

**Impact of mathematics Courses for Prospective Teachers on their Mathematical
Knowledge for Teaching**

Thesis

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Abstract

This project examines the impact of a mathematics course for prospective elementary teachers and a mathematics course for prospective middle school teachers on those enrolled in their respective courses using a pre-post test methodology. Prospective teachers were asked to take tests, designed by the Learning Mathematics for Teaching project, which claim to measure mathematical knowledge for teaching. Results indicate that the courses positively impacted the mathematical knowledge of prospective teachers. Examination of the results on clusters of items covering specific topics provides additional insight into how these courses are impacting prospective teachers and how they might be modified as a part of ongoing course improvement efforts.

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Chapter 1: Objective

One might consider it trivial to observe that the mathematical preparation of K-12 students is important and worthwhile. As such, it is reasonable to take steps to ensure that students have the best possible chance at getting the best possible mathematical education. One potential way of supporting positive outcomes with students is provide prospective teachers with high quality mathematical preparation. This study seeks to quantitatively examine the efficacy of two mathematics content courses for prospective teachers offered at a large Midwestern university. The goal of such examination is to determine areas in which these courses are successful in nurturing mathematical knowledge, so as to inform ongoing course improvement efforts. The following questions guided data collection and analysis:

1. To what extent will the first semester Math for Elementary Teachers content course result in growth as measured by the Learning Mathematics for Teaching assessments of Mathematical Knowledge for Elementary Teachers? How will observed growth in Number Concepts and Operations compare to observed growth in Patterns Functions and Algebra? In what ways might the design of the Math for Elementary Teachers content course have led to these growth trends?
2. To what extent will the first semester Math for Middle School Teachers content course result in growth as measured by the Learning Mathematics for Teaching assessments of Mathematical Knowledge for Middle School Teachers? How will

observed growth in Number Concepts and Operations compare to observed growth in Patterns Functions and Algebra? In what ways might the design of the Math for Middle School Teachers content course have led to these growth trends?

3. What growth trends occur on specific mathematical topics (e.g. division of fractions) that are covered by at multiple items on the assessments of Mathematical Knowledge for Elementary Teachers? How might the design of the Math for Elementary Teachers course have led to these growth trends?
4. What growth trends occur on specific mathematical topics (e.g. division of fractions) that are covered by multiple items on the assessments of Mathematical Knowledge for Middle School Teachers? How might the design of the Math for Middle School Teachers course have led to these growth trends?

Rationale

Along with its clear applicability to the specific courses under investigation, this study has broad applicability to similar courses at other institutions. In particular, the elementary mathematics content classes used in this project follow an elementary teacher preparation framework that is common across the United States.

This study contributes to a growing body of literature on mathematical knowledge for teaching and ways such knowledge may be nurtured in teacher preparation. A review of articles in the *Journal of Teacher Education*, the *Journal of Mathematics Teacher Education*, and the *Mathematics Teacher Educator* from the year 2014 to the present found only seven articles (Bleiler, Thompson, & Krajčevski, 2014; Subramaniam, 2014; Thanheizer, 2015; Yeh & Santagata, 2015; Whitacre, 2015; Turner & Drake, 2016;

Whitacre & Nickerson, 2016) dealing with the state of mathematical knowledge of prospective K-12 teachers. Of these, only two (Whitacre, 2015; Whitacre & Nickerson, 2016) dealt with the growth of prospective teacher mathematical knowledge due to mathematics content courses. Furthermore, all seven used qualitative methods in their investigations of prospective teacher mathematical knowledge. As such, a quantitative investigation into the growth of prospective teachers' mathematical knowledge due to educational experiences provided for them in mathematics content courses is warranted.

In selecting a tool to gather such data, there was only one reasonable choice. An ideal tool needed to be specific to the population under study and to have established acceptable levels of reliability/validity. Furthermore, it must be in a quantitative format, allow for pre- and post-assessment, and work within practical constraints (e.g. time). The only available assessments that fulfill all of these important criteria are the Learning Mathematics for Teaching assessments.

Definition of Terms

- *Prospective Teacher (PT)*: A student enrolled in a teacher preparation program.
- *Mathematical Knowledge for Teaching (MKT)*: The mathematical knowledge that is required by teachers of mathematics in order to exhibit the highest possible levels of performance at their job.
- *Pedagogical Content Knowledge (PCK)*: Knowledge at the intersection of content knowledge and pedagogical knowledge. This includes, but is not limited to, knowledge of the best examples and representations to use when teaching, as

well as knowledge of the prior understandings and (mis)conceptions that students bring with them.

- *Common Content Knowledge (CCK)*: This is comprised of mathematical knowledge that is common in the United States in the sense that it appears in K-12 curricula. Examples include the standard algorithms for multi-digit addition, subtraction, multiplication, and division.
- *Specialized Content Knowledge (SCK)*: This is comprised of mathematical knowledge that is not common amongst laypeople or non-teacher users of mathematics, but is needed by teachers of mathematics. Examples include the ability to recognize nonstandard algorithms for addition, subtraction, multiplication, and division.
- *Knowledge of Content and Students (KCS)*: This is the subset of pedagogical content knowledge that combines knowledge of mathematics with knowledge of students. It includes the ability to determine what aspects of an example students would find interesting and motivating, and anticipating how easy or difficult a given task might be for students.
- *Knowledge of Content and Teaching (KCT)*: This is the subset of pedagogical content knowledge that combines knowledge of mathematics with knowledge of teaching. This sort of knowledge includes teachers understanding of how to sequence tasks in order lead students to deeper understanding, as well as knowledge of the strengths and weaknesses of various representations.
- *Learning Mathematics for Teaching (LMT)*: The Learning Mathematics for Teaching project is a research initiative focused on investigating mathematical

knowledge for teaching. This goal is pursued through the writing, piloting, and analysis of assessments that are meant to capture such knowledge.

- *Number Concepts and Operations (NCOP)*: This is a subset of mathematical knowledge that encompasses knowledge of the basic facts and underlying structure of number and arithmetic.
- *Patterns, Functions, and Algebra (PFA)*: This is a subset of mathematical knowledge that focuses on basic algebraic topics and structure, as well as on how to identify and describe patterns using algebra and functions.

Chapter 2: Literature Review

This chapter will offer a review of literature in key areas pertaining to current study. First, some basic historical context will be given in order to provide greater insight into the motivations underlying this work. This historical review is not meant to be exhaustive, but it is intended to trace some of the major developments in research on teacher education from the past 30 years, tracing a rough trajectory. Next, the specific assessment materials used for this study will be described, along with details on the theoretical construct underlying these tools and the existing literature on their validity and interpretation.

Historical Context

Debate over what mathematics to teach and how to teach it is enduring. Buried in this debate is the question of what factors influence mathematical achievement of students. Teachers, broadly speaking, are viewed as one of the most influential factors. Thus, it is not surprising that teacher preparation occupies a key role in discussions surrounding how to ensure that our nation's children are mathematically prepared for success in the modern world.

In keeping with this trend, one of the potentially critical variables in mathematics education identified by Edward Begle (1979) in his massive overview of empirical literature on mathematics education was indeed the mathematical preparation of teachers.

In this work, Begle aimed to provide a survey of all of the empirical studies in mathematics education carried out from 1960-1976, and to compile a complete list of all of the variables which had been studied for their impacts on mathematics education. However, Begle did not find that the mathematical preparation of teachers correlated strongly and consistently with student outcomes. Such findings did little to dispel the common-sense notion that more mathematically facile teachers should be able to produce more mathematically facile students. This might be the result of widespread ignorance of Begle's findings, but it is worth noting that Begle's proxies for measuring the mathematical preparation of teachers were very coarse, looking simply at degrees obtained or mathematics courses taken. While such proxies were not unreasonable, they were also not sufficiently fine to allow one to strongly conclude that mathematical preparation of teachers was not impacting future student outcomes.

In the years following Begle's work, many new metrics and assessments for capturing the mathematical knowledge of teachers and prospective teachers have been proposed. Shulman (1986) famously proposed that in addition to pure content knowledge, teachers needed some sort of specialized knowledge in order to perform their duties at the highest level of proficiency. In particular, Shulman proposes the construct of "pedagogical content knowledge" (PCK) to describe knowledge of "the most useful forms of representation... the most powerful analogies, illustrations, examples, explanations, and demonstrations" as well as knowledge of "the conceptions and preconceptions that students of different ages and backgrounds bring with them to the learning of those most frequently taught topics and lessons." (p. 9) Shulman's framework certainly suggests components of teaching that are missed by many measures of

mathematical knowledge, though it problematizes ways to usefully define and capture PCK.

Shulman's proposal that mathematical knowledge for teaching (MKT) might be a unique construct from mathematical knowledge in general inspired many researchers to try to conceptualize theoretical models of what MKT might look like. While an exhaustive overview of these is not warranted here, a brief overview of them is necessary in order to give some insight into the landscape of research in this area. Three specific examples of such efforts will be detailed here since they provide, collectively, an overview of the landscape. These include the conceptualization proposed by the Diagnostic Teacher Assessment of Mathematics and Science, the conceptualization used by the Learning Mathematics for Teaching project, and the conceptualization proposed by Tim Rowland and Fay Turner (2008).

The Diagnostic Teacher Assessment of Mathematics and Science conceptualization of MKT

The Diagnostic Teacher Assessment of Mathematics and Science is based on a model of MKT that grew out of Shulman's proposal. This model splits MKT into four areas: declarative knowledge, conceptual knowledge, problem solving and reasoning, and PCK. In effect, Shulman's categorization was taken as a starting point, and then non-pedagogical content knowledge was split into three layers of depth. Declarative knowledge is taken to be knowledge which can be learned through memorization. Conceptual knowledge is meant to describe knowledge which is rich in relationships, where pieces of knowledge are well-connected to other pieces of knowledge. Problem

solving and reasoning is intended to encompass how people can use their existing knowledge in order to reason inductively and deductively with mathematical ideas (Saderholm, Ronau, Brown, & Collins, 2010). It is worth noting briefly that the declarative/conceptual distinction was inspired by the work of Skemp (1976) as well as Hiebert and Lefevre (1986), while the problem solving category was inspired by Polya (1957) and Schoenfeld (1985).

The Learning Mathematics for Teaching Project conceptualization of MKT

The Learning Mathematics for Teaching (LMT) project took a substantively different view of what MKT looks like. Though Shulman's suggestion is again taken as a jumping-off point, this model splits content knowledge and PCK each into three distinct categories. Content knowledge is divided amongst common content knowledge (CCK), specialized content knowledge (SCK), and knowledge at the mathematical horizon. PCK is split into knowledge of content and students (KCS), knowledge of content and teaching (KCT), and knowledge of curriculum. The four of these areas that have been given abbreviations will be revisited later when discussing the theoretical underpinnings of the assessments used in this study. For now, it shall be assumed that the titles of all six domains are illustrative of their intended meaning, with the possible exception of SCK. SCK is meant to indicate the sorts of knowledge a teacher might have need for that are strictly mathematical rather than pedagogical in nature, but that other professional users of mathematics might not have need for (Hill, Schilling, & Ball, 2004; Hill, Sleep, Lewis, & Ball, 2007; Thames & Ball, 2010).

Tim Rowland and Fay Turner's conceptualization of MKT

Tim Rowland and Fay Turner (2008) offer an alternative model of MKT in direct response to the 6-domain model just described. Rowland and Turner retain the broad category of PCK, but critique SCK as being inadequately distinguished from CCK. Instead, they propose dividing subject matter knowledge into substantive subject matter knowledge and syntactic subject matter knowledge. The former “encompasses the key facts, concepts, principles, structures, and explanatory frameworks” (p. 92) in the discipline while the latter “concerns the rules of evidence and warrants of truth” (p. 92) within the discipline. This method of distinguishing content knowledge was inspired by Schwab (1978).

All of these new conceptualizations of MKT reopened the question that Begle (1979) wrestled with during the 70's. Although Begle's overview didn't find a substantial relationship between the mathematical preparation of teachers and the mathematical achievement of their students, all these new theoretical constructs provide new ways of investigating and measuring the mathematical preparation of teachers. Furthermore, there has been a shift in American colleges toward offering mathematics courses that cater to PT's, which often focus specifically on the mathematics that these future teachers will be expected to teach. These invitations to continue examining the mathematical preparation of PT's are further bolstered by more recent projects which have found a correlation between teacher MKT and future student outcomes (e.g. Monk, 1994; Hill, Ball, Blunk, Goffney, & Rowan, 2007).

The Learning Mathematics for Teaching Assessments

The LMT assessments for MKT were designed and piloted during the 2000's. These tests are developed based on a theoretical framework that conceptualizes MKT to consist of six domains spanning the gamut from knowledge that is primarily content oriented and knowledge that is primarily pedagogical. These six domain include:

- Common Content Knowledge (CCK) and Horizon Content Knowledge (HCK)
- Specialized Content Knowledge (SCK)
- Knowledge of Content and Students (KCS) and Knowledge of Content and Teaching (KCT)
- Knowledge of Content and Curriculum

The above list is ordered according to the aforementioned spectrum. The actual items on the LMT assessments focus only on CCK, SCK, KCS, and KCT. The KCT items are sectioned off from the other items, so the multiple-choice items appearing on these assessments focuses in on only CCK, SCK, and KCS (Hill, Schilling, & Ball, 2004).

The LMT assessments are comprised of multiple tests, separated according to both broad content area and the grade level that the test-takers teach or intend to teach.

The following tests are available:

- Number Concepts and Operations (Grades K-6 or Grades 6-8)
- Patterns, Functions, and Algebra (Grades K-6 or Grades 6-8)
- Geometry (Grades 4-8)
- Rational Number (Grades 4-8)
- Proportional Reasoning (Grades 4-8)
- Data, Probability, and Statistics (Grades 4-8)

All of these assessments are multiple choice and were designed in close cooperation with psychometricians (Hill, Schilling, & Ball, 2004). They are norm-referenced rather than criterion-referenced; consequently, scores indicate only how well test-takers have done compared to others rather than indicating how much they know compared to what they are expected to know. In practice, this has led to them being used to compare test-takers to themselves at two different points in time, thereby capturing growth that occurred due to coursework or professional development (e.g. Hill & Ball, 2004).

Although the LMT assessments lack the reliability index needed to be used towards making any substantive claims about individuals, they are sufficiently reliable to allow for making claims about populations of PT's or teachers. Hill, Schilling, and Ball (2004) found that the early NCOP and PFA assessments had Cronbach's alpha values between 0.845 and 0.89 (p. 25).

The question of validity is more complicated, but there is some empirical evidence supporting the validity of these assessments. The theoretical construct of MKT, as described above guided item development process for the LMT measures. Heather Hill, Stephen Schilling, and Deborah Ball (2004) clearly state that one of their central assumptions is that teachers possess some form of specialized content knowledge that distinguishes them from both laypeople and other professional users of mathematics (p. 12). This belief is manifested, at least superficially, in the nature of the questions as listed on the tests (LMT, 2008b). This is perhaps best seen by contrasting these items with those from other assessments used to measure teacher knowledge. ETS' Praxis mathematics content test (Educational Testing Service, 2015) is highly indicative of some

of the most common historical trends in such assessments, as elaborated on by Hill, Sleep, Lewis, and Ball (2007) who provide many such points of comparison. Whereas ETS' items involve exactly the same sorts of questions we might reasonably expect to be asked of a K-12 student, the LMT items appear to be more specific to the actual experiences of teachers.

These sorts of comparisons lend the LMT measures face validity. However, face validity is subjective by definition. Consequently, more extensive consideration of the assessments' capacity to capture the knowledge it claims to capture is certainly warranted. Schilling and Hill (2007) describe how Kane's (2001; 2004) approach to validity testing was modified and used to examine the LMT assessments in a sequence of papers (Hill, Ball, Blunk, Goffney, & Rowan, 2007; Hill, Dean, & Goffney, 2007; Schilling, 2007; Schilling, Blunk, & Hill, 2007). In short, they lay out a sequence of assumptions and inferences, they then perform a sequence of experiments/investigations in order to evaluate and potentially reformulate these assumptions and inferences.

Schilling and Hill (2007) make the "elemental assumption" that the items on the LMT assessments reflect teachers' MKT rather than alternate skills such as test taking strategies. Their consequent inference is that "teachers' reasoning for a particular item will be consistent with the multiple choice answer they selected." (p. 79) To assess this assumption and inference, Hill, Dean, and Goffney (2007) engaged teachers, non-teachers, and mathematicians in clinical interviews using a subset of the LMT test items. They found evidence to support the elemental assumption with pure CK items. For these, the vast majority of participants relied on mathematical justification or reasoning in order to answer the questions. However, they found substantial reason to question the validity

of KCS items. For these, approximately 50% of study participants used purely mathematical reasoning rather than knowledge of students, and approximately 20% relied on test-taking strategies. Furthermore, some items received a large number of correct responses from non-teachers who used purely mathematical reasoning.

