ON THE DETERMINATION OF SPECTRAL PROPERTIES OF CERTAIN FAMILIES OF OPERATORS

DISSERTATION

Presented in Partial Fulfillment of the Requirements for the Degree Doctor of Philosophy in the Graduate School of the Ohio State University

By

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2015

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ABSTRACT

In this paper, we discuss the perturbations of the Harmonic Oscillator and Parabolic Cylinder Operators by an odd pair of point interactions. We prove that there is a convenient formula for the eigenvalues, and show that if the point interactions are purely imaginary in addition to being odd, that nonreal eigenvalues exist as the size of the perturbation grows.

ACKNOWLEDGMENTS

We are grateful to Özge Özcakir for use of her LaTeX class file. We are also grateful to Prof. Boris Mityagin for his advice and support in suggesting the problems and suggesting some approaches, especially regarding the proof of Theorem 1. We are grateful to the user `TrialandError` of `math.stackexchange.com` for his suggestions to simplify the proof of Lemma A.2.2.

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PUBLICATIONS

Baker, Charles, On Determination of Positive-Definiteness of an Anisotropic Operator, *Positivity*, DOI:10.1007/s11117-015-0342-8

FIELDS OF STUDY

Major Field: Graduate Program in Mathematics

Specialization: Spectral Theory

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INTRODUCTION

The *harmonic oscillator operator* is the unbounded, densely-defined, closed, self-adjoint, positive, compact-resolvent operator denoted by

$$\mathfrak{D}\left(L_{\mathrm{HO}}^{0}\right) = \{u \in \mathcal{H}^2(\mathbb{R}) : x^2 u \in L^2(\mathbb{R})\} \tag{1.0.1a}$$

$$L_{\mathrm{HO}}^{0} w(x) := -w''(x) + x^2 w(x), \quad w \in \mathfrak{D}\left(L_{\mathrm{HO}}^{0}\right), \tag{1.0.1b}$$

where $\mathcal{H}^m(\mathbb{R}) = \mathcal{W}^{m,2}(\mathbb{R})$ is the set of L^2 functions with distributional derivatives up to the mth order in L^2, and w'' is interpreted as a distributional derivative.

Since L_{HO}^{0} is an operator of compact resolvent, its spectrum is entirely composed of eigenvalues. Indeed, the form of the eigenvalues and eigenfunctions are well-known (see, e.g., Folland's real analysis text [Fol99, Exercise 8.23, pp. 256–257]): defining the *Hermite polynomials*

$$H_n(x) := (-1)^n e^{x^2} \frac{d^n}{dx^n}\left(e^{-x^2}\right), \quad n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\} \tag{1.0.2}$$

and the *Hermite functions*

$$h_n(x) := \frac{1}{\sqrt{2^n n! \sqrt{\pi}}} e^{-x^2/2} H_n(x), \quad n \in \mathbb{N}_0 \tag{1.0.3}$$

we have that

$$L^0_{\mathrm{HO}} h_n = (2n+1)h_n, \quad n \in \mathbb{N}_0. \tag{1.0.4}$$

Since the Hermite functions form an orthonormal basis of $L^2(\mathbb{R})$ (e.g., [Fol99, Exercise 8.23(g), p. 257]), and L^0_{HO} is self-adjoint, the spectrum $\mathrm{Sp}(L^0_{\mathrm{HO}})$ is understood:

$$\mathrm{Sp}(L^0_{\mathrm{HO}}) = \{2n+1\}_{n=0}^{\infty}$$

and any eigenfunction for eigenvalue $2n+1$, $n \in \mathbb{N}_0$, is a scalar multiple of $h_n(x)$.

Several papers — B. Mityagin's and P. Siegl's text [MS13], B. Mityagin's preprint [Mit14] and paper [Mit15], and the work of Haag, Cartarius, and Wunner in [HCW14] — discuss the perturbation of L^0_{HO} by a pair of point interactions; e.g., they define the closed operator

$$L_{\mathrm{HO}}(\zeta, \beta) w = L^0_{\mathrm{HO}} w + \mathcal{A}_{\zeta,\beta}(x) w, \quad \beta > 0, \quad \zeta \in \mathbb{C}$$

where

$$\mathcal{A}_{\zeta,\beta}(x) = \zeta[\delta\,(x-\beta) - \delta\,(x+\beta)], \quad \beta > 0, \quad \zeta \in \mathbb{C}.$$

(The interpretation on the point-mass multiplication is given by

$$\delta\,(x-p)\,u(x) = u(p)\delta\,(x-p)\,, \quad p \in \mathbb{R}, \quad u \in C(\mathbb{R}); \tag{1.0.5}$$

evaluation at a point is justified because the operator is defined on a subset of $\mathcal{H}^2(\mathbb{R})$, and every $f \in \mathcal{H}^2(\mathbb{R})$ has a continuous representative f_0, i.e., there exists $f_0 \in C(\mathbb{R})$ such that $f = f_0$ Lebesgue-a.e. on $\mathbb{R}$ [Fol99, Thm. 9.17, pp. 303–4].)

The sum is interpreted as a form-sum as guaranteed by the KLMN Theorem, as shown in T. Kato's text [Kat95, Section VI.2]. The operator is shown to be of compact resolvent (essentially, [Kat95, Chapter VI, Section 4.2, Thm. 4.3, p. 396]), and the behavior of the eigenvalues are studied. A particular result is that when z is purely imaginary, i.e., $z = ir$ for $r \in \mathbb{R}$, the number of nonreal eigenvalues is finite and bounded in terms of $|r|$. The question we seek to answer is if for $z = ir$, r real, *any* eigenvalues are nonreal, and more generally, what else can be said about the behaviour of the eigenvalues.

When attempting to study this question in more depth, however, we reach a practical difficulty. It is not hard to show that any eigenfunction of $L_{\mathrm{HO}}(\zeta, \beta)$, with eigenvalue λ, is not only continuous on $\mathbb{R}$, but also C^∞ on the intervals $(-\infty, -b)$, $(-b, b)$, and (b, ∞), because it must be a solution of the differential equation

$$-\frac{d^2w}{dx^2} + x^2 w(x) = \lambda w(x) \tag{1.0.6}$$

on each of the aforementioned intervals. Although the $L^2(\mathbb{R})$ requirement enforces decay conditions as $x \to \pm\infty$, a priori any solution could be the solution on $(-b, b)$, regardless of its growth or decay at ∞ – in particular, since two linearly independent solutions must exist on any interval (as found in any differential equation text; in particular, in F. Olver's asymptotics text at [Olv74, Thm. 5.1.1, p. 139]), it could be any combination of any given basis of solutions. It would therefore be useful to have a known basis of solutions for each $\lambda \in \mathbb{C}$. We do not know of any such explicitly computed list for the above differential equation, but we do have one for the *Weber parabolic cylinder equation*, one of whose guises is

$$-\frac{d^2y}{dx^2} + \frac{x^2}{4} y(x) = \left(\nu + \frac{1}{2}\right) y(x), \tag{1.0.7}$$

which is quite similar: indeed, we will prove that the conversion

$$\begin{aligned} w(x) &:= y\left(x\sqrt{2}\right) \\ \lambda &:= 2\nu + 1. \end{aligned} \tag{1.0.8}$$

transforms solutions of (1.0.7) to solutions of (1.0.6), invertibly (on appropriate intervals).

We prefer, however, to rewrite (1.0.7) as

$$-\frac{d^2y}{dx^2} + \left[\frac{x^2}{4} - \frac{1}{2}\right] y(x) = \nu y(x), \tag{1.0.9}$$

and we therefore define

$$\mathfrak{D}\left(L^0_{\mathrm{PC}}\right) = \{u \in \mathcal{H}^2(\mathbb{R}) : x^2 u \in L^2(\mathbb{R})\} \quad \left(= \mathfrak{D}\left(L^0_{\mathrm{HO}}\right)\right) \tag{1.0.10a}$$

$$L^0_{\mathrm{PC}} y := -y''(x) + \left(\frac{1}{4}x^2 - \frac{1}{2}\right) y(x), \quad y \in \mathfrak{D}\left(L^0_{\mathrm{PC}}\right). \tag{1.0.10b}$$

Then we can formally write the eigenvalue problem as

$$-y'' + \left[\frac{x^2}{4} - \frac{1}{2}\right] y(x) = \nu y(x), \quad y \in \mathfrak{D}\left(L^0_{\mathrm{PC}}\right),$$

so that the parameter ν can be taken to be synonymous with the eigenvalue.

Also, it will follow that

$$\operatorname{Sp} L^0_{\mathrm{PC}} = \mathbb{N}_0 := \mathbb{N} \cup \{0\}, \tag{1.0.11}$$

and that the corresponding normalized eigenfunction is $\frac{1}{\sqrt[4]{2}} h_n\left(\frac{x}{\sqrt{2}}\right)$. The collection of eigenfunctions $\left\{\frac{1}{\sqrt[4]{2}} h_n\left(\frac{x}{\sqrt{2}}\right)\right\}_{n=0}^{\infty}$ is again an orthonormal basis.

Thus, analogously to the above, we will define the perturbation of L^0_{PC} by

$$L_{\mathrm{PC}}(z,b)y(x) = -y''(x) + \left[\frac{x^2}{4} - \frac{1}{2}\right] y(x) + z[\delta(x-b) - \delta(x-b)], \quad b>0, \quad z \in \mathbb{C} \tag{1.0.12}$$

and search for its eigenvalues; the formal conversion is as follows.

Proposition 1 (Folklore)**.** *Fix $b > 0$ and $z \in \mathbb{C}$. Let $Sx = x\sqrt{2}$ denote the linear transformation on $\mathbb{R}$, and let $Tf(x) = f \circ S(x)$ denote its extension to a bounded operator on $L^2(\mathbb{R})$. Then we have that*

$$L_{PC}(z,b) = \frac{1}{2} T^{-1} \circ L_{HO}\left(z\sqrt{2}, \frac{b}{\sqrt{2}}\right) \circ T - \frac{1}{2} I. \tag{1.0.13}$$

In particular,

$$\mathrm{Sp}(L_{PC}(z,b)) = \frac{\mathrm{Sp}\, L_{HO}\left(z\sqrt{2}, \frac{b}{\sqrt{2}}\right) - 1}{2}. \tag{1.0.14}$$

Hence, answering questions about the eigenvalues of $L_{\mathrm{PC}}(z,b)$ will enable us to discuss the eigenvalues of $L_{\mathrm{HO}}(\zeta,\beta)$.

In the case $z = ir$, r real, the eigenvalues of $L_{\mathrm{PC}}(z,b)$ were discussed by E. Demiralp in the paper [Dem05]. (Demiralp takes the weight on the point-mass at $-b$ to be the *conjugate* of the weight on the point-mass at b, but in the case $z = ir$, his convention and ours coincide.) In particular, [Dem05] gives the following formula for $\nu \notin \mathbb{N}_0$ to be an eigenvalue of $L_{\mathrm{PC}}(ir,b)$:

$$1 + \frac{|r|^2}{W^2} D_\nu^2(b) \left(D_\nu^2(-b) - D_\nu^2(b)\right) = 0, \tag{1.0.15}$$

where $D_\nu(x)$ is the *parabolic cylinder function* that is a standard solution to (1.0.7) decaying as $x \to \infty$ (see [Olv74, Chapter 6, Section 6, pp. 206–208]), and $W = \mathrm{Wr}\left[D_\nu, D_\nu(-\cdot)\right](b) = \mathrm{Wr}\left[D_\nu, D_\nu(-\cdot)\right](-b)$ is the Wronskian of $D_\nu(x)$ and $D_\nu(-x)$

in x at the point b (or $-b$). It is known (see N. Temme's contribution to the Digital Library of Mathematical Functions, in particular [Tem14, Section 12.2(iii), (12.2.11)]) that

$$W = \frac{\sqrt{2\pi}}{\Gamma(-\nu)}, \tag{1.0.16}$$

so the failure of the formula for $\nu \in \mathbb{N}_0$ is caused by a term $\Gamma^2(-\nu)$ implicit in the second term on the left-hand side of (1.0.15).

We are now in position to state the new results of this dissertation. Our first result is a small extension of the critical formula in (1.0.15), that works for all $z \in \mathbb{C}$ and all $\nu \in \mathbb{C}$. The formula uses the fact that (1.0.7) has a distinguished even solution and a distinguished odd solution, called $y_1(\nu; x)$ and $y_2(\nu; x)$, respectively. More details about these solutions are in Section 3.1.

Theorem 1. *Fix $b > 0$, and $z \in \mathbb{C}$. Then $\nu \in \mathbb{C}$ is an eigenvalue of $L_{PC}(z, b)$ if and only if*

$$\frac{\sqrt{\pi}}{\sqrt{2}\Gamma(-\nu)} - z^2 D_\nu^2(b) y_1(\nu; b) y_2(\nu; b) = 0. \tag{1.0.17}$$

In particular, we find that after appropriate changes of variables, (1.0.17) and (1.0.15) are the same up to a factor of $\dfrac{2\Gamma^2(\nu)}{\pi}$ — primarily because

$$D_\nu^2(-b) - D_\nu^2(b) = -\left(D_\nu^2(b) - D_\nu(-b)\right) \cdot \left(D_\nu(b) + D_\nu(-b)\right)$$

is essentially the evaluation at b of an *even* solution and an *odd* solution to (1.0.7) by linearity of the differential equation. Hence, it is a (ν-dependent) multiple of $y_1(\nu; b) y_2(\nu; b)$; moreover, the constant multiple contains a factor of $\dfrac{1}{\Gamma(-\nu)}$, which effectively cancels one of the $\Gamma(-\nu)$ factors in (1.0.15). We have:

Theorem 2. *Fix $b > 0$, $z \in \mathbb{C}$, and $\nu \in \mathbb{C} \setminus \mathbb{N}_0$. Then $\nu \in \operatorname{Sp} L_{PC}(z,b)$ if and only if*

$$1 - z^2 M(\nu; b) = 0, \tag{1.0.18}$$

where

$$M(\nu; b) := \frac{\Gamma(-\nu)\sqrt{2}}{\sqrt{\pi}} D_\nu^2(b) y_1(\nu; b) y_2(\nu; b). \tag{1.0.19}$$

Alternatively, for $z \neq 0$, (5.4.7) can be rewritten as

$$M(\nu; b) = \frac{1}{z^2}, \tag{1.0.20}$$

which allows us to separate the variable z out if it is nonzero.

The variation (1.0.20) has several advantages. First, if for $\nu \notin \mathbb{N}_0$, $M(\nu; b) = 0$, then ν is not itself an eigenvalue of $L_{\mathrm{PC}}(z,b)$; however, as $|z| \to \infty$, $\frac{1}{z^2} \to 0$, so the (noninteger) zeroes of $\nu \mapsto M(\nu; b)$ become relevant. The second technical advantage is reducing the powers of Γ in the numerator. This aids our recognition of the fact that if, say, $\nu \mapsto D_\nu(b)$ has a zero at $\nu = n \in \mathbb{N}_0$, $\nu \mapsto M(\nu; b)$ has a removable discontinuity, and in fact the extended function has a zero at $\nu = n$. We can treat the extended equation much like the original, so at the cost of doing the work twice, we may remove certain genericity condtions and get the following absolute result, answering the question posed on page 2.

