# An Introduction to the Bergman Projection and Kernel

A Thesis

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## Abstract

In this thesis, we discuss the fundamental concepts of the Bergman projection and kernel. In particular, we prove that the Bergman kernel on a given domain $\Omega$ can be characterized in terms of an orthonormal basis of $A^2(\Omega)$. This result is central to the theory of the Bergman kernel and will therefore be shown in full detail. We calculate the Bergman kernel on several domains such as the unit disk, the unit ball in $\mathbb{C}^2$ and $\mathbb{C}^n$, and the annulus. In addition, we show that given two conformally equivalent domains $\Omega_1, \Omega_2 \subseteq \mathbb{C}$, one can represent the Bergman kernel of $\Omega_1$ in terms of the Bergman kernel of $\Omega_2$.

I dedicate this thesis to my wife, my parents, and my brothers.

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# Chapter 1: Motivation

We start by stating our topic of study: integral operators and their associated kernels over $\mathbb{C}$ (and later we will look into those over $\mathbb{C}^n$). In particular we begin with the Cauchy integral formula. A list of notation used in this thesis appears at the end of this chapter.

**Theorem 1.1.** *Let $\Omega \subseteq \mathbb{C}$ be a bounded domain with piecewise smooth boundary. If $f$ is holomorphic on $\Omega$ and $f$ extends smoothly to the boundary of $\Omega$, then*

$$f(z) = \frac{1}{2\pi i} \int_{\partial\Omega} \frac{f(w)}{w - z}\, dw \tag{1.1}$$

*for any point $z \in \Omega$. [1, p. 113]*

The Cauchy integral formula is an amazing fact and the very thing that piqued this author's interest into complex analysis. We ask ourselves "what else can be said about this?" For one thing, it turns out for each $f \in C(\partial\Omega)$, the Cauchy integral is holomorphic in $\Omega$, which we'll show in Corollary 1.3 below.

**Theorem 1.2.** *Let $F(z, s)$ be defined for $(z, s) \in \Omega \times [0, 1]$, where $\Omega$ is an open set in $\mathbb{C}$. Suppose $F$ satisfies the following properties:*

*(i) $F(z, s)$ is holomorphic in $z$ for each $s$.*

*(ii) $F$ is continuous on $\Omega \times [0, 1]$.*

*Then the function defined on $\Omega$ by*

$$f(z) = \int_0^1 F(z,s)\, ds$$

*is holomorphic. [3, p. 56]*

**Corollary 1.3.** *Let $\Omega \subseteq \mathbb{C}$ be a bounded open subset with smooth boundary and let $f \in C(\partial\Omega)$. Then the Cauchy transform of $f$,*

$$\mathcal{C}(f)(z) = \frac{1}{2\pi i} \int_{\partial\Omega} \frac{f(w)}{w-z}\, dw \tag{1.2}$$

*is holomorphic in $\Omega$.*

Thus the Cauchy transform, $\mathcal{C}$, can be thought of as an integral operator with associated kernel $C(z,w) = \dfrac{1}{2\pi i(w-z)}$.

**Proof:** Let $\gamma : [0,1] \to \partial\Omega$ be a smooth parameterization of $\partial\Omega$ and let $z \in \Omega$. Then, by setting $w = \gamma(t)$ in (1.2),

$$\mathcal{C}(f)(z) = \frac{1}{2\pi i} \int_{\partial\Omega} \frac{f(w)}{w-z}\, dw = \frac{1}{2\pi i} \int_0^1 F(z,t)\, dt, \tag{1.3}$$

where $F(z,t) = \dfrac{f(\gamma(t))}{\gamma(t)-z}\gamma'(t)$. We claim that $F$ is holomorphic in $z$ for each $t$ and that $F$ is continuous on $\Omega \times [0,1]$ in order to apply Theorem 1.2.

Notice that for any $t \in [0,1]$, $z \neq \gamma(t)$ for all $z \in \Omega$. For if so, then there exists $t_0 \in [0,1]$ such that $z = \gamma(t_0) \in \partial\Omega$. This would imply that $z \notin \Omega^\circ$. Since $\Omega$ is open, we would therefore have $z \notin \Omega$. Hence we have a contradiction. And so, $\gamma(t) - z \neq 0$ for all $t \in [0,1]$ and $z \in \Omega$.

Therefore, for each $t \in [0,1]$, $F(z,t) = \dfrac{f(\gamma(t))}{\gamma(t)-z}\gamma'(t)$ is holomorphic in $z$. Further, as $f \in C(\partial\Omega)$, $F(z,t)$ is continuous on $\Omega \times [0,1]$. Thus, by Theorem 1.2, $\mathcal{C}(f)(z)$ is holomorphic. $\square$

With all the wonderful results that complex analysis holds we have to ask ourselves several questions. Was the Cauchy integral formula a fluke or are there more equations like this? If so, what properties do they possess? In addition, are they generalizable?

We plan to address each of these questions. In the coming chapters we will discuss the latter two, but for now we draw our attention to the first. Before doing so we recall two classic results in complex analysis mentioned below.

**Theorem 1.4.** *Let $f$ be a holomorphic function on an open set $\Omega \subseteq \mathbb{C}$. Then for any $z_o \in \Omega$ and $0 < r < dist(z_0, \partial\Omega)$*

$$f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{i\theta})\, d\theta = \frac{1}{\pi r^2} \int_{\mathbb{D}_r(z_0)} f(z)\, dV(z). \tag{1.4}$$

The first equality of (1.4) is the mean value property for circles and the second is the mean value property for disks. We will utilize the latter in the proof of an important result in chapter 2. Whereas, for our purposes, the former will only be used in obtaining the property for the disk.

**Proof:** Let $f$ be a holomorphic function on an open set $\Omega \subseteq \mathbb{C}$ and let $z_0 \in \Omega$. Then for any $0 < r < \text{dist}(z_0, \partial\Omega)$, $\mathbb{D}_r(z_0) \subseteq \Omega$. Hence, by the Cauchy integral formula,

$$f(z_0) = \frac{1}{2\pi i} \int_{\partial\mathbb{D}_r(z_0)} \frac{f(w)}{w - z_0}\, dw. \tag{1.5}$$

By the substitution $w = z_0 + re^{i\theta}$ for $0 \le \theta \le 2\pi$, $dw = ire^{i\theta}$ and

$$f(z_0) = \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(z_0 + re^{i\theta})}{re^{i\theta}} ire^{i\theta}\, d\theta = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{i\theta})\, d\theta \tag{1.6}$$

thereby establishing the first equality. Next, by setting $z = z_0 + se^{i\theta}$ for $0 \leq s \leq r$ and $0 \leq \theta \leq 2\pi$ we have

$$\begin{aligned}\frac{1}{2\pi}\int_{\mathbb{D}_r(z_0)} f(z)\, dV(z) &= \frac{1}{2\pi}\int_0^r \int_0^{2\pi} f(z_0 + se^{i\theta}) s\, d\theta ds \\ &= \int_0^r f(z_0) s\, ds \\ &= f(z_0)\left(\frac{r^2}{2}\right).\end{aligned} \tag{1.7}$$

Note that the middle equality was due to the mean value property for circles established in (1.6). Therefore, by dividing by $r^2/2$, we obtain the second equality. □

The answer to the first question is that "no, this wasn't some fluke." There are many different equations like the Cauchy integral formula. In particular, we will explore one in detail but do note that there are others to be studied as well.

**Theorem 1.5** (The Bergman kernel on the unit disk)**.** *For all $f \in A^2(\mathbb{D})$ and $z \in \mathbb{D}$,*

$$f(z) = \frac{1}{\pi}\int_{\mathbb{D}} \frac{f(w)}{(1 - z\overline{w})^2}\, dV(w). \tag{1.8}$$

Notice that the integral defined above is a convergent integral. Indeed, as $|z| < 1$ and $|w| < 1$, $|z\overline{w}| = |z||w| < |z|$ and $1 - |z| > 0$. Then, by the reverse triangle inequality, $|1 - z\overline{w}| \geq 1 - |z\overline{w}| > 1 - |z|$. Thus,

$$\int_{\mathbb{D}} \left|\frac{1}{(1 - z\overline{w})^2}\right|^2 dV(w) \leq \int_{\mathbb{D}} \left|\frac{1}{(1 - |z|)^2}\right|^2 dV(w) < \infty, \tag{1.9}$$

and we have the convergence of (1.8) by means of the Cauchy-Schwarz inequality.

**Proof:** Let $f \in A^2(\mathbb{D})$ and fix $z \in \mathbb{D}$. By setting $w = re^{i\theta}$ with $0 \leq r \leq 1$ and $0 \leq \theta \leq 2\pi$, we have $dV(w) = dxdy = rdrd\theta$. Hence,

$$\int_{\mathbb{D}} \frac{f(w)}{(1 - z\overline{w})^2}\, dV(w) = \int_0^1 \int_0^{2\pi} \frac{f(re^{i\theta})}{(1 - zre^{-i\theta})^2} r\, d\theta dr. \tag{1.10}$$

Since $w = re^{i\theta}$, $dw = ire^{i\theta}d\theta = iwd\theta$. Therefore $d\theta = \dfrac{-idw}{w}$, $e^{-i\theta} = \frac{r}{w}$, and

$$\begin{aligned}\int_0^1 \int_0^{2\pi} \frac{f(re^{i\theta})}{(1 - zre^{-i\theta})^2} r \, d\theta dr &= \int_0^1 \int_{\partial\mathbb{D}} \frac{f(w)}{(1 - z\frac{r^2}{w})^2} r \, \frac{-idw}{w} dr \\ &= \int_0^1 \int_{\partial\mathbb{D}} \frac{-irf(w)}{(\frac{w - zr^2}{w})^2} \, \frac{dw}{w} dr \\ &= \int_0^1 \int_{\partial\mathbb{D}} \frac{-irwf(w)}{(w - zr^2)^2} \, dwdr.\end{aligned} \tag{1.11}$$

.

Notice that as $|z| < 1$ and $r \leq 1$, $|zr^2| < 1$. Thus $\dfrac{-irwf(w)}{(w - zr^2)^2}$ has a double pole at $zr^2 \in \mathbb{D}$. By residue calculus, the inside integral of our last equality in (1.11) becomes

$$\int_{\partial\mathbb{D}} \frac{-irwf(w)}{(w - zr^2)^2} \, dw = 2\pi i \cdot \operatorname{Res}\left[\frac{-irwf(w)}{(w - zr^2)^2}, zr^2\right]. \tag{1.12}$$

Since $zr^2$ is a double pole,

$$\begin{aligned}\operatorname{Res}\left[\frac{-irwf(w)}{(w - zr^2)^2}, zr^2\right] &= \lim_{w \to zr^2} \frac{d}{dw} \left( (w - zr^2)^2 \frac{-irwf(w)}{(w - zr^2)^2} \right) \\ &= \lim_{w \to zr^2} \frac{d}{dw} \left( -irwf(w) \right) \\ &= \lim_{w \to zr^2} \left( -irf(w) - irwf'(w) \right) \\ &= -irf(zr^2) - ir^3 zf'(zr^2).\end{aligned} \tag{1.13}$$

This calculation transforms (1.12) into

$$\begin{aligned}\int_{\partial\mathbb{D}} \frac{-irwf(w)}{(w - zr^2)^2} \, dw &= 2\pi i(-irf(zr^2) - ir^3 zf'(zr^2)) \\ &= \pi(2rf(zr^2) + 2zr^3 f'(zr^2)).\end{aligned} \tag{1.14}$$

At this stage we notice that $2rf(zr^2) + 2zr^3 f'(zr^2) = \frac{d}{dr}\left(r^2 f(zr^2)\right)$. Hence,

$$\begin{aligned}\int_0^1 \int_{\partial\mathbb{D}} \frac{-irwf(w)}{(w - zr^2)^2} \, dwdr &= \int_0^1 \pi(2rf(zr^2) + 2zr^3 f'(zr^2)) \, dr \\ &= \int_0^1 \pi \frac{d}{dr} \left( r^2 f(zr^2) \right) \, dr \\ &= \pi f(z).\end{aligned} \tag{1.15}$$

Therefore $f(z) = \dfrac{1}{\pi} \displaystyle\int_{\mathbb{D}} \frac{f(w)}{(1 - z\overline{w})^2} \, dV(z)$. $\square$

The function $B(z, w) = \dfrac{1}{\pi(1 - z\overline{w})^2}$ is called the Bergman kernel for the unit disk. We note that $B(z, w)$ is a conjugate symmetric function which is holomorphic in $z$ and anti-holomorphic in $w$. Further, the integral operator corresponding to $B$ reproduces holomorphic functions. These facts turn out to be true in a generalized setting as well and we will discuss this in the next chapter.

The properties of the Bergman kernel are well known. However, in our opinion there does not seem to be a suitable reference which goes through the proofs of these properties in full detail. In particular, the proof of the result we present in Theorem 2.16 will require several preliminary results that we intend to highlight. As such, it is our goal in Chapter 2 to carefully show and prove many of the results corresponding to the Bergman kernel.

The remainder of this chapter is dedicated to notation that will be used throughout.

**Notation:**

1. $\mathbb{N} = \{0, 1, \dots\}$.

2. $dV(z)$ denotes Lebesgue measure in $\mathbb{C}^n$.

3. $\mathbb{R}_{>0} = \{r \in \mathbb{R} \mid r > 0\}$ and $\mathbb{R}^n_{>0} = \{(r_1, \dots, r_n) \in \mathbb{R}^n \mid r_i > 0 \text{ for all } 1 \leq i \leq n\}$.

4. $\langle \cdot, \cdot \rangle_{\mathcal{H}} : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$ denotes the inner product on a given Hilbert space, $\mathcal{H}$. If the space is clear from the context, we will omit the subscript and simply write $\langle \cdot, \cdot \rangle$. Note that this inner product is linear in the first argument and conjugate linear in the second.

5. $||\cdot||_{\mathcal{H}} : \mathcal{H} \to [0,\infty)$ denotes the norm on a given Hilbert space, $\mathcal{H}$. If the space is clear from the context, we will omit the subscript and simply write. $||\cdot||$.

6. $\mathbb{D}_r(a) = \{z \in \mathbb{C} \mid |z-a| < r\}$ with $a \in \mathbb{C}$ and $r \in \mathbb{R}_{>0}$. For the unit disk centered at the origin we use $\mathbb{D}$.

7. $\mathbb{A}_\sigma^\rho = \{z \in \mathbb{C} \mid \sigma < |z| < \rho\}$ with $0 < \sigma < \rho < \infty$.

8. $\mathbb{B}^n = \{z \in \mathbb{C}^n \mid |z| < 1\}$

9. $C(\Omega) = \{f : \Omega \to \mathbb{C} \mid f \text{ is continuous}\}$.

10. $\mathcal{O}(\Omega) = \{f : \Omega \to \mathbb{C} \mid f \text{ is holomorphic}\}$.