Schilling and Hill (2007) then make the “structural assumption” that MKT can be distinguished by both subject matter (e.g. geometry, algebra, number concepts, etc.) and by knowledge type (CCK, SCK, and KCS). They propose four inferences that follow from this assumption: Items reflecting the same subject matter and type of knowledge will have stronger inter-item correlation than items that differ in one or both of these, “teachers can be reliably distinguished by unidimensional scores reflecting this organization by subject matter and types of knowledge,” teachers will typically answer non-CCK items using knowledge specific to teaching while non-teacher rely on other skills, and “teachers’ reasoning for a particular item will reflect the type of reasoning that the item was designed to reference.” (pp. 79-80)

These assumptions/inferences are analyzed by both Schilling (2007) and by Hill, Dean, and Goffney (2007) in a series of related studies. The latter study, described earlier, briefly notes that some evidence was found for multidimensionality. In particular, it was observed that some of the teachers were able to correctly answer KCS items using their knowledge of students whilst mathematicians and non-teachers tended to reason using other, “more circuitous” (p. 92) means. It is further noted that the mathematicians were less likely to correctly answer the KCS items than the CK items.

Schilling (2007) took a different approach to investigating the structural assumption. Rather than making use of clinical interviews, he took roughly 1235 results

from a large pilot project of the LMT measures (see Hill & Ball, 2004) and then performed exploratory full-information item factor analysis using the tools of psychometrics. Strong evidence was found that CK and KCS items were measuring different constructs. However, the distinction between CCK and SCK was much weaker. It was also found that CCK was indeed essentially unidimensional, but Schilling was unable to construct a reliable and essentially unidimensional scale for the KCS items. With regard to the SCK items, Schilling goes so far as to claim that “there is probably no way to construct a unidimensional scale” for such items (p. 106).

The final assumption that Schilling and Hill (2007) make is the “ecological assumption,” that the LMT measures of MKT capture knowledge which teachers need in order to effectively teach mathematics. The inferences made following this assumption are that higher scores these measures will correlate positively with higher quality mathematics instruction and with improved student learning. Hill, Ball, Blunk, Goffney, and Rowan (2007) tackled this problem in two ways:

- First, they pulled data from the Study of Instructional Improvement that attached MKT scores to teachers and Terra Nova standardized test scores to their students. They found that this data showed positive correlation between teacher scores on the LMT items and the gains of their students on the Terra Nova standardized test of mathematics following a year of instruction, and that this effect persisted even when various student, classroom, and teacher characteristics were controlled for (e.g. the SES of students or the number of years of teaching experience possessed by the teacher). “Comparing a teacher who achieved an average MKT score and a teacher who was in the top quartile of scores, we saw that the above-average

teacher “added” an effect equivalent to that of 2 to 3 extra weeks of instruction to her students’ gain scores” (p. 109).

- Second, they analyzed videotapes of lessons taught by teachers participating in California’s Mathematical Professional Development Institutes and rated the quality of the lessons, finding a positive correlation between these scores and the teachers’ scores on the LMT measures.

In short, there is some evidence that the LMT measures are usefully valid, if not perfectly so (for critique of these validity studies, see Alonzo, 2007; Fisher, 2007; Garner, 2007; Gearhart, 2007; Kane, 2007; Kulikowich, 2007; Lawrenz & Toal, 2007; Schoenfeld, 2007). It is also worth pointing out that the test is a cultural construct, and that any shift from the culture in which it was constructed can be problematic (e.g. Delaney, Ball, Hill, Schilling, & Zopf, 2008; Ng, 2011; Ng, Mosvold, & Fauskanger, 2012). This issue should have minimal impact on the data I am analyzing here, thanks to the fact that it was collected from PT’s in an American university. However, several PT’s are from other countries, and this could have impacted their scores.

Chapter 3: Methods and Procedure

This study took place at a large Midwestern university. At this institution, prospective elementary teachers are required to take two semesters of mathematics content for elementary teachers followed by one semester of mathematical methods course. Prospective middle school teachers are required to take general mathematics courses through pre-calculus and introductory statistics as well as four semesters of mathematics content for middle school teachers and one semester long course in of methods of teaching middle school.

Data was collected during the first semester of content course sequence for elementary teachers as well as the first semester of the pure content course for middle school teachers. Students in each class were asked to take two of the LMT assessments of MKT. All data collection took place during the fall semester of 2015, with PT's taking a pre-test within the first two weeks of the semester and a post-test during the last two weeks of the semester. Thus, any demonstrated growth from the pre-test to the post-test could likely be attributed to the given content course.

Students were asked to take those LMT modules that corresponded to the materials covered in their respective courses. In both cases, this meant that PT's were asked to complete a number concepts and operations assessment (LMT, 2006; LMT, 2007a) as well as a patterns, functions, and algebra assessment (LMT, 2007b; LMT, 2008a). Participation was purely voluntary, although participants did receive a minor

course credit incentive; elementary PT's were given one day's worth of attendance credit each for the pre-test and post-test while middle school PT's were given a homework grade. The scores of specific students were not revealed to their course instructors and had no bearing on their receipt of this participation incentive. PT's took the assessments online via the Teacher Knowledge Assessment System that was developed by the LMT project.

The Elementary Course and Elementary PT's Under Study

Three sections of the Mathematics for Elementary Teachers course participated in this study. These sections had 43, 44, and 46 students enrolled, respectively, for a total of 133 PT's. 111 of these took the pre-test, 84 took the post-test, and 83 took both.

This course is intended specifically to prepare people to teach elementary students. The course focuses on number concepts and operations and number theory, but also includes work with expressions, equations, sequences, and series. Additional algebraic and geometric topics are not covered until the second semester of the course sequence.

The course instructors believe that “knowing the mathematics for yourself is not the same as knowing the mathematics for teaching” (Course Instructors, 2015a). To that end, they place more emphasis on explanation than on simply arriving at correct answers. All problems that appear on homework, quizzes, or tests requires PT's to explain how they arrived at their solution using narrative complemented by pictures.

Beckmann's (2014) text is used for the class. Students complete chapters 1 – 9 of this textbook. PT's are expected to read the text, and some of the homework items are

drawn from the homework sets contained therein. Classes meet five times a week, three times with the course instructor in groups of approximately 50 students and twice a week with a teaching associate in groups of approximately 25 students. All class sessions focus on group activities, taken either from Beckmann's text or from the supplementary materials that the course instructors have developed over years of working together. These supplementary activities feature activities in the vein of the class activities in Beckmann. Some of these supplementary activities cover topics not covered in Beckmann's text (e.g. the development of place value numbering systems), while others are meant to augment or replace activities from Beckmann's text.

In addition to the content-oriented learning objectives, course instructors also make explicit five other learning goals for PT's:

- Persevere in problem solving
- Make connections among mathematical topics to deepen understanding
- Develop the meaning underlying definitions, formulas, and algorithms
- Use correct and precise mathematical language
- Evaluate the spoken and written work of others to improve correctness and clarity

It is reasonable to suggest that these broader conceptual goals match the desired mathematical practices future teachers are expected to nurture among school learners.

The Middle School Course and Middle School PT's Under Study

Two sections of Mathematics for Middle School Teachers participated in this study. These sections had 24 and 18 students enrolled, respectively, for a total of 42 PT's. 40 of these participated in the pre-test, 38 in the post-test, and 37 in both.

This course is designed to prepare people to teach mathematics to middle school students. The course covers topics such as number concepts and operations, arithmetic, algebra, number theory, ratios and proportions, sequences, series, and polynomials. Additional topics such as geometry, calculus, and the history of mathematics are covered in later courses. Additional algebraic topics also appear in later courses.

There is substantial overlap between how the two content courses for prospective elementary and middle school teachers are designed. Common methodologies include the use of group work, emphasis on meaning making and conceptual understanding, and deepening understanding of connections among mathematical topics. Course instructors share the beliefs of their elementary associates regarding the importance of explaining thinking and reasoning. The course syllabus indicates that the course focuses on deep understanding of school mathematics, “which involves being able to explain (in multiple ways) why facts are true and why procedures work.” (Course Instructors, 2015b)

Class meetings occur five times a week. One section of the course has all five meetings with a course instructor, while the other section of the course has three meetings with a course instructor and two with a teaching associate. Class meetings focus on group work and sharing out reasons and arguments. Rather than having a textbook, students use course notes which were coauthored by several of the course instructors.

Chapter 4: Results and Analysis

This chapter details the results of the elementary and middle school assessments of MKT. First, the results of the Elementary NCOP, Elementary PFA, Middle School NCOP, and Middle School PFA assessments are shared. Next, topic specific breakdowns are offered for each test, respectively. The topic specific breakdowns isolate clusters of three or more items that test the same specific piece of content (e.g. division of fractions) and look at growth trends within that topic.

Elementary and Middle School NCOP and PFA Results

I begin by considering some broad results. A good place to start is examining the spread of scores on the various pre and post-tests. Figure 1 shows score frequency histograms for the elementary PT's, while figure 2 shows the score frequency histograms for the middle school PT's.

Note briefly that the scores seem to be normally distributed, as one might expect, with the middle PFA post-test showing a distinct skew to the left. There is also a visible tendency for post-test scores to be centered further to the right as compared to pre-test scores, though this shift is very weak in the case of the elementary PFA scores. This suggests that PT's demonstrated growth on all assessments, but most weakly on the elementary PFA assessment.

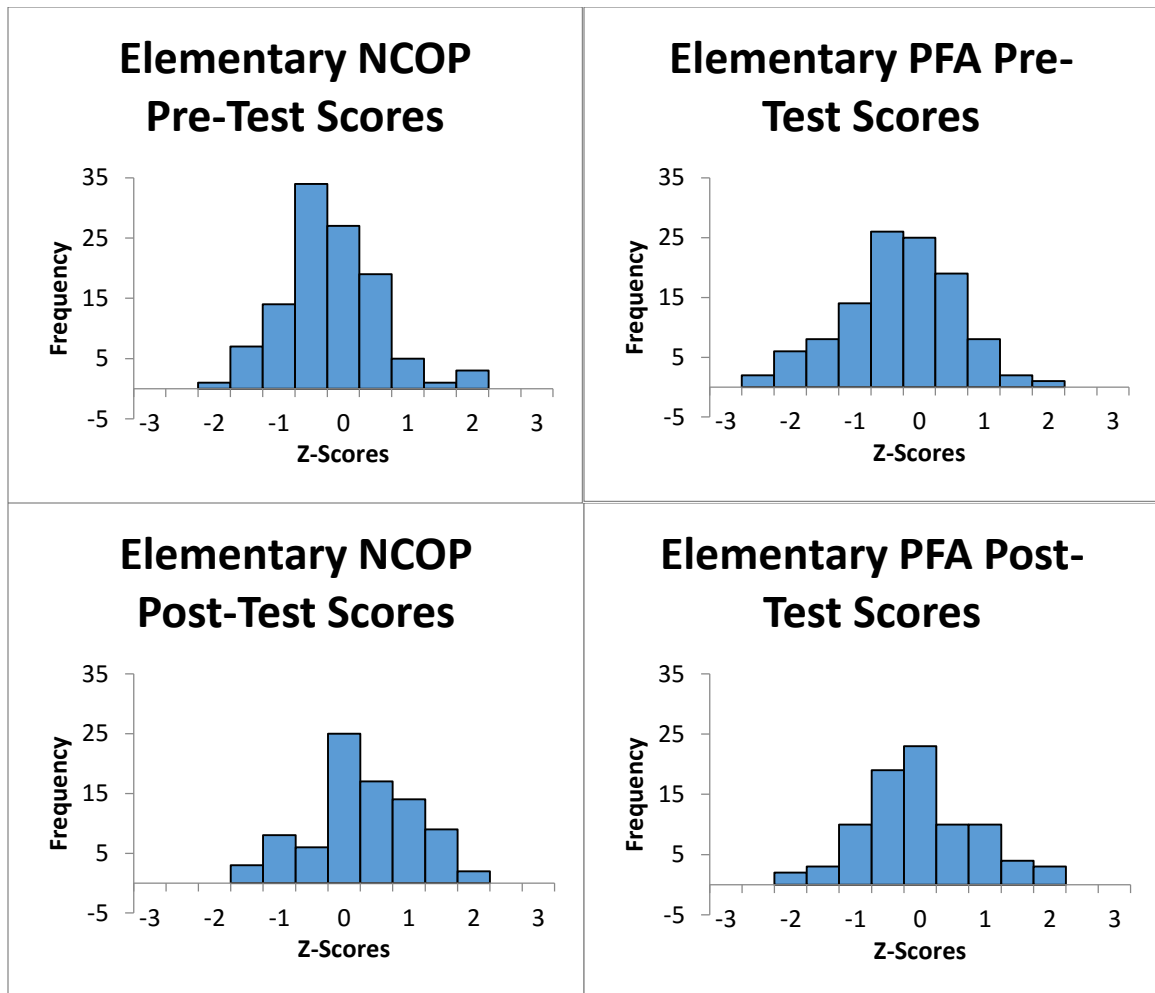


Figure 1: Histograms of elementary pre ($n=111$) and post ($n=84$) score frequencies

The next, and perhaps most important, piece of data to examine is the growth data. Recall that pre- and post-test scores were paired, so that we could compare each individual PT's growth in Z-score from the start of the semester to the end of the semester. Figures 3 and 4 contain the frequency histograms for these score changes for elementary and middle PT's, respectively. It is apparent that the majority of these results are at or above zero, again suggesting that PT's demonstrated growth on all four assessments.

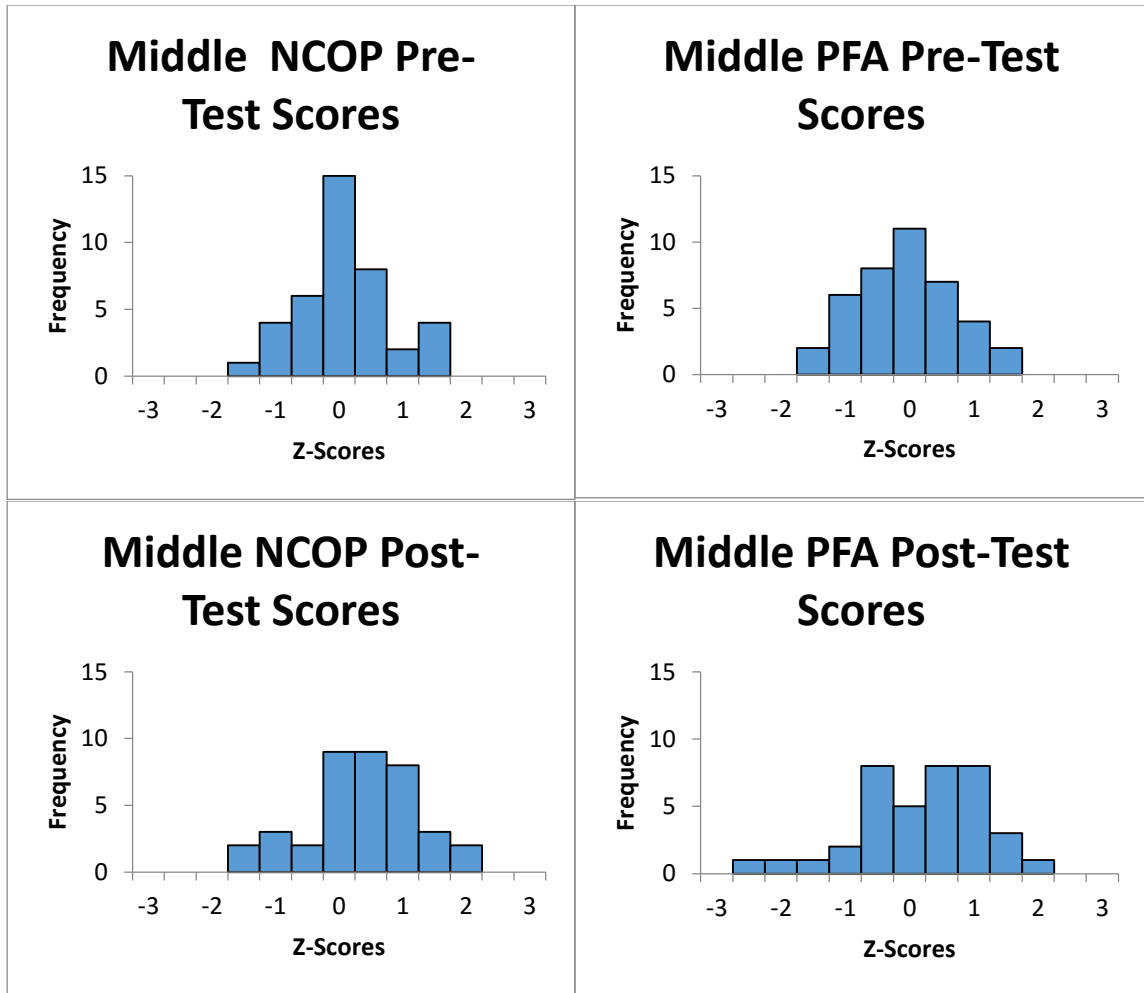


Figure 2: Histograms of middle pre (n=40) and post (n=38) score frequencies

Table's 1 and 2 provide summary statistics on key learning objectives. Table 1 provides the five number summary for growth scores in each test as well as the mean and standard deviation. Table 2 shows statistical measures of the growth data relating to effect size and statistical significance. With all of these materials in hand, we are now well-prepared to make some inferences based on the data.