Theorem 3. *Fix $b > 0$. Then for sufficiently large $r > 0$, $L_{PC}(ir, b)$ has nonreal eigenvalues. Moreover, if $\mathcal{N}(r)$ is the counting-function for the number of nonreal eigenvalues of $L_{PC}(ir, b)$, then*

$$\lim_{r \to \infty} \mathcal{N}(r) = \infty. \tag{1.0.21}$$

The rest of the paper is organized as follows. Chapter 2 reminds the reader of some background theory and some technical lemmas. Chapter 3 constructs the differential equation and operators we will need, though most of the constructions are hardly original, emanating from [Kat95], [MS13], and S. Albeverio et al's book [Alb+05]. For completeness, Chapter 4 deals with the proof of Proposition 1, although the proof is elementary. Chapter 5 deals with the proof of Theorem 1 and Theorem 2. Chapter 6 concerns the proof of Theorem 3. Chapter 7 discusses some partial results towards how the non-real eigenvalues are created.

CHAPTER 2

TECHNICAL PRELIMINARIES

In this chapter we recall various definitions and theorems that will be useful in the sequel.

2.1 Fourier Transforms, Tempered Distributions, and Sobolev Spaces

Our convention on the Fourier Transform is that for $f \in L^1(\mathbb{R})$,

$$\mathcal{F}[f](\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x) e^{-i\pi x \cdot \xi} \, dx. \tag{2.1.1}$$

We abbreviate $\mathcal{F}[f](\xi)$ by $\widehat{f}(\xi)$ when appropriate.

Similarly, the Inverse Fourier Transform is that for $g \in L^1(\mathbb{R})$,

$$\mathcal{F}^{-1}[g]x = \mathcal{F}[g](-x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} g(\xi) e^{i\pi \xi \cdot x} \, d\xi, \tag{2.1.2}$$

and we abbreviate $\mathcal{F}^{-1}[g]x$ by $\check{g}(x)$. We know (see, e.g., [Fol99, Thm. 8.29, p. 252] that the Fourier Transform on $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ extends uniquely to an isomorphism on $L^2(\mathbb{R})$, denoted with the same variables.

We also recall the theory of Tempered Distributions and the Fourier Transform;

see, e.g., [RS72, Section V.3, pp. 133-134] and [RS75, Section IX.1]. The *tempered distributions* in $\mathbb{R}$ are the dual space of $\mathcal{S}$; we denote the pairing between a tempered distribution F and a test-function $\varphi \in \mathcal{S}$ by

$$\langle F, \varphi \rangle_{\mathcal{S}', \mathcal{S}}. \tag{2.1.3}$$

We know that for all $F \in \mathcal{S}'$, the nth distributional derivative $F^{(n)}$ satisfies

$$\widehat{F^{(n)}} = (i\xi)^n \widehat{F}, \tag{2.1.4}$$

which leads to the consequence that $F^{(n)} \in L^2(\mathbb{R})$ if and only if $\xi^n \widehat{F} \in L^2(\mathbb{R})$.

We also recall the definition of the L^2-Sobolev spaces. For $s \in \mathbb{R}$, we define the L^2-Sobolev space $\mathcal{H}^s(\mathbb{R})$ to be the set of tempered distributions such that

$$(1 + \xi^2)^{s/2} \widehat{f}(\xi) \in L^2(\mathbb{R});$$

the inner product and norm are defined by

$$(f, g)_{\mathcal{H}^s(\mathbb{R})} := \int_{\mathbb{R}} (1 + \xi^2)^s \widehat{f}(\xi) \overline{\widehat{g}(\xi)}\, d\xi \tag{2.1.5a}$$

$$\|f\|_{\mathcal{H}^s(\mathbb{R})} := \left[\int_{\mathbb{R}} (1 + \xi^2)^s |\widehat{f}(\xi)|^2\, d\xi \right]^{1/2}. \tag{2.1.5b}$$

It is known (see, e.g., [Fol99, p. 302]) that $(\cdot, \cdot)_{\mathcal{H}^s(\mathbb{R})}$ gives $\mathcal{H}^s(\mathbb{R})$ the structure of a Hilbert space, and if $t < s$, $\mathcal{H}^s(\mathbb{R})$ is a dense subspace of $\mathcal{H}^t(\mathbb{R})$ in $\mathcal{H}^t(\mathbb{R})$ norm. Also, $\mathcal{H}^0(\mathbb{R}) = L^2(\mathbb{R})$ by the Plancherel Theorem, so for $s \geq 0$, $\mathcal{H}^s(\mathbb{R})$ is composed of L^2 functions.

2.2 Hermite Polynomials and Hermite Functions

We remark on some properties of Hermite polynomials, following Szegő's text [Sze75].

Proposition 2.2.1 (Properties of the Hermite Polynomials).

Orthogonal Polynomials. *The Hermite Polynomials are orthogonal with respect to the weight-function* e^{-x^2}*; i.e.,*

$$\text{if} j \neq k, j, k \in \mathbb{N}_0, \quad \int_{\mathbb{R}} H_j(x) H_k(x) e^{-x^2/2}\, dx = 0. \tag{2.2.1}$$

For more details, see [Sze75, Section II.2.4].

Recurrence Relation. *The Hermite polynomials can be computed by the following initial conditions and recurrence relation:*

$$H_0(x) = 1, \tag{2.2.2a}$$

$$H_1(x) = 2x, \tag{2.2.2b}$$

$$H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x), \quad n \in \mathbb{N}. \tag{2.2.2c}$$

Multiplication-by-x. *Certainly, multiplication by x gives that*

$$xH_0(x) = x = \frac{1}{2} H_1(x),$$

and for $n \in \mathbb{N}$, rewriting (2.2.2c) *gives that*

$$xH_n(x) = \frac{1}{2} H_{n+1}(x) + nH_{n-1}(x), \quad n \in \mathbb{N}. \tag{2.2.3}$$

Note that (2.2.3) *holds even for $n = 0$, if we define $H_{-1}(x)$ to be the zero polynomial, for simplicity.*

Parity. *Being orthogonal polynomials on an interval symmetric with respect to the origin, according to a weight-function symmetric with respect to the origin, a Hermite polynomials is an even function (resp. odd function) if their index is even (resp. odd); i.e.,*

$$H_n(-x) = (-1)^n H_n(x), \quad n \in \mathbb{N} \cup \{0\} \tag{2.2.4}$$

Moreover, $H_n(x)$ *only contains those powers of* x *which are congruent to* n *modulo* 2. *For both of these results, see [Sze75, Section II.2.3.2, p. 28].*

We now note some other properties of the Hermite Functions:

Proposition 2.2.2 (Properties of the Hermite functions)**.**

Orthonormality. *The Hermite functions form an orthonormal basis of* $L^2(\mathbb{R})$*, as noted in [Fol99, Exercise 8.23, pp. 256–7] and elsewhere.*

Fourier Transform. *Under our convention on the Fourier Transform, the Hermite Functions are eigenfunctions of the Fourier transform (e.g., [Fol99, Exercise 8.23h, p. 257]):*

$$\widehat{h_n}(\xi) = (-i)^n h_n(\xi), n \in \mathbb{N}_0 \tag{2.2.5}$$

Schwarz Class. *For all* $n \in \mathbb{N}_0$, $h_n(x) \in \mathcal{S}$*, the class of rapidly-decaying* C^∞ *functions on* $\mathbb{R}$*. This follows since for all* $N \in \mathbb{N}_0$, $(1 + x^2)^N h_n(x)$ *is a polynomial times* $e^{-x^2/4}$*, hence is bounded, and differentation preserves the class of functions that are polynomials times* $e^{-x^2/4}$*.*

Multiplication by powers of x. *Starting with* (2.2.2c)*, multiplying both sides by* $e^{-x^2/2}$*, and normalizing as in* (1.0.3)*, we see that*

$$xh_n(x) = \frac{\sqrt{n+1}}{\sqrt{2}} h_{n+1}(x) + \frac{\sqrt{n}}{\sqrt{2}} h_{n-1}(x), n \in \mathbb{N}. \tag{2.2.6}$$

Of course, if $n = 0$*, one verifies that* $xh_0(x) = \frac{1}{\sqrt{2}} h_1(x)$*, which is again consistent with* (1.0.3) *under the convention* $h_{-1}(x) = 0$*. As we have* x^2 *appearing in our operators, we go ahead and calculate the effects of multiplication by* x^2*:*

$$\begin{aligned} x^2 h_n(x) = {} & \frac{\sqrt{(n+1)(n+2)}}{2} h_{n+2}(x) + \frac{(2n+1)}{2} h_n(x) \\ & + \frac{\sqrt{n(n-1)}}{2} h_{n-2}(x), \quad n \geq 2,\, n \in \mathbb{N}, \end{aligned} \tag{2.2.7}$$

where again the formulas extend to $n = 1$ *and* $n = 0$ *under the convention* $h_{-1}(x) = h_{-2}(x) = 0$.

Parity. *Of course,* $e^{-x^2/2}$ *is even, and positive constants do not change oddness or evenness, so* $h_n(x)$ *is odd or even according to whether* n *is odd or even.*

2.3 Unbounded Operators, Quadratic Forms

We review some simple facts from the theory of unbounded linear operators on Hilbert spaces; see, e.g., [RS72, Chapter 8] or [Kat95, Chapter V, Section 3] In particular, we wish to remind the reader that even if A is unbounded, for any bounded invertible operator B, $B^{-1}AB$ shares most of the properties of B. This will be useful once we have (1.0.13), as it will allow us to transfer what we need.

Lemma 2.3.1. *Let* $A : \mathfrak{D}(A) \to \mathcal{H}$ *be a closed, densely defined (possibly unbounded) linear operator with* $\mathfrak{D}(A) \subseteq \mathcal{H}$*, and let* $B : \mathcal{H} \to \mathcal{H}$ *be a bounded, invertible linear operator. Then* AB*,* BA*, and* $B^{-1}AB$ *are also closed and densely defined.*

Lemma 2.3.2. *Let $A : \mathfrak{D}(A) \to \mathcal{H}$ be a closed, densely defined, and unbounded linear operator on a domain $\mathfrak{D}(A) \subseteq \mathcal{H}$, and let $B : \mathcal{H} \to \mathcal{H}$ be a bounded, invertible linear operator. Then:*

- $\operatorname{Sp} A = \operatorname{Sp} B^{-1}AB$
- *A has compact resolvent if and only if $B^{-1}AB$ has compact resolvent.*
- *A is self-adjoint if and only if $B^{-1}AB$ is.*

In addition to writing operators as conjugates of known operators, we will create operators from quadratic forms. We here note some of the theory from, e.g., [Kat95, Chapter VI, Sections 1 – 3]

Definition 2.3.3. Fix $\mathcal{H}$ a Hilbert space, and $\mathcal{L}$ a linear manifold in $\mathcal{H}$. A *sesquilinear quadratic form on $\mathcal{L}$* is a map $\mathfrak{t} : \mathcal{L} \times \mathcal{L} \to \mathcal{H}$, linear in the first argument and conjugate-linear in the second argument. Often, we omit the word "sesquilinear" and explicit reference to the domain, and talk about quadratic forms when we mean sesquilinear quadratic forms on a linear manifold $\mathcal{L}$. Also, we may use the alternative notation $\mathfrak{D}(\mathfrak{t})$ for $\mathcal{L}$.

We defer most of the quadratic-form theory to Appendix 3, where it is used in the formal construction of the operators $L_{\mathrm{PC}}(z, b)$, but we mention one result coming from this theory in particular.

Lemma 2.3.4. *There exists a positive, self-adjoint square root of L^0_{HO}, which we call $(L^0_{HO})^{1/2}$. Moreover, $\mathfrak{D}\left((L^0_{HO})^{1/2}\right) = \mathfrak{D}_1$, where*

$$\mathfrak{D}_1 := \{f \in \mathcal{H}^1(\mathbb{R}) : xf \in L^2(\mathbb{R})\}. \tag{2.3.1}$$

Moreover,

$$(L^0_{HO})^{1/2} h_k(x) = \sqrt{2k+1}\, h_k(x). \tag{2.3.2}$$

2.4 Decay Lemmata

We now show that to some extent, the decay of the operators is embedded in the decay of the Hermite functions.

Definition 2.4.1. Let ℓ_2 denote $L^2(\mathbb{N}_0)$, with the implied measure being counting measure. We define the space $\mathfrak{L}_N$, $N \in \mathbb{N}_0$, to be the space of square-summable functions on $\mathbb{N}$ with weight $(1+k)^{N/2}$:

$$\begin{aligned} \mathfrak{L}_N &:= \left\{ (c_k)_{k=0}^{\infty} : \sum_{k=0}^{\infty} (1+k)^N |c_k|^2 < \infty \right\} \\ &= \big\{ (c_k)_{k=0}^{\infty} : (k^{j/2} c_k)_{k=0}^{\infty} \in L^2(\mathbb{N}_0) \text{ for all } j \in \{0, 1, \ldots, N\} \big\}. \end{aligned} \tag{2.4.1}$$

Of course, $\ell_2 = \mathfrak{L}_0$, and for all $N \in \mathbb{N}_0$, $\mathfrak{L}_{N+1} \subseteq \mathfrak{L}_N$.

Definition 2.4.2. We define the domains $\mathfrak{D}_N$, $N \in \mathbb{N}_0$, to be the space of $\mathcal{H}^N(\mathbb{R})$ functions whose (inverse) Fourier transform is also in $\mathcal{H}^N(\mathbb{R})$:

$$\begin{aligned} \mathfrak{D}_N &:= \{ f(x) \in \mathcal{H}^N(\mathbb{R}) : \check{f}(\xi) \in \mathcal{H}^N(\mathbb{R}) \} \\ &= \{ f(x) \in L^2(\mathbb{R}) : (1+x^2)^{N/2} f(x) \in L^2(\mathbb{R}), (1+\xi^2)^{N/2} \widehat{f}(\xi) \in L^2(\mathbb{R}) \} \\ &= \big\{ f(x) \in L^2(\mathbb{R}) : x^j f(x) \in L^2(\mathbb{R}) \text{ for all } j \in \{0, 1, \ldots, N\}, \\ &\qquad \xi^j \widehat{f}(\xi) \in L^2(\mathbb{R}) \text{ for all } h \in \{0, 1, \ldots, N\} \big\} \end{aligned} \tag{2.4.2}$$

Of course, $\mathfrak{D}_0 = L^2(\mathbb{R})$, and $\mathfrak{D}_{N+1} \subseteq \mathfrak{D}_N$.

Since we will need it later, we interpose a quick lemma.

Lemma 2.4.3. *For all $N \in \mathbb{N}_0$, $\mathcal{S} \subseteq \mathfrak{D}_N$.*

Proof. We note that the Schwartz class $\mathcal{S}$ is a subset of every Sobolev space $\mathcal{H}^s(\mathbb{R})$, $s \in \mathbb{R}$. Morever, the (inverse) Fourier transform is a bijection on $\mathcal{S}$. Hence, for all $\varphi \in \mathcal{S}$, and any $N \in \mathbb{N}_0$, $\varphi \in \mathcal{H}^N(\mathbb{R})$ and $\check{\varphi}(\xi) \in \mathcal{H}^N(\mathbb{R})$, so $\varphi \in \mathfrak{D}_N$. □

Now we prove the desired lemma connecting the decay of the Hermite-function-basis coefficients to the decay of the functions.

