Perfect Numbers and Perfect Polynomials: Motivating Concepts From Kindergarten to College

A Thesis

Presented in Partial Fulfillment of the Requirements for the Degree Master of Mathematical Science in the Graduate School of The Ohio State University

By

Hiba Abu-Arish, B.S.

Graduate Program in Mathematics

The Ohio State University

2016

Master's Examination Committee:

Dr. Bart Snapp, Research Advisor Dr. Rodica Costin, Advisor Dr. Herb Clemens © Copyright by Hiba Abu-Arish 2016

Abstract

In this thesis we will show that perfect numbers can be used as a medium for conveying fundamental ideas from school mathematics. Guided by the Common Core Standards, we present activities designed for students from pre-kindergarten through high school. Additionally, we show how perfect numbers can be used in college level courses. These activities aim for students to gain a deeper understanding of the different mathematical concepts that are related to perfect numbers. This will bring students closer to unsolved problems in mathematics. This is dedicated to all who helped and supported me during my college years, and who wished me success from the bottom of their hearts: my teachers, my colleagues, my friends and my family. In particular I would like to thank those who took care of my daughters and treated them with love in order for me to pass this stage of my life.

Vita

2004	High School, Palestine
2004-2008	B.C. Applied Mathematics, Palestine Polytechnic University.
2007-2008	Teacher, University Graduates Union, Palestine
2014-present	Graduate Teaching Associate, The Ohio State University.

Fields of Study

Major Field: Mathematics

Table of Contents

	Pa	ıge
Abst	act	ii
Dedi	eation	iii
Vita		iv
Chaj	ters:	
1.	Introduction	1
2.	Perfect Numbers and Perfect Polynomials over $\mathbb{Z}_2[x]$	3
	 2.1 Perfect Numbers	3 11 14 17 19 20 21
3.	Pre-Kindergarten to Third Grade	33
	3.1Making Rectangles3.2How many ways3.3Pick a Number3.4Sharing cookies	33 41 45 48

4.	Four	th to Ninth Grade
	$4.1 \\ 4.2$	Prime Numbers 53 Divisors 55
	$\begin{array}{c} 4.3\\ 4.4 \end{array}$	The sum of the reciprocal divisors of a number60Number of Factors64
5.	Seve	nth to Twelfth Grade
	$5.1 \\ 5.2 \\ 5.3$	(" σ " Function)67Perfect Numbers Formula71Find Perfect Numbers76
6.	Colle	ege
	$6.1 \\ 6.2$	Even Perfect Numbers78Perfect polynomials (mod 2)83
App	endice	es:
А.	Pre-]	Kindergarten to Third Grade Activities
	A.1 A.2 A.3	Making Rectangles 87 How many ways 89 Pick a Number 90 Ol 90
_	A.4	Sharing cookies
В.	Four	th to Ninth Grade Activities
	B.1 B.2 B.3 B.4	Prime Numbers95Divisors96The sum of the reciprocal divisors of a number96Number of Factors96
C.	Seve	nth to Twelfth Grade Activities
	C.1 C.2 C.3	" σ " Function101Perfect Numbers Formula103Find Perfect Numbers105
D.	Colle	ege
	D.1	Even Perfect Numbers

D.2 Perfect polynomials	$\pmod{2} \dots \dots$	109
Bibliography		110

Chapter 1: Introduction

A natural number is called *perfect* if it equals the sum of its positive proper divisors. Euclid was the first person who discussed perfect numbers in his book *The Elements.* In his book, he proved the following: If n is perfect, then $n = 2^{p-1}(2^p - 1)$ where $(2^p - 1)$ is a prime and $p \in \mathbb{N}$.

This thesis is designed for advanced high school teachers. The purpose of this thesis is to show how perfect numbers can be used as a medium for conveying fundamental ideas that arise in school mathematics. We do this by designing activities for different school levels: Pre-Kindergarten through Third grade, Fourth through Ninth grade, Seventh grade through High School, and College.

Our activities cover the ideas of finding divisors, adding divisors, finding prime factorization, and other skills. We work with both natural numbers and polynomials depending on the level of students and the corresponding common core standards [11]. We have four chapters of activities in this thesis. These activities are based on common core standards relevant to the appropriate grade level.

 Chapter three of this thesis is for students from pre-kindergarten to third grade. Activities in this chapter make students work with objects to practice counting. We also introduce students to different categories of numbers depending on the different ways students can organize the objects in different shapes.

- Chapter four is for students from fourth to ninth grade. The activities in this chapter cover the idea of using multiplication and division in order to find divisors. Students also practice addition.
- Chapter five is for students from seventh grade to high school. Students work with the sum of the divisors function and are introduced to theorems and proofs.
- Chapter six is designed for college students. The activities walk students through the proof of Euclid formula and covers the idea of perfect polynomials.

In this thesis, we also discuss perfect polynomials in $\mathbb{Z}_2[x]$, as introduced by Canady in his Ph.D thesis *The Sum of The Divisors of a Polynomial* [3]. A perfect polynomial in $\mathbb{Z}_2[x]$ is one that is the sum of its divisors.

Example 1.0.1. The polynomial $x^2 + x$ is a perfect in $\mathbb{Z}_2[x]$.

Answer. The factors of $x^2 + x$ are

1,
$$x$$
, $x + 1$, $x^2 + x$.

Summing these in $\mathbb{Z}_2[x]$ we find

$$1 + x + x + 1 + x^2 + x \equiv x^2 + x \pmod{2}$$

In this thesis, we use perfect numbers to motivate ways for students to gain a deeper understanding for school mathematics. We also show the connection between perfect numbers and perfect polynomials in $\mathbb{Z}_2[x]$ and the parallels between the different theorems and proofs, to show how higher level mathematical topic can be taught by analogies.

Chapter 2: Perfect Numbers and Perfect Polynomials over $\mathbb{Z}_2[x]$

2.1 Perfect Numbers

The set of natural numbers

$$\mathbb{N} = \{1, 2, 3, 4, 5, \dots\}$$

is the set of positive whole numbers. Natural numbers can be partitioned into three categories: *perfect*, *deficient*, or *abundant*. These three categories and other definitions are described below.

Definition. We call $d \in \mathbb{N}$ a **divisor** of the number $c \in \mathbb{N}$ if there exist a number $b \in \mathbb{N}$ such that $c = b \times d$. In this case, both d and b are divisors of c.

Example 2.1.1. {1, 2, 3, 4, 6, 12} are the divisors of 12.

Definition. A proper divisor of $n \in \mathbb{N}$ is any divisor of n, other than n itself.

Example 2.1.2. $\{1, 2, 3, 4, 6\}$ are the proper divisors of 12.

Definition. A natural number is called **composite** if it has at least one positive divisor other than 1 and the number itself.

Example 2.1.3. $\{8, 15, 30, 46, 63, \ldots\}$ are some examples of composite numbers.

Definition. A natural number is called **prime** if it only has two positive divisors; 1 and the number itself. It has exactly one proper divisor, and every non-prime number has at least two proper divisors.

Example 2.1.4. $\{2, 3, 5, 7, 11, \ldots\}$ are the first five positive prime numbers.

Definition. The number 1 is neither a prime nor a composite. It is called a **unit**.

Definition. A natural number is called **deficient**, if the sum of its proper divisors is less than the number.

Example 2.1.5. $\{4, 8, 9, 15, \ldots\}$ are some examples of deficient numbers. Indeed, the proper divisors of 4 are 1 and 2 and 1 + 2 < 4.

Definition. A natural number is called **abundant**, if the sum of its proper divisors is greater than the number.

Example 2.1.6. {12, 18, 20, 24, ...} are some examples of abundant numbers. Indeed, the proper divisors of 12 are {1, 2, 3, 4, 6} and 1+2+3+4+6 = 16 > 12.

Definition. A natural number is called **perfect**, if the number equals the sum of its proper divisors. All even perfect numbers have the form $2^{n-1}(2^n - 1)$ where $n \in \mathbb{N}$ and $n \geq 2$. In fact, by 2016 there were only 49 known perfect numbers. The last known perfect number was discovered by (Cooper, Woltman, Kurowski, Blosser, et al.) and has 44,677,235 digits [19].

Example 2.1.7. {6, 28, 496, 8128, 33550336, 8589869056, 137438691328, 2305843008139952128, 265845599156...615953842176, ... } are some examples of perfect numbers.

Indeed, the proper divisors of 6 are 1, 2, 3 and 1 + 2 + 3 = 6.

Lemma 2.1.1 (Bezout's Identity). If $a, b \in \mathbb{Z}$ such that a, b are not both zero, and gcd(a, b) = d. Then there exist $x, y \in \mathbb{Z}$ such that ax + by = d.

Proof. Let $a, b \in \mathbb{Z}$ such that a, b are not both zero. Let

$$M = \{n \in \mathbb{Z}, n > 0 : n = ax + by : x, y \in \mathbb{Z}\}$$
 where $M \neq \emptyset$

M has a least element let it d. Then

$$d = au + bv$$
 for some $u, v \in \mathbb{Z}$

By the division theorem. Let $n \in M$

$$n = qd + r$$
 where $0 < r < d$.

Suppose that $d \not| n$. But there exist $x, y \in \mathbb{Z}$ such that n = ax + by and we have

$$r = n - qd$$

= $ax + by - q(au + bv)$
= $ax - qau + by - qbv$
= $a(x - qu) + b(y - qv)$

This means that $r \in M$ and r < d contradicting the fact that d is the least element in M. Therefore $\forall n \in M : d | n$, this means that d | a and d | b. So $1 \leq d \leq \gcd(a, b)$. But $\gcd(a, b) | a$ and $\gcd(a, b) | b$. Therefore

$$gcd(a, b)|au + bv = d$$

 $gcd(a, b) \le d$
 $gcd(a, b) = d$
 $gcd(a, b) = d = au + d$

bv

Lemma 2.1.2 (Euclid's Lemma).

If p is a prime number and p|ab where $a, b, p \in \mathbb{N}$, then p|a or p|b (or both).

Proof. Suppose that $p \not| a$, then gcd (a, p) = 1 by (Bezout's Identity 2.1.1,) there are r and s so that

$$pr + as = 1.$$

Multiply both sides by b

$$pbr + abs = b.$$

The term pbr is divisible by p, and the term abs is divisible by ab which by hypothesis is divisible by p. Therefore their sum, b, is also divisible by p.

Theorem 2.1.1. (Unique Factorization in the natural numbers.)([9]) Every natural number larger than 1 can be factored into a product of prime numbers that is unique upto the order of the prime factors.

Proof. Seeking a contradiction, suppose there is a number greater than 1 that <u>cannot</u> be factored into primes. Let

 $B = \{ all natural numbers that cannot be factored into primes. \}$

Note $B \subseteq \mathbb{N}$ and by assumption $B \neq \emptyset$.

Hence by the Well-Ordering Principle B has a least element, call it b.

First, note b is not prime, else it cannot be in B.

Then, note b must be composite, as it is larger than 1 and it is not prime.

$$\therefore b = x \cdot y$$
 where $x, y \in \mathbb{N}$ and $x, y \notin B$

therefore $x = p_1 \cdot p_2 \cdot \ldots \cdot p_n$

$$y = q_1 \cdot q_2 \cdot \ldots q_m$$

 $x \cdot y = p_1 \cdot p_2 \dots p_n \cdot q_1 \cdot q_2 \dots q_m = b$

This is a prime factorization of b. Contradiction.

We now show that the prime factorization is unique. Let

 $T = \{a > 1 : a \text{ has at least two different prime factorization}\}.$

If $T \neq \emptyset$, let a_0 be the smallest element of T. Take two distinct prime factorization

$$a_0 = p_1 \dots p_n = p'_1 \dots p'_r$$

where all p_i and p'_j are positive primes. Since $p_1 \dots p_n | a_0$ and $a_0 = p'_1 \dots p'_r$, then

$$p_1 \dots p_n | p'_1 \dots p'_r \Longrightarrow p_i = p'_j$$
 for some i, j
 $p_1 | p'_1 \dots p'_r \Longrightarrow p_1 = p'_j$ for some j (Lemma 2.1.2)

If we reorder or re-number the p'_i we can assume $p_1 = p'_1$. Cancel them $\implies p_2 \dots p_n = p'_2 \dots p'_r = \frac{a_0}{p_1} = b_0$. But $1 < b_0 < a_0 \implies b_0 \notin T$ it has a prime factorization that is unique.

$$p_2 \dots p_n = p'_2 \dots p'_r = \frac{a_0}{p_1} = b_0$$

$$\implies r = n$$
 and the $p_2 \dots p_n = p'_2 \dots p'_r = \frac{a_0}{p_1}$

up to ordering.

 \implies The factorization of a_0 were the same. Contradiction.

Definition. A **geometric series** is a series with a constant ratio between successive terms.

Lemma 2.1.3.

$$1 + r + r^{2} + r^{3} + r^{4} + r^{5} + \dots + r^{n} = \frac{r^{n+1} - 1}{r - 1}$$

Proof. Note that this is a geometric series. The ratio between successive terms is r. Proof by induction. First if n = 0, then

$$r^0 = \frac{r-1}{r-1} = 1$$

Assuming that

$$\sum_{i=1}^{k} r^{i} = \frac{r^{k+1} - 1}{r - 1}$$

we need to show

$$\sum_{i=1}^{k+1} r^i = \frac{r^{k+2} - 1}{r - 1}.$$

Adding r^{k+1} and multiplying we obtain:

$$\sum_{i=1}^{k} r^{i} + r^{k+1} = \frac{r^{k+1} - 1}{r - 1} + r^{k+1}$$
$$\sum_{i=1}^{k+1} r^{i} = \frac{r^{k+1} - 1 + r^{k+1}(r - 1)}{r - 1}$$
$$= \frac{r^{k+1} - 1 + r^{k+2} - r^{k+1}}{r - 1}$$
$$= \frac{r^{k+2} - 1}{r - 1}$$

Therefor, for all $n \in \mathbb{N}$,

$$\sum_{i=1}^{n} r^{i} = \frac{r^{n+1} - 1}{r - 1}.$$

Euclid was the first person to write about perfect numbers. In his book *The Elements* around 300BCE he stated [5]:

If as many numbers as we please beginning from a unit be set out continuously in double proportion, until the sum of all becomes a prime, and if the sum multiplied into the last make some number, the product will be perfect[5].

Let's see if we can understand what Euclid was telling us.

set out ... in double proportion Starting with 1, create a sequence by doubling the number each time.

$$2^0 = 1, \quad 2^1, \quad 2^2, \quad 2^3, \quad \dots$$

until the sum of all becomes a prime Consider the sum of the sequence above, and keep adding terms until the sum becomes a prime number.

$$2^0 + 2^1 + 2^2 + 2^3 + \dots = a$$
 prime

Building off of Euclid's Thm, our next question is: When is

$$\sum_{k=0}^{n-1} 2^k \text{ prime?}$$

We need the following lemma:

Lemma 2.1.4. If $(2^n - 1)$ is prime for some integer $n \ge 2$, then n is prime.