11. $L^2(\Omega) = \left\{ f : \Omega \to \mathbb{C} \,\middle|\, \int_\Omega |f(z)|^2 \, dV(z) < \infty \right\}$.

12. $A^2(\Omega) = \{f \in L^2(\Omega) \mid f \in \mathcal{O}(\Omega)\}$ is the subspace of $L^2(\Omega)$ of holomorphic functions.

13. $\mathbb{D}_r^n(a) = \mathbb{D}_{r_1}(a_1) \times \cdots \times \mathbb{D}_{r_n}(a_n)$ with $a \in \mathbb{C}^n$ and $r \in \mathbb{R}_{>0}^n$.

14. $T(a,r) = \partial\mathbb{D}_{r_1}(a_1) \times \cdots \times \partial\mathbb{D}_{r_n}(a_n)$ with $a \in \mathbb{C}^n$ and $r \in \mathbb{R}_{>0}^n$.

**Multi-index notation:**

For the $n$-tuple $(\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$, we write

1. $\alpha = (\alpha_1, \ldots, \alpha_n)$

2. $|\alpha| = \alpha_1 + \ldots + \alpha_n$

3. $\alpha! = \alpha_1! \cdots \alpha_n!$

For $z = (z_1, \ldots, z_n), w = (w_1, \ldots, w_n) \in \mathbb{C}^n$ and $n$-tuples $\alpha$, we write

1. $z^\alpha = z_1^{\alpha_1} \cdots z_n^{\alpha_n}$

2. $\langle z, w \rangle = z_1 w_1 + \ldots + z_n w_n$

3. $z - w = (z_1 - w_1) \cdots (z_n - w_n)$

# Chapter 2: The Bergman Projection and Kernel

In this chapter we develop some of the theory of the Bergman projection and its associated integral kernel in one complex variable. In particular, we intend to develop the theory in an arbitrary domain and utilize these results to calculate the Bergman kernel for the unit disk in a different manner than in Theorem 1.5. The two complex variable and the $n$-complex variable cases will be addressed in the next chapter. For this chapter we will work solely with functions from $A^2(\Omega)$ (or $A^2(\mathbb{D})$ depending on the situation).

Evidently $A^2(\Omega)$ is a subspace of $L^2(\Omega)$, which follows immediately from the linearity of differentiation. Indeed, if $f, g$ are holomorphic and $\alpha, \beta \in \mathbb{C}$, then $\alpha f + \beta g$ is also holomorphic. Further, $\alpha f + \beta g$ is square integrable since $L^2(\Omega)$ is a Hilbert space. In fact, we will show below that $A^2(\Omega)$ is a closed subspace of $L^2(\Omega)$. In order to do so we will require the following estimate.

**Lemma 2.1.** *Let $f \in A^2(\Omega)$. Then for all $z \in \Omega$, $|f(z)| \leq \frac{1}{\sqrt{\pi}} \cdot \frac{1}{dist(z,\Omega^c)} \cdot ||f||_{L^2(\Omega)}$.*

**Proof:** Let $f \in A^2(\Omega)$ and $z_0 \in \Omega$ and set $r = \text{dist}(z_0, \Omega^c)$. Then $\mathbb{D}_r(z) \subseteq \Omega$ and by the mean value property for disks in Theorem 1.4,

$$\begin{aligned} |f(z_0)| &= \left| \frac{1}{\pi r^2} \int_{\mathbb{D}_r(z_0)} f(z)\, dV(z) \right| \\ &\leq \frac{1}{\pi r^2} \int_{\mathbb{D}_r(z_0)} |f(z)|\, dV(z). \end{aligned} \tag{2.1}$$

By applying the Cauchy-Schwarz inequality to (2.1), we see

$$\begin{aligned} |f(z_0)| &\leq \frac{1}{\pi r^2}\left(\int_{\mathbb{D}_r(z_0)} |f(z)|^2\, dV(z)\right)^{1/2}\left(\int_{\mathbb{D}_r(z_0)} |1|^2\, dV(z)\right)^{1/2} \\ &= \frac{1}{\pi r^2}\left(\int_{\mathbb{D}_r(z_0)} |f(z)|^2\, dV(z)\right)^{1/2}(\pi r^2)^{1/2} \\ &= \frac{1}{\sqrt{\pi}\, r}\left(\int_{\mathbb{D}_r(z_0)} |f(z)|^2\, dV(z)\right)^{1/2}. \end{aligned} \tag{2.2}$$

Note that since $|f(z)|^2 \geq 0$ and $\mathbb{D}_r(z_0) \subseteq \Omega$,

$$\begin{aligned} \frac{1}{\sqrt{\pi}\, r}\left(\int_{\mathbb{D}_r(z_0)} |f(z)|^2\, dV(z)\right)^{1/2} &\leq \frac{1}{\sqrt{\pi}\, r}\left(\int_{\Omega} |f(z)|^2\, dV(z)\right)^{1/2} \\ &= \frac{1}{\sqrt{\pi}\, r}||f||_{L^2(\Omega)}. \end{aligned} \tag{2.3}$$

Hence, as $f \in A^2(\Omega)$ and $z_0 \in \Omega$ were arbitrary, $|f(z)| \leq \frac{1}{\sqrt{\pi}} \cdot \frac{1}{\operatorname{dist}(z, \Omega^c)} \cdot ||f||_{L^2(\Omega)}$ for every $f \in A^2(\Omega)$ and $z \in \Omega$. □

This estimate is crucial to the proof of Theorem 2.3 below. The following fact is a basic result in the theory of holomorphic functions.

**Theorem 2.2.** *If $\{f_n\}$ is a sequence of holomorphic functions on $\Omega$ and $f_n \to f$ uniformly on compact subsets of $\Omega$, then $f$ is holomorphic on $\Omega$. [3, p. 53]*

**Theorem 2.3.** *Let $\Omega \subseteq \mathbb{C}$ be an open set. Then $A^2(\Omega)$ is a closed subspace of $L^2(\Omega)$.*

**Proof:** Let $\{f_n\} \subseteq A^2(\Omega)$ such that $f_n \to f$ for some $f \in L^2(\Omega)$. We wish to show that $f \in A^2(\Omega)$. As $f_n$ is convergent in $L^2(\Omega)$, $f_n$ must also be Cauchy in $L^2(\Omega)$. Further, as $\{f_n\} \subseteq A^2(\Omega)$, $f_n - f_m \in A^2(\Omega)$ for all $m, n \in \mathbb{N}$. Let $E \subseteq \Omega$ be a compact subset and observe that for all $z \in E$, $0 < \operatorname{dist}(E, \Omega^c) \leq \operatorname{dist}(z, \Omega^c)$.

Let $\epsilon > 0$. Since $\{f_n\}$ is Cauchy in $L^2(\Omega)$, we must have the existence of an $N \in \mathbb{N}$ such that $||f_n - f_m||_{L^2(\Omega)} < \epsilon(\sqrt{\pi} \cdot \operatorname{dist}(E, \Omega^c))$ for all $n, m \geq N$. By Lemma 2.1, we have that for all $z \in E$ and $n, m \geq N$,

$$\begin{aligned} |f_n(z) - f_m(z)| &\leq \frac{1}{\sqrt{\pi}} \frac{1}{\text{dist}(z, \Omega^c)} ||f_n - f_m||_{L^2(\Omega)} \\ &\leq \frac{1}{\sqrt{\pi}} \frac{1}{\text{dist}(E, \Omega^c)} ||f_n - f_m||_{L^2(\Omega)} \\ &< \epsilon. \end{aligned} \tag{2.4}$$

Thus, $f_n \to f$ uniformly on $E$. As $E$ was arbitrary, we have by Theorem 2.2 that $f$ is holomorphic. Hence, $f \in A^2(\Omega)$ and $A^2(\Omega)$ must be a closed subspace of $L^2(\Omega)$. □

**Remark 1.** *It is important to note the following implication from the proof above: convergence in $A^2(\Omega)$ implies pointwise convergence in $\mathbb{C}$. That is, if $\{f_n\}$ is a sequence of functions in $A^2(\Omega)$ such that $\{f_n\}$ converges in the norm, then $\{f_n\}$ converges pointwise as a function in $\mathbb{C}$.*

Since $A^2(\Omega)$ is a closed subspace, we have $L^2(\Omega) = A^2(\Omega) \oplus A^2(\Omega)^\perp$ and there exists an orthogonal projection onto $A^2(\Omega)$. This is called the *Bergman projection*, which we will denote by $\mathcal{B}$. It will turn out that the integral operator defined by the Bergman kernel is equal to the Bergman projection, where an integral operator is defined as follows:

**Definition 2.1.** *Suppose $T_K : L^2(\Omega) \to L^2(\Omega)$ is a linear operator given by the formula*

$$T_K(f)(z) = \int_\Omega K(z, w) f(w)\, dV(w) \tag{2.5}$$

*for $f \in L^2(\Omega)$ and $z \in \Omega$. We call $T_K$ an **integral operator** and $K$ is its associated **(integral) kernel**. [4, p. 187]*

In order to prove that the two Bergman operators actually coincide we will first construct the Bergman kernel and establish several of its key properties. This will require the following theorem.

**Theorem 2.4** (The Riesz representation theorem)**.** *Let $\ell$ be a continuous linear functional on a Hilbert space $\mathcal{H}$. Then there exists a unique $g \in \mathcal{H}$ such that*

$$\ell(f) = \langle f, g\rangle_{\mathcal{H}} \text{ for all } f \in \mathcal{H}.$$

*Moreover, $||\ell||_{\mathcal{H}} = ||g||_{\mathcal{H}}$. [4, p. 182]*

**Lemma 2.5.** *Let $\Omega \subseteq \mathbb{C}$ be an open subset. Then for each fixed $z \in \Omega$, there exists a unique $g_z \in A^2(\Omega)$ such that $f(z) = \langle f, g_z\rangle$ for all $f \in A^2(\Omega)$.*

**Proof:** Fix $z \in \Omega$ and consider the linear operator $\ell_z : A^2(\Omega) \to \mathbb{C}$ defined by $\ell_z(f) = f(z)$ for all $f \in A^2(\Omega)$. Notice that for any $f \in A^2(\Omega)$,
$|\ell_z(f)| = |f(z)| \leq C_z||f||_{L^2(\Omega)}$ by Lemma 2.1 and so $\ell_z$ is therefore bounded. Thus $\ell_z$ is a continuous linear functional. Hence, by Theorem 2.4, there exists a unique $g_z \in A^2(\Omega)$ such that $\ell_z(f) = \langle f, g_z\rangle$ for all $f \in A^2(\Omega)$. But $\ell_z(f) = f(z)$. Thus $f(z) = \langle f, g_z\rangle$. □

**Lemma 2.6.** *Set $K(z, w) = \overline{g_z(w)}$, where $g_z$ is as defined above. Then $K$ is conjugate symmetric and is a reproducing kernel for $A^2(\Omega)$. That is, for all $f \in A^2(\Omega)$ and $z \in \Omega$,*

$$f(z) = \int_\Omega K(z, w)f(w)\, dV(w). \tag{2.6}$$

**Proof:** If $K(z, w) = \overline{g_z(w)}$, then for each $f \in A^2(\Omega)$ we have from Lemma 2.5,

$$f(z) = \langle f, g_z\rangle = \int_\Omega f(w)\overline{g_z(w)}\, dV(w) = \int_\Omega K(z, w)f(w)\, dV(w). \tag{2.7}$$

Next, let $w \in \Omega$ be fixed. We claim that $\overline{K(w, \cdot)} \in A^2(\Omega)$. To show this, notice that $\overline{K(w, \cdot)} = \overline{(\overline{g_w(\cdot)})} = g_w(\cdot) \in A^2(\Omega)$. Hence, by Lemma 2.5,

$$\overline{K(w, z)} = g_w(z) = \langle g_w, g_z\rangle = \overline{\langle g_z, g_w\rangle} = \overline{g_z(w)} = K(z, w)$$

and so $K$ is conjugate symmetric. $\square$

Thus by virtue of the Riesz representation theorem, we have created an integral kernel on $A^2(\Omega)$. We show next that this function is the only such function which satisfies the hypotheses of the following lemma.

**Lemma 2.7.** *Assume $\tilde{K}(\cdot, w) \in A^2(\Omega)$ for each $w \in \Omega$, $\tilde{K}(z, w)$ is a reproducing kernel, and that $\tilde{K}$ is conjugate symmetric. Then $\tilde{K} = K$.*

**Remark 2.** *For simplicity of notation and clarity we will sometimes denote $\tilde{K}(\cdot, w)$ or $K(\cdot, w)$ by $\tilde{K}_w(\cdot)$ or $K_w(\cdot)$, respectively, for fixed $w \in \Omega$.*

**Proof:** Suppose $\tilde{K}$ is as described above. Since $\tilde{K}(z, w) \in A^2(\Omega)$ for each fixed $w \in \Omega$, we see that by Lemma 2.5

$$\begin{aligned} \tilde{K}(z, w) = \tilde{K}_w(z) &= \langle \tilde{K}_w, g_z \rangle \\ &= \int_\Omega \tilde{K}(\zeta, w)\overline{g_z(\zeta)}\, dV(\zeta). \end{aligned} \tag{2.8}$$

But $\overline{g_z(\zeta)} = K(z, \zeta)$ by definition. So (2.8) becomes

$$\begin{aligned} \tilde{K}_w(z) &= \int_\Omega \tilde{K}(\zeta, w)K(z, \zeta)\, dV(\zeta) \\ &= \overline{\int_\Omega \overline{\tilde{K}(\zeta, w)}\,\overline{K(z, \zeta)}\, dV(\zeta)}. \end{aligned} \tag{2.9}$$

By assumption $\tilde{K}$ is conjugate symmetric. Thus, as $\tilde{K}$ is an integral kernel and $\overline{K(z, \zeta)}$ is holomorphic in $\zeta$,

$$\begin{aligned} \tilde{K}_w(z) &= \overline{\int_\Omega \overline{K(z, \zeta)}\tilde{K}(w, \zeta)\, dV(\zeta)} \\ &= \overline{(\overline{K(z, w)})} \\ &= K(z, w) \end{aligned} \tag{2.10}$$

and $\tilde{K} = K$. $\square$

We now have the machinery necessary to show that the Bergman projection is identified by the Bergman kernel. Indeed, the four properties of $K$ (holomorphic in the first variable, a reproducing kernel, conjugate symmetry, and uniqueness) mentioned above will prove sufficient.

**Theorem 2.8.** *Let $T_K : L^2(\Omega) \to L^2(\Omega)$ denote the integral operator with associated kernel $K$, given by Lemma 2.6. Then $T_K = \mathcal{B}$, where $\mathcal{B} : L^2(\Omega) \to L^2(\Omega)$ denotes the Bergman projection.*

**Proof:** To show equality it suffices to show that $T_K$ is an orthogonal projection onto $A^2(\Omega)$. For if so, we have by uniqueness that $T_K = \mathcal{B}$. To begin we note that by Lemma 2.6, $K$ reproduces square integrable holomorphic functions. Thus, if $f \in A^2(\Omega)$, then $T_K(f)(z) = f(z)$ for all $z \in \Omega$. Next we show that $T_K(g) = 0$ for all $g \in A^2(\Omega)^\perp$.