Lemma 2.4.4. *Let ι denote the isomorphism $\ell_2 \to L^2(\mathbb{R})$ given by*

$$\iota\left((c_k)_{k=0}^{\infty}\right) := \sum_{k=0}^{\infty} c_k h_k(x). \tag{2.4.3}$$

For $N \in \{0, 1, 2\}$, $\iota(\mathfrak{L}_N) = \mathfrak{D}_N$.

Proof, $\mathfrak{D}_N \subseteq \iota(\mathfrak{L}_N)$. The case $N = 0$ merely reiterates that ι is an isomorphism. It behooves us to next prove the case $N = 2$. Suppose $f \in \mathfrak{D}_2$. Then $f \in \mathfrak{D}\left(L^0_{\text{HO}}\right)$, so $L^0_{\text{HO}} f \in L^2$. Yet by Parseval, and by L^0_{HO} symmetric, if $f = \sum c_k h_k$, then

$$\begin{aligned} L^0_{\text{HO}} f &= \sum_{k=0}^{\infty} \left(L^0_{\text{HO}} f, h_k\right)_{L^2(\mathbb{R})} h_k \\ &= \sum_{k=0}^{\infty} \left(f, L^0_{\text{HO}} h_k\right)_{L^2(\mathbb{R})} h_k \\ &= \sum_{k=0}^{\infty} (f, (2k+1) h_k)_{L^2(\mathbb{R})} h_k \\ &= \sum_{k=0}^{\infty} (2k+1) c_k h_k, \end{aligned} \tag{2.4.4}$$

and hence

$$\|L^0_{\text{HO}} f\|^2 = \sum_{k=0}^{\infty} (2k+1)^2 |c_k|^2. \tag{2.4.5}$$

Hence, if $f \in \mathfrak{D}_2$, then $\{(2k+1)c_k\}_{k=0}^{\infty} \in \ell^2$, so $\{(k+1)c_k\}_{k=0}^{\infty} \in \ell^2$, so $f \in \iota(\mathfrak{L}_2)$.

Now we prove the case $N = 1$. Fortunately, by Lemma A.1.9, we have the positive, self-adjoint operator $(L^0_{\text{HO}})^{1/2}$ with domain $\mathfrak{D}_1$; moreover, by (2.3.2), $(L^0_{\text{HO}})^{1/2}$ satisfies

$$(L^0_{\text{HO}})^{1/2} h_k = \sqrt{2k+1}\, h_k.$$

Hence, for any $f \in \mathfrak{D}_1$, $(L^0_{\text{HO}})^{1/2} f \in L^2(\mathbb{R})$, so if $f(x) = \sum c_k h_k(x)$,

$$\begin{aligned}(L^0_{\text{HO}})^{1/2} f &= \sum_{k=0}^{\infty} \big((L^0_{\text{HO}})^{1/2} f, h_k\big)_{L^2(\mathbb{R})} h_k \\ &= \sum_{k=0}^{\infty} \big(f, (L^0_{\text{HO}})^{1/2} h_k\big)_{L^2(\mathbb{R})} h_k \\ &= \sum_{k=0}^{\infty} \Big(f, \sqrt{2k+1} h_k\Big)_{L^2(\mathbb{R})} h_k \\ &= \sum_{k=0}^{\infty} c_k \sqrt{2k+1} h_k,\end{aligned} \tag{2.4.6}$$

Hence, $\{c_k\sqrt{2k+1}\}_{k=0}^{\infty} \in \ell^2$, so $\{c_k\sqrt{k+1}\}_{k=0}^{\infty} \in \ell^2$, so $f \in \iota(\mathfrak{L}_1)$. □

Proof, $\iota(\mathfrak{L}_N) \subseteq \mathfrak{D}_N$. The case $N = 0$ again follows from ι being an isomorphism between ℓ^2 and $L^2(\mathbb{R})$, so let us assume $N = 1$. Fix $f(x) \in \iota(\mathfrak{L}_1)$, so that $f(x) = \sum_{k=0}^{\infty} c_k h_k(x)$ with $(\sqrt{1+k} c_k)_{k=0}^{\infty} \in \ell^2$. For $J \in \mathbb{N}_0$, let S_J denote the orthogonal projection onto $\text{span}\langle h_0(x), h_1(x), \dots h_J(x)\rangle$.

For any $J \in \mathbb{N}_0$, $S_J f \in \mathfrak{D}(M_x)$, since by (2.2.6),

$$\begin{aligned}M_x S_J(f) &= \sum_{k=0}^{J} c_k M_x h_k(x) \\ &= c_0 \frac{1}{\sqrt{2}} h_1(x) + \sum_{k=1}^{J} c_k \left(\sqrt{\frac{k+1}{2}} h_{k+1}(x) + \sqrt{\frac{k}{2}} h_{k-1}(x)\right) \\ &= \sum_{k=0}^{J} c_k \sqrt{\frac{k+1}{2}} h_{k+1}(x) + \sum_{k=1}^{J} c_k \sqrt{\frac{k}{2}} h_{k-1}(x).\end{aligned} \tag{2.4.7}$$

We wish to justify that multiplication-by-x is well-described by the Hermite-function basis, i.e., we wish to show that $\lim_{J\to\infty} M_x S_J(f) = M_x f$. Fortunately, it is clear that for any $P, Q \in \mathbb{N}$, $2 \leq P \leq Q$, we have that by (2.4.7),

$$\begin{aligned}
&\|M_x S_Q f - M_x S_P f\|^2_{L^2(\mathbb{R})} \\
&= \left\| \sum_{k=P+1}^{Q} c_k \sqrt{\frac{k+1}{2}} h_{k+1}(x) + \sum_{k=P+1}^{Q} c_k \sqrt{\frac{k}{2}} h_{k-1}(x) \right\|^2_{L^2(\mathbb{R})} \\
&\leq \left(\left\| \sum_{k=P+1}^{Q} c_k \sqrt{\frac{k+1}{2}} h_{k+1}(x) \right\|_{L^2(\mathbb{R})} + \left\| \sum_{k=P+1}^{Q} c_k \sqrt{\frac{k}{2}} h_{k-1}(x) \right\|_{L^2(\mathbb{R})} \right)^2 \\
&\leq 2 \left(\left\| \sum_{k=P+1}^{Q} c_k \sqrt{\frac{k+1}{2}} h_{k+1}(x) \right\|^2_{L^2(\mathbb{R})} + \left\| \sum_{k=P+1}^{Q} c_k \sqrt{\frac{k}{2}} h_{k-1}(x) \right\|^2_{L^2(\mathbb{R})} \right) \\
&= 2 \left(\sum_{k=P+1}^{Q} |c_k|^2 \frac{k+1}{2} + \sum_{k=P+1}^{Q} |c_k| \frac{k}{2} \right) \\
&= 2 \cdot \frac{2}{2} \left(\sum_{k=P+1}^{Q} (k+1)|c_k|^2 \right),
\end{aligned} \tag{2.4.8}$$

where of course if $P = Q$, the sum is empty. Now by $f \in \iota(\mathfrak{L}_1)$, $\sum_{k=0}^{\infty}(k+1)|c_k|^2 < \infty$, so it is clear that

$$\sum_{k=P+1}^{Q} (k+1)|c_k|^2 \leq \sum_{k=P+1}^{\infty} (k+1)|c_k|^2, \tag{2.4.9}$$

ans as $P \to \infty$, the right-hand side of (2.4.9) tends to 0. By the Squeeze Theorem, as $\min\{P, Q\} \to \infty$, $\|M_x S_Q f - M_x S_P f\|^2_{L^2(\mathbb{R})}$ tends to 0. In other words,

$$(M_x S_J f)_{J=0}^{\infty} \quad \text{is Cauchy in } L^2(\mathbb{R}). \tag{2.4.10}$$

Since $L^2(\mathbb{R})$ is complete, then clearly $\{M_x S_J f(x)\}_{J=0}^{\infty}$ is a convergent sequence, so there exists

$$g(x) := \lim_{J \to \infty} M_x S_J f(x).. \tag{2.4.11}$$

Moreover, M_x is a self-adjoint operator, hence closed (see, e.g., Reed/Simon Volume

I, specifically [RS72, Section VIII.3, Proposition 1, pp. 259-260]). We of course have $\lim_{J\to\infty} S_J f(x) = f(x)$ in $L^2(\mathbb{R})$, and (2.4.11) holds, so by closure of M_x, $f \in \mathfrak{D}(M_x)$ and

$$xf(x) = M_x f(x) = \lim_{J\to\infty} M_x S_J f(x), \tag{2.4.12}$$

so $xf(x) \in L^2(\mathbb{R})$. We note that by (2.2.5),

$$\widehat{f}(\xi) = \sum_{k=0}^{\infty} (-i)^k c_k h_k(x), \tag{2.4.13}$$

and of course $((-i)^l c_k\sqrt{1+k})_{k=0}^{\infty} \in \ell^2$ if and only if $(c_k\sqrt{1+k})_{k=0}^{\infty} \in \ell^2$, so by similar logic, one shows that $\xi\widehat{f}(\xi) \in L^2(\mathbb{R})$. Hence, $f \in \mathfrak{D}_1$, as required. This works for all $f \in \iota(\mathfrak{L}_1)$, so $\iota(\mathfrak{L}_1) \subseteq \mathfrak{D}_1$.

The case $N = 2$ is similar. Of course, $\mathfrak{L}_2 \subseteq \mathfrak{L}_1$ and $\iota(\mathfrak{L}_2) \subseteq \iota(\mathfrak{L}_1) = \mathfrak{D}_1$, so $xf(x) \in L^2(\mathbb{R})$ and $\xi\widehat{f}(\xi) \in L^2(\mathbb{R})$ for all $f(x) \in \iota(\mathfrak{L}_2)$. To show $x^2 f(x) \in L^2(\mathbb{R})$, one replaces the use of (2.2.6) by (2.2.7), but essentially the same proof shows that $x^2 S_J f(x) \xrightarrow{L^2(\mathbb{R})} x^2 f(x)$ for $f(x) \in \iota(\mathfrak{L}_2)$. Again, the Fourier transform shows that $\xi^2\widehat{f}(\xi) \in L^2(\mathbb{R})$ in the same way. We leave the details to the interested reader. $\square$

2.5 Theory of Linear Homogeneous ODEs of Second Order

In this section, we remind the reader of various facts in the theory of ordinary differential equations. Our presentation follows Chapter 5 of F. Olver's text [Olv74].

The results are stated for holomorphic functions over open sets in the complex plane; analogous statements for continuous functions (with appropriate numbers of derivatives) on open intervals of the real line follow in the same way.

Proposition 2.5.1 ([Olv74, Chap. V, Thm. 3.1, p. 145]). *Let $f(x)$ and $g(x)$ be holomorphic in a simply connected domain Ω. Then the equation*

$$\frac{d^2w}{dx^2} + f(x)\frac{dw}{dx} + g(x)w = 0 \tag{2.5.1}$$

has an infinity of solutions which are holomorphic in Ω. If the values of w and $dw/d\zeta$ are prescribed at any point, then the solution is unique.

In addition, we have the following enhancement of the previous proposition to the case of a parameter.

Proposition 2.5.2 ([Olv74, Chap. V., Thm. 3.2, p. 146–7]). *Fix Ω and U open, simply connected subsets of $\mathbb{C}$, and define $f, g \in \mathrm{Hol}(U \times \Omega)$, and consider, for each $u \in U$, the differential equation*

$$\frac{d^2w}{dx^2} + f(u,x)\frac{dw}{dx} + g(u,x)w = 0, \quad x \in \Omega. \tag{2.5.2}$$

Suppose that at some fixed $x_0 \in \Omega$, the values of w and $\dfrac{\partial w}{\partial z}$ are holomorphic functions of u in U. Then at each $x \in \Omega$, the solution $w(u,x)$ of (2.5.2) and its first two partial x derivatives are holomorphic functions of u.

Recall that given two functions $f, g \in \mathrm{Hol}(\Omega)$ for some open, simply connected subset Ω of $\mathbb{C}$, we define their Wronskian $\mathrm{Wr}\,[f,g]\,(x) : \Omega \to \mathbb{C}$ by

$$\mathrm{Wr}\,[f,g]\,(x) := f(x)\frac{dg}{dx}(x) - g(x)\frac{df}{dx}(x), \quad x \in \Omega. \tag{2.5.3}$$

The elementary properties of the Wronskian are listed below for completeness.

Lemma 2.5.3. *Fix Ω an open, simply connected subset of $\mathbb{C}$. Then:*

1. *The Wronskian is linear in each argument: for* $f, \varphi, g, \psi \in \mathrm{Hol}(\Omega)$*,* $c, d \in \mathbb{C}$*, and* $x \in \Omega$*,*

$$\mathrm{Wr}\left[cf + \varphi, g\right](x) = c\,\mathrm{Wr}\left[f, g\right](x) + \mathrm{Wr}\left[\varphi, g\right](x)\,, \quad \textit{and} \tag{2.5.4a}$$

$$\mathrm{Wr}\left[f, dg + \psi\right](x) = d\,\mathrm{Wr}\left[f, g\right](x) + \mathrm{Wr}\left[f, \psi\right](x)\,. \tag{2.5.4b}$$

2. *The Wronskian is an alternating function at every point: for* $f, g \in \mathrm{Hol}(\Omega)$ *and* $x \in \Omega$*,*

$$\mathrm{Wr}\left[g, f\right](x) = -\,\mathrm{Wr}\left[f, g\right](x) \tag{2.5.5}$$

In particular, $\mathrm{Wr}\left[f, f\right](x) = 0$ *for all* $f \in \mathrm{Hol}(\Omega)$*.*

The Wronskian is useful for determining linear independence of solutions to a second-order, linear, homogeneous ODE as discussed below.

Proposition 2.5.4 ([Olv74, Thm. 5.1.2, pp. 141–143, and p. 146])**.** *Let* Ω *be an open, simply connected subset of* $\mathbb{C}$*, and let* w_1 *and* w_2 *be two solutions of the differential equation*

$$-\frac{d^2y}{dx^2} + f(x)\frac{d^1y}{dx^1} + g(x)y(x) = 0, \quad f(x), g(x) \textit{ holomorphic over } \Omega \tag{2.5.6}$$

Then the following three statements are equivalent.

1. *Any solution* w *of* (2.5.6) *is a linear combination of* w_1 *and* w_2*.*

2. *The Wronskian* $\mathrm{Wr}\left[w_1, w_2\right](x)$ *does not vanish at any* $x \in \Omega$*. In particular, if* $f(x)$ *is the zero-function, then the Wronskian is a constant function on* Ω*.*

3. w_1 *and* w_2 *are linearly independent on* Ω*.*

Similar statements hold for open, connected subsets of the real line, where f*,* g *need only be continuous.*

We finish the section by explicitly recalling the effects of rescaling the input of functions on their derivatives and Wronskians; the proof is left to the reader.

Lemma 2.5.5. *Fix $f, g \in \mathrm{Hol}(\mathbb{C})$, and for some $c \in \mathbb{C}$, define $\mathfrak{f}, \mathfrak{g} \in \mathrm{Hol}(\mathbb{C})$ by*

$$\mathfrak{f}(x) := f(cx), x \in \mathbb{C}$$
$$\mathfrak{g}x := g(cx), x \in \mathbb{C}.$$

Then the following statements hold.