Proof. Let's prove the contrapositive. We want to show that if $n \ge 2$ is not prime, then $2^n - 1$ also not prime. Suppose $n \ge 2$ is not a prime, then there exist $r, s \in \mathbb{N}$ with r, s > 1 such that n = rs. Write

$$2^{n} - 1 = 2^{rs} - 1$$
$$= (2^{s} - 1)(2^{s(r-1)} + 2^{s(r-2)} + \dots + 2^{s \cdot 2} + 2^{s \cdot 1} + 1)$$

so we see that $(2^s - 1)$ divides $2^n - 1$. Since s > 1, then $(2^s - 1) > 1$. Also since r > 1, then

$$(2^{s} - 1) < (2^{r \cdot s} - 1) = (2^{n} - 1).$$

This means that $(2^{s}-1)$ is a proper divisor for $(2^{n}-1)$, and thus $(2^{n}-1)$ is composite.

The converse of the Lemma is not true. For example, n = 11 is a prime , but $2^{11} - 1 = 2,047 = 23 \cdot 89$ is not a prime.

and if sum multiplied into the last make some number multiply the prime number (the result of adding double proportions) by the largest term in the sum. This product will be perfect.

Example 2.1.8. Lets add double proportions to get a prime.

$$2^0 + 2^1 + 2^2 = 1 + 2 + 4 = 7.$$

Now we multiply the largest term in the sum, 4, by the prime to find 28. Since

$$28 = 1 + 2 + 4 + 7 + 14.$$

28 is a perfect number.

Now we will write Euclid's statement in modern notation and give a proof.

Theorem 2.1.2. ([8]) If $2^p - 1$ is a prime number, then $2^{p-1}(2^p - 1)$ is a perfect number.

Proof. Let $q = 2^p - 1$ be a prime. We need to check that $2^{p-1}q$ is a perfect number. The proper divisors of $2^{p-1}q$ are:

1, 2, 4, 8, ...,
$$2^{p-1}$$
, and, q , $2q$, $4q$, $8q$, ..., $2^{p-2}q$.

Now lets add up the proper divisors of $2^{p-1}q$

$$\sum_{n=0}^{p-1} 2^n + \sum_{n=0}^{p-2} 2^n q = (2^p - 1) + q(2^{p-1} - 1) \qquad (Lemma \ 2.1.3)$$
$$= (2^p - 1) + q(2^{p-1} - 1)$$
$$= q + q(2^{p-1} - 1)$$
$$= q(2^{p-1})$$
$$= 2^{p-1}(2^p - 1)$$

This means that if $(2^p - 1)$ a prime, then $2^{p-1}(2^p - 1)$ is perfect.

2.1.1 The Sigma Function

Definition. The sum of the positive divisors function is denoted by:

 $\sigma(n) = \text{sum of all positive divisors of } n \text{ (including 1 and } n).$

$$\sigma(n) = \sum_{d|n} d$$

Then clearly we have,

if n is perfect, then $\sigma(n) = 2n$.

Theorem 2.1.3. If p is a prime and k is a natural number with $k \ge 1$, then

$$\sigma(p) = p + 1.$$

$$\sigma(p^k) = \frac{p^{k+1} - 1}{p - 1}.$$

Proof. if p is a prime then p has only two divisors which are 1 , p, this means that $\sigma(p) = 1 + p$.

Now the divisors of a prime power p^k are:

$$d = 1, \ p, \ p^2, \ p^3, \ \dots, \ p^{k-1}, \ p^k$$

$$\sigma(p^k) = \sum_{d|p^k} d$$

$$= 1 + p + p^2 + p^3 + \dots + p^{k-1} + p^k = \frac{p^{k+1} - 1}{p - 1} \quad (Lemma \ 2.1.3)$$

Theorem 2.1.4. If gcd(m, n) = 1, then

$$\sigma(m \cdot n) = \sigma(m) \cdot \sigma(n)$$

Proof. If gcd(m, n) = 1

$$\sigma(nm) = \sum_{d \mid mn} d$$

Since gcd(m, n) = 1 then d can be written $d = r \cdot s$ where r|m and s|n

$$\sum_{d|mn} d = \sum_{rs|mn} (rs)$$
$$= \sum_{r|m,s|n} (rs)$$
$$= \sum_{r|m} r \cdot \sum_{s|n} s \quad (\gcd(r,s) = 1)$$
$$= \sigma(m) \cdot \sigma(n)$$

- 6		

The following theorem shows that all the *even* perfect numbers have the form of (Theorem 2.1.3).

Theorem 2.1.5. ([8]) If n is an even perfect number, then

$$n = 2^{p-1}(2^p - 1)$$

where $2^p - 1$ is a prime.

Proof. Let $n = 2^k m$ an even perfect number, where $k \ge 1$ and m is odd. Now compute

$$\sigma(n) = \sigma(2^k m)$$

= $\sigma(2^k)\sigma(m)$, $gcd(2^k, m) = 1$
= $(2^{k+1} - 1)\sigma(m)$, (Theorem 2.1.3)

But since n is also perfect then $\sigma(n) = 2n = 2^{k+1}m$. Now we have two different expressions for $\sigma(n)$ that are equal

$$2^{k+1}m = (2^{k+1} - 1)\sigma(m)$$

The number $2^{k+1} - 1$ is odd. This means that 2^{k+1} divides $\sigma(m)$, thus

$$\sigma(m) = 2^{k+1}a$$

for some number a. But

$$2^{k+1}m = (2^{k+1} - 1)\sigma(m) = (2^{k+1} - 1)2^{k+1}a$$

Canceling 2^{k+1} we will get $m = (2^{k+1} - 1)a$. Now we have

$$m = (2^{k+1} - 1)a$$
 and $\sigma(m) = 2^{k+1}a$

We need to show that a = 1 by contradiction. Assume that a > 1, then $m = (2^{k+1} - 1)a$. This means that m have for sure the distinct factors 1, a, m. Where a and m are different sine n was even and $k \ge 1$. Also m can be divisible by many other this means that

$$\sigma(m) \ge 1 + a + m =$$

= 1 + a + (2^{k+1})a
= 1 + 2^{k+1}a

However, $\sigma(m) = 2^{k+1}a$, so

$$2^{k+1}c \ge 1 + 2^{k+1}a$$

Contradiction, since $0 \leq 1$ which shows that c = 1. Now we have

$$m = (2^{k+1} - 1)$$

and

$$\sigma(m) = m + 1$$

But from (Thmeorem 2.1.3) (if p is a prime, then $\sigma(p) = p + 1$). This means that $m = (2^{k+1} - 1)$ is a prime number.

$$n = 2^k m = 2^k (2^{k+1} - 1)$$

Where $(2^{k+1} - 1)$ is a prime. And n is perfect. By (Lemma 2.1.4) this means that k+1 is a prime let us call it p = k+1. So every even perfect number can be written in the form $n = 2^{p-1}(2^p - 1)$ with $(2^p - 1)$ is prime.

2.1.2 Mersenne Primes

Definition. A Mersenne primes is a prime of the form $2^n - 1$ where $n \in \mathbb{N}$ and $n \geq 2$.

The appearance and interest in Mersenne primes came because of their association with perfect numbers. "Every Mersenne prime gives rise to a perfect number. The historical justification for this nomenclature seems rather weak, since several perfect numbers and their corresponding primes have been known since antiquity and occur in almost every medieval numerological speculation." [12]

Proposition 2.1.1. If $a^n - 1$ is prime for some numbers $a \ge 2$ and $n \ge 2$, then a must be 2 and n is prime.

Proof. lets look at different cases for a

- 1. If a is odd, then $a^n 1$ is even, so it cannot be a prime.
- 2. It is always true that $a^n 1$ is divisible by (a 1), and it is proved by using geometric series.

$$1 + a + a^{2} + a^{3} + a^{4} + \dots + a^{n-1} = \frac{a^{n} - 1}{a - 1}$$
$$(1 + a + a^{2} + a^{3} + a^{4} + \dots + a^{n-1})(a - 1) = a^{n} - 1$$

This means that $a^n - 1$ is always composite unless a - 1 = 1, that is, unless a = 2. Then $2^n - 1$ is prim if n is prim (Lemma 2.1.4).

- 3. If a = 2, the number $2^n 1$ can be a prime or a composite. let n = m.k then $2^n = (2^m)^k$
 - $2^{n} 1 = (2^{m})^{k} 1$ $= [(2^{m} 1)][(2^{m})^{k-1} + (2^{m})^{k-2} + \dots + (2^{m})^{2} + (2^{m}) + 1] \text{ geometric series}$

This shows that if n is composite then $2^n - 1$ is composite.

Let us look at the following table:

n		1	2	3	4	5	6	7	8	9	10	
2^{n} -	- 1	1	3	7	35	31	$3^2 \times 7$	127	$3 \times 5 \times 17$	7×73	$3 \times 11 \times 31$	

This table suggest the following:

- 1. When n is even, the number $2^n 1$ is divisible by 3.
- 2. When n is divisible by 3, the number $2^n 1$ is divisible by 7.
- 3. When n is divisible by 5, $2^n 1$ is divisible by 31.

"Father Marin Mersenne was one who discovered the fact of Mersenne numbers and they were called after him. It is not known how Mersenne discovered these facts. In 1644 he listed all primes less than 258 that makes $2^p - 1$ be a prime. His list was

p = 2, 3, 5, 7, 13, 17, 19, 31, 67, 127, 257.

This list was not correct. The complete list of primes less than 10000 for which $2^p - 1$ is Mersenne prime is:

 $p=2,\ 3,\ 5,\ 7,\ 13,\ 17,\ 19,\ 31,\ 61,\ 89,\ 107,\ 127,\ 521,\ 607,\ 1279,$

2203, 2281, 3217, 4253, 4423, 9689, 9941."

[8]

"It was only with the advent of computing machines that it became possible to check numbers with hundreds of digits for primality. Indeed, it was not until 1876 that E. Lucas proved conclusively that $2^{127} - 1$ is prime [8]." Recently the "GIMPS" is a project where a group of mathematicians uses soft wares in order to find Mersenne primes [19].

2.1.3 Perfect Numbers Are Triangular and Hexagonal.

Definition. A triangular number is a number that can be written on the form $\frac{k(k+1)}{2}$. This number is called triangular because it can be represented in a triangular form when arranged in rows of $1, 2, 3, \ldots$ points that starts with a point and each subsequent row contains one more element than the previous one [10].

Example 2.1.9. The first few triangular numbers are:

 $\{1, 3, 6, 10, 15, 21, 28, 36, 45, 55, 66, 78, 91,$

 $105, 120, 136, 153, 171, 190, 210, 231, 253, 276, 300, 325, 351, 378, 406, \dots$



Theorem 2.1.6. Every even perfect number is a triangular number.

Proof. Let

$$T_k = \frac{k(k+1)}{2}$$

be a triangular number. If we let

$$k = 2^n - 1$$

for any positive integer n, then

$$k + 1 = 2^{n}$$
$$T_{k} = \frac{(2^{n} - 1)2^{n}}{2} = 2^{n-1}(2^{n} - 1)$$

which is Euclid's form for perfect numbers.

One may ask "Is every triangular number a perfect number?" This is not true since in particular 3 is a triangular number but 3 is not a perfect number.

Definition. A hexagonal number is a number that can be written on the form k(2k-1). This number is called hexagonal because it can be represented with points that consist the outlines of hexagons [14].

Example 2.1.10. The first few hexagonal numbers are:

 $\{1, 6, 15, 28, 45, 66, 91, 120, 153, 190, 231, 276, 325, 378, 435, 496, 561, 630, 703, 780, 861, 946, \ldots\}$



Theorem 2.1.7. Every even perfect number is a hexagonal number.

Proof. Let

$$H_k = k(2k - 1)$$

be a hexagonal number. let

$$k = 2^{n-1}.$$

Then

$$(2k-1) = (2^n - 1).$$

 $H_k = 2^{n-1}(2^n - 1)$

which is Euclid's form for perfect numbers.

We can also prove that a **hexagonal number** is also a **triangular number**.

Proof. Let

$$m = 2k$$
 then $k = \frac{m}{2}$,

also

$$(2k - 1) = (m - 1).$$

Now the hexagonal number

$$H_k = k(2k - 1) = \frac{m(m - 1)}{2}$$

which is a triangular number.

But not every **triangular number** is a **hexagonal number**. The same not every **hexagonal number** is a **perfect number**.

2.1.4 The Reciprocals of the Divisors of a Perfect Number

For a perfect number when we are adding the reciprocals of the divisors; we are adding different unit fractions and we are always getting 2.

Theorem 2.1.8. If n is a perfect number, then

$$\sum_{d|n} \frac{1}{d} = 2.$$

Proof. To start, we claim

$$\sum_{d|n} \frac{1}{d} = \sum_{d|n} \frac{d}{n}.$$

To see this it suffices to show that if $X = \{\frac{1}{d} : d|n\}$ and $Y = \{\frac{d}{n} : d|n\}$, then X = Y. We will do this by showing $X \subseteq Y$ and $X \subseteq Y$.

 $(X \subseteq Y)$ Consider $\frac{1}{d} \in X$. Since d|n, n = dq. So

$$\frac{1}{d} = \frac{q}{d \cdot q} = \frac{q}{n} \in Y.$$

 $(X \supseteq Y)$ Consider $\frac{d}{n} \in Y$. Since d|n, n = dq. So

$$\frac{d}{n} = \frac{d}{d \cdot q} = \frac{1}{q} \in X.$$

Now, if n is perfect, then $\sigma(n) = \sum_{d|n} d = 2n$. Write

$$2 = \frac{1}{n} \sum_{d|n} d$$
$$= \sum_{d|n} \frac{d}{n}$$
$$= \sum_{d|n} \frac{1}{d}.$$

One may ask "What is the relation between the sum of the reciprocals of the divisors of a perfect number and the **Egyptian Fractions**?"

Definition. A **unit fraction** is the reciprocal of a natural number, $\frac{1}{n}$ where $n \in \mathbb{N}$. **Definition.** An **Egyptian fraction** is a fraction represented by a finite sum of unit fractions where all unit fractions are each distinct.

2.1.5 Odd Perfect Numbers

All the theorems and formulas for perfect numbers suggest that perfect numbers are only even. Does this means that there are no odd perfect numbers?

Although the technology and the different softwares that mathematitions are using to find perfect numbers, to this day they have found none [13].

