of (x,y).

Theorem 2.2.1 ([EM93], [GOS10])**.** *We have* $|\Gamma e_1\cap S(l,\theta_1,\theta_2)|\sim l^2(\theta_2-\theta_1)$ *as* $l\to\infty$.

Lemma 2.2.3. *Fix* $C>0$ *as in Lemma 2.2.2 and let* $\kappa\geq 1$*. There exists a constant* $C_0>0$ *with the following property: for any* $(\alpha,\beta)\in\Gamma e_1$ *with* $0<\frac{\alpha}{\beta}<1$*, there exists a large constant* $L_{(\alpha,\beta)}>0$ *such that for any* $l>L_{(\alpha,\beta)}$ *the interval* $\left[\frac{\alpha}{\beta}-\frac{C}{18}\cdot\frac{1}{\beta^{\kappa+1}},\frac{\alpha}{\beta}+\frac{C}{18}\cdot\frac{1}{\beta^{\kappa+1}}\right]$ *contains at least* $C_0l^2/\beta^{\kappa+1}$ *many disjoint subintervals* $\left[\frac{\tilde{\alpha}}{\tilde{\beta}}-\frac{C}{18}\cdot\frac{1}{\tilde{\beta}^{\kappa+1}},\frac{\tilde{\alpha}}{\tilde{\beta}}+\frac{C}{18}\cdot\frac{1}{\tilde{\beta}^{\kappa+1}}\right]$ *where* $(\tilde{\alpha},\tilde{\beta})\in\Gamma e_1\cap S(l,\frac{\pi}{4},\frac{\pi}{2})$.

Proof. Suppose that $\left[\frac{\alpha}{\beta}-\frac{C}{18}\cdot\frac{1}{\beta^{\kappa+1}},\frac{\alpha}{\beta}+\frac{C}{18}\cdot\frac{1}{\beta^{\kappa+1}}\right]$ contains two subintervals

$$\left[\frac{\tilde{\alpha}}{\tilde{\beta}}-\frac{C}{18}\cdot\frac{1}{\tilde{\beta}^{\kappa+1}},\frac{\tilde{\alpha}}{\tilde{\beta}}+\frac{C}{18}\cdot\frac{1}{\tilde{\beta}^{\kappa+1}}\right]\text{ and }\left[\frac{\tilde{\gamma}}{\tilde{\delta}}-\frac{C}{18}\cdot\frac{1}{\tilde{\delta}^{\kappa+1}},\frac{\tilde{\gamma}}{\tilde{\delta}}+\frac{C}{18}\cdot\frac{1}{\tilde{\delta}^{\kappa+1}}\right]$$

where $(\tilde{\alpha},\tilde{\beta})$ and $(\tilde{\gamma},\tilde{\delta})$ are two distinct points in $\Gamma e_1\cap S(l,\frac{\pi}{4},\frac{\pi}{2})$. By Lemma 2.2.2, we have

$$\begin{aligned}\left|\frac{\tilde{\alpha}}{\tilde{\beta}}-\frac{\tilde{\gamma}}{\tilde{\delta}}\right|=\frac{|\tilde{\alpha}\tilde{\delta}-\tilde{\beta}\tilde{\gamma}|}{|\tilde{\beta}\tilde{\delta}|}&\geq\frac{C}{|\tilde{\beta}\tilde{\delta}|}\geq\frac{C}{4l^2}\\ =\ &\frac{C}{16}\left(\frac{1}{(l/\sqrt{2})^2}+\frac{1}{(l/\sqrt{2})^2}\right)\geq\frac{C}{16}\left(\frac{1}{\tilde{\beta}^2}+\frac{1}{\tilde{\delta}^2}\right)\\ \geq\ &\frac{C}{16}\left(\frac{1}{\tilde{\beta}^{\kappa+1}}+\frac{1}{\tilde{\delta}^{\kappa+1}}\right).\end{aligned}$$

This implies that any two such subintervals are disjoint, and hence to prove the lemma it suffices to prove that in the interval $\left[\frac{\alpha}{\beta} - \frac{C}{18} \cdot \frac{1}{\beta^{\kappa+1}}, \frac{\alpha}{\beta} + \frac{C}{18} \cdot \frac{1}{\beta^{\kappa+1}}\right]$ there are at least $C_0 l^2/\beta^{\kappa+1}$ many points of the form $\tilde{\alpha}/\tilde{\beta}$ where $(\tilde{\alpha}, \tilde{\beta}) \in \Gamma e_1 \cap S(l, \frac{\pi}{4}, \frac{\pi}{2})$. We have

$$\begin{aligned}
& \frac{\tilde{\alpha}}{\tilde{\beta}} \in \left[\frac{\alpha}{\beta} - \frac{C}{18} \cdot \frac{1}{\beta^{\kappa+1}}, \frac{\alpha}{\beta} + \frac{C}{18} \cdot \frac{1}{\beta^{\kappa+1}}\right] \\
\iff \quad & \arg(\tilde{\alpha}, \tilde{\beta}) \in \left[\operatorname{arccot}\left(\frac{\alpha}{\beta} + \frac{C}{18} \cdot \frac{1}{\beta^{\kappa+1}}\right), \operatorname{arccot}\left(\frac{\alpha}{\beta} - \frac{C}{18} \cdot \frac{1}{\beta^{\kappa+1}}\right)\right].
\end{aligned}$$

Since $\left|\operatorname{arccot}\left(\frac{\alpha}{\beta} + \frac{C}{18} \cdot \frac{1}{\beta^{\kappa+1}}\right) - \operatorname{arccot}\left(\frac{\alpha}{\beta} - \frac{C}{18} \cdot \frac{1}{\beta^{\kappa+1}}\right)\right| \sim \frac{1}{\beta^{\kappa+1}}$, by Theorem 2.2.1 we know that the number of points in $S\left(l, \operatorname{arccot}\left(\frac{\alpha}{\beta} + \frac{C}{18} \cdot \frac{1}{\beta^{\kappa+1}}\right), \operatorname{arccot}\left(\frac{\alpha}{\beta} - \frac{C}{18} \cdot \frac{1}{\beta^{\kappa+1}}\right)\right)$ is asymptotically equal to $l^2/\beta^{\kappa+1}$ up to a constant. Note that the implicit constant is absolute since $0 < \alpha/\beta < 1$. This completes the proof of the lemma. □

2.3 Hausdorff Dimension

In this section, we will give a proof of Theorem 2.1.1. We need some preparations.

Definition 2.3.1. We say that $x \in \mathbb{R}$ is Diophantine of type κ with respect to Γe_1 if there exists a constant $\tilde{C} > 0$ such that for any $(\alpha, \beta) \in \Gamma e_1$ with $\beta \neq 0$ we have

$$|\beta|^\kappa |x\beta - \alpha| \geq \tilde{C}.$$

We denote by S_κ the subset of $\mathbb{R}$ of all Diophantine numbers of type κ with respect to Γe_1.

Lemma 2.3.1. *Let* $p = \Gamma \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \backslash \operatorname{PSL}(2, \mathbb{R})$ *with* $c \neq 0$. *Then* $p \in S_{j,\kappa}^c$ *if and only if* $a/c \in S_\kappa^c$.

Proof. We have

$$m_j(\pi_j^{-1}(p)) = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \Gamma e_1 = \left\{ \begin{pmatrix} d\alpha - b\beta \\ -c\alpha + a\beta \end{pmatrix} \middle| \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \in \Gamma e_1 \right\}.$$

By the definition of $S_{j,\kappa}$, if $p \in S_{j,\kappa}^{\mathrm{c}}$, then there exist infinitely many $(\alpha, \beta) \in \Gamma e_1$ such that

$$|a\beta - c\alpha| \to 0 \text{ and } |d\alpha - b\beta|^\kappa |a\beta - c\alpha| \to 0.$$

By the discreteness of Γe_1, this implies that $|\beta| \to \infty$. Note that

$$|d\alpha - b\beta| = \frac{|cd\alpha - cb\beta|}{|c|} = \frac{|cd\alpha - (ad-1)\beta|}{|c|} = \frac{|d(c\alpha - a\beta) + \beta|}{|c|}.$$

Therefore we have $|d\alpha - b\beta| \sim |\beta|$ and $a/c \in S_\kappa^{\mathrm{c}}$. Here the implicit constant in $\sim$ depends on p.

Conversely, if $a/c \in S_\kappa^{\mathrm{c}}$, then there exist infinitely many $(\alpha, \beta) \in \Gamma e_1$ with $\beta \neq 0$ such that

$$|\beta|^\kappa \left| \frac{a}{c}\beta - \alpha \right| \to 0.$$

By Lemma 2.2.1, this implies that

$$|a\beta - c\alpha| \to 0$$

and consequently

$$|\beta| \to \infty \text{ and } |d\alpha - b\beta| = \frac{|d(c\alpha - a\beta) + \beta|}{c} \sim |\beta|.$$

Hence we have

$$|a\beta - c\alpha| \to 0, \quad |d\alpha - b\beta|^\kappa |a\beta - c\alpha| \to 0$$

and $p \in S^{\mathrm{c}}_{j,\kappa}$. This completes the proof of the lemma. $\square$

Proof of Theorem 2.1.1. From the discussions above, we know that in order to prove Theorem 2.1.1 it is enough to show that

$$\dim_H S^{\mathrm{c}}_{j,\kappa} = 2 + \frac{2}{\kappa+1}.$$

By Lemma 1.4.1 and the fact that the subset

$$\left\{ \Gamma \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \middle| a, b \in \mathbb{R} \right\} \subset \Gamma \backslash \mathrm{PSL}(2,\mathbb{R})$$

has dimension 2, it suffices to prove that

$$\dim_H S^{\mathrm{c}}_{\kappa} = \frac{2}{\kappa+1}.$$

In the rest of this section we will prove this formula.

Since $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in \Gamma$, for any $n \in \mathbb{Z}$ we have

$$S^{\mathrm{c}}_{\kappa} \cap (n, n+1) = n + S^{\mathrm{c}}_{\kappa} \cap (0,1).$$

Therefore, we only need to compute the Hausdorff dimension of $S^{\mathrm{c}}_{\kappa} \cap (0,1)$. For the upper bound, by the definition of S_κ, we can construct an open cover

$$\left\{ I_{(\alpha,\beta)} = \left(\frac{\alpha}{\beta} - \frac{1}{\beta^{\kappa+1}}, \frac{\alpha}{\beta} + \frac{1}{\beta^{\kappa+1}} \right) \middle| (\alpha,\beta) \in \Gamma e_1, \alpha/\beta \in (0,1) \right\} \supseteq S^{\mathrm{c}}_{\kappa} \cap (0,1).$$

For $\delta > 0$ by Theorem 2.2.1 we have

$$\begin{aligned}
&\sum_{\substack{(\alpha,\beta)\in\Gamma e_1\\ \alpha/\beta\in(0,1)}} \operatorname{diam}(I_{(\alpha,\beta)})^\delta \\
\ll\ & \sum_{n=1}^{\infty} \sum_{(\alpha,\beta)\in\Gamma e_1\cap S(2^n C,\frac{\pi}{4},\frac{\pi}{2})} \frac{1}{\beta^{\delta(\kappa+1)}} \\
\ll\ & \sum_{n=1}^{\infty} \frac{2^{2n}}{2^{n\delta(\kappa+1)}} = \sum_{n=1}^{\infty} \frac{1}{2^{n(\delta(\kappa+1)-2)}}.
\end{aligned}$$

If $\delta > 2/(\kappa+1)$, then $\sum_{\substack{(\alpha,\beta)\in\Gamma e_1\\ \alpha/\beta\in(0,1)}} \operatorname{diam}(I_{(\alpha,\beta)})^\delta$ converges and hence by properties of Hausdorff dimension we have

$$\dim_H S_\kappa^c \cap (0,1) \leq \frac{2}{\kappa+1}.$$

For the lower bound, let $\epsilon > 0$ be fixed and we construct a tree-like set in $S_\kappa^c \cap (0,1)$ as the intersection of closed subsets in $[0,1]$ by induction. Let $\mathcal{A}_0 = \{[0,1]\}$ and $\mathbf{A}_0 = [0,1]$. Let l_1 be a sufficiently large number and define

$$\mathcal{A}_1 = \left\{ \left[\frac{\alpha}{\beta} - \frac{C}{18}\cdot\frac{1}{\beta^{\kappa+\epsilon+1}}, \frac{\alpha}{\beta} + \frac{C}{18}\cdot\frac{1}{\beta^{\kappa+\epsilon+1}}\right] \middle| (\alpha,\beta)\in\Gamma e_1 \cap S(l_1,\frac{\pi}{4},\frac{\pi}{2}) \right\}$$

and $\mathbf{A}_1 = \bigcup \mathcal{A}_1$. Suppose that we find $l_1 < l_2 < \cdots < l_j$ and construct families $\mathcal{A}_j, \mathcal{A}_{j-1}, \ldots, \mathcal{A}_0$ and closed subsets $\mathbf{A}_j \subseteq \mathbf{A}_{j-1} \subseteq \cdots \subseteq \mathbf{A}_1 \subseteq \mathbf{A}_0$. Now by Lemma 2.2.3, we can find a sufficiently large $l_{j+1} > 0$ such that

1. $\log l_{j+1} \geq j^2 \log(l_j l_{j-1} \ldots l_1)$.

2. For every $\left[\frac{\alpha}{\beta} - \frac{C}{18}\cdot\frac{1}{\beta^{\kappa+\epsilon+1}}, \frac{\alpha}{\beta} + \frac{C}{18}\cdot\frac{1}{\beta^{\kappa+\epsilon+1}}\right] \in \mathcal{A}_j$, it contains at least $C_0\frac{l_{j+1}^2}{l_j^{\kappa+\epsilon+1}}$ subintervals (since $\beta \sim l_j$) of the form $\left[\frac{\tilde\alpha}{\tilde\beta} - \frac{C}{18}\cdot\frac{1}{\tilde\beta^{\kappa+\epsilon+1}}, \frac{\tilde\alpha}{\tilde\beta} + \frac{C}{18}\cdot\frac{1}{\tilde\beta^{\kappa+\epsilon+1}}\right]$ with $(\tilde\alpha,\tilde\beta)\in\Gamma e_1\cap S(l_{j+1},\frac{\pi}{4},\frac{\pi}{2})$.

We denote the family of all these new subintervals by $\mathcal{A}_{j+1}$ as

$$\left[\frac{\alpha}{\beta} - \frac{C}{18} \cdot \frac{1}{\beta^{\kappa+\epsilon+1}}, \frac{\alpha}{\beta} + \frac{C}{18} \cdot \frac{1}{\beta^{\kappa+\epsilon+1}}\right]$$

runs through all the intervals in $\mathcal{A}_j$ and let $\mathbf{A}_{j+1} = \bigcup \mathcal{A}_{j+1}$. Here C and C_0 are as in Lemma 2.2.3.

Now we take $\mathbf{A}_\infty = \bigcap_{j=0}^{\infty} \mathbf{A}_j$ and $\mathcal{A} = \bigcup_{j=0}^{\infty} \mathcal{A}_j$. From the construction of $\mathbf{A}_j$'s and the definition of S_κ, we know that $\mathbf{A}_\infty \subseteq S_\kappa^c \cap (0,1)$. Also we have

$$\Delta_j(\mathcal{A}) \sim \frac{l_{j+1}^2}{l_j^{\kappa+\epsilon+1}} \cdot \frac{1}{l_{j+1}^{\kappa+\epsilon+1}} \text{ and } d_j(\mathcal{A}) \sim \frac{1}{l_j^{\kappa+\epsilon+1}}.$$

Therefore by Theorem 2.1.2, we have

$$\begin{aligned}
& \dim_H S_\kappa^c \cap (0,1) \\
\geq\ & \dim_H \mathbf{A}_\infty \\
\geq\ & 1 - \limsup_{j\to\infty} \frac{-\sum_{i=1}^{j} \log(l_{i+1}^2/(l_i l_{i+1})^{\kappa+\epsilon+1})}{\log l_{j+1}^{\kappa+\epsilon+1}} \\
=\ & 1 - \limsup_{j\to\infty} \frac{(\kappa+\epsilon+1)\log l_1 + \sum_{i=2}^{j} 2(\kappa+\epsilon)\log l_i + (\kappa+\epsilon-1)\log l_{j+1}}{(\kappa+\epsilon+1)\log l_{j+1}} \\
=\ & 1 - \frac{\kappa+\epsilon-1}{\kappa+\epsilon+1} = \frac{2}{\kappa+\epsilon+1}.
\end{aligned}$$

Since this is true for any $\epsilon > 0$, we obtain that

$$\dim_H S_\kappa^c \cap (0,1) \geq \frac{2}{\kappa+1}.$$

This completes the proof of Theorem 2.1.1. $\square$

Chapter 3

Effective Equidistribution of Abelian Horospherical Orbits

3.1 Introduction

In this part, we will consider the effective equidistribution of horospherical orbits in homogeneous spaces. This topic has been studied well, and the present work is motivated by [S13] and [V10]. To be precise, let $\{a_t\} = \{\exp(tX)\}_{t\in\mathbb{R}}$ be a one parameter $\mathbb{R}$-diagonalizable subgroup in a semisimple Lie group G, Γ a lattice in G and μ the Haar measure on $\Gamma\backslash G$. Let $\mathrm{Ad}(g)$ be the adjoint action of G on $\mathrm{Lie}(G)$ induced by the action of conjugation $x \mapsto gxg^{-1}(x \in G)$. Let U be the horospherical subgroup of $\{a_t\}$, i.e.

$$U = \{g \in G | a_{-t} g a_t \to e\}.$$

The decomposition of $\mathrm{Lie}(U)$ with respect to $\{a_t\}$ under the adjoint action is

$$\mathrm{Lie}(U) = \mathfrak{g}_{\alpha_1} \oplus \mathfrak{g}_{\alpha_2} \oplus \cdots \oplus \mathfrak{g}_{\alpha_n}$$

where α_i are the roots of $\{a_t\}$ in U, that is,

$$\mathrm{Ad}(a_t)X_i = \alpha_i(a_t)X_i$$

for any $X_i \in \mathfrak{g}_{\alpha_i}$. Without loss of generality, we can assume that each $\mathfrak{g}_{\alpha_i}$ is one-dimensional and some of these α_i's may be identical. We denote the exponential map from Lie(G) to G by exp and we fix a norm $\|\cdot\|$ on the Lie algebra $\mathfrak{g}$. For each i, fix $v_i \in \mathfrak{g}_{\alpha_i}$ with norm 1 and let $B(T_1, T_2, \dots, T_n)$ be the parametrized box in U, i.e.

$$B(T_1, T_2, \dots, T_n) = \{\exp(t_1 v_1 + t_2 v_2 + \cdots + t_n v_n) | 0 \le t_i \le T_i (1 \le i \le n)\}.$$

For any $t > 0$ we can find a number $\alpha > 0$ such that for any $t > 0$

$$e^{\alpha t} = \alpha_1(a_t)\alpha_2(a_t)\cdots\alpha_n(a_t)$$

and then

$$t^{\alpha} = \alpha_1(a_{\ln t})\alpha_2(a_{\ln t})\cdots\alpha_n(a_{\ln t}).$$

Also we define

$$\begin{aligned} B(t) : &= B(\alpha_1(a_{\ln t}), \alpha_2(a_{\ln t}), \cdots, \alpha_n(a_{\ln t})) \\ &= a_{\ln t} B(1, 1, \dots, 1) a_{-\ln t}. \end{aligned}$$

We will denote by B_r the open ball of radius $r > 0$ around e in G. Here the distance on G is induced by the norm $\|\cdot\|$. Also we will write $BC_l^{\infty}(\Gamma\backslash G)$ for the set of bounded smooth functions on $\Gamma\backslash G$ with bounded Lie derivatives up to order l.