1. *For all $n \in \mathbb{N}_0$,*
$$\left.\frac{d^n \mathfrak{f}}{dx^n}\right|_{x=x_0} = c^n \left.\frac{d^n \mathfrak{f}}{dx^n}\right|_{x=ax_0}, n \in \mathbb{N} \cup \{0\}. \tag{2.5.7}$$

2.
$$\mathrm{Wr}\left[\mathfrak{f}, \mathfrak{g}\right](x) = c\,\mathrm{Wr}\left[f, g\right](cx), \quad x \in \mathbb{C} \tag{2.5.8}$$

2.6 Zeroes of Analytic Functions of One Variable

Here we remind the reader of some details of zeroes of functions of one complex variable. We mostly follow S. Lang's complex analysis text, specifically [Lan85, Chapter 2, Section 5].

In the theory of functions of one complex variable, there is a theorem as follows:

Proposition 2.6.1 (e.g., J. B. Conway's text, [Con78, Thm. IV.7.4, pp. 98–99]; L. Ahlfors's text, [Ahl78, Thm. 4.11, p. 131])**.** *Suppose that $f(\zeta)$ is analytic in a neighborhood of $\zeta = \alpha$, $f(\zeta) = \beta$, and $f(\zeta) - \beta$ has a zero of order m at $\zeta = \alpha$. If $\epsilon > 0$ is sufficiently small, there exists a corresponding $\delta > 0$ such that for all γ with $|\gamma - \beta| < \delta$, the equation $f(\zeta) = \gamma$ has exactly m roots in the disk $|z - \alpha| < \epsilon$.*

Since $D_\nu(b)$, $y_1(\nu; b)$ and $y_2(\nu; b)$ are holomorphic in ν, and $\frac{1}{\Gamma(-\nu)}$ is meromorphic in ν with poles at the nonnegative integers, it is clear from (1.0.20) that one can apply the theorem to the map $\nu \mapsto M(\nu; b)$ for any $b > 0$. The following corollary is clear.

Corollary 2.6.2. *Fix $b > 0$. Suppose that $\mu \in \mathbb{C} \setminus \mathbb{N}_0$ satisfies $M(\mu; b) = 0$, and let m be the degree of the zero in ν. Then there exists $C = C(\nu_0; b) > 0$ such that $|z| > C$ implies that* (5.4.9) *has m solutions in a neighborhood of μ; equivalently, by Theorem 2, $L_{PC}(z, b)$ has m eigenvalues in an neighborhood of μ.*

Thus, whenever we will find zeros of $\nu \mapsto M(\nu; b)$, we will discern the long-term behavior of some of the eigenvalues of $L_{\mathrm{PC}}(z, b)$, as $|z| \to \infty$.

Corollary 2.6.2 is too weak for our purposes, however. First, it takes some work (using, e.g., the Maximum Modulus Principle) to show that as $|z| \to \infty$, i.e., as $\left|\frac{1}{z^2}\right| \to 0$, that the eigenvalues tend to ν_0. Moreover, as our sources, e.g., [Mit15], are interested in the reality or non-reality of the eigenvalues as $z = ir$, $|r| \to \infty$, we need some information about the asymptotic direction of approach. Standard proofs of Proposition 2.6.1, e.g., [Con78, p. 98–99] are obtained by various zero-counting theorems employing line integrals over contours wrapping around the zero; it is not easy to extract information about the phase/argument of the zeros from this setting. Therefore, we use the Lagrange interpolation Theorem, as in [Lan85, Chapter II, Sections 1–5] or to discuss the power-series approach to Proposition 2.6.1, to allow us to find a relevant asymptotic. Our results are as follows.

Proposition 2.6.3. *Fix $\alpha \in \mathbb{C}$, U open in $\mathbb{C}$ with $\alpha \in U$, and let $f \in \mathrm{Hol}(U)$ such that α is a zero of order m of $f(\zeta)$. Let the power-series centered at α be given by*

$$f(\zeta) = \sum_{j=m}^{\infty} a_j (\zeta - \alpha)^j, \quad a_m \neq 0. \tag{2.6.1}$$

Then if $\gamma \in \mathbb{C}$ with $|\gamma| > 0$ small enough , then there exist exactly m solutions $\{\zeta_k\}_{k=0}^{m-1} = \{\zeta_k(\gamma)\}_{k=0}^{m-1}$ to $f(\zeta) = \gamma$ for ζ in some neighborhood of α. Moreover, *if* $\gamma = re^{i\theta}$ *and* $a_m = \rho e^{i\psi}$, $r, \rho \in \mathbb{R}^+$, $\theta, \psi \in [-\pi, \pi)$, *then let* $\mathbf{c} = r^{1/m}e^{i\theta/m}$ *and* $\mathfrak{a} = \rho^{1/n}e^{i\psi/m}$ *be specific mth roots of* γ *and* a_m*, respectively. Then the leading-order expansion of the* ζ_k *is*

$$\zeta_k = \alpha + \frac{\mathbf{c}}{\mathfrak{a}}\exp\left(\frac{2\pi ik}{m}\right) - \frac{\mathbf{c}^2}{m\mathfrak{a}^2}\frac{a_{m+1}}{a_m}\exp\left(\frac{4\pi ik}{m}\right) + O\left(\mathbf{c}^3\right), 0 \le k \le m-1. \quad (2.6.2)$$

In particular, as $|\gamma| \to 0$, $\zeta_k \to 0$ *for all* k, $0 \le k \le m-1$*; indeed,*

$$|\zeta_k(\gamma)| = \Theta(|\gamma|^{1/m}), \quad 0 \le k \le m \quad (2.6.3)$$

(recall that for u, g *positive functions,* $u(t) = \Theta(g(t))$ *as* $t \to 0$ *if and only if there exists* $0 < c < C$ *such that for* t *small enough,*

$$cg(t) < u(t) < Cg(t).) \quad (2.6.4)$$

If θ *is fixed and* $r \to 0$*,*

$$\lim_{r\to 0^+}\frac{\zeta_k - \alpha}{|\zeta_k - \alpha|} = \exp\left(\frac{i(\theta - \psi + 2\pi k)}{m}\right), \quad 0 \le k \le m-1 \quad (2.6.5)$$

We further have use to note that the inversion of a real series for real inputs is real.

Lemma 2.6.4. *Fix* $\alpha \in \mathbb{R}$, U *open in* $\mathbb{C}$ *with* $\alpha \in U$*, and let* $f \in \mathrm{Hol}(U)$ *such that* α *is a zero of order* 1 *of* $f(\zeta)$*. Let the power-series at* $\zeta = \alpha$ *have real coefficients; i.e.,*

$$f(\zeta) = \sum_{j=1}^{\infty} a_j(\zeta - \alpha)^j, \quad a_j \in \mathbb{R} \text{ for all } j \in \mathbb{N}, a_1 \neq 0. \quad (2.6.6)$$

Then if $\gamma \in \mathbb{R}$, $|\gamma|$ small enough, then the unique solution ζ_0 to $f(\zeta) = \gamma$, as guaranteed by Proposition 2.6.3, is real.

Proof. Since $\left.\dfrac{df}{d\zeta}\right|_{\zeta=\alpha} \neq 0$ by α a zero of order 1, and since α is real, and $f(\zeta) = 0$, we may simply use the real Inverse Function Theorem to say that for all $\gamma \in \mathbb{R}$ sufficiently close to 0, there exists f^{-1} mapping a neighborhood of 0 back to a neighborhood of α. Since γ and α are real, and all the coefficients in (2.6.6) are real, the inverse function must give real values. □

CHAPTER 3

CONSTRUCTION OF THE OPERATOR

3.1 The Weber Parabolic Cylinder Equation and its Properties

The Weber parabolic cylinder equation is given under either of the variations

$$-\frac{d^2y}{dx^2} + \left(\frac{1}{4}x^2 - \left[\nu + \frac{1}{2}\right]\right) y(x) = 0, \quad x \in \mathbb{C}, \quad \nu \in \mathbb{C}, \tag{3.1.1a}$$

$$-\frac{d^2y}{dx^2} + \left(\frac{1}{4}x^2 + a\right) y(x) = 0, \quad x \in \mathbb{C}, \quad a \in \mathbb{C}. \tag{3.1.1b}$$

These variations are obviously equivalent under the rule

$$a = -\nu - \frac{1}{2} \tag{3.1.2}$$

and (3.1.2) is assumed throughout the rest of the paper. We will primarily use (3.1.1a), as it will be the choice of coordinates used by the relevant reference [Dem05], and because of this clearer connection of this set of coordinates to the harmonic oscillator, as stated above. We mention (3.1.1b) because of the frequent use of this variation in the literature (e.g., [Olv74, Section 6.6], [Tem14], and [Dea66]).

Lemma 3.1.1 (Symmetries of Weber parabolic cylinder equation)**.** *The Weber parabolic cylinder equation obeys the following symmetries.*

Rotation by $\frac{\pi}{2}$. *If for some $\mu \in \mathbb{C}$, $y_0(x)$ is a solution to the $\nu = \mu$ case of* (3.1.1a)*, then $\mathfrak{y}(x) := y_0(ix)$ is a solution to the $\nu = -\mu - 1$ case of* (3.1.1a)*.*

Rotation by $-\frac{\pi}{2}$. *If for some $\mu \in \mathbb{C}$, $y_0(x)$ is a solution to the $\nu = \mu$ case of* (3.1.1a)*, then we claim that $\mathfrak{y}_-(x) := y_0(-ix)$ is a solution to the $\nu = -\mu - 1$ case of* (3.1.1a)*.*

Reflection. *If for some $\mu \in \mathbb{C}$, $y_0(x)$ is a solution to the $\nu = \mu$ case of* (3.1.1a)*, then the reflection $\widetilde{y_0}(x)$, defined by $\widetilde{y_0}(x) := y(-x)$, is a solution of the $\nu = \mu$ case of* (3.1.1a)*.*

Conjugation. *If for some $\mu \in \mathbb{C}$, $y_0(x) \in \mathrm{Hol}(\mathbb{C})$ is a solution to the $\nu = \mu$ case of* (3.1.1a)*, then $\mathfrak{y}(x) := \overline{y_0(\overline{x})}$ is a solution to* (3.1.1a) *with $\nu = \overline{\mu}$.*

Proof, Rotation by $\frac{\pi}{2}$. Suppose that for some $\mu \in \mathbb{C}$, $y_0(x)$ is a solution to the $\nu = \mu$ case of (3.1.1a). We wish to show that $\mathfrak{y}(x) := y_0(ix)$ is a solution to the $\nu = -\mu - 1$ case of (3.1.1a). For if the above holds, i.e,

$$-\frac{d^n y_0}{dx^n} + \left[x^2 - \left(\mu + \frac{1}{2}\right)\right] y(x) = 0, \tag{3.1.3}$$

then evaluating at $x = ia$, $a \in \mathbb{C}$ gives that

$$\begin{aligned} -\left.\frac{d^2 y_0}{dx^2}\right|_{x=\pm ia} + \left[(ia)^2 - \left(\mu + \frac{1}{2}\right)\right] y(\pm ia) = 0 \\ -\left.\frac{d^2 y_0}{dx^2}\right|_{x=\pm ia} + \left[-a^2 - \left(\mu + \frac{1}{2}\right)\right] \mathfrak{y}(a) = 0. \end{aligned} \tag{3.1.4}$$

Yet by Lemma 2.5.5, part 1

$$\left.\frac{d^2 \mathfrak{y}}{dx^2}\right|_{x=a} = (i)^2 \left.\frac{d^2 y_0}{dx^2}\right|_{x=ia}$$

and since $i^2 = -1$, we may substitute for $-\left.\frac{d^2 y_0}{dx^2}\right|_{x=ia}$ in (3.1.4), and get

$$\left.\frac{d^2\mathfrak{y}}{dx^2}\right|_{x=a} + \left[-a^2 - \left(\mu + \frac{1}{2}\right)\right]\mathfrak{y}(a) = 0.$$

Multiplying by -1, we get

$$-\left.\frac{d^2\mathfrak{y}}{dx^2}\right|_{x=a} + \left[a^2 + \left(\mu + \frac{1}{2}\right)\right]\mathfrak{y}(a) = 0.$$

Finally, we note that

$$\mu + \frac{1}{2} = -\left(-\mu - \frac{1}{2}\right) = -\left(-[-\mu - 1] + \frac{1}{2}\right),$$

so this becomes

$$-\left.\frac{d^2\mathfrak{y}}{dx^2}\right|_{x=a} + \left[a^2 - \left([-\mu - 1] + \frac{1}{2}\right)\right]\mathfrak{y}(a) = 0. \tag{3.1.5}$$

This works for all $a \in \mathbb{C}$, so the the $y = \mathfrak{y}$, $\nu = -\mu - 1$ case of (3.1.1a) is satisfied. $\square$

The proof for rotation by $-\frac{\pi}{2}$ follows analogously, so we proceed to the proof of the reflective symmetry.

Proof of Reflection Symmetry. Fix $\mu \in \mathbb{C}$, such that $y_0(x)$ is a solution to the $\nu = \mu$ case of (3.1.1a). We wish to show that the reflection $\widetilde{y_0}$, defined by $\widetilde{y_0}(x) := y(-x)$, is a solution of the $\nu = \mu$ case of (3.1.1a). This follows because $(-x)^2 = x^2$, and because by Lemma 2.5.5,

$$\left.\frac{d^2\widetilde{y_0}}{dx^2}\right|_{x=a} = (-1)^2 \left.\frac{d^n y}{dx^n}\right|_{x=-a}, \tag{3.1.6}$$

so the second derivative will retain its sign. [This could also have been done be repeating either of the above rotation symmetries, as the map $\mu \mapsto -\mu - 1$ is an involution.] $\square$

Proof of Conjugation Symmetry. Suppose that for some $\mu \in \mathbb{C}$, $y_0(x) \in \mathrm{Hol}(\mathbb{C})$ is a solution to the $\nu = \mu$ case of (3.1.1a). We wish to show that $\mathfrak{y}(x) := \overline{y_0(\overline{x})}$ is a solution to (3.1.1a) with $\nu = \overline{\mu}$. For evaluating at $x = \overline{a}$, $a \in \mathbb{C}$,

$$-\left.\frac{d^2 y_0}{dx^2}\right|_{x=\overline{a}} + \left(\overline{a}^2 - \left[\mu + \frac{1}{2}\right]\right) y(\overline{a}) = 0 \tag{3.1.7}$$

and then conjugating both sides,

$$\begin{aligned} -\overline{\left.\frac{d^2 y_0}{dx^2}\right|_{x=\overline{a}}} + \left(a^2 - \left[\overline{\mu} + \frac{1}{2}\right]\right) \overline{y(\overline{a})} = 0 \\ -\overline{\left.\frac{d^2 y_0}{dx^2}\right|_{x=\overline{a}}} + \left(a^2 - \left[\overline{\mu} + \frac{1}{2}\right]\right) \mathfrak{y}(a) = 0. \end{aligned} \tag{3.1.8}$$

The only point left is to show that

$$\left.\frac{d^2 \mathfrak{y}}{dx^2}\right|_{x=a} = \overline{\left.\frac{d^2 y_0}{dx^2}\right|_{x=\overline{a}}} \tag{3.1.9}$$

This is best shown with power-series expansions. If we have that in the expansion about $x = \overline{a}$,

$$y_0(x) = \sum_{k=0}^{\infty} c_k (x - \overline{a}))^k, \quad c_k \in \mathbb{C} \tag{3.1.10}$$

then we have that

$$\begin{aligned} y_0(\overline{x}) &= \sum_{k=0}^{\infty} c_k(\overline{x}-\overline{a})^k \\ y_0(\overline{x}) &= \sum_{k=0}^{\infty} c_k\overline{(x-a)^k} \\ \overline{y_0(\overline{x})} &= \sum_{k=0}^{\infty} \overline{c_k}(x-a)^k \\ \mathfrak{y}(x) &= \sum_{k=0}^{\infty} \overline{c_k}(x-a)^k. \end{aligned} \tag{3.1.11}$$

Moreover, by the standard theory of functions of one complex variable, we know that if for some $f \in \mathrm{Hol}(\mathbb{C})$, if the power series around some $\alpha \in \mathbb{C}$ is written as

$$f(\xi) = \sum_{j=0}^{\infty} t_j(\xi-\alpha)^j,$$

then

$$\left.\frac{d^j f}{d\xi^j}\right|_{\xi=\alpha} = j!t_j; \tag{3.1.12}$$

see, e.g., [Con78, Prop. III.2.5(c), p. 35] or [Lan85, p. 84].