2.2 Perfect Polynomials over $\mathbb{Z}_2[x]$

After we introduced perfect numbers and Mersenne primes in the set of natural numbers, we are going to introduce similar ideas in the set of polynomials in $\mathbb{Z}_2[x]$.

Definition. We denote by $\mathbb{Z}_2[x]$ the ring of all polynomials with coefficients in \mathbb{Z}_2 . This means that the coefficients are 0 or 1, and addition and multiplication are done modulo 2.

Example 2.2.1. $(x+1)^2 = x^2 + 2x + 1 \equiv x^2 + 1 \pmod{2}$

Definition. A set \mathbb{F} equipped with two operators, denoted $+, \cdot$, is a field if:

- 1. $\mathbb F$ is closed under $+,\cdot.$
- 2. $+, \cdot$ are both associative and commutative operations.
- 3. There is $0 \in \mathbb{F}$ such that a + 0 = a.
- 4. There is $1 \in \mathbb{F}$ such that $a \cdot 1 = a$.
- 5. For all $a \in \mathbb{F}$, there is $-a \in \mathbb{F}$ such that a + (-a) = 0.
- 6. For all $a \in \mathbb{F} 0$, there is a^{-1} such that $a \cdot a^{-1} = 1$.
- 7. a(b+c) = ab + ac.

Definition. A polynomial in $\mathbb{Z}_2[x]$ is called **reducible** if the polynomial can be factored as a product of nonconstant polynomials in $\mathbb{Z}_2[x]$.

Example 2.2.2. $x^2 + 1$ is reducible in $\mathbb{Z}_2[x]$. If x = 1, then $x^2 + 1 \equiv 0 \pmod{2}$

Definition. A polynomial over $\mathbb{Z}_2[x]$ is called **irreducible or prime** if the polynomial cannot be factored in $\mathbb{Z}_2[x]$ into polynomials of lower degree.

Example 2.2.3. $x^2 + x + 1$ is an irreducible polynomial is $\mathbb{Z}_2[x]$.

Let us agree in this thesis to call a polynomial over $\mathbb{Z}_2[x]$ even if it has roots in \mathbb{Z}_2 and odd other wise [7].

Lemma 2.2.1 (Bezout's identity). Let a(x), $b(x) \in \mathbb{F}[x]$ such that a(x), b(x) are not both zero, and gcd(a(x), b(x)) = d(x). Then there exist $n(x), m(x) \in \mathbb{F}[x]$ such that a(x)n(x) + b(x)m(x) = d(x).

Proof. Let $a(x), b(x) \in \mathbb{F}[x]$ such that a(x), b(x) are not both zero. Let

$$D = \{z(x) \in \mathbb{F}[x], \deg(z(x)) \ge 1 : z(x) = a(x)n(x) + b(x)m(x) : n(x), m(x) \in \mathbb{F}[x]\}$$

where $D \neq \emptyset$. D has a least element let it d(x). Then

$$d(x) = a(x)u(x) + b(x)v(x)$$
 for some $u(x), v(x) \in \mathbb{F}[x]$

By the division theorem

$$z(x) = q(x)d(x) + r(x) \text{ where } 0 < \deg r(x) < \deg d(x).$$

Suppose that $d(x) \not| z(x)$. But there exist $n(x), m(x) \in \mathbb{Z}$ such that

$$z(x) = a(x)n(x) + b(x)m(x)$$

and we have

$$\begin{aligned} r(x) &= z(x) - q(x)d(x) \\ &= a(x)n(x) + b(x)m(x) - q(x)(a(x)u(x) + b(x)v(x)) \\ &= a(x)n(x) - q(x)a(x)u(x) + b(x)m(x) - q(x)b(x)v(x) \\ &= a(x)(n(x) - q(x)u(x)) + b(x)(m(x) - q(x)v(x)) \end{aligned}$$

This means that $r(x) \in D$ and $0 < \deg r(x) < \deg d(x)$ contradicting the fact that d(x) is the least element in D.

Therefore $\forall z(x) \in D : d(x)|z(x)$, this means that d(x)|a(x) and d(x)|b(x). So $1 \leq d(x) \leq \gcd(a(x), b(x))$. But $\gcd(a(x), b(x))|a(x)$ and $\gcd(a(x), b(x))|b(x)$. Therefor

$$gcd (a(x), b(x))|a(x)u(x) + b(x)v(x) = d(x)$$
$$gcd (a(x), b(x)) \le d(x)$$
$$gcd (a(x), b(x)) = d(x)$$
$$gcd (a(x), b(x)) = d(x)$$
$$= a(x)u(x) + b(x)v(x)$$

	-	-	-	

Lemma 2.2.2. [Euclid's Lemma] If a(x), b(x), $p(x) \in \mathbb{Z}_2[x]$, and p(x) is an irreducible polynomial such that p(x)|a(x)b(x). Then p(x)|a(x) or p(x)|b(x).

Proof. Let p(x)|a(x)b(x), and assume that a(x) and p(x) be relatively prime. By Bezout's identity, there are r(x) and s(x) making

$$r(x)p(x) + s(x)a(x) = 1.$$

Multiply both sides by b(x):

$$r(x)p(x)b(x) + s(x)a(x)b(x) = b(x).$$

The term r(x)p(x)b(x) is divisible by p(x), and the term s(x)a(x)b(x) is divisible by a(x)b(x) which by hypothesis is divisible by p(x). Therefore their sum, b(x), is also divisible by p(x).

Theorem 2.2.1. (Unique Factorization in $\mathbb{F}[x]$) Every polynomial $f(x) \in F[x]$ with degree $f \geq 1$ can be written uniquely as a finite product of irreducible polynomials.

Proof. Seeking a contradiction, suppose there is a polynomial f(x) that cannot be written as a finite product of irreducible polynomials. Let

 $B = \{g(x) \in F[x] : deg(g) \ge 1 \text{ and } g \text{ is not the product of irreducible polynomials} \}$

By assumption, $B\neq \emptyset$.

Let

$$D = \{ d = \deg(g) : g(x) \in B \} \subset \mathbb{N} \text{ so } D \neq \emptyset$$

by the well-ordering principle, D has a smallest element, so B has an element of smallest degree, let it be h(x). Then h(x) is not irreducible since $h(x) \in B$. So h(x)factors non-trivially as

$$h(x) = a(x)b(x)$$

where $\deg(a) \ge 1$ and $\deg(b) \ge 1$. Then

$$\deg(h) = \deg(a) + \deg(b)$$

 \mathbf{SO}

$$1 \le \deg(a) < \deg(h)$$
 and $1 \le \deg(b) < \deg(h)$

This means $a(x), b(x) \notin B$. So they factor as a product of irreducible polynomials in $\mathbb{Z}_2[x]$

$$a(x) = p_1(x) \dots p_r(x)$$
 with irreducible $p_i(x)$
 $b(x) = q_1(x) \dots q_s(x)$ with irreducible $q_j(x)$

Then

$$h(x) = a(x) \cdot b(x) = p_1(x) \dots p_i(x) \cdot q_1(x) \dots q_j(x)$$

has a factorization into irreducible polynomials. So $h(x) \notin B$. This is a contradiction. So every polynomial $f(x) \in F[x]$ with degree $f \ge 1$ can be written as a finite product of irreducible polynomials

To show that the factorization is unique, let

 $T = \{a(x) \in F[x] : \deg(a) \ge 1 \text{ and } a \text{ has at least two different prime factorization}\}.$

 $T \neq \emptyset$, let $a_0(x)$ be the smallest element of T. Take two distinct prime factorization

$$a_0(x) = p_1(x) \dots p_n(x) = p'_1(x) \dots p'_r(x)$$

where all $p_i(x)$ and $p_j(x)$ are irreducible polynomials in F[x]. Then by (lemma 2.2.2)

$$p_i(x)|p'_1(x)\dots p'_r(x)$$
 for some *i* let it be 1

with out lose of generosity let i = 1

$$p_1(x)|p'_1(x)\dots p'_r(x)$$
 for some j

If we reorder or re-number the $p'_i(x)$ we can assume $p_1(x) = p'_1(x)$. Cancel them

$$p_2(x) \dots p_r(x) = p'_2(x) \dots p'_n(x) = \frac{a_0(x)}{p_1(x)} = b_0(x).$$

But $1 \ge \deg(b_0(x)) < \deg(a_0(x))$ then $b_0(x) \notin T$ it has a prime factorization that is unique.

$$p_2(x) \dots p_r(x) = p'_2(x) \dots p'_n(x) = \frac{a_0(x)}{p_1(x)} = b_0(x)$$

$$r = n \text{ and the } p_2(x) \dots p_r(x) = p'_2(x) \dots p'_n(x) = \frac{a_0(x)}{p_1(x)}$$

up to ordering.

The factorization of $a_0(x)$ were the same. Contradiction.

Thus, $T = \emptyset$ and the factorization is unique.

If we recall the proof of unique factorization in natural numbers, we can see that we are using the same technique and the same steps in proving unique factorization in $\mathbb{Z}_2[x]$.

Definition. The sum of the divisors of a polynomial A(x) is given by

$$\sigma(A(x)) = \sum_{D|A(x)} D.$$

Proposition 2.2.1. The σ function is multiplicative in the sense that whenever gcd(A(x), B(x)) = 1 we have

$$\sigma(A(x)B(x)) = \sigma(A(x)) \times \sigma(B(x))$$

Proof. By the definition of σ ,

$$\sigma(A(x)B(x)) = \sum_{D|A(x)B(x)} D$$

but gcd (A(x), B(x)) = 1. This means that D can be written as $D = r(x) \cdot s(x)$ where r(x)|A(x) and s(x)|B(x)

$$\sum_{D|A(x)B(x)} D = \sum_{r(x)s(x)|A(x)B(x)} (r(x)s(x))$$

$$= \sum_{r(x)|A(x), s(x)|B(x)} (r(x)s(x))$$

$$= \sum_{r(x)|A(x)} r(x) \cdot \sum_{s(x)|B(x)} s(x) \quad (\gcd(r(x), s(x)) = 1)$$

$$= \sigma(A(x)) \cdot \sigma(B(x))$$

Note, this is the same proof for σ being multiplicative over the natural numbers.

Definition. If $A(x) = \sigma(A(x))$ where $A(x) \in \mathbb{Z}_2[x]$ we call A(x) a **perfect polynomial** over $\mathbb{Z}_2[x]$.

Definition. A polynomial over $\mathbb{Z}_2[x]$ is called **Mersenne polynomials** if it can be written in the form $x^a(x+1)^b + 1$. An irreducible Mersenne polynomial is called **Mersenne prime** polynomial [6].

Example 2.2.4. Why is $x^{a}(x+1)^{b} + 1$ a Mersenne polynomials?

Answer. Because this is the only form that give us irreducible polynomials (mod 2) when we raise the polynomial to a power like n. Polynomials in $\mathbb{Z}_2[x]$ can have the following forms:

$$x^{a}, (x+1)^{b}, x^{a}(x+1)^{b}, x^{a}(x+1)^{b}+1$$

and if we look at the first three forms, they are always reducible in $\mathbb{Z}_2[x]$ when raised to a power. The only form that can make irreducible polynomials in $\mathbb{Z}_2[x]$ is the form $x^a(x+1)^b + 1$.

The first few Mersenne primes in $\mathbb{Z}_2[x]$ are [6]:

$$M_1 = 1 + x + x^2$$
, $M_2 = 1 + x + x^3$, $M_3 = 1 + x^2 + x^3$,
 $M_4 = 1 + x + x^2 + x^3 + x^4$, $M_5 = 1 + x^3 + x^4$.

If we look closely at these Mersenne prime polynomials and plug in x = 2, we see the following:

 $M_1 = 1 + x + x^2 \longleftrightarrow 7 \longleftrightarrow (2^3 - 1)$ Mersenne prime $M_2 = 1 + x + x^3 \longleftrightarrow 11 \longleftrightarrow$ not Mersenne prime $M_3 = 1 + x^2 + x^3 \longleftrightarrow 13 \longleftrightarrow$ not Mersenne prime $M_4 = 1 + x + x^2 + x^3 + x^4 \longleftrightarrow 31 \longleftrightarrow (2^5 - 1)$ Mersenne prime $M_5 = 1 + x^3 + x^4 \longleftrightarrow 25 \longleftrightarrow$ not Mersenne prime So two of these Mersenne prime polynomials correspond to two Mersenne primes in the natural numbers, which are $(2^3 - 1)$ and $(2^5 - 1)$. And if we look at the rest they correspond to odd numbers and some of them are primes. If we recall Euclid's: "If as many numbers as we please beginning from a unit be set out **continuously** in double proportion, until the sum of all becomes a prime" in M_1 and M_4 we are just adding powers of 2 continuously this is why they matched Mersenne primes in \mathbb{N} .

The study of perfect polynomials over $\mathbb{Z}_2[x]$ was first introduced by Canaday in his Ph.D. thesis *The Sum of The Divisors of a Polynomial* [3]. He treated only polynomials that split into linear factors over $\mathbb{Z}_2[x]$.

Definition. A trivial perfect polynomial is an even perfect polynomial over $\mathbb{Z}_2[x]$ with exactly two prime divisors. Those polynomials are of the form $(x(x+1))^{2^n-1}$ for some $n \in \mathbb{N}$ [6].

The first trivial perfect polynomial is x(x + 1) and if we plug x = 2 then this polynomial correspond to 6 which is the first perfect number in N.

Example 2.2.5. The polynomial $A(x) = (x(x+1))^3$ is perfect in $\mathbb{Z}_2[x]$. Write

$$\sigma(x(x+1))^3 = \sigma(x^3) \times \sigma((x+1)^3)$$
since gcd(x, x + 1) = 1 and A(x) factors in $\mathbb{Z}_2[x]$ in to linear factors we write

$$\begin{aligned} \sigma(x^3) \times \sigma((x+1)^3) &= (x^3 + x^2 + x + 1)((x+1)^3 + (x+1)^2 + (x+1) + 1) \\ &= (x^3 + x^2 + x + 1)((x^3 + 3x^2 + 3x + 1) + (x^2 + 2x + 1) \\ &+ (x+1) + 1) \\ &= (x^3 + x^2 + x + 1)((x^3 + x^2 + x + 1) + (x^2 + 2x + 1) \\ &+ (x+1) + 1) \\ &= (x^3 + x^2 + x + 1)(x^3 + 2x^2 + 4x + 4) \pmod{2} \\ &= (x+1)^3 \cdot x^3 \pmod{2} \\ &= A(x) \end{aligned}$$

Definition. A nontrivial (sporadic) perfect polynomial in $\mathbb{Z}_2[x]$ is a polynomial in $\mathbb{Z}_2[x]$ that has factorization with Mersenne primes as odd divisors [6].