Definition 3.1.1. For any $x \in \Gamma\backslash G$, we define the injectivity radius at x by the largest number $\eta > 0$ with the property that the map

$$B_\eta \to x B_\eta \subset \Gamma\backslash G$$

by sending $g \in B_\eta$ to $xg \in \Gamma\backslash G$ is injective. We will denote the injectivity radius at x by $\eta(x)$.

We will follow the proof of Lemma 9.5 in [V10] and prove the following theorem.

Theorem 3.1.1. *Suppose that U is abelian. There exist constants $a, b > 0$ such that for any $f \in BC_l^\infty(\Gamma\backslash G)$, we have*

$$\left|\frac{1}{T^\alpha}\int_{B(T)} f(xu)du - \int_{\Gamma\backslash G} f d\mu\right| \ll \frac{1}{T^a\eta^b}\|f\|_{\infty,l}$$

for some large constant $l > 0$. Here $\eta = \eta(a_{\ln T}x)$ is the injectivity radius at $a_{\ln T}x$ and $\|\cdot\|_{\infty,l}$ is the L^∞-Sobolev norm involving Lie derivatives of orders up to l. The implicit constant depends only on $\Gamma\backslash G$.

Remark 3.1.1. We will always assume that $\|f\|_{\infty,l}$ is defined and finite, and l is large enough so that all the theorems and arguments in this note would hold. Readers may refer to [KM99] for more details about the Sobolev norm and the number l.

Remark 3.1.2. Theorem 3.1.1 is weaker than the theorem proved by Strömbergsson [S13] in the case of $\Gamma\backslash \mathrm{PSL}(2,\mathbb{R})$. But the proof would involve only mixing property of a semisimple flow and give a result for a general homogeneous space. Readers may compare $T\eta^{\frac{b}{a}}$ and the r-factor in the main theorem of [S13].

Using the same arguments as in the proof of Theorem 3.1.1, we can prove the following

Theorem 3.1.2. *Suppose that U is abelian. Let $h(u)$ be a compactly supported smooth function on U. Then there exist constants $a, b > 0$ such that for any $f \in BC_l^\infty(\Gamma\backslash G)$ we have*

$$\left|\frac{1}{T^\alpha}\int_U f(xu)h(a_{-\ln T}ua_{\ln T})du - \int_{\Gamma\backslash G} f d\mu \int_U h(u)du\right| \ll \frac{1}{T^a\eta^b}\|f\|_{\infty,l}.$$

Here η, l and $\|\cdot\|_{\infty,l}$ are the same as in Theorem 3.1.1. The implicit constant depends only on $h(u)$ and $\Gamma\backslash G$.

Definition 3.1.2. A point $p \in \Gamma\backslash G$ is called Diophantine of type μ ($\mu \in \mathbb{R}$) with respect to $\{a_t\}$ if there exists a constant $C > 0$ such that

$$\eta(pa_t) \geq Ce^{-\mu t}$$

for all $t > 0$. Also we say that an orbit $\{pa_t\}_{t\geq 0}$ in $\Gamma\backslash G$ is non-divergent of order μ if there exists a constant $C > 0$ such that

$$\eta(pa_{t_k}) \geq Ce^{-\mu t_k}$$

for infinitely many $t_k \to \infty$.

The following is an immediate corollary of Theorem 3.1.1 and [Sh94].

Corollary 3.1.1. *Assume the conditions in Theorem 3.1.1. If x is Diophantine of type $\mu < a/b$ with respect to $\{a_t\}$, or $\{xa_t\}_{t\geq 0}$ is non-divergent of order $\mu < a/b$, then*

$$\frac{1}{T^\alpha}\int_{B(T)} f(xu)du \to \int f d\mu.$$

Here the constants a, b and the function f are as in Theorem 3.1.1.

Acknowledgement. I would like to thank Professor Andreas Strömbergsson and Samuel Edwards for many discussions. I was told that they had results about the effective equidistribution of horocycle orbits in homogeneous spaces using number theoretic tools. Here what we prove in Theorem 3.1.1 is much weaker than [S13] and our purpose is just to show how to use mixing property only to get such results.

3.2 Preliminaries

In the proof of Theorem 3.1.1, we will need the following exponential mixing property.

Theorem 3.2.1 (Kleinbock and Margulis [KM99])**.** *There exists $\kappa > 0$ such that for any $f, g \in BC^\infty(\Gamma\backslash G)$, we have*

$$\left|(a_t \cdot f, g) - \int_{\Gamma\backslash G} f \int_{\Gamma\backslash G} g\right| \ll e^{-\kappa t}\|f\|_{\infty,l}\|g\|_{\infty,l}.$$

Here $(a_t \cdot f)(x) = f(xa_{-t})$ is the right translation of f by a_t and $\|\cdot\|_{\infty,l}$ is the same Sobolev norm as in Theorem 3.1.1.

Throughout this chapter, we will assume that U and U^+ are abelian.

3.3 Some Lemmas

In this section, we will use the same arguments in the proof of Lemma 9.5 in [V10] and prove some lemmas which will be used in the proof of Theorem 3.1.1 and Theorem 3.1.2.

Lemma 3.3.1. *Let x be any point in $\Gamma\backslash G$. Then for every point $y \in xB(1)$ we have*

$$\eta(y) \sim \eta(x).$$

Here the implicit constant depends only on G. Generally, if $y \in xB$ for some bounded subset $B \subset U$, then the same result holds with the implicit constant depending only on B and G.

Proof. This follows from the compactness of $B(1)$. $\square$

Now we fix a positive compactly supported smooth function $g(x)$ with integral one on $\mathbb{R}$, and for any $n \in \mathbb{N}$, $\gamma > 0$ and $\delta > 0$ define

$$
\begin{aligned}
& g_{\delta,n,\gamma}(u) \\
= \; & \frac{1}{\delta^n} \int_0^{\gamma} \int_0^{\gamma} \cdots \int_0^{\gamma} g\left(\frac{u_1 - t_1}{\delta}\right) g\left(\frac{u_2 - t_2}{\delta}\right) \cdots g\left(\frac{u_n - t_n}{\delta}\right) dt_1 dt_2 \ldots dt_n
\end{aligned}
$$

for $u \in \mathbb{R}^n$. The following lemma is an immediate consequence from calculations.

Lemma 3.3.2. *We have*

1. $\int_{\mathbb{R}^n} g_{\delta,n,\gamma}(u) du = \gamma^n$.

2. $g_{\delta,n,\gamma}(u)$ *is supported in a neighborhood of the box* $[0,\gamma] \times [0,\gamma] \times \cdots \times [0,\gamma]$.

3. $\int_{\mathbb{R}^n} |g_{\delta,n,\gamma}(u) - \chi_{[0,\gamma]^n}(u)| du \ll \delta(\gamma + \delta)^{n-1}$.

Lemma 3.3.3. *Let* $y \in \Gamma\backslash G$ *and* $f \in C^\infty(\Gamma\backslash G)$. *Assume that* $\int_{\Gamma\backslash G} f d\mu = 0$. *Then there exist constants* $a, b > 0$ *such that for any* $t > 0$ *and* $\gamma < \frac{\eta(y)}{2}$ *we have*

$$\left| \int_{B(\gamma,\gamma,\ldots,\gamma)} f(yua_{-t}) du \right| \ll \frac{1}{e^{at}\gamma^b} \|f\|_{\infty,l}.$$

The implicit constant depends only on $\Gamma\backslash G$.

Proof. Now let U^+ be the unstable horospherical subgroup of $\{a_t\}$ and $Z = Z(a_t)$ be the central subgroup of $\{a_t\}$ in G. Then we know that

$$\mathrm{Lie}(U) \oplus \mathrm{Lie}(U^+) \oplus \mathrm{Lie}(Z) = \mathrm{Lie}(G).$$

Let $\dim U = \dim U^+ = n$ and $\dim Z(a_t) = m$. By Lemma 3.3.2 and the same arguments as in Lemma 9.5 of [V10] we have

$$
\begin{aligned}
&\int_{B(\gamma,\gamma,\dots,\gamma)} f(yua_{-t})du \\
= \;& \int_{\mathrm{Lie}(U)} f(y\exp(u)a_{-t})g_{\delta,n,\gamma}(u)du + O(\|f\|_{\infty,l})\delta(\gamma+\delta)^{n-1} \\
= \;& \frac{1}{\delta^m\gamma^n}\iiint_{\mathrm{Lie}(U)\times\mathrm{Lie}(Z)\times\mathrm{Lie}(U^+)} f(y\exp(u)a_{-t})g_{\delta,n,\gamma}(u)g_{\delta,m,\delta}(z)g_{\delta,n,\gamma}(v)dudzdv \\
& +O(\|f\|_{\infty,l})\delta(\gamma+\delta)^{n-1} \\
= \;& \frac{1}{\delta^m\gamma^n}\iiint_{\mathrm{Lie}(G)} f(y\exp(u)\exp(z)\exp(v)a_{-t})g_{\delta,n,\gamma}(u)g_{\delta,m,\delta}(z)g_{\delta,n,\gamma}(v)dudzdv \\
& +O(\|f\|_{\infty,l})(\delta(\gamma+\delta)^{n-1}+\gamma^n\max\{\delta,\gamma/e^{qt}\}) \\
= \;& \frac{1}{\delta^m\gamma^n}\int_{\Gamma\backslash G} f(xa_{-t})g_{\delta,y}(x)d\mu(x) + O(\|f\|_{\infty,l})(\delta(\gamma+\delta)^{n-1}+\gamma^n\max\{\delta,\gamma/e^{qt}\}) \\
= \;& \frac{1}{\delta^m\gamma^n}(a_t\cdot f, g_{\delta,y}) + O(\|f\|_{\infty,l})(\delta(\gamma+\delta)^{n-1}+\gamma^n\max\{\delta,\gamma/e^{qt}\}).
\end{aligned}
$$

Here $q>0$ is a positive constant. Also we know that there exists a function depending only on G such that

$$d\mu = F(u,z,v)dudzdv$$

and $g_{\delta,y}$ is the function

$$g_{\delta,y}(y\exp(u)\exp(z)\exp(v)) = g_{\delta,n,\gamma}(u)g_{\delta,m,\delta}(z)g_{\delta,n,\gamma}(v)F(u,z,v)$$

supported on the ball of radius $\eta(y)$ at y in $\Gamma\backslash G$. Note that all injectivity radii have a common upper bound depending only on $\Gamma\backslash G$. By the definition of Lie derivatives, we can compute $\|g_{\delta,y}\|_{\infty,l}$ and there exists a constant $p>0$ such that

$$\|g_{\delta,y}\|_{\infty,l} \ll 1/\delta^p.$$

Therefore, by exponential mixing of semisimple flow (Theorem 3.2.1), we have

$$\begin{aligned} & \left| \int_{B(\gamma,\gamma,\ldots,\gamma)} f(yua_{-t}) du \right| \\ \ll \; & \frac{1}{\delta^m \gamma^n} \frac{1}{e^{\kappa t} \delta^p} \|f\|_{\infty,l} + \|f\|_{\infty,l} (\delta(\gamma+\delta)^{n-1} + \max\{\delta, \gamma/e^{qt}\}). \end{aligned}$$

Let $\delta = \gamma e^{-\epsilon t} < \gamma$ for some small $\epsilon > 0$ and this completes the proof of the lemma. □

Lemma 3.3.4. *Assume the conditions in Lemma 3.3.3. Let $h(u)$ be a smooth compactly supported function on U. Then there exist constants $a, b > 0$ such that for any $t > 0$ and $\gamma < \frac{\eta(y)}{2}$ we have*

$$\left| \int_{B(\gamma,\gamma,\ldots,\gamma)} f(yua_{-t}) h(u) du \right| \ll \frac{1}{e^{at} \gamma^b} \|f\|_{\infty,l} \|h\|_{\infty,l}.$$

Here $\|h(u)\|_{\infty,l}$ is the L^∞-Sobolev norm involving partial derivatives of orders up to l on U. The implicit constant depends only on $\Gamma \backslash G$.

Proof. By Lemma 3.3.2, we have

$$\begin{aligned} & \int_{B(\gamma,\gamma,\ldots,\gamma)} f(yua_{-t}) h(u) du \\ = \; & \int_{\mathrm{Lie}(U)} f(y \exp(u) a_{-t}) h(\exp(u)) g_{\delta,n,\gamma}(u) du + O(\|f\|_{\infty,l} \|h\|_{\infty,l}) \delta(\gamma+\delta)^{n-1}. \end{aligned}$$

Now the lemma follows from the same arguments as in Lemma 3.3.3. (In this case, we have $\|g_{\delta,y}\|_{\infty,l} \ll 1/\delta^p \|h\|_{\infty,l}$ for some $p > 0$.) □

3.4 Effective Equidistribution of Abelian Horospherical Orbits

In this section, we will prove Theorem 3.1.1 and Theorem 3.1.2.

Proof of Theorem 3.1.1. Without loss of generality, assume that $\int f d\mu = 0$. We know that

$$\frac{1}{T^\alpha}\int_{B(T)} f(xu)du = \int_{B(1)} f(xa_{\ln T}ua_{-\ln T})du.$$

By Lemma 3.3.1 and the assumption that U is abelian, we can find $\gamma > 0$ with the following properties

1. We can devide $B(1)$ into small boxes $\{B_j\}$. For each j, there exists $y_j \in B(1)$ such that $B_j = y_j B(\gamma, \gamma, \ldots, \gamma)$.
2. For each j, we have $\gamma < \eta(xa_{\ln T}y_j)/2$.
3. $\gamma \sim \eta(xa_{\ln T})$ and the implicit constant in $\sim$ depends only on $\Gamma\backslash G$.

In fact, we can take such γ by first taking the infimum of $\{\eta(xa_{\ln T}y)/2 | y \in B(1)\}$ and then modifying it so that $1/\gamma$ is an integer. Note that the number of these boxes B_j is $1/\gamma^n$. Now by Lemma 3.3.3 we have

$$\begin{aligned} \left|\frac{1}{T^\alpha}\int_{B(T)} f(xu)du\right| &= \left|\int_{B(1)} f(xa_{\ln T}ua_{-\ln T})du\right| \\ \leq \ & \sum_j \left|\int_{B_j} f(xa_{\ln T}ua_{-\ln T})du\right| \\ = \ & \sum_j \left|\int_{B(\gamma,\ldots,\gamma)} f((xa_{\ln T}y_j)ua_{-\ln T})du\right| \\ \ll \ & \frac{1}{\gamma^n}\frac{1}{T^a\gamma^b}\|f\|_{\infty,l} \ll \frac{1}{T^a\eta(xa_{\ln T})^{b+n}}\|f\|_{\infty,l}. \end{aligned}$$

This completes the proof of Theorem 3.1.1. $\square$

Proof of Theorem 3.1.2. The proof is similar to that of Theorem 3.1.1. We assume that $\int f d\mu = 0$. We have

$$\begin{aligned}
& \frac{1}{T^\alpha}\int_U f(xu)h(a_{-\ln T}ua_{\ln T})du \\
= & \int_U f(xa_{\ln T}ua_{-\ln T})h(u)du = \int_B f(xa_{\ln T}ua_{-\ln T})h(u)du
\end{aligned}$$

for some box $B \subset U$ since $h(u)$ is compactly supported. Using the same arguments as in the proof of Theorem 3.1.1, we can find $\gamma > 0$ with the following properties

1. We can devide B into small boxes $\{B_j\}$. For each j, there exists $y_j \in B$ such that $B_j = y_j B(\gamma, \gamma, \ldots, \gamma)$.

2. For each j, we have $\gamma < \eta(xa_{\ln T}y_j)/2$.

3. $\gamma \sim \eta(xa_{\ln T})$ and the implicit constant in $\sim$ depends only on B and $\Gamma\backslash G$.

By Lemma 3.3.4, we obtain that

$$\begin{aligned}
& \left|\frac{1}{T^\alpha}\int_U f(xu)h(a_{-\ln T}ua_{\ln T})du\right| = \left|\int_B f(xa_{\ln T}ua_{-\ln T})h(u)du\right| \\
\leq & \sum_j \left|\int_{B_j} f(xa_{\ln T}ua_{-\ln T})h(u)du\right| \\
= & \sum_j \left|\int_{B(\gamma,\ldots,\gamma)} f((xa_{\ln T}y_j)ua_{-\ln T})h(y_j u)du\right| \\
\ll & \sum_j \frac{1}{T^a\gamma^b}\|f\|_{\infty,l}\|h\|_{\infty,l} \ll \frac{\mathrm{Vol}(B)}{\gamma^n}\frac{1}{T^a\gamma^b}\|f\|_{\infty,l}\|h\|_{\infty,l} \\
\ll & \frac{1}{T^a\eta(xa_{\ln T})^{b+n}}\|f\|_{\infty,l}.
\end{aligned}$$

Here the implicit constant depends on $h(u)$. This completes the proof of Theorem 3.1.2. □

Chapter 4

Diophantine Points in Rank One Homogeneous Spaces

4.1 Introduction

The Diophantine approximation of numbers is a well-developed subject. One of the classical theorems in this subject is Jarnik-Besicovitch Theorem, which gives a formula for the Hausdorff dimensions of Diophantine numbers of different orders. Later, this theorem was generalized by Dodson [D92], which describes Hausdorff dimensions of Diophantine matrices of different orders.