Thus, we have that by (3.1.10), resp. (3.1.11),

$$\begin{aligned} \left.\frac{d^2 y_0}{dx^2}\right|_{x=\overline{a}} &= 2! \cdot c_2 \\ \left.\frac{d^2 \mathfrak{y}}{dx^2}\right|_{x=a} &= 2! \cdot \overline{c_2}, \end{aligned} \tag{3.1.13}$$

from whence (3.1.9) follows smoothly. Thus, for any $a \in \mathbb{C}$,

$$-\left.\frac{d^2\mathfrak{y}}{dx^2}\right|_{x=a} + \left(a^2 - \left[\overline{\mu} + \frac{1}{2}\right]\right)\mathfrak{y}(a) = 0. \tag{3.1.14}$$

Since this holds for all $a \in \mathbb{C}$, the $y = \mathfrak{y}$, $\nu = \mu$ case of (3.1.1a) is satisfied. □

In either of its forms, (3.1.1a) or (3.1.1b), the Weber parabolic cylinder equation, being a perfectly good second-order differential equation with analytic–over–$\mathbb{C}$ coefficients, the Weber parabolic cylinder equation has two linearly independent holomorphic solutions over $\mathbb{C}$, by Proposition 2.5.1. Moreover, if the initial conditions at a fixed point are holomorphic in the parameter ν, the solutions are holomorphic in both x and ν, by Proposition 2.5.2. There are several ways to choose a reasonable pair of linearly independent solutions, as we discuss below.

A solution with good asymptotics at ∞, and a guaranteed linearly independent complement. It is known from standard theory (e.g., [Olv74, Section 6.6.1] that there exist solutions of (3.1.1a) as $x \to \infty$ with leading asymptotic $x^\nu e^{-x^2/4}$ and $x^{-\nu-1} e^{x^2/4}$, respectively. The solution $D_\nu(x)$ (or $U(a,x)$, under the convention (3.1.2)) is specified by the requirement that $D_\nu(x) \sim x^\nu e^{-x^2/4}$ as $x \to \infty$ (here, $p(x) \sim q(x)$ means that $\lim\limits_{x\to\infty} \dfrac{p(x)}{q(x)} = 1$).

Since the map $\nu \mapsto -\nu - 1$ is an involution on $\mathbb{C}$, we see by the above rotation symmetries that either of $D_{-\nu-1}(\pm ix)$ will form a linearly independent set with $D_\nu(x)$; indeed, it is known (see [Tem14, Section 2.iii, (12.2.12)]) that

$$\mathrm{Wr}\left[D_\nu, D_{-\nu-1}(\pm i\cdot)\right](x) = \mp i e^{\mp i\nu\pi}, \quad x \in \mathbb{C} \tag{3.1.15}$$

which guarantees the linear independence of the sets $\{D_\nu(x), D_{-\nu-1}(ix)\}$ and $\{D_\nu(x), D_{-\nu-1}(-ix)\}$ by Proposition 2.5.4.

A solution with good asymptotics at $-\infty$, and a guaranteed linearly independent complement. It is clear that since $D_\nu(x)$ decays as $x \to \infty$, $D_\nu(-x)$ decays as $x \to -\infty$. To find a linearly independent complement, we may use the $c = -1$ case of Lemma 2.5.5, part 2, and we have that by (3.1.15),

$$\begin{aligned}\operatorname{Wr}\left[D_\nu(-\cdot), D_{-\nu-1}(\pm i\cdot)\right](x) &= -\operatorname{Wr}\left[D_\nu, D_{-\nu-1}(\mp i\cdot)\right](-x) \\ &= \pm i e^{\pm i\nu\pi}, \quad x \in \mathbb{C}.\end{aligned} \tag{3.1.16}$$

By Proposition 2.5.4, the sets $\{D_\nu(-x), D_{-\nu-1}(ix)\}$ and $\{D_\nu(-x), D_{-\nu-1}(-ix)\}$ are also linearly independent.

A solution with good asymptotics at $+\infty$, and a solution with good asymptotics at $-\infty$. We again start with $D_\nu(x)$ as the first solution; however, by the reflection-symmetry noted in Lemma 3.1.1, $D_\nu(-x)$ is also a solution to (3.1.1a), and it is known that

$$\operatorname{Wr}\left[D_\nu, D_\nu(-\cdot)\right](x) = \frac{\sqrt{2\pi}}{\Gamma(-\nu)}, \quad x \in \mathbb{C} \tag{3.1.17}$$

[Tem14, Section 2.iii], where Γ denotes the standard Gamma function. Since the entire function $\frac{1}{\Gamma(\nu)}$ only has zeros at the nonpositive integers, $\frac{1}{\Gamma(-\nu)}$ only has zeros at the nonnnegative integers. Therefore, by Proposition 2.5.4, $D_\nu(x)$ and $D_\nu(-x)$ are linearly independent functions of x if and only if their Wronskian is nonzero, which happens if and only if $\nu \notin \mathbb{N}_0$. Thus, for ν not a nonnegative integer, $D_\nu(x)$ and $D_\nu(-x)$ are linearly independent.

Power-Series Solutions. Given the reflection symmetry of the linear differential equation (3.1.1a), there are presumably even and odd solutions to (3.1.1a). To prove this, and given the asymptotics of solutions at ∞ above, one may try the change-of-variable $y = e^{-x^2/4}\mathfrak{w}$, and equation (3.1.1a) becomes

$$-\frac{d^2\mathfrak{w}}{dx^2} + x\frac{d\mathfrak{w}}{dx} - \nu\mathfrak{w}(x) = 0 \tag{3.1.18}$$

This is quite amenable to a power series solution $\mathfrak{w}(x) = \sum_{k=0}^{\infty} c_k x^k$; formally manipulating, the recurrence relation for the coefficients is

$$c_{k+2} = \frac{-\nu + k}{(k+1)(k+2)}\, c_k, \quad k \geq 0.$$

This recurrence notably skips over c_{k+1}, and so choosing $c_0 = 1$, $c_1 = 0$ gives an even solution to (3.1.18), and $c_0 = 0$, $c_1 = 1$ gives an odd solution to (3.1.18). (The proof that the series actually converge everywhere in $\mathbb{C}$ is left as an tedious exercise.) Converting back to terms of the original y, we have that (3.1.1a) has the solutions

$$y_1(\nu; x) := e^{-x^2/4} \left[1 + (-\nu)\frac{x^2}{2!} + (-\nu)(-\nu+2)\frac{x^4}{4!} + \cdots\right] \tag{3.1.19a}$$

and

$$y_2(\nu; x) := e^{-x^2/4} \left[x + (-\nu+1)\frac{x^3}{3!} + (-\nu+1)(-\nu+3)\frac{x^5}{5!} + \cdots\right]. \tag{3.1.19b}$$

For future reference, we note that from (3.1.19a) and (3.1.19b), one sees that

$$y_1(\nu; 0) = 1, \qquad \frac{\partial}{\partial x}\left(y_1(\nu; x)\right)\bigg|_{x=0} = 0 \tag{3.1.20a}$$

$$y_2(\nu; 0) = 0, \qquad \frac{\partial}{\partial x}\left(y_2(\nu; x)\right)\bigg|_{x=0} = 1 \tag{3.1.20b}$$

Thus, $y_1(\nu; x)$ and $y_2(\nu; x)$ are linearly independent, since it follows that

$$\mathrm{Wr}\left[y_1(\nu; \cdot), y_2(\nu; \cdot)\right](0) = 1 \cdot 1 - 0 \cdot 0 = 1, \tag{3.1.21}$$

and we may invoke Proposition 2.5.4.

The equation (3.1.20) also allows us to determine the relationship of $D_\nu(x)$ to the odd and even solutions; as noted in [Tem14, (12.2.6–7) and Section 4], we see that

$$D_\nu(x) = \frac{2^{\nu/2}\sqrt{\pi}}{\Gamma\left(-\frac{\nu}{2}+\frac{1}{2}\right)} y_1(\nu; x) - \frac{2^{(\nu+1)/2}\sqrt{\pi}}{\Gamma\left(-\frac{\nu}{2}\right)} y_2(\nu; x). \tag{3.1.22}$$

From (3.1.22) we may recover (3.1.17). We also note that for $\nu \notin \mathbb{N} \cup \{0\}$, $D_\nu(x)$ is unbounded as $x \to -\infty$; this essentially follows from formula 12.2.15 of [Tem14].

We note one last property of $y_1(\nu; b)$ and $y_2(\nu; b)$.

Lemma 3.1.2. *Fix $b > 0$. Then for all $\nu \in \mathbb{C}$, $y_1(\nu; b)$ and $y_2(\nu; b)$ are not simultaneously* 0.

Proof. First, we note that if two solutions $u_1(x)$, $u_2(x)$ to a linear, second-order, homogeneous ODE have a common zero, then they have a zero Wronskian at that point, for if $x = a$ is the common zero, then

$$\begin{aligned} \operatorname{Wr}\left[u_1, u_2\right](a) &= u_1(a) \left.\frac{du_2(x)}{dx}\right|_{x=a} - u_2(a) \left.\frac{du_1(x)}{dx}\right|_{x=a} \\ &= 0 \left.\frac{du_2(x)}{dx}\right|_{x=a} - 0 \left.\frac{du_1(x)}{dx}\right|_{x=a} = 0. \end{aligned} \tag{3.1.23}$$

By the contrapositive, $y_1(\nu; b)$ and $y_2(\nu; b)$, being solutions to (3.1.1a), cannot be simultaneously zero, since by (3.1.21) their Wronskian is nonzero at a point, and by Proposition 2.5.4 it is nonzero at all points. □

3.2 Construction of Unperturbed Operator

The construction of $\widetilde{L}^0_{\mathrm{PC}}$ or L^0_{PC} is quite akin to that of L^0_{HO}. $\mathfrak{D}_2$ is a natural space in which both $-y''$ and $x^2 y(x)$ are defined, since $f \in \mathfrak{D}_2$ implies $x^2 f(x) \in L^2(\mathbb{R})$

and $f \in \mathcal{H}^2(\mathbb{R})$, so $f''(x) \in L^2(\mathbb{R})$. Certainly, on $\mathfrak{D}_2$ one may define, as in the introduction,

$$\mathfrak{D}\left(\widetilde{L}^0_{\text{PC}}\right) = \mathfrak{D}_2 \tag{3.2.1a}$$

$$\widetilde{L}^0_{\text{PC}} y := -y''(x) + \frac{1}{4}x^2 y(x), \quad y \in \mathfrak{D}\left(\widetilde{L}^0_{\text{PC}}\right), \tag{3.2.1b}$$

and

$$\mathfrak{D}\left(L^0_{\text{PC}}\right) = \mathfrak{D}_2 \tag{3.2.2a}$$

$$L^0_{\text{PC}} y := -y''(x) + \left(\frac{1}{4}x^2 - \frac{1}{2}\right) y(x), \quad y \in \mathfrak{D}\left(\widetilde{L}^0_{\text{PC}}\right), \tag{3.2.2b}$$

and of course $L^0_{\text{PC}} = \widetilde{L}^0_{\text{PC}} - \frac{1}{2}I$, and these operators are densely defined.

To check the other properties of these operators – closure, self-adjointness, etc. – we use the fact to be proven later (see (4.2.11))

$$\widetilde{L}^0_{\text{PC}} = \frac{1}{2}T^{-1}L^0_{\text{HO}}T \tag{3.2.3a}$$

$$L^0_{\text{PC}} = \frac{1}{2}T^{-1}L^0_{\text{HO}}T - \frac{1}{2}I. \tag{3.2.3b}$$

By Lemma 2.3.1, $T^{-1}L^0_{\text{HO}}T$ is closed, and by Lemma 2.3.2, $T^{-1}L^0_{\text{HO}}T$ is self-adjoint, has compact resolvent, and the same spectrum as the unperturbed operator. Since multiplication by $\frac{1}{2}$ and subtraction of a real multiple of the identity has no effect on the closure, self-adjointess, etc., and has a clear effect on the spectrum, we have the following.

Proposition 3.2.1. *$\widetilde{L}^0_{PC}$ and L^0_{PC} are closed, self-adjoint, and have compact resolvent. In addition*

$$\operatorname{Sp} \widetilde{L}^0_{PC} = \left\{ n + \frac{1}{2} \right\}_{n=0}^{\infty}, \tag{3.2.4a}$$

$$\operatorname{Sp}(L^0_{PC}) = \{n\}_{n=0}^{\infty}. \tag{3.2.4b}$$

A normalized eigenfunction for the eigenvalue $n + \frac{1}{2}$ of $\widetilde{L}^0_{PC}$, or n of L^0_{PC}, $n \in \mathbb{N}_0$, is $\frac{1}{\sqrt[4]{2}} h_n\left(\frac{x}{\sqrt{2}}\right)$.

Therefore, appropriately stretched Hermite functions are indeed solutions to the equation (3.1.1a), and eigenfunctions of $\widetilde{L}^0_{\mathrm{PC}}$. Indeed, we have that for $n \in \mathbb{N}_0$,

$$\begin{aligned} D_n(x) &= 2^{-n/2} e^{-x^2/4} H_n\left(\frac{x}{\sqrt{2}}\right) \\ &= \sqrt{n!\sqrt{\pi}} \cdot h_n\left(\frac{x}{\sqrt{2}}\right), \quad n \in \mathbb{N}_0, \end{aligned} \tag{3.2.5}$$

(see, e.g., [Tem14, Section 7.i]), and so $D_n(x)$ is a (non-normalized) eigenfunction of L^0_{PC} with eigenvalue n, $n \in \mathbb{N}_0$. Hence, the exceptional behavior of the nonnegative integers in the Wronskian (3.1.17) no longer surprises us; as the Hermite polynomials are odd or even depending on the parity of n, $H_n(-x) = (-1)^n H_n(x)$, the reflection operators give back a multiple of the original function. We restate this parity condition very explicitly for use later:

$$D_n(-x) = (-1)^n D_n(x), \quad n \in \mathbb{N}_0, \quad x \in \mathbb{C} \tag{3.2.6}$$

3.3 Construction of Perturbed Operator, and Characterization of Eigenfunctions

Our operator is constructed as follows.