Canady discovered 11 nontrivial even perfect polynomials [7]. Nine of them have four or fewer prime factors. The following are the nontrivial even perfect polynomials with their correspondence values after plugging x = 2

$$\begin{aligned} x(x+1)^2(x^2+x+1) &= 126, \ x^2(x+1)(x^2+x+1) = 84, \\ x^3(x+1)^4(x^4+x^3+1) &= 16200, \ x^4(x+1)^3(x^4+x^3+x^2+x+1) = 13392, \\ x(x+1)^2(x^2+x+1)^2(x^4+x+1) &= 16758, \ x^2(x+1)(x^2+x+1)^2(x^4+x+1) = 1596, \\ x^3(x+1)^6(x^3+x+1)(x^3+x^2+1) &= 833976, \ x^6(x+1)^3(x^3+x+1)(x^3+x^2+1) = 247104, \\ x^4(x+1)^4(x^4+x^3+1)(x^4+x^3+x^2+x+1) &= 51706512, \end{aligned}$$

and 2 of the nontrivial even perfect polynomials have 5 prime factors

$$x^{4}(x+1)^{6}(x^{3}+x+1)(x^{3}+x^{2}+1)(x^{4}+x^{3}+x^{2}+x+1) = 48370608$$

$$x^{6}(x+1)^{4}(x^{3}+x+1)(x^{3}+x^{2}+1)(x^{4}+x^{3}+1) = 18532800$$

We can see that all of the 11 nontrivial perfect polynomials have even values after plugging x = 2.

I am going to use the following lemma with out proving it.

Lemma 2.2.3. (See [6, Lemma 2.6] and [3, Theorem 8]) Let $n \in \mathbb{N}$. If any irreducible factor of $1 + x + \cdots + x^{2n}$ is of the form $x^a(x+1)^b + 1$, then $n \in \{1, 2, 3\}$.

Lemma 2.2.4. (See [6, Lemma 3.1]) Let $A = x^a (x+1)^b \prod_{i=1}^r P_i^{h_i}$ be an even perfect polynomial in $\mathbb{Z}_2[x]$, where each P_i is a Mersenne prime and $h_i = 2^{n_i} - 1$, $n_i \in \mathbb{N}$. Then:

- 1. if a is even, $a \in \{2, 4, 6\}$.
- 2. if *a* is odd *a* is of the form $2^{t}u 1$, where $u \in \{1, 3, 5, 7\}$.

Proof. 1. If A is perfect

$$A = \sigma(A),$$

hence A and $\sigma(A)$ have the same factor.

We know

$$\sigma(x^a) = 1 + x + \dots + x^a$$

And since x, (x + 1), P_i are all relatively prime we see that one of these must divide $\sigma(x^a)$.

However, nither x nor (x+1) divide $\sigma(x^a)$.

 \therefore One of the Mersenne polynomials $P_i | \sigma(x^a)$

By (Lemma 2.2.3) $a \in \{2, 4, 6\}$.

2. If a is odd, then a + 1 is even. Put $a + 1 = 2^t u$, with u odd and $t \ge 1$. We have

$$\sigma(x^{a}) = 1 + x + \dots + x^{a}$$
$$= (1+x)^{2^{t}-1}(1+x+\dots+x^{u-1})^{2^{t}}$$

If $u \ge 3$, then by part (a), u - 1 must be $\{2, 4, 6\}$

Lemma 2.2.5. ([6, Corollary 3.2]) If $A = x^a(x+1)^b \prod_{i=1}^{h_i} P_i^{h_i}$ is perfect, with each P_i Mersenne prime and $h_i \in 1$, 3, then *a* is even or *b* is even.

Proof. If a and b are both odd, then by (Lemma 2.2.4), $a = 2^t u - 1$, $b = 2^s v - 1$ for some $t, s \in \mathbb{N}$ and $u, v \in \{1, 3, 5, 7\}$. Hence

$$\sigma(x^{a}) = (x + 1)^{2^{t}-1}(1 + x + \dots + x^{u-1})^{2^{t}}$$
$$\sigma((x+1)^{b}) = x^{2^{s}-1}(1 + (x+1) + \dots + (x+1)^{v-1})^{2^{s}}$$

Furthermore, for any i, j, P_i does not divide $\sigma(P_j^{h_j}) = (1 + P_j)^{h_j}$.

If $u \ge 3$ and if some $P_i | 1 + x + ... + x^{u-1}$ and $P_i \not| 1 + (x+1) + ... + (x+1)^{v-1}$ then $2^t = h_i = 2^{n_i} - 1$, which is impossible.

If $P_i|1 + x + \ldots + x^{u-1}$ and $P_i|1 + (x+1) + \ldots + (x+1)^{v-1}$ then $2^t + 2^s = h_i = 2^{n_i} - 1$, which is also impossible.

So u = 1 and the same v = 1.

it follows that

$$\sigma(x^a) = (x + 1)^a$$
, $\sigma((x+1)^b) = x^b$.

Also a = b and $x^a(x+1)^b$ is perfect. This means that $\prod_{i=1}^r P_i^{h_i}$ is also perfect. This is a contradiction since $P_i^{h_i}$ is a Mersenne prime.

Proposition 2.2.2. ([6]) If A(x) is an even perfect polynomial over $\mathbb{Z}_2[x]$ such that the number of prime factors are 2, then $A = (x(x+1))^{2^n-1}$ for some $n \in \mathbb{N}$.

Proof. Let $A = (x(x+1))^{2^n-1}$ where $n \in \mathbb{N}$

$$\sigma(A) = \sigma((x(x+1))^{2^{n}-1})$$

= $\sigma(x^{2^{n}}) \cdot \sigma((x+1)^{2^{n}-1})$ since gcd $(x, x+1) = 1$.

Since $2^n - 1$ is odd, then by (lemma 2.2.5)

$$= (x+1)^{2^{n}-1}x^{2^{n}-1}$$
$$= A$$

	7
	- 1
	- 1
	- 1

Chapter 3: Pre-Kindergarten to Third Grade

This chapter contains activities aim to introduce pre-kindergarten to third grade students to the idea of perfect numbers. These activities address skills that the students are familiar with like counting, addition and the very basic division. Appendix A contains an unsolved version of these activities.

Our goal in designing these activities is to meet the common core standards.

3.1 Making Rectangles

This activity is designed to introduce students to prime numbers, separated into two parts. The first part is to have students count and build rectangles out of squares for composite numbers. The second part is to have students count and build rectangles out of squares for prime numbers. After working through both parts, the goal is to have students see that there are two types of numbers: those with only 1 representation, and those with more than 1. Thus, students gain an intuitive understanding of the difference between prime (one representation) and composite numbers (more than one representation). Given the following 12 squares:



${Part 1}$

This part is designed for pre-Kindergarten through first grade. Students in this stage are still learning how to count and are learning about basic shapes. The responses for the following questions might depend on students; some will consider a vertical line and horizontal line of the same length to be one representation, and others will consider it to be two. However, we should encourage students to consider these as the same rectangle so that the difference between prime and composite numbers will be clear after part 2.

Example 3.1.1. Take 4 squares. Arrange them into a rectangle. How many different rectangles can you make?

One rectangle:

Another rectangle:



Answer. We can make 2 rectangles.

1) Take 6 squares. Arrange them in to a rectangle. How many different rectangles can you make?

Answer. There are 3 rectangles.

2) Take 10 squares. Arrange them in to a rectangle. How many different rectangles can you make?

Answer. There are 3 rectangles.

 $\{ \text{ Part } 2 \}$

This part is more appropriate for students in first through third grade. Students at this level are able to apply more reasoning skills.

3) Take 2 squares. Try to put them in a rectangular form. How many different rectangles can you make?

Answer. There is only one representation.

4) Take 3 squares. Try to put them in a rectangular form. How many different rectangles can you make?

Answer. There is only 1 representation.

5) How many different rectangles can you make if you have 5 squares?

Answer. There is only 1 representation.

6) How many different rectangles can you make if you have 7 squares?

Answer. There is only 1 representation.

7) How many different rectangles can you make if you have 9 squares?

Answer. There are 2 representations.

8) How many different rectangles can you make if you have 11 squares?

Answer. There is 1 representation.

9) How many different rectangles can you make if you have 12 squares?

Answer. There are 3 representations.

10) Which numbers had only 1 representation?

Answer. $\{2, 3, 5, 7, 11\}.$

11) Why do these numbers have only 1 representation?

Answer. They only have two divisors; 1 and the number itself.

12) which numbers had more than one representation?

Answer. 4, 6, 8, 9, 10, 12. These numbers are called Composite numbers.

This activity adresses the following common core standers:

CCSS.MATH.CONTENT.K.CC.A.2

Count forward beginning from a given number within the known sequence (instead of having to begin at 1).

If we go back to our activity we can see that this is addressed when we ask students to count squares.

CCSS.MATH.CONTENT.K.CC.A.3

Write numbers from 0 to 20. Represent a number of objects with a written numeral 0 - 20 (with 0 representing a count of no objects).

When students connect counting to object, it is easier on them to see how many units are there.

CCSS.MATH.CONTENT.K.CC.B.4.B

Understand that the last number name said tells the number of objects counted. The number of objects is the same regardless of their arrangement or the order in which they were counted.

This is addressed when students are asked to count 5 squares. They start with one square, until they reach the number five they will stop taking squares. The last number counted (5) means that they have 5 total squares.

CCSS.MATH.CONTENT.K.CC.B.4.C

Understand that each successive number name refers to a quantity that is one larger.

This is addressed When we ask students first to take 2 squares, and then we ask them to take 3 squares for the next question. They already have 2, so they will take 1 more square only.

CCSS.MATH.CONTENT.K.CC.C.6

Identify whether the number of objects in one group is greater than, less than, or equal to the number of objects in another group. When students are asked to make rectangles they will trying to break the squares into equal groups. So if they are working with 5 squares and try to arrange them into two groups, they will always have a group with more square in it.

CCSS.MATH.CONTENT.K.OA.A.1

Represent addition and subtraction with objects, fingers, mental images, drawings, sounds (e.g., claps), acting out situations, verbal explanations, expressions, or equations.

When we ask students first to take 2 squares, then we ask them to take 4 squares on the following question, they already have 2 squares so they add 2 more squares to get 4 squares.

CCSS.MATH.CONTENT.K.OA.A.3

Decompose numbers less than or equal to 10 into pairs in more than one way, e.g., by using objects or drawings, and record each decomposition by a drawing or equation (e.g., 5 = 2 + 3 and 5 = 4 + 1).

This appears when we ask students: "Take 10 squares. Arrange them into a rectangle. How many different rectangles can you make?"

CCSS.MATH.CONTENT.K.G.B.6

Compose simple shapes to form larger shapes. For example, "Can you join these two triangles with full sides touching to make a rectangle?"

Approximately all the problems in this activity are asking to compose squares in order to make rectangles.

CCSS.MATH.CONTENT.1.OA.A.2

Solve word problems that call for addition of three whole numbers whose sum is

less than or equal to 20, e.g., by using objects, drawings, and equations with a symbol for the unknown number to represent the problem.

When students are asked to arrange 6 squares into a rectangle, they will make groups and add the squares in each group to make sure that they have 6 = 2 + 2 + 2.

CCSS.MATH.CONTENT.1.OA.B.3

Apply properties of operations as strategies to add and subtract 2. Examples: If 8 + 3 = 11 is known, then 3 + 8 = 11 is also known. (Commutative property of addition.) To add 2 + 6 + 4, the second two numbers can be added to make a ten, so 2 + 6 + 4 = 2 + 10 = 12. (Associative property of addition.)

Add 1's 6 times. Add 2's three times. Add 3's two times all of them are representations for 6.

CCSS.MATH.CONTENT.1.OA.C.5

Relate counting to addition and subtraction (e.g., by counting on 2 to add 2).

This is addressed in problems when students are asked to count a number of squares then are asked to count more squares.

CCSS.MATH.CONTENT.1.G.A.2

Compose two-dimensional shapes (rectangles, squares, trapezoids, triangles, halfcircles, and quarter-circles)

Help to find prime numbers. Prime numbers have only one representation which is a line of squares.

CCSS.MATH.CONTENT.3.OA.A.1

Interpret products of whole numbers, e.g., interpret 5×7 as the total number of

objects in 5 groups of 7 objects each. For example, describe a context in which a total number of objects can be expressed as 5×7 .

Making sense that 6 can be represented in 2 rows \times 3 columns, 3 rows \times 2 columns, 1 rows \times 6 columns, and 6 rows \times 1 columns

3.2 How many ways

This activity is designed to introduce students to the factors of a number using multiplication properties.

1) How many ways can you write the number 4 as a product of two numbers?

Answer. There are 3 ways.

$$1 \times 4 = 4$$
$$2 \times 2 = 4$$
$$4 \times 1 = 4$$

2) How many way can you write the number 6 as a product of two numbers?

Answer. There are 4 ways.

$$1 \times 6 = 6$$
$$2 \times 3 = 6$$
$$3 \times 2 = 6$$
$$6 \times 1 = 6$$

3) How many way can you write the number 7 as a product of two numbers?

Answer. There are 2 ways.

$$1 \times 7 = 7$$
$$7 \times 1 = 7$$

4) How many way can you write the number 8 as a product of two numbers?

Answer. There are 4 ways:

$$1 \times 8 = 8$$
$$2 \times 4 = 8$$
$$4 \times 2 = 8$$
$$8 \times 1 = 8$$

5) How many way can you write the number 9 as a product of two numbers?

Answer. There are 3 ways:

$$1 \times 9 = 9$$
$$3 \times 3 = 9$$
$$9 \times 1 = 9$$

6) How many way can you write the number 10 as a product of two numbers?

Answer. There are 4 ways:

$$1 \times 10 = 10$$
$$2 \times 5 = 10$$
$$5 \times 2 = 10$$
$$10 \times 1 = 10$$

7) How many ways can you write the number 12 as a product of two numbers?

Answer. There are 6 ways:

1 >	< 1	2	=	12
2	×	6	=	12
3	×	4	=	12
4	×	3	=	12
6	×	2	=	12
12	×	1	=	12

Now can you write down all the numbers in order, without repeating any, and when you multiply them together they give you the following:

Number	Ordered products
4	1, 2, 4
6	1, 2, 3, 6
7	1, 7
8	1, 2, 4, 8
9	1, 3, 9
10	1, 2, 5, 10
12	1, 2, 3, 4, 6, 12

8) Two factors of 20 are 1 and 20, because $1 \times 20 = 20$. Find four more factors of 20.

Answer. The other four factors are: $\{2, 4, 5, 10\}$

The common core standers that are addressed in the activity is:

CCSS.MATH.CONTENT.3.OA.A.1

Interpret products of whole numbers, e.g., interpret 5×7 as the total number of objects in 5 groups of 7 objects each. For example, describe a context in which a total number of objects can be expressed as 5×7 .