Let $g \in A^2(\Omega)^\perp$ and fix $z \in \Omega$. Then

$$\begin{aligned} T_K(g)(z) &= \int_\Omega K(z,w)g(w)\,dV(w) \\ &= \int_\Omega g(w)\overline{K(w,z)}\,dV(w) \end{aligned} \tag{2.11}$$

by the conjugate symmetry of $K$ (as shown in Lemma 2.6). By definition of the inner production on $L^2(\Omega)$ we see from above that $T_K(g)(z) = \langle g, K_z \rangle$. Since $K_z = K(\cdot, z) \in A^2(\Omega)$, we must have that $T_K(g)(z) = 0$. Thus, as $z \in \Omega$ was arbitrary, $g = 0$.

Now, let $f \in L^2(\Omega)$. Since $A^2(\Omega)$ is a closed subspace $f = f_1 + f_2$ where $f_1 \in A^2(\Omega)$ and $f_2 \in A^2(\Omega)^\perp$. Then, by the linearity of $T_K$,

$$T_K(f) = T_K(f_1 + f_2) = T_K(f_1) + T_K(f_2) = f_1. \tag{2.12}$$

As this $f \in L^2(\Omega)$ was arbitrary we see from (2.12) that $\mathrm{Im}(T_K) = A^2(\Omega)$. We additionally have that for all $f \in L^2(\Omega)$,

$$T_K(T_K(f)) = T_K(f_1) = f_1 = T_K(f). \tag{2.13}$$

Thus $T_K$ is a projection onto $A^2(\Omega)$ and the only thing left to verify is that $T_K$ is self-adjoint. However, this is immediate. Indeed, recall that the kernel of $T_K^*$ is $\overline{K(w,z)}$. However, by the conjugate symmetry of $K$, $\overline{K(w,z)} = K(z,w)$. Thus $T_K$ and $T_K^*$ are identified by the same kernel and we have $T_K = T_K^*$. Hence, $T_K$ is an orthogonal projection onto $A^2(\Omega)$ and we have $T_K = \mathcal{B}$. □

So far we have only described $K$ abstractly. Now we will give an explicit way of calculating $K$ given an orthonormal basis of $A^2(\Omega)$ as in Theorem 2.16 below. In order to do so will require some results about sequences of functions which are uniformly bounded. The next four items are taken from [3, pp. 225–227] (though the notation will be changed in some cases).

**Definition 2.2.** *A sequence $\{E_n\}_{n=1}^{\infty}$ of compact subsets of $\Omega \subseteq \mathbb{C}$ is called an **exhaustion** if:*

*(i) $E_n \subseteq E_{n+1}^{\circ}$ for all $n = 1, 2, \ldots$*

*(ii) Any compact set $E \subseteq \Omega$ is contained in $E_n$ for some $n$. In particular,*

$$\Omega = \bigcup_{n=1}^{\infty} E_n.$$

If such a sequence $\{E_n\}_{n=1}^{\infty}$ exists we say that $\Omega$ has an exhaustion.

**Lemma 2.9.** *Any open subset $\Omega \subseteq \mathbb{C}$ has an exhaustion.*

**Remark 3.** *The proof amounts to examining two cases: whether $\Omega \subseteq \mathbb{C}$ is bounded or unbounded. When $\Omega$ is bounded, set $E_n = \{z \in \Omega \mid dist(z, \partial\Omega) \geq 1/n\}$. In the event that $\Omega$ is unbounded, set $E_n = \{z \in \Omega \mid dist(z, \partial\Omega) \geq 1/n \text{ and } |z| \leq n\}$. However, this restriction to $\mathbb{C}$ is not necessary since the same argument holds in the case where $\Omega \subseteq \mathbb{C}^n$. Therefore, any open set $\Omega \subseteq \mathbb{C}^n$ has an exhaustion.*

**Definition 2.3.** *A family $\mathcal{F}$ of holomorphic functions on $\Omega$ is said to be **normal** if every sequence in $\mathcal{F}$ has a subsequence that converges uniformly on every compact subset $K \subseteq \Omega$.*

**Theorem 2.10** (Montel's theorem)**.** *Suppose $\mathcal{F}$ is a family of holomorphic functions on $\Omega \subseteq \mathbb{C}$ that is uniformly bounded on compact subsets of $\Omega$. Then:*

*(i) $\mathcal{F}$ is equicontinuous on every compact subset of $\Omega$.*

*(ii) $\mathcal{F}$ is a normal family.*

We include a few details of the proof as they will be relevant in a moment. For the full proof, see page 226 of [3]. Now, to prove equicontinuity requires a clever argument involving the Cauchy integral formula. Indeed if $E \subseteq \Omega$ is a compact subset, then there exists $r > 0$ so that for $z, w \in E$ with $|z - w| < r$,

$$\begin{aligned} |f(z) - f(w)| &= \left| \frac{1}{2\pi i} \int_{\partial\mathbb{D}_{2r}(w)} f(\zeta) \left( \frac{1}{\zeta - z} - \frac{1}{\zeta - w} \right) \, d\zeta \right| \\ &\leq \frac{1}{2\pi} \frac{2\pi(2r)}{2r^2} C_E |z - w|, \end{aligned} \tag{2.14}$$

where $C_E$ is the uniform bound of $\mathcal{F}$ on $E$. Therefore, for any $\epsilon > 0$, choose $\delta > 0$ so that $\delta < \min\left\{ r, \frac{r\epsilon}{C_E} \right\}$. Then (2.14) implies that $|f(z) - f(w)| < \epsilon$ whenever $|z - w| < \delta$, thereby proving equicontinuity.

To prove $\mathcal{F}$ is a normal family requires two diagonalization arguments. During the first digaonalization argument, we choose a sequence $\{w_i\}$ which is dense in $\Omega$. With this dense subset, we inductively construct subsequences $\{f_{j,n}\}_{n\in\mathbb{N}}$ so that $f_{j,n}(w_k)$ converges for all $k \leq j$. We extract the diagonal subsequence $\{g_n\} = \{f_{n,n}\}_{n\in\mathbb{N}}$, which will converge uniformly on a compact subset $E$ of $\Omega$ (equicontinuity is essential to prove this). To create the subsequence which converges uniformly on every compact subset of $\Omega$ requires pairing this result and Lemma 2.9 with a somewhat similar diagonalization argument.

**Remark 4.** *As mentioned in Remark 3, every open $\Omega \subseteq \mathbb{C}^n$ also has an exhaustion. Therefore, proving that a family, $\mathcal{F}$, of holomorphic functions defined on $\Omega \subseteq \mathbb{C}^n$ is normal only amounts to proving equicontinuity. To the point, this means that Montel's theorem is true in n-dimensions as well. As we will require this generalization for Corollary 2.14, we present the argument to show equicontinuity below. To do so will require the use of the Cauchy integral formula in n-dimensions.*

**Theorem 2.11** (The Cauchy integral formula for the polydisc)**.** *Let $f(z) = f(z_1, \dots, z_n)$ be continuous on $\Omega \subseteq \mathbb{C}^n$ and holomorphic with respect to each variable separately. Then for every closed polydisc $\overline{\mathbb{D}_r^n(a)} \subseteq \Omega$,*

$$f(z) = \frac{1}{(2\pi i)^n} \int_{T(a,r)} \frac{f(\zeta)}{(\zeta_1 - z_1)\dots(\zeta_n - z_n)}\, d\zeta_1 \dots d\zeta_n \tag{2.15}$$

*for all $z \in \mathbb{D}_r^n(a)$. [2, p. 7]*

We must also prove a small algebraic result before moving forward with the $n$-dimensional case of Montel's theorem. As it will improve readability (at least in our opinion), we will use the following notation: For $\zeta, z, w \in \mathbb{C}^n$, $i \leq j$ and $k \leq l$, set

$$\Delta_{z_i^j}^{w_k^l} = (\zeta_i - z_i)\dots(\zeta_j - z_j)(\zeta_k - w_k)\dots(\zeta_l - w_l). \tag{2.16}$$

In the event that $z$ or $w$ does not occur, we write $\Delta_{z_0^0}^{w_k^l}$ or $\Delta_{z_i^j}^{w_0^0}$, respectively. In addition, if $i > j$, then we set $\Delta_{z_i^j}^{w_k^l} = \Delta_{z_0^0}^{w_k^l}$. For the following lemma, recall the multi-index notation $z - w = (z_1 - w_1) \cdots (z_n - w_n)$ mentioned in Chapter 1 (item 3).

**Lemma 2.12.** *Let $\zeta, z, w \in \mathbb{C}^n$. Then*

$$\frac{1}{\zeta - z} - \frac{1}{\zeta - w} =$$
$$= \frac{\Delta_{z_2^n}^{w_0^0}(z_1 - w_1) + \Delta_{z_3^n}^{w_1^1}(z_2 - w_2) + \ldots + \Delta_{z_n^n}^{w_1^{n-2}}(z_{n-1} - w_{n-1}) + \Delta_{z_0^0}^{w_1^{n-1}}(z_n - w_n)}{\Delta_{z_1^n}^{w_1^n}}. \tag{2.17}$$

**Proof:** Let $\zeta, z, w \in \mathbb{C}^n$. Then

$$\begin{aligned} \frac{1}{\zeta - z} - \frac{1}{\zeta - w} = \frac{1}{\Delta_{z_1^n}^{w_0^0}} - \frac{1}{\Delta_{z_0^0}^{w_1^n}} &= \frac{\Delta_{z_0^0}^{w_1^n} - \Delta_{z_1^n}^{w_0^0}}{\Delta_{z_1^n}^{w_1^n}} \\ &= \frac{\Delta_{z_0^0}^{w_1^n} - \Delta_{z_2^n}^{w_1^1} + \Delta_{z_2^n}^{w_1^1} - \Delta_{z_1^n}^{w_0^0}}{\Delta_{z_1^n}^{w_1^n}} \\ &= \frac{\Delta_{z_0^0}^{w_1^1}(\Delta_{z_0^0}^{w_2^n} - \Delta_{z_2^n}^{w_0^0}) + \Delta_{z_2^n}^{w_0^0}(z_1 - w_1)}{\Delta_{z_1^n}^{w_1^n}}. \end{aligned} \tag{2.18}$$

Notice that

$$\begin{aligned} \Delta_{z_0^0}^{w_2^n} - \Delta_{z_2^n}^{w_0^0} &= \Delta_{z_0^0}^{w_2^n} - \Delta_{z_3^n}^{w_2^2} + \Delta_{z_3^n}^{w_2^2} - \Delta_{z_2^n}^{w_0^0} \\ &= \Delta_{z_0^0}^{w_2^2}(\Delta_{z_0^0}^{w_3^n} - \Delta_{z_3^n}^{w_0^0}) + \Delta_{z_3^n}^{w_0^0}(z_2 - w_2). \end{aligned} \tag{2.19}$$

Hence, (2.18) expands to

$$\frac{\Delta_{z_0^0}^{w_1^2}(\Delta_{z_0^0}^{w_3^n} - \Delta_{z_3^n}^{w_0^0}) + \Delta_{z_3^n}^{w_1^1}(z_2 - w_2) + \Delta_{z_2^n}^{w_0^0}(z_1 - w_1)}{\Delta_{z_1^n}^{w_1^n}}. \tag{2.20}$$

Inductively we iterate the process done in (2.19) to $\Delta_{z_0^0}^{w_k^n} - \Delta_{z_k^n}^{w_0^0}$ for each $k \leq n$. That is, for each $k \leq n$,

$$\begin{aligned} \Delta_{z_0^0}^{w_k^n} - \Delta_{z_k^n}^{w_0^0} &= \Delta_{z_0^0}^{w_k^n} - \Delta_{z_{k+1}^n}^{w_k^k} + \Delta_{z_{k+1}^n}^{w_k^k} - \Delta_{z_k^n}^{w_0^0} \\ &= \Delta_{z_0^0}^{w_k^k}(\Delta_{z_0^0}^{w_{k+1}^n} - \Delta_{z_{k+1}^n}^{w_0^0}) + \Delta_{z_{k+1}^n}^{w_0^0}(z_k - w_k). \end{aligned} \tag{2.21}$$

Upon doing so, we will obtain our desired result. $\square$

**Corollary 2.13.** *If $\mathcal{F}$ is a family of holomorphic functions on $\Omega \subseteq \mathbb{C}^n$ that is uniformly bounded on compact subsets of $\Omega$, then $\mathcal{F}$ is equicontinuous on every compact subset of $\Omega$.*

**Proof:** Let $F \subseteq \Omega$ be a compact subset and let $f \in \mathcal{F}$. Choose $r > 0$ sufficiently small so that $T_n(z, 3r) \subseteq \Omega$ for all $z \in F$ (here $3r = (3r, \ldots, 3r) \in \mathbb{R}^n$ and $T_n(z, 3r) = \partial\mathbb{D}_{3r}(z_1) \times \ldots \times \partial\mathbb{D}_{3r}(z_n)$). Let $z, w \in F$ so that $|z - w| < r$ and consider $T_n(w, 2r)$. Then $|z_i - w_i| < r$ for $1 \leq i \leq n$ (since $\max\{|z_i - w_i| \mid 1 \leq i \leq n\} \leq |z - w|$). Further, if $\zeta \in T_n(w, 2r)$, then $|\zeta_i - w_i| = 2r$ and $r \leq |\zeta_i - z_i| \leq 3r$ for any $1 \leq i \leq n$. Therefore by the Cauchy integral formula on the polydisc (Theorem 2.11), we have

$$\begin{aligned} |f(z) - f(w)| &= \left| \frac{1}{(2\pi i)^n} \int_{T_n(w,2r)} f(\zeta) \left( \frac{1}{\zeta - z} - \frac{1}{\zeta - w} \right) d\zeta \right| \\ &\leq \frac{1}{(2\pi)^n} \int_{T_n(w,2r)} |f(\zeta)| \left| \frac{1}{\zeta - z} - \frac{1}{\zeta - w} \right| d\zeta \\ &\leq \frac{1}{(2\pi)^n} C_F \frac{(3r)^{n-1}|z_1 - w_1| + \ldots + (3r)^{n-1}|z_n - w_n|}{(2r)^n r^n} V(T_n(w, 2r)), \end{aligned} \tag{2.22}$$

where $V(T_n(w, 2r))$ denotes the volume of $T_n(w, 2r)$ and $C_F$ is the uniform bound of $\mathcal{F}$ on $F$. Note the last inequality was due to Lemma 2.12 and the fact that $|\zeta_i - z_i|, |\zeta_i - w_i| \leq 3r$ for any $1 \leq i \leq n$.