Proposition 3.3.1. *Fix $b > 0$ and $z \in \mathbb{C}$. There exists a closed, densely defined operator $L_{PC}(z,b)$ with compact resolvent, where*

$$\mathfrak{D}\left(L_{PC}(z,b)\right) := \mathfrak{D}_1 \tag{3.3.1a}$$

$$\begin{aligned} L_{PC}(z,b)y(x) := -y''(x) + \left[\frac{x^2}{4} - \frac{1}{2}\right] y(x) \\ + z[u(b)\delta\left(x-b\right) - u(-b)\delta\left(x-b\right)] \end{aligned} . \tag{3.3.1b}$$

The eigenvalues of $L_{PC}(z,b)$ are contained in some shifted sector opening to the right, of aperture less than $\frac{\pi}{2}$: i.e., for some τ in $\mathbb{R}$ and $\delta \in (0, \frac{\pi}{4})$,

$$\operatorname{Sp}\left(L_{PC}(z,b)\right) \subset \{\mu \in \mathbb{C} : |\arg(\mu - \tau)| \leq \delta\} \tag{3.3.2}$$

If z is real, however, $L_{PC}(z,b)$ is self-adjoint and semibounded below.

We note that the details of the construction do not affect the later work, save the specification of the domain in (3.3.1a). Nor is the proof particularly original; such operators have been constructed, for example, in [MS13]. Hence, we defer the construction to Appendix 3.

More importantly for us is the following characterization of the eigenvalues of $L_{\mathrm{PC}}(z,b)$:

Proposition 3.3.2 (Folklore)**.** *Fix $b > 0$ and $z \in \mathbb{C}$. Then $y \in \mathfrak{D}\left(L_{PC}(z,b)\right)$ is* *an eigenfunction of $L_{PC}(z, b)$ with eigenvalue $\nu \in \mathbb{C}$ if and only if it is continuous, satisfies the differential equation*

$$-\left.\frac{d^2y}{dt^2}\right|_{t=x} + \left[\frac{x^2}{4} - \frac{1}{2}\right] y(x) = \nu y(x) \tag{3.3.3}$$

on the intervals $(-\infty, -b)$, $(-b, b)$, and (b, ∞) [hence is C^∞ on these intervals], and satisfies the jump conditions

$$y'(-b+) - y'(-b-) = -zy(-b), \tag{3.3.4a}$$

$$y'(b+) - y'(b-) = zy(b), \tag{3.3.4b}$$

where for $p \in \mathbb{R}$, $y'(p+)$ (resp. $y'(p-)$) denotes $\lim_{x \to p^+} y'(x)$ (respectively, $\lim_{x \to p^-} y'(x)$).

These conditions for point-perturbations of the operators associated to differential equations are not particularly surprising; for real weights on perturbations of the Laplacian, they appear in [Alb+05, Chapter I.3, Thm. 3.1.1, pp. 76, and Chapter II.2, p. 142], and of course [Dem05] uses this to get his criterion. Again, we provide a proof, but again defer to a later appendix, specifically Appendix B.

CHAPTER 4

COMPARISON OF PARABOLIC CYLINDER AND HARMONIC OSCILLATOR OPERATORS

We now wish to prove Proposition 1, which compares the operators

$$L_{\mathrm{HO}}(\zeta,\beta)w(x) = -w''(x) + x^2 w(x) + \zeta w(\beta)\delta\left(x-\beta\right) - \zeta w(-\beta)\delta\left(x+\beta\right), \quad (4.0.1a)$$

$$\widetilde{L}_{\mathrm{PC}}(z,b)y(x) = -y''(x) + \frac{x^2}{4}y(x) + zy(b)\delta\left(x-b\right) - zy(-b)\delta\left(x+b\right), \quad (4.0.1b)$$

$$L_{\mathrm{PC}}(z,b)y(x) = -y''(x) + \left(\frac{x^2}{4} - \frac{1}{2}\right)y(x) + zy(b)\delta\left(x-b\right) - zy(-b)\delta\left(x+b\right). \quad (4.0.1c)$$

Of course,

$$L_{\mathrm{PC}}(z,b) = \widetilde{L}_{\mathrm{PC}}(z,b) - \frac{1}{2}I,$$

so the only issue is how to compare $L_{\mathrm{HO}}(\zeta,\beta)$ and $\widetilde{L}_{\mathrm{PC}}(z,b)$.

To do so, we first discuss the comparison in the case of the differential equations, then bootstrap to the comparison between the unperturbed operators L^0_{HO} and L^0_{PC}, then finally come back to $L_{\mathrm{HO}}(\zeta,\beta)$ and $L_{\mathrm{PC}}(z,b)$.

4.1 Comparison of Differential Equations

We now discuss the relationship between the differential equations

$$-\frac{d^2y}{dx^2} + \frac{1}{4}x^2y(x) = \left(\nu + \frac{1}{2}\right)y(x), \qquad x \in (a, c) \tag{4.1.1a}$$

and

$$-\frac{d^2w}{dx^2} + x^2w(x) = \lambda w(x), \qquad x \in (\alpha, \gamma); \tag{4.1.1b}$$

where $-\infty \le a < c \le \infty$ and $-\infty \le \alpha < \gamma \le \infty$. Of course, by Proposition 2.5.1, there are C^∞ solutions on the appropriate intervals.

Proposition 4.1.1. *The solutions of* (4.1.1a) *(equivalently,* (3.1.1a)*) and* (4.1.1b) *correspond via the correspondence*

$$\begin{aligned} w(x) &= y\left(x\sqrt{2}\right) \\ \lambda &= 2\nu + 1 \\ \alpha &= \frac{a}{\sqrt{2}} \\ \gamma &= \frac{c}{\sqrt{2}} \end{aligned} \tag{4.1.2}$$

In other words, if $y(x)$ *is a* $C^\infty(a, c)$ *solution to* (4.1.1a), $k \ge 2$*, then* $w(x) = y\left(x\sqrt{2}\right)$ *is a* $C^\infty(\alpha, \gamma)$ *solution to* (4.1.1b) *with* $\lambda = 2\nu + 1$, $\alpha = \frac{a}{\sqrt{2}}$, $\gamma = \frac{c}{sqrt2}$*. Similarly, if* $w(x)$ *is a* $C^\infty(\alpha, \gamma)$ *solution to* (4.1.1a)*, then* $y(x) = w\left(\frac{x}{\sqrt{2}}\right)$ *is a* $C^\infty(\mathbb{R})$ *solution to* (4.1.1a) *with* $\nu = \frac{\lambda - 1}{2}$, $a = \alpha\sqrt{2}$, $c = \gamma\sqrt{2}$*. Equivalent statements hold with* $C^\infty(a, c)$ *replaced by* $\mathrm{Hol}(\mathbb{C})$.

Proof. We demonstrate the path from solutions of (4.1.1a) to solutions of (4.1.1b); the other direction is similar.

We evaluate (4.1.1a) at some $p \in (a, c)$, but we let $p = q \cdot \sqrt{2}$, so $q \in \left(\frac{\alpha}{\sqrt{2}}, \frac{\gamma}{\sqrt{2}}\right)$; we get

$$-\left.\frac{d^2y}{dx^2}\right|_{x=q\sqrt{2}} + \frac{1}{2}q^2 y(q\sqrt{2}) = \left(\nu + \frac{1}{2}\right) y(q\sqrt{2}). \tag{4.1.3}$$

Now, let $w(x) := y(x\sqrt{2})$, for $x \in \left(\frac{\alpha}{\sqrt{2}}, \frac{\gamma}{\sqrt{2}}\right)$; then by Lemma 2.5.5, part 1, we have that

$$\left.\frac{d^2w}{dx^2}\right|_{x=q} = (\sqrt{2})^2 \left.\frac{d^2y}{dx^2}\right|_{x=q\sqrt{2}} \tag{4.1.4}$$

or, turning it around,

$$\left.\frac{d^2y}{dx^2}\right|_{x=q\sqrt{2}} = \frac{1}{2} \left.\frac{d^2w}{dx^2}\right|_{x=q}.$$

Thus, substituting into (4.1.3),

$$-\frac{1}{2} \left.\frac{d^2w}{dx^2}\right|_{x=q} + \frac{1}{2}q^2 w(q) = \left(\nu + \frac{1}{2}\right) w(q), \tag{4.1.5}$$

$$-\left.\frac{d^2w}{dx^2}\right|_{x=q} + q^2 y(q) = (2\nu + 1)w(q). \tag{4.1.6}$$

Since this works for all $q \in (\alpha, \gamma) := \left(\frac{\alpha}{\sqrt{2}}, \frac{\gamma}{\sqrt{2}}\right)$, we have that

$$-\frac{d^2w}{dx^2} + x^2 w(x) = [2\nu + 1]w(x), \quad x \in (\alpha, \gamma),$$

so $w(x)$ is a solution to (4.1.1b) on (α, γ) with $\lambda = 2\nu + 1$.

Of course, in $\mathrm{Hol}(\mathbb{C})$, the analogous proof works as well. □

We wish to go over the proof again, in the case that $y, w \in L^2(\mathbb{R}) \cap C^\infty(\mathbb{R})$, $\alpha = a = -\infty$, and $\gamma = c = \infty$. We do so to use the formalism S, T used in the statement of Proposition 1. As a reminder, $Sx = x\sqrt{2}$ is a linear operator on $\mathbb{R}$, and

$Tf(x) = f(Sx) = f(x\sqrt{2})$ defines an operator on $L^2(\mathbb{R})$, clearly bounded with norm $\frac{1}{\sqrt[4]{2}}$: for all $f \in L^2(\mathbb{R})$,

$$\int_{\mathbb{R}} |Tf(x)|^2 \, dx = \int_{\mathbb{R}} |f(x\sqrt{2})|^2 \, dx, \tag{4.1.7}$$

and making the substitution $t = x\sqrt{2}$, $dt = \sqrt{2}dx$, we get

$$\frac{1}{\sqrt{2}} \int_{\mathbb{R}} |f(t)|^2 \, dt, \tag{4.1.8}$$

so $\|Tf\|^2_{L^2(\mathbb{R})} = \frac{1}{\sqrt{2}} \|f\|^2_{L^2(\mathbb{R})}$. Also, of course, $T^{-1}f(x) = f\left(\frac{x}{\sqrt{2}}\right)$ is also a bounded operator on $L^2(\mathbb{R})$.

Now, suppose $y \in L^2(\mathbb{R}) \cap C^\infty(\mathbb{R})$ is a solution to (4.1.1a). Then letting $w(x) = [Ty](x)$, (4.1.3) is essentially applying T to the entire equation, so we have

$$\begin{aligned} -T(y''(x)) + T\left(\frac{x^2}{4} y(x)\right) &= T\left[\left(\nu + \frac{1}{2}\right) y(x)\right] \\ -T(y''(x)) + \frac{x^2}{2}[Ty](x) &= \left(\nu + \frac{1}{2}\right)[Ty](x). \end{aligned} \tag{4.1.9}$$

By the Chain Rule, however, we have that

$$\begin{aligned} \frac{d^2}{dx^2}\left[y(x\sqrt{2})\right] &= (\sqrt{2})^2 y''(x\sqrt{2}) \\ \frac{d^2}{dx^2}[Ty](x) &= 2Ty''(x), \end{aligned} \tag{4.1.10}$$

so we have that

$$-\frac{1}{2}\frac{d^2[Ty](x)}{dx^2} + \frac{x^2}{2}[Ty](x) = \left(\nu + \frac{1}{2}\right)[Ty](x). \tag{4.1.11}$$

Multiplying both sides by 2,

$$-\frac{d^2[Ty](x)}{dx^2} + x^2[Ty](x) = (2\nu+1)[Ty](x), \tag{4.1.12}$$

so defining $w(x) = Ty(x)$ and $\lambda = 2\nu + 1$, we are done. This gives us an idea of how to extend to the operator case.

4.2 Comparison of Unperturbed Operators

We now wish to compare the unperturbed operators

$$\mathfrak{D}\left(L^0_{\mathrm{HO}}\right) = \mathfrak{D}_2 \tag{4.2.1a}$$

$$L^0_{\mathrm{HO}} w(x) := -w''(x) + x^2 w(x), \quad w \in \mathfrak{D}\left(L^0_{\mathrm{HO}}\right), \tag{4.2.1b}$$

and

$$\mathfrak{D}\left(\widetilde{L}^0_{\mathrm{PC}}\right) = \mathfrak{D}_2 \tag{4.2.2a}$$

$$\widetilde{L}^0_{\mathrm{PC}} y := -y''(x) + \frac{1}{4}x^2 y(x), \quad y \in \mathfrak{D}\left(\widetilde{L}^0_{\mathrm{PC}}\right), \tag{4.2.2b}$$

An observation makes the comparison much easier. Namely, we note that T extends naturally to a linear operator on $\mathcal{S}'$, with (see [Fol99, p.285])

$$\langle TF, \varphi\rangle_{\mathcal{S}',\mathcal{S}} = \frac{1}{\sqrt{2}}\left\langle F, T^{-1}\varphi\right\rangle_{\mathcal{S}',\mathcal{S}} \tag{4.2.3}$$

Standard calculations with distributions give the following rules to move differentiation and multiplication-by-x past T, confirming that intuition holds in these cases.

Lemma 4.2.1 ([Fol99, p. 284–285, 295])**.** *For all $F \in \mathcal{S}'$, $T[xF] = \sqrt{2}xTF$, $(TF)' =$*

$\sqrt{2}T(F')$, $\mathcal{F}[TF] = \dfrac{1}{\sqrt{2}}T^{-1}\widehat{\varphi}$. *In particular, for all* $u \in L^2(\mathbb{R})$, $Txu(x) = \sqrt{2}xTu(x)$, $(Tu)'(x) = \sqrt{2}T[u'](x)$, *and* $\mathcal{F}[u(\sqrt{2}\cdot)](\xi) = \dfrac{1}{\sqrt{2}}\widehat{u}\left(\dfrac{\xi}{\sqrt{2}}\right)$.

Corollary 4.2.2. *For all* $N \in \mathbb{N}_0$, T *restricts to a bijection on* $\mathfrak{D}_N$.