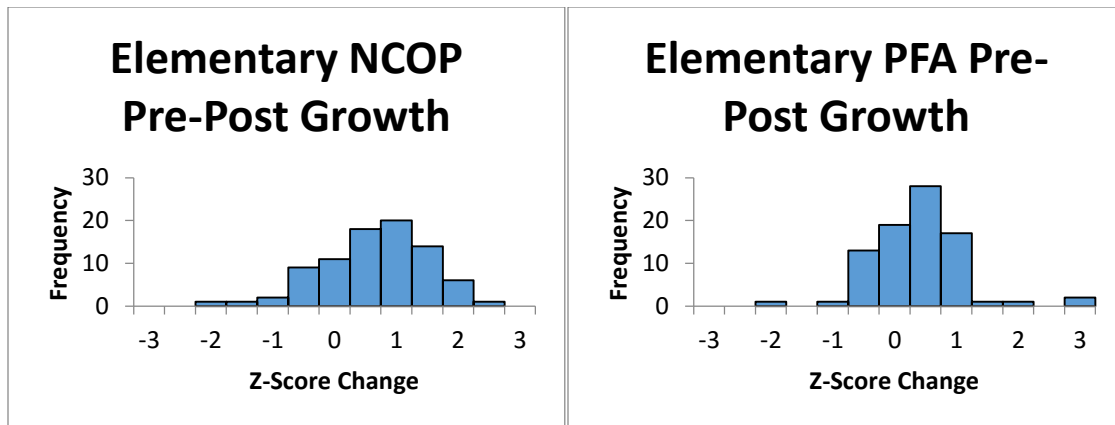


Figure 3: Elementary score change frequencies (n=83)

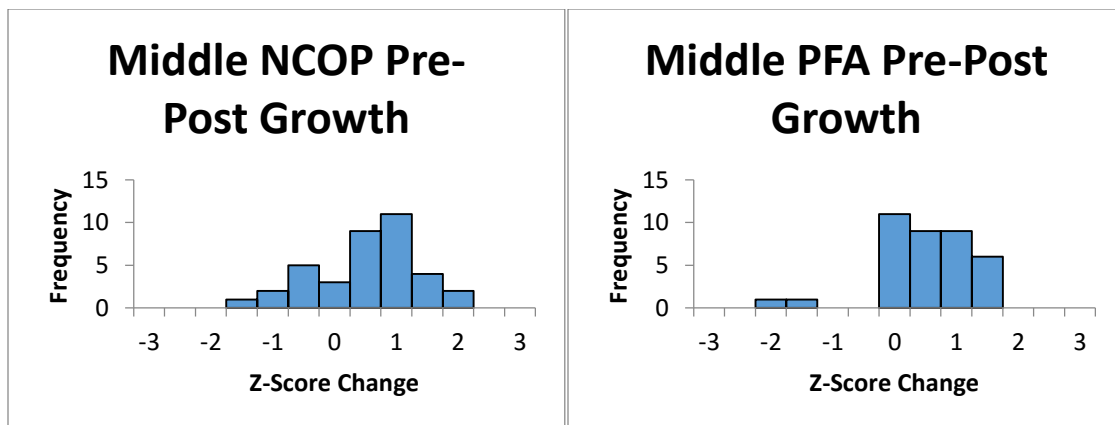


Figure 4: Middle score change frequencies (n=37)

There is strong evidence that PT's experienced growth in their MKT, as captured by these assessments, as a result of taking their respective courses. The greatest amount of growth was observed in the Elementary NCOP test, and the least in the Elementary PFA test. This trend is visible in the histograms illustrated above, but is made very clear by the statistical measures in tables 1 and 2. The former table shows all assessments having median and mean growths in Z-Score greater than 0, with the highest/lowest mean and median reflecting the trend suggested above.

Test	n	Max	Q3	Median	Q1	Min	Mean	SD
Elementary NCOP	83	2.0878	0.9728	0.4998	-0.2393	-2.1156	0.3627	0.8111
Elementary PFA*	81	1.6452	0.4864	0.0712	-0.2377	-2.2192	0.0813	0.6110
Middle NCOP	37	1.6616	0.8133	0.4304	-0.2987	-1.7913	0.2520	0.8737
Middle PFA	37	1.4778	0.7595	0.3195	-0.1446	-2.1496	0.2571	0.7713

Table 1: Summary statistics for pre- to post-test growth scores

Test	n	Cohen's d	T (95% CI)	Significance (2-tailed)	Mean Difference	Lower Bound	Upper Bound
Elementary NCOP	83	0.4861	4.0735	0.0001	0.3627	0.1856	0.5398
Elementary PFA	83	0.2078	1.8574	0.0668	0.1747	-0.0124	0.3618
Middle NCOP	37	0.3232	1.7544	0.0879	0.2520	-0.0393	0.5433
Middle PFA	37	0.3063	2.0280	0.0500	0.2571	0.0000	0.5143

Table 2: Effect size and significance statistics for pre- to post-test growth scores

Cohen's D, a statistical measure of the mean difference in pre- and post-test scores expressed in standard deviation units, indicates small to moderate positive effect sizes in all four areas as shown in table 2. For the elementary course, the NCOP effect size is nearly 0.5 while the PFA effect size is approximately 0.2, suggesting again that their course resulted in far greater improvement in NCOP than in PFA. The effect sizes for middle NCOP and PFA are both roughly 0.3, suggesting that those PT's saw comparable improvement in both areas.

To look at the statistical significance of the observed growth, we make use of a 2-

tailed t-test with the standard 95% confidence interval. The elementary NCOP score improvement is highly significant, with a p-value of 0.0001. The middle PFA score improvement is statistically significant, with a p-value of almost exactly 0.05. The improvement on the elementary PFA and middle NCOP show weak evidence of significance, with p-values of approximate 0.067 and 0.088, respectively.

These results are in keeping with the content emphasis in their respective courses. The first semester of the mathematics course for future elementary teachers has a much stronger emphasis on number concepts and operations than it does on algebra. In fact, algebra is not even introduced as a topic until the last few weeks of the course. In the mathematics course for middle school teachers, on the other hand, algebra and number concepts receive a somewhat more equitable treatments, and algebra is introduced much earlier.

Trends in Growth on Individual Topics: Elementary NCOP

While it is useful to see how PT's performed on the NCOP and PFA assessments, these content umbrellas are too broad to offer specific guides towards establishing course impact. In order to give more specific feedback about courses under study, it is necessary to see how PT's experienced growth in finer detail. Here, test items that cover similar content have been collected for discussion. Only topics that are covered by at least three items will be discussed here, with the exception of one topic covered by only two items that stood out as requiring discussion for other reasons. Although the LMT terms of use preclude the inclusion of copies of LMT assessment items in this document, descriptions of the items will be provided.

For each item, five pieces of data will be provided: The percent of PT's who responded correctly on the pre-test, the percent of PT's who responded correctly on the post-test, the difference of these two numbers, the items official reference information, and an informal number to reference during discussion of the items. Looking at the difference of the percent correct values will allow for conjecture regarding how the course impacted PT knowledge of the given topic, while the original percentages will give some indication of how difficult that item was for PT's. Since the reliability indices of these assessments allow for conclusions at the level of a population but not at the level of an individual, care must be taken in deciding whether or not a change in the correct answers percentage is simply due to imperfect reliability. To account for this, differences within the interval (-10%, 10%) will not be assumed to be meaningful.

In order to make it easier to see how PT performance changed from the start of the course to the end of the course on items, a standardized color scheme has been adopted for the tables below:

- Orange indicates differences in the interval (-20%, 10%]
 - Yellow indicates differences in the interval (-10%, 10%)
 - Light green indicates differences in the interval [10%, 20%)
 - Green indicates differences in the interval [20%, 30%)
 - Blue indicates differences greater than 30%
- Place Value

This test included 9 items that dealt purely with place value, outside the context of any operation. Items 1a – 1d all asked PT's to identify whether a proposed non-prototypical decomposition of a three digit number into ones, tens, and hundreds was

correct or incorrect. All four items used the same three digit number, but had a different proposed decomposition. Items 2a – 2e presented pictures of base-ten blocks (cubes, rods of 10 cubes, “flats” of 100 cubes, and “blocks” of 1000 cubes), then proposed a number of different ways of representing a decimal comprised of three digits with these manipulatives. Students then had to determine whether each proposed representation was correct or incorrect.

From a purely mathematical perspective, questions 1 and 2 are very similar. However, student performance on the two questions differed markedly. Table 3 shows the percentage of students who got each item correct on the pre-test, the percent who got the item correct on the post-test, and the difference of these two values. For all four parts of problem 1, PT’s failed to improve on two parts and actually performed *less well* during their post-assessment on the other two parts. For all five parts of problem 2, *more* students answered the item correctly on the post-test. In fact, the increase in students answering 2 correctly was greater than 20% for three parts.

Item	1a	1b	1c	1d	2a	2b	2c	2d	2e
Pre- Correct	49%	46%	51%	52%	51%	71%	53%	29%	62%
Post- Correct	43%	39%	37%	39%	81%	88%	81%	52%	76%
Difference	-6%	-7%	-14%	-13%	30%	17%	28%	23%	14%
Item Reference	EI_NCOP 2002A_1	EI_NCOP 2002A_1	EI_NCOP 2002A_1	EI_NCOP 2002A_1	EI_NCOP 2008A_27	EI_NCOP 2008A_27	EI_NCOP 2008A_27	EI_NCOP 2008A_27	EI_NCOP 2008A_27

Table 3: Percent correct for Elementary NCOP items dealing with place value

How, then, do these problems differ? There are a few key differences. First, although both problems use numbers comprised of three digits, problem 1 has no digit past the decimal. Second, problem 27 includes visuals/manipulatives while problem 1 doesn't. Third, problem 27 uses decompositions that are open to interpretation (e.g. 13 rods and 2 cubes could mean 1320, 132, 13.2, etc.) while problem 1 uses decompositions that are not open to interpretation (e.g. 13 tens and 2 ones).

- The Algorithm for Division

This test included 2 items dealing specifically with the division algorithm. Item 3 shows student work wherein the student uses a (valid) nonstandard division algorithm, then asks PT's to respond based on whether they think the method is always valid, sometimes valid, or never valid. An analogous item exists in the set of released items (LMT, 2008b) and can be found in figure 5 for reference. Item 4 displays an example of the conventional division algorithm and asks students to select an answer that best indicates *why* it works.

Table 4 shows the percentage of students who answered each item correctly on the pre-test, the percentage who answered correctly on the post-test, and the difference in these two values. The starred column will be elaborated on in the next paragraph. Item 3 stands out as having the greatest raw increase in percent correct on the post-test as compared to the pre-test out of the entire Elementary NCOP assessment. Item 4, on the other hand, saw no increase in the percentage of students answering it correctly.

This may seem counterintuitive at first, since being able to recognize valid nonstandard algorithms implies a greater understanding of division in general and the prototypical algorithm in particular. However, this could be partially accounted for by

2. Imagine that you are working with your class on multiplying large numbers. Among your students' papers, you notice that some have displayed their work in the following ways:

Student A	Student B	Student C
$\begin{array}{r} 35 \\ \times 25 \\ \hline 125 \\ +75 \\ \hline 875 \end{array}$	$\begin{array}{r} 35 \\ \times 25 \\ \hline 175 \\ +700 \\ \hline 875 \end{array}$	$\begin{array}{r} 35 \\ \times 25 \\ \hline 25 \\ 150 \\ 100 \\ +600 \\ \hline 875 \end{array}$

Which of these students would you judge to be using a method that could be used to multiply any two whole numbers?

	Method would work for all whole numbers	Method would NOT work for all whole numbers	I'm not sure
a) Method A	1	2	3
b) Method B	1	2	3
c) Method C	1	2	3

Figure 5: Released test item showing nonstandard multiplication algorithms (LMT, 2008b)

Item	3	4	4*
Pre-Correct	22%	22%	32%
Post-Correct	55%	22%	38%
Difference	33%	0%	6%
Item Reference	EL_NCOP_2004B_11	MS_NCOP_2005B_4	MS_NCOP_2005B_4

Table 4: Percent correct for Elementary NCOP items dealing with algorithm for division

the fact that this problem includes two *correct* answers, even if only one of them is the *best* answer as required by the directions. Both answers accurately describe division as repeated subtraction, but the *best* answer opts for a more efficient form of subtraction that ties in more closely with the algorithm. The starred column in Table 4 shows what the results look like if you accept both correct explanations. Unfortunately, even after taking this into account, any growth in item 4 is still insignificant.

- Partitive vs. Quotative Division

This test included 4 items that focus exclusively on requiring PT's to distinguish partitive division from quotative division. Problem 5 starts by giving PT's a story problem which would be solved by division. Parts a – d then each give another story problem and ask if it uses the same interpretation of division.

Table 5 shows that there was basically no change in the percent of students answering parts a, c, and d correctly. Part b, however, saw the percent of students answering correctly double, increasing from 22% to 44%. Part b also stands out as having been far more challenging for PT's, with only 1/3 as many answering it correctly in the pre-test as compared to parts a, c, and d.

It is difficult to explain why PT performance on part b differed so markedly from their performance on parts a, c, and d. Parts a, b, and d all used a different interpretation of division than the original story problem; consequently, that does not distinguish b from the other parts. Parts a, b, c, and d all used exactly the same two numbers as the dividend and divisor, so that does not distinguish b from the rest. However, there does seem to be a subtle but important difference amongst these test items. Parts a and d use situations where “groups” and “objects” may be meaningful mathematically, but they are not

Item	5a	5b	5c	5d
Pre-Correct	67%	22%	60%	65%
Post-Correct	67%	44%	62%	63%
Difference	0%	22%	2%	-2%
Item Reference	EL_NCOP_2008B_22	EL_NCOP_2008B_22	EL_NCOP_2008B_22	EL_NCOP_2008B_22

Table 5: Percent correct for Elementary NCOP items dealing with distinguishing partitive and quotative division

subject to casual interpretation of those words. For example, when given the area of a rectangle and dividing by the length of one of the sides in order to get the remaining side, casual interpretation of the words group and object have no obvious meaning. However, the original example, as well as parts b and c, feature story problems that are subject to casual interpretation of the words group and object. Story problems involving exemplify this sort of problem. This similarity in the original problem and parts b and c may explain why so many PT's claimed both b and c used the same interpretation of division as the original.

- Dividing by Fractions

This test included 9 items dealing division by fractions. Items 6 a – d present a story problem which would be solved by dividing a natural number by a fraction. Parts a – d each suggest an answer to this problem, and PT's must decide if they are correct or not. Answering correctly requires students to both carry out the calculation correctly, obtaining a mixed number, and then to interpret what that fractional portion of the answer is a fraction of. Problem 7 presents an arithmetic expression where a natural number is

being divided by a fraction. Parts 7a – 7d each present a story problem that would supposedly be solved by evaluating the arithmetic expression, and PT’s are tasked with determining whether or not this is true. Item 8 presents PT’s with a valid nonstandard method of dividing fractions that students are known to discover on their own, and asks PT’s if the method is never valid, conditionally valid, or always valid.

Table 6 presents the student performance on problems 6, 7, and 8. Note that there was steady improvement on these items. The only item that saw a decrease in the percentage of students who answered it correctly was item 7c, but this item also began with a rather imposing 81% of students responding correctly. It is also notable that roughly half of students answered each part of problem 6 incorrectly on both the pre- and post-assessment. Furthermore, item 8 proved to be very difficult for PT’s, but also saw the percentage answering it correctly more than double.