Since

$$|z_1 - w_1| + \ldots + |z_n - w_n| \leq n(\max\{|z_i - w_i| \mid 1 \leq i \leq n\}) \leq n|z - w|, \tag{2.23}$$

we have from (2.22) that

$$\begin{aligned} |f(z) - f(w)| &\leq \frac{C_F}{(2\pi)^n} \frac{(3r)^{n-1} n}{(2r)^n r^n} V(T_n(W, 2r))|z - w| \\ &= C'_F |z - w|. \end{aligned} \tag{2.24}$$

Thus, if $\epsilon > 0$, we may choose $\delta < \min\left\{r, \frac{\epsilon}{C'_F}\right\}$ so that $|f(z) - f(w)| < \epsilon$ whenever $|z - w| < \delta$. Therefore $\mathcal{F}$ is an equicontinuous family. □

Thus we have the $n$-dimensional case of Montel's theorem. As such, we obtain a very useful corollary.

**Corollary 2.14.** *Let $\Omega \subseteq \mathbb{C}^n$ be an open subset and $\{f_n\}$ be a sequence of holomorphic functions converging pointwise to a function $f : \Omega \to \mathbb{C}$. Suppose $\{f_n\}$ is uniformly bounded on every compact subset $E \subseteq \Omega$. Then $f_n \to f$ uniformly on compact subsets of $\Omega$.*

**Proof:** Let $\Omega \subseteq \mathbb{C}^n$ be an open set and $\{f_n\}$ be as defined above. Further let $E \subseteq \Omega$ be compact and $\{f_{n_k}\}$ be a subsequence of $\{f_n\}$. Since $\{f_n\}$ is uniformly bounded on compact subsets, so must be $\{f_{n_k}\}$. Therefore, by Corollary 2.13 and our remark to Theorem 2.10, $\{f_{n_k}\}$ is a normal family. Hence, there exists a subsequence of $\{f_{n'_k}\}$ of $\{f_{n_k}\}$ such that $f_{n'_k} \to g$ uniformly on $E$ for some $g$. However, as $f_n \to f$ pointwise we must have that $g = f$ by uniqueness. Thus $f_{n'_k} \to f$ uniformly on $E$.

Since $\{f_{n_k}\}$ was an arbitrary subsequence, it follows that every subsequence of $\{f_n\}$ has a sub-subsequence converging uniformly to $f$. Now, to show that $f_n \to f$ uniformly on $E$, we suppose to the contrary. Then there exists $\epsilon_0 > 0$ and $z_0 \in E$ such that for every $M \in \mathbb{N}$, there exists a $n_M \geq N$ such that $|f_{n_M}(z_0) - f(z_0)| \geq \epsilon_0$. Then $\{f_{n_M}\}$ defines a subsequence of $\{f_n\}$ which does not converge to $f$. This would mean that $\{f_{n_M}\}$ could not have a subsequence converging uniformly to $f$, which leads us to a contradiction. Hence $f_n \to f$ uniformly on $E$. As this $E \subseteq \Omega$ was arbitrary, $f_n \to f$ uniformly on compact subsets of $\Omega$. □

Below we will show that, given an orthonormal basis $\{\phi_n\} \subset A^2(\Omega)$, $K$ can be represented as the series $\sum_{n=0}^{\infty} \phi_n(z)\overline{\phi_n(w)}$, which will converge uniformly on compact subsets of $\Omega \times \Omega$. Considering this, we see the necessity for Corollary 2.14 (since $K$ depends on more than one complex variable). The pointwise convergence of the series above will almost be immediate. However, to prove that the series is uniformly bounded will require a different expression of the inner product on $L^2(\Omega)$.

**Lemma 2.15.** *Let $\mathcal{H}$ be a Hilbert space. Then for all $g \in \mathcal{H}$,*

$$||g||_{\mathcal{H}} = \sup_{||f||_{\mathcal{H}}=1} \{|\langle f, g\rangle_{\mathcal{H}}|\}$$

**Proof:** Let $g \in \mathcal{H}$. If $g = 0$, the result is immediate. So suppose $g \in \mathcal{H}$ such that $g \neq 0$. Let $f \in \mathcal{H}$ such that $||f||_{\mathcal{H}} = 1$. Then, by the Cauchy-Schwarz inequality, $|\langle f, g\rangle|_{\mathcal{H}} \leq ||f||_{\mathcal{H}} \cdot ||g||_{\mathcal{H}} = ||g||_{\mathcal{H}}$. Therefore, by taking the supremum over all $f \in \mathcal{H}$ such that $||f||_{\mathcal{H}} = 1$, we see $\sup_{||f||_{\mathcal{H}}=1} \{|\langle f, g\rangle_{\mathcal{H}}|\} \leq ||g||_{\mathcal{H}}$. Second, as $g \neq 0$, set $f = \frac{g}{||g||_{\mathcal{H}}}$. Then $||f||_{\mathcal{H}} = 1$ and

$$|\langle f, g\rangle_{\mathcal{H}}| = \left|\left\langle \frac{g}{||g||_{\mathcal{H}}}, g\right\rangle_{\mathcal{H}}\right| = \frac{||g||_{\mathcal{H}}^2}{||g||_{\mathcal{H}}} = ||g||_{\mathcal{H}}. \tag{2.25}$$

Hence, $\sup_{||f||_{\mathcal{H}}=1} \{|\langle f, g\rangle_{\mathcal{H}}|\} \geq ||g||_{\mathcal{H}}$ and we have equality. □

The lemma above gives us an alternate characterization of the norm in any given Hilbert space. This will prove useful when we consider the fact that $|\langle f, g_z\rangle| = |f(z)|$ for the $g_z$ as described in Lemma 2.5.

Now we are finally ready to characterize $K$ with respect to an orthonormal basis.

**Theorem 2.16.** *For any choice of orthonormal basis $\{\phi_n\}$ of $A^2(\Omega)$,*

$$K(z, w) = \sum_{n=0}^{\infty} \phi_n(z)\overline{\phi_n(w)}, \tag{2.26}$$

*which converges uniformly and absolutely on compact subsets $F \subseteq \Omega \times \Omega$.*

**Proof:** Notice that as $A^2(\Omega)$ is a closed subspace of $L^2(\Omega)$ and $L^2(\Omega)$ is a Hilbert space, $A^2(\Omega)$ must also be a Hilbert space. As such, $A^2(\Omega)$ admits at least one orthonormal basis. Let $\{\phi_n\}$ be such an orthonormal basis and fix $w \in \Omega$. Therefore, since $K(\cdot, w) \in A^2(\Omega)$,

$$K(\cdot, w) = \sum_{n=0}^{\infty} \langle K(\cdot, w), \phi_n(\cdot) \rangle \phi_n(\cdot). \tag{2.27}$$

However, notice that $\langle K(\cdot, w), \phi_n(\cdot) \rangle = \overline{\langle \phi_n(\cdot), K(\cdot, w) \rangle} = \overline{\phi_n(w)}$ by Lemma 2.7. That is to say,

$$K(\cdot, w) = \sum_{n=0}^{\infty} \phi_n(\cdot)\overline{\phi_n(w)} \tag{2.28}$$

for fixed $w \in \Omega$ and so $K(\cdot, w)$ converges in the norm. Recall that pointwise convergence is dominated by $L^2(\Omega)$ convergence in $A^2(\Omega)$. Therefore $K(\cdot, w)$ converges pointwise to $\sum_{n=0}^{\infty} \phi_n(\cdot)\overline{\phi_n(w)}$. Thus, in accordance with Corollary 2.14, it suffices to show that our summation in (2.26) is uniformly bounded on compact subsets of $\Omega$.

Next, we note that

$$||K(\cdot, w)||^2_{L^2(\Omega)} = \sum_{n=0}^{\infty} |\langle \phi_n(\cdot), K(\cdot, w) \rangle|^2 = \sum_{n=0}^{\infty} |\phi_n(w)|^2. \tag{2.29}$$

Now, by Lemma 2.15 and the conjugate symmetry of $K$,

$$\begin{aligned} ||K(\cdot, w)||_{L^2(\Omega)} = ||\overline{K(w, \cdot)}||_{L^2(\Omega)} &= ||g_w(\cdot)||_{L^2(\Omega)} \\ &= \sup_{||f||_{L^2(\Omega)}=1} \{|\langle f, g_w \rangle|\} \\ &= \sup_{||f||_{L^2(\Omega)}=1} \{|f(w)|\}, \end{aligned} \tag{2.30}$$

where the last equality is due to Lemma 2.5. Therefore, by Lemma 2.1 we must have

$$\begin{aligned} ||K(\cdot, w)||_{L^2(\Omega)} &= \sup_{||f||_{L^2(\Omega)}=1} \{|f(w)|\} \\ &\leq \sup_{||f||_{L^2(\Omega)}=1} \left\{ \frac{1}{\sqrt{\pi}} \cdot \frac{1}{\text{dist}(w, \Omega^c)} \cdot ||f||_{L^2(\Omega)} \right\} \\ &= \frac{1}{\sqrt{\pi}} \cdot \frac{1}{\text{dist}(w, \Omega^c)}. \end{aligned} \tag{2.31}$$

Thus, if $E$ is a compact subset of $\Omega$, then there exists a constant $C_E \geq 0$ so that $||K(\cdot, w)||_{L^2(\Omega)} \leq C_E$ for all $w \in E$. From (2.29) this implies that $\sum_{n=0}^{\infty} |\phi_n(w)|^2 \leq C_E^2$. Therefore, on compact subsets $F \subseteq \Omega \times \Omega$, we have by the Cauchy-Schwarz inequality, that

$$\left| \sum_{n=0}^{\infty} \phi_n(z)\overline{\phi_n(w)} \right| \leq \left( \sum_{n=0}^{\infty} |\phi_n(z)|^2 \right)^{1/2} \left( \sum_{n=0}^{\infty} |\phi_n(w)|^2 \right)^{1/2} \leq C_F^2. \tag{2.32}$$

That is, $\sum_{n=0}^{\infty} \phi_n(z)\overline{\phi_n(w)}$ is uniformly bounded on compact subsets of $\Omega \times \Omega$. Note that we may not apply Corollary 2.14 since this series is not holomorphic on $\Omega \times \Omega$. However if we make the following observation, we will obtain our result.

Let $\Omega^* = \{\overline{w} \mid w \in \Omega\}$ and for each $m \in \mathbb{N}$ define $f_m : \Omega \times \Omega^* \to \mathbb{C}$ by

$$f_m(z, w) = \sum_{n=0}^{m} \phi_n(z)\overline{\phi_n(\overline{w})} \tag{2.33}$$

Then $\{f_m\}$ is a sequence of holomorphic functions that is uniformly bounded and converges pointwise to $K(z, \overline{w})$ by the arguments presented above. Therefore, by Corollary 2.14, $\{f_m\}$ converges uniformly to $K$ on compact subsets $F' \subseteq \Omega \times \Omega^*$. However, for all $(z, w) \in \Omega \times \Omega$ and for each $m \in \mathbb{N}$,

$$\sum_{n=0}^{m} \phi_n(z)\overline{\phi_n(w)} = f_m(z, \overline{w}), \tag{2.34}$$

which converges uniformly to $K(z, \overline{\overline{w}}) = K(z, w)$. Hence, $\sum_{n=0}^{\infty} \phi_n(z)\overline{\phi_n(w)}$ converges uniformly to $K(z, w)$ on compact subsets $F \subseteq \Omega \times \Omega$. $\square$

Thus (2.26) provides a formula for the integral kernel on any bounded open set $\Omega \subseteq \mathbb{C}$. However, we note that in general this can prove quite cumbersome to write explicitly depending on the domain $\Omega$ we reside in. We do not go into details in this chapter but even for "nice" domains, such as an annulus, or an ellipse centered at the origin, the Bergman kernel turns out to be difficult to calculate.

For most domains, it is not possible to determine an explicit orthonormal basis. Even when we can find an orthonormal basis, it is usually difficult to say whether or not the summation defined in (2.26) will simplify to a simple equation. However, when $\Omega = \mathbb{D}$ the result does turn out to be simple as we saw in Theorem 1.5. We will now calculate the Bergman kernel by means of Theorem 2.16 above, which will require finding an orthonormal basis for $A^2(\mathbb{D})$.

**Theorem 2.17.** *Let $\mathcal{H}$ be a Hilbert space. The following properties of an orthonormal set $\{\phi_n\}$ are equivalent:*

*(i) Finite linear combinations of elements in $\{\phi_n\}$ are dense in $\mathcal{H}$.*

*(ii.) If $f \in \mathcal{H}$ and $\langle f, \phi_n \rangle = 0$ for all $n$, then $f = 0$.*

*(iii.) If $f \in \mathcal{H}$ and $S_N(f) = \sum_{n=0}^{N} a_n \phi_n$, where $a_n = \langle f, \phi_n \rangle$, then $S_N(f) \to f$ as $N \to \infty$ in the norm.*

*(iv.) If $a_n = \langle f, \phi_n \rangle$, then $||f||^2 = \sum_{n=0}^{\infty} |a_n|^2$. [4, p. 165]*

**Lemma 2.18.** *Let $\phi_n(z) = \sqrt{\frac{n+1}{\pi}} z^n$ for all $n \geq 0$. Then $\{\phi_n\}$ is an orthonormal basis for $A^2(\mathbb{D})$.*

For this proof if will be important to recall that for $m, n \in \mathbb{Z}$,

$$\int_0^{2\pi} e^{i(m-n)\theta} d\theta = \begin{cases} 0 & m \neq n \\ 2\pi & m = n. \end{cases} \tag{2.35}$$

**Proof:** We begin by showing the orthonormality of the set $\{\phi_n\}$. For distinct $m, n \in \mathbb{N}$,

$$\langle \phi_n, \phi_m \rangle = \int_{\mathbb{D}} \left( \frac{\sqrt{(n+1)(m+1)}}{\pi} \right) z^n \overline{z}^m \, dV(z). \tag{2.36}$$

Setting $z = re^{i\theta}$ for $0 \leq r \leq 1$ and $0 \leq \theta \leq 2\pi$, (2.36) becomes

$$\begin{aligned} \langle \phi_n, \phi_m \rangle &= \int_0^1 \int_0^{2\pi} \left( \frac{\sqrt{(n+1)(m+1)}}{\pi} \right) r^n e^{in\theta} r^m e^{-im\theta} \, r d\theta dr \\ &= \int_0^1 \int_0^{2\pi} \left( \frac{\sqrt{(n+1)(m+1)}}{\pi} \right) r^{n+m+1} e^{i(n-m)\theta} \, d\theta dr. \end{aligned} \tag{2.37}$$

With (2.35) in mind, we see that $\langle \phi_n, \phi_m \rangle = 0$ if $n \neq m$. However, if $n = m$, then (2.36) becomes

$$\begin{aligned} \langle \phi_n, \phi_n \rangle &= \int_0^1 \int_0^{2\pi} \frac{n+1}{\pi} r^{2n+1} \, d\theta dr \\ &= \int_0^1 2(n+1) r^{2n+1} \, dr \\ &= 1. \end{aligned} \tag{2.38}$$

Thus $\{\phi_n\}$ forms an orthonormal set and it remains to show that this set forms a basis for $A^2(\mathbb{D})$.