It turns out that numbers and matrices with Diophantine conditions are closely related to points in homogeneous spaces with excursion rates under semisimple flows. A detailed description of this connection could be found in [K01]. For example, by reformulation, Dodson's work in [D92] actually gives a formula for Hausdorff dimensions of points with different excursion rates under the semisimple flow $a_t = \operatorname{diag}(e^{t/m}, \ldots, e^{t/m}, e^{-t/n}, \ldots, e^{-t/n})$ on $\operatorname{PSL}(m+n, \mathbb{R})/\operatorname{PSL}(m+n, \mathbb{Z})$. Also Melián and Pestana [MP93] get a formula for Hausdorff dimensions of points with different geodesic excursion rates in a hyperbolic manifold. As a consequence, their result implies a generalized version of Jarnik-Besicovitch Theorem on Diophantine approximation by numbers in some number fields of degree 2. In [D85], Dani associates badly approximable $m \times n$ matrices with bounded orbits under

$$a_t = \operatorname{diag}(e^{t/m}, \ldots, e^{t/m}, e^{-t/n}, \ldots, e^{-t/n})$$

on $\mathrm{PSL}(m+n,\mathbb{R})/\mathrm{PSL}(m+n,\mathbb{Z})$ and shows that the subset of points with bounded orbits under this flow has full Hausdorff dimension. In this direction, Dani [D86] also shows that such result holds for a non-quasiunipotent flow on G/Γ where G is a semisimple Lie group of rank one and Γ is a lattice in G. Finally in [KM96], Kleinbock and Margulis give a complete proof of this result for a non-quasiunipotent flow on any homogeneous space.

Moreover, this topic has also been developed in the case of negatively curved manifolds. For example, in [HP01], they define a Diophantine condition for geodesics starting from a point p and exponentially acculmulating at another point q. Then they get a sharp estimate on Hausdorff dimensions of such Diophantine geodesics. Later in [HP04] they study the case of $q=\infty$ and obtain a Kintchine-Sullivan-type theorem about such Diophantine geodesics. Readers may also refer to [HP02A] and [HP02B] for more details.

In this note, we will consider a similar question in the homogeneous space G/Γ where G is a semisimple Lie group of real rank one and Γ is a non-uniform lattice in G. Let $\{a_t\}$ denote a semisimple flow (i.e., every element in $\{a_t\}$ is Ad-semisimple) on the homogeneous space G/Γ. We will define Diophantine points in G/Γ and we would like to obtain a formula similar to Jarnik-Besicovitch Theorem. In the last section, we will see that the Diophantine condition defined in this paper is equivalent to that in the hyperbolic case [MP93] and that in the complex hyperbolic case in [HP02B].

Before stating the main theorem, we need some notations. For any $p \in G/\Gamma$, we will denote by $\mathrm{Stab}(p)$ the stabilizer of p in G. If $p = g\Gamma$, then $\mathrm{Stab}(p) = g\Gamma g^{-1}$ is a lattice conjugate to Γ. We will fix a norm $\|\cdot\|_{\mathfrak{g}}$ on the Lie algebra $\mathfrak{g}$ of G and denote by $d(\cdot,\cdot)_G$ and $d(\cdot,\cdot)_{G/\Gamma}$ the induced distances on G and G/Γ respectively. We will denote by $\mathfrak{a}$ the Lie algebra of the one parameter subgroup $\{a_t\}$.

For the semisimple Lie group G of rank one, we can write the Cartan decomposition with respect to a Cartan involution θ by

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$$

where $\mathfrak{k}$ and $\mathfrak{p}$ are the 1-eigenspace and (-1)-eigenspace of θ respectively, and we may assume that $\mathfrak{a} \subset \mathfrak{p}$ is a maximal abelian subalgebra of $\mathfrak{p}$. Let K be the maximal compact subgroup with the Lie algebra $\mathfrak{k}$. We write the root space decompostion of $\mathfrak{g}$ with respect to the adjoint action of $\{a_t\}$ as

$$\mathfrak{g} = \mathfrak{g}_{-2\alpha} \oplus \mathfrak{g}_{-\alpha} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_\alpha \oplus \mathfrak{g}_{2\alpha}.$$

Here α is a simple root, but we will think of this simple root as a positive number via the identification

$$\mathfrak{a}^* \cong \mathbb{R}.$$

In other words, we have $\alpha > 0$ and

$$\mathrm{Ad}a_t(v) = e^{\alpha t}v \quad (\forall v \in \mathfrak{g}_\alpha), \qquad \mathrm{Ad}a_t(v) = e^{2\alpha t}v \quad (\forall v \in \mathfrak{g}_{2\alpha})$$

$$\mathrm{Ad}a_t(v) = e^{-\alpha t}v \quad (\forall v \in \mathfrak{g}_{-\alpha}), \qquad \mathrm{Ad}a_t(v) = e^{-2\alpha t}v \quad (\forall v \in \mathfrak{g}_{-2\alpha}).$$

Note that the root spaces $\mathfrak{g}_{-2\alpha}$ and $\mathfrak{g}_{2\alpha}$ may be empty.

Definition 4.1.1. For any $p \in G/\Gamma$, we define the injectivity radius at p by

$$\eta(p) = \inf_{v \in \mathrm{Stab}(p) \setminus \{e\}} d(v, e)_G.$$

Definition 4.1.2. A point $p \in G/\Gamma$ is Diophantine of type γ if there exists a constant

$C > 0$ such that

$$\eta(a_t p) \geq Ce^{-\gamma t} \text{ for all } t > 0.$$

We will denote by S_γ the subset of all Diophantine points of type γ in G/Γ, and by S_γ^c the complement of S_γ in G/Γ.

Remark 4.1.1. Note that by [D86] and [KM96], the subset S_0 has full Hausdorff dimension.

Now we can state the main theorem in this chapter.

Theorem 4.1.1 (Main Theorem). *If $\mathfrak{g}_{2\alpha} = \emptyset$, then the Hausdorff dimension of S_γ^c $(0 \leq \gamma < \alpha)$ is*

$$\dim \mathfrak{g}_{-\alpha} + \dim \mathfrak{g}_0 + \frac{\alpha - \gamma}{\alpha} \dim g_\alpha.$$

If $g_{2\alpha} \neq \emptyset$, then the Hausdorff dimension of S_γ^c $(0 \leq \gamma < 2\alpha)$ is

$$\dim \mathfrak{g}_{-2\alpha} + \dim \mathfrak{g}_{-\alpha} + \dim \mathfrak{g}_0 + \frac{4\alpha - \gamma}{4\alpha} \dim g_\alpha + \frac{2\alpha - \gamma}{2\alpha} \dim g_{2\alpha}.$$

Remark 4.1.2. We will see in section 8 that the definition of S_γ^c is equivalent to the Diophantine condition in [MP93], and Theorem 4.1.1 generalizes Theorem 1 in [MP93].

Here we obtain an exact formula for the Hausdorff dimension of S_γ^c which, to the best of the author's knowledge, was only known in the case of hyperbolic manifold, i.e. $G = SO(n, 1)$ (e.g. [MP93]). Also, our proof is based on the theories of Lie groups and dynamical systems. So this note would be a new proof in the hyperbolic case. (See section 8 for more details.)

The following theorem is a finer version of Theorem 4.1.1. First we need some notations. Let $\xi_1, \ldots, \xi_k$ be the inequivalent cusps of G/Γ, and we fix sufficiently small neighborhoods Y_i of ξ_i in G/Γ $(1 \leq i \leq k)$ such that these Y_i's are pairwise disjoint.

Definition 4.1.3. A point $p \in G/\Gamma$ is Diophantine of type $(\gamma_1, \dots, \gamma_k)$ if there exists a constant $C > 0$ such that for any $i \in \{1, 2, \dots, k\}$ and any $t > 0$, we have

$$\eta(a_t p)\chi_{Y_i}(a_t p) \geq Ce^{-\gamma_i t}\chi_{Y_i}(a_t p).$$

We will denote by $S_{\gamma_1,\dots,\gamma_k}$ the subset of Diophantine points of type $(\gamma_1, \dots, \gamma_k)$, and by $S^c_{\gamma_1,\dots,\gamma_k}$ the complement of $S_{\gamma_1,\dots,\gamma_k}$ in G/Γ.

Remark 4.1.3. This definition measures different excursion rates γ_i of the orbit $\{a_t p\}$ near the cusps ξ_i $(1 \leq i \leq k)$.

Theorem 4.1.2. *If $\mathfrak{g}_{2\alpha} = \emptyset$, then the Hausdorff dimension of $S^c_{\gamma_1,\dots,\gamma_k}$ $(0 \leq \gamma_i < \alpha, 1 \leq i \leq k)$ is*

$$\dim \mathfrak{g}_{-\alpha} + \dim \mathfrak{g}_0 + \frac{\alpha - \min_{1\leq i\leq k} \gamma_i}{\alpha} \dim g_\alpha.$$

If $g_{2\alpha} \neq \emptyset$, then the Hausdorff dimension of $S^c_{\gamma_1,\dots,\gamma_k}$ $(0 \leq \gamma_i < \alpha, 1 \leq i \leq k)$ is

$$\begin{aligned} \dim \mathfrak{g}_{-2\alpha} &+ \dim \mathfrak{g}_{-\alpha} + \dim \mathfrak{g}_0 \\ &+ \frac{4\alpha - \min_{1\leq i\leq k} \gamma_i}{4\alpha} \dim g_\alpha + \frac{2\alpha - \min_{1\leq i\leq k} \gamma_i}{2\alpha} \dim g_{2\alpha}. \end{aligned}$$

Remark 4.1.4. We will show that Theorem 4.1.2 generalizes Theorem 1.7.1 in Chapter 1.

As an application, we will deduce a Jarnik-Besicovitch Theorem on Diophantine approximation in Heisenberg groups in the setting of [HP02B]. We will follow the definitions and notations in [HP02B], which readers may refer to for more details.

The real Heisenberg group $\mathcal{H}_{2n-1}(\mathbb{R})$ is the manifold $\mathbb{C}^{n-1} \times \mathbb{R}$ in which the group multiplication is given by

$$(\zeta, v)(\zeta', v') = \left(\zeta + \zeta', v + v' + 2\Im\left(\zeta \cdot \overline{\zeta'}\right)\right)$$

where $\zeta, \zeta' \in \mathbb{C}^{n-1}$ and $v, v' \in \mathbb{R}$. We write $d_{\mathcal{H}_{2n-1}(\mathbb{R})}$ for the right invariant distance on $\mathcal{H}_{2n-1}(\mathbb{R})$. $\mathcal{H}_{2n-1}(\mathbb{R})$ is the $\mathbb{R}$-points of a connected algebraic group $\mathcal{H}_{2n-1}$ defined over $\mathbb{Q}$ with $\mathbb{Q}$-points $\mathcal{H}_{2n-1}(\mathbb{Q}) \cong \mathbb{Q}[i]^{n-1} \times \mathbb{Q}$. For any $r \in \mathcal{H}_{2n-1}(\mathbb{Q})$, the height $h(r)$ of r is defined as the absolute value of the least common multiple of the denominators of the rationals in the components of r. Let $|\cdot| : \mathcal{H}_{2n-1}(\mathbb{R}) \to \mathbb{R}$ be defined by

$$|(\zeta, v)| = \left(|\zeta|^4 + v^2\right)^{\frac{1}{4}}$$

and the Cygan distance on $\mathcal{H}_{2n-1}(\mathbb{R})$ is defined by

$$d_{\mathrm{Cyg}}((\zeta, v), (\zeta' v')) = |(\zeta, v)(\zeta', v')^{-1}|.$$

Note that this distance is invariant under the right multilplication of $\mathcal{H}_{2n-1}(\mathbb{R})$. See [G99] and [HP02B] for more details.

Definition 4.1.4. A point $\alpha \in \mathcal{H}_{2n-1}(\mathbb{R})$ is Diophantine of type γ ($\gamma \in \mathbb{R}$) if there exists a constant $C > 0$ such that

$$d_{\mathrm{Cyg}}(\alpha, r) \geq \frac{C}{(h(r))^{\gamma}}$$

for any $r \in \mathcal{H}_{2n-1}(\mathbb{Q})$. We will denote by L_γ the subset of all Diophantine points of type γ in $\mathcal{H}_{2n-1}(\mathbb{R})$, and by L_γ^c the complement of L^γ in $\mathcal{H}_{2n-1}(\mathbb{R})$.

Remark 4.1.5. Note that by Theorem 3.4 in [HP02B], we have $\gamma \geq 1$.

Theorem 4.1.3. *The Hausdorff dimension of L_γ^c ($\gamma \geq 1$) with respect to the right invariant distance $d_{\mathcal{H}_{2n-1}(\mathbb{R})}$ is equal to*

$$\frac{\gamma+1}{\gamma}n - 1.$$

Remark 4.1.6. Note that Hersonsky and Paulin give a Kintchine-Sullivan type theorem on Diophantine Approximation in Heisenberg groups (Theorem 3.5 in [HP02B]). Here Theorem 4.1.3 can be thought of as a Jarnik-Besicovitch theorem in this setting.

In the proof of Theorem 4.1.1, the counting problem (see section 4.5) will be crucial which involves the mixing property of $\{a_t\}$ acting on G/Γ. Actually it will play an important role in calculating both the upper bound and the lower bound of the Hausdorff dimension. In section 4.2, we will list concepts and theorems needed in this paper. In section 4.3, we will reduce Theorem 4.1.1 (see Theorem 4.3.1). In section 4.4, we will give some Lie group facts which will be used often throughout this note. In section 4.5, we will give a definition of rational points in G/Γ and define the denominator of a rational point. With the help of the mixing property of $\{a_t\}$, we will be able to count the rational points with their denominators between two large numbers. This counting result will be used to calculate the Hausdorff dimension of a tree-like subset. In section 4.6, we will closely study the meaning of a point in G/Γ being Diophantine and give sufficient condition and necessary condition for it. Combining all the results in this paper, we will give the proof of Theorem 4.1.1 in section 4.7. The necessary condition in section 4.6 will be used for the upper bound of the Hausdorff dimension and the sufficient condition will be used for the lower bound. In section 4.8, we will prove Theorem 4.1.2. In the last section, we will discuss the relations between Theorem 4.1.1 and [MP93], Theorem 4.1.2 and [Z16], and give a proof of Theorem 4.1.3.

4.2 Notations and Preliminaries

Let exp be the exponential map from $\mathfrak{g}$ to G. For any Lie subgroup $H \subseteq G$, we will denote by Lie(H) the Lie subalgebra of H and μ_H the Haar measure on H.

Let

$$\mathfrak{n}_+ = \mathfrak{g}_\alpha \oplus \mathfrak{g}_{2\alpha}, \quad \mathfrak{n}_- = \mathfrak{g}_{-\alpha} \oplus \mathfrak{g}_{-2\alpha}$$

and N_+ and N_- be the corresponding unipotent subgroups. We will denote by $A = \{a_t\}$ and

$$A_{s_1,s_2} = \{a_t \in A : s_1 \leq t \leq s_2\}$$

and

$$N_+(S) = \{n \in N_+ : n \in S\} \quad N_-(S) = \{n \in N_- : n \in S\}$$

for any subset $S \subset N_+$ or N_-. We will fix two bases in $\mathfrak{g}_\alpha$ and $\mathfrak{g}_{2\alpha}$, and we will write

$$\mathfrak{B}_{\mathfrak{g}_\alpha}(r) \text{ and } \mathfrak{B}_{\mathfrak{g}_{2\alpha}}(r)$$

for the open cubes along these bases of equal side length r in $\mathfrak{g}_\alpha$ and $\mathfrak{g}_{2\alpha}$ respectively

If $\mathfrak{g}_{2\alpha} = \emptyset$, we have that $\mathfrak{n}_+ = \mathfrak{g}_\alpha$ and $\mathfrak{n}_- = \mathfrak{g}_{-\alpha}$. We will denote by

$$B_{N_+}(r) = \exp(\mathfrak{B}_{\mathfrak{g}_\alpha}(r))$$

the open cube centered at e with side length r in N_+. If $\mathfrak{g}_{2\alpha} \neq \emptyset$, then $\mathfrak{n}_+ = \mathfrak{g}_\alpha \oplus \mathfrak{g}_{2\alpha}$ and we will denote by

$$B_{N_+}(r_1, r_2) = \exp(\mathfrak{B}_{\mathfrak{g}_\alpha}(r_1) + \mathfrak{B}_{\mathfrak{g}_{2\alpha}}(r_2))$$

for the open box centered at e with length r_1 in $\mathfrak{g}_\alpha$-direction and r_2 in $\mathfrak{g}_{2\alpha}$-direction.

The Bruhat decomposition in the real rank one case has the following simple form

$$G = MAN_{-} \cup MAN_{-}\omega MAN_{-}$$

where M is the centralizer of $\mathfrak{a}$ in K and ω is a representative of the non-trivial element in the Weyl group.

In the following, we will need Theorem 2.1.2 to compute Hausdorff dimension. For convenience, we list it here.

Theorem 4.2.1 ([M87], [U91] or [KM96]). *Let (X, m) be a Riemannian manifold. Assume that there exist constants $D > 0$ and $k > 0$ such that*

$$m(B(x,r)) \leq Dr^k$$

for any $x \in X$ and any ball $B(x,r)$ of radius r around x. Then for any tree-like collection $\mathcal{A}$ of subsets of E

$$\dim_H(\mathbf{A}_\infty) \geq k - \limsup_{j\to\infty} \frac{\sum_{i=0}^{j} \log(\frac{1}{\Delta_i(\mathcal{A})})}{\log(\frac{1}{d_{j+1}(\mathcal{A})})}$$

We also need the following theorem about the fundamental domain of a non-uniform lattice in G of real rank one. We write the Siegel set

$$\Omega(s,V) = KA_{s,\infty}N_{-}(V)$$

for some $s \in \mathbb{R}$ and some compact subset $V \subset N_{-}$.

Theorem 4.2.2 ([GR70], [D84]). *There exist $s_0 > 0$, a compact subset V_0 of N and a finite subset Σ of G such that the following assertions hold:*

1. *$G = \Omega(s_0, V_0)\Sigma\Gamma$.*

2. *For all $\sigma \in \Sigma$, $\Gamma \cap \sigma^{-1} N_- \sigma$ is a cocompact lattice in $\sigma^{-1} N_- \sigma$.*

3. *For all compact subsets V of N the set*

$$\{\gamma \in \Gamma | \Omega(s_0, V)\Sigma\gamma \cap \Omega(s_0, V) \neq \emptyset\}$$

   is finite.

4. *Give a compact subset V of N containing V_0, there exists $s_1 \in (0, s_0)$ such that whenever $\sigma, \tau \in \Sigma$ are such that $\Omega(s_0, V)\sigma\gamma \cap \Omega(s_1, V)\tau$ is non-empty for some γ then $\sigma = \tau$ and $\sigma\gamma\sigma^{-1} \in (K \cap Z) \cdot N_- \subset P$.*

Here Z is the centralizer of $A = \{a_t\}$ and $P = ZN_-$.

Remark 4.2.1. Note that the subset Σ corresponds to the cusp set $\{\xi_1, \ldots, \xi_k\}$.