Item	6a	6b	6c	6d	7a	7b	7c	7d	8
Pre-Correct	30%	30%	47%	38%	56%	60%	81%	42%	12%
Post-Correct	43%	43%	57%	40%	62%	71%	69%	57%	27%
Difference	13%	13%	10%	2%	6%	11%	-12%	15%	15%
Item Reference	EI_NCOP 2008A_20	EI_NCOP 2008A_20	EI_NCOP 2008A_20	EI_NCOP 2008A_20	EI_NCOP 2004A_11	EI_NCOP 2004A_11	EI_NCOP 2004A_11	EI_NCOP 2004A_11	EI_NCOP 2001C_17

Table 6: Percent correct for Elementary NCOP items dealing with dividing by fractions

- Borrowing in the Subtraction Algorithm

This test included 4 items that focused on borrowing in the subtraction algorithm. Items 9 a – c present an example of the standard multi-digit subtraction algorithm having been carried out correctly. Parts a, b, and c then provide student explanations for the borrowing that occurred in the algorithm, and PT’s are asked to decide if each student explanation suggests correct understanding of the algorithm or not. Item 10 presented a classroom activity where base 10 blocks are used to model subtraction of two-digit numbers. PT’s are then given 4 student responses to this activity, and asked to select which one would be the best segue into discussing the comparison meaning of subtraction.

Table 7 shows the results on items 9a – 9c and item 10. Item 9b is a correct explanation for what borrowing does, and PT’s improved substantially at identifying it as such. Item 9c proved difficult for PT’s at both the start and end of the course; it proposes a student explanation that correctly describes the rote procedure of borrowing but conveys no underlying meaning. Item 10, which is very difficult to answer using purely

Item	9a	9b	9c	10
Pre-Correct	43%	46%	20%	9%
Post-Correct	55%	69%	21%	14%
Difference	12%	23%	1%	5%
Item Reference	EL_NCOP_2008B_15	EL_NCOP_2008B_15	EL_NCOP_2008B_15	EL_NCOP_2008A_32

Table 7: Percent correct for Elementary NCOP items dealing with borrowing in the subtraction algorithm

mathematical reasoning, proved extremely difficult for PT's.

- Subtraction of Negatives

This test included 4 items focusing on subtraction of negatives. All four of them shared a prompt describing a real-life situation in which one might need to find the difference between a positive integer and a negative integer. Parts a – d each provided a students proposed arithmetic expression meant to evaluate the difference (e.g. $m + (-n)$) as well as the students explanation for why their method made sense. PT's are thus tasked with deciding whether each student's explanation reflects correct thinking or not.

Table 8 summarizes the results. It seems that PT's failed to show substantial improvement on any item other than 11b, an item which shows the correct prototypical arithmetic expression. It is curious that even on the post-test, 30% of PT's still marked as correct items that fail to even yield a correct numeric result.

Item	11a	11b	11c	11d
Pre-Correct	63%	65%	66%	58%
Post-Correct	70%	78%	68%	67%
Difference	7%	13%	2%	9%
Item Reference	EL_NCOP_2008B_23	EL_NCOP_2008B_23	EL_NCOP_2008B_23	EL_NCOP_008B_23

Table 8: Percent correct for Elementary NCOP items dealing with subtracting negatives

Trends in Growth on Individual Topics: Elementary PFA

Below, topics which are reflected in three or more items in the elementary PFA test are identified, and PT performance on those items is summarized and discussed. Growth will be measured by looking at the change in the percent of PT's answering correctly on the pre-test to the percent answering correctly on the post-test. Changes in the range (-10%, 10%) will not be treated as significant. Since the LMT terms of use do not allow for the inclusion of items in this document, item descriptions will be provided.

- **Arithmetic Sequences**

This test included 18 items that dealt with arithmetic sequences and the n th values of such sequences. Although some of these problems are phrased as dealing with linear functions in general, they all deal with discrete rather than continuous situations, and consequently all qualify as dealing with arithmetic sequences.

Problem 1 presents a shape iterated in order to form a chain. Parts 1a – 1d each suggest an arithmetic expression for the perimeter of a 100-chain, and PT's must decide if each expression would work or not. Problem 2 presents a table of n -values and associated x -values. Parts 2a – 2d each provide claims about the n th term or proposals for how to find it, expressed in words rather than put in the form of an algebraic expression. PT's then decide if each claim/method is correct or not. Problems 3 and 5 are virtually identical to problem 1, presenting a shape chain constructed from toothpicks. Problem 3 then asks PT's to choose from amongst several proposed ways of finding how many toothpicks are required to build an n -chain, while problem 5 fixes a large value of n and asks PT's to select the specific number of toothpicks required. Problem 4 shows four arithmetic sequences, one expressed as a visual pattern, one expressed via a story

problem, and two presented as explicit formulae for an n th term. PT's must then decide which of the four represent the same sequence. Problem 6 introduces an explicit formula for n th terms of a sequence, then each of parts 6a – 6c feature a story problem meant to be modeled by the explicit formula. In each case, PT's decide if the story problem is genuinely modeled by the original explicit formula. Problem 7 is analogous to problem 1, but it features a chain of 3-D figures rather than 2-D shapes and asks for the surface area of an n -chain rather than perimeter of a 100-chain.

Item	1a	1b	1c	1d	2a	2b	2c	2d	3
Pre-Correct	41%	29%	33%	44%	45%	74%	34%	71%	34%
Post-Correct	65%	35%	44%	79%	38%	79%	35%	76%	41%
Difference	24%	6%	11%	35%	7%	5%	1%	5%	7%
Item Reference	EI_PFA 2001A_29	EI_PFA 2001A_29	EI_PFA 2001A_29	EI_PFA 2001A_29	EI_PFA 2006A_9	EI_PFA 2006A_9	EI_PFA 2006A_9	EI_PFA 2006A_9	EI_PFA 2001A_33
Item	4	5	6a	6b	6c	7a	7b	7c	7d
Pre-Correct	13%	25%	33%	38%	60%	67%	55%	64%	60%
Post-Correct	12%	58%	44%	50%	56%	70%	62%	68%	68%
Difference	-1%	33%	11%	12%	-4%	3%	7%	4%	8%
Item Reference	EI_PFA 2004A_29	EI_PFA 2001C_30	EI_PFA 2004A_30	EI_PFA 2004A_30	EI_PFA 2004A_30	EI_PFA 2001B_35	EI_PFA 2001B_35	EI_PFA 2001B_35	EI_PFA 2001B_35

Table 9: Percent correct for Elementary PFA items dealing with arithmetic sequences

PT's failed to show growth on items that required them to identify a correct explicit formula for the n^{th} term of an arithmetic sequence. However, they did show consistent growth in items that fixed large values of n . In item 1, PT's showed the most growth at identifying the most intuitive correct expression and the most intuitively-tempting incorrect expression. Item 5, which required only that test-takers identify a correct numeric answer rather than a correct arithmetic expression, PT's more than doubled their rate of correct responses from 25% to 58%. PT's also showed significant, but less substantial, improvement on the items that required them to identify stories that would be modeled by a given linear equation.

- Interpreting a Graph

This test included 4 items that required PT's to interpret a graph. All of problem 8 shared a common graph as well as a common context for the graph. Each of the Parts a – d made a claim based on the graph, and PT's had to decide if the claim was true or not. Item 8c, the only item that saw significant growth, claimed that the graph has the

Item	8a	8b	8c	8d
Pre-Correct	89%	60%	46%	84%
Post-Correct	91%	59%	68%	79%
Difference	2%	-1%	22%	-5%
Item Reference	EL_PFA_2001C_32	EL_PFA_2001C_32	EL_PFA_2001C_32	EL_PFA_2001C_32

Table 10: Percent correct for Elementary PFA items dealing with interpreting a graph

qualities of a function. The remaining items dealt with the slope of the graph, specific values of the graph, or comparisons between two specific points on the graph.

- Equivalent Algebraic Expressions, Distribution, and Area Models

This test included 9 items that relate to equivalent algebraic expressions, the distributive property, and area models for the distributive property. Problem 9 shows PT's an area model for a specific instance of the distributive property; parts 9a – 9d then each suggest algebraic expressions for the area of the figure. PT's must decide if each expression correctly represents the given area or not. Problem 10 reverses problem 9 in some sense. It provides students with an expression of the form $(x+c)(x+d)$ where c and d are constant integers, and shows them proposed ways of representing this multiplication with base-10 blocks. For each part of problem 10, PT's decide whether or not a given base-10 representation correctly models the original expression or not. Item 11 doesn't deal with distribution or area, but continues to deal with algebraic equivalence. PT's are

Item	9a	9b	9c	9d	10a	10b	10c	10d	11
Pre-Correct	71%	74%	56%	63%	38%	38%	28%	30%	71%
Post-Correct	65%	74%	74%	79%	42%	54%	44%	62%	68%
Difference	-6%	0%	18%	16%	4%	16%	16%	32%	-3%
Item Reference	EI_PFA 2006A_20	EI_PFA 2006A_20	EI_PFA 2006A_20	EI_PFA 2006A_20	MS_PFA 2005B_15	MS_PFA 2005B_15	MS_PFA 2005B_15	MS_PFA 2005B_15	EI_PFA 2001A_32

Table 11: Percent correct for Elementary PFA items dealing with equivalent algebraic expressions, distribution, and area models

shown a 2-dimensional figure and several proposed methods of expressing the perimeter of the figure algebraically. PT's need to select the only *incorrect* expression.

In item 9, PT's improved at identifying the correct expressions of the area but not at identifying incorrect expressions of the area. In item 10, the only part that PT's failed to improve in was the part that used the correct base-10 blocks, but in the wrong quantities.

- Ratios and Cross-Multiplication

This test included 7 items whose sole foci were ratio and cross-multiplication. Items 12a – 12d each provide a story problem and ask PT's whether the problem could be correctly answered using cross-multiplication or not. Item 13 shows a specific algebraic equation of one unknown which could be solved with cross multiplication and provides several proposed explanations for *why* the algorithm works. PT's then select the *best* explanation. Item 14 shows two student attempts at applying cross-multiplication to a

Item	12a	12b	12c	12d	13	14	15
Pre-Correct	69%	29%	45%	68%	15%	25%	38%
Post-Correct	85%	18%	59%	62%	18%	29%	38%
Difference	16%	-11%	14%	-6%	3%	4%	0%
Item Reference	EI_PFA 2006A_22	EI_PFA 2006A_22	EI_PFA 2006A_22	EI_PFA 2006A_22	MS_NCOP 2005A_5	MS_NCOP 2005A_7	MS_NCOP 2005A_3

Table 12: Percent correct for Elementary PFA items dealing with ratios and cross multiplication

story problem which can genuinely be solved via cross multiplication, and tasks PT's with determining the validity of these student methods. Item 15 shows a student attempt to apply cross-multiplication to a problem it doesn't necessarily apply to, and asks PT's about the validity/efficiency of the method and the correctness of the student's answer.

In most of these items, PT's showed no significant improvement. In problem 12, it is not clear what might have led to growth on two items but not on the other two. PT's improved in one item that could be solved with cross-multiplication and one item that couldn't, but also failed to improve in one item that could be solved with cross-multiplication and one item that couldn't. Furthermore, the two items that can be solved with cross-multiplication are structurally identical.

- Identifying Functions

This test included 5 items that required PT's to identify functions. Items 16a – 16e each present the cloud diagram (e.g. *figure 6*) for a relation and ask PT's whether the relation is a function or not.

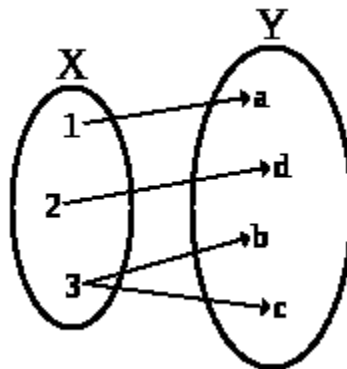


Figure 6: A relation represented with a cloud diagram

Item	16a	16b	16c	16d	16e
Pre-Correct	89%	79%	73%	26%	77%
Post-Correct	86%	74%	72%	28%	74%
Difference	3%	-5%	-1%	2%	-3%
Item Reference	MS_PFA_2005B_21	MS_PFA_2005B_21	MS_PFA_2005B_21	MS_PFA_2005B_21	MS_PFA_2005B_21

Table 13: Percent correct for Elementary PFA items dealing with identifying functions

PT's showed no significant growth on any of these items, though the majority of PT's correctly answers parts a, b, c, and e. Part d, which was correctly answered at much lower rates than the other parts, features a non-injective function.

- Algebraically Expressing a Proportional Relationship

This test included 4 items that required PT's to identify correct algebraic expressions for a proportional relationship. Problem 17 presents a story describing a proportional relationship, then proposes four algebraic representations of the relationship. PT's showed no significant growth on any of these items.

Item	17a	17b	17c	17d
Pre-Correct	53%	40%	42%	31%
Post-Correct	53%	47%	48%	33%
Difference	0%	7%	6%	2%
Item Reference	EL_PFA_2001A_30	EL_PFA_2001A_30	EL_PFA_2001A_30	EL_PFA_2001A_30

Table 14: Percent correct for Elementary PFA items requiring an algebraic expression of a proportional relationship

Trends in Growth on Individual Topics: Middle NCOP

Below, topics which are reflected in three or more items are identified, and PT performance on those items is summarized and discussed. As before, since the LMT terms of use prevent the inclusion of actual LMT items in this document, item descriptions are provided. Additionally, since there were roughly half as many middle school PT's involved in this study as elementary PT's, there is greater risk for differences in percent correct on post- and pre-tests being due to imperfect reliability. To help cope with this problem, the intervals of differences deemed insignificant is being expanded to (-15%, 15%), and the color scheme is being adjusted:

- Orange indicates differences in the interval (-30%, 15%]
 - Yellow indicates differences in the interval (-15%, 15%)
 - Light green indicates differences in the interval [15%, 30%)
 - Green indicates differences in the interval [30%, 45%)
- Properties of Rational and Irrational Numbers

The middle NCOP assessment included 8 items relating to properties of rational and irrational numbers. Each part of problems 1 and 2 states claim about rational numbers, and PT's must decide if the claim is true or not.

The items that showed improvement deal with sums/products of rational numbers, sums of rational numbers with irrational numbers, and methods of generating irrational numbers. The items revealed no significant improvement deal with finding minimal rational/irrational values, identifying rational numbers between other rational numbers, products of rational numbers with irrational numbers, and decimal expansions of rational and irrational numbers.

Item	1a	1b	1c	1d	2a	2b	2c	2d
Pre- Correct	37%	84%	47%	47%	62%	76%	62%	48%
Post- Correct	37%	79%	74%	84%	68%	84%	84%	47%
Difference	0%	-5%	27%	37%	6%	8%	22%	-1%
Item Reference	MS_NCOP 2005A_2	MS_NCOP 2005A_2	MS_NCOP 2005A_2	MS_NCOP 2005A_2	MS_NCOP 2005B_1	MS_NCOP 2005B_1	MS_NCOP 2005B_1	MS_NCOP 2005B_1

Table 15: Percent correct for Middle NCOP items dealing with properties of rational and irrational numbers

- Area Models of Decimal Multiplication

The middle NCOP assessment included 5 items focusing on base-10 models for decimal multiplication. Item 3 presents the multiplication of two decimals as well as the correct area model of the multiplication. A portion of this model dealing with one part of the multiplication is highlighted, and PT's must decide which part of the multiplication is represented by the highlighted area. Problem 4 also starts with the multiplication of two decimals and a correct area model of the multiplication. Parts 4a – 4d each make a claim about what part of that model represents, and PT's must decide if the claim is correct or not.

PT's improved in four out of these five items, and improved significantly on several of them. Notably, on Item 3 the responses shifted from 53% of PT's answering incorrectly to only 11% of PT's answering incorrectly, while item 4a almost tripled the percent of PT's answering correctly from 15% to 42%.

Item	3	4a	4b	4c	4d
Pre-Correct	47%	15%	48%	48%	30%
Post-Correct	89%	42%	63%	68%	32%
Difference	42%	27%	15%	20%	2%
Item Reference	EL_PV 2006_17	MS_NCOP 2005A_1	MS_NCOP 2005A_1	MS_NCOP 2005A_1	MS_NCOP 2005A_1

Table 16: Percent correct for Middle NCOP items dealing with area models of decimal multiplication

- Division of Fractions

The middle NCOP assessment included 9 items dealing with division of fractions. Problem 5 gives an arithmetic expression that divides one fraction by another. Parts 5a – 5d then provide narrative interpretations of this expression, and ask PT’s to determine if these interpretations are valid or not. Problem 6 gives an arithmetic expression that divides a natural number by a fraction. Parts 6a – 6d relate story problems, and PT’s have to decide if the story problems reflect the given division problem or not. Item 7 has a student share a valid (though this is not revealed), non-prototypical way to divide fractions that students are known to discover on their own. PT’s are then asked if the method is never valid, conditionally valid, or always valid.