Making sense that 6 can be represented in 2 rows \times 3 columns, 3 rows \times 2 columns, 1 rows \times 6 columns, and 6 rows \times 1 columns

CCSS.MATH.CONTENT.3.OA.B.5

Apply properties of operations as strategies to multiply and divide. Examples: If $6 \times 4 = 24$ is known, then $4 \times 6 = 24$ is also known. (Commutative property of multiplication.) $3 \times 5 \times 2$ can be found by $3 \times 5 = 15$, then $15 \times 2 = 30$, or by $5 \times 2 = 10$, then $3 \times 10 = 30$. (Associative property of multiplication.) Knowing that $8 \times 5 = 40$ and $8 \times 2 = 16$, one can find 8×7 as $8 \times (5+2) =$ $(8 \times 5) + (8 \times 2) = 40 + 16 = 56$. (Distributive property.)

When we ask students to find factors of a number.

3.3 Pick a Number

This activity is designed for students to practice multiplication and division, aiming to find the divisors of any number, keeping in mind that multiplication is the inverse operation of division.

10

Consider the following:

$$1 \times 10 = \underline{10}$$
$$2 \times 5 = \underline{10}$$
$$5 \times 2 = \underline{10}$$
$$10 \times 1 = 10$$

$$10 \div 1 = \underline{10}$$
$$10 \div 2 = \underline{5}$$
$$10 \div 5 = \underline{2}$$
$$10 \div 10 = \underline{1}$$

Here the numbers 1, 2, 5 and 10 are called the divisors of the number 10.

1) List all the divisors of the number 6

Answer. The divisors are 1, 2, 3 and 6.

2) Can you give a number that has only two positive divisors?

Answer. 2, 3, 5, ... All prime numbers have only two divisors, which are one and the number itself.

3) Give me a number that has four positive divisors?

Answer. 6 has four divisors which are 1, 2, 3, and 6. 8 has four divisors which are 1, 2, 4, and 8

4) Can you find the number between 1 and 50 that has the greatest number of positive divisors?

Answer. 48 is the number between 1 and 50 that has the greatest number of divisors. The divisors of 48 are 1, 2, 3, 4, 6, 8, 12, 16, 24 and 48

5) Can you find two numbers that have the same positive divisors?

Answer. No, due to unique factorization.

6) Can you find two numbers that have the same number of positive divisors?

Answer. Yes 6 and 8 have the same number of divisors.

7) Can you find a number that is equal to the sum of its divisors?

Answer. No, we can not find a number that is equal to the sum of all divisors.

8) Can you find a number that is equal to half the sum of its divisors?

Answer. Yes. 6 and 28 both are equal to half the sum of their divisors.

9) Can you find a number that is greater than the sum of its divisors?

Answer. No. We can not find a number that is greater than the sum of all divisors, since we are adding the number to at least 1, which makes the sum of the divisors always greater than the number itself; n < n + 1.

CCSS.MATH.CONTENT.3.OA.B.6

Understand division as an unknown-factor problem. For example, find $32 \div 8$ by finding the number that makes 32 when multiplied by 8. Multiply and divide within 100.

All the exercises in this activity are asking to use multiplication and division to find the factors of any number less than 100.

CCSS.MATH.CONTENT.3.OA.C.7

Fluently multiply and divide within 100, using strategies such as the relationship between multiplication and division (e.g., knowing that $40 \div 5 = 8$) or properties of operations. By the end of Grade 3, know from memory all products of two one-digit numbers.

In this activity we are using properties of multiplication and division to find the factors of any number. For example, when we ask students to find the factors of 12, they should know that $12 \div 4 = 3$ and $2 \times 6 = 12$. So by using multiplication or division, students are able to find the divisors.

3.4 Sharing cookies

This activity is a brief overview of division. It covers the concept of sharing in equal amounts. Our goal is that students at the end will be able to find divisors and proper divisors. Another goal for this activity is to introduce students to the different categories of numbers depending on the sum of their divisors.

You have some cookies that you want to share with your friends.

1. You must offer each of your friends more than one whole cookie.

2. You must offer each of your friends an equal number of whole cookies.

3. None of the cookies can be cut into parts.

1) Suppose you have 3 cookies. How many different friends could you share these cookies with?

Answer. One friend. You can only give one friend because you can not give one cookie to a friend.

2) Suppose you have 4 cookies. How many different friends could you share these cookies with?

Answer. One or two friends. The proper divisors of 4 are 1 or 2 since $1 \times 4 = 4$ and $2 \times 2 = 4$

3) Suppose you have 5 cookies. How many different friends could you share these cookies with?

Answer. One friend only. The only way we can give each friend an equal amount of whole cookies with each receiving more than one cookie is to give all five cookies to one friend.

4) Suppose you have 6 cookies. How many different friends could you share these cookies with?

Answer. We can give cookies to one, two or three friends.

1 friend \times 6 cookies = 6 2 friends \times 3 cookies = 6 3 friends \times 2 cookies = 6

5) Suppose you have 7 cookies. How many different friends could you share these cookies with?

Answer. One friend for the same reason as question 3 (7 is a prime number).

6) Suppose you have 8 cookies. How many different friends could you share these cookies with?

Answer. One, two and four friends.

1 friend \times 8 cookies = 8 2 friends \times 4 cookies = 8 4 friends \times 2 cookies = 8

Let's record these facts in the table below.

Number of cookies	Number of friends to share with
2	1
3	1
4	1 or 2
5	1
6	1 or 2 or 3
7	1
8	1 or 2 or 4

For this table our goal is for students to apply what they learned in finding divisors and practice their skills in adding numbers.

Number of cookies	Add possible numbers of friends
6	1 + 2 + 3 = 6
8	1 + 2 + 4 = 7
9	1 + 3 = 4
10	1 + 2 + 5 = 8
12	1 + 2 + 3 + 4 + 6 = 16
15	1 + 3 + 5 = 9
16	1 + 2 + 4 + 8 = 15
20	1 + 2 + 4 + 5 + 10 = 22
22	1 + 2 + 11 = 14
25	1 + 5 = 6
28	1 + 2 + 4 + 7 + 14 = 28
30	1 + 2 + 3 + 5 + 6 + 10 + 15 = 42

Now try to complete the following table.

7) Sally noticed if you add the proper divisors they are always less than the number itself, while Kim noticed that value is always larger. Can you help me to figure out which one of them is right?

Answer. Neither are correct. As we can see from our table, for some numbers of cookies if we add all possible friends, we may have the same number of cookies 6 = 1 + 2 + 3. For some other numbers, the sum of the divisors is less than the number: for example, the sum of the divisors of 9 is 4. For other numbers, the sum of the divisors is greater than the number: for example, the sum of the divisors of 12 is 16

8) Does the sum of all possible numbers of friends ever equal the number of cookies?

Answer. Yes. look at 6 = 1 + 2 + 3 also 28 = 1 + 2 + 4 + 7 + 14. We call these numbers "Perfect Numbers"

The common core standers that are addressed in this activity:

CCSS.MATH.CONTENT.1.OA.B.3

Apply properties of operations as strategies to add and subtract 2. Examples: If 8 + 3 = 11 is known, then 3 + 8 = 11 is also known. (Commutative property of addition.) To add 2 + 6 + 4, the second two numbers can be added to make a ten, so 2 + 6 + 4 = 2 + 10 = 12. (Associative property of addition.)

Students are asked to add more than two numbers, so they use associative property.

CCSS.MATH.CONTENT.1.OA.A.2

Solve word problems that call for addition of three whole numbers whose sum is less than or equal to 20, e.g., by using objects, drawings, and equations with a symbol for the unknown number to represent the problem.

CCSS.MATH.CONTENT.2.OA.B.2

Fluently add and subtract within 20 using mental strategies. By end of Grade 2, know from memory all sums of two one-digit numbers.

In all the exercises, students are asked to add divisors.

CCSS.MATH.CONTENT.3.OA.A.2

Interpret whole-number quotients of whole numbers, e.g., interpret $56 \div 8$ as the number of objects in each share when 56 objects are partitioned equally into 8 shares, or as a number of shares when 56 objects are partitioned into equal shares of 8 objects each. For example, describe a context in which a number of shares or a number of groups can be expressed as $56 \div 8$.

Here we have cookies and we asked students to partition them among their friends under some constraints.

Chapter 4: Fourth to Ninth Grade

This chapter contains activities for students from fourth through ninth graders. Answers for these activities are also provided in this chapter. These activities introduce students to perfect numbers.

Appendix B contains an unsolved version of these activities.

4.1 Prime Numbers

The goal of this activity is to introduce students to prime and composite numbers. Students needs to know prime numbers to be able to apply Euclid's formula.

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	50
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

Finding Primes. Look at the following table

Given that 2 is a prime number, cross out every other number.

1) Is 3 crossed out or not? If 3 is crossed out then move to the next number. If 3 is not crossed out, 3 is a prime, so circle it and cross out every third number.

Answer. Yes 3 is a prime.

2) Is 4 crossed out or not? If 4 is crossed out, then move to the next number. If 4 is not crossed out, 4 is a prime, so circle it and cross out every fourth number.

Answer. 4 is already crossed out, so it is not prime.

3) Repeat the process until you get all the prime numbers between 1 and 100. List all of the prime numbers found between 1 and 100. How many are there?

Answer. After this process you will get the set of all 25 prime numbers between 1 and 100, which is:

 $\{2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, \dots$ $47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97\}$

This activity addresses the following common core standards:

CCSS.MATH.CONTENT.4.OA.B.4

Find all factor pairs for a whole number in the range 1 - 100. Recognize that a whole number is a multiple of each of its factors. Determine whether a given whole number in the range 1 - 100 is a multiple of a given one-digit number. Determine whether a given whole number in the range 1 - 100 is prime or composite.

When students are crossing out the numbers in this activity, they are trying to factor each number and ask themselves does it have more than two factors or not? Is it composite or prime?

4.2 Divisors

The goal of this activity is for students to practice properties of multiplication and division to be able to find the divisors of a number. This helps students understand the idea of the sigma function. Also, the activity introduces students to different categories of numbers: perfect, deficient, and abundant.

Number	Divisors
4	1, 2, 4
6	
8	
9	
12	
15	
17	
20	
22	
25	
28	
32	
100	
202	
236	
496	

1) Find all the divisors of the following numbers:

	Number	Divisors
	4	1, 2, 4
	6	1, 2, 3, 6
	8	1, 2, 4, 8
	9	1, 2, 3, 9
	12	1, 2, 3, 4, 6, 12
	15	1, 3, 5, 15
	17	1, 17
Answer.	20	1, 2, 4, 5, 10, 20
	22	1, 2, 11, 22
	25	1, 5, 25
	28	1, 2, 4, 7, 14, 28
	32	1, 2, 4, 8, 16, 32
	100	1, 2, 4, 5, 10, 20, 25, 50, 100
	202	1, 2, 101, 202
	236	1, 2, 4, 59, 118, 236
	496	1, 2, 4, 8, 16, 31, 62, 124, 248, 496

The *proper divisors* of a number are all the divisors except the number itself.

2) Add the proper divisors for each number in the previous exercise.

Number	Sum of Proper Divisors
4	1 + 2 = 3
6	
8	
9	
12	
15	
17	
20	
22	
25	
28	
32	
100	
202	
236	
496	

	Number	Divisors
	4	1 + 2 = 3
	6	1 + 2 + 3 = 6
	8	1 + 2 + 4 = 7
	9	1 + 3 = 4
	12	1 + 2 + 3 + 4 + 6 = 16
	15	1 + 3 + 5 = 9
	17	1 = 1
Answer.	20	1 + 2 + 4 + 5 + 10 + 20 = 42
	22	1 + 2 + 11 = 14
	25	1 + 5 = 6
	28	1 + 2 + 4 + 7 + 14 = 28
	32	1 + 2 + 4 + 8 + 16 = 31
	100	1 + 2 + 4 + 5 + 10 + 20 + 25 + 50 = 117
	202	1 + 2 + 101 = 104
	236	1 + 2 + 4 + 59 + 118 = 184
	496	1 + 2 + 4 + 8 + 16 + 31 + 62 + 124 + 248 = 496

3) Which numbers have the sum of their proper divisors less than the number itself?

Answer.

 $\{4, 8, 9, 15, 17, 22, 25, 32, 202, 236\}$

4) Which numbers have the sum of their proper divisors equal to the number itself?

Answer.

 $\{6, 28, 496\}$

5) Which numbers have the sum of their proper divisors greater than the number itself?

Answer.

 $\{12, 20, 100\}$

This activity addresses the following common core standards:

CCSS.MATH.CONTENT.4.OA.A.1

Interpret a multiplication equation as a comparison, e.g., interpret $35 = 5 \times 7$ as a statement that 35 is 5 times as many as 7 and 7 times as many as 5. Represent verbal statements of multiplicative comparisons as multiplication equations.

This will appear when students are trying to find the divisors of any number.

CCSS.MATH.CONTENT.4.OA.C.5

Generate a number or shape pattern that follows a given rule. Identify apparent features of the pattern that were not explicit in the rule itself. For example, given the rule "Add 3" and the starting number 1, generate terms in the resulting sequence and observe that the terms appear to alternate between odd and even numbers. Explain informally why the numbers will continue to alternate in this way.

This appears when students find the sum of proper divisors.

CCSS.MATH.CONTENT.4.NBT.B.5

Multiply a whole number of up to four digits by a one-digit whole number, and multiply two two-digit numbers, using strategies based on place value and the properties of operations. Illustrate and explain the calculation by using equations, rectangular arrays, and/or area models.

This appears when students are asked to find the divisors of large numbers like 202.

CCSS.MATH.CONTENT.5.NBT.B.6

Find whole-number quotients of whole numbers with up to four-digit dividends

and two-digit divisors, using strategies based on place value, the properties of operations, and/or the relationship between multiplication and division. Illustrate and explain the calculation by using equations, rectangular arrays, and/or area models.

We use division to find the divisors of numbers; for example find the divisors of 242.

CCSS.MATH.CONTENT.6.NS.B.2

Fluently divide multi-digit numbers using the standard algorithm.

Using this strategies to find the divisors of large numbers.

CCSS.MATH.CONTENT.6.EE.A.2.B

Identify parts of an expression using mathematical terms (sum, term, product, factor, quotient, coefficient); view one or more parts of an expression as a single entity. For example, describe the expression 2(8+7) as a product of two factors; view (8+7) as both a single entity and a sum of two terms.

Using multiplication to find factors.

4.3 The sum of the reciprocal divisors of a number

1) If d is a divisor of a number, find the sum of the reciprocal of all divisors of the following numbers:

1. n = 2.

Answer.

$$d = 1, 2.$$

 $\sum (1/d) = 1 + \frac{1}{2} = \frac{3}{2}$

2. n = 3.