4.3 Reductions

By the property of Hausdorff dimension, to prove Theorem 4.1.1, it suffices to prove that for any point $x = g\Gamma$ and any small open neighborhood $B_G(r)x \subset G/\Gamma$ of $x = g\Gamma$ with a sufficiently small $r < \eta(x)$, the subset $S_\gamma^c \cap B_G(r)x$ has the same Hausdorff dimension as that in the main theorem. Furthermore, for any element g in $B_G(r)$, we can write

$$g = n_- a n_+$$

for some $n_- \in N_-$, $a \in A$ and $n_+ \in N_+$. By definition, $g \in S_\gamma$ if and only if $n_+ \in S_\gamma$, and hence it is enough to prove that for any small open ball U_0 at e in N_+ we have

$$\dim_H S_\gamma^c \cap U_0 x = \begin{cases} \frac{\alpha-\gamma}{\alpha} \dim \mathfrak{g}_\alpha & \text{if } g_{2\alpha} = \emptyset \\ \frac{4\alpha-\gamma}{4\alpha} \dim \mathfrak{g}_\alpha + \frac{2\alpha-\gamma}{2\alpha} \dim \mathfrak{g}_{2\alpha} & \text{if } \mathfrak{g}_{2\alpha} \neq \emptyset \end{cases}.$$

Replacing the lattice Γ by $g\Gamma g^{-1}$, we can assume without loss of generality that $x = e\Gamma$ and hence to prove the main theorem, it suffices to prove the following

Theorem 4.3.1. *Let U_0 be a small open ball at e in N_+ of radius $r < \eta(e\Gamma)$. Then we have*

$$\dim_H S_\gamma^c \cap U_0(e\Gamma) = \begin{cases} \frac{\alpha-\gamma}{\alpha} \dim \mathfrak{g}_\alpha & \text{if } \mathfrak{g}_{2\alpha} = \emptyset \\ \frac{4\alpha-\gamma}{4\alpha} \dim \mathfrak{g}_\alpha + \frac{2\alpha-\gamma}{2\alpha} \dim \mathfrak{g}_{2\alpha} & \text{if } g_{2\alpha} \neq \emptyset \end{cases}.$$

Here $0 \leq \gamma < \alpha$ if $\mathfrak{g}_{2\alpha} = \emptyset$ and $0 \leq \gamma < 2\alpha$ if $\mathfrak{g}_{2\alpha} \neq \emptyset$.

In the following sections, we will fix this open ball $U_0 \subset N_+$ and study the problem of Diophantine points in the space $U_0(e\Gamma)$ instead of G/Γ. Since U_0 is isomorphic to $U_0(e\Gamma)$, we will still write μ_{N_+} for the N_+-invariant measure on $U_0(e\Gamma)$, i.e.

$$\mu_{N_+}(B(e\Gamma)) = \mu_{N_+}(B)$$

for any Borel subset $B \subset U_0 \subset N_+$, and for any point $n_+ \in U_0$, the subset $B_{N_+}(r)n_+\Gamma$ or $B_{N_+}(r_1, r_2)n_+\Gamma$ will be an open box at $n_+\Gamma \in U_0(e\Gamma)$. In other words, we will use the notations in U_0 and $U_0(e\Gamma)$ interchangeably.

4.4 Some Lie Group Facts

In this section, we will prove some Lie group facts which will be used often in this note.

Proposition 4.4.1. *For any $u \in \mathfrak{n}_- \setminus \{0\}$ and $v \in \mathfrak{n}_+ \setminus \{0\}$ we have*

$$\|[u, v]\|_{\mathfrak{g}} \sim \|u\|_{\mathfrak{g}} \|v\|_{\mathfrak{g}}.$$

Here the implicit constant depends only on G.

Proof. It suffices to prove that for any $u \in \mathfrak{n}_- \setminus \{0\}$ and $v \in \mathfrak{n}_+ \setminus \{0\}$

$$[u, v] \neq 0$$

and this follows immediately from Lemma 3.4 in [BZ16]. □

Proposition 4.4.2. *Let $u \in G$ be a unipotent element. Then there exists a unique element n in $N_+ \cup \{\omega\}$ such that $Adn(u) \in N_-$. Moreover, if $u \notin N_+$, then this $n \in N_+$.*

Proof. We know that there is an element $g \in G$ such that $\mathrm{Ad}g(u) \in N_-$. By the Bruhat decomposition, g is either $ma\bar{n}\omega$ or $ma\bar{n}n$ for some $m \in M, a \in A, n \in N_+$ and $\bar{n} \in N_-$. Since $ma\bar{n}$ stablizes N_-, we have either $\mathrm{Ad}\omega(u) \in N_-$ or $\mathrm{Ad}n(u) \in N_-$.

Suppose that there are two elements $n_1, n_2 \in N_+ \cup \{\omega\}$ such that $\mathrm{Ad}n_i(u) \in N_-$. Then $\mathrm{Ad}n_2 n_1^{-1} N_- \cap N_- \neq \{0\}$. By Lemma 3.4 in [BZ16], this implies that $n_2 n_1^{-1} \in MAN_-$ and hence by the Bruhat decomposition, $n_1 = n_2$. The second part follows immediately from the first. This completes the proof of the proposition. □

Proposition 4.4.3. *Suppose that $\mathfrak{g}_{-2\alpha} \neq \emptyset$. Then for any $\sigma \in \Sigma$, we have $\sigma\Gamma\sigma^{-1} \cap$ $\exp(\mathfrak{g}_{-2\alpha})$ is a lattice in $\exp(\mathfrak{g}_{-2\alpha})$.*

Proof. Let $u \in \mathfrak{g}_{-\alpha} \setminus \{0\}$. By Lemma 7.73 (a) in [K02], we know that the map

$$ad(u) : \mathfrak{g}_{-\alpha} \to \mathfrak{g}_{-2\alpha}$$

is surjective and hence

$$[\mathfrak{g}_{-\alpha}, \mathfrak{g}_{-\alpha}] = \mathfrak{g}_{-2\alpha}.$$

This implies that

$$[N_-, N_-] = \exp(\mathfrak{g}_{-2\alpha}).$$

On the other hand, since $\sigma\Gamma\sigma^{-1} \cap N_-$ is a lattice in N_-, by Corollary 1 of Theorem 2.3 in [R87], we know that $\sigma\Gamma\sigma^{-1} \cap [N_-, N_-]$ is a lattice in $[N_-, N_-]$. This completes the proof of the proposition. $\square$

4.5 Counting Rational Points

In this section, we will define rational points in G/Γ and their denominators. Then we will count rational points, which will be crucial to the study of Diophantine points in the following sections.

Definition 4.5.1. A point $p \in G/\Gamma$ is called rational if $\mathrm{Stab}(p) \cap N_- \neq \{e\}$.

Note that if p is rational, then $\eta(a_t p) \to 0$ and $\{a_t p\}$ diverges as $t \to \infty$. By Corollary 6.2 in [D85], we immediately get the following

Proposition 4.5.1 (Corollary 6.2 [D85])**.** *$p \in G/\Gamma$ is rational if and only if $p \in \bigcup_{\sigma\in\Sigma} MAN_-\sigma\Gamma$.*

Definition 4.5.2. A point p is called σ-rational for some $\sigma \in \Sigma$ if $p \in MAN_-\sigma\Gamma$.

Proposition 4.5.2. *Let $p \in G/\Gamma$ be σ-rational and suppose that $p = m_1a_1n_1\sigma\Gamma = m_2a_2n_2\sigma\Gamma$. Then $a_1 = a_2$.*

Proof. Since $m_1a_1n_1\sigma\Gamma = m_2a_2n_2\sigma\Gamma$, the lattices of N_-

$$m_1a_1n_1\sigma\Gamma\sigma^{-1}n_1^{-1}a_1^{-1}m_1^{-1} \cap N_-, \quad m_2a_2n_2\sigma\Gamma\sigma^{-1}n_2^{-1}a_2^{-1}m_2^{-1} \cap N_-$$

coincide and hence they have the same co-volume in N. This implies that $a_1 = a_2$. $\square$

Definition 4.5.3. We define the σ-denominator of a σ-rational point $p \in G/\Gamma$ by

$$d_\sigma(p) = e^{-\alpha t_0}$$

where $p = m a_{t_0} n \sigma \Gamma$ for some $t_0 \in \mathbb{R}$.

Remark 4.5.1. Note that by Proposition 4.5.2, this definition is well-defined.

Definition 4.5.4. For any $U \subset N_+$, we will denote by $S_\sigma(U(e\Gamma), l_1, l_2)$ the subset of σ-rational points in $U(e\Gamma)$ whose σ-denominators are between l_1 and l_2.

Proposition 4.5.3. *Suppose that $\mathfrak{g}_{2\alpha} = \emptyset$. Then for any open subset $U \subset U_0 \subset N_+$, we have that $S_\sigma(U(e\Gamma), l_1, l_2)$ is finite and*

$$|S_\sigma(U(e\Gamma), C/2, C)| \sim \mu_{N_+}(U) C^{\dim \mathfrak{g}_\alpha}$$

for any sufficiently large $C > 0$. Here the implicit constant depends only on G and Γ.

Proof. Since $\mathfrak{g}_{2\alpha} = \emptyset$, we have that $\mathfrak{n}_+ = \mathfrak{g}_\alpha$ and $\mathfrak{n}_- = \mathfrak{g}_{-\alpha}$. Recall that $B_{N_+}(r)$ denotes the open box centered at e with length r in N_+.

Let $n_+\Gamma \in N_+\Gamma$ be a σ-rational point and there exist $m \in M, a_{t_0} \in A$ and $n \in N_-$ such that $n_+\Gamma = m a_{t_0} n \sigma \Gamma$. By definition, the σ-denominator of $n_+\Gamma$ being between $C/2$ and C is equivalent to the condition that

$$C/2 \le e^{-\alpha t_0} \le C.$$

This implies that $a_{\ln C/\alpha} n \Gamma \in M A_{-\ln 2/\alpha, 0} N(\Omega) \sigma \Gamma$.

Since $M A_{-\ln 2/\alpha, 0} N(\Omega) \sigma \Gamma$ is compact, we can find $\delta > 0$ such that

$$\delta < \eta(y)$$

for any $y \in MA_{-\ln 2/\alpha,0}N_\Omega\sigma\Gamma$. Now by thickening the subset $MA_{-\ln 2/\alpha,0}N_\Omega\sigma\Gamma$ along the N_+ direction, we would like to study the following integral

$$\int_U \chi_{B_{N_+}(\delta)MA_{-\ln 2/\alpha,0}N(\Omega)\sigma\Gamma}(a_{\ln C/\alpha}n\Gamma)d\mu_{N_+}(n). \tag{4.1}$$

On the one hand, by the mixing property of the action a_t, integral (4.1) is asymptotically equal to

$$\mu_{N_+}(U)\int_{G/\Gamma} \chi_{B_{N_+}(\delta)MA_{-\ln 2/\alpha,0}N(\Omega)\sigma\Gamma}(g)d\mu_G(g) \sim \mu_{N_+}(U)\delta^{\dim \mathfrak{g}_\alpha}. \tag{4.2}$$

On the other hand, if $n_q\Gamma$ is a σ-rational point in $U(e\Gamma)$, then any $n\Gamma \in B_{N_+}(\delta/C)n_q\Gamma$ satisfies the following

$$a_{\ln C/\alpha}n\Gamma \in B_{N_+}(\delta)MA_{-\ln 2/\alpha,0}N(\Omega)\sigma\Gamma.$$

Conversely, if a point $n\Gamma \in U(e\Gamma)$ satisfies

$$a_{\ln C/\alpha}n\Gamma \in B_{N_+}(\delta)MA_{-\ln 2/\alpha,0}N_\Omega\sigma\Gamma,$$

then there is a point $n_q\Gamma$ with

$$n_q\Gamma \in B_{N_+}(\delta/C)n\Gamma$$

such that

$$a_{\ln C/\alpha}n_q\Gamma \in MA_{-\ln 2/\alpha,0}N(\Omega)\sigma\Gamma$$

which means that n_q is a σ-rational point. This implies that integral (4.1) is also asymptotically equal to

$$|S_\sigma(U, C/2, C)|(\delta/C)^{\dim \mathfrak{g}_\alpha}. \tag{4.3}$$

Hence, by comparing equations (4.2) and (4.3), we know that

$$|S_\sigma(U, C/2, C)| \sim \mu_{N_+}(U)\delta^{\dim \mathfrak{g}_\alpha}/(\delta/C)^{\dim \mathfrak{g}_\alpha} = \mu_{N_+}(U)C^{\dim \mathfrak{g}_\alpha}.$$

This completes the proof of the proposition □

Proposition 4.5.4. *Suppose that $\mathfrak{g}_{2\alpha} \neq 0$. Then for any open subset $U \subset U_0 \subset N_+$, we have that $S_\sigma(U(e\Gamma), l_1, l_2)$ is finite and*

$$|S_\sigma(U(e\Gamma), C/2, C)| \sim \mu(U)C^{\dim \mathfrak{g}_\alpha + 2\dim \mathfrak{g}_{2\alpha}}$$

for any sufficiently large $C > 0$. Here the implicit constant depends only on G and Γ.

Proof. In this case, the proof is almost identical to that in the case of $\mathfrak{g}_{2\alpha} = 0$ but computations involved will be more complicated. Recall that $B_{N_+}(r_1, r_2)$ denotes the open box centered at e with length r_1 in $\mathfrak{g}_\alpha$-direction and r_2 in $\mathfrak{g}_{2\alpha}$-direction.

Let $n\Gamma \in N_+\Gamma$ be σ-rational and there exist $m \in M$, $a_{t_0} \in A$ and $n \in N_-$ such that $n\Gamma = ma_{t_0}\bar{n}\sigma\Gamma$. The σ-denominator of $n\Gamma$ being between $C/2$ and C is equivalent to the condition that

$$C/2 \leq e^{-\alpha t_0} \leq C.$$

This implies that $a_{\ln C/\alpha}n\Gamma \in MA_{-\ln 2/\alpha,0}N(\Omega)\sigma\Gamma$.

Since $MA_{-\ln 2/\alpha,0}N(\Omega)\sigma\Gamma$ is compact, we can find $\delta > 0$ such that

$$\delta < \eta(y)$$

for any $y \in MA_{-\ln 2/\alpha,0}N(\Omega)\sigma\Gamma$. Now by thickening the subset $MA_{-\ln 2/\alpha,0}N(\Omega)\sigma\Gamma$

along the N_+ direction, we would like to study the following integral

$$\int_U \chi_{B_{N_+}(\delta,\delta)MA_{-\ln 2/\alpha,0}N(\Omega)\sigma\Gamma}(a_{\ln C/\alpha}n\Gamma)dn. \tag{4.4}$$

On the one hand, by the mixing property of the action a_t, integral (4.4) is asymptotically equal to

$$\mu_{N_+}(U)\int_{G/\Gamma} \chi_{B_{N_+}(\delta,\delta)MA_{-\ln 2/\alpha,0}N(\Omega)\sigma\Gamma}(g)dg \sim \mu_{N_+}(U)\delta^{\dim \mathfrak{g}_\alpha + \dim \mathfrak{g}_{2\alpha}}. \tag{4.5}$$

On the other hand, if $n_q\Gamma$ is a σ-rational point in $U(e\Gamma)$, then any

$$n\Gamma \in B_{N_+}(\delta/C, \delta/C^2)n_q\Gamma$$

satisfies the following

$$a_{\ln C/\alpha}n\Gamma \in B_{N_+}(\delta,\delta)MA_{-\ln 2/\alpha,0}N(\Omega)\sigma\Gamma.$$

Conversely, if a point $n\Gamma \in U(e\Gamma)$ satisfies

$$a_{\ln C/\alpha}n\Gamma \in B_{N_+}(\delta,\delta)MA_{-\ln 2/\alpha,0}N(\Omega)\sigma\Gamma,$$

then there is a point $n_q\Gamma$ with

$$n\Gamma \in B_{N_+}(\delta/C, \delta/C^2)n_q\Gamma$$

such that

$$a_{\ln C/\alpha}n_q\Gamma \in MA_{-\ln 2/\alpha,0}N(\Omega)\sigma\Gamma$$

which means that $n_q\Gamma$ is a σ-rational point. This implies that integral (4.4) is also asymptotically equal to

$$|S_\sigma(U, C/2, C)|(\delta/C)^{\dim \mathfrak{g}_\alpha}(\delta/C^2)^{\dim \mathfrak{g}_{2\alpha}}. \tag{4.6}$$

Hence, by comparing equations (4.5) and (4.6), we know that

$$\begin{aligned} |S_\sigma(U, C/2, C)| &\sim \mu_{N_+}(U)\delta^{\dim \mathfrak{g}_\alpha + \dim \mathfrak{g}_{2\alpha}}/(\delta/C)^{\dim \mathfrak{g}_\alpha}(\delta/C^2)^{\dim \mathfrak{g}_{2\alpha}} \\ &= \mu_{N_+}(U)C^{\dim \mathfrak{g}_\alpha + 2\dim \mathfrak{g}_{2\alpha}}. \end{aligned}$$

This completes the proof of the proposition □

Definition 4.5.5. We define the denominator of a rational point p by

$$d(p) = \inf_{v \in \mathrm{Stab}(p) \cap \exp(\mathfrak{g}_{-\beta}) \setminus \{e\}} \|v\|^{\frac{\alpha}{\beta}}$$

where $\beta = \alpha$ if $\mathfrak{g}_{2\alpha} = \emptyset$ and $\beta = 2\alpha$ if $\mathfrak{g}_{2\alpha} \neq \emptyset$.

Remark 4.5.2. Note that by Proposition 4.4.3, $\mathrm{Stab}(p) \cap \exp(\mathfrak{g}_{-2\alpha}) \neq \{e\}$ is a lattice in $\exp(\mathfrak{g}_{-2\alpha})$ if $\mathfrak{g}_{2\alpha} \neq \emptyset$ and so $d(p)$ is well-defined.

Definition 4.5.6. For any $U \subset N_+$, we will denote by $S(U(e\Gamma), l_1, l_2)$ the subset in $U(e\Gamma)$ of rational points whose denominators are between l_1 and l_2.

Proposition 4.5.5. *Let $p \in G/\Gamma$ be a rational point. Then*

$$d(p) \sim d_\sigma(p)$$

whenever p is a σ-rational point for some $\sigma \in \Sigma$. Here the implicit constant depends only on G and Γ.