Let $f \in A^2(\mathbb{D})$ and suppose $\langle f, \phi_n \rangle = 0$ for all $n$. Note that as $f \in A^2(\mathbb{D})$, $f \in \mathcal{O}(\mathbb{D})$ as well. Thus, $f$ has a unique power series representation, $f = \sum_{k=0}^{\infty} a_k z^k$ with $a_k \in \mathbb{C}$ for all $k$. Fix $n \in \mathbb{N}$. Then, by converting to polar,

$$\begin{aligned} 0 = \langle f, \phi_n \rangle &= \int_{\mathbb{D}} f(z) \overline{\phi_n(z)} \, dV(z) \\ &= \int_0^1 \int_0^{2\pi} f(re^{i\theta}) \overline{\phi_n(re^{i\theta})} r \, d\theta dr \\ &= \int_0^1 \int_0^{2\pi} \left( \sum_{k=0}^{\infty} a_k r^k e^{ik\theta} \right) \left( \sqrt{\frac{n+1}{\pi}} r^n e^{-in\theta} \right) r \, d\theta dr \\ &= \sqrt{\frac{n+1}{\pi}} \int_0^1 \sum_{k=0}^{\infty} a_k r^{n+k+1} \int_0^{2\pi} e^{i(k-n)\theta} \, d\theta dr. \end{aligned} \tag{2.39}$$

However, as seen in (2.35), all terms of the summation will vanish save for the $n^{th}$ term of the power series. So, continuing with the calculation,

$$\begin{aligned} 0 = \langle f, \phi_n \rangle &= \sqrt{\frac{n+1}{\pi}} \int_0^1 2\pi a_n r^{2n+1} \, dr \\ &= a_n \sqrt{\frac{n+1}{\pi}} \frac{\pi}{n+1} \end{aligned} \tag{2.40}$$

So by necessity, $a_n = 0$. As $n \in \mathbb{N}$ was arbitrary, it follows from (2.40) that $a_n = 0$ for all $n$. That is, $f = 0$. Thus by Theorem 2.17, $\{\phi_n\}$ forms an orthonormal basis for $A^2(\mathbb{D})$. $\square$

If we may make a small remark, the calculations done in the proof above provide us with a sufficient condition to determine whether a given holomorphic function on $\mathbb{D}$ will be square integrable.

**Theorem 2.19.** *Let $f \in \mathcal{O}(\mathbb{D})$. Then $f \in A^2(\mathbb{D})$ if the series $\sum_{n=0}^{\infty} \frac{|a_n|^2}{n+1}$ is convergent, where $a_n$ represent the coefficients of the power series representation of $f$.*

**Proof:** Let $f \in \mathcal{O}(\mathbb{D})$. The $f$ has a power series representation, $f = \sum_{n=0}^{\infty} a_n z^n$, which converges uniformly on compact subsets of $\mathbb{D}$. In order for $f \in A^2(\mathbb{D})$, we require $\int_{\mathbb{D}} |f(z)|^2 \, dV(z) < \infty$. Then, by converting to polar,

$$\begin{aligned} \int_{\mathbb{D}} |f(z)|^2 \, dV(z) &= \int_{\mathbb{D}} f(z)\overline{f(z)} \, dV(z) \\ &= \int_0^1 \int_0^{2\pi} \left( \sum_{n=0}^{\infty} a_n r^n e^{in\theta} \right) \left( \sum_{m=0}^{\infty} \overline{a_m} r^m e^{-im\theta} \right) r \, d\theta dr. \end{aligned} \tag{2.41}$$

Recalling (2.35) we see that all terms of the (2.41) vanish save for when $n = m$. Hence,

$$\begin{aligned} \int_0^1 \int_0^{2\pi} \left( \sum_{n=0}^{\infty} a_n r^n e^{in\theta} \right) \left( \sum_{m=0}^{\infty} \overline{a_m} r^m e^{-im\theta} \right) \, dV(z) &= \int_0^1 \int_0^{2\pi} \sum_{n=0}^{\infty} |a_n|^2 r^{2n+1} \, d\theta dr \\ &= 2\pi \int_0^1 \sum_{n=0}^{\infty} |a_n|^2 r^{2n+1} \, dr. \end{aligned} \tag{2.42}$$

Since $|a_n|^2 \geq 0$ for all $n \in \mathbb{N}$, we have by Tonelli's theorem that

$$\begin{aligned} 2\pi \int_0^1 \sum_{n=0}^{\infty} |a_n|^2 r^{2n+1} \, dr &= 2\pi \sum_{n=0}^{\infty} \int_0^1 |a_n|^2 r^{2n+1} \, dr \\ &= \pi \sum_{n=0}^{\infty} \frac{|a_n|^2}{n+1}. \end{aligned} \tag{2.43}$$

That is, $\int_{\mathbb{D}} |f(z)|^2 \, dV(z) = \pi \sum_{n=0}^{\infty} \frac{|a_n|^2}{n+1}$. Thus, if $f \in \mathcal{O}(\mathbb{D})$, then it is sufficient to require that $\sum_{n=0}^{\infty} \frac{|a_n|^2}{n+1} < \infty$ in order for $f \in A^2(\mathbb{D})$. □

Continuing on, we recall that for $|z| < 1$, the function $f(z) = \frac{1}{1-z}$ is holomorphic and has power series representation

$$\sum_{n=0}^{\infty} z^n = \frac{1}{1-z}. \tag{2.44}$$

Then we may differentiate this series term by term to obtain the following lemma.

**Lemma 2.20.** *If* $|z| < 1$, *then* $\sum_{n=0}^{\infty} (n+1) z^n = \frac{1}{(1-z)^2}$.

We are now ready to calculate the Bergman kernel on $\mathbb{D}$ by means of Theorem 2.16. Indeed, if we use the orthonormal basis found in Lemma 2.18 we see that

$$\begin{aligned} B(z,w) = \quad \sum_{n=0}^{\infty} \phi_n(z)\overline{\phi_n(w)} &= \sum_{n=0}^{\infty} \left(\sqrt{\frac{n+1}{\pi}} z^n\right)\left(\sqrt{\frac{n+1}{\pi}} \overline{w}^n\right) \\ &= \frac{1}{\pi} \sum_{n=0}^{\infty} (n+1)(z\overline{w})^n. \end{aligned} \tag{2.45}$$

Therefore, by Lemma 2.20, we have precisely that

$$B(z,w) = \frac{1}{\pi(1-z\overline{w})^2}. \tag{2.46}$$

Notice that in our construction of the Bergman kernel, there was nothing that required $\Omega$ to be specifically a subset of $\mathbb{C}$. In fact, our argument is generalizable for $\Omega \subseteq \mathbb{C}^n$. Thus, as alluded to in the beginning of the chapter we will now calculate the Bergman kernel on the unit ball for the two complex variable and $n$-complex variable case.

# Chapter 3: The Bergman Kernel in $\mathbb{C}^2$ and $\mathbb{C}^n$

In this chapter we will construct the Bergman kernel on $A^2(\mathbb{B}^2)$ and then on $A^2(\mathbb{B}^n)$. In each case we will create an orthonormal basis and utilize Theorem 2.16. For the two dimensional case, we note that $\{z_1^n z_2^m\}_{m,n\in\mathbb{N}}$ form an orthogonal basis on $A^2(\mathbb{B}^2)$, but it is not orthonormal. Thus, we must find coefficients $c_{mn} \in \mathbb{C}$ so that $\{c_{mn} z_1^n z_2^m\}_{m,n\in\mathbb{N}}$ forms an orthonormal basis, which we will require calculating $||z_1^m z_2^n||_{L^2(\mathbb{B}^2)}$. Before doing so, we recall the *Beta function* which is defined by:

$$\beta(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\,dt \quad = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}, \tag{3.1}$$

where $\Gamma$ denotes the Gamma function and $x, y > 0$. We will utilize this function on several occasions in calculating the integrals below, especially when we investigate the $n$-dimensional case.

Let $m, n \in \mathbb{N}$. Then,

$$\begin{aligned} ||z_1^m z_2^n||^2_{L^2(\mathbb{B}^2)} &= \int_{\mathbb{B}^2} |z_1|^{2m}|z_2|^{2n}\,dV(z_1)dV(z_2) \\ &= \int_{|z_1|\le 1} |z_1|^{2m} \left( \int_{|z_2|\le\sqrt{1-|z_1|^2}} |z_2|^{2n}\,dV(z_2) \right) dV(z_1). \end{aligned} \tag{3.2}$$

First set $z_2 = r_2 e^{i\theta_2}$ for $0 \leq r_2 \leq \sqrt{1-|z_1|^2}$ and $0 \leq \theta_2 \leq 2\pi$. Then the inner integral becomes

$$\begin{aligned}\int_{|z_2|\leq\sqrt{1-|z_1|^2}} |z_2|^{2n}\, dV(z_2) &= \int_0^{2\pi}\int_0^{\sqrt{1-|z_1|^2}} r_2^{2n}\, r_2 dr_2 d\theta_2 \\ &= 2\pi \int_0^{\sqrt{1-|z_1|^2}} r_2^{2n+1}\, dr_2 \\ &= \frac{2\pi}{2n+2} r_2^{2n+2}\Big|_0^{\sqrt{1-|z_1|^2}} \\ &= \frac{\pi}{n+1}\left(1-|z_1|^2\right)^{n+1}.\end{aligned} \tag{3.3}$$

Thus, the equation in (3.2) becomes

$$||z_1^m z_2^n||^2_{L^2(\mathbb{B}^2)} = \frac{\pi}{n+1}\int_{|z_1|\leq 1} |z_1|^{2m}\left(1-|z_1|^2\right)^{n+1}\, dV(z_1). \tag{3.4}$$

Next by setting $z_1 = r_1 e^{i\theta_1}$ with $0 \leq r_1 \leq 1$ and $0 \leq \theta_1 \leq 2\pi$,

$$\begin{aligned}\int_{|z_1|\leq 1} |z_1|^{2m}\left(1-|z_1|^2\right)^{n+1}\, dV(z_1) &= \int_0^{2\pi}\int_0^1 (r_1^2)^m (1-r_1^2)^{n+1}\, r_1 dr_1 d\theta_1 \\ &= \frac{1}{2}\int_0^{2\pi}\int_0^1 t^m (1-t)^{n+1}\, dt d\theta_1 \\ &= \pi \int_0^1 t^m (1-t)^{n+1}\, dt.\end{aligned} \tag{3.5}$$

Note that the second equality above came from making a change of variables with $t = r_1^2$ (and hence, $dt = 2r_1 dr_1$ and $0 \leq t \leq 1$). Therefore,

$$\int_0^1 t^m (1-t)^{n+1}\, dt = \beta(m+1, n+2) \quad = \frac{\Gamma(m+1)\Gamma(n+2)}{\Gamma(m+n+3)}. \tag{3.6}$$

Thus, by combining (3.4), (3.5), and the result directly above, we have

$$||z_1^m z_2^n||^2_{L^2(\mathbb{B}^2)} = \left(\frac{\pi}{n+1}\right)\left(\pi\frac{\Gamma(m+1)\Gamma(n+2)}{\Gamma(m+n+3)}\right). \tag{3.7}$$

Recalling that for all $n \in \mathbb{N}$, $\Gamma(n+1) = n!$, we have

$$\begin{aligned}||z_1^m z_2^n||^2_{L^2(\mathbb{B}^2)} &= \left(\frac{\pi}{n+1}\right)\left(\pi\frac{m!(n+1)!}{(m+n+2)!}\right) \\ &= \frac{\pi^2 m! n!}{(m+n+2)!}.\end{aligned} \tag{3.8}$$

Thus we have found our constants $c_{mn}$ and our corresponding orthonormal basis $\{\phi_{mn}(z_1, z_2)\}_{m,n\in\mathbb{N}}$, where for each $m, n \in \mathbb{N}$,

$$\phi_{mn}(z_1, z_2) = \sqrt{\frac{(m+n+2)!}{\pi^2 m!n!}} z_1^m z_2^n. \tag{3.9}$$

Now, before we move on we recall Lemma 2.20 and realize that this formula can actually be generalized. The proof follows immediately from induction on $k$ and we therefore omit it.

**Lemma 3.1.** *If* $|z| < 1$, *then* $\sum_{n=0}^{\infty}(n+1)\cdots(n+k)z^n = \frac{k!}{(1-z)^{k+1}}$

This lemma will prove useful in our calculations of the Bergman kernel in both the two dimensional case and the $n$-dimensional case.

**Theorem 3.2.** *The Bergman kernel on* $\mathbb{B}^2$ *is given by the equation*

$$B(z, w) = B(z_1, z_2, w_1, w_2) \quad = \frac{2}{\pi^2 \left(1 - (z_1\overline{w_1} + z_2\overline{w_2})\right)^3} \tag{3.10}$$

*for* $z = (z_1, z_2), w = (w_1, w_2) \in \mathbb{B}^2 \subseteq \mathbb{C}^2$.

**Proof:** Let $\phi_{m,n} \in A^2(\mathbb{B}^2)$ be as defined in (3.9). Then $\{\phi_{mn}\}_{m,n\in\mathbb{N}}$ forms an orthonormal basis for $A^2(\mathbb{B}^2)$. Therefore, by Theorem 2.16,

$$\begin{aligned} B(z, w) &= \sum_{m,n\in\mathbb{N}} \phi_{mn}(z_1, z_2)\overline{\phi_{mn}(w_1, w_2)} \\ &= \sum_{m,n\in\mathbb{N}} \frac{(m+n+2)!}{\pi^2 m!n!}(z_1^m z_2^n)(\overline{w_1}^m \overline{w_2}^n) \\ &= \sum_{m,n\in\mathbb{N}} \frac{(m+n+2)!}{\pi^2 m!n!}(z_1\overline{w_1})^m (z_2\overline{w_2})^n. \end{aligned} \tag{3.11}$$

Now, we may rewrite (3.11) by indexing as follows:

$$B(z, w) = \sum_{k=0}^{\infty} \sum_{m+n=k} \frac{(m+n+2)!}{\pi^2 m!n!}(z_1\overline{w_1})^m (z_2\overline{w_2})^n. \tag{3.12}$$

We note that we may represent $n$ in terms of $k$ and $m$, which transforms (3.12) to

$$\begin{aligned} B(z,w) &= \sum_{k=0}^{\infty}\sum_{m=0}^{k} \frac{(k+2)!}{\pi^2 m!(k-m)!}(z_1\overline{w_1})^m(z_2\overline{w_2})^{k-m} \\ &= \sum_{k=0}^{\infty} \frac{(k+2)(k+1)}{\pi^2}\left(\sum_{m=0}^{k}\frac{k!}{m!(k-m)!}(z_1\overline{w_1})^m(z_2\overline{w_2})^{k-m}\right), \end{aligned} \tag{3.13}$$

with the last equality due to the inner sum depending on $m$. But the inside summation is precisely the binomial expansion of $(z_1\overline{w_1}+z_2\overline{w_2})^k$. Hence, by Lemma 3.1 and this observation,

$$\begin{aligned} B(z,w) &= \sum_{k=0}^{\infty} \frac{(k+2)(k+1)}{\pi^2}(z_1\overline{w_1}+z_2\overline{w_2})^k \\ &= \frac{2}{\pi^2(1-(z_1\overline{w_1}+z_2\overline{w_2}))^3}. \end{aligned} \tag{3.14}$$

and we have our desired result. □

It is natural and pleasantly surprising to see some similarity between the Bergman kernel on $\mathbb{D}$ and on $\mathbb{B}^2$. In fact, this will continue on to $\mathbb{B}^n$ as well. However, notice that this notation will only become more complex (pun intended!) as we move to the $n$-dimensional case. Indeed, $B(z_1,\ldots,z_n,w_1,\ldots,w_n)$ becomes painful to read and write! As such, we will use the multi-index notation as mentioned in Chapter 1 when necessary.