Answer.

$$d = 1, 3.$$

 $\sum (1/d) = 1 + \frac{1}{3} = \frac{4}{3} = 1\frac{1}{3}$

3. n = 4.

Answer.

$$d = 1, 2, 4.$$

$$\sum (1/d) = 1 + \frac{1}{2} + \frac{1}{4} = \frac{7}{4} = 1\frac{3}{4}$$

4. n = 6.

Answer.

$$d = 1, 2, 3, 6.$$

$$\sum (1/d) = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{6} = \frac{12}{6} = 2$$

5. n = 8.

Answer.

$$d = 1, 2, 4, 8.$$
$$\sum (1/d) = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{15}{8} = 1\frac{7}{8}$$

6. n = 28.

Answer.

$$d = 1, 2, 4, 7, 14, 28.$$
$$\sum (1/d) = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{7} + \frac{1}{14} + \frac{1}{28} = \frac{56}{28} = 2$$

7. n = 496.

Answer.

$$d = 1, 2, 4, 8, 16, 31, 62, 124, 248, 496.$$
$$\sum(1/d) = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{31} + \frac{1}{62} + \frac{1}{124} + \frac{1}{248} + \frac{1}{496} = \frac{992}{496} = 2$$

2) If you know that 6, 28 and 496 are perfect numbers, can you see a relation between $\sum (1/d)$ and a perfect number?

Answer. Yes, there is a relation $\sum (1/d)$ for a perfect number always equals 2

3) What is the relation between the sum of the divisors of n and the sum of the reciprocal of the divisors of n, if n is a perfect number?

Answer. Yes there is a relation. Look at

$$\sigma(n) = 2n$$

$$\sum (1/d) = 2$$

so if we multiply $\sum (1/d)$ by n we get that

$$\sigma(n) = n \sum (1/d)$$

This activity addresses the following common core standards:

CCSS.MATH.CONTENT.5.OA.A.2

Write simple expressions that record calculations with numbers, and interpret numerical expressions without evaluating them. For example, express the calculation "add 8 and 7, then multiply by "2" as $2 \times (8 + 7)$. Recognize that $3 \times (18932 + 921)$ is three times as large as 18932 + 921, without having to calculate the indicated sum or product.

CCSS.MATH.CONTENT.5.NF.A.1

Add and subtract fractions with unlike denominators (including mixed numbers) by replacing given fractions with equivalent fractions in such a way as to produce an equivalent sum or difference of fractions with like denominators. For example, 2/3 + 5/4 = 8/12 + 15/12 = 23/12. (In general, a/b + c/d = (ad + bc)/bd.)

Practice finding the greatest common divisor in order to make the denominators the same for all fractions in order to add them.

CCSS.MATH.CONTENT.6.EE.A.2.B

Identify parts of an expression using mathematical terms (sum, term, product, factor, quotient, coefficient); view one or more parts of an expression as a single entity. For example, describe the expression 2(8+7) as a product of two factors; view (8+7) as both a single entity and a sum of two terms.

Be able to use the function $\sum \frac{1}{d}$ fluently.

CCSS.MATH.CONTENT.7.NS.A.1

Apply and extend previous understandings of addition and subtraction to add and subtract rational numbers; represent addition and subtraction on a horizontal or vertical number line diagram.

This is addressed when students add the reciprocal of the divisors of a number; they are adding rational numbers.

4.4 Number of Factors

Jessica is a clever student. The teacher asked her to find the number of divisors of the number $2^3 \times 3^2 \times 7$. So she started as following:



Consider the diagram she made:

1) Find the number of divisors of the number $2^3 \times 3^2 \times 7$.

Answer. By counting the divisors from the diagram, the number of divisors is 24.

2) Can you write the divisors of the number $2^3 \times 3^2 \times 7$?

Answer.

3) Can you help Jessica find an easier way to find the number of divisors?

Answer. The number of divisors $= \prod_{i=1}^{n} (n_i + 1)$ where n_i is the power of the *i*th term.

The number of divisors of $2^3 \times 3^2 \times 7 = (3+1) \times (2+1) \times (1+1) = 24$.
4) Can you make a similar diagram for 48 and find the number of divisors?

Answer.



The number of divisors is $(4+1) \times (1+1) = 10$.

5) Find the number of divisors of 240.

Answer.

$$240 = 2^4 \times 3 \times 5$$

the number of divisors $= 5 \times 2 \times 2 = 20$.

6) If p is a prime number, find the number of divisors of p^n .

Answer. The number of divisors is (n + 1).

7) If p is a prime. Find the number of divisors of $2p^5$.

Answer. The number of divisors $2 \times 6 = 12$.

This activity addresses the following common core standards:

CCSS.MATH.CONTENT.6.NS.B.4

Find the greatest common factor of two whole numbers less than or equal to 100 and the least common multiple of two whole numbers less than or equal to 12. Use the distributive property to express a sum of two whole numbers 1-100 with a common factor as a multiple of a sum of two whole numbers with no common factor. For example, express 36 + 8 as 4(9 + 2). Try to follow the tree in order to find all factors.

CCSS.MATH.CONTENT.8.EE.A.1

Know and apply the properties of integer exponents to generate equivalent numerical expressions. For example, $3^2 \times 3^{-5} = 3^{-3} = 1/33 = 1/27$.

Being able to work with expressions like 2^n and know how to find divisors of numbers written in the following expression form $2^3 \times 3^2 \times 7$.

CCSS.MATH.CONTENT.HSA.SSE.A.1.A

Interpret parts of an expression, such as terms, factors, and coefficients.

Find the factors of any number.

CCSS.MATH.CONTENT.HSA.SSE.A.1.B

Interpret complicated expressions by viewing one or more of their parts as a single entity. For example, interpret $P(1+r)^n$ as the product of P and a factor not depending on P.

Being able to find all the factors and find an expression or a rule that allows students to find the numbers of factors.

Chapter 5: Seventh to Twelfth Grade

This chapter contains activities to introduce student seventh grade through high school to perfect numbers. The answers and the common core standards are also included.

Appendix C contains an unsolved version of these activities.

5.1 (" σ " Function)

The goal of this activity is for students to practice factoring, using functions and trying to prove some small statements like sigma function being multiplicative.

1) If $\sigma(6) = 1 + 2 + 3 + 6 = 12$. Find the following:

- 1. $\sigma(2) = 1 + 2 = 3$.
- 2. $\sigma(3) = 1 + 3 = 4$.
- 3. $\sigma(4) = 1 + 2 + 4 = 7$.
- 4. $\sigma(5) = 1 + 5 = 6.$
- 5. $\sigma(8) = 1 + 2 + 4 + 8 = 15.$
- 6. $\sigma(9) = 1 + 3 + 9 = 13.$

- 7. $\sigma(10) = 1 + 2 + 5 + 10 = 18.$
- 8. $\sigma(12) = 1 + 2 + 3 + 4 + 6 + 12 = 28.$
- 9. $\sigma(20) = 1 + 2 + 4 + 5 + 10 + 20 = 42$.
- 10. $\sigma(25) = 1 + 5 + 25 = 31.$
- 11. $\sigma(28) = 1 + 2 + 4 + 7 + 14 + 28 = 56$
- **2)** Look a $\sigma(10)$ does it equal $\sigma(2) \times \sigma(5)$?

Answer. They are equal

$$\sigma(10) = 18$$
$$\sigma(2) \cdot \sigma(5) = 3 \times 6$$
$$= 18$$

so they are equal.

3) Does $\sigma(12)$ equals $\sigma(3) \times \sigma(4)$?

Answer. Lets see

$$\sigma(12) = 28$$
$$\sigma(3) \times \sigma(4) = 4 \times 7$$
$$= 28$$

so $\sigma(12) = \sigma(3) \times \sigma(4)$ are equal.

4) What about $\sigma(20)$ does it equal $\sigma(4) \times \sigma(5)$?

Answer. Let's see

$$\sigma(20) = 42$$
$$\sigma(4) \times \sigma(5) = 7 \times 6$$
$$= 42$$

so they are equal.

5) If m, n are two different integers can we say that $\sigma(m.n) = \sigma(m) \times \sigma(n)$?

Answer. If gcd(m, n) = 1, then $\sigma(m \cdot n) = \sigma(m) \times \sigma(n)$. (Theorem 2.1.3)

6) If your answer was yes look at $\sigma(4)$ does it equal $\sigma(2) \times \sigma(2)$

Answer. Since $gcd(2,2) \neq 1$, then $\sigma(4) = 7 \neq \sigma(2) \times \sigma(2) = 9$.

7) If the gcd (a, b) = 1 and $\sigma(a) = 5$, $\sigma(b) = 9$. Find $\sigma(a \cdot b)$.

Answer. since gcd(a, b) = 1 them

$$\sigma(a \cdot b) = \sigma(a) \times \sigma(b)$$
$$= 5 \times 9$$
$$= 45$$

This activity addresses the following common core standards:

CCSS.MATH.CONTENT.7.EE.A.1

Apply properties of operations as strategies to add, subtract, factor, and expand linear expressions

Being able to apply the sigma function.

CCSS.MATH.CONTENT.8.EE.A.1

Know and apply the properties of integer exponents to generate equivalent numerical expressions. For example, $3^2 \times 3^{-5} = 3^{-3} = 1/33 = 1/27$.

Being able to apply sigma function and find some of its properties.

CCSS.MATH.CONTENT.HSF.BF.A.1.A

Determine an explicit expression, a recursive process, or steps for calculation from a context.

All the problems in this activity apply this standard in order to find the value of $\sigma(n)$.

CCSS.MATH.PRACTICE.MP1

Make sense of problems and persevere in solving them.

Making sense when proving that the sigma function is multiplicative.

5.2 Perfect Numbers Formula

The following activity helps students walk through the proof of Euclid's formula: $2^{n-1}(2^n - 1)$.

The following are perfect numbers:

$$6 = 1 + 2 + 3 = 2^{0} + 2^{1} + 3$$

$$28 = 1 + 2 + 4 + 7 + 14$$

$$= 2^{0} + 2^{1} + 2^{2} + 7 + 7 \times 2$$

$$496 = 1 + 2 + 4 + 8 + 16 + 31 + 62 + 124 + 248$$

$$= 2^{0} + 2^{1} + 2^{2} + 2^{3} + 2^{4} + 31 + 31 \times 2 + 31 \times 2^{2} + 31 \times 2^{3}$$

1) Do you see any pattern?

Answer. There is a pattern which is adding powers of 2's until you reach a prime number.

	Sum	Prime	Calculation	Perfect Number
1 + 2	3	Prime	2×3	6
1 + 2 + 4				
1+2+4+8	15	not a Prime	-	-

2) Complete the following table

Answer.

	Sum	Prime	Calculation	Perfect Number
1 + 2	3	Prime	2×3	6
1 + 2 + 4	7	Prime	4×7	28
1 + 2 + 4 + 8	15	not a Prime	-	-
1+2+4+8+16	31	Prime	31×16	496
1+2+4+8+16+32	63	not Prime	-	-
1+2+4+8+16+32+64	127	Prime	127×128	16256
1 + 2 + 4 + 8 + 16 + 32 + 64 + 128	255	not Prime	-	-

While you were adding up the powers of 2, did you observe any pattern in the sequence of sums obtained? It may be more obvious if we express each number in terms of powers of 2, as we shall see in the follow table.

3) Complete the table.

Series	Sum	Prime
$1+2^1$	$2^2 - 1$	prime
$1+2^1+2^2$		
$1+2^1+2^2+2^3$		not prime
$1 + 2^1 + 2^2 + 2^3 + 2^4$		
$1 + 2^1 + 2^2 + 2^3 + 2^4 + 2^5$		
$1 + 2^1 + 2^2 + 2^3 + 2^4 + 2^5 + 2^6$		
$1 + 2^1 + 2^2 + 2^3 + 2^4 + 2^5 + 2^6 + 2^7$		

Answer.

Series	Sum	Prime
$1+2^1$	$2^2 - 1$	prime
$1 + 2^1 + 2^2$	$2^3 - 1$	prime
$1 + 2^1 + 2^2 + 2^3$	$2^4 - 1$	not prime
$1 + 2^1 + 2^2 + 2^3 + 2^4$	$2^5 - 1$	prime
$1 + 2^1 + 2^2 + 2^3 + 2^4 + 2^5$	$2^6 - 1$	not a prime
$1 + 2^1 + 2^2 + 2^3 + 2^4 + 2^5 + 2^6$	$2^7 - 1$	prime
$1 + 2^1 + 2^2 + 2^3 + 2^4 + 2^5 + 2^6 + 2^7$	$2^8 - 1$	not a prime

4) Use the table you just completed to compute the following sums:

$$1 + 2^{1} + 2^{2} + 2^{3} + 2^{4} + \dots + 2^{8}$$
$$1 + 2^{1} + 2^{2} + 2^{3} + 2^{4} + \dots + 2^{8} + 2^{9}$$
$$1 + 2^{1} + 2^{2} + 2^{3} + 2^{4} + \dots + 2^{9} + 2^{10}$$
$$1 + 2^{1} + 2^{2} + 2^{3} + 2^{4} + \dots + 2^{10} + 2^{11}$$

Answer.

$$1 + 2^{1} + 2^{2} + 2^{3} + 2^{4} + \dots + 2^{8} = 2^{9} - 1$$

$$1 + 2^{1} + 2^{2} + 2^{3} + 2^{4} + \dots + 2^{8} + 2^{9} = 2^{10} - 1$$

$$1 + 2^{1} + 2^{2} + 2^{3} + 2^{4} + \dots + 2^{9} + 2^{10} = 2^{11} - 1$$

$$1 + 2^{1} + 2^{2} + 2^{3} + 2^{4} + \dots + 2^{10} + 2^{11} = 2^{12} - 1$$

5) Can you find a formula for finding the sums of 2's?

Answer. The formula is $2^n - 1$.

6) Can you find a relation between n and the formula you found?

Answer. If n is not a prime, then $(2^n - 1)$ is not a prime.

7) What values of n make the sum of 2's a prime number?

Answer. Prime values

This activity addresses the following common core standards:

CCSS.MATH.CONTENT.7.EE.B.4.A

Solve word problems leading to equations of the form px+q = r and p(x+q) = r, where p, q, and r are specific rational numbers. Solve equations of these forms fluently.

CCSS.MATH.CONTENT.8.EE.A.1

Know and apply the properties of integer exponents to generate equivalent numerical expressions. For example, $3^2 \times 3^{-5} = 3^{-3} = 1/33 = 1/27$.

Students need to know how to find sums of double proportions $2^0 + 2^1 + 2^2 = 1 + 2 + 4 = 7$

CCSS.MATH.CONTENT.HSA.SSE.A.1.B

Interpret complicated expressions by viewing one or more of their parts as a single entity. For example, interpret $P(1+r)^n$ as the product of P and a factor not depending on P.