No significant growth occurred in problem 6. Performance on Problem 7, changed from no PT’s answering correctly to more than a third responding correctly. In problem 5, PT’s failed to show growth on two parts, one of which required them to recognize as correct a description of division as repeated subtraction. The other part where no growth occurred uses language that could be interpreted in two ways, one

Item	5a	5b	5c	5d	6a	6b	6c	6d	7
Pre- Correct	76%	29%	52%	29%	76%	81%	76%	62%	0%
Post- Correct	68%	63%	68%	32%	79%	89%	79%	53%	37%
Difference	8%	34%	16%	3%	3%	8%	3%	-9%	37%
Item Reference	MS_NCOP 2005B_10	MS_NCOP 2005B_10	MS_NCOP 2005B_10	MS_NCOP 2005B_10	EL_NCOP 2001C_22	EL_NCOP 2001C_22	EL_NCOP 2001C_22	EL_NCOP 2001C_22	EL_NCOP 2001C_17

Table 17: Percent correct for Middle NCOP items dealing with division of fractions

leading to two different possible answers. The part that of problem 5 that saw mild growth is a prototypical incorrect interpretation of division of fractions, while the part that saw more substantial growth interprets division as it occurs in missing multiplicand reasoning.

- Subtraction of Fractions

The middle NCOP assessment included 3 items that concerned subtraction of fractions. Problem 8 begins by showing an arithmetic expression that subtracts one unit fraction from another. Parts 8a – 8c each state a word problem, and PT’s must decide if the given word problem reflects the original expression or not. No significant growth was observed on any of these items.

Item	8a	8b	8c
Pre-Correct	26%	63%	68%
Post-Correct	32%	74%	79%
Difference	6%	11%	11%
Item Reference	EL_NCOP 2001B_20	EL_NCOP 2001B_20	EL_NCOP 2001B_20

Table 18: Percent correct for Middle NCOP items dealing with subtraction of fractions

- Greatest Common Factor and Least Common Multiple

The middle NCOP assessment included 5 items dealing with greatest common factors and least common multiples. Problem 9 opens by proposing that we consider a fixed pair of counting numbers. Parts 9a – 9e each make some claim about the greatest common factor or least common multiple of these numbers under some set of conditions, and PT’s must decide if the claim is false, sometimes true, or always true. No significant improvement was observed on any of these items.

Item	9a	9b	9c	9d	9e
Pre-Correct	32%	42%	63%	21%	21%
Post-Correct	42%	47%	42%	32%	11%
Difference	10%	5%	-21%	11%	-10%
Item Reference	EL_PV 2006_14	EL_PV 2006_14	EL_PV 2006_14	EL_PV 2006_14	EL_PV 2006_14

Table 19: Percent correct for Middle NCOP items dealing with greatest common factors and least common multiples

- Proportion, Ratio, and Percent

The middle NCOP assessment included 8 items covering ratio, proportion, and percent. Each part of problem 5 gives a fraction, a decimal, and a percent. PT's are asked to decide if the three values are equivalent or not. Problem 6 presents a story involving two mixture situations. Parts 6a – 6c each give a student an argument for why one mixture or the other is more concentrated, and PT's are asked to decide if the student reasoning is correct or incorrect.

Item 5d, the only part of problem 5 to show significant improvement, was the only item to use values less than 1% (e.g. 1/10000, 0.0001, 0.01%). However, it is worth noting that the other parts of problem 5 had been solved correctly at very high rates on the pre-test. PT's did not show any significant improvement on any part of problem 6.

Item	5a	5b	5c	5d	5e	6a	6b	6c
Pre- Correct	95%	90%	86%	57%	81%	67%	32%	84%
Post- Correct	89%	84%	79%	79%	74%	58%	21%	68%
Difference	-6%	-6%	-7%	22%	-7%	-9%	-11%	-16%
Item Reference	MS_NCOP 2005B_8	MS_NCOP 2005B_8	MS_NCOP 2005B_8	MS_NCOP 2005B_8	MS_NCOP 2005B_8	MS_NCOP 2007A_14	MS_NCOP 2007A_14	MS_NCOP 2007A_14

Table 20: Percent correct for Middle NCOP items dealing with proportion, ratio, and percent

Trends in Growth on Individual Topics: Middle PFA

Below, topics which are reflected in three or more items in the middle school PFA test are identified, and PT performance on those items is summarized and discussed. Growth will be measured by looking at the change in the percent of PT's answering correctly on the pre-test to the percent answering correctly on the post-test. Changes in the range (-15%, 15%) will not be treated as significant. Since the LMT terms of use do not allow for the inclusion of LMT items in this document, item descriptions are provided.

- **Linear Functions and Arithmetic Sequences**

This test included 6 items that focused on linear functions and arithmetic sequences. Problem 1 opens with a linear equation, bereft of context. Each of parts 1a – 1c relate a story problem, and PT's must decide if the situation in the story corresponds to the original linear equation or not. Item 2 shows several story problems and asks PT's to select the only one which is nonlinear. Item 3 presents PT's with a shape iterated to make a chain, made out of toothpicks, and asks them to select the correct explicit function for how many toothpicks are required to make an n-chain. Item 4 presents a visual pattern modeled by a linear equation, a story modeled by a linear equation, and two explicit forms for the nth term of an arithmetic sequence. PT's are asked to decide which of these four things represent the same linear function.

PT's showed growth on the two items that dealt with explicit formulas for the nth term of a sequence, but failed to show growth in items that focused on associating story problems with specific growth functions.

Item	1a	1b	1c	2	3	4
Pre-Correct	59%	64%	77%	40%	46%	28%
Post-Correct	56%	75%	75%	49%	63%	43%
Difference	-3%	11%	-2%	9%	16%	15%
Item	EL_PFA	EL_PFA	EL_PFA	MS_PFA	EL_PFA	MS_PFA
Reference	2004A_30	2004A_30	2004A_30	2005A_12	2001A_33	2005B_14

Table 21: Percent correct for Middle PFA items dealing with linear functions and arithmetic sequences

- Nonlinear Functions

This test included 12 items that focused on nonlinear functions. Item 5 visually presents a nonlinear pattern as well as a student's description of the pattern. PT's then select, from amongst several correct explicit formulas, the one which structurally corresponds to the student's description of the pattern. Item 6 presents a few terms of a sequence and asks in parts 6a – 6c if the sequence could be constant, linear, or quadratic. Item 7 visually presents 4 sequences and asks PT's to determine whether each is linear, quadratic, or exponential. Item 8 presents four story problems and asks PT's to decide whether each represents exponential growth or not.

There was no significant improvement in performance on any of the items that dealt with exponential growth. However, PT's performance did improve in items showing linear or quadratic growth.

Item	5	6a	6b	6c	7a	7b
Pre-Correct	10%	23%	73%	36%	28%	67%
Post-Correct	20%	31%	94%	56%	59%	82%
Difference	10%	8%	21%	20%	31%	15%
Item Reference	MS_PFA 2007A_1	MS_PFA 2007A_7	MS_PFA 2007A_7	MS_PFA 2007A_7	MS_PFA 2005A_16	MS_PFA 2005A_16
Item	7c	7d	8a	8b	8c	8d
Pre-Correct	50%	33%	56%	78%	44%	22%
Post-Correct	55%	36%	55%	55%	50%	18%
Difference	5%	3%	-1%	-23%	6%	-4%
Item Reference	MS_PFA 2005A_16	MS_PFA 2005A_16	EL_PFA 2004A_30	MS_PFA 2005A_12	EL_PFA 2001A_33	MS_PFA 2005B_14

Table 22: Percent correct for Middle PFA items dealing with nonlinear functions

- Defining and Identifying Functions

This test included 6 items that required PT's to define or identify functions. Each part of problem 9 shows a relation expressed as a cloud diagram (e.g. *figure 6*) and asks PT's to decide if it is a function or not. Problem 10 shows six possible definitions of "function" and asks PT's to select the one which best suits a middle school teacher's need for a definition which is mathematically accurate but will minimize confusion for their students.

Item	9a	9b	9c	9d	9e	10
Pre- Correct	100%	86%	86%	29%	81%	6%
Post- Correct	100%	94%	94%	56%	75%	27%
Difference	0%	8%	8%	27%	-6%	21%
Item	MS_PFA	MS_PFA	MS_PFA	MS_PFA	MS_PFA	MS_PFA
Reference	2005B_21	2005B_21	2005B_21	2005B_21	2005B_21	2007B_4

Table 23: Percent correct for Middle PFA items dealing with defining and identifying functions

There was no significant improvement on most of these problems but PT's did show improvement in performance on the part that featured a non-injective function. PT's also showed improvement on problem 10. A close inspection of their answers indicates that on the pre-test, most selected an answer that focused only on the graphical properties of a function. On the post-test, more PT's selected an answer that applied to functions more broadly.

- Number of Solutions

This test included 7 items whose focus was deciding how many solutions existed to one or more equations or inequalities. Items 11a – 11f as well as items 12 and 13 all feature one or more equations or inequalities and ask PT's to decide how many solutions exist for the system. PT's failed to show significant improvement on the items that dealt with systems of two or more equations/inequalities. They also showed no significant improvement on an item that had a single equation with infinitely many solutions or on

Item	11a	11b	11c	11d	11e	11f	12	13
Pre- Correct	28%	56%	17%	33%	50%	50%	56%	48%
Post- Correct	23%	73%	18%	50%	41%	73%	73%	44%
Difference	-5%	17%	1%	18%	-9%	23%	17%	-4%
Item	MS_PFA	MS_PFA	MS_PFA	MS_PFA	MS_PFA	MS_PFA	MS_PFA	MS_PFA
Reference	2007B_30	2007B_30	2007B_30	2007B_30	2007B_30	2007B_30	2005A_15	2007A_9

Table 24: Percent correct for Middle PFA items dealing with how many solutions exist for equations and inequalities

an item that included an absolute value. The items on which PT's improved all had a single equation or inequality which could be reasoned about without the use of any algebraic simplification.

- The Distributive Property

This test included 12 items dealing with the distributive property. Item 14 opens with an area model for a specific multiplication, as well as a correct, simplified expression for the total area of the diagram. PT's are then asked to find an alternative expression that correctly gives the total area in the form "length times width". Each part of 14 suggests an answer, and PT's either accept or reject it. The focus in this problem is on whether PT's need to multiply or add together the various components present in the original diagram. Problem 15 presents an expression analogous to $2a-(3b+4c)$ and asks PT's to justify why this expression is equivalent to $2a-3b-4c$. Each part of problem 15 gives some explanation, and PT's must either accept or reject it. Problem 16 opens by giving

Item	14a	14b	14c	14d	15a	15b
Pre-Correct	86%	82%	41%	86%	27%	45%
Post-Correct	81%	75%	38%	69%	25%	56%
Difference	-5%	-7%	-3%	-17%	-2%	11%
Item Reference	MS_PFA 2005A_13	MS_PFA 2005A_13	MS_PFA 2005A_13	MS_PFA 2005A_13	MS_PFA 2007A_3	MS_PFA 2007A_3
Item	15c	15d	16a	16b	16c	16d
Pre-Correct	59%	59%	44%	61%	56%	61%
Post-Correct	56%	56%	67%	50%	57%	76%
Difference	-3%	-3%	23%	-11%	1%	15%
Item Reference	MS_PFA 2007A_3	MS_PFA 2007A_3	MS_PFA 2005B_15	MS_PFA 2005B_15	MS_PFA 2005B_15	MS_PFA 2005B_15

Table 25: Percent correct for Middle PFA items dealing with the distributive property

an expression of the form $(x+c)(x+d)$, where c and d are constant integers). Each part of the problem proposes a way of representing this multiplication with base-10 blocks, and PT's must decide if the proposed representation correctly represents the expression.

PT's did not show significant improvement on ten out of these twelve items. The two items that they did show improvement on, 16a and 16d, are base-10 representations that area nearly correct. Improvement was not evidenced on 16b or 16c, items that

featured, respectively, a correct representation and a thoroughly incorrect representation.

- Interpreting $-x$ for Real x

This test included 5 items that concerned properties of the term $-x$ when x is allowed to be any real number. Each of these items makes a claim (e.g. $-x < 0$) and requires the PT to decide if the claim is true always, sometimes, or never (e.g. $-x < 0$ sometimes). PT's showed no significant improvement on any of these items. They struggled the most with an item that utilized an absolute value symbol, and struggled least with an item comparable to the one I gave moments ago as an example.

Item	17a	17b	17c	17d	17e
Pre-Correct	5%	86%	55%	50%	64%
Post-Correct	13%	87%	56%	50%	56%
Difference	8%	1%	1%	0%	-8%
Item Reference	EL_PV 2006_14	EL_PV 2006_14	EL_PV 2006_14	EL_PV 2006_14	EL_PV 2006_14

Table 26: Percent correct for Middle PFA items dealing properties of $-x$ for real x

Chapter 5: Conclusions and Directions for Future Research

This project examines the impact of a mathematics course for prospective elementary teachers and a mathematics course for prospective middle school teachers on those enrolled in their respective courses using a pre-post test methodology. Prospective teachers were asked to take tests, designed by the Learning Mathematics for Teaching project, which claim to measure mathematical knowledge for teaching. Analysis of the results was guided by the following research questions:

1. To what extent will the first semester Math for Elementary Teachers content course result in growth as measured by the Learning Mathematics for Teaching assessments of Mathematical Knowledge for Elementary Teachers? How will observed growth in Number Concepts and Operations compare to observed growth in Patterns Functions and Algebra? In what ways might the design of the Math for Elementary Teachers content course have led to these growth trends?
2. To what extent will the first semester Math for Middle School Teachers content course result in growth as measured by the Learning Mathematics for Teaching assessments of Mathematical Knowledge for Middle School Teachers? How will observed growth in Number Concepts and Operations compare to observed growth in Patterns Functions and Algebra? In what ways might the design of the Math for Middle School Teachers content course have led to these growth trends?

3. What growth trends occur on specific mathematical topics (e.g. division of fractions) that are covered by multiple test items on the assessments of Mathematical Knowledge for Elementary Teachers? How might the design of the Math for Elementary Teachers course have led to these growth trends?
4. What growth trends occur on specific mathematical topics (e.g. division of fractions) that are covered by multiple items on the assessments of Mathematical Knowledge for Middle School Teachers? How might the design of the Math for Middle School Teachers course have led to these growth trends?

Research Question 1: Impact of the First Semester Math for Elementary Teachers Content Course on NCOP and PFA

For the elementary course, the Cohen's D effect size on the NCOP test was nearly 0.5. Two-tailed t-tests with 95% confidence intervals revealed a p-value of 0.0001. These values indicate that PT's averaged growth of one half a standard deviation on the NCOP assessment, with this growth demonstrating strong evidence of statistical significance.

The Cohen's D effect size on the elementary PFA test was roughly 0.2. A two-tailed t-test with a 95% confidence interval resulted in a p-value of 0.088. These values indicate that PT's averaged growth of one fifth of a standard deviation on the PFA assessment, with this growth showing weak evidence of statistical significance.

There are several aspects of the Math for Elementary Teachers course design that may have contributed to the more substantial and significant growth of these PT's in NCOP than PFA. NCOP topics are introduced on the very first day of the course, and are

covered explicitly during class for 10 out of the 15 weeks that the class meets (weeks 1-9 and week 11). PFA topics, on the other hand, are first introduced during a single class in week 5, then don't reappear until week 10. Overall, PFA topics are covered during only 5 of the 15 weeks that class meets (week 10 and weeks 12-15). This stronger emphasis on and earlier introduction to NCOP than PFA likely contributed to the differences observed in PT growth on those assessments.

Looking more carefully at the topics covered by test items, every item on the NCOP assessment focuses on topics that were covered during the course, while 14 of the 53 items on the PFA assessment focus on topics that PT's had not encountered yet since they are not covered until the second semester of the Math for Elementary Teachers course sequence (e.g. functions and linear functions). An additional 17 of the 53 items on the PFA assessment focus on topics that are covered at the very end of the course. PT's had not yet taken a summative assessment in the course on these topics at the time PT's took their LMT post-test. This lack of exposure to some PFA topics and lack of summative feedback on others was probably an additional contributing factor to the differences in PT growth on those assessments.

It is worth noting that there are a few areas of conceptual overlap between NCOP and PFA that are relevant to the LMT assessments and to the course under study. The LMT assessments include the topics of ratio, proportion, cross-multiplication, and the distributive property as elements of PFA, though there are strong arguments for deeming these to be in the realm of NCOP. The LMT assessments also include questions on divisibility in the NCOP test, while questions about factors and multiples are deemed to fall in the domain of PFA. It is not wrong, nor even uncommon, to identify these topics

in this way, but it is also not trivial that they would be classified as such.