Using multi-index notation, the orthonormal basis in (3.9) can be written as $\{\phi_\alpha\}_{\alpha\in\mathbb{N}^2}$, where

$$\phi_\alpha(z) = \sqrt{\frac{(|\alpha|+2)!}{\pi^2\alpha!}}z^\alpha. \tag{3.15}$$

This can of course be extended to $\mathbb{N}^n$ for any $n$, and we do so below.

Much like in the two complex variable case, $\{z^\alpha\}_{\alpha\in\mathbb{N}^n}$ will form an orthogonal basis on $A^2(\mathbb{B}^n)$. So, once again we find $c_\alpha$ to normalize this basis. We will proceed in the same fashion as before - by computing $||z^\alpha||_{L^2(\mathbb{B}^n)}$. When relevant we shall use the following notation: for $k > 1$, $z = (z_1, \dots, z_n) \in \mathbb{B}^n$ we denote $A_k = \sqrt{1 - |z_1|^2 - \dots - |z_{k-1}|^2}$. Further, notice that

$$A_k^2 = A_{k-1}^2 - |z_{k-1}|^2 \text{ for all } k > 2. \tag{3.16}$$

For $\alpha \in \mathbb{N}^n$ we have

$$\begin{aligned} ||z^\alpha||^2_{L^2(\mathbb{B}^n)} &= \int_{\mathbb{B}^n} |z|^{2\alpha}\, dV(z) \\ &= \int_{|z_1|\le 1} |z_1|^{2\alpha_1} \int_{|z_2|\le A_2} \cdots \int_{|z_n|\le A_n} |z_n|^{2\alpha_n}\, dV(z_n)\cdots dV(z_1) \\ &= \int_{|z_1|\le 1} |z_1|^{2\alpha_1} \int_{|z_2|\le A_2} \cdots \int_0^{2\pi}\int_0^{A_n} r_n^{2\alpha_n+1}\, dr_n d\theta_n dV(z_{n-1})\cdots dV(z_1), \end{aligned} \tag{3.17}$$

where the conversion to polar $z_n = r_n e^{i\theta_n}$ with $0 \le r_n \le A_n$ and $0 \le \theta_n \le 2\pi$ was made in the last step. Hence, the inner integral (with respect to $z_{n-1}$) in (3.17) becomes

$$\frac{\pi}{\alpha_n + 1} \int_{|z_{n-1}|\le A_{n-1}} |z_{n-1}|^{2\alpha_{n-1}} (A_{n-1}^2 - |z_{n-1}|^2)^{\alpha_n+1}\, dV(z_{n-1}), \tag{3.18}$$

which by setting $z_{n-1} = r_{n-1}e^{i\theta_{n-1}}$ with $0 \le r_{n-1} \le A_{n-1}$ and $0 \le \theta_{n-1} \le 2\pi$, will yield

$$\frac{\pi}{\alpha_n + 1} \int_0^{2\pi} \int_0^{A_{n-1}} r_{n-1}^{2\alpha_{n-1}+1} (A_{n-1}^2 - r_{n-1}^2)^{\alpha_n+1}\, dr_{n-1} d\theta_{n-1}. \tag{3.19}$$

Now we make a substitution again. This time we set $r_{n-1} = A_{n-1}u$ with $0 \le u \le 1$. Then $dr_{n-1} = A_{n-1}du$ and (3.19) becomes

$$\frac{2\pi^2}{\alpha_n + 1} \int_0^1 A_{n-1}^{2\alpha_{n-1}+1} u^{2\alpha_{n-1}+1} (A_{n-1}^2 - A_{n-1}^2 u^2)^{\alpha_n+1} A_{n-1}\, du, \tag{3.20}$$

which simplifies to

$$\frac{2\pi^2}{\alpha_n + 1} \int_0^1 A_{n-1}^{2(\alpha_n+\alpha_{n-1}+2)} (u^2)^{\alpha_{n-1}} (1 - u^2)^{\alpha_n+1}\, udu. \tag{3.21}$$

If we set $t = u^2$ and recall the Beta function in (3.1), we are left with

$$\begin{aligned} \frac{\pi^2}{\alpha_n+1}\int_0^1 A_{n-1}^{2(\alpha_n+\alpha_{n-1}+2)} t^{\alpha_{n-1}}(1-t)^{\alpha_n+1}\,dt &= \frac{\pi^2 A_{n-1}^{2(\alpha_n+\alpha_{n-1}+2)}}{\alpha_n+1}\beta(\alpha_{n-1}+1,\alpha_n+2) \\ &= \frac{\pi^2 A_{n-1}^{2(\alpha_n+\alpha_{n-1}+2)}}{\alpha_n+1}\frac{\Gamma(\alpha_{n-1}+1)\Gamma(\alpha_n+2)}{\Gamma(\alpha_n+\alpha_{n-1}+3)} \\ &= \frac{\pi^2\alpha_n!\alpha_{n-1}!}{(\alpha_n+\alpha_{n-1}+2)!}A_{n-1}^{2(\alpha_n+\alpha_{n-1}+2)} \\ &= C_{n-1}A_{n-1}^{2(\alpha_n+\alpha_{n-1}+2)}, \end{aligned} \tag{3.22}$$

where $C_{n-1} = \dfrac{\pi^2\alpha_n!\alpha_{n-1}!}{(\alpha_n+\alpha_{n-1}+2)!}$.

Going back to (3.17), we have reduced our integral to

$$C_{n-1}\int_{|z_1|\le 1}|z_1|^{2\alpha_1}\int_{|z_2|\le A_2}\cdots\int_{|z_{n-2}|\le A_{n-2}}|z_{n-2}|^{2\alpha_{n-2}}A_{n-1}^{2(\alpha_n+\alpha_{n-1}+2)}\,dV(z_{n-2})\cdots dV(z_1). \tag{3.23}$$

If we recall (3.16), then we have $A_{n-1}^2 = A_{n-2}^2 - |z_{n-2}|^2$. Thus, the inner integral of (3.23) becomes

$$\int_{|z_{n-2}|\le A_{n-2}}|z_{n-2}|^{2\alpha_{n-2}}(A_{n-2}^2-|z_{n-2}|^2)^{\alpha_n+\alpha_{n-1}+2}\,dV(z_{n-2}) \tag{3.24}$$

and now we see a pattern forming. Indeed, the conversion to polar $z_{n-2} = r_{n-2}e^{i\theta_{n-2}}$ with $0 \le r_{n-2} \le A_{n-2}$ and $0 \le \theta_{n-2} \le 2\pi$ converts (3.24) to

$$\int_0^{2\pi}\int_0^{A_{n-2}} r_{n-2}^{2\alpha_{n-2}+1}(A_{n-2}^2 - r_{n-2}^2)^{\alpha_n+\alpha_{n-1}+2}\,dr_{n-2}d\theta_{n-2}. \tag{3.25}$$

Set $r_{n-2} = A_{n-2}u$ with $0 \le u \le 1$. Then $dr_{n-2} = A_{n-2}du$ and (3.25) is equivalent to

$$2\pi\int_0^1 A_{n-2}^{2\alpha_{n-2}+1}u^{2\alpha_{n-2}+1}(A_{n-2}^2 - A_{n-2}^2u^2)^{\alpha_n+\alpha_{n-1}+2}\,A_{n-2}du, \tag{3.26}$$

which simplifies to

$$2\pi\int_0^1 A_{n-2}^{2(\alpha_n+\alpha_{n-1}+\alpha_{n-2}+3)}(u^2)^{\alpha_{n-2}}(1-u^2)^{\alpha_n+\alpha_{n-1}+2}\,udu. \tag{3.27}$$

By setting $t = u^2$ and utilizing the Beta function again, this integral is equivalent to

$$\begin{aligned} &\pi A_{n-2}^{2(\alpha_n+\alpha_{n-1}+\alpha_{n-2}+3)}\beta(\alpha_{n-2}+1, \alpha_n+\alpha_{n-1}+3) = \\ &= \pi A_{n-2}^{2(\alpha_n+\alpha_{n-1}+\alpha_{n-2}+3)} \frac{\alpha_{n-2}!(\alpha_n+\alpha_{n-1}+2)!}{(\alpha_n+\alpha_{n-1}+\alpha_{n-2}+3)!}. \end{aligned} \tag{3.28}$$

Thus, we have reduced (3.23) to

$$C_{n-2}\int_{|z_1|\leq 1} |z_1|^{2\alpha_1} \int_{|z_2|\leq A_2} \cdots \int_{|z_{n-3}|\leq A_{n-3}} |z_{n-3}|^{2\alpha_{n-3}} (A_{n-3}^2 - |z_{n-3}|^2)^{\alpha_n+\alpha_{n-1}+\alpha_{n-2}+3} \\ dV(z_{n-3})\cdots dV(z_1), \tag{3.29}$$

where $C_{n-2} = \dfrac{\pi^3 \alpha_n!\alpha_{n-1}!\alpha_{n-2}!}{(\alpha_n+\alpha_{n-1}+\alpha_{n-2}+3)!}$. Iterating this process, we continue until we reach

$$\frac{\pi^{n-1}\alpha_n!\cdots\alpha_2!}{(\alpha_n+\cdots+\alpha_2+(n-1))!}\int_{|z_1|\leq 1} |z_1|^{2\alpha_1}(1-|z_1|^2)^{\alpha_n+\cdots+\alpha_2+(n-1)}\, dV(z_1). \tag{3.30}$$

Then, by doing one last conversion to polar with $z_1 = r_1 e^{i\theta_1}$, we are left with

$$\frac{\pi^{n-1}\alpha_n!\cdots\alpha_2!}{(\alpha_n+\cdots+\alpha_2+(n-1))!}\int_0^{2\pi}\int_0^1 r_1^{2\alpha_1}(1-r_1^2)^{\alpha_n+\cdots+\alpha_2+(n-1)}\, r dr_1 d\theta_1. \tag{3.31}$$

Last we set $t = r^2$ with $0 \leq t \leq 1$ and $dt = 2rdr$, which takes our double integral to

$$\begin{aligned} \pi\int_0^1 t^{\alpha_1}(1-t)^{\alpha_n+\cdots+\alpha_2+(n-1)}\, dt &= \pi\beta(\alpha_1+1, \alpha_n+\cdots+\alpha_2+n) \\ &= \frac{\pi\Gamma(\alpha_1+1)\Gamma(\alpha_n+\cdots+\alpha_2+n)}{\Gamma(\alpha_n+\cdots+\alpha_1+(n+1))} \\ &= \frac{\pi\alpha_1!(\alpha_n+\cdots\alpha_2+(n-1))!}{(\alpha_n+\cdots+\alpha_1+n)!}. \end{aligned} \tag{3.32}$$

Hence, in accordance with our multi-index notation,

$$||z^\alpha||^2_{L^2(\mathbb{B}^n)} = \frac{\pi^n \alpha!}{(|\alpha|+n)!}. \tag{3.33}$$

Thus, we have found our constants $c_\alpha$ and our orthonormal basis for $\mathbb{B}^n$ is $\{\phi_\alpha\}_{\alpha\in\mathbb{N}^n}$, where

$$\phi_\alpha = \sqrt{\frac{(|\alpha|+n)!}{\pi^n\alpha!}}z^\alpha \tag{3.34}$$

and we are finally ready to calculate the Bergman kernel in the $n$-dimensional case. Before we begin we recall the multinomial theorem.

**Theorem 3.3.** *For any positive integer $m$ and nonnegative integer $n$,*

$$(z_1 + \cdots + z_m)^n = \sum_{|\alpha|=n} \binom{n}{\alpha_1, \ldots, \alpha_m} z^\alpha, \tag{3.35}$$

*where* $\binom{n}{\alpha_1, \ldots, \alpha_n} = \dfrac{n!}{\alpha!}$.

**Theorem 3.4.** *The Bergman kernel on $\mathbb{B}^n$ is given by the equation*

$$B(z, w) = \frac{n!}{\pi^n (1 - \langle z, \overline{w} \rangle)^{n+1}} \tag{3.36}$$

*for $z, w \in \mathbb{B}^n \subseteq \mathbb{C}^n$.*

**Proof:** Let $\phi_\alpha \in A^2(\mathbb{B}^n)$ be as defined in (3.34). Then $\{\phi_\alpha\}_{\alpha \in \mathbb{N}^n}$ forms an orthonormal basis for $A^2(\mathbb{B}^n)$. Hence, by Theorem 2.16, for $z, w \in \mathbb{B}^n$

$$\begin{aligned} B(z, w) &= \sum_{\alpha \in \mathbb{N}^n} \phi_\alpha(z) \overline{\phi_\alpha(w)} \\ &= \sum_{\alpha \in \mathbb{N}^n} \frac{(|\alpha| + n)!}{\pi^n \alpha!} z^\alpha \overline{w}^\alpha. \end{aligned} \tag{3.37}$$

We then re-index as follows:

$$\begin{aligned} B(z, w) &= \sum_{k=0}^{\infty} \sum_{|\alpha|=k} \frac{(|\alpha| + n)!}{\pi^n \alpha!} (z\overline{w})^\alpha \\ &= \sum_{k=0}^{\infty} \sum_{|\alpha|=k} \frac{(k + n)!}{\pi^n \alpha!} (z\overline{w})^\alpha \\ &= \sum_{k=0}^{\infty} \frac{(k+1) \cdots (k+n)}{\pi^n} \sum_{|\alpha|=k} \frac{k!}{\alpha!} (z\overline{w})^\alpha. \end{aligned} \tag{3.38}$$

Thus, by the multinomial theorem in conjunction with Lemma 3.1

$$\begin{aligned} B(z, w) &= \frac{1}{\pi^n} \sum_{k=0}^{\infty} (k+1) \cdots (k+n) \langle z, \overline{w} \rangle^k \\ &= \frac{n!}{\pi^n (1 - \langle z, \overline{w} \rangle)^{n+1}}, \end{aligned} \tag{3.39}$$

which is our desired result. □

# Chapter 4: The Bergman Kernel and Conformal Mappings

In this chapter we shall calculate the Bergman kernel on several other domains. We mentioned previously that finding and explicit formula for the Bergman kernel can be a daunting task. This is mostly due to the difficulty in obtaining an orthonormal basis for a given domain. In some circumstances however, we may bypass this necessity altogether. The concept revolves around conformal mappings and the Riemann mapping theorem, which we will state below.