To know how to find the divisors of $2^{p-1}q$ given that q is a prime.

CCSS.MATH.CONTENT.HSA.SSE.A.1.A

Interpret parts of an expression, such as terms, factors, and coefficients.

To understand what is a divisor, and to be able to find divisors of different numbers and expressions.

CCSS.MATH.CONTENT.HSA.SSE.B.3.C

Use the properties of exponents to transform expressions for exponential functions.

To show that some observations are true. For example, if n is even then $2^n - 1$ is not prime.

let
$$n = m \cdot k$$
 then $2^n = (2^m)^k$ let $n = m \cdot k$ then $2^n = (2^m)^k$

$$2^{n} - 1 = (2^{m})^{k} - 1$$
$$= [(2^{m} - 1)][(2^{m})^{k-1} + (2^{m})^{k-2} + \dots + (2^{m})^{2} + (2^{m}) + 1] \text{ geometric series}$$

CCSS.MATH.CONTENT.HSA.SSE.B.4

Derive the formula for the sum of a finite geometric series (when the common ratio is not 1), and use the formula to solve problems.

It is always true that $a^n - 1$ is divisible by (a - 1), and it is proved by using geometric series.

$$1 + a + a^{2} + a^{3} + a^{4} + \dots + a^{n-1} = \frac{a^{n} - 1}{a - 1}$$
$$(1 + a + a^{2} + a^{3} + a^{4} + \dots + a^{n-1})(a - 1) = a^{n} - 1$$

5.3 Find Perfect Numbers

The goal is to walk students through the proof of Euclid's formula, and find perfect numbers.

Series	Sum	Sum × last 2^{n-1} in the series
$1+2^{1}$	$2^2 - 1$	$3 \times 2^1 = 6$
$1 + 2^1 + 2^2$		
$1 + 2^1 + 2^2 + 2^3$		
$1 + 2^1 + 2^2 + 2^3 + 2^4$		
$1 + 2^1 + 2^2 + 2^3 + 2^4 + 2^5$		
$1 + 2^1 + 2^2 + 2^3 + 2^4 + 2^5 + 2^6$		
$1 + 2^1 + 2^2 + 2^3 + 2^4 + 2^5 + 2^6 + 2^7$		

Complete the following table

1) Find a pattern.

Answer. Since we are multiplying the sum, which we figured out in the previous activity is $2^n - 1$ by the last in the series, which is 2^{n-1} . Then the pattern is $2^{n-1}n(2^n - 1)$

2) Find perfect numbers.

Answer. When

$$n = 2 \Rightarrow 2^{2-1}(2^2 - 1) = 6$$

 $n = 3 \Rightarrow 2^{3-1}(2^3 - 1) = 28$

both 6 and 28 are perfect numbers.

3) Can you give a formula to find perfect numbers?

Answer. To get a perfect number, n should be prime.

4) Are all perfect numbers even, odd, or both?

Answer. All the perfect numbers that we found are even.

This activity addresses the following common core standards:

CCSS.MATH.CONTENT.HSA.SSE.A.1

Interpret expressions that represent a quantity in terms of its context

CCSS.MATH.CONTENT.HSA.SSE.A.1.B

Interpret complicated expressions by viewing one or more of their parts as a single entity. For example, interpret $P(1+r)^n$ as the product of P and a factor not depending on P.

To know how to find the divisors of $2^{p-1}q$ given that q is a prime.

CCSS.MATH.CONTENT.HSA.SSE.A.2

Use the structure of an expression to identify ways to rewrite it. For example, see $x^4 - y^4$ as $(x^2)^2 - (y^2)^2$, thus recognizing it as a difference of squares that can be factored as $(x^2 - y^2)(x^2 + y^2)$.

To be familiar with ways to factor a polynomial and find its divisors.

CCSS . <code>MATH</code> . <code>CONTENT</code> . <code>HSA</code> . <code>APR</code> . <code>C</code> . <code>5</code>

Know and apply the Binomial Theorem for the expansion of $(x+y)^n$ in powers of x and y for a positive integer n, where x and y are any numbers, with coefficients determined for example by Pascal's Triangle.1

Students should be able to use Binomial Theorem to prove some parts of theorems like:

"If $(2^n - 1)$ is prime for some integer $n \ge 2$, then n is prime."

Chapter 6: College

This chapter contains activities to introduce college students to perfect numbers and polynomials. These activities are easy to follow. They are provided with a key answer and followed by some standards that we wish a student to accomplish by going through these activities. Appendix 4 of this paper contains an unsolved version of these activities.

6.1 Even Perfect Numbers

The goal of this activity is to walk students through the proof of Euclid's formula for perfect numbers.

Consider:

$$2^{0} = \underline{1} = \underline{2^{1} - 1}$$

$$2^{0} + 2^{1} = \underline{3} = \underline{2^{2} - 1}$$

$$2^{0} + 2^{1} + 2^{2} = \underline{7} = \underline{2^{3} - 1}$$

$$1 + 2^{1} + 2^{2} + 2^{3} = \underline{15} = \underline{2^{4} - 1}$$

$$1 + 2^{1} + 2^{2} + 2^{3} + 2^{4} = \underline{31} = \underline{2^{5} - 1}$$

$$1 + 2^{1} + 2^{2} + 2^{3} + 2^{4} + 2^{5} = \underline{63} = \underline{2^{6} - 1}$$

$$1 + 2^{1} + 2^{2} + 2^{3} + 2^{4} + 2^{5} = \underline{63} = \underline{2^{6} - 1}$$

$$1 + 2^{1} + 2^{2} + 2^{3} + 2^{4} + 2^{5} + 2^{6} = \underline{127} = \underline{2^{7} - 1}$$

Use the equations above to answer the following questions:

1) When is $2^n - 1$ a prime?

Answer. $2^{n} - 1$ is prime when n = 2, 3, 5, 7.

2) When $2^n - 1$ is a prime number, we call it Mersenne prime. Can you give me the first 3 Mersenne primes?

Answer. The First three Mersenne primes are 3, 7, 31.

3) When is $2^n - 1$ is composite?

Answer. when n = 4, 6.

4) Look at the following table:

 $(n \text{ verses } 2^n - 1)$

1	2	3	4	5	6	7	8	9	10	11	•••	n
1	3	$\overline{7}$	15	31	63	127	255	511	1,023	2,047	•••	$2^{n} - 1$

5) Write 28 in the form $2^{n-1}(2^n - 1)$. Then write all the factors of 28.

Answer.

$$28 = 2^2(2^3 - 1)$$

The factors of 28 are

1, 2,
$$2^2$$
, $(2^3 - 1)$, $2(2^3 - 1)$, $2^2(2^3 - 1)$.

6) Write 496 in the form $2^{n-1}(2^n - 1)$. Then write all the factors of 496.

Answer.

$$496 = 2^4(2^5 - 1)$$

The factors are

1, 2,
$$2^2$$
, 2^3 , 2^4 , $(2^5 - 1)$, $2(2^5 - 1)$, $2^2(2^5 - 1)$, $2^3(2^5 - 1)$, $2^4(2^5 - 1)$.

7) Write 8128 in the form $2^{n-1}(2^n - 1)$. Then write all the factors of 8128.

Answer.

$$8128 = 2^6(2^7 - 1)$$

The factors are

1, 2,
$$2^2$$
, 2^3 , 2^4 , 2^5 , 2^6 , $(2^7 - 1)$, $2(2^7 - 1)$, $2^2(2^7 - 1)$, ...
 $2^3(2^7 - 1)$, $2^4(2^6 - 1)$, $2^5(2^7 - 1)$, $2^6(2^7 - 1)$

8) Each Mersenne prime give rise to a perfect number. Can you show how? And can you give the three perfect numbers associated with the first three Mersenne primes?

Answer. After looking at the table and the exercises, we can see that a perfect number is given by $2^{n-1}(2^n - 1)$ where $(2^n - 1)$ is a Mersenne prime. So the first 3 perfect numbers are 6, 28, 496.

9) Let p have the form $2^{n-1}(2^n - 1)$ where $(2^n - 1)$ is a prime. Write all the factors of p.

Answer. The factors are:

1, 2,
$$2^2$$
, ..., $(2^n - 1)$, $2(2^n - 1)$, ..., $2^{n-1}(2^n - 1)$

10) Add the factors of p.

Answer.

$$(1+2+2^2+2^3+\ldots+2^{n-1})+(2^n-1)(1+2+2^2+2^3+\ldots+2^{n-2})$$
$$(1+2+2^2+2^3+\ldots+2^{n-1})(1+2^n-1)=2^n(1+2+2^2+2^3+\ldots+2^{n-1})$$
$$=2.2^{n-1}(1+2+2^2+2^3+\ldots+2^{n-1})=2p$$

11) Since the sum of the divisors of p equals 2p, what do we call p?

Answer. p is called a perfect number. This means any number with the form $2^{n-1}(2^n - 1)$ where $(2^n - 1)$ is prime is called a perfect number.

These activities address the following standards:

CCSS.MATH.PRACTICE.MP1

Make sense of problems and persevere in solving them.

CCSS.MATH.PRACTICE.MP2

Reason abstractly and quantitatively.

CCSS.MATH.PRACTICE.MP3

Construct viable arguments and critique the reasoning of others.

CCSS.MATH.PRACTICE.MP5

Use appropriate tools strategically.

CCSS.MATH.PRACTICE.MP6

Attend to precision.

CCSS.MATH.PRACTICE.MP7

Look for and make use of structure.

CCSS.MATH.PRACTICE.MP8

Look for and express regularity in repeated reasoning.

6.2 Perfect polynomials (mod 2)

1) Find all real zeros of the polynomial, and write the polynomial in factored form:

1. $x^2 + x$ Answer. $x^2 + x = x(x+1)$

The zeros are 0, 1.

2. $x^4 + x^2$

Answer. $x^4 + x^2 = x^2(x^2 + 1)$

The is only one real zero which is 0 with multiplicity 2.

3. $x^4(x+1)^4$

Answer. $x^4(x+1)^4 = x \cdot x \cdot x \cdot x(x+1)(x+1)(x+1)(x+1)$

There are two zeros, 0 and 1, with multiplicity 4 for each zero.

2) If you know that $\sigma(A(x))$ is the sum of all the factors of the polynomial A(x), what is $\sigma(A(x))$ of the previous polynomials if we are summing mod 2, or $\mathbb{Z}_2[x]$.

Answer. 1.

$$\sigma(x^2 + x) = \sigma((x)(x+1)) \pmod{2}$$

Since gcd((x), (x+1)) = 1

$$= \sigma(x) \cdot \sigma(x+1) \pmod{2}$$

= $(x+1)((x+1)+1) \pmod{2}$
= $(x+1)(x+2) \pmod{2} \qquad (x+2) \equiv x \pmod{2}$
 $\equiv (x+1)x \pmod{2}$

$$\sigma(x^4 + x^2) = \sigma((x^2)(x^2 + 1)) \pmod{2}$$

Since $gcd((x^2), (x^2 + 1)) = 1$

$$= \sigma(x^2) \cdot \sigma(x^2 + 1) \pmod{2}$$
$$= ((x^2) + 1)((x^2 + 1) + 1) \pmod{2}$$
$$= (x^2 + 1)(x^2) \pmod{2}$$

3.

 $\sigma(x^4(x+1)^4)$

Since gcd((x), (x+1)) = 1,

$$= \sigma(x^4) \cdot \sigma((x+1)^4) \pmod{2}$$

= $(x+1)^4(x)^4 \pmod{2}$

3) If a Mersenne prime polynomial is a polynomial of the form $x^a(x+1)^b + 1$ that is irreducible, can you give a polynomial that is a Mersenne prime polynomial?

Answer. Some Mersenne priem polynomials are $1 + x + x^2$, $1 + x + x^3$.

4) Do all perfect polynomials have a Mersenne prime polynomial as a factor?

Answer. Perfect polynomials are two types. One is called trivial perfect polynomial and it does not have Mersenne prime polynomials in them like the first two of the examples in this activity. Another type is called a nontrivial perfect polynomial and it has Mersenne prime polynomials as an odd divisor. There are only 11 polynomials of the nontrivial perfect polynomials. Thus, not all perfect polynomials have a Mersenne prime polynomial as a divisor.

2.

5) Give an example of a perfect polynomial that does *not* have a Mersenne prime as a factor.

Answer. All polynomials on the form $(x(x+1))^{2^n-1}$ where $n \in \mathbb{N}$ are trivial perfect polynomials. Thus, the first two polynomials of the first question does not have a Mersenne prime as a factor.

CCSS.MATH.PRACTICE.MP2

Reason abstractly and quantitatively.

Make reason about the pattern and why or why not Mersenne prime polynomial is a factor of a perfect polynomials or not.

CCSS.MATH.PRACTICE.MP3

Construct viable arguments and critique the reasoning of others.

Do all perfect polynomials have a Mersenne prime polynomial as a factor? Why or why not

CCSS.MATH.PRACTICE.MP6

Attend to precision.

Factoring and finding zeros.

CCSS.MATH.PRACTICE.MP7

Look for and make use of structure.

Make sense of the last two problems.

CCSS.MATH.PRACTICE.MP8

Look for and express regularity in repeated reasoning.

Last two questions.

Appendix A: Pre-Kindergarten to Third Grade Activities

In this appendix we give our activity collection for pre-kindergarten to third Grade students.

A.1 Making Rectangles

Given the following 12 squares:



$\{Part 1\}$

1) Take 4 squares. Arrange them in to a rectangle. How many different rectangles can you make?

2) Take 6 squares. Arrange them in to a rectangle. How many different rectangles can you make?

3) Take 10 squares. Arrange them in to a rectangle. How many different rectangles can you make?

$\{ \text{ Part } 2 \}$

4) Count 2 squares. Try to put them in a rectangular form. How many different rectangles can you make?

5) Count 3 squares. Try to put them in a rectangular form. How many different rectangles can you make?

6) How many different rectangles can you make if you have 5 squares?

7) How many different rectangles can you make if you have 7 squares?

- 8) How many different rectangles can you make if you have 9 squares?
- 9) How many different rectangles can you make if you have 11 squares?
- 10) How many different rectangles can you make if you have 12 squares?
- 11) Which numbers had only 1 representation?
- 12) Why do these numbers have only 1 representation?
- 13) Which numbers had more than one representation?

A.2 How many ways

How many ways can you write the number 4 as a product of two numbers?
 How many way can you write the number 6 as a product of two numbers?
 How many way can you write the number 7 as a product of two numbers?
 How many way can you write the number 8 as a product of two numbers?
 How many way can you write the number 9 as a product of two numbers?
 How many way can you write the number 10 as a product of two numbers?
 How many ways can you write the number 12 as a product of two numbers?