Proof. The case $N = 0$ merely restates that T is a bijection on $L^2(\mathbb{R})$, so assume $N \in \mathbb{N}_0$. We now show that if for some $j \in \mathbb{N}$, $x^j f(x) \in L^2(\mathbb{R})$, then $x^j(Tf)(x) \in L^2(\mathbb{R})$. Yet this is simple, since

$$\begin{aligned} x^j Tf(x) &= x^j f(x\sqrt{2}) \\ &= \frac{1}{2^{j/2}}\left(x\sqrt{2}\right)^j f(x\sqrt{2}) \\ &= \frac{1}{2^{j/2}}Tg(x), \quad g(x) := x^j f(x) \end{aligned} \tag{4.2.4}$$

and since $g(x) \in L^2(\mathbb{R})$ by hypothesis, so is $x^j Tf(x)$. Hence, T maps $\{u \in L^2(\mathbb{R}) : x^j u \in L^2(\mathbb{R})\}$ inside itself; since similar logic works for T^{-1}, T is a linear, continuous bijection on $\{u \in L^2(\mathbb{R}) : x^j u \in L^2(\mathbb{R})\}$. In particular, for any $N \in \mathbb{N}$, T is a linear, continuous bijection on

$$\{u \in L^2(\mathbb{R}) : x^j u \in L^2(\mathbb{R}) \text{ for all } j \in \{0, 1, \dots, N\}\}. \tag{4.2.5}$$

Moreover, for all $u \in L^2(\mathbb{R})$, we mentioned that $\mathcal{F}[Tu] = \dfrac{1}{\sqrt{2}}T^{-1}\widehat{u}$, so to show that if $\xi^j \widehat{f}(\xi) \in L^2(\mathbb{R})$, $\xi^j \mathcal{F}[Tf] \in L^2(\mathbb{R})$, we repeat with T^{-1} instead of T:

$$\begin{aligned} \xi^j \mathcal{F}[Tf](x) &= \xi^j \frac{1}{\sqrt{2}}\widehat{f}\left(\frac{\xi}{\sqrt{2}}\right) \\ &= 2^{(j-1)/2}\left(\frac{\xi}{\sqrt{2}}\right)^j \widehat{f}\left(\frac{\xi}{\sqrt{2}}\right) \\ &= 2^{(j-1)/2}T^{-1}\left(\xi^j \widehat{f}(\xi)\right) \end{aligned} \tag{4.2.6}$$

and since T, hence T^{-1}, preserves L^2, we see that $\xi^j \mathcal{F}[Tf](\xi) \in L^2(\mathbb{R})$ if and only if $\xi^j \widehat{f}(\xi) \in L^2(\mathbb{R})$. In particular, for any $N \in \mathbb{N}$, T is a linear, continuous bijection on

$$\{u \in L^2(\mathbb{R}) : \xi^j \widehat{u}(\xi) \in L^2(\mathbb{R}) \text{ for all } j \in \{0, 1, \ldots, N\}\}. \tag{4.2.7}$$

Since $\mathfrak{D}_N$ is the intersection of the domains in (4.2.5) and (4.2.7), we are done. □

Proposition 4.2.3. *As closed operators with domain* $\mathfrak{D}_2$*,*

$$L^0_{HO} \circ T = 2T \circ \widetilde{L}^0_{PC} \tag{4.2.8}$$

Proof. First, a comment on the domains. Certainly $\mathfrak{D}\left(2T \circ \widetilde{L}^0_{\mathrm{PC}}\right) = \mathfrak{D}\left(\widetilde{L}^0_{\mathrm{PC}}\right) = \mathfrak{D}_2$, since $2T$ is a bounded operator on all of $L^2(\mathbb{R})$. $\mathfrak{D}\left(L^0_{\mathrm{HO}} \circ T\right)$ is automatically $T^{-1}\mathfrak{D}\left(L^0_{\mathrm{HO}}\right)$, since $u \in \mathfrak{D}\left(L^0_{\mathrm{HO}} \circ T\right)$ if and only if $Tu \in \mathfrak{D}\left(L^0_{\mathrm{HO}}\right)$, but by Corollary 4.2.2, $Tu \in \mathfrak{D}\left(L^0_{\mathrm{HO}}\right) = \mathfrak{D}_2$ if and only if $u \in \mathfrak{D}\left(L^0_{\mathrm{HO}}\right)$, so there is no issue letting the domain be $\mathfrak{D}_2$. So $\mathfrak{D}\left(L^0_{\mathrm{HO}} \circ T\right) = \mathfrak{D}\left(2T \circ \widetilde{L}^0_{\mathrm{PC}}\right) = \mathfrak{D}_2$.

To verify the equality, we start with the left-hand-side: for all $y \in \mathfrak{D}_2$,
$Ty \in \mathfrak{D}_2$ by Corollary 4.2.2, so we may write

$$L^0_{\mathrm{HO}} \circ Ty(x) = -[Ty]''(x) + x^2[Ty](x). \tag{4.2.9}$$

Yet by repeated use of Lemma 4.2.1, $-[Ty]''(x) = -2T[y''](x)$, and by (4.2.4),

$x^2[Ty](x) = \dfrac{1}{2}T[x^2y(x)]$. Therefore, we have

$$\begin{aligned} L^0_{\mathrm{HO}} \circ Ty(x) &= -[Ty]''(x) + x^2[Ty](x) \\ &= -2T[y''(x)] + \frac{1}{2}T[x^2y(x)] \\ &= 2T\left[-y''(x) + \frac{x^2}{4}y(x)\right] \\ &= 2T \circ \widetilde{L}^0_{\mathrm{PC}}y(x). \end{aligned} \tag{4.2.10}$$

This holds for all $y \in \mathfrak{D}_2$, so we have the equality on $\mathfrak{D}_2$.

As for closure, suppose that $y_n(x) \in \mathfrak{D}_2$, $y_n(x) \xrightarrow{L^2(\mathbb{R})} y(x)$ and $L^0_{\mathrm{HO}} \circ T[y_n](x) \xrightarrow{L^2(\mathbb{R})} w(x)$. Then since T is bounded, surely $u_n(x) := T[y_n](x) \xrightarrow{L^2(\mathbb{R})} u(x) := T[y](x)$, and so $L^0_{\mathrm{HO}}[u_n](x) \xrightarrow{L^2(\mathbb{R})} w(x)$. L^0_{HO}, being self-adjoint, is closed, so $u \in \mathfrak{D}\left(L^0_{\mathrm{HO}}\right) = \mathfrak{D}_2$ and $L^0_{\mathrm{HO}}u_n(x) \xrightarrow{L^2(\mathbb{R})} u(x)$, or $L^0_{\mathrm{HO}} \circ Ty_n(x) \xrightarrow{L^2(\mathbb{R})} L^0_{\mathrm{HO}} \circ Ty(x)$. Hence, $L^0_{\mathrm{HO}} \circ T$ is closed. □

We note the following consequence. Post-composing both sides of (4.2.8) by $\dfrac{1}{2}T^{-1}$, we have that

$$\widetilde{L}^0_{\mathrm{PC}} = \frac{1}{2}T^{-1}L^0_{\mathrm{HO}}T, \tag{4.2.11}$$

or, noting that $L^0_{\mathrm{PC}} = \widetilde{L}^0_{\mathrm{PC}} - \dfrac{1}{2}I$,

$$L^0_{\mathrm{PC}} = \frac{1}{2}T^{-1}L^0_{\mathrm{HO}}T - \frac{1}{2}I. \tag{4.2.12}$$

4.3 Comparison of Perturbed Operators

We now discuss the comparison between the operators

$$L_{\mathrm{HO}}(\zeta,\beta)w(x) = -w''(x) + x^2w(x) + \zeta w(\beta)\delta\left(x-\beta\right) - \zeta w(-\beta)\delta\left(x+\beta\right), \quad (4.3.1a)$$

$$\widetilde{L}_{\mathrm{PC}}(z,b)y(x) = -y''(x) + \frac{x^2}{4}y(x) + zy(b)\delta\left(x-b\right) - zy(-b)\delta\left(x+b\right), \quad (4.3.1b)$$

$$L_{\mathrm{PC}}(z,b)y(x) = -y''(x) + \left(\frac{x^2}{4} - \frac{1}{2}\right)y(x) + zy(b)\delta\left(x-b\right) - zy(-b)\delta\left(x+b\right). \quad (4.3.1c)$$

Again,

$$L_{\mathrm{PC}}(z,b) = \widetilde{L}_{\mathrm{PC}}(z,b) - \frac{1}{2}I,$$

so the only issue is how to compare $L_{\mathrm{HO}}(\zeta,\beta)$ and $\widetilde{L}_{\mathrm{PC}}(z,b)$.

Of course, by Proposition 4.1.1, a solution to (4.1.1a) on $(-b,b)$ translates to a solution to (4.1.1b) on $\left(-\frac{b}{\sqrt{2}},\frac{b}{\sqrt{2}}\right)$. Therefore, the correspondence $\beta = \frac{b}{\sqrt{2}}$ suggests itself. What is less obvious is what the correspondence between ζ and z should be. To build intuition, we have several reasonable avenues, but the easiest is probably to start with the criterion of Proposition 3.3.2. If $y(x)$ is an eigenfunction for $\widetilde{L}_{\mathrm{PC}}(z,b)$, i.e. a solution to

$$-y''(x) + \frac{x^2}{4}y(x) + zy(b)\delta\left(x-b\right) - zy(-b)\delta\left(x+b\right) = \nu y(b),$$

then the jump condition at $x = b$, (3.3.4), is that

$$y'(b+) - y'(b-) = zy(b). \quad (4.3.2)$$

Letting $w(x) = y(\sqrt{2}x)$, by (2.5.5) we know that

$$w'(x) = \sqrt{2}y'(\sqrt{2}x),$$

and evaluating at $x = \dfrac{b}{\sqrt{2}}$,

$$\begin{aligned} w'\left(\frac{b}{\sqrt{2}}\right) &= \sqrt{2}y(b) \\ \frac{w'\left(\frac{b}{\sqrt{2}}\right)}{\sqrt{2}} &= \sqrt{2}y(b) \end{aligned} \tag{4.3.3}$$

Since the same logic will apply to the one-sided limits, we apply the above to (3.3.4b) and get that

$$\begin{aligned} \frac{w'\left(\frac{b}{\sqrt{2}}+\right)}{\sqrt{2}} - \frac{w'\left(\frac{b}{\sqrt{2}}-\right)}{\sqrt{2}} &= zw\left(\frac{b}{\sqrt{2}}\right) \\ w'\left(\frac{b}{\sqrt{2}}+\right) - w'\left(\frac{b}{\sqrt{2}}-\right) &= \sqrt{2}zw\left(\frac{b}{\sqrt{2}}\right) \end{aligned} \tag{4.3.4}$$

Since the transformation from y to w is precisely what we used in Proposition 4.1.1 to convert solutions of (4.1.1a) to (4.1.1b), it suggests the correspondence $\zeta = \sqrt{2}z$ in addition to $\beta = \dfrac{b}{\sqrt{2}}$. This is indeed the case; to prove it, however, we note a less obvious property of the map T and distributions.

Lemma 4.3.1. *For all* $p \in \mathbb{R}$, $T\delta\left(x - p\right) = \dfrac{1}{\sqrt{2}}\delta\left(x - \dfrac{p}{\sqrt{2}}\right)$.

Proof. It follows from the standard computation: for any $\varphi(x) \in \mathcal{S}$,

$$\begin{aligned}\langle T\delta(x-p), \varphi(x)\rangle_{\mathcal{S}',\mathcal{S}} &= \frac{1}{\sqrt{2}}\langle \delta(x-p), T^{-1}\varphi(x)\rangle_{\mathcal{S}',\mathcal{S}} \\ &= \frac{1}{\sqrt{2}}\left\langle \delta(x-p), \varphi\left(\frac{x}{\sqrt{2}}\right)\right\rangle_{\mathcal{S}',\mathcal{S}} \\ &= \frac{1}{\sqrt{2}}\varphi\left(\frac{p}{\sqrt{2}}\right) \\ &= \frac{1}{\sqrt{2}}\left\langle \delta\left(x-\frac{p}{\sqrt{2}}\right), \varphi(x)\right\rangle_{\mathcal{S}',\mathcal{S}}\end{aligned} \tag{4.3.5}$$

□

Proposition 4.3.2. *Fix $b > 0$ and $z \in \mathbb{C}$. Then $L_{HO}\left(z\sqrt{2}, \frac{b}{\sqrt{2}}\right) \circ T$ and $T \circ \widetilde{L}_{PC}(z, b)$ have the same domain, and*

$$L_{HO}\left(z\sqrt{2}, \frac{b}{\sqrt{2}}\right) \circ T = 2T \circ \widetilde{L}_{PC}(z, b). \tag{4.3.6}$$

Proof. Take any $w \in \mathcal{H}^1(\mathbb{R})$. Then we discuss the distribution that would formally be $L_{\mathrm{HO}}\left(z\sqrt{2}, \frac{b}{\sqrt{2}}\right) \circ T(w(x))$, namely,

$$\begin{aligned}&-(Tw)''(x) + x^2 Tw(x) \\ &\quad + z\sqrt{2}[Tw]\left(\frac{b}{\sqrt{2}}\right)\delta\left(x-\frac{b}{\sqrt{2}}\right) \\ &\quad - z\sqrt{2}[Tw]\left(-\frac{b}{\sqrt{2}}\right)\delta\left(x+\frac{b}{\sqrt{2}}\right)\end{aligned} \tag{4.3.7}$$

As expressed in the proof of Proposition 4.2.3, $[Tw]''(x) = 2T[w''(x)]$ and $x^2[Tw](x) =$

$2T\left[\frac{x^2}{4}w(x)\right]$, so these terms convert as before. As for the weight at $\frac{b}{\sqrt{2}}$ we see that

$$\begin{aligned} &z\sqrt{2}[Tw]\left(\frac{b}{\sqrt{2}}\right)\delta\left(x-\frac{b}{\sqrt{2}}\right) \\ &= z\sqrt{2}w(b)\delta\left(x-\frac{b}{\sqrt{2}}\right) \\ &= 2zw(b)\frac{1}{\sqrt{2}}\delta\left(x-\frac{b}{\sqrt{2}}\right) \end{aligned} \tag{4.3.8}$$

and by Lemma 4.3.1 this is simply

$$2zw(b)T\delta\left(x-b\right) = 2T[zw(b)\delta\left(x-b\right)]. \tag{4.3.9}$$

Similarly, we have that

$$-z\sqrt{2}[Tw]\left(-\frac{b}{\sqrt{2}}\right)\delta\left(x+\frac{b}{\sqrt{2}}\right) = 2T[-zw(-b)\delta\left(x+b\right) \tag{4.3.10}$$

Altogether, then, we have that for all $w \in \mathcal{H}^1(\mathbb{R})$, as elements of $\mathcal{S}'$,

$$\left\{\begin{array}{c} -(Tw)''(x) + x^2 Tw(x) \\ +z\sqrt{2}[Tw]\left(\frac{b}{\sqrt{2}}\right)\delta\left(x-\frac{b}{\sqrt{2}}\right) \\ -z\sqrt{2}[Tw]\left(-\frac{b}{\sqrt{2}}\right)\delta\left(x+\frac{b}{\sqrt{2}}\right) \end{array}\right\} = 2T\left\{\begin{array}{c} -w''(x) + \frac{x^2}{4}w(x) \\ +zw(b)\delta\left(x-b\right) \\ -zw(-b)\delta\left(x+b\right) \end{array}\right\} \tag{4.3.11}$$

Now, we know by Proposition 3.3.1 that $\mathfrak{D}\left(\widetilde{L}_{\mathrm{PC}}(z,b)\right) \subseteq \mathfrak{D}_1 \subseteq \mathcal{H}^1(\mathbb{R})$, and by T bounded invertible,

$$\mathfrak{D}\left(2T \circ \widetilde{L}_{\mathrm{PC}}(z,b)\right) = \mathfrak{D}\left(\widetilde{L}_{\mathrm{PC}}(z,b)\right) \subseteq \mathfrak{D}_1 \subseteq \mathcal{H}^1(\mathbb{R}). \tag{4.3.12}$$