Proof. Let $p = m a_{t_0} n \sigma \Gamma$ for some $\sigma \in \Sigma$. Suppose that $\mathfrak{g}_{2\alpha} = \emptyset$. Then

$$\mathrm{Stab}(p) \cap \mathfrak{g}_{-\alpha} = a_{t_0}(\mathrm{Stab}(mn\sigma\Gamma) \cap \mathfrak{g}_{-\alpha}) a_{t_0}^{-1}.$$

Since m, n, σ are all in compact subsets in G, this implies that

$$d(p) \sim e^{-\alpha t_0} = d_\sigma(p).$$

The proof in the case $\mathfrak{g}_{2\alpha} \neq \emptyset$ is similar. This completes the proof of the proposition. □

Proposition 4.5.6. *Let $U \subset U_0 \subset N_+$ be an open box in N_+. For any sufficiently large $C > 0$ we have that $S(U(e\Gamma), C/2, C)$ is finite and*

1. *if $\mathfrak{g}_{2\alpha} = \emptyset$, then*
$$|S(U(e\Gamma), C/2, C)| \sim \mu_{N_+}(U) C^{\dim \mathfrak{g}_\alpha}.$$

2. *if $\mathfrak{g}_{2\alpha} \neq \emptyset$, then*

$$|S(U(e\Gamma), C/2, C)| \sim \mu_{N_+}(U) C^{\dim \mathfrak{g}_\alpha + 2 \dim \mathfrak{g}_{2\alpha}}.$$

Here the implicit constants depend only on G and Γ.

Proof. This follows immediately from Proposition 4.5.1, Proposition 4.5.3, Proposition 4.5.4 and Proposition 4.5.5. □

4.6 Diophantine Points

In this section, we will study the Diophantine points and prove some propositions which will be used in the proof of Theorem 4.3.1.

Proposition 4.6.1. *Let $p \in G/\Gamma$ be a non-rational point. If p is not Diophantine of type γ, then there exists a sequence $t_n \to \infty$ satisfing the following conditions*

1. *for each $t_n > 0$, there is a $v_n \in Stab(a_{t_n}p)$ such that v_n is unipotent and $d(v_n, e)_G = C_p e^{-\gamma t_n}$ for some constant $C_p > 0$ depending only on p*

2. *for each $t_n > 0$, there exists $\epsilon_n > 0$ such that for any $t \in (0, \epsilon_n)$, we have*

$$d(Ad(a_{-t})v_n, e)_G > C_p e^{-\gamma(t_n - t)}.$$

Proof. By definition, we know that if p is not Diophantine of type γ, then there exist a constant $C_p > 0$ and a sequence $t_n \to \infty$ such that

$$\eta(a_{t_n}p) = C_p e^{-\gamma t_n}$$

and for each $t_n > 0$, there exists $\epsilon_n > 0$ such that for any $t \in (0, \epsilon_n)$ we have

$$\eta(a_{t_n - t}p) > C_p e^{-\gamma(t_n - t)}.$$

This implies that there exists $v_n \in \mathrm{Stab}(a_{t_n}p)$ such that

$$d(v_n, e)_G = C_p e^{-\gamma t_n}, \quad d(Ad(a_{-t})v_n, e)_G > C_p e^{-\gamma(t_n - t)} (\forall t \in (0, \epsilon_n)).$$

It follows from Corollary 11.18 in [R87] that v_n is unipotent for sufficiently large $t_n > 0$. This completes the proof of the proposition. $\square$

Proposition 4.6.2. *Suppose that $\mathfrak{g}_{2\alpha} = \emptyset$. Let $p \in U_0(e\Gamma) \subset G/\Gamma$ be non-rational, $t_0 \in \mathbb{R}_+$ a sufficiently large number and $v_0 \in Stab(a_{t_0}p)$ such that t_0 and v_0 satisfy the conditions in Proposition 4.6.1. Then there is a rational point $q \in U_0(e\Gamma)$ with*

$d(q) \sim e^{(\alpha-\gamma)t_0}$ (the implicit constant depending only on p) such that

$$p \in B_{N_+}(Cd(q)^{-\frac{\alpha}{\alpha-\gamma}})q$$

for some constant $C > 0$ depending only on p.

Proof. Since $\{a_{t_0}\}$ expands N_+, the condition 2 in Proposition 4.6.1 implies that $v_0 \notin N_+$. Let $v_0 = \exp(v)$ for some $v \in \mathfrak{g}$. By Proposition 4.4.2, there is a unique $n \in N_+$ such that

$$\mathrm{Ad}n(v) = z \in \mathfrak{n}_-.$$

Now set $n^{-1} = \exp(u)$ for some $u \in \mathfrak{n}_+$. Then we have

$$z + [u, z] + [u, [u, z]]/2 = v$$

$$z \in \mathfrak{n}_-, \quad [u, z] \in \mathfrak{g}_0, \quad [u, [u, z]] \in \mathfrak{n}_+.$$

Since $\|v\|_{\mathfrak{g}} \sim d(v_0, e)_G = C_p e^{-\gamma t_0}$ for some C_p as in Proposition 4.6.1, by the condition 2 in Proposition 4.6.1, we have

$$\|z\|_{\mathfrak{g}} \sim C_p e^{-\gamma t_0}, \quad \|[u, z]\|_{\mathfrak{g}} \leq C_p e^{-\gamma t_0}$$

and by Proposition 4.4.1, this implies that u and n are bounded.

Now by definition, we know that $na_{t_0}p$ is rational and hence $a_{-t_0}na_{t_0}p$ is rational. Let $q = (a_{-t_0}na_{t_0})p$ and we have that

$$d(a_{-t_0}na_{t_0}, e)_G \leq \|a_{-t_0}ua_{t_0}\|_{\mathfrak{g}} \ll e^{-\alpha t_0}.$$

Also we have that

$$d(na_{t_0}p) \sim C_p e^{-\gamma t_0},$$

and hence the denominator of $q = a_{-t_0} n a_{t_0} p$ is equal to

$$d(q) = e^{\alpha t_0} d(na_{t_0}p) \sim C_p e^{(\alpha-\gamma)t_0}$$

and hence

$$d(a_{-t_0} n a_{t_0}, e)_G \leq C d(q)^{-\frac{\alpha}{\alpha-\gamma}}$$

for some constant $C > 0$. Note that C depends only on p. This completes the proof of the proposition. □

Proposition 4.6.3. *Suppose that $\mathfrak{g}_{2\alpha} = \emptyset$. Let $p \in U_0(e\Gamma) \subset G/\Gamma$ be a non-rational point. If p is not Diophantine of type γ, then there exist a constant $C > 0$ and a sequence of distinct rational points $q_n \in U_0(e\Gamma)$ with $d(q_n) \to \infty$ such that*

$$p \in B_{N_+}(Cd(q_n)^{-\frac{\alpha}{\alpha-\gamma}})q_n.$$

Proof. Suppose that $p \in U_0(e\Gamma)$ is not Diophantine of type γ. By Proposition 4.6.1, there exist infinitely many $t_n \to \infty$ and $v_n \in \mathrm{Stab}(a_{t_n}p)$ satisfying the conditions in Proposition 4.6.1 and hence by Proposition 4.6.2, there exist infinitely many rational points $q_n \in U_0(e\Gamma)$ with $d(q_n) \sim e^{(\alpha-\gamma)t_n}$ such that

$$p \in B(Cd(q_n)^{-\frac{\alpha}{\alpha-\gamma}})q_n$$

for some constant C depending only on p. This completes the proof of the proposition.

□

Proposition 4.6.4. *Suppose that $\mathfrak{g}_{2\alpha} = \emptyset$. Let $p \in U_0(e\Gamma)$ and let $\epsilon > 0$ be a* *sufficiently small number. If there exist a constant $C > 0$ and a sequence $q_n \in U_0(e\Gamma)$ of distinct rational points with $d(q_n) \to \infty$ such that*

$$p \in B_{N_+}(Cd(q_n)^{-\frac{\alpha}{\alpha-(\gamma+\epsilon)}})q_n,$$

then p is not Diophantine of type γ.

Proof. Suppose that there exist a constant $C > 0$ and a sequence $q_n \in G/\Gamma$ of rational points converging to p such that

$$p \in B_{N_+}(Cd(q_n)^{-\frac{\alpha}{\alpha-(\gamma+\epsilon)}})q_n.$$

Let $t_n = \ln d(q_n)^{\frac{1}{\alpha-(\gamma+\epsilon)}}$. Then we have that

$$d(a_{t_n}q_n, a_{t_n}p)_{G/\Gamma} \leq C$$

and

$$\eta(a_{t_n}q_n) \sim d(q_n)/e^{\alpha t_n} = e^{-(\gamma+\epsilon)t_n}.$$

This implies that

$$\eta(a_{t_n}p) \leq C'e^{-(\gamma+\epsilon)t_n}$$

for some constant $C' > 0$ and infinitely many $t_n \to \infty$. Hence by definition p is not Diophantine of type γ. $\square$

Proposition 4.6.5. *Suppose that $\mathfrak{g}_{2\alpha} \neq \emptyset$. Let $p \in U_0(e\Gamma) \subset G/\Gamma$ be non-rational, $t_0 \in \mathbb{R}_+$ a sufficiently large number and $v_0 \in Stab(a_{t_0}p)$ such that t_0 and v_0 satisfy the conditions in Proposition 4.6.1. Then there exists $n \in N_+$ such that*

$$Adn(v_0) \in \exp(\mathfrak{g}_{-2\alpha}).$$

Proof. By Theorem 4.2.2, we write

$$a_{t_0}p = ka_s n\sigma\Gamma$$

for some $k \in K$, $a_s \in A$, $n_0 \in N_-(V_0)$ and $\sigma \in \Sigma$. Since

$$v_0 \in \mathrm{Stab}(a_{t_0}p) = ka_s n_0(\sigma\Gamma\sigma^{-1})n_0^{-1}a_{-s}k^{-1} \text{ and } d(v_0, e)_G = C_p e^{-\gamma t_0}$$

we know that s is a sufficiently large number. Also note that k, n_0 and σ are all in compact subsets and by Proposition 4.4.3, $\sigma\Gamma\sigma^{-1} \cap \exp(\mathfrak{g}_{-2\alpha})$ is a lattice in $\exp(\mathfrak{g}_{-2\alpha})$. Therefore, we have that

$$v_0 \in ka_s n_0(\sigma\Gamma\sigma^{-1} \cap \exp(\mathfrak{g}_{-2\alpha}))n_0^{-1}a_{-s}k^{-1}.$$

In other words, we can find $n \in G$ such that

$$\mathrm{Ad}n(v_0) \in \exp(\mathfrak{g}_{-2\alpha}). \tag{4.7}$$

By Proposition 4.4.2 (or repeating the proof of Proposition 4.4.2), we can assume, without loss of generality, that $n \in N_+ \cup \{\omega\}$. Since $\{a_t\}$ expands N_+, condition 2 in Proposition 4.6.1 implies that $v_0 \notin N_+$, and hence by equation (4.7), we have $n \in N_+$. This completes the proof of the proposition. $\square$

Proposition 4.6.6. *Suppose that $\mathfrak{g}_{2\alpha} \neq \emptyset$. Let $p \in U_0(e\Gamma) \subset G/\Gamma$ be non-rational, $t_0 \in \mathbb{R}_+$ a sufficiently large number and $v_0 \in Stab(a_{t_0}p)$ such that t_0 and v_0 satisfy the conditions in Proposition 4.6.1. Then there is a rational point $q \in U_0(e\Gamma)$ with*

$d(q) \sim e^{(2\alpha-\gamma)t_0/2}$ *(the implicit constant depending only on* p*) such that*

$$p \in B_{N_+}(Cd(q)^{-\frac{2\alpha}{2\alpha-\gamma}}, Cd(q)^{-\frac{4\alpha}{2\alpha-\gamma}})q$$

for some constant $C > 0$ *depending only on* p*.*

Proof. By Proposition 4.6.5, we know that there exists $n \in N_+$ such that

$$\mathrm{Ad}(n)(v_0) \in \exp(\mathfrak{g}_{-2\alpha}).$$

Let $v_0 = \exp(v)$ for some $v \in \mathfrak{g}$. Then we have

$$\mathrm{Ad}n(v) = z \in \mathfrak{g}_{-2\alpha}.$$

Now set $n^{-1} = \exp(u)$ for some $u = u_1 + u_2 \in \mathfrak{n}_+$ with $u_1 \in \mathfrak{g}_\alpha$ and $u_2 \in \mathfrak{g}_{2\alpha}$. Then we have

$$\begin{aligned} z &+ [u_1, z] + ([u_2, z] + [u_1, [u_1, z]]/2) \\ &+ ([u_1, [u_1, [u_1, z]]]/6 + [u_1, [u_2, z]]/2 + [u_2, [u_1, z]]/2) + \cdots = v \end{aligned}$$

$$a = [u_1, z] \in \mathfrak{g}_{-\alpha}, b = [u_2, z] + [u_1, [u_1, z]]/2 \in \mathfrak{g}_0$$

$$\begin{aligned} c &= [u_1, [u_1, [u_1, z]]]/6 + [u_1, [u_2, z]]/2 + [u_2, [u_1, z]]/2 \\ &= [u_1, [u_1, [u_1, z]]]/6 + [u_2, [u_1, z]] \\ &= [u_1, [u_1, [u_1, z]]]/6 + [u_1, [u_2, z]] \\ &= -[u_1, [u_1, [u_1, z]]]/3 + [u_1, b] \in \mathfrak{g}_\alpha. \end{aligned}$$

Since $\|v\|_{\mathfrak{g}} \sim d(v_0, e)_G = C_p e^{-\gamma t_0}$, we have

$$\|a\|_{\mathfrak{g}}, \|b\|_{\mathfrak{g}}, \|c\|_{\mathfrak{g}} \ll C_p e^{-\gamma t_0}$$

and by the condition 2 in Proposition 4.6.1, we have

$$\text{either } \|z\|_{\mathfrak{g}} \sim C_p e^{-\gamma t_0} \text{ or } \|a\|_{\mathfrak{g}} = \|[u_1, z]\|_{\mathfrak{g}} \sim C_p e^{-\gamma t_0}.$$

If $\|z\|_{\mathfrak{g}} \sim C_p e^{-\gamma t_0}$, then we have

$$\|u_1\|_{\mathfrak{g}} \|z\|_{\mathfrak{g}} \ll \|a\|_{\mathfrak{g}} \leq C_p e^{-\gamma t_0}, \quad \|u_1\|_{\mathfrak{g}} \leq C_1$$

for some constant $C_1 > 0$ and

$$\|u_2\|_{\mathfrak{g}} \|z\|_{\mathfrak{g}} \ll \|b\|_{\mathfrak{g}} + \|u_1\|_{\mathfrak{g}}^2 \|z\|_{\mathfrak{g}} \ll C_p e^{-\gamma t_0}, \quad \|u_2\|_{\mathfrak{g}} \leq C_2$$

for some constant $C_2 > 0$.

If $\|a\|_{\mathfrak{g}} \sim C_p e^{-\gamma t_0}$, then

$$\|a\|_{\mathfrak{g}} \ll \|u_1\|_{\mathfrak{g}} \|z\|_{\mathfrak{g}} \leq \|u_1\|_{\mathfrak{g}} C_p e^{-\gamma t_0}, \quad \|u_1\|_{\mathfrak{g}} \geq C_3$$

for some constant $C_3 > 0$ and

$$\|u_1\|_{\mathfrak{g}}^2 \|a\|_{\mathfrak{g}} \ll \|u_1\|_{\mathfrak{g}} \|b\|_{\mathfrak{g}} + \|c\|_{\mathfrak{g}}, \quad \|u_1\|_{\mathfrak{g}} \ll 1 + \frac{1}{\|u_1\|_{\mathfrak{g}}} \leq C_4$$

for some constant $C_4 > 0$ and

$$\|u_2\|_{\mathfrak{g}} \|a\|_{\mathfrak{g}} \ll \|c\|_{\mathfrak{g}} + \|u_1\|_{\mathfrak{g}}^3 \|z\|_{\mathfrak{g}}, \quad \|u_2\|_{\mathfrak{g}} \leq C_5$$

for some constant $C_5 > 0$. Either case, we have that u and n are bounded.

Now by definition, we have that $na_{t_0}p$ is rational and hence $a_{-t_0}na_{t_0}p$ is rational. Let $q = (a_{-t_0}na_{t_0})p$ and we have that

$$a_{-t_0}na_{t_0} \in B_{N_+}(C_6e^{-\alpha t_0}, C_6e^{-2\alpha t_0})$$

for some constant $C_6 > 0$. By definition we know that

$$d(na_{t_0}p) \sim (C_pe^{-\gamma t_0})^{\frac{1}{2}}$$

and hence the denominator of $q = a_{-t_0}na_{t_0}p$ is equal to

$$d(q) = e^{\alpha t_0}d(na_{t_0}p) \sim C_p^{\frac{1}{2}}e^{(2\alpha-\gamma)t_0/2}.$$

So we have that

$$a_{-t_0}na_{t_0} \in B_{N_+}(Cd(q)^{-\frac{2\alpha}{2\alpha-\gamma}}, Cd(q)^{-\frac{4\alpha}{2\alpha-\gamma}})$$

for some $C > 0$. Note that C depends only on p. This completes the proof of the proposition. □

Proposition 4.6.7. *Suppose that $\mathfrak{g}_{2\alpha} \neq \emptyset$. Let $p \in U_0(e\Gamma)$ be a non-rational point. If p is not Diophantine of type γ, then there exist a constant $C > 0$ and a sequence $q_n \in U(e\Gamma)$ of distinct rational points with $d(q_n) \to \infty$ such that*

$$p \in B_{N_+}(Cd(q_n)^{-\frac{2\alpha}{2\alpha-\gamma}}, Cd(q_n)^{-\frac{4\alpha}{2\alpha-\gamma}})q_n.$$

Proof. Suppose that $p \in U(e\Gamma)$ is not Diophantine of type γ. Then there exist infinitely many $t_n \to \infty$ and $v_n \in \mathrm{Stab}(a_{t_n}p)$ satisfying the conditions in Proposition 4.6.1 and hence by Proposition 4.6.6, there exist infinitely many rational points $q_n \in$

$U(e\Gamma)$ with $d(q_n) \sim e^{(2\alpha-\gamma)t_n/2}$ such that

$$p \in B_{N_+}(Cd(q_n)^{-\frac{2\alpha}{2\alpha-\gamma}}, Cd(q_n)^{-\frac{4\alpha}{2\alpha-\gamma}})q_n$$

for some constant $C > 0$ depending only on p. This completes the proof of the proposition. $\square$

Proposition 4.6.8. *Suppose that $\mathfrak{g}_{2\alpha} \neq \emptyset$. Let $p \in U_0(e\Gamma)$ and let $\epsilon > 0$ be a sufficiently small number. If there exist a constant $C > 0$ and a sequence $q_n \in U_0(e\Gamma)$ of distinct rational points with $d(q_n) \to \infty$ such that*

$$p \in B_{N_+}(Cd(q_n)^{-\frac{2\alpha}{2\alpha-(\gamma+\epsilon)}}, Cd(q_n)^{-\frac{4\alpha}{2\alpha-(\gamma+\epsilon)}})q_n,$$

then p is not Diophantine of type γ.