Now can you write down all the numbers in order, without repeating any, and when you multiply them together they give you the following:

The Number	Ordered products
4	1, 2, 4
6	
7	
8	
9	
10	
12	

A.3 Pick a Number

Consider the following:

$$1 \times 10 = \underline{10}$$
$$2 \times 5 = \underline{10}$$
$$5 \times 2 = \underline{10}$$
$$10 \times 1 = \underline{10}$$
$$10 \div 1 = \underline{10}$$
$$10 \div 2 = \underline{5}$$
$$10 \div 5 = \underline{2}$$
$$10 \div 10 = \underline{1}$$

10

Here the numbers 1, 2, 5 and 10 are called the divisors of the number 10.

- 1) List all the divisors of the number 6.
- 2) Can you give a number that has only two positive divisors?
- 3) Give me a number that has four positive divisors?

4) Can you find the number between 1 and 50 that has the greatest number of positive divisors?

- 5) Can you find two numbers that have the same positive divisors?
- 6) Can you find two numbers that have the same number of positive divisors?

- 7) Can you find a number that is equal to the sum of its divisors?
- 8) Can you find a number that is equal to half the sum of its divisors?
- 9) Can you find a number that is greater than the sum of its divisors?

A.4 Sharing cookies

You have some cookies that you want to share with your friends.

1. You must give each of your friends **more** than one whole cookie.

2. You must give each of your friends an equal number of whole cookies.

3. Non of the cookies can be cut into parts.

1) Suppose you have 3 cookies. How many different friends could you share these cookies with?

2) Suppose you have 4 cookies. How many different friends could you share these cookies with?

3) Suppose you have 5 cookies. How many different friends could you share these cookies with?

4) Suppose you have 6 cookies. How many different friends could you share these cookies with?

5) Suppose you have 7 cookies. How many different friends could you share these cookies with?

6) Suppose you have 8 cookies. How many different friends could you share these cookies with?

Let's record these facts in the table below.

Number of cookies	Number of friends to share with
2	
3	
4	1 or 2
5	
6	
7	
8	

For this table our goal is for students to apply what they learned in finding divisors,

and practice their skills in adding numbers.

Number of cookiesAdd possible numbers of friends689910101215162022252830

Now try to complete the following table

7) Sally noticed if you add the proper divisors they are always less than the number itself, while Kim noticed that value is always larger. Can you help me to figure out which one of them is right?

8) Does the sum of all possible numbers of friends ever equal the number of cookies?

Appendix B: Fourth to Ninth Grade Activities

This appendix contains activities for students from fourth grade to ninth grade

B.1 Prime Numbers

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	50
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

Finding Primes. Look At the following table.

Given that 2 is two a prime number, cross out every other number.

1) Is 3 crossed out or not? If 3 is crossed out then move to the next number. If 3 is not crossed out, 3 is a prime, so circle it and cross out every third number.

2) Is 4 crossed out or not? If 4 is crossed out, then move to the next number. If 4 is not crossed out, 4 is a prime, so circle it and cross out every fourth number.

3) Repeat the process until you get all the prime numbers between 1 and 100. List all of the prime numbers found between 1 and 100. How many are there?

B.2 Divisors

Number	Divisors
4	1, 2, 4
6	
8	
9	
12	
15	
17	
20	
22	
25	
28	
32	
100	
202	
236	
496	

1) Find all the divisors of the following numbers:

The proper divisors of a number are all the divisors except the number itself.

2) Add the proper divisors for each number in the previous exercise.

Number	Sum of Proper Divisors
4	1 + 2 = 3
6	
8	
9	
12	
15	
17	
20	
22	
25	
28	
32	
100	
202	
236	
496	

3) Which numbers have the sum of their proper divisors less than the number itself?
4) Which numbers have the sum of their proper divisors equal to the number itself?
5) Which numbers have the sum of their proper divisors greater than the number itself?

B.3 The sum of the reciprocal divisors of a number

1) If d is a divisor of a number, find the sum of the reciprocal of all divisors of the following numbers:

1. n = 2. 2. n = 3. 3. n = 4. 4. n = 6. 5. n = 8. 6. n = 28. 7. n = 496.

2) If you know that 6, 28 and 496 are perfect numbers, can you see a relation between $\sum (1/d)$ and a perfect number?

3) What is the relation between the sum of the divisors of n and the sum of the reciprocal of the divisors of n, if n is a perfect number?

B.4 Number of Factors

Jessica is a clever student. The teacher asked her to find the number of divisors of the number $2^3 \times 3^2 \times 7$. So she started as following:



Consider the diagram she made:

- 1) Find the number of divisors of the number $2^3 \times 3^2 \times 7$.
- **2)** Can you write the divisors of the number $2^3 \times 3^2 \times 7$?
- 3) Can you help Jessica find an easier way to find the number of divisors?
- 4) Can you make a similar diagram for 48 and find the number of divisors?
- 5) Find the number of divisors of 240.
- 6) If p is a prime number, find the number of divisors of p^n .
- 7) If p is a prime. Find the number of divisors of $2p^5$

Appendix C: Seventh to Twelfth Grade Activities

This appendix contains activities for students from seventh grade to high school.
C.1 " σ " Function

If $\sigma(6) = 1 + 2 + 3 + 6 = 12$, so σ is a function that adds up all the divisors of a function. Find the following:

- 1. $\sigma(2)$ 2. $\sigma(3)$ 3. $\sigma(4)$ 4. $\sigma(5)$ 5. $\sigma(8)$ 6. $\sigma(9)$ 7. $\sigma(10)$ 8. $\sigma(12)$ 9. $\sigma(20)$ 10. $\sigma(25)$ 11. $\sigma(28)$ 1) Now look at $\sigma(10)$ does it equal $\sigma(2) \times \sigma(5)$?
- **2)** Dose $\sigma(12)$ equal $\sigma(3) \times \sigma(4)$?
- **3)** What about $\sigma(20)$ does it equal $\sigma(4) \times \sigma(5)$?
- 4) If m, n are two different natural numbers can we say that $\sigma(m.n) = \sigma(m) \times \sigma(n)$?

5) If your answer was yes for the previous questions, look at $\sigma(4)$ does it equal $\sigma(2) \times \sigma(2)$

6) If the gcd (a, b) = 1 and $\sigma(a) = 5$, $\sigma(b) = 9$. Find $\sigma(a \cdot b)$.

C.2 Perfect Numbers Formula

A perfect number is a positive integer number, that equals to the sum of it's proper divisors. The following are perfect numbers

6 = 1 + 2 + 3 28 = 1 + 2 + 4 + 7 + 14 496 = 1 + 2 + 4 + 8 + 16 + 31 + 62 + 124 + 2488128 = 1 + 2 + 4 + 8 + 16 + 32 + 64 + 127 + 254 + 508 + 1016 + 2032 + 4064

1) Do you see any pattern?

Complete the following table

	Sum	Prime	Calculation	Perfect Number		
1+2	3	Prime	2×3	6		
1 + 2 + 4						
1+2+4+8	15	not a Prime	-	-		

While you were adding up the powers of two, did you observe any pattern in the sequence of sums obtained? It may be more obvious if we express each number in terms of powers of two, as we shall see in the follow table:

Series	Sum	Prime
$1+2^1$	$2^2 - 1$	prime
$1+2^1+2^2$		
$1+2^1+2^2+2^3$		not prime
$1 + 2^1 + 2^2 + 2^3 + 2^4$		
$1 + 2^1 + 2^2 + 2^3 + 2^4 + 2^5$		
$1 + 2^1 + 2^2 + 2^3 + 2^4 + 2^5 + 2^6$		
$1 + 2^1 + 2^2 + 2^3 + 2^4 + 2^5 + 2^6 + 2^7$		

2) Use the table you just completed to compute the following sums:

$$1 + 2^{1} + 2^{2} + 2^{3} + 2^{4} + \dots + 2^{8}$$
$$1 + 2^{1} + 2^{2} + 2^{3} + 2^{4} + \dots + 2^{8} + 2^{9}$$
$$1 + 2^{1} + 2^{2} + 2^{3} + 2^{4} + \dots + 2^{9} + 2^{10}$$
$$1 + 2^{1} + 2^{2} + 2^{3} + 2^{4} + \dots + 2^{10} + 2^{11}$$

3) Can you find a formula for finding the sums of 2's?

4) Can you find a relation between n and the formula you found?

5) What values of n make the sum of 2's a prime number?

C.3 Find Perfect Numbers

Complete the following table

Series	Sum	Sum \times last 2^n in the series
$1+2^{1}$	$2^2 - 1$	$3 \times 2^1 = 6$
$1+2^1+2^2$		
$1 + 2^1 + 2^2 + 2^3$		
$1 + 2^1 + 2^2 + 2^3 + 2^4$		
$1 + 2^1 + 2^2 + 2^3 + 2^4 + 2^5$		
$1 + 2^1 + 2^2 + 2^3 + 2^4 + 2^5 + 2^6$		
$1 + 2^1 + 2^2 + 2^3 + 2^4 + 2^5 + 2^6 + 2^7$		

1) Find a pattern.

- 2) Find perfect numbers.
- 3) Can you give a formula to find perfect Numbers?
- 4) Are all perfect numbers even, odd, or both?

Appendix D: College

In this appendix we give our activity collection for college students.

D.1 Even Perfect Numbers

Consider:

$$2^{0} = \underline{1} = \underline{2^{1} - 1}$$

$$2^{0} + 2^{1} = \underline{3} = \underline{2^{2} - 1}$$

$$2^{0} + 2^{1} + 2^{2} = \underline{7} = \underline{2^{3} - 1}$$

$$1 + 2^{1} + 2^{2} + 2^{3} = \underline{15} = \underline{2^{4} - 1}$$

$$1 + 2^{1} + 2^{2} + 2^{3} + 2^{4} = \underline{31} = \underline{2^{5} - 1}$$

$$1 + 2^{1} + 2^{2} + 2^{3} + 2^{4} + 2^{5} = \underline{63} = \underline{2^{6} - 1}$$

$$1 + 2^{1} + 2^{2} + 2^{3} + 2^{4} + 2^{5} = \underline{63} = \underline{2^{6} - 1}$$

$$1 + 2^{1} + 2^{2} + 2^{3} + 2^{4} + 2^{5} + 2^{6} = \underline{127} = 2^{7} - 1$$

Use the equations above to answer the following questions:

1) When is $2^n - 1$ a prime?

2) When $2^n - 1$ is a prime number, we call it Mersenne prime. Can you give me the first 3 Mersenne primes?

3) When is $2^n - 1$ is composite?

4) Look at the following table:

 $(n \text{ verses } 2^n - 1)$

1	2	3	4	5	6	7	8	9	10	11	• • •	n
 1	3	7	15	31	63	127	255	511	1,023	2,047		$2^n - 1$

5) Write 28 in the form $2^{n-1}(2^n - 1)$. Then write all the factors of 28.

6) Write 496 in the form $2^{n-1}(2^n - 1)$. Then write all the factors of 496.

7) Write 8128 in the form $2^{n-1}(2^n - 1)$. Then write all the factors of 8128.

8) Each Mersenne prime give rise to a perfect number. Can you show how? And can you give the three perfect numbers associated with the first three Mersenne primes?

9) Let p have the form $2^{n-1}(2^n - 1)$ where $(2^n - 1)$ is a prime. Write all the factors of p.

- 10) Add the factors of p.
- 11) Since the sum of the divisors of p equals 2p, what do we call p?

D.2 Perfect polynomials (mod 2)

1) Find all real zeros of the polynomial, and write the polynomial in factored form:

- 1. $x^2 + x$
- 2. $x^4 + x^2$
- 3. $x^4(x+1)^4$

2) If you know that $\sigma(A(x))$ is the sum of all the factors of the polynomial A(x), what is $\sigma(A(x))$ of the previous polynomials if we are summing mod 2, or $\mathbb{Z}_2[x]$.

3) If a Mersenne prime polynomial is a polynomial of the form $x^a(x+1)^b + 1$ that is irreducible, can you give a polynomial that is a Mersenne prime polynomial?

4) Do all perfect polynomials have a Mersenne prime polynomial as a factor?

5) Give an example of a perfect polynomial that does *not* have a Mersenne prime as a factor.

Bibliography

- Areaa I. and Godoyb E. and Ronveauxc A. and Zarzod A. Extensions of some results of P. Humbert on Bezouts identity for classical orthogonal polynomials 2005: Journal of Computational and Applied Mathematics 196 (2006) 212–228.
- [2] Cajori, F. A History Of Mathematical Notations 1929: The Open Court Publishing Company. Chicago.Illinois.
- [3] Canaday E. F. The sum of the divisors of a polynomial 1941: Duke Math.Journal 8, 721–737.
- [4] Department of Higher Education. Retrieved from https://www.ohiohighered.org
- [5] Euclid's. The Thirteen Books Of Euclid's Elements. Translated by Heath .T 1956: Dover Publications, INC. New York.
- [6] Gallardo L. H. and Rahavandrainy O. On even (unitary) perfect polynomials over F2 2012: Finite Fields and Their Applications 18(5). 920932.
- [7] Gallardo L. and Pollack P. and Rahavandrainy O. On a conjecture of Beard, O'Connell and West concerning perfect polynomials 2008: Finite Fields and Their Applications 14(1). 242-249
- [8] Joseph H. Silverman A Friendly Introduction to Number Theory, Third Edition 1997: Brown University.
- [9] LeVeque W. J. Fundamentals of Number Theory 1977: Dover Publications, INC. New York.
- [10] Maor E. and Eugen J. Beautiful Geometry 2013: Princeton University Press. New Jersey 08540.
- [11] National Governors Association Center for Best Practices and Council of Chief State School Officers. (2010). Common Core State Standards. Washington, DC: National Governors Association Center for Best Practices and Council of Chief State School Officers.

- [12] Oystein Ore Number Theory And Its History 1948: Sterling Professor Of Mathematics, Yale University.
- [13] Pearlman J. Necessary Conditions For the Non-existence of Odd Perfect Numbers 2005.
- [14] Richard W. Shoemaker *Perfect Numbers* 1906: University of Toledo, Toledo, Ohio.
- [15] Richardson. L. A Mathematics Activity Curriculum for Early Childhood and Special Education 1980: Macmillan, New York.
- [16] Solomon R. An Introduction to Abstract Algebra 2013.
- [17] Stanley Bezuszka and Lou D'Angelo and Margaret J. Kenney and Stephen Kokoska *Perfect Numbers* 1980: Boston College Press.
- [18] William J. Leveque Fundamentals of Number Theory 1977: Dover Publications, INC. New York.
- [19] Great Internet Mersenne Prime Search PrimeNet: www.mersenne.org