Similarly, we know that $\mathfrak{D}\left(L_{\text{HO}}\left(z\sqrt{2}, \dfrac{b}{\sqrt{2}}\right)\right) \subseteq \mathfrak{D}_1 \subseteq \mathcal{H}^1(\mathbb{R})$; since T preserves $\mathfrak{D}_1$ by Lemma 2.4.4, it follows that

$$\mathfrak{D}\left(L_{\text{HO}}\left(z\sqrt{2}, \frac{b}{\sqrt{2}}\right) \circ T\right) \subseteq \mathfrak{D}_1 \subseteq \mathcal{H}^1(\mathbb{R}). \tag{4.3.13}$$

Therefore, $\mathfrak{D}\left(L_{\text{HO}}\left(z\sqrt{2}, \dfrac{b}{\sqrt{2}}\right) \circ T\right)$ is the set of $w \in \mathcal{H}^1(\mathbb{R})$ for which the left-hand side of (4.3.11) is in $L^2(\mathbb{R})$, and hence the right-hand side of (4.3.11) is in $L^2(\mathbb{R})$ as well. Hence, $L_{\text{HO}}\left(z\sqrt{2}, \dfrac{b}{\sqrt{2}}\right) \circ T \subseteq 2T \circ \widetilde{L}_{\text{PC}}(z, b)$. Similarly, the reverse inclusion holds. Thus, the operators are equal. □

Post-composing $\dfrac{1}{2}T^{-1}$ to both sides of (4.3.6), we have that

$$\widetilde{L}_{\text{PC}}(z, b) = \frac{1}{2}T^{-1} \circ L_{\text{HO}}\left(z\sqrt{2}, \frac{b}{\sqrt{2}}\right) \circ T, \tag{4.3.14}$$

and recalling that $L_{\text{PC}}(z, b) = \widetilde{L}_{\text{PC}}(z, b) - \dfrac{1}{2}I$, subtracting $\dfrac{1}{2}I$ from both sides,

$$L_{\text{PC}}(z, b) = \frac{1}{2}T^{-1} \circ L_{\text{HO}}\left(z\sqrt{2}, \frac{b}{\sqrt{2}}\right) \circ T - \frac{1}{2}I, \tag{4.3.15}$$

i.e., (1.0.13) holds. Thus, the main part of Proposition 1 is proven. Our objective is now to show that the expected relation on the spectrum holds. Fortunately, by Lemma 2.3.2, in the case $A = L_{\text{HO}}\left(z\sqrt{2}, \dfrac{b}{\sqrt{2}}\right)$, $B = T$, we have that

$$\operatorname{Sp}\left(T^{-1}L_{\text{HO}}\left(z\sqrt{2}, \frac{b}{\sqrt{2}}\right)T\right) = \operatorname{Sp} L_{\text{HO}}\left(z\sqrt{2}, \frac{b}{\sqrt{2}}\right). \tag{4.3.16}$$

Noting that $\mathrm{Sp}(\frac{1}{2}A) = \frac{1}{2}\mathrm{Sp}\, A$ and $\mathrm{Sp}(A - \frac{1}{2}I) = \mathrm{Sp}(A) - \frac{1}{2}$,

$$\mathrm{Sp}\left(L_{\mathrm{PC}}(z, b)\right) = \frac{\mathrm{Sp}\left(L_{\mathrm{HO}}\left(z\sqrt{2}, \frac{b}{\sqrt{2}}\right)\right) - 1}{2},$$

i.e., (1.0.14) holds. Proposition 1 is proven.

CHAPTER 5
CRITERION FOR EIGENVALUES

We now attempt to use Propositions 3.3.1 and 3.3.2 to discuss when $\nu \in \mathbb{C}$ is an eigenvalue of $L_{\mathrm{PC}}(z, b)$.

5.1 L^2 Requirement

We know from Proposition 3.3.2 that eigenfunctions must be solutions to the Weber parabolic cylinder equation (3.3.3) on various subintervals; yet eigenfunctions must be $L^2(\mathbb{R})$ functions by construction, so we see what the $L^2(\mathbb{R})$ constraint does for us. To begin, we have the following lemma.

Lemma 5.1.1. *Fix $b > 0$ and $\nu \in \mathbb{C}$.*

(a) If for some $y(x) \in \mathrm{Hol}(\mathbb{C})$, $\{D_\nu(x), y(x)\}$ is a linearly independent pair of solutions to (3.1.1a) over $\mathbb{C}$, then $y(x) \notin L^2((b, \infty))$.

(b) If for some $y(x) \in \mathrm{Hol}(\mathbb{C})$, $\{D_\nu(-x), y(x)\}$ is a linearly independent pair of solutions to (3.1.1a) over $\mathbb{C}$, then $y(x) \notin L^2((-\infty, -b))$.

Proof. To prove part (a), suppose that for some $b > 0$, $\nu \in \mathbb{C}$, and $y(x) \in \mathrm{Hol}(\mathbb{C})$, $\{D_\nu(x), y(x)\}$ is a linearly independent set of solutions to (3.1.1a). We know that

$D_\nu(x)$ is that solution that must decay as $x^\nu e^{-x^2/4}$ as $x \to \infty$, and that another solution, call it $V(x)$, must be proportional to $x^{-\nu-1}e^{x^2/4}$ as $x \to \infty$, in the sense that

$$\lim_{x\to\infty} \frac{V(x)}{x^{-\nu-1}e^{x^2/4}} = 1. \tag{5.1.1}$$

By the distinct asymptotic behaviours $\{D_\nu(x), B(x)\}$ is a linearly independent set of solutions to (3.1.1a) on some interval of the form (M, ∞), $M > 0$, and so y must be a linear combination of D_ν and V; say,

$$y(x) = c_1 D_\nu(x) + c_2 V(x), \quad x > M. \tag{5.1.2}$$

Note that by $\{D_\nu, y\}$ a linearly independent set of solutions on all of $\mathbb{C}$, $c_2 \neq 0$. Then of course, since D_ν decays as $x \to \infty$,

$$\begin{aligned} \lim_{x\to\infty} \frac{y(x)}{x^{-\nu-1}e^{x^2/4}} &= c_1 \lim_{x\to\infty} \frac{D_\nu(x)}{x^{-\nu-1}e^{x^2/4}} + c_2 \lim_{x\to\infty} \frac{V(x)}{x^{-\nu-1}e^{x^2/4}} \\ &= c_1 \cdot 0 + c_2 \cdot 1 = c_2 \end{aligned} \tag{5.1.3}$$

Since it follows that

$$\lim_{x\to\infty} \left| \frac{y(x)}{x^{-\nu-1}e^{x^2/4}} \right| = |c_2| \neq 0,$$

then for some $M_1 \geq M$,

$$\left| \frac{y(x)}{x^{-\nu-1}e^{x^2/4}} \right| \geq \frac{|c_2|}{2}. \tag{5.1.4}$$

Therefore, we have that for $M_2 = \max\{M_1, b+1\}$,

$$\begin{aligned}\int_b^\infty |y(x)|^2\,dx &\geq \int_{M_2}^\infty |y(x)|^2\,dx \\ &\geq \int_{M_2}^\infty |c_2 x^{-\nu-1} e^{x^2/4}|^2\,dx \\ &\geq |c_2|^2 \int_{M_2}^\infty \frac{e^{x^2/2}}{x^{2\nu+2}}\,dx \\ &= \infty,\end{aligned} \tag{5.1.5}$$

and so y is not square-integrable on (b, ∞).

To prove part (b), we reduce to part (a). For if $\{D_\nu(-x), y(x)\}$ is a linearly independent set of solutions to (3.1.1a), then $\{D_\nu(x), -y(-x)\}$ is certainly a set of solutions to (3.1.1a) by the refleciton symmetry, and we claim that it is a linearly independent set of solutions, since if

$$a_1 D_\nu(x) + a_2 y(-x) \equiv 0, \quad x \in \mathbb{C} \tag{5.1.6}$$

then letting $t = -x$, we have that

$$a_1 D_\nu(t) + a_2 y(-t) \equiv 0, \quad t \in \mathbb{C} \tag{5.1.7}$$

Yet by linear independence of D_ν and $y(-\cdot)$, $a_1 = a_2 = 0$. Hence, $D_\nu(x)$ and $y(-x)$ are linearly independent over $\mathbb{C}$. By part (a), then, $y(-x)$ is not square–integrable on (b, ∞), which means that y is not square–integrable on $(-\infty, -b)$. □

We now show that any eigenvector of $L_{\mathrm{PC}}(z, b)$ must be of a particular fixed form.

Corollary 5.1.2. *Fix $b > 0$, $z \in \mathbb{C}$, and $\nu \in \mathbb{C} \setminus \mathbb{N}_0$. Then if y is an eigenfunction* *of $L_{PC}(z,b)$ with eigenvalue ν, then it must be of the form*

$$y(x) = \begin{cases} \beta D_\nu(-x), & x \leq -b \\ \gamma y_1(\nu,x) + \delta y_2(\nu,x), & -b \leq x \leq b \\ \alpha D_\nu(x), & x \geq b \end{cases} . \tag{5.1.8}$$

Proof. By Proposition 3.3.1, we know that any eigenfunction must satisfy (3.3.3) on $(-\infty,-b)$, $(-b,b)$, and (b,∞). It remains to choose an appropriate fundamental basis of solutions on each interval. On the basis of our discussion in Section 3.1, we now choose a basis of solutions to (3.3.3) on each interval, working right-to-left.

(b,∞). Since we want to use the proof independent of the condition of whether or not $\nu \in \mathbb{N}_0$, we choose the guaranteed basis of $\{D_\nu(x), D_{-\nu-1}(ix)\}$ on this interval.

$(-b,b)$. To take maximal advantage of the symmetry in the domain, it makes sense to use $\{y_1(\nu;x), y_2(\nu;x)\}$ as our basis here.

$(-\infty,-b)$. Again, to put the appropriately decaying function in our domain, and a guaranteed complement, we choose the basis of $\{D_\nu(-x), D_{-\nu-1}(ix)\}$ on this interval.

Since the above fundamental sets of solutions are solutions over $\mathbb{C}$, their restrictions to any open subinterval of $\mathbb{R}$ satisfy (3.3.3) on that subinterval. By Proposition 3.3.1, eigenfunctions of $L_{\mathrm{PC}}(z,b)$ must satisfy (3.3.3) on $(-\infty,-b)$, $(-b,b)$, and (b,∞), and must be continuous on $\mathbb{R}$. Hence, any eigenfunction must be of the form

$$y(x) = \begin{cases} \beta D_\nu(-x) + t D_{-\nu-1}(ix), & x \le -b \\ \gamma y_1(\nu; x) + \delta y_2(\nu; x), & -b \le x \le b \\ \alpha D_\nu(x) + s D_{-\nu-1}(ix), & x \ge b \end{cases}. \tag{5.1.9}$$

Suppose, by way of contradiction, that $s \neq 0$. Then since $\{D_\nu(x), D_{-\nu-1}(ix)\}$ is a linearly independent set of solutions to (3.1.1a), and $s \neq 0$, $\{D_\nu(-x), \alpha D_\nu(x) + sD_{-\nu-1}(ix)\}$ is a linearly independent set of solution to (3.1.1a). By Lemma 5.1.1, part (a), $\alpha D_\nu(x) + sD_\nu(-x)$ is not square-integrable on $(-\infty, -b)$, so $y(x)$, restricted to (b, ∞), is not in $L^2((b,\infty))$. Yet $y(x)$ is an eigenfunction of $L_{\text{PC}}(z, b)$, an operator defined on a subset of $L^2(\mathbb{R})$, so the restriction of $y(x)$ to (b, ∞) must be in $L^2(b, \infty)$. Contradiction. Thus, $s = 0$. Similarly, using Lemma 5.1.1, part (b), $t = 0$. □

5.2 Jump Conditions Requirement

Fix $b > 0$, $z \in \mathbb{C}$, and $\nu \in \mathbb{C} \setminus \mathbb{N}_0$. We now discuss the remaining conditions for y as in (5.1.8) to be a ν-eigenfunction of $L_{\text{PC}}(z, b)$.

Continuity at $x = \pm b$. The continuity condition at $x = b$ is

$$\begin{gathered} y(b-) = y(b+), \quad \text{or} \\ \gamma y_1(\nu; b) + \delta y_2(\nu; b) = \alpha D_\nu(b) \end{gathered} \tag{5.2.1}$$

Similarly, the continuity condition at $x = -b$ is

$$\begin{gathered} y(-b-) = y(-b+), \quad \text{or} \\ \beta D_\nu(-[-b]) = \gamma y_1(\nu; -b) + \delta y_2(\nu; -b) \end{gathered} \tag{5.2.2}$$

and since $y_1(\nu, x)$ is even in x and $y_2(\nu; x)$ is odd in ix, this becomes

$$\beta D_\nu(b) = \gamma y_1(\nu; b) - \delta y_2(\nu; b) \tag{5.2.3}$$

Jump conditions on the derivative. In light of (5.1.8), the left-hand side of, say, (3.3.4b) becomes

$$\alpha \left.\frac{dD_\nu}{dx}\right|_{x=b} - \left(\gamma \left.\frac{d^1}{dx^1}\left(y_1(\nu; x)\right)\right|_{x=b} + \delta \left.\frac{d^1}{dx^1}\left(y_2(\nu; x)\right)\right|_{x=b}\right) \tag{5.2.4}$$

By (5.2.1), however, the right-hand side of (3.3.4b) becomes

$$zy(b) = z\alpha D_\nu(b).$$

Hence, altogether, (3.3.4b) becomes

$$\alpha \left.\frac{dD_\nu}{dx}\right|_{x=b} - \left(\gamma \left.\frac{\partial}{\partial x}\left(y_1(\nu; x)\right)\right|_{x=b} + \delta \left.\frac{\partial}{\partial x}\left(y_2(\nu; x)\right)\right|_{x=b}\right) = z\alpha D_\nu(b) \tag{5.2.5}$$

Similarly, (3.3.4a) becomes

$$\begin{aligned}\left(\gamma \left.\frac{\partial}{\partial x}\left(y_1(\nu; x)\right)\right|_{x=-b} + \delta \left.\frac{\partial}{\partial x}\left(y_2(\nu; x)\right)\right|_{x=-b}\right) & \\ -\beta \left.\frac{d}{dx} D_\nu(-x)\right|_{x=-b} & = -zy(-b)\end{aligned} \tag{5.2.6}$$

Yet by the $c = -1$ case of Lemma 2.5.5,part 1 we know that for all even differentiable functions $f(t) \in C^1(\mathbb{R})$,

$$\begin{aligned}\left.\frac{df}{dx}\right|_{x=-b} &= -\left.\frac{d}{dx}\left(f(-x)\right)\right|_{x=b} \\ &= -\left.\frac{d}{dx}\left(f(x)\right)\right|_{x=b} \quad \text{by } f \text{ even.}\end{aligned} \tag{5.2.7}$$

Therefore,

$$\frac{\partial}{\partial x}\left(y_1(\nu;x)\right)\bigg|_{x=-b} = -\frac{\partial}{\partial x}\left(y_1(\nu;x)\right)\bigg|_{x=b} \tag{5.2.8}$$

Similarly, if $g(t) \in C^1(\mathbb{R})$ is an odd function,

$$\frac{dg