Research Question 2: Impact of the First Semester Math for Middle School Teachers Content Course on NCOP and PFA

The Cohen's D effect sizes for the middle school NCOP and PFA tests were 0.3233 and 0.3063, respectively. Two-tailed t-tests with 95% confidence intervals revealed p-values for these two tests of 0.067 and 0.05, respectively. These values indicate that PT's in the Math for Middle School Teachers course averaged scores roughly three tenths of a standard deviation higher on the post-test than they did on the pre-test, and that these results show weak to moderate evidence of statistical significance.

In contrast to the elementary teacher course, Math for Middle School Teachers introduces both NCOP topics and PFA topics at the very start of the course (the former in section 1 of 17, the latter in section 3 of 17). In total, 8 of the 17 sections in the course notes cover NCOP topics while 7 of them cover PFA topics. The remaining 2 sections in the course notes cover modular arithmetic, is not a topic that appears on the LMT assessments. This treatment of NCOP and PFA topics is far more equitable than the treatment given in the elementary teacher course, and likely contributes to the far more equitable growth results on the LMT assessments. It would also seem to account for the fact that the middle school PT's showed weaker growth than the elementary PT's in NCOP, but more growth than the elementary PT's in PFA.

Research Question 3: Impact of the First Semester Math for Elementary Teachers Content Course on Specific Topics in Number Concepts and Operations

- Place Value

The elementary NCOP test included 9 items that dealt purely with place value, outside the context of any operation. PT's showed no significant improvement on four items that required them to identify whether non-prototypical decompositions of a number into ones, tens, and hundreds were correct or not. However, PT's did show significant improvement on five items that asked them to decide whether collections of base-ten blocks could be used to represent a given number or not.

Place value is a recurring theme throughout the Mathematics for Elementary Teachers course. It is the main focus of the first three days of class, a time when it is discussed on its own in the context of learning about modern place-value numbering systems. It is brought up again explicitly and repeatedly during the ensuing weeks of the course in order to discuss other topics such as: Decimals (week 3), the addition and subtraction algorithms (week 4), multiplication by ten (week 5), the multiplication algorithm (week 6), the division algorithm (week 8), and divisibility (week 11).

Given PT's lack of significant improvement on the items dealing with non-prototypical decomposition into ones, tens, and hundreds, it seems that PT's who entered the course with inflexible place-value decomposition skills failed to show improvement in that area. During the course, PT's are called upon to regroup numbers into non-prototypical place-value decompositions (e.g. class activity 3K from Beckmann, 2014). However, this only occurs during discussions of how the subtraction algorithm works. This low level of emphasis on non-prototypical place-value representation, in conjunction

with its lack of introduction during sections that focus on place value outside the context of algorithms, likely contributed to the absence of significant PT growth on these items.

Given PT's improvement on the five items dealing with base-ten blocks, it seems that the Mathematics for Elementary Teachers course increased their familiarity with these manipulatives and also increased their flexibility of representation when using them (e.g. 13 rods and 2 cubes could mean 1320, 132, 13.2, etc.). Base-ten blocks and other analogous representations (e.g. sticks and bundles of sticks) are used repeatedly and persistently throughout the course. Students physically use such representations during several class meetings, and are asked to draw pictures of them on a number of homework assignments and summative assessments. Several class activities focus entirely on how such representations can be flexibly interpreted to mean more than one number (e.g. class activity 1D and 1E from Beckmann, 2014)

- The Algorithm for Division

This test included 2 items dealing specifically with the division algorithm. PT's showed significant improvement on one item that asked them to consider the validity of a nonstandard division algorithm. However, they showed no significant improvement on an item that asked them to select the best explanation for why the standard division algorithm works, even when the existence of a second lower quality but still correct answer was taken into account.

Division is another prominent topic in the Mathematics for Elementary Teachers course, with roughly two full weeks of class devoted solely to considering aspects/types of division (parts of weeks 7 and 8, then all of week 9). Division is also implicitly built into several other topics scattered throughout the rest of the semester, including: Ratio

and proportion (week 10), divisibility (week 11), factors and multiples (week 12), and rational numbers (week 13). However, there is only one day of class that is devoted specifically to discussing the long division algorithm.

Based on the observed growth on the item dealing with a nonstandard division algorithm, it seems that PT's did gain more substantial understanding of what division is and more flexibility with their reasoning about division. Given the lack of growth on an item that required PT's to select a reason for why the standard division algorithm works, it also seems that they still struggle to put into words *why* the standard algorithm works, or perhaps they simply still struggle to interpret the purely written explanation of another for why it works. In the Mathematics for Elementary Teachers course, many days are spent reasoning about division and interpreting the results of division. Division is reasoned about in many ways, some narrative and some pictorial. This extended focus on reasoning about division likely contributed to PT's substantially improved ability to recognize a valid but nonstandard algorithm. However, only one day is devoted solely to understanding the standard division algorithm. This is pursued by first discussing scaffold division, an extended version of the standard long division algorithm (class activity 6G from Beckmann, 2014), and tracing the analogy between the extended algorithm and the standard one. Thus, even understanding of the standard division algorithm is pursued via exploration of a nonstandard algorithm. Furthermore, the LMT item on the standard division algorithm focuses on the "repeated subtraction" interpretation of division, while the Mathematics for Elementary Teachers course tends to focus on interpretations like "how many groups if each group has x objects" or "how many objects if there are x groups".

- Partitive vs. Quotative Division

This test included 4 items that focus exclusively on requiring PT's to distinguish partitive division from quotative division. All four items require PT's to decide if two story problems use the same interpretation of division. PT's improved significantly on one of these four items, but not on the other three. Careful consideration of the story problems for the four items indicates that PT's may be identifying "type of division" based on superficial features of the stories. However, the improvement on one of the items suggests that some PT's were able to begin moving past purely superficial classification.

As described when discussing PT growth relating to the algorithm for division, division is a prominent and recurring topic throughout the latter two-thirds of the Mathematics for elementary Teachers course. There are also four days that are largely devoted to interpreting division story problems as quotative or partitive. Most of the examples provided in Beckmann (2014) and in the course materials are examples where groups and objects have clear everyday meaning (e.g. inches as objects contain in groups called feet, or cups of flour being the objects in groups called batches), as is the case in the one item of this sort that PT's showed growth on. However, two of the items that PT's did not show growth in are not so clearly subject to groups and objects interpretations (e.g. consider dividing a rectangle's area by one side length in order to obtain the other side length). Such items are much less prominent in Beckmann (2014) and in the course materials.

- Dividing by Fractions

This test included 9 items dealing division by fractions. Four items required PT's

to identify correct answers to problems requiring the division of a natural number by a fraction, requiring PT's to both calculate a correct numeric answer and to correctly interpret what the fractional portion of the answer was a fraction of. Four items asked PT's to decide whether a given word problem was associated with a given division of fractions arithmetic expression or not. The remaining item asked PT's to consider the validity of a nonstandard method of dividing fractions.

PT's showed significant growth on 6 of these 9 items, indicating that the Mathematics for Elementary Teachers course positively impacted their ability to divide fractions and interpret the fractions appearing in mixed number results. Division is a prominent and recurring theme in the Mathematics for Elementary Teachers course, as discussed when considering results related to the algorithm for division. Out of the two weeks devoted to talking solely about division, two days are devoted to division of fractions. PT's also spend an entire week near the start of the course (week 2) learning about fractions. It seems that this heavy emphasis on both topics resulted in measurable growth.

- Borrowing in the Subtraction Algorithm

This test included 4 items that focused on borrowing in the subtraction algorithm. Three items presented student explanations of what took place in a specific example of the subtraction algorithm being used on values that required borrowing, and asked PT's to decide if the student explanations demonstrated understanding of the algorithm or not. PT's showed significant improvement on two of these three items. The item on which they showed no growth featured a correct mechanical description of the subtraction algorithm and borrowing, but no underlying conceptual explanation. The fourth item

focusing on borrowing in the subtraction algorithm required PT's to identify which student response to an activity involving base-ten blocks would serve as the best segue into discussing the comparison meaning of subtraction; PT's showed no significant growth on this item.

Subtraction is discussed in the Mathematics for Elementary Teachers course at the same time as addition. These two topics are the focus of week 4, with one day of that week devoted solely to understanding the standard subtraction algorithm. It seems that this time positively impacted PT ability to recognize conceptually correct explanations of borrowing. However, PT's did not improve in their ability to recognize a mechanically correct but conceptually empty student explanation, nor did they improve in their ability to identify the most efficient student-prompted segue into discussing a topic of interest. Both of these skills are pedagogical in nature, and the Mathematics for Elementary Teachers course is not designed to cover pedagogical material.

- Subtraction of Negatives

This test included 4 items focusing on subtraction of negatives. All four required PT's to decide whether or not a student's proposed arithmetic expression (e.g. $8 + (-7)$) correctly corresponded to a real-life situation in which one would find the difference between a positive integer and a negative integer. PT's only showed significant growth on one of these four items, one which showed the correct prototypical arithmetic expression. This indicates that the Mathematics for Elementary Teachers course had a weakly positive impact on PT understanding of the subtraction of negatives. However, fully 30% of PT's failed to recognize as incorrect student answers that don't even result in the correct numeric result, suggesting that this topic would be worth additional

attention.

Although subtraction (in conjunction with addition) is the focus of class for an entire week, there is very little focus on negative integers. Negative integers are a key part of class discussion for only one day during week 4 (addition and subtraction of integers) and one day during week 7 (multiplication of negative numbers). The discussion of adding and subtracting negative numbers makes heavy use of red and black chips as ways of visualizing integers, while the discussion of multiplication of negatives is really grounded in an argument based on patterns. This low level of focus on negatives, in conjunction with them being discussed first through a brand new physical representation and later through abstract discussion of patterns, likely contributed to weak PT growth in this area.

Research Question 3: Impact of the First Semester Math for Elementary Teachers Content Course on Specific Topics in Patterns, Functions, and Algebra

- Arithmetic Sequences

This test included 18 items that dealt with arithmetic sequences and the n th values of such sequences. PT's failed to show growth on items that required them to identify a correct explicit formula for the n^{th} term of an arithmetic sequence. However, they did show growth in items that fixed large values of n , and also on items that required them to associate story problems with linear equations.

The Mathematics for Elementary Teachers course only really covers sequences in general, and arithmetic sequences in particular, during the final week of classes. There is one class activity in week 10 that requires PT's to investigate a sequence (the locker

problem), but that activity is really a means to investigate factors rather than a means to investigate sequences. At the time that PT's took their LMT post-test, they had not yet even received summative feedback on their work with sequences in the course.

Consequently, it is very reasonable that they did not show growth on the test items requiring them to work with general n th terms of sequences. However, given PT's abilities to find n th terms when a large value of n was specified as well as their ability to associate story problems with sequences, it seems that they did improve in their ability to reason about arithmetic sequences. Their issue, then, is one of algebraic representation. Algebraic representation is covered during the last two weeks of the course, but it seems to be an area of ongoing weakness.

- Interpreting a Graph

This test included 4 items that required PT's to interpret and describe a graph. PT's showed significant growth in performance on an item that asked them if the graph had the qualities of a function, but showed no growth on items dealing with slope, specific values of the graph, or comparison between two specific points on the graph.

Functions do not actually appear as a topic of discussion in the first semester of Mathematics for Elementary Teachers, although the work with sequences strongly foreshadows them (e.g. class activity 9BB from Beckmann, 2014). Thus, it is quite reasonable that PT's failed to show growth on three of these four items. It is more difficult to explain why students might have improved on the remaining item. One possible contributing factor to that improvement is that PT's, at the time of the LMT post-test, had recently done an activity (class activity 9BB from Beckmann, 2014) that asked them to graph the points of an arithmetic sequence.

- Equivalent Algebraic Expressions, Distribution, and Area Models

This test included 9 items that relate to equivalent algebraic expressions, the distributive property, and area models for the distributive property. In these items, PT's improved at identifying correct and very incorrect answers, but did not improve at identifying nearly correct answers as incorrect.

Although PT's aren't introduced to algebraic expressions until week 14 of the course, they are exposed to base-ten blocks and area models of distribution much earlier. For example, distribution and the underlying meaning of the multiplication algorithm are both examined via area models and work with base-ten manipulatives (e.g. class activities 4K, 4N, and 4T in Beckmann, 2014). It seems that this work resulted in measurable growth, but that PT's continue to struggle with algebraic representation. The struggle of PT's with algebraic representation was encountered earlier when discussing arithmetic sequences.

- Ratios and Cross-Multiplication

This test included 7 items whose sole foci were ratio and cross-multiplication. Four items provide a story problem and ask PT's whether the problem could be correctly answered using cross-multiplication or not. One item asks PT's to identify the best proposed explanations for *why* the cross-multiplication algorithm works. Two items ask PT's to consider the validity of student attempts to use cross-multiplication

PT's showed no improvement in five of these seven items, and failed to improve on the items that required them to analyze the validity of student answers or the item that proposes reasons for *why* the cross-multiplication algorithm works. The Mathematics for Elementary Teachers Course focuses on ratio and proportion for three class meetings

during week 10, one of which centers the discussion on cross-multiplication. Given this relatively low level of emphasis, as well as the fact that the course has no aims to cover pedagogical topics (like anticipating and interpreting students answers), this lack of improvement is reasonable. Furthermore, Beckmann's explanation for the cross-multiplication algorithm differs from the one that appears on the LMT assessment. Given the equation $\frac{a}{b} = \frac{c}{d}$, one might describe cross-multiplication as a shortcut for multiplying both sides by bd , or one might describe it as a shortcut for multiplying the left by d/d and the right by $\frac{b}{b}$. Although these explanations are equivalent, their superficial difference could certainly have contributed to the lack of PT growth on the item requiring them to explain the algorithm.

- Identifying Functions

This test included 5 items that required PT's to identify functions. Each item presented the cloud diagram for a relation and ask PT's whether the relation is a function or not. PT's showed no significant growth on any of these items, though the majority of PT's correctly answers parts a, b, c, and e. Part d, which was correctly answered at much lower rates than the other parts, features a non-injective function. The Mathematics for Elementary Teachers Course does not discuss functions during the first semester of the course sequence, nor does it introduce cloud diagrams. As such, this lack of growth is reasonable.

- Algebraically Expressing a Proportional Relationship

This test included 4 items that required PT's to identify correct algebraic expressions for a proportional relationship. They present a story describing a

proportional relationship, then each propose an algebraic representation of the relationship which PT's must identify as correct or incorrect. PT's showed no significant growth on any of these items. Although proportions are covered during week 10 and algebraic representations are discussed and used during weeks 14 and 15 of the Mathematics for Elementary Teachers course, there is strong evidence that PT's still struggle heavily with the latter as noted in the section above on arithmetic series and the section on equivalent algebraic expressions.

Research Question 4: Impact of the First Semester Math for Middle School Teachers Content Course on Specific Topics in Number Concepts and Operations

- Properties of Rational and Irrational Numbers

The middle NCOP assessment included 8 items relating to properties of rational and irrational numbers. Each part of problems 1 and 2 states claim about rational numbers, and PT's are required to decide if the claim is true or not. The items that showed improvement deal with sums/products of rational numbers, sums of rational numbers with irrational numbers, and methods of generating irrational numbers. The items that revealed no significant improvement deal with finding minimal rational/irrational values, identifying rational numbers between other rational numbers, products of rational numbers with irrational numbers, and decimal expansions of rational and irrational numbers. It is noteworthy that the items that saw improvement all deal with finite processes/situations, while the items that didn't see improvement primarily involved infinite processes/situations and some concept of density. In particular, realizing that there is no minimal positive rational number, or that there are infinitely

many rational numbers between any two real numbers, requires PT's to see rational numbers as dense in the set of real numbers. Discussing decimal expansions of rational and irrational numbers also requires one to consider infinite cases.

One of the seventeen sections in the Math for Middle School Teachers' course notes focuses solely on rational numbers, and another focuses on decimal representation. The topics on which PT's improved are indeed covered in these sections. There is also some discussion of infinite processes, but only the surface of this topic is covered; a note in that section indicates that infinite processes will be covered in more depth in calculus. The question of finding a rational number between two rational numbers appears prominently in the section on rational numbers, but it is restricted to a single finite case. The concept of infinity is implicitly present in two later sections of the notes (the ones discussing sequences and series), but the clear focus in these sections is on finite pieces of infinite things (e.g. finding an n th term of a sequence). This instructional choice to focus on finite situations likely contributed to the presence of PT growth on items requiring them to reason about finite situations, and the lack of PT growth on items requiring them to consider infinite processes/situations.