**Definition 4.1.** *Let $U, V \subseteq \mathbb{C}$. A bijective holomorphic function $f : U \to V$ is called a* ***conformal map*** *and we say that $U$ and $V$ are* ***conformally equivalent****. [3, p. 206]*

The simplest examples of conformal mappings are translations ($f(z) = z + a$ for some $a \in \mathbb{C}$) and dilations ($f(z) = cz$ for some $c \in \mathbb{C}\backslash\{0\}$). Take for example the function $f : \mathbb{D}_r \to \mathbb{D}$ defined by

$$f(z) = \frac{1}{r}z, \tag{4.1}$$

where $r > 0$. Then $f$ is a conformal mapping with inverse $f^{-1}(z) = rz$.

It is useful to note that if $f : U \to V$ is a conformal map, then $f'(z) \neq 0$ for all $z \in U$. Further, we also have that $f^{-1}$ must also be a conformal map. They will be of particular use to us because we will show that the existence of a conformal map between two domains allows you to write the Bergman kernel of one domain in terms of the other.

**Theorem 4.1.** *Let $\Omega_1, \Omega_2 \subseteq \mathbb{C}$ and suppose there exists a conformal map $f : \Omega_1 \to \Omega_2$. Then for all $(z, w) \in \Omega_1 \times \Omega_1$,*

$$K_{\Omega_1}(z, w) = f'(z) K_{\Omega_2}(f(z), f(w)) \overline{f'(w)}, \tag{4.2}$$

*where $K_{\Omega_1}$ and $K_{\Omega_2}$ represent the Bergman kernels on $\Omega_1$ and $\Omega_2$, respectively.*

**Proof:** Let $\{\phi_n\}$ be an orthonormal basis for $A^2(\Omega_2)$. For each $n \in \mathbb{N}$ and $z \in \Omega_1$, set $\psi_n(z) = f'(z)(\phi_n \circ f)(z)$. We show that $\{\psi_n\}$ is an orthonormal basis for $A^2(\Omega_1)$. For if so, then we will have our result immediately from Theorem 2.16.

First let $m, n \in \mathbb{N}$. Then

$$\begin{aligned} \langle \psi_n, \psi_m \rangle_{A^2(\Omega_1)} &= \int_{\Omega_1} \psi_n(z) \overline{\psi_m(z)} \, dV(z) \\ &= \int_{\Omega_1} f'(z)(\phi_n \circ f)(z) \overline{f'(z)(\phi_m \circ f)(z)} \, dV(z) \\ &= \int_{\Omega_1} \phi_n(f(z)) \overline{\phi_m(f(z))} \, |f'(z)|^2 \, dV(z). \end{aligned} \tag{4.3}$$

We make the change of variables $w = f(z)$. Since $f$ is bijective and $f'(z) \neq 0$ for any $z \in \Omega_1$, $dV(w) = |f'(z)|^2 dV(z)$ and

$$\begin{aligned} \int_{\Omega_1} \phi_n(f(z)) \overline{\phi_m(f(z))} \, |f'(z)|^2 \, dV(z) &= \int_{\Omega_2} \phi_n(w) \overline{\phi_m(w)} \, dV(w) \\ &= \langle \phi_n, \phi_m \rangle_{A^2(\Omega_2)}. \end{aligned} \tag{4.4}$$

Since $\{\phi_n\}$ is an orthonormal basis, we therefore have

$$\langle \psi_n, \psi_m \rangle_{A^2(\Omega_1)} = \begin{cases} 0 & m \neq n \\ 1 & m = n, \end{cases} \tag{4.5}$$

meaning $\{\psi_n\}$ is an orthonormal set in $A^2(\Omega_1)$.

Next suppose $g \in A^2(\Omega_1)$ such that $\langle g, \psi_n \rangle_{A^2(\Omega_1)} = 0$ for all $n \in \mathbb{N}$. Then

$$\int_{\Omega_1} g(z)\overline{f'(z)(\phi_n \circ f)(z)}\, dV(z) = 0. \tag{4.6}$$

By making the same change of variables as (4.4) we have $w = f(z)$ and $dV(w) = |f'(z)|^2 dV(z)$. Further, as $f$ is bijective, $f^{-1}(w) = z$. Thus,

$$\begin{aligned} \int_{\Omega_1} g(z)\overline{f'(z)(\phi_n \circ f)(z)}\, dV(z) &= \int_{\Omega_2} \overline{(f' \circ f^{-1})(w)} \frac{(g \circ f^{-1})(w)}{|(f' \circ f^{-1})(w)|^2} \overline{\phi_n(w)}\, dV(w) \\ &= \int_{\Omega_2} \frac{(g \circ f^{-1})(w)}{(f' \circ f^{-1})(w)} \overline{\phi_n(w)}\, dV(w) \\ &= \left\langle \frac{g \circ f^{-1}}{f' \circ f^{-1}}, \phi_n \right\rangle_{A^2(\Omega_2)}. \end{aligned} \tag{4.7}$$

Therefore $\left\langle \frac{g \circ f^{-1}}{f' \circ f^{-1}}, \phi_n \right\rangle_{A^2(\Omega_2)} = 0$ for all $n \in \mathbb{N}$. Since $\{\phi_n\}$ is an orthonormal basis for $A^2(\Omega_2)$, we must have that $\frac{g \circ f^{-1}}{f' \circ f^{-1}} = 0$. Thus, $(g \circ f^{-1})(w) = g(f^{-1}(w)) = 0$ for every $w \in \Omega_2$. Therefore, as $f$ is bijective, $g(z) = 0$ for all $z \in \Omega_1$. That is, $g = 0$. As $g$ was arbitrary under this assumption we must have that $\{\psi_n\}$ is an orthonormal basis for $A^2(\Omega_1)$.

By Theorem 2.16,

$$\begin{aligned} K_{\Omega_1}(z, w) &= \sum_{n=0}^{\infty} \psi_n(z)\overline{\psi_n(w)} \\ &= \sum_{n=0}^{\infty} f'(z)(\phi_n \circ f)(z)\overline{(\phi_n \circ f)(w)}\,\overline{f'(w)} \\ &= f'(z) \left( \sum_{n=0}^{\infty} \phi_n(f(z))\overline{\phi_n(f(w))} \right) \overline{f'(w)}. \end{aligned} \tag{4.8}$$

However, as $\{\phi_n\}$ is an orthonormal basis for $A^2(\Omega_2)$, $K_{\Omega_1}(z, w) = \sum_{n=0}^{\infty} \phi_n(z)\overline{\phi_n(w)}$ by Theorem 2.16 once again. Hence,

$$K_{\Omega_1}(z, w) = f'(z) K_{\Omega_2}(f(z), f(w))\overline{f'(w)}. \ \square \tag{4.9}$$

For a simple example of Theorem 4.1 we recall $f : \mathbb{D}_r \to \mathbb{D}$ defined (4.1). Then $f'(z) = \dfrac{1}{r}$ and we therefore have, for all $z, w \in \mathbb{D}_r$,

$$\begin{aligned} K_{\mathbb{D}_r}(z,w) &= f'(z) K_{\mathbb{D}}(f(z), f(w)) \overline{f'(w)} \\ &= \frac{1}{r} \left( \frac{1}{\pi \left(1 - \frac{z\overline{w}}{r^2}\right)^2} \right) \frac{1}{r} \\ &= \frac{r^2}{\pi (r^2 - z\overline{w})^2}. \end{aligned} \tag{4.10}$$

Thus in some occasions we may forgo finding an orthonormal basis to compute the Bergman kernel. Indeed, if we have an explicit formula for the Bergman kernel of $\Omega_2$ and a conformal mapping $f : \Omega_1 \to \Omega_2$, then we immediately have an explicit formula for the Bergman kernel of $\Omega_1$. Of course finding a conformal mapping can be difficult, but it at least opens up another method of calculating the Bergman kernel.

Considering that we know the formula for the Bergman kernel on $\mathbb{D}$, Theorem 4.1 becomes even more valuable when we consider the Riemann mapping theorem.

**Theorem 4.2** (The Riemann mapping theorem)**.** *Suppose $\Omega \subsetneq \mathbb{C}$ is proper and simply connected. If $z_0 \in \Omega$, then there exists a unique conformal map $F : \Omega \to \mathbb{D}$ such that $F(z_0) = 0$ and $F'(z_0) > 0$. [3, p. 224]*

Thus if we know $\Omega \subsetneq \mathbb{C}$ is proper and simply connected, then there must exists a conformal map between $\Omega$ and $\mathbb{D}$. So calculating the Bergman kernel on $\Omega$ amounts to finding a conformal map between $\Omega$ and $\mathbb{D}$. In many cases the conformal maps are known explicitly.

**Corollary 4.3** (The Bergman kernel for the upper half plane)**.** *The Bergman kernel on the upper half plane $\mathbb{H} = \{z \in \mathbb{C} \mid Im(z) > 0\}$ is given by the equation*

$$K_{\mathbb{H}}(z,w) = \frac{-1}{\pi (z - \overline{w})^2} \tag{4.11}$$

*for* $z, w \in \mathbb{H}$.

**Proof:** Notice that the function $f : \mathbb{H} \to \mathbb{D}$ defined by $f(z) = \dfrac{i-z}{i+z}$ is a conformal map with inverse $f^{-1}(z) = i\left(\dfrac{1-z}{1+z}\right)$. For a proof of this, we suggest [3] on pg. 208. Then $f'(z) = \dfrac{-2i}{(i+z)^2}$ and $\overline{f'(z)} = \dfrac{2i}{(-i+\overline{z})^2}$. Thus, by Theorem 4.1,

$$\begin{aligned} K_{\mathbb{H}}(z,w) &= f'(z)K_{\mathbb{D}}(f(z),f(w))\overline{f'(w)} \\ &= \left(\frac{-2i}{(i+z)^2}\right)\left(\frac{1}{\pi(1-\frac{i-z}{i+z}\frac{-i-\overline{w}}{-i+\overline{w}})^2}\right)\left(\frac{2i}{(-i+\overline{w})^2}\right). \end{aligned} \tag{4.12}$$

Carrying out the simplification we see

$$\begin{aligned} K_{\mathbb{H}}(z,w) &= \left(\frac{-2i}{(i+z)^2}\right)\left(\frac{1}{\pi\left(\frac{(i+z)(-i+\overline{w})-(i-z)(-i-\overline{w})}{(i+z)(-i+\overline{w})}\right)^2}\right)\left(\frac{2i}{(-i+\overline{w})^2}\right) \\ &= \left(\frac{-2i}{(i+z)^2}\right)\left(\frac{1}{\pi\left(\frac{-2i(z-\overline{w})}{(i+z)(-i+\overline{w})}\right)^2}\right)\left(\frac{2i}{(-i+\overline{w})^2}\right) \\ &= \frac{1}{-\pi(z-\overline{w})^2}. \end{aligned} \tag{4.13}$$

Hence, $K_{\mathbb{H}}(z,w) = \dfrac{-1}{\pi(z-\overline{w})^2}$. □

**Corollary 4.4** (The Bergman kernel for a sector)**.** *Let* $n \in \mathbb{N}$ *and* $\mathcal{S} = \{z \in \mathbb{C} \mid 0 < arg(z) < \pi/n\}$ *denote a sector in the complex plane. Then the Bergman kernel on* $\mathcal{S}$ *is given by the equation*

$$K_{\mathcal{S}}(z,w) = \frac{-n^2(z\overline{w})^{n-1}}{\pi(z^n - \overline{w}^n)^2} \tag{4.14}$$

*for* $z, w \in \mathcal{S}$.

**Proof:** The function $g : \mathcal{S} \to \mathbb{H}$ defined by $g(z) = z^n$ is a conformal map with inverse $g^{-1}(z) = z^{1/n}$. Then, by Theorem 4.1 and Corollary 4.3,

$$\begin{aligned} K_{\mathcal{S}}(z,w) &= g'(z)K_{\mathbb{H}}(g(z),g(w))\overline{g'(w)} \\ &= nz^{n-1}\left(\frac{-1}{\pi(z^n-\overline{w}^n)^2}\right)n\overline{w}^{n-1}. \end{aligned} \tag{4.15}$$

Thus, $K_{\mathcal{S}}(z,w) = \dfrac{-n^2(z\overline{w})^{n-1}}{\pi(z^n - \overline{w}^n)^2}$. $\square$

We finish with an example which showcases the fact the Bergman kernel is not always given by a simple formula. In particular, we will now compute the Bergman kernel for an arbitrary annulus (centered at the origin). Indeed, let $0 < \sigma < \rho < \infty$ and consider the annulus $\mathbb{A}_\sigma^\rho$. Note that $\mathbb{A}_\sigma^\rho$ is not simply connected and therefore we cannot utilize the Riemann mapping theorem to our advantage. Instead we must find an orthonormal basis.

For each $k \in \mathbb{Z}$ define $\phi_k(z) = c_k z^k$, where

$$c_k = \begin{cases} \sqrt{\frac{k+1}{\pi(\rho^{2(k+1)} - \sigma^{2(k+1)})}} & k \neq -1 \\ \frac{1}{\sqrt{2\pi \ln(\frac{\rho}{\sigma})}} & k = -1 \end{cases} \tag{4.16}$$

Note that if $k < -1$, then $k+1 < 0$. However, as $\sigma < \rho$, we will have $\rho^{2(k+1)} - \sigma^{2(k+1)} < 0$ when $k < -1$. Thus $c_k$ is well defined for $k < -1$.

**Theorem 4.5.** *The set $\{\phi_k\}_{k\in\mathbb{Z}}$ defined above forms an orthonormal basis for $A^2(\mathbb{A}_\sigma^\rho)$.*

**Proof:** We first show that $\{\phi_k\}_{k\in\mathbb{Z}}$ forms an orthonormal set. Indeed, for distinct $k, l \in \mathbb{Z}$,

$$\langle \phi_k, \phi_l \rangle = \int_{\mathbb{A}_\sigma^\rho} c_k c_l z^k \overline{z}^l \, dV(z). \tag{4.17}$$

Setting $z = re^{i\theta}$ with $\sigma \leq r \leq \rho$ and $0 \leq \theta \leq 2\pi$ yields

$$\int_{\mathbb{A}_\sigma^\rho} c_k c_l z^k \overline{z}^l \, dV(z) = \int_\sigma^\rho \int_0^{2\pi} c_k c_l r^{k+l+1} e^{i(k-l)\theta} \, d\theta dr. \tag{4.18}$$

However, since $k \neq l$, we have from (2.35) that $\langle \phi_k, \phi_l \rangle = 0$. Next we show normality.