Proof. Suppose that there exist a constant $C > 0$ and a sequence $q_n \in G/\Gamma$ of rational points converging to p such that

$$p \in B_{N_+}(Cd(q_n)^{-\frac{2\alpha}{2\alpha-(\gamma+\epsilon)}}, Cd(q_n)^{-\frac{4\alpha}{2\alpha-(\gamma+\epsilon)}})q_n.$$

Let $t_n = \ln d(q_n)^{\frac{2}{2\alpha-(\gamma+\epsilon)}}$. Then we have that

$$d(a_{t_n}q_n, a_{t_n}p)_{G/\Gamma} \leq C$$

and

$$\eta(a_{t_n}q_n) \sim d(q_n)^2/e^{2\alpha t_n} = e^{-(\gamma+\epsilon)t_n}.$$

This implies that

$$\eta(a_{t_n}p) \leq C'e^{-(\gamma+\epsilon)t_n}$$

for some constant $C' > 0$ and infinitely many $t_n \to \infty$. Hence by definition p is not Diophantine of type γ.

$\square$

4.7 Proof of Theorem 4.3.1

Before we prove Theorem 4.3.1, we need the following propositions.

Proposition 4.7.1. *Suppose that $\mathfrak{g}_{2\alpha} = \emptyset$. There exist r_0 and $C_0 > 0$ with the following property: for any rational point $q \in U_0(e\Gamma)$, there exists a large constant $L_q > 0$ such that for any $l > L_q$ the open box $B_{N_+}(r_0 d(q)^{-\frac{\alpha}{\alpha-\gamma}})q$ contains at least $C_0 l^{\dim \mathfrak{g}_\alpha} \mu_{N_+}(B_{N_+}(r_0 d(q)^{-\frac{\alpha}{\alpha-\gamma}})q)$ many disjoint sub open boxes $B_{N_+}(r_0 d(\tilde{q})^{-\frac{\alpha}{\alpha-\gamma}})\tilde{q}$ where $\tilde{q}$'s are rational points with denominator between l and $2l$.*

Proof. We fix a sufficiently small $r_0 > 0$. By Proposition 4.5.6, for the open box $B_{N_+}(r_0 d(q)^{-\frac{\alpha}{\alpha-\gamma}})q$, there exists a large constant L_q such that for $l > L_q$ there are at least $C_0 l^{\dim \mathfrak{g}_\alpha} \mu_{N_+}(B_{N_+}(r_0 d(q)^{-\frac{\alpha}{\alpha-\gamma}})q)$ many rational points in $B_{N_+}(r_0 d(q)^{-\frac{\alpha}{\alpha-\gamma}})q$ with denominator between l and $2l$ for some absolute constant $C_0 > 0$. For each such rational point $\tilde{q}$, we construct an open box $B_{N_+}(r_0 d(\tilde{q})^{-\frac{\alpha}{\alpha-\gamma}})\tilde{q}$ around $\tilde{q}$, and to prove the proposition, we only need to prove that these open boxes are disjoint. Let q_1 and q_2 be two such rational points. Suppose that $B_{N_+}(r_0 d(q_1)^{-\frac{\alpha}{\alpha-\gamma}})q_1$ and $B_{N_+}(r_0 d(q_2)^{-\frac{\alpha}{\alpha-\gamma}})q_2$ are not disjoint. Then there exists an element $n \in N_+$ such that

$$nq_1 = q_2 \text{ and } n \in B_{N_+}(r_0 d(q_1)^{-\frac{\alpha}{\alpha-\gamma}} + r_0 d(q_2)^{-\frac{\alpha}{\alpha-\gamma}}).$$

By applying a_{t_0} with $t_0 = \ln(l/r_0)/\alpha$ on both sides, we have

$$(a_{t_0} n a_{-t_0}) a_{t_0} q_1 = a_{t_0} q_2 \tag{4.8}$$

and by calculations, we have that

$$d(a_{t_0}q_1) \sim d(a_{t_0}q_2) \sim r_0 \text{ and } d(a_{t_0}na_{-t_0}, e)_G \ll l^{-\frac{\gamma}{\alpha-\gamma}}.$$

Since r_0 is sufficiently small and $a_{t_0}na_{-t_0}$ is bounded, from the equation (4.8) we know that there exist nonzero elements $u_1 \in Stab(a_{t_0}q_1) \cap \mathfrak{n}_-$ and $u_2 \in Stab(a_{t_0}q_2) \cap \mathfrak{n}_-$ such that

$$\mathrm{Ad}(a_{t_0}na_{-t_0})u_1 = u_2$$

and hence

$$\mathrm{Ad}(a_{t_0}na_{-t_0})\mathfrak{n}_- \cap \mathfrak{n}_- \neq \{0\}.$$

So by Lemma 3.4 in [BZ16], we have $a_{t_0}na_{-t_0} = e$, $n = e$ and $q_1 = q_2$. This completes the proof of the proposition. □

Proposition 4.7.2. *Suppose that $\mathfrak{g}_{2\alpha} \neq \emptyset$. There exist r_0 and $C_0 > 0$ with the following property: for any open box $U(e\Gamma) \subset U_0(e\Gamma)$, there exists a large constant $L > 0$ such that for any $l > L$ the open box $U(e\Gamma)$ contains at least $C_0 l^{\dim \mathfrak{g}_\alpha + 2\dim \mathfrak{g}_{2\alpha}} \mu_{N_+}(U)$ many disjoint sub open boxes of the form $B_{N_+}(r_0 d(q)^{-\frac{2\alpha}{2\alpha-\gamma}}, r_0 d(q)^{-\frac{4\alpha}{2\alpha-\gamma}})q$ where q's are rational points with denominator between l and $2l$.*

Proof. We fix a sufficiently small $r_0 > 0$. By Proposition 4.5.6, for any $U(e\Gamma)$, there exists a large constant L such that for $l > L$ there are at least $C_0 l^{\dim \mathfrak{g}_\alpha + 2\dim \mathfrak{g}_{2\alpha}} \mu_{N_+}(U)$ many rational points in $U(e\Gamma)$ with denominators between l and $2l$ for some absolute constant $C_0 > 0$. For each such rational point q, we construct an open box $B_{N_+}(r_0 d(q)^{-\frac{2\alpha}{2\alpha-\gamma}}, r_0 d(q)^{-\frac{4\alpha}{2\alpha-\gamma}})q$ around q, and to prove the proposition, we only need to prove that these open boxes are disjoint. Let q_1 and q_2 be two such rational points.

Suppose that

$$B_{N_+}(r_0 d(q_1)^{-\frac{2\alpha}{2\alpha-\gamma}}, r_0 d(q_1)^{-\frac{4\alpha}{2\alpha-\gamma}})q_1 \text{ and } B_{N_+}(r_0 d(q_2)^{-\frac{2\alpha}{2\alpha-\gamma}}, r_0 d(q_2)^{-\frac{4\alpha}{2\alpha-\gamma}})q_2$$

are not disjoint. Then there exists an element $n \in N_+$ such that $nq_1 = q_2$ and

$$\begin{aligned} n \in B_{N_+}(& r_0 d(q_1)^{-\frac{2\alpha}{2\alpha-\gamma}} + r_0 d(q_2)^{-\frac{2\alpha}{2\alpha-\gamma}}, \\ & r_0 d(q_1)^{-\frac{4\alpha}{2\alpha-\gamma}} + r_0 d(q_2)^{-\frac{4\alpha}{2\alpha-\gamma}} + r_0 d(q_1)^{-\frac{2\alpha}{2\alpha-\gamma}} r_0 d(q_2)^{-\frac{2\alpha}{2\alpha-\gamma}}). \end{aligned}$$

By applying a_{t_0} with $t_0 = \ln(l/r_0^{\frac{1}{2}})/\alpha$ on both sides, we have

$$(a_{t_0} n a_{-t_0}) a_{t_0} q_1 = a_{t_0} q_2 \tag{4.9}$$

and by calculations, we have that

$$d(a_{t_0} q_1) \sim d(a_{t_0} q_2) \sim r_0^{\frac{1}{2}} \text{ and } d(a_{t_0} n a_{-t_0}, e)_G \ll l^{-\frac{\gamma}{2\alpha-\gamma}}.$$

Then the rest of the proof is identical to that in Proposition 4.7.1. □

Proof of Theorem 4.3.1 in the case of $\mathfrak{g}_{2\alpha} = \emptyset$. By Proposition 4.6.3, we can construct an open cover of $S_\gamma^c \cap U_0(e\Gamma)$

$$\begin{aligned} S_\gamma^c \cap U_0(e\Gamma) &= S_\gamma^c \cap U_0(e\Gamma) \cap \bigcup_q \bigcup_C \{B_{N_+}(Cd(q)^{-\frac{\alpha}{\alpha-\gamma}})q\} \\ &= \bigcup_C \left(S_\gamma^c \cap U_0(e\Gamma) \cap \bigcup_q \{B_{N_+}(Cd(q)^{-\frac{\alpha}{\alpha-\gamma}})q\} \right) \end{aligned}$$

where q runs through all rational points in $U_0(e\Gamma)$ and C runs through all the positive rational numbers. By the countability of the set of rational numbers, to get an upper bound for the Hausdorff dimension of $S_\gamma^c \cap U_0(e\Gamma)$, it suffices to get an upper bound for the Hausdorff dimension of

$$S_\gamma^c \cap U_0(e\Gamma) \cap \bigcup_q \{B_{N_+}(Cd(q)^{-\frac{\alpha}{\alpha-\gamma}})q\}$$

for each $C \in \mathbb{Q}_+$. Fix $C > 0$ and let $\delta > 0$. By Proposition 4.5.6, we have that

$$\begin{aligned}
& \sum_q \operatorname{diam}^\delta(B_{N_+}(Cd(q)^{-\frac{\alpha}{\alpha-\gamma}})q) \\
= & \sum_q (Cd(q))^{-\delta\frac{\alpha}{\alpha-\gamma}} \\
\sim & \sum_{n\in\mathbb{N}} \sum_{2^n \le d(q) \le 2^{n+1}} d(q)^{-\delta\frac{\alpha}{\alpha-\gamma}} \\
\sim & \sum_{n\in\mathbb{N}} (2^n)^{\dim \mathfrak{g}_\alpha} (2^n)^{-\delta\frac{\alpha}{\alpha-\gamma}} \\
= & \sum_{n\in\mathbb{N}} (2^n)^{\dim \mathfrak{g}_\alpha - \delta\frac{\alpha}{\alpha-\gamma}}.
\end{aligned}$$

This implies that the Hausdorff dimension of

$$S_\gamma^c \cap U_0(e\Gamma) \cap \bigcup_q \{B_{N_+}(Cd(q)^{-\frac{\alpha}{\alpha-\gamma}})q\}$$

is less than or equal to $\frac{\alpha-\gamma}{\alpha} \dim \mathfrak{g}_\alpha$ for each $C \in \mathbb{Q}_+$, and hence

$$\dim S_\gamma^c \cap U_0(e\Gamma) \le \frac{\alpha-\gamma}{\alpha} \dim \mathfrak{g}_\alpha.$$

For the lower bound, we fix a sufficiently small $\epsilon > 0$ and construct a tree-like set in $U_0(e\Gamma)$ by induction. Let $\mathcal{A}_0 = \{U_0(e\Gamma)\}$ and $\mathbf{A}_0 = U_0(e\Gamma)$. Let r_0 be as in

Proposition 4.7.1 and pick a sufficiently large number l_1. Define

$$\mathcal{A}_1 = \left\{ B_{N_+}(r_0 d(q)^{-\frac{\alpha}{\alpha-(\gamma+\epsilon)}})q \middle| q \in S(U, l_1/2, l_1) \right\}$$

and $\mathbf{A}_1 = \bigcup \mathcal{A}_1$. Suppose that we find $l_1 < l_2 < \cdots < l_j$ and construct families $\mathcal{A}_j, \mathcal{A}_{j-1}, \ldots, \mathcal{A}_0$ and subsets $\mathbf{A}_j \subseteq \mathbf{A}_{j-1} \subseteq \cdots \subseteq \mathbf{A}_1 \subseteq \mathbf{A}_0$. Now by Proposition 4.7.1, we can find a sufficiently large $l_{j+1} > 0$ such that

1. $\log l_{j+1} \geq j^2 \log(l_j l_{j-1} \ldots l_1)$.
2. For every $B_{N_+}(r_0 d(q)^{-\frac{\alpha}{\alpha-(\gamma+\epsilon)}})q \in \mathcal{A}_j$, it contains at least

$$C_0 l_{j+1}^{\dim \mathfrak{g}_\alpha} \mu_{N_+}(B_{N_+}(r_0 d(q)^{-\frac{\alpha}{\alpha-(\gamma+\epsilon)}})q)$$

   sub-open boxes of the form $B_{N_+}(r_0 d(\tilde{q})^{-\frac{\alpha}{\alpha-(\gamma+\epsilon)}})\tilde{q}$ with

$$\tilde{q} \in S(B_{N_+}(r_0 d(q)^{-\frac{\alpha}{\alpha-(\gamma+\epsilon)}})q, l_{j+1}/2, l_{j+1}).$$

We denote the family of all these new sub-open boxes by $\mathcal{A}_{j+1}$ as

$$B_{N_+}(r_0 d(q)^{-\frac{\alpha}{\alpha-(\gamma+\epsilon)}})q$$

runs through all the open boxes in $\mathcal{A}_j$ and let $\mathbf{A}_{j+1} = \bigcup \mathcal{A}_{j+1}$.

Now we take $\mathbf{A}_\infty = \bigcap_{j=0}^\infty \mathbf{A}_j$ and $\mathcal{A} = \bigcup_{j=0}^\infty \mathcal{A}_j$. By the construction of $\mathbf{A}_j$'s and Proposition 4.6.4, we know that $\mathbf{A}_\infty \subset S_\gamma^c$. Also we have that

$$\Delta_j(\mathcal{A}) \sim l_{j+1}^{\dim \mathfrak{g}_\alpha} l_{j+1}^{-\frac{\alpha}{\alpha-(\gamma+\epsilon)} \dim \mathfrak{g}_\alpha} \text{ and } d_j(\mathcal{A}) = r_0 l_j^{-\frac{\alpha}{\alpha-(\gamma+\epsilon)}}.$$

By Theorem 4.2.1, we know that

$$\dim_H(\mathbf{A}_\infty) \geq \dim \mathfrak{g}_\alpha - \limsup_{j\to\infty} \frac{\sum_{i=0}^{j} \log\left(\frac{1}{l_{i+1}^{-\frac{\gamma+\epsilon}{\alpha-(\gamma+\epsilon)}\dim \mathfrak{g}_\alpha}}\right)}{\log\left(\frac{1}{l_{j+1}^{-\frac{\alpha}{\alpha-(\gamma+\epsilon)}}}\right)} = \dim \mathfrak{g}_\alpha \left(1 - \frac{\gamma+\epsilon}{\alpha}\right).$$

Let $\epsilon \to 0$ and we have

$$\dim_H S_\gamma^c \cap U_0(e\Gamma) \geq \dim_H(\mathbf{A}_\infty) \geq \frac{\alpha-\gamma}{\alpha} \dim \mathfrak{g}_\alpha.$$

This completes the proof of the theorem if $\mathfrak{g}_{2\alpha} = \emptyset$. $\square$

Proof of Theorem 4.3.1 in the case of $\mathfrak{g}_{2\alpha} \neq 0$. By Proposition 4.6.7, we can build an open cover of $S_\gamma^c \cap U_0(e\Gamma)$

$$\begin{aligned} S_\gamma^c \cap U_0(e\Gamma) &= S_\gamma^c \cap U_0(e\Gamma) \cap \bigcup_C \bigcup_q \{B_{N_+}(Cd(q)^{-\frac{2\alpha}{2\alpha-\gamma}}, Cd(q)^{-\frac{4\alpha}{2\alpha-\gamma}})q\} \\ &= \bigcup_C \left(S_\gamma^c \cap U_0(e\Gamma) \cap \bigcup_q \{B_{N_+}(Cd(q)^{-\frac{2\alpha}{2\alpha-\gamma}}, Cd(q)^{-\frac{4\alpha}{2\alpha-\gamma}})q\} \right) \end{aligned}$$

where q runs through all rational points in $U_0(e\Gamma)$ and C runs through all the positive rational numbers. By the countability of the set of rational numbers, to get an upper bound for the Hausdorff dimension of $S_\gamma^c \cap U_0(e\Gamma)$, it suffices to get an upper bound for the Hausdorff dimension of

$$S_\gamma^c \cap U_0(e\Gamma) \cap \bigcup_q \{B_{N_+}(Cd(q)^{-\frac{2\alpha}{2\alpha-\gamma}}, Cd(q)^{-\frac{4\alpha}{2\alpha-\gamma}})q\}$$

for each $C \in \mathbb{Q}_+$. Fix $C \in \mathbb{Q}_+$. For each rational $q \in U_0(e\Gamma)$, we devide

$$B_{N_+}(Cd(q)^{-\frac{2\alpha}{2\alpha-\gamma}}, Cd(q)^{-\frac{4\alpha}{2\alpha-\gamma}})q$$

into small cubes of equal sides of length $Cd(q)^{-\frac{4\alpha}{2\alpha-\gamma}}$, and there are $(d(q)^{\frac{2\alpha}{2\alpha-\gamma}})^{\dim \mathfrak{g}_\alpha}$ such small cubes. Let $\mathcal{F}_C$ be the collection of all these small cubes of equal sides of length $Cd(q)^{-\frac{4\alpha}{2\alpha-\gamma}}$ where q runs through all rational points in $U_0(e\Gamma)$. For any $\delta > 0$, by Proposition 4.7.2, we have that

$$\begin{aligned}
& \sum_{B \in \mathcal{F}_C} \mathrm{diam}(B)^\delta \\
\sim\ & \sum_q d(q)^{-\delta\frac{4\alpha}{2\alpha-\gamma}} (d(q)^{\frac{2\alpha}{2\alpha-\gamma}})^{\dim \mathfrak{g}_\alpha} \\
=\ & \sum_{n \in \mathbb{N}} \sum_{2^n \le d(q) \le 2^{n+1}} d(q)^{-\delta\frac{4\alpha}{2\alpha-\gamma}} (d(q)^{\frac{2\alpha}{2\alpha-\gamma}})^{\dim \mathfrak{g}_\alpha} \\
\sim\ & \sum_{n \in \mathbb{N}} (2^n)^{\dim \mathfrak{g}_\alpha + 2\dim \mathfrak{g}_{2\alpha}} (2^n)^{-\delta\frac{4\alpha}{2\alpha-\gamma}} ((2^n)^{\frac{2\alpha}{2\alpha-\gamma}})^{\dim \mathfrak{g}_\alpha} \\
=\ & \sum_{n \in \mathbb{N}} (2^n)^{\frac{4\alpha-\gamma}{2\alpha-\gamma}\dim \mathfrak{g}_\alpha + 2\dim \mathfrak{g}_{2\alpha} - \delta\frac{4\alpha}{2\alpha-\gamma}}.
\end{aligned}$$