- Area Models of Decimal Multiplication

The middle NCOP assessment included 5 items focusing on base-10 models for decimal multiplication. All of these items require PT's to connect a subset of an area model to what part of decimal multiplication it represents (e.g. tenths times tenths). PT's improved in four out of these five items, with substantial improvement on several of them.

The second of the seventeen sections of the course notes used in Math for Middle

School Teachers is about arithmetic and focuses heavily on the standard algorithms for the four basic arithmetic operations. The eighth of the seventeen sections focuses on decimal representations, and includes some work with performing operations on numbers in decimal form. Area models are not emphasized in either section, though one area model does appear in the latter. However, PT's are repeatedly required to give algebraic justification for the standard algorithms of the four basic arithmetic operations. This justification requires PT's to think carefully about the part played by place value in these operations, and it seems that sort of understanding has translated to improved interpretation of area models of decimal multiplication.

- Division of Fractions

The middle NCOP assessment included 9 items dealing with division of fractions. PT's showed growth in recognizing a non-prototypical division of fractions algorithm as valid. PT's showed no significant growth in items that asked them to associate story problems with corresponding division of fraction problems. When called upon to narratively interpret division, PT's showed growth in recognizing an incorrect interpretation and in recognizing a missing multiplicand interpretation. However, PT's did not show growth in recognizing a repeated subtraction interpretation as valid.

Division is covered as a part of the Course Notes' section on arithmetic, and fractions are mostly covered in a later section on rational numbers. However, division of fractions does not have a strong presence in the course notes, being largely relegated to a single class activity (A.20) and to implicit coverage when discussing division of decimals. Furthermore, quotative and partitive interpretation of division receive some attention, but only once in the context of dividing fractions. It seems that focus on

algorithms in general, and division in particular, throughout several sections of the Math for Middle School Teachers course had a positive impact on PT flexibility in identifying non-prototypical algorithms as well as in their interpretation of division. However, the relatively minimal focus on dividing fractions resulted in lack of significant growth on PT's ability to associate story problems with corresponding division of fractions expressions.

- Subtraction of Fractions

The middle NCOP assessment included 3 items that concerned subtraction of fractions. All three require PT's to associate a story problem with an arithmetic expression consisting of the subtraction of one fraction from another. No significant growth was observed on any of these items.

The Math for Middle School Teachers course notes discuss subtraction in a section on arithmetic, though that topic is restricted to subtraction of natural numbers. The course notes also discuss fractions in a section on rational numbers. The course activities feature situations where fractions are added, multiplied, or divided. However, I have not found any instances of fraction subtraction. This absence may account for the lack of growth on items dealing with the subtraction of fractions.

- Greatest Common Divisor and Least Common Multiple

The middle NCOP assessment included 5 items dealing with greatest common factors and least common multiples. These items show a fixed pair of counting numbers along with some claim about the greatest common divisor or least common multiple of the numbers under some set of conditions, and PT's need to decide if the claim is false, sometimes true, or always true. No significant improvement was observed on any of

these items.

Both greatest common divisor and least common multiple are introduced in the fourth of the seventeen sections of the course notes. However, greatest common divisor appears only in one of the problems at the end of the section, and in none of the activities featured in the appendices of the notes. Least common multiple appears in none of the exercises and none of the activities. Furthermore, the one exercise featuring greatest common divisors asks only that PT's calculate the greatest common divisor of a pair of numbers. The low level of emphasis that course materials place on these topics certainly accounts for the lack of significant PT growth in this area.

- Proportion, Ratio, and Percent

The middle NCOP assessment included 8 items covering ratio, proportion, and percent. Five items require PT's to decide whether a fraction, a decimal, and a percent are equivalent or not, while three present student arguments for why one mixture is more concentrated than another and ask PT's to decide if the student reasoning is correct or incorrect. PT's showed no growth on the mixture items, and showed growth on only one of the fraction/decimal/percent items. The item that showed growth was the only one to use values smaller than 1%.

One of the seventeen sections in the course notes focuses on rational numbers, another focuses on decimal representations, and a third focuses on ratios and proportional relationships. The section on rational number does ask students to translate decimal representations of rational numbers into fractional representations (and vice versa). Percentages, however, appear only in a single activity in the appendices of the notes, and that activity is really about geometric series rather than connecting different

representations of rational numbers. Three mixture problems appear at the end of the section on ratio and proportion, and another in one of the activities collected in the appendices, but only one explicitly requires PT's to interpret someone else's explanation for why one mixture is more concentrated than another. The minimal focus on percentages accounts for the lack of PT growth on the decimal/fraction/percent items, while the minimal focus on interpreting mixture explanations of others accounts for the lack of PT growth on the mixture items.

Research Question 4: Impact of the First Semester Math for Middle School Teachers Content Course on Specific Topics in Patterns, Functions, and Algebra

- Linear Functions and Arithmetic Sequences

This test included 6 items that focused on linear functions and arithmetic sequences. PT's showed growth on the two items that dealt with explicit formulas for the n th term of a sequence, but failed to show growth in items that focused on associating story problems with specific growth functions.

One of the seventeen sections in the Math for Middle School Teachers course notes is devoted to sequences and functions, and another covers the closely related topic of series. Many of the activities and exercises included in the course notes for these sections do require PT's to find n^{th} values of sequences and series (sometimes for a specific large value of n), but I have not found any that require PT's to attach a story to a specified sequence. This one-directional focus could certainly have contributed to the observed pattern of growth by PT's on test items dealing with linear functions and arithmetic sequences.

- Nonlinear (or potentially nonlinear) Functions

This test included 12 items that focused on nonlinear functions. There was no significant improvement in performance on any of the items that dealt with exponential growth. However, PT's performance did improve in items showing linear or quadratic growth. PT's also showed no significant growth on items requiring them to associate types of growth with story problems.

The Math for Middle School Teachers course notes introduce the topic of both linear and nonlinear functions in the context of sequences. Linear functions are associated with arithmetic sequences, while exponential functions are associated with geometric sequences. Quadratic functions are also mentioned in this section. Linear and quadratic functions are brought up again in a later section that focuses on polynomials, since polynomials of degree one and two are linear and quadratic, respectively. Exponentials, on the other hand, don't reappear in any sections other than the section in which they are introduced. The added focus on linear and quadratic functions, along with these topics having been brought up again closer to the end of the course, likely accounts for PT's having shown growth on those items and not on those dealing with exponential functions.

- Defining and Identifying Functions

This test included 6 items that required PT's to define or identify functions. PT's showed no significant improvement on most of these problems, but did show improvement on an item that required them to recognize a non-injective function as still being a function. PT's also showed improvement on the item that required them to choose an appropriate definition of function, with PTs shifting from selecting an answer

that focused only on the graphical properties of a function to one that was more holistic.

Functions are defined and discussed in one of the seventeen sections contained in the course notes, a section that covers both sequences and functions. Functions are still present (even if the word is not used) in a later section on series, and yet again in a section on polynomials. Functions are defined formally, and the words “vertical line test” appear nowhere in the course notes. This could certainly account for the shift in how PT’s responded to proposed definitions of function. However, the exercises regarding functions focused heavily on finding n^{th} terms of sequences/series rather than on deciding if some given relation was a function or not, and this could account for the lack of growth on the items requiring PT’s to do just that.

- Number of Solutions

This test included 7 items whose focus was deciding how many solutions existed to one or more equations or inequalities. PT’s failed to show significant improvement on the items that dealt with systems of two or more equations/inequalities. They also showed no significant improvement on an item that had a single equation with infinitely many solutions or on an item that included an absolute value. The items on which PT’s improved all had a single equation or inequality which could be reasoned about without the use of any algebraic simplification.

In keeping with these results, the course notes don’t cover systems of multiple equations or inequalities, nor do they cover absolute value equations. However, the section on polynomials does broach the question of how many solutions polynomials can have. These trends in topic coverage account for the trends in PT growth.

- The Distributive Property

This test included 12 items dealing with the distributive property. PT's showed no improvement on 10 out of these 12 items. PT's did not show improvement in items that required them to explain distribution or in items that required them to select an algebraic expression of distribution that matched a given area model. However, in a problem focusing on base-10 block representations of distribution, PT's did show improvement at identifying nearly-correct items as incorrect.

The distributive property is covered in the course notes as a part of the section on arithmetic. However, it is certainly not a focal point. Over the course of 220 pages, "distributive" appears four times, "distribute" three times, and "distribution" zero times. This low level of focus accounts for the lack of growth in most items covering this topic.

- Interpreting $-x$ for Real x

This test included 5 items that concerned properties of the term $-x$ when x is allowed to be any real number. PT's showed no significant improvement on any of these items. They struggled the most with an item that utilized an absolute value symbol, and struggled least with an item comparable to the one I gave moments ago as an example.

This is not a topic that is covered explicitly by the course notes. Absolute value functions are also not covered. This absence certainly accounts for the lack of PT growth in this area. Most of the definitions provided in the course notes use formal mathematical language (e.g. "A rational number can be written as $\frac{a}{b}$ where $a \in \mathbb{Z}$, $b \in \mathbb{Z}$, and $b \neq 0$ "), which one might expect to translate into improved ability to reason about $-x$. However, none of these formal definitions actually require PT's to reason about $-x$.

Recommendations for Practice

Here I will make some recommendations for practice based on the PT growth results observed in the assessments they were asked to take. The content covered in the course, as evidenced by course materials (calendars, textbooks, course notes, syllabi, etc.), also plays a part in these recommendations. However, classroom observations were *not* part of the data collected, so no firm claims can be made about what was covered in these courses or how it was covered. As such, these recommendations should be viewed as tentative but informed.

I make the following tentative recommendations for Mathematics for Elementary Teachers:

- Emphasis on prototypical place value decompositions in the courses early development of place value numbering systems, with non-prototypical decompositions appearing only in the context of borrowing in the subtraction algorithm, may have given PT's an overly rigid and inflexible view of place value. It may be worthwhile to introduce the flexibility of place-value systems during the early development of that topic, rather than instantly "bundling" the moment enough items are present to be represented by a higher place value.
- Improvement of PT's on several items dealing with base-ten blocks or analogous area models suggests that the use of these manipulatives in the course may have had a positive impact on PT's. Consequently, use of these manipulatives/visuals should be continued.
- Extensive focus on division in the course seemed to have a positive impact on PT knowledge of several aspects of that topic. However, lack of growth on just under

half of the division items reviewed above indicates the potential for more substantial improvement in this topic.

- The course's work with subtraction resulted in substantial improvement on several items. However, it did not result in observed improvement on subtraction items that were pedagogical in nature. The course was not intended to cover the pedagogical side of mathematical content, but the question of whether it should be is an open research question.
- The course had very little focus on negative integers, and PT performance on the assessment showed only weakly positive growth on this topic. Given the large role negative numbers play in post-elementary math courses, it could be valuable to find opportunities to give the topic additional attention.
- Algebraic topics are only covered at the end of the first semester Mathematics for Elementary Teachers course, with the topic continued during the second semester. Consequently, PT's showed little or no growth on many algebraic topics. Functions and graphing in the Euclidean plane, for example, are topics that are not covered until the second semester of the course, and PT's showed no growth in nearly every item relating to these topics. There is some opportunity to foreshadow these topics (notably when discussing sequences/series), and it is possible that more growth could have been elicited by doing so.
- Proportions and cross-multiplication are given only brief attention in the course, and PT's only showed growth on two of the eleven items dealing with these topics. These could be areas worthy of additional attention.

I make the following recommendations for Mathematics for Middle School

Teachers:

- Although the attention given to rational numbers and decimal representation in the course seemed to improve PT performance on some items dealing with properties of rational numbers, PT's did not show measurable growth on items requiring a view of rational numbers as being dense within the set of real numbers. Combining the existing discussion of how to find a rational number between two rational numbers with the existing discussion of limiting processes could help develop this idea.
- The course's existing algebraic investigation of place value seemed to contribute to strong improvement in PT performance on test items that utilized area models of place value. However, the course has minimal focus on distribution, and PT improvement on area models of place value did not extend to area models of distribution.
- Relative to the course for elementary teachers, the course for middle school teachers placed very little emphasis on associating story problems with arithmetic expressions. As a result, the middle PT's showed little to no growth on items requiring such association across several mathematical topics (e.g. division of fractions, subtraction of fractions, and exponential growth). Given the need for teachers to be able to make such associations, this type of task deserves additional attention in the course.
- The course gave minimal attention to the topic of greatest common factor and least common multiple, and PT's showed no significant growth on items dealing

with these topics. Given the role primes play as the building blocks of integers, and the role greatest common factor and least common multiple play in describing key aspects of prime decomposition, this could be an area worthy of additional attention.

- The course has a visible focus on fractions and on decimals, but not on percentages. Given that PT's showed growth on only one out of five items dealing with equivalent decimals/fractions/percents, incorporating percentages in the course may be the most efficient way to foster further growth on this topic.
- Mixture tasks receive minimal emphasis in the course, and no significant PT growth was observed on such items. Given how tightly these sorts of tasks can tie in with ideas of proportion and ratio, they could be worthy of additional attention.
- The course features discussion of linear and quadratic functions in several contexts (e.g. sequences, series, and polynomials), and PT's showed consistent growth in items dealing with these topics.
- Systems of equations/inequalities are not covered in this course, and this was reflected in the lack of PT growth on items dealing with this topic. Since such systems appear in secondary mathematics and are fundamental to higher mathematics, they should be present in one of the semesters of this course.

All of the above recommendations for practice are rather local in nature. This invites the question of the extent to which they are generalizable and applicable to courses other than those that took part in this study. Briefly, we might observe that one of the key trends across topics in this study seems to be that increased course focus on a topic tended to invite more substantial growth on that topic. It certainly seems likely that

this broad trend would extend to other courses. However, a more interesting answer to the question of generalizability is that, to the extent that other courses are similar to these courses, the results can be generalized much more finely.

In the case of the Mathematics for Elementary Teachers course, this means that the results on individual topics may generalize to many similar courses across the nation. The content, order, and pacing of the course is largely driven by Beckmann (2014), a textbook that is commonly used in similar courses. Furthermore, the course's emphasis on activities and explanation is completely in line with the predominant philosophies underlying such courses at other institutions.

The results of the Mathematics for Middle School Teachers course may be less generalizable at the level of specific topics than the results of the Mathematics for Elementary Teachers course were. Although the focus on activities and explanation remains, the course for Middle School teachers primarily uses instructor designed materials.

Recommendations for Research

As described at the outset of this paper, a review of articles in the *Journal of Teacher Education*, the *Journal of Mathematics Teacher Education*, and the *Mathematics Teacher Educator* from the year 2014 to the present found only seven articles (Bleiler, Thompson, & Krajčevski, 2014; Subramaniam, 2014; Thanheizer, 2015; Yeh & Santagata, 2015; Whitacre, 2015; Turner & Drake, 2016; Whitacre & Nickerson, 2016) dealing with the state of mathematical knowledge of prospective K-12 teachers. Of these, only two (Whitacre, 2015; Whitacre & Nickerson, 2016) dealt with the growth of

prospective teacher mathematical knowledge due to mathematics content courses.

Furthermore, all seven used qualitative methods in their investigations of prospective teacher mathematical knowledge.

The Whitacre (2015) and Whitacre & Nickerson (2016) studies are the ones most comparable in aim to this study, so it is worth looking at them more closely. They both took place at a large, urban university and examined the impact of a first semester mathematics course for prospective elementary teachers. In this sense, these two studies had much in common with what has been described in this paper. However, along with using qualitative methods, these two studies each focused very tightly on one specific mathematical topic. Whitacre (2015) focused on mental addition and subtraction of multi-digit whole numbers while Whitacre & Nickerson (2016) focused on fraction comparison.

Whitacre (2015) opens by pointing out a body of literature suggesting that PT's are limited in the flexibility of their mathematical thinking and tend to rely on the standard algorithms, and he claims that the PT's he studied initially fit this description. Although he was having PT's compute sums and differences mentally rather than on paper, their mental methods were analogues of the standard algorithms. After the course, Whitacre found that PT flexibility increased. Whitacre suggests that these results are nonstandard, pointing to Newton's (2008) study of PT knowledge of fraction arithmetic which found that PT's under study did not become more flexible in their arithmetic. However, this study provides some evidence to support Whitacre's results, showing that both elementary and middle school PT's showed growth in items requiring them to consider the validity of nonstandard algorithms.

Whitacre & Nickerson (2016) also opens by discussing how PT's tend to have limited flexibility, this time with regard to comparing fractions. Again, it was found that PT flexibility increased in this area after taking the course. This study did not look at tasks covering an analogous topic, and neither supports nor refutes Whitacre & Nickerson's results.