We have two cases to consider. First suppose $k \neq -1$. Then

$$\begin{aligned} ||\phi_k||^2_{L^2(\mathbb{A}_\sigma^\rho)} &= \int_\sigma^\rho \int_0^{2\pi} \frac{k+1}{\pi(\rho^{2(k+1)} - \sigma^{2(k+1)})} r^{2k+1} \, d\theta dr \\ &= \frac{2(k+1)}{\rho^{2(k+1)} - \sigma^{2(k+1)}} \int_\sigma^\rho r^{2k+1} \, dr \\ &= 1. \end{aligned} \tag{4.19}$$

Second, suppose $k = -1$. Then

$$\begin{aligned} ||\phi_{-1}||^2_{L^2(\mathbb{A}^\rho_\sigma)} &= \int_\sigma^\rho \int_0^{2\pi} \frac{1}{2\pi \ln(\frac{\rho}{\sigma})} r^{-1}\, dr \\ &= \frac{1}{\ln(\frac{\rho}{\sigma})} \int_\sigma^\rho r^{-1}\, dr \\ &= 1. \end{aligned} \tag{4.20}$$

Thus, $||\phi_k||_{L^2(\mathbb{A}^\rho_\sigma)} = 1$ for all $k \in \mathbb{Z}$, thereby proving the orthonormality of $\{\phi_k\}_{k\in\mathbb{Z}}$. Next we prove that $\{\phi_k\}_{k\in\mathbb{Z}}$ forms a basis for $A^2(\mathbb{A}^\rho_\sigma)$, which will require the use of Laurent series.

Suppose $f \in A^2(\mathbb{A}^\rho_\sigma)$ such that $\langle f, \phi_k\rangle = 0$ for all $k \in \mathbb{Z}$. Then, as $f \in A^2(\mathbb{A}^\rho_\sigma)$, $f \in \mathcal{O}(\mathbb{A}^\rho_\sigma)$. Thus $f$ has a Laurent series expansion, $f(z) = \sum_{-\infty}^{\infty} a_m z^m$, which converges uniformly within $\mathbb{A}^\rho_\sigma$. Fix $k \in \mathbb{Z}$ and let $\epsilon > 0$ such that $\sigma + \epsilon < \rho - \epsilon$. Then, by converting to polar,

$$\begin{aligned} \int_{\mathbb{A}^{\rho-\epsilon}_{\sigma+\epsilon}} f(z)\overline{\phi_k(z)}\, dV(z) &= \int_{\mathbb{A}^{\rho-\epsilon}_{\sigma+\epsilon}} \left(\sum_{-\infty}^{\infty} a_m z^m\right) \overline{c_k z^k}\, dV(z) \\ &= \int_{\sigma+\epsilon}^{\rho-\epsilon} \int_0^{2\pi} \left(\sum_{-\infty}^{\infty} a_m r^m e^{im\theta}\right) c_k r^k e^{-ik\theta}\, r d\theta dr. \end{aligned} \tag{4.21}$$

Since $f$ will converge uniformly on $\mathbb{A}^{\rho-\epsilon}_{\sigma+\epsilon}$ to its Laurent series, we have

$$\int_{\sigma+\epsilon}^{\rho-\epsilon}\int_0^{2\pi}\left(\sum_{-\infty}^{\infty} a_m r^m e^{im\theta}\right) c_k r^k e^{-ik\theta}\, r d\theta dr = \int_{\sigma+\epsilon}^{\rho-\epsilon} \sum_{-\infty}^{\infty} a_m c_k r^{m+k+1} \int_0^{2\pi} e^{i(m-k)\theta}\, d\theta dr. \tag{4.22}$$

By (2.35) all terms vanish save for when $m = k$ and we are left with

$$\begin{aligned} \int_{\sigma+\epsilon}^{\rho-\epsilon} \int_0^{2\pi} a_k c_k r^{2k+1}\, d\theta dr &= 2\pi a_k c_k \int_{\sigma+\epsilon}^{\rho-\epsilon} r^{2k+1}\, dr \\ &= \frac{\pi a_k c_k}{k+1} \left((\rho-\epsilon)^{2(k+1)} - (\sigma+\epsilon)^{2(k+1)}\right). \end{aligned} \tag{4.23}$$

Note that in the event that $k = -1$, we will instead have $2\pi a_k c_k \ln(\dfrac{\rho-\epsilon}{\sigma+\epsilon})$. In any case, as $\epsilon > 0$ was arbitrary, we may take $\epsilon \to 0$ to have

$$\langle f, \phi_k\rangle_{A^2(\mathbb{A}^\rho_\sigma)} = \begin{cases} \frac{\pi a_k c_k}{k+1}\left(\rho^{2(k+1)} - \sigma^{2(k+1)}\right) & k \neq -1 \\ 2\pi a_k c_k \ln(\frac{\rho}{\sigma}) & k = -1. \end{cases} \tag{4.24}$$

By assumption $\langle f, \phi_k \rangle_{A^2(\mathbb{A}_\sigma^\rho)} = 0$, which implies $a_k = 0$. As $k \in \mathbb{Z}$ was arbitrary, it follows that $a_k = 0$ for all $k \in \mathbb{Z}$. Thus $f = 0$ and $\{\phi_k\}_{k \in \mathbb{Z}}$ is an orthonormal basis for $A^2(\mathbb{A}_\sigma^\rho)$. $\square$

Therefore, by Theorem 2.16,

$$K_{\mathbb{A}_\sigma^\rho}(z, w) = \sum_{k \in \mathbb{Z}} |c_k|^2 z^k \overline{w}^k, \tag{4.25}$$

where $z, w \in \mathbb{A}_\sigma^\rho$ and the $c_k$ are as defined in (4.16). By looking at this summation it is very difficult to determine whether or not this can simplify to "nice" equation such as on $\mathbb{D}$ or any of the other domains we have computed previously. While it may not be as elegant as the unit disk, there is a way to represent the Bergman kernel for the annulus in a clever way. The first step will utilize conformal mappings.

Given any annulus $\mathbb{A}_\sigma^\rho$, there exists $\tau > 0$ such that $\mathbb{A}_\sigma^\rho$ is conformally equivalent to $\mathbb{A}_1^\tau$. Indeed, set $\tau = \dfrac{\rho}{\sigma}$ and define $f : \mathbb{A}_1^\tau \to \mathbb{A}_\sigma^\rho$ by $f(z) = \sigma z$. Then $f$ is a conformal mapping between the two annuli. Thus given any annulus, we may assume without loss of generality that it has inner radius 1.

Therefore our constants $c_k$ reduce to

$$c_k = \begin{cases} \sqrt{\frac{k+1}{\pi(\rho^{2(k+1)}-1)}} & k \neq -1 \\ \frac{1}{\sqrt{2\pi \ln(\rho)}} & k = -1, \end{cases} \tag{4.26}$$

which will simplify calculations slightly. Next we break up our summation into parts. Namely,

$$K_{\mathbb{A}_1^\rho}(z, w) = \sum_{k=-\infty}^{-2} \frac{k+1}{\pi(\rho^{2(k+1)} - 1)}(z\overline{w})^k + \frac{1}{2\pi \ln(\rho) z\overline{w}} + \sum_{k=0}^{\infty} \frac{k+1}{\pi(\rho^{2(k+1)} - 1)}(z\overline{w})^k. \tag{4.27}$$

For the first summation, we add and subtract $\dfrac{k+1}{\pi}$ to obtain

$$\begin{aligned}\sum_{k=-\infty}^{-2} \frac{k+1}{\pi(\rho^{2(k+1)}-1)}(z\overline{w})^k &= \sum_{k=-\infty}^{-2} \left( \frac{k+1}{\pi(\rho^{2(k+1)}-1)} + \frac{k+1}{\pi} - \frac{k+1}{\pi} \right) (z\overline{w})^k \\ &= \sum_{k=-\infty}^{-2} \frac{(k+1)\rho^{2(k+1)}}{\pi(\rho^{2(k+1)}-1)}(z\overline{w})^k - \sum_{k=-\infty}^{-2} \frac{k+1}{\pi}(z\overline{w})^k.\end{aligned} \tag{4.28}$$

Notice that

$$-\sum_{k=-\infty}^{-2} \frac{k+1}{\pi}(z\overline{w})^k = \sum_{k=-\infty}^{-2} \frac{-k-1}{\pi} \left(\frac{1}{z\overline{w}}\right)^{-k} = \sum_{k=2}^{\infty} \frac{k-1}{\pi} \left(\frac{1}{z\overline{w}}\right)^{k}. \tag{4.29}$$

Then, as $z, w \in \mathbb{A}_1^\rho$, $|z|, |\overline{w}| > 1$. Thus $\dfrac{1}{|z\overline{w}|} < 1$ and we have by Lemma 2.20 that

$$\begin{aligned}\sum_{k=2}^{\infty} \frac{k-1}{\pi} \left(\frac{1}{z\overline{w}}\right)^{k} &= \sum_{k=0}^{\infty} \frac{k+1}{\pi} \left(\frac{1}{z\overline{w}}\right)^{k+2} \\ &= \frac{1}{\pi(z\overline{w})^2} \sum_{k=0}^{\infty} (k+1) \left(\frac{1}{z\overline{w}}\right)^{k} \\ &= \frac{1}{\pi(z\overline{w})^2} \frac{1}{\left(1-\frac{1}{z\overline{w}}\right)^2}\end{aligned} \tag{4.30}$$

To make a small computational aside, we note that

$$\frac{1}{\pi(z\overline{w})^2} \frac{1}{\left(1-\frac{1}{z\overline{w}}\right)^2} = \frac{1}{\pi(1-z\overline{w})^2}, \tag{4.31}$$

which is the same formula for the Bergman kernel on the unit disk only now applied to $z, w \in \mathbb{A}_1^\rho$. In any case, since $z, w \in \mathbb{A}_1^\rho$, $\dfrac{1}{z}, \dfrac{1}{w} \in \mathbb{D}$. Thus we have from (4.30) that

$$-\sum_{k=-\infty}^{-2} \frac{k+1}{\pi}(z\overline{w})^k = \frac{1}{(z\overline{w})^2} K_{\mathbb{D}}\left(\frac{1}{z}, \frac{1}{w}\right) \tag{4.32}$$

and our first summation in (4.27) is equivalent to

$$\frac{1}{(z\overline{w})^2} K_{\mathbb{D}}\left(\frac{1}{z}, \frac{1}{w}\right) + \sum_{k=-\infty}^{-2} \frac{(k+1)\rho^{2(k+1)}}{\pi(\rho^{2(k+1)}-1)}(z\overline{w})^k. \tag{4.33}$$

For the second summation in (4.27), we add and subtract $\frac{k+1}{\pi\rho^{2(k+1)}}$ to obtain

$$\begin{aligned}\sum_{k=0}^{\infty}\frac{k+1}{\pi(\rho^{2(k+1)}-1)}(z\overline{w})^k &= \sum_{k=0}^{\infty}\left(\frac{k+1}{\pi(\rho^{2(k+1)}-1)}-\frac{k+1}{\pi\rho^{2(k+1)}}+\frac{k+1}{\pi\rho^{2(k+1)}}\right)(z\overline{w})^k \\ &= \sum_{k=0}^{\infty}\frac{k+1}{\pi\rho^{2(k+1)}(\rho^{2(k+1)}-1)}(z\overline{w})^k+\sum_{k=0}^{\infty}\frac{k+1}{\pi\rho^{2(k+1)}}(z\overline{w})^k.\end{aligned} \tag{4.34}$$

However, note that

$$\sum_{k=0}^{\infty}\frac{k+1}{\pi\rho^{2(k+1)}}(z\overline{w})^k=\frac{1}{\rho}\left(\sum_{k=0}^{\infty}\frac{k+1}{\pi}\left(\frac{z\overline{w}}{\rho^2}\right)^k\right)\frac{1}{\rho}. \tag{4.35}$$

As $z, w \in \mathbb{A}_1^{\rho}$, $\left|\frac{z\overline{w}}{\rho^2}\right| < 1$. In addition, we also have that $z, w \in \mathbb{D}_r$. Recalling the Bergman kernel on $\mathbb{D}_r$ mentioned in (4.10), we utilize Lemma 2.20 again to see that

$$\begin{aligned}\frac{1}{\rho}\left(\sum_{k=0}^{\infty}\frac{k+1}{\pi}\left(\frac{z\overline{w}}{\rho^2}\right)^k\right)\frac{1}{\rho} &= \frac{1}{\rho}\left(\frac{1}{\pi\left(1-\frac{z\overline{w}}{\rho^2}\right)^2}\right)\frac{1}{\rho} \\ &= K_{\mathbb{D}_\rho}(z,w).\end{aligned} \tag{4.36}$$

Thus our second summation in (4.27) is equivalent to

$$K_{\mathbb{D}_\rho}(z,w)+\sum_{k=0}^{\infty}\frac{k+1}{\pi\rho^{2(k+1)}(\rho^{2(k+1)}-1)}(z\overline{w})^k. \tag{4.37}$$

To bring everything together we now have that for all $z, w \in \mathbb{A}_1^{\rho}$

$$K_{\mathbb{A}_1^{\rho}}(z,w)=\frac{1}{(z\overline{w})^2}K_{\mathbb{D}}\left(\frac{1}{z},\frac{1}{w}\right)+K_{\mathbb{D}_\rho}(z,w)+R(z,w), \tag{4.38}$$

where

$$R(z,w)=\frac{1}{2\pi\ln(\rho)z\overline{w}}+\sum_{k=-\infty}^{-2}\frac{(k+1)\rho^{2(k+1)}}{\pi(\rho^{2(k+1)}-1)}(z\overline{w})^k+\sum_{k=0}^{\infty}\frac{k+1}{\pi\rho^{2(k+1)}(\rho^{2(k+1)}-1)}(z\overline{w})^k. \tag{4.39}$$

Unfortunately, the series in $R$ cannot be summed explicitly. However, $R(z,w)$ has a few important properties. For example, each of the three pieces in the sum is bounded. Therefore, there exists $C > 0$ such that $|R(z,w)| \leq C$ for all $z, w \in \overline{\mathbb{A}_1^\rho}$. So $R$ can be thought of as a "remainder" of sorts. To the point, this implies that the Bergman kernel on the annulus $\mathbb{A}_1^\rho$ is directly related to the Bergman kernels on the unit disk and disk of radius $\rho$.

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