This implies that the Hausdorff dimension of

$$S_\gamma^c \cap U_0(e\Gamma) \cap \bigcup_q \{B_{N_+}(Cd(q)^{-\frac{2\alpha}{2\alpha-\gamma}}, Cd(q)^{-\frac{4\alpha}{2\alpha-\gamma}})q\}$$

is less than or equal to $\frac{4\alpha-\gamma}{4\alpha}\dim \mathfrak{g}_\alpha + \frac{2\alpha-\gamma}{2\alpha}\dim \mathfrak{g}_{2\alpha}$ for each $C \in \mathbb{Q}_+$, and hence

$$\dim S_\gamma^c \cap U_0(e\Gamma) \le \frac{4\alpha-\gamma}{4\alpha}\dim \mathfrak{g}_\alpha + \frac{2\alpha-\gamma}{2\alpha}\dim \mathfrak{g}_{2\alpha}.$$

For the lower bound, we fix a sufficiently small $\epsilon > 0$ and construct a tree-like set in U_0 by induction. Let $\mathcal{A}_0 = \{U_0(e\Gamma)\}$ and $\mathbf{A}_0 = U_0(e\Gamma)$. Let r_0 be as in Proposition

4.7.2 and pick a sufficiently large number l_1. Define

$$\mathcal{A}'_1 = \left\{ B_{N_+}(r_0 d(q)^{-\frac{2\alpha}{2\alpha-(\gamma+\epsilon)}}, r_0 d(q)^{-\frac{4\alpha}{2\alpha-(\gamma+\epsilon)}})q \middle| q \in S(U_0(e\Gamma), l_1/2, l_1) \right\}.$$

For each $q \in S(U_0(e\Gamma), l_1/2, l_1)$, we devide $B_{N_+}(r_0 d(q)^{-\frac{2\alpha}{2\alpha-(\gamma+\epsilon)}}, r_0 d(q)^{-\frac{4\alpha}{2\alpha-(\gamma+\epsilon)}})q$ in $\mathcal{A}'_1$ into small cubes of equal sides of length $r_0 d(q)^{-\frac{4\alpha}{2\alpha-(\gamma+\epsilon)}}$. Let $\mathcal{A}_1$ be the family of all these small cubes and $\mathbf{A}_1 = \bigcup \mathcal{A}_1$. Suppose that we find $l_1 < l_2 < \cdots < l_j$ and construct families $\mathcal{A}_j, \mathcal{A}_{j-1}, \ldots, \mathcal{A}_0$ and subsets $\mathbf{A}_j \subseteq \mathbf{A}_{j-1} \subseteq \cdots \subseteq \mathbf{A}_1 \subseteq \mathbf{A}_0$. Now by Proposition 4.7.2, we can find a sufficiently large $l_{j+1} > 0$ such that

1. $\log l_{j+1} \geq j^2 \log(l_j l_{j-1} \ldots l_1)$.

2. For every $B \in \mathcal{A}_j$, it contains at least

   $$C_0 l_{j+1}^{\dim \mathfrak{g}_\alpha + 2\dim \mathfrak{g}_{2\alpha}} \mu_{N_+}(B)$$

   sub-open boxes of the form $B_{N_+}(r_0 d(\tilde{q})^{-\frac{2\alpha}{2\alpha-(\gamma+\epsilon)}}, r_0 d(\tilde{q})^{-\frac{4\alpha}{2\alpha-(\gamma+\epsilon)}})\tilde{q}$ with

   $$\tilde{q} \in S(B, l_{j+1}/2, l_{j+1}).$$

For each $\tilde{q} \in S(B, l_{j+1}/2, l_{j+1})$, we devide $B_{N_+}(r_0 d(\tilde{q})^{-\frac{2\alpha}{2\alpha-(\gamma+\epsilon)}}, r_0 d(\tilde{q})^{-\frac{4\alpha}{2\alpha-(\gamma+\epsilon)}})\tilde{q}$ into small cubes of equal sides of length $r_0 d(\tilde{q})^{-\frac{4\alpha}{2\alpha-(\gamma+\epsilon)}}$. We denote the family of all these small cubes by $\mathcal{A}_{j+1}$ as B runs through all the cubes in $\mathcal{A}_j$ and let $\mathbf{A}_{j+1} = \bigcup \mathcal{A}_{j+1}$.

Now we take $\mathbf{A}_\infty = \bigcap_{j=0}^\infty \mathbf{A}_j$ and $\mathcal{A} = \bigcup_{j=0}^\infty \mathcal{A}_j$. By the construction of $\mathbf{A}_j$'s and Proposition 4.6.8, we know that $\mathbf{A}_\infty \subset S_\gamma^c$. Also we have that

$$\Delta_j(\mathcal{A}) \sim l_{j+1}^{\dim \mathfrak{g}_\alpha + 2\dim \mathfrak{g}_{2\alpha}} l_{j+1}^{-\frac{2\alpha}{2\alpha-(\gamma+\epsilon)} \dim \mathfrak{g}_\alpha} l_{j+1}^{-\frac{4\alpha}{2\alpha-(\gamma+\epsilon)} \dim \mathfrak{g}_{2\alpha}} \text{ and } d_j(\mathcal{A}) = r_0 l_j^{-\frac{4\alpha}{2\alpha-(\gamma+\epsilon)}}.$$

By Theorem 4.2.1, we know that

$$\begin{aligned}
& \dim_H(\mathbf{A}_\infty) \\
\geq\ & \dim \mathfrak{g}_\alpha + \dim \mathfrak{g}_{2\alpha} \\
& - \limsup_{j\to\infty} \frac{\sum_{i=0}^{j} \log\left(l_{i+1}^{\dim \mathfrak{g}_\alpha + 2\dim \mathfrak{g}_{2\alpha}} l_{i+1}^{-\frac{2\alpha}{2\alpha-(\gamma+\epsilon)}\dim \mathfrak{g}_\alpha} l_{i+1}^{-\frac{4\alpha}{2\alpha-(\gamma+\epsilon)}\dim \mathfrak{g}_{2\alpha}} \right)}{\log\left(\frac{1}{l_{j+1}^{-\frac{4\alpha}{2\alpha-(\gamma+\epsilon)}}} \right)} \\
=\ & \left(1 - \frac{\gamma+\epsilon}{2\alpha}\right) \dim \mathfrak{g}_{2\alpha} + \left(1 - \frac{\gamma+\epsilon}{4\alpha}\right) \dim \mathfrak{g}_\alpha.
\end{aligned}$$

Let $\epsilon \to 0$ and we have

$$\begin{aligned}
\dim_H S_\gamma^c \cap U_0(e\Gamma) &\geq \dim_H \mathbf{A}_\infty \\
&\geq \left(1 - \frac{\gamma}{2\alpha}\right) \dim \mathfrak{g}_{2\alpha} + \left(1 - \frac{\gamma}{4\alpha}\right) \dim \mathfrak{g}_\alpha.
\end{aligned}$$

This completes the proof of the theorem. □

4.8 Proof of Theorem 4.1.2

In this section we will prove Theorem 4.1.2. Let

$$S(i,\gamma) = \{p \in G/\Gamma | \exists C > 0 \text{ s.t. } \eta(a_t p)\chi_{Y_i}(a_t p) \geq Ce^{-\gamma t}\chi_{Y_i}(a_t p)\ (\forall t > 0)\}.$$

Then by definition, we know that

$$S_{\gamma_1,\ldots,\gamma_k} = S(1,\gamma_1) \cap \cdots \cap S(k,\gamma_k)$$

and hence

$$\dim_H S^c_{\gamma_1,\ldots,\gamma_k} = \max_{1\le i\le k} \dim_H S(i,\gamma_i)^c.$$

So in order to prove Theorem 4.1.2, it is enough to prove the following

Theorem 4.8.1. *Let $i \in \{1,\ldots,k\}$. If $\mathfrak{g}_{2\alpha} = \emptyset$, then the Hausdorff dimension of $S(i,\gamma)^c$ $(0 \le \gamma < \alpha)$ is*

$$\dim \mathfrak{g}_{-\alpha} + \dim \mathfrak{g}_0 + \frac{\alpha-\gamma}{\alpha} \dim g_\alpha.$$

If $g_{2\alpha} \neq \emptyset$, then the Hausdorff dimension of $S(i,\gamma)^c$ $(0 \le \gamma < 2\alpha)$ is

$$\dim \mathfrak{g}_{-2\alpha} + \dim \mathfrak{g}_{-\alpha} + \dim \mathfrak{g}_0 + \frac{4\alpha-\gamma}{4\alpha} \dim g_\alpha + \frac{2\alpha-\gamma}{2\alpha} \dim g_{2\alpha}.$$

Now we will fix a cusp ξ_i for some $i \in \{1,\ldots,k\}$ and let $\sigma_i \in \Sigma$ be the element corresponding to ξ_i.

Proof of Theorem 4.8.1. The proof for $S(i,\gamma)^c$ is almost identical to the proof of Theorem 4.1.1 (or equivalently Theorem 4.3.1), except that we replace rational points by σ_i-rational points, denominators by σ_i-denominators. In fact, our discussion in section 5 is cuspwise, and we can use Proposition 4.5.3 and Proposition 4.5.4 to count σ_i-rational points instead of Proposition 4.5.6. Hence Proposition 4.7.1 and Proposition 4.7.2 holds also for σ_i-rational points with σ_i-denominators. The same happens in Proposition 4.6.4 and Proposition 4.6.8. The only thing we need to do is to prove that after replacing by σ_i-rational points and σ_i-denominators in Proposition 4.6.2 and Proposition 4.6.6 with the assumption that $p \in S(i,\gamma)^c$, the rational $q \in U_0(e\Gamma)$ we obtain is actually a σ_i-rational point. We will prove this for the case of $\mathfrak{g}_{2\alpha} = \emptyset$. The case of $\mathfrak{g}_{2\alpha} \neq \emptyset$ is similar.

Now assume that $\mathfrak{g}_{2\alpha} = \emptyset$ and $p \in S(i,\gamma)^c$. By the proof of Proposition 4.6.2, we know that

$$na_{t_0}p = a_{t_0}q$$

for some bounded $n \in N_+$, and we have $a_{t_0}p \in Y_i$. This implies that

$$\eta(a_{t_0}q) \sim \eta(a_{t_0}p) \sim C_p e^{-\gamma t_0}$$

and the rational point $a_{t_0}q \in Y_i'$ for a small neighborhood Y_i' of ξ_i in G/Γ. This happens if and only if q is a σ_i-rational point. This completes the proof of Theorem 4.8.1. □

4.9 Further Discussions

4.9.1 The Hyperbolic Case

In this subsection, we will discuss the relation between the definition of Diophantine points in this paper and the work by Melián and Pestana [MP93] where they deal with the hyperbolic case. We will assume $G = SO(d+1,1)$. First, we need some notations in [MP93]. Let $\mathcal{M}^{d+1} = \mathbf{H}^{d+1}/\Gamma$ be a complete non-compact Riemannian manifold of constant negative curvature and of finite volume, where Γ is a non-uniform lattice in G. Denote by $d_{\mathbf{H}^{d+1}}$ the distance on $\mathcal{M}^{d+1}$. For any point $x \in \mathcal{M}^{d+1}$, let $S(x)$ be the unit ball in the tangent space of $\mathcal{M}^{d+1}$ at x. Let $g_v(t)$ be the geodesic starting from x in the direction of $v \in S(x)$. For any $\gamma \geq 0$, we write

$$E_\gamma(x) = \left\{ v \in S(x) : \limsup_{t\to\infty} \frac{d_{\mathbf{H}^{d+1}}(g_v(t), x)}{t} \geq \gamma \right\}.$$

Note that the maximal compact subgroup $K \cong SO(d+1)$, $M \cong SO(d)$ and $\{a_t\}$ is the geodesic flow on G/Γ. Note that here the speed of the geodesic flow $\alpha = 1$. We will denote by π_K the projection from G/Γ to $\mathcal{M}^{d+1} = K\backslash G/\Gamma$ and π_M the projection from G/Γ to $T^1\mathcal{M}^{d+1} = M\backslash G/\Gamma$. Let $x_0 = \pi_K(e\Gamma) \in \mathcal{M}^{d+1}$.

Proposition 4.9.1. *For any $p \in G/\Gamma$ we have*

$$\eta(p) \sim e^{-d_{\mathbf{H}^{d+1}}(\pi_K(p), x_0)}.$$

Here the implicit constant depends only on G and Γ.

Proof. By Theorem 4.2.2, for any $p \in G/\Gamma$, we can write

$$p = k a_{t_0} n \sigma \Gamma$$

for some $k \in K = SO(n)$, $a_{t_0} \in A$ with $t_0 \geq s_0$, $n \in N_-(V_0)$ and $\sigma \in \Sigma$. On the one hand, since A contracts N_- and k, n, σ are all in compact subsets of G, by calculations, we have

$$\eta(p) \sim e^{-\alpha t_0} = e^{-t_0}.$$

On the other hand, since k, n, σ are all in compact subsets of G, by calculations, we have

$$|d_{\mathbf{H}^{d+1}}(\pi_K(p), x_0) - t_0| \leq C$$

for some constant $C > 0$. This implies that

$$\eta(p) \sim e^{-d_{\mathbf{H}^{d+1}}(\pi_K(p), x_0)}.$$

Note that all the implicit constants here depend only on G and Γ according to Theorem 4.2.2. This completes the proof of the proposition. $\square$

Remark 4.9.1. Note that we can prove a similar version for any semisimple Lie group of rank one.

Proposition 4.9.2. *Fix $x = \pi_K(p)$ with $p \in G/\Gamma$ and $x \in \mathcal{M}^{d+1}$. For any $\epsilon > 0$, we have*

$$\pi_M(Kp \cap S_\gamma^c) \subset \{(x,v) : v \in E_\gamma(x)\} \subset \pi_M(Kp \cap S_{\gamma-\epsilon}^c).$$

Proof. Let $p' \in Kp \cap S_\gamma^c$. Let $\pi_M(p') = (x, v')$ for $v' \in T_x^1\mathcal{M}^{d+1}$. By definition, there exist a constant $C_1 > 0$ and a sequence $t_n \to \infty$ such that

$$\eta(a_{t_n}p') \leq Ce^{-\gamma t_n}.$$

By Proposition 4.9.1, this implies that

$$d_{\mathbf{H}^{d+1}}(g_{v'}(t_n), x_0) \geq \gamma t_n + C_2$$

for $t_n \to \infty$ and some constant C_2 and hence $v' \in E_\gamma(x)$. This proves the first inclusion.

Now we prove the second inclusion. Let $v \in E_\gamma(x)$. Let $p' \in G/\Gamma$ with $\pi_M(p') = (x, v)$. Then $p' \in Kp$. Since $v \in E_\gamma(x)$, for any $\epsilon > 0$, there exists a sequence $t_n \to \infty$ such that

$$d_{\mathbf{H}^{d+1}}(g_v(t_n), x) \geq \left(\gamma - \frac{\epsilon}{2}\right) t_n$$

and hence there exists a constant C_3 such that

$$d_{\mathbf{H}^{d+1}}(g_v(t_n), x_0) \geq \left(\gamma - \frac{\epsilon}{2}\right) t_n + C_3.$$

By Proposition 4.9.1, there exists a constant $C_4 > 0$ such that

$$\eta(a_t p') \leq C_4 e^{-(\gamma - \frac{\epsilon}{2})t_n} = (Ce^{-\frac{\epsilon}{2}t_n})e^{-(\gamma-\epsilon)t_n}.$$

Therefore $p' \in S^c_{\gamma-\epsilon}$. This completes the proof of the proposition. $\square$

Now we show that in the case of hyperbolic spaces, our theorem coincides with Theorem 1 in [MP93].

Theorem 4.9.1 (Melián and Pestana, Theorem 1 in [MP93])**.** *We have*

$$\dim_H E_\gamma(x) = (1-\gamma)d.$$

Proof. Let $x = \pi_K(p_0) \in \mathcal{M}^{d+1}$ for some $p_0 \in G/\Gamma$. By the discussions in section 3 and Theorem 4.3.1, we know that for any $p \in G/\Gamma$ and any small neighborhood W of e in G

$$\dim_H((Wp) \cap S^c_\gamma) = \dim \mathfrak{g}_{-\alpha} + \dim \mathfrak{g}_0 + (1-\gamma)\dim \mathfrak{g}_\alpha.$$

Now let W_K be a neighborhood of e in K and W_{AN_-} a neighborhood of e in AN_-. Then $W_{AN_-}W_K$ is a neighborhood of e in G, and hence

$$\dim_H((W_{AN_-}W_K p) \cap S^c_\gamma) = \dim \mathfrak{g}_{-\alpha} + \dim \mathfrak{g}_0 + (1-\gamma)\dim \mathfrak{g}_\alpha.$$

Since $\{a_t\}$ commutes with A and contracts N_-, by definition, a point $(hk)p$ with $h \in W_{AN_-}$ and $k \in W_K$ belongs to S_γ if and only if kp belongs to S_γ. Therefore we have

$$(W_{AN_-}W_K p) \cap S^c_\gamma = W_{AN_-}\left((W_K p) \cap S^c_\gamma\right).$$

This implies that

$$\dim_H(W_K p \cap S_\gamma^c) = \dim M + (1-\gamma)\dim \mathfrak{g}_\alpha.$$

Let p vary in the K-orbit Kp_0, we get

$$\dim_H(Kp_0 \cap S_\gamma^c) = \dim M + (1-\gamma)\dim \mathfrak{g}_\alpha$$

and by Proposition 4.9.2, we have

$$\dim_H E_\gamma(x) = (1-\gamma)\dim \mathfrak{g}_\alpha = (1-\gamma)d.$$

This completes the proof of Theorem 1 in [MP93]. $\square$

4.9.2 The Case of $PSL(2, \mathbb{R})$

In this subsection, we discuss the relation between Theorem 4.1.2 and Theorem 1.7.1 in Chapter 1. Suppose that $G = PSL(2, \mathbb{R})$ and the speed of the geodesic flow $\alpha = 1$. We will reuse the notations in the previous subsection.

Proof of Theorem 1.7.1. By Proposition 4.9.1, we know that $p \in G/\Gamma$ is Diophantine of type $(\gamma_1, \ldots, \gamma_k)$ if and only if for each $i \in \{1, \ldots, k\}$ we have

$$\limsup_{t\to\infty} \left(d_{\mathbf{H}^2}(\pi_K(a_t p), x_0) - \gamma_i t\right) \chi_{Y_i}(a_t p) < \infty$$

or equivalently

$$\limsup_{t\to\infty} \left(d_{\mathbf{H}^2}(\pi_K(a_t p), x_0)\chi_{Y_i}(a_t p) - \gamma_i t\right) < \infty.$$

By Lemma 1.3.4 in Chapter 1, this is equivalent to say that p is Diophantine of type

$(\kappa_1, \ldots, \kappa_k)$ in the sense of Definition 1.1.1 in Chapter 1. Here

$$\gamma_i = \frac{\kappa_i - 1}{\kappa_i + 1}.$$

Then we complete the proof of Theorem 1.7.1 by applying Theorem 4.1.2. $\square$

4.9.3 Diophantine Approximation in Heisenberg Groups

In this subsection, we will give an application to Diophantine approximation in Heisenberg groups. We will follow the notations and results in [HP02B].