The remaining articles dealing with the state of mathematical knowledge of prospective K-12 teachers did not look at the impact of content courses for those PT's. Thanheiser (2015) looked at the impact of two tasks on PT conception of multi-digit whole numbers and place value, with these tasks being given as part of a teaching experiment to 6 PT's and as part of a math methods course to 33 PT's. Subramaniam (2014) looked at the state of secondary PT's pedagogical knowledge of length estimation by examining the benchmarks PT's used to estimate lengths and the extent to which these benchmarks manifested in pedagogical contexts. Bleiler, Thompson, & Krajčevski (2014) looked at how secondary PT's provided feedback on student proofs. Yeh and Santagata (2015) looked at how PT's generated hypotheses about the impact of mathematical teaching on student learning both before and after a methods course, as well as before and after another specialized course. Turner and Drake (2016) propose a theoretical framework for combining existing work on prospective teachers learning about children's mathematical thinking with existing work on children's cultural funds of knowledge. Rather than tying in directly with work done for this project, these articles draw attention to what this project offers that these articles did not offer: quantitative investigation of the impact of math content courses on PT's across several topics. There is clear need for additional research on the impact of math content courses on PT's,

including studies that cover the impact of courses on multiple topics in order to identify patterns and trends in growth, as well as studies that focus finely on individual topics.

In 2008, a report released by the National Mathematics Advisory Panel made the following recommendations for mathematics teacher preparation and research based on a review of existing research literature:

- Teachers must “know in detail the mathematical content they are responsible for teaching and its connections to other important mathematics (p. 37).
- “More precise measures should be developed to uncover in detail the relationships among teachers’ knowledge, their instructional skills, and students’ learning, and to identify the mathematical and pedagogical knowledge needed for teaching.” (p. 38)
- The mathematical preparation of future elementary and middle school teachers must be strengthened.
- “A well-designed program of research and evaluation, meeting standards permitting the generalization of results, should be undertaken to create a sound basis for the education of teachers of mathematics.” (p. 40)

This research project contributed to the advancement of these goals in its examination of the impact that math courses for prospective teachers had on the mathematical knowledge of those enrolled. It found measurable growth occurred as a result of PT involvement in the given mathematical content courses for teachers, and also characterized some of the particular topics in which growth was either observed or not observed.

However, additional research is still needed in all of the areas mentioned by the National Mathematics Advisory Panel back in 2008. Notably, although this study made

use of the LMT model of MKT in designing measures of MKT, there is a need for the development and study of new models of MKT. In addition to developing such models, new methods of measuring MKT need to be developed and researched. In particular, there is need for the development of a criterion-based measure of MKT (the measures used in this study were norm-referenced) that correlates strongly with future student outcomes and could be proctored and graded efficiently at large scale. Such a tool would help correct an issue mentioned in the introduction of this paper, namely that the only existing tools that could reasonably be utilized in a study of the sort undertaken here were the LMT measures of MKT.

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Appendix A: Mathematics for Elementary Teachers Syllabus

Lecturer, TA, and other identifying information removed to preserve anonymity.

Texts

- *Mathematics for Elementary Teachers and Activities Manual, 4th edition*) by Sybilla Beckmann. (These are packaged together. The ISBN is: 0321836715).
- Student Packet (posted on website – please print out each activity page as needed for use in class)

Note: Used texts are not recommended for this class. The same texts will be used for Mathematics for Elementary Teachers II so the expense covers both classes.

Course Description and Expectations

This is the first course in a two-semester sequence. Mathematics for Elementary Teachers I focuses on concepts of number systems and operations, number theory, and some work with expressions, equations, sequences, and series.

The goal of this course is to prepare you to become teachers of elementary and middle school students. Knowing the mathematics for yourself is not the same as knowing the math for teaching. To that end, we emphasize explanations of mathematical ideas. To make this point very clear: Full credit will NOT be given for correct mathematical answers without an explanation that is clear and complete.

Attendance and participation 5 days a week is critical to your success in this class.

Each class (lecture and recitation) will consist of doing an activity in a small group and

discussing it with the whole class. You are expected to participate actively in all phases, so please bring the Activities Manual pages to every class. Explaining your thinking verbally in small and large groups will prepare you to explain mathematics to your students. It will also help you clarify your own ideas and/or questions.

Reading is crucial because we do not teach using the traditional lecture format. Reading assignments are designed to provide the explanation and summary of material that are not provided in class. You are expected to complete all reading assignments. You will find the Practice Problems and their solutions particularly helpful.

MOST IMPORTANTLY, the activities and assignments we complete as part of this course, both inside and outside of class time, are designed to help you achieve the following goals. Your grade for the course should also reflect your personal progress towards these goals.

1. Persevere in solving mathematical problems using problem-solving strategies, including the “explore, conjecture, justify” model.
2. Make connections and comparisons between different subjects to deepen understanding and help with solving problems.
3. Develop the meaning of mathematical definitions, formulas, and algorithms.
4. Use correct and precise mathematical language.
5. Evaluate spoken and written mathematical work to improve correctness and clarity.

These goals will not only aid you in being a better student of mathematics, they will also help you as future teachers. Our aim is for the things you learn in this course to not only be useful in your future classrooms, but outside of them as well.

Exams

This course will have TWO midterms (one hour each) and a final exam (one hour, 45 minutes). These will be weighted equally. The midterms are common exams held in the evenings to give you maximum time and quiet.

If you have a university-sanctioned conflict with an exam, be sure to alert your lecturer. All makeups require written documentation of the conflict (e.g., illness, religious holiday, another university commitment). For family emergencies, speak with your lecturer.

Homework

There will be weekly homework assignments. Homework assignments will receive a score out of 15 points: you will receive 5 points for completing all of the problems (less if you do not) and ONE randomly selected problem will be graded on the 10-point scale following this paragraph. The graded problem will be assessed on both the quality of your explanation and the correctness of your solution.

Grading Rubric

Points	Description	Characteristics
10	Excellent	Correct mathematics that is carefully thought out and thoroughly explained.
8	Good	Correct mathematics with an emerging but incomplete explanation.
6	Basic	Correct mathematics but little or no explanation OR largely correct mathematics with an emerging explanation that shows understanding.
4	Emerging	Work that has some merit but also has significant shortcomings in the mathematics and/or explanation.
2	Credit for effort	Work that shows some <i>relevant</i> effort but is seriously flawed.
0	No credit	No work submitted or no relevant effort shown.

Table 27: Mathematics for Elementary Teachers Grading Rubric

Occasionally, a score will be given that is not on the rubric (e.g., a “1” or a “7”). This indicates that your work is between two scores.

Homework Revision Policy:

Any graded homework problem earning less than an “8” may be revised and resubmitted according to the following requirements:

- Resubmit the original homework assignment with your TA’s comments.

- Submit a complete revision of the problem solution.
- Return the revision to your TA no later than 1 week after the graded papers have been returned.
- You may earn up to half the missed points. These will be added to the original score for your final score.

Quizzes

This course will have 7 in-class quizzes. See calendar for dates. Each 20 minute quiz will cover material taken from the activities completed since the previous quiz in lecture and in recitation. Quizzes will be graded according to the same 10-point rubric applied to the homework problem. The lowest quiz grade will be dropped allowing you to miss a Friday class without excuse and without penalty. *Makeup Policy:* If you have an excused absence (e.g., illness with doctor's note, religious holiday, documented university conflict), a makeup quiz will be permitted. In most cases, written documentation will be required for a makeup. For other emergencies, speak to your lecturer.

Overall Grading Scheme

Participation in Class:	5%
Homework:	20%
Quizzes:	15%
Exams (3 at 20% each):	60%

Side Note:

As part of our ongoing efforts to refine this course and the program, you will be asked to take pretest and posttest versions of on-line assessments of ``mathematical knowledge for teaching." You will receive attendance/participation points for completing these assessments, but your scores on these assessments will have no influence on your course grade. Furthermore, you may choose whether to participate in the research involving these assessments, and that choice will have no influence on your course grade.

Semester Grades

These will be determined roughly according to the standard University scheme:

Letter Grade	%		Letter Grade	%
A	93 – 100%		C	73 – 76
A-	90 – 92		C-	70 – 72
B+	87 – 89		D+	67 – 69
B	83 – 86		D	60 – 66
B-	80 – 82		E	0 – 59
C+	77 – 79			

Table 28: University Grading Scheme

***You are encouraged to work with other students and with tutors in the Tutor Room; however, you must submit your own individual written work.

***You may use the Internet as an additional resource, HOWEVER, any use of examples or text taken from any Internet website must be cited as with any other outside materials.

Disability Statement

Students with disabilities that have been certified by the Office for Disability Services will be appropriately accommodated, and should inform the instructor as soon as possible of their needs. The Office for Disability Services is located at ADDRESS; WEBSITE.

Academic Misconduct Statement.

It is the responsibility of the Committee on Academic Misconduct to investigate or establish procedures for the investigation of all reported cases of student academic misconduct. The term “academic misconduct” includes all forms of student academic misconduct wherever committed; illustrated by, but not limited to, cases of plagiarism and dishonest practices in connection with examinations. Instructors shall report all instances of alleged academic misconduct to the committee. For additional information, see the Code of Student Conduct (WEBSITE)

Appendix B: Mathematics for Middle School Teachers Syllabus

Lecturer, TA, and other identifying information removed to preserve anonymity.

Greetings. Welcome to the first course in UNIVERSITY'S mathematics sequence for future middle school mathematics teachers! We are very pleased to offer the following 4-course mathematics sequence designed especially for your professional needs.

Mathematics for Middle School Teachers I (Number and Algebra)

Mathematics for Middle School Teachers II (Geometry and Algebra)

Calculus for Middle School Teachers

History of Mathematics for Middle School Teachers

These four courses, designed to be taken over two years, focus on deep understanding of school mathematics, which involves being able to explain (in multiple ways) why facts are true and why procedures work. Thus, we hope that the ideas and materials from these courses will become important mathematical resources throughout your professional life. Later in your program, you will take mathematics methods courses, which are about how this mathematics can be taught.

Course Overview

The course notes, Numbers and Algebra can be purchased cheaply at any of the book stores. The course will consist of the following 5 chapters: (1) Arithmetic and Algebra (2) Numbers (3) Ratios, Functions, and Beyond (4) Solving Equations (5)

Harmony of Numbers

Grading.

Your grade will be based 40% on homework and 60% on exams, with the following grading scheme:

100%–93% A

92%–90% A–

89%–87% B+

86%–83% B

82%–80% B–

79%–77% C+

76%–73% C

72%–70% C–

69%–67% D+

66%–60% D

59%–0% E

You can check your grades at anytime using WEBSITE, which serves as the class website. As grades are posted, you have 2 weeks from the posting date to notify us concerning any errors or irregularities. Note: Credit for college-level calculus can replace Calculus for Middle School Teachers. If you have a choice, we strongly recommend that you take Mathematics for Middle School Teachers, which shows how the big ideas of calculus have important roots in middle grades mathematics.

Homework

Homework assignments typically will consist of three (3) difficult problems that you are to solve and write up formally. Students typically find that they spend about 10 hours per week on homework. You are encouraged to work with your classmates on the homework problems, but when you sit down to write up your solution, you are expected to work on your own. Homework assignments will often be due on Fridays.

Office Hours

Because students typically find office hours to be necessary for completing the homework, we have scheduled regular office hours, as follows. Except when stated otherwise, office hours will be held in ROOM.

Office Hour Times

Be sure that your schedule allows you to attend at least one scheduled office hour. Office hours are, of course, also available by appointment.

Exams

Midterm exams are scheduled for DATE, during regular class time. The common final exam is scheduled for DATE, TIME, ROOM.

Course Improvement

As part of our ongoing efforts to refine this course and the program, you will be asked to take pretest and posttest versions of on-line assessments of “mathematical knowledge for teaching.” You will receive homework points for completing these

assessments, but your scores on these assessments will have no influence on your course grade. Furthermore, you may choose whether to participate in the research involving these assessments, and that choice will have no influence on your course grade.

Students with Disabilities

Students with disabilities that have been certified by the Office for Disability Services will be appropriately accommodated, and should inform the instructor as soon as possible of their needs. The Office for Disability Services is located in ROOM; TELEPHONE; webpage: WEBPAGE

Academic Misconduct Statement

It is the responsibility of the Committee on Academic Misconduct to investigate or establish procedures for the investigation of all reported cases of student academic misconduct. The term academic misconduct includes all forms of student academic misconduct wherever committed; illustrated by, but not limited to, cases of plagiarism and dishonest practices in connection with examinations. Instructors shall report all instances of alleged academic misconduct to the committee. For additional information, see the Code of Student Conduct: WEBPAGE

Appendix C: Institutional Review Board Approval



Office of Responsible Research Practices

300 Research Administration Building
1960 Kenny Road
Columbus, OH 43210-1063

Phone (614) 688-8457
Fax (614) 688-0366
www.orrp.osu.edu

Protocol Title: Impact of Math Courses for Prospective Teachers on their Mathematical Knowledge for Teaching
Protocol Number: 2015E0472
Principal Investigator: Azita Manouchehri
Determination: The Office of Responsible Research Practices has determined the above referenced project exempt from IRB review.
Date of Determination: 08/21/2015
Qualifying Category: 02
Attachments: None

Dear Investigators,
Please note the following about the above determination:

- Retain a copy of this correspondence for your records.
- Only the Ohio State staff and students named on the application are approved as Ohio State investigators and/or key personnel for this study.
- Simple changes to personnel that do not require changes to materials can be submitted for review and approval through Buck-IRB.
- No other changes may be made to exempt research (e.g., to recruitment procedures, advertisements, instruments, protocol, etc.). If changes are needed, a new application for exemption must be submitted for review and approval prior to implementing the changes.
- Records relating to the research (including signed consent forms) must be retained and available for audit for at least 5 years after the study is closed. For more information, see university policies, [Institutional Data](#) and [Research Data](#).
- It is the responsibility of the investigators to promptly report events that may represent unanticipated problems involving risks to subjects or others.

This determination is issued under The Ohio State University's OHRP Federalwide Assurance #00006378. Human research protection program policies, procedures, and guidance can be found on the [ORRP website](#).

Please feel free to contact the Office of Responsible Research Practices with any questions or concerns.

QI Specialist, ORRP

Cheri Pettey, MA CIP
Quality Improvement Specialist
Regulatory & Exempt Determinations
The Ohio State University
Office of Research Office of Responsible Research Practices
300 Research Administration Building, 1960 Kenny Road, Columbus, OH 43221
614-688-0389 Office / 614-688-0366 Fax
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Figure 8: Institutional Review Board Approval

Appendix D: Information for Study Participants

Dear Students:

I am conducting a research project that seeks to investigate UNIVERSITY's teacher preparatory math courses. Mathematics teachers require specialized mathematical knowledge in order to do their job, and I'm seeking to find out how well the teacher preparatory math courses here are succeeding in building this knowledge. Are these courses benefitting their students? Are they benefitting all students equally? Are there any particular topics that the course impacts more/less positively than others?

In order to begin answering these questions, you will be asked to take a test twice during this semester as a regular component of the course. You will be tested once near the start of the semester and once near the end, thus allowing me to see how well the course helped to improve your skills. Each time you take the test will require roughly one hour. Your test scores will NOT be shared with your instructor, and you will be assigned a random code number so that your name won't be directly connected to your scores. I am analyzing the COURSE, not those taking it.

You will earn attendance points for completing the test, but your specific scores on the test will have NO IMPACT on your course grade.

Although you will all be asked to take the test as a regular part of the course, you have full control over whether your scores are actually used in the study or not. Your instructor will not be told who gives consent and who doesn't. Your grade will not be

impacted by whether you consent to be a part of this study or not. You may change your mind about allowing me to include your scores in the study AT ANY TIME prior to its public release, and you will not be penalized in any way if you choose not to participate.

In order to keep your scores confidential, you will be assigned a random identification number that you will use the first time you login to the testing system. The key connecting your names and ID's will be kept in a locked desk by the administrator until the conclusion of the study, at which time it will be destroyed. It will NOT be shared with your course instructors, nor anyone else outside of the research team. As a part of the test, you will be asked to disclose your sex and race since we want to know if all students are being benefitted equally by this course. In order to protect your confidentiality, results may be published across broad populations (e.g. "males") but the race/sex of specific participants will not be revealed.

If you choose to participate, nothing will be required of you other than taking the pre and post-test. You will not be interviewed or videotaped, and your involvement will not extend beyond when you take the post-test.

If you have any questions, please feel free to contact me. Thank you for your help and cooperation!

Sincerely,

David Bowers

Masters Degree Candidate

Department of Mathematics

(614) 292-7071

bowers.353@osu.edu

For questions about your rights as a participant in this study or to discuss other study-related concerns or complaints with someone who is not part of the research team, you may contact PERSON in the Office of Responsible Research Practices at PHONE NUMBER.