Let $G = SU(n,1)$ and $\Gamma = G(\mathbb{Z}[i])$. Let $\mathcal{M} = K\backslash G/\Gamma \cong \mathbf{H}^n_{\mathbb{C}}/\Gamma$ where K is the maximal compact subgroup of G. Here we use the model of Siegel domain for $\mathbb{H}^n_{\mathbb{C}}$ (see section 3.8 in [HP02B]). Note that $\mathcal{M}$ is not of constant curvature. Let $\{a_t\}$ be the geodesic flow on G/Γ. For simplicity, we will assume that $\mathcal{M}$ has only one cusp $\xi = \infty$; for general case, we only need to replace rational points by σ-rational points in the following arguments as we do in section 8 and apply Theorem 4.8.1. Also we will assume that the orbit $\{a_t(e\Gamma)\}$ diverges to the cusp $\xi = \infty$ in $\partial\mathbb{H}^n_{\mathbb{C}}$ as $t \to -\infty$. We will denote by π_K the projection from G/Γ to $\mathcal{M}$ and by π_M the projection from G/Γ to $M\backslash G/\Gamma$. Let $x_0 = \pi_K(e\Gamma)$ and $U_0 \subset N_+$ be a small open subset. The speed of the geodesic flow is the simple root $\alpha = 1$. Note that $N_+ \cong \mathcal{H}_{2n-1}(\mathbb{R})$ acts naturally on $\partial\mathbf{H}_{\mathbb{C}^n}$ (see section 3.10 in [HP02B]). We will write

$$B_{\mathrm{Cyg}}(\alpha, R) = \{\beta \in H_{2n-1}(\mathbb{R}) : d_{\mathrm{Cyg}}(\beta, \alpha) < R\}.$$

We recall some definitions and notations from [HP02B]. The geodesic lines starting from $\xi = \infty$ and diverging to itself in $\mathcal{M}$ are called rational lines, and the geodesic lines starting from $\xi = \infty$ but not diverging to it are called irrational lines. Here we can identify the geodesic lines starting from $\xi = \infty$ with $\partial\mathbf{H}^n_{\mathbb{C}} \setminus \{\infty\} \cong \mathcal{H}_{2n-1}(\mathbb{R})$, and the rational lines starting from $\xi = \infty$ with rational points in a subset of $\mathcal{H}_{2n-1}(\mathbb{Q})$ (see section 3.10 in [HP02B]). The height function β on $\mathcal{M}$, the Hamenstädt distance d_∞ on $\partial\mathbf{H}^n_{\mathbb{C}} \cong \mathcal{H}_{2n-1}(\mathbb{R}) \cup \{\infty\}$ and the depth $D(r)$ of a rational geodesic line r in $\mathcal{M}$ are defined in section 2 in [HP02B]. For simplicity, we will not list these concepts here. Readers may refer to [HP02B] for more details. Note that by Proposition 3.14 in [HP02B], we have

$$D(r) = \ln h(r) \text{ and } d_\infty = \frac{1}{\sqrt{2}} d_{\mathrm{Cyg}}.$$

Also note that by our assumption, for any $p \in U_0(e\Gamma)$, $\pi_K(a_t p)$ is a geodesic line starting from $\xi = \infty$ which could be identified with a point in the Heisenberg group.

Proposition 4.9.3. *Let $p \in U_0(e\Gamma)$. Then p is rational if and only if the geodesic line $\pi_K(a_t p)$ is rational.*

Proof. Suppose that p is rational. Then by definition, $a_t p$ diverges in G/Γ to the cusp $\xi = \infty$ as $t \to \infty$ and hence so does $\pi_K(a_t p)$ in $\mathcal{M}$. Since $a_t(e\Gamma)$ diverges to the cusp $\xi = \infty$ as $t \to -\infty$ and $p \in U_0(e\Gamma)$, we know that $a_t p$ also diverges to the cusp as $t \to -\infty$. This implies that $\pi_K(a_t p)$ is a rational geodesic line in the sense of [HP02B].

Conversely, if $\pi_K(a_t p)$ is a rational line, then $a_t p$ diverges to the cusp as $t \to \infty$ and hence by Corollary 6.2 in [D85] and Proposition 4.5.1, we know that p is rational. This completes the proof of the proposition. $\square$

Remark 4.9.2. This proposition implies that the rational points in $U_0(e\Gamma)$ are in $\mathcal{H}_{2n-1}(\mathbb{Q})$ via the identification. See section 3.10 in [HP02B].

Proposition 4.9.4. *Let $p \in U_0(e\Gamma)$ be rational. Then we have*

$$h(p) \sim d(p).$$

Here we consider p as a rational point in $\mathcal{H}_{2n-1}(\mathbb{Q})$ and the implicit constant depends only on U_0 and G/Γ.

Proof. We will consider the rational line $\pi_K(\{a_t p\})$ and its depth $D(\pi_K(\{a_t p\}))$. Fix a level set $\beta^{-1}(l) \subset \mathcal{M}$ for some l. Then there exists a constant $\epsilon > 0$ such that for any $p' \in \pi_K^{-1}(\beta^{-1}(l))$ we have

$$\eta(p') \sim \epsilon.$$

Let $s_0 > 0$ be the last time such that $a_{s_0} p \in \pi_K^{-1}(\beta^{-1}(l))$. Then by definition, we have

$$\epsilon \sim \eta(a_{s_0} p) \sim e^{-2s_0} d(p)^2. \tag{4.10}$$

On the other hand, since $a_t(e\Gamma)$ diverges as $t \to -\infty$, there exists $t_0 > 0$ such that t_0 is the largest number with $a_{-t_0}(e\Gamma) \in \pi_K^{-1}(\beta^{-1}(l))$ and we have

$$\epsilon \sim \eta(a_{-t_0}(e\Gamma)) \sim e^{2t_0}. \tag{4.11}$$

As $p \in U_0(e\Gamma)$ and a_{-t} contracts N_+ as $t \to \infty$, $a_{-t_0} p$ is near the subset $\pi_K^{-1}(\beta^{-1}(l))$. By definition of the depth function, this implies that there exists a constant $C_1 > 0$ such that

$$|D(\pi_K(\{a_t p\})) - (s_0 + t_0)| \leq C_1.$$

Also by equations (4.10) and (4.11), we know that

$$d(p)^2 \sim e^{2(s_0+t_0)}.$$

So there exists a constant $C > 0$ such that

$$d(p) \sim e^{D(\pi_K(\{a_t p\}))}.$$

This implies that

$$d(p) \sim h(p).$$

Note that the implicit constant depends only on U_0 and G/Γ. This completes the proof of the proposition. $\square$

Proposition 4.9.5. *Let $p \in U_0(e\Gamma)$ with $p = g\Gamma$ for some $g \in U_0$ and let $\epsilon > 0$ be a sufficiently small number. If there exist a constant $C > 0$ and a sequence $q_j \in U_0(e\Gamma)$ of distinct rational points with $d(q_j) \to \infty$ such that*

$$p \in B_{N_+}(Cd(q_j)^{-(\gamma+\epsilon)}, Cd(q_j)^{-2(\gamma+\epsilon)})q_j,$$

then $g \in N_+ \cong \mathcal{H}_{2n-1}(\mathbb{R})$ is not Diophantine of type γ.

Proof. By the definition of the Cygan distance, Proposition 4.9.3 and Proposition 4.9.4, the condition in the proposition is equivalent to the condition that there exist a constant $C > 0$ and a sequence of rational points $r_j \in U_0 \subset N_+ \cong \mathcal{H}_{2n-1}(\mathbb{R})$ with $h(r_j) \to \infty$ such that

$$d_{\mathrm{Cyg}}(g, r_j) \leq \frac{C}{h(r_j)^{\gamma+\epsilon}}.$$

By definition, this implies that g is not Diophantine of type γ. $\square$

Proof of Theorem 4.1.3. The proof is similar to that in Theorem 4.1.1. For the upper bound of the Hausdorff dimension of L_γ^c, it is enough to get an upper bound for $L_\gamma^c \cap U_0$ for every small open subset U_0 in $N_+ \cong \mathcal{H}_{2n-1}(\mathbb{R})$. Now fix a small open subset U_0. By definition, we can construct an open cover of $L_\gamma^c \cap U_0$ as follows

$$\begin{aligned} L_\gamma^c \cap U_0 &= \left(\bigcup_r \bigcup_C B_{\mathrm{Cyg}}(r, Ch^{-\gamma}(r)) \right) \cap L_\gamma^c \cap U_0 \\ &= \bigcup_C \left(\bigcup_r B_{\mathrm{Cyg}}(r, Ch^{-\gamma}(r)) \right) \cap L_\gamma^c \cap U_0. \end{aligned}$$

where C runs through all the positive rational numbers and r runs through all the rational points in $\mathcal{H}_{2n-1}(\mathbb{Q})$. So it is enough to get an upper bound for the Hausdorff dimension of

$$\left(\bigcup_r B_{\mathrm{Cyg}}(r, Ch^{-\gamma}(r))\right) \cap L_\gamma^c \cap U_0.$$

For each rational r, the $B_{\mathrm{Cyg}}(r, Ch^{-\gamma}(r))$ could be thought of as a box with lengths $Ch^{-\gamma}(r)$ in the first $2(n-1)$ real coordinates and length $C^2h^{-2\gamma}(r)$ in the last coordinate, with respect to the right invariant distance $d_{\mathcal{H}_{2n-1}(\mathbb{R})}$. So we can devide $B_{Cyg}(r, Ch^{-\gamma}(r))$ into small cubes of equal sides of length $C^2h^{-2\gamma}(r)$, and there are $(h(r)^\gamma/C)^{2n-2}$ such small cubes. Let $\mathcal{F}_C$ be the collection of all these small cubes of equal sides of length

$$C^2h(r)^{-2\gamma}$$

where r runs through all rational points in $\mathcal{H}_{2n-1}(\mathbb{Q})$.

Fix $C > 0$ and let $\delta > 0$. By Theorem 3.7 in [HP02B], we have

$$\begin{aligned} & \sum_{B\in\mathcal{F}_C} \operatorname{diam}^\delta(B) \\ \sim \quad & \sum_{r\in\mathcal{H}_{2n-1}(\mathbb{Q})} h(r)^{-2\delta\gamma} h(r)^{\gamma(2n-2)} \\ = \quad & \sum_{j\in\mathbb{N}} \sum_{2^{j-1}\le h(r)\le 2^j} h(r)^{-2\delta\gamma} h(r)^{\gamma(2n-2)} \\ \ll \quad & \sum_{j\in\mathbb{N}} 2^{2nj} 2^{-2j\delta\gamma} 2^{j\gamma(2n-2)} \end{aligned}$$

This implies that

$$\dim_H L_\gamma^c \le \frac{1+\gamma}{\gamma} n - 1.$$

For the lower bound of the Hausdorff dimension, by Proposition 4.9.5, we only need to repeat the proof in Theorem 4.1.1, and we can get

$$\dim_H L_\gamma^c \cap U_0 \geq \frac{1+\gamma+\epsilon}{2(\gamma+\epsilon)} 2(n-1) + \frac{1}{\gamma+\epsilon}$$

for any $\epsilon > 0$. Let $\epsilon \to 0$ and we have

$$\dim_H L_\gamma^c \geq \frac{1+\gamma}{\gamma}(n-1) + \frac{1}{\gamma}.$$

This completes the proof of Theorem 4.1.3. $\square$

Bibliography

[BZ16] C.D.Buenger and C.Zheng, *Non-Divergence of Unipotent Flows on Quotients of Rank One Semisimple Groups*, Ergod. Th. and Dynam. Sys., FirstView, 1-26, May 2016

[D86] S.G.Dani, *Bounded orbits of flows on homogeneous spaces*, Comment. Math. Helv. 61, 636-660, 1986

[D85] S.G.Dani, *Divergent trajectories of flows on homogeneous spaces and Diophantine approximation*, J.Reine. Angew. Math, 359, 55-89, 1985

[D78] S. G. Dani, *Invariant measures of horospherical flows on noncompact homogeneous spaces*, Invent. Math. 47, no. 2, 101–138, 1978.

[D81] S. G. Dani, *Invariant measures and minimal sets of horospherical flows*, Invent. Math. 64, no. 2, 357–385, 1981.

[D84] S. G. Dani, *On orbits of unipotent flows on homogeneous spaces*, Ergod. Th. and Dynam. Sys., 4, 25-34, 1984

[DS84] S.G.Dani and J.Smillie, *Uniform distribution of horocycle orbits for fuchsian groups*, Duke Math. J., 51(1), 185-194, 1984

[D92] M.M.Dodson, *Hausdorff dimension, lower order and Khintchine's theorem in metric Diophantine approximation*, J. reine angew. Math. 432, 69-76, 1992

[DE14] M.Dodson and B. Everitt, *Metrical Diophantine approximation for quaternions*, Math. Proc. Cambridge Philos. Soc., 157, 513-542, 2014

[EW10] M.Einsiedler and T.Ward, *Ergodic Theory: with a view towards Number Theory*, Graduate Texts in Mathematics, Springer, 2010

[EM93] A.Eskin and C.McMullen, *Mixing, counting and equidistribution in Lie groups*, Duke Mathematical Journal 71, 143-180, 1993

[F73] H. Furstenberg, *The unique ergodicity of the horocycle flow*, Lecture Notes in Mathematics, 318, 95-115,1973

[GR70] H.Garland and M.S.Raghunathan, *Fundamental domains for lattices in* $\mathbb{R}$-*rank 1 semisimple Lie groups*, Ann. of Math., 92(2), 279-326,1970

[G99] W.M.Goldman, *Complex hyperbolic geometry*, Oxford Univ. Press, 1999

[GOS10] A.Gorodnik, Hee Oh and N.A.Shah, *Strong wavefront lemma and counting lattice points in sectors*, Israel Journal of Mathematics 176, 419-444, 2010

[HP04] S. Hersonsky and F. Paulin, *Counting orbit points in coverings of negatively curved manifolds and Hausdorff dimension of cusp excursions*, Ergod. Th and Dynam. Sys., 24, 803-824, 2004

[HP02A] S. Hersonsky and F. Paulin, *Diophantine approximation for negatiely curved manifolds*, Math. Z. 241, 181-226, 2002

[HP02B] S. Hersonsky and F. Paulin, *Diophantine approximation in negatively curved manifolds and in the Heisenberg group*, Rigidity in Dynamics and Geometry (Cambridge, 2000), Eds. M. Burger and A. Iozzi. Springer, Berlin, 203-226, 2002

[HP01] S. Hersonsky and F. Paulin, *Hausdorff dimension of diophantine geodesics in negatively curved manifolds*, J. reine angew. Math. 539, 29-43, 2001

[K01] D.Y.Kleinbock, *Some applications of homogeneous dynamics to number theory*, in Smooth ergodic theory and its application, Proc. Sympos. Pure Math. 69, AMS, Providence, RI, 639-660, 2001

[KM96] D.Y.Kleinbock and G.A.Margulis, *Bounded orbits of nonquasiunipotent flows on homogeneous spaces*, Sinai's Moscow Seminar on Dynamical Systems, 141-172, Amer. Math. Soc. Transl. Ser. 2,171, Amer. Math. Soc., Providence, RI, 1996

[KM98] D.Y.Kleinbock and G.A.Margulis, *Flows on homogeneous spaces and Diophantine approximation on manifolds*, Ann. of Math., 148(1), 339-360, 1998

[KM99] D.Y.Kleinbock and G.A.Margulis, *Logarithm laws for flows on homogeneous spaces*, Invent. Math., 138(3), 451-494, 1999

[K02] A.W.Knapp, *Lie groups beyond an introduction*, second edition, Progress in mathematics 140, Birkhauser, 2002

[M89] G.A. Margulis, *Discrete subgroups and ergodic theory*, In: Number theory, trace formulas and discrete groups (Oslo, 1987), 377–398, Academic Press, Boston, MA, 1989.

[M87] C. McMullen, *Area and Hausdorff dimension of Julia sets of entire functions*, Trans. Amer. Math. Soc. 300, 329-342, 1987

[MP93] M.V.Melián and D.Pestana, *Geodesic excursions into cusps in finite-volume hyperbolic manifolds*, Michigan Math. J., 40, 77-93, 1993

[R72] M.S.Raghunathan, *Discrete subgroups of Lie groups*, Springer-Verlag, 1972

[R90b] M.Ratner, *On measure rigidity of unipotent subgroups of semisimple Lie groups*, Acta Math., 165, 229-309, 1990

[R91a] M.Ratner, *On Raghunathan's measure conjecture*, Ann. of Math., 134, 545-607,1991

[R91b] M.Ratner, *Raghunathan's topological conjecture and distributions of unipotent flows*, Duke Math. J., 63, 235-280, 1991

[R90a] M.Ratner, *Strict measure rigidity for unipotent subgroups of solvable groups*, Invent. Math., 101, 449-482, 1990

[R87] M.Ratner, *The rate of mixing for geodesic and horocycle flow*, Ergodic Theory Dyn. Syst., 7,267-288, 1987

[SU15] P.Sarnak and A.Ubis, *The horocycle flow at prime numbers*, J. Math. Pures Appl. (9) 103 (2015), no. 2, 575-618.

[Sh91] N.A.Shah, *Uniformly distributed orbits of certain flows on homogeneous spaces*, Math. Ann. 289, no. 2, 315-334, 1991.

[Sh94] N.A.Shah, *Limit distributions of polynomial trajectories on homogeneous spaces*, Duke Math. J., 75(3), 711-732, 1994

[S13] A.Strömbergsson, *On the deviation of ergodic averages for horocycle flows*, J. Mod. Dyn., 7(2), 291-328, 2013

[TV15] Tanis and Vishe, *Uniform bounds for period integrals and sparse equidistribution*, arXiv:1501.05228

[U91] M. Urbanski, *The Hausdorff dimension of the set of points with nondense orbit under a hyperbolic dynamical system*, Nonlinearity 2, 385-397, 1991

[V10] A.Venkatesh, *Sparse equidistribution problems, period bounds, and subconvexity*, Ann. of Math., 172(2), 989-1094, 2010

[Z16] C.Zheng, *Spare equidistribution of unipotent orbits in finite-volume homogeneous spaces*, J. Mod. Dyn., 10(2), 1-21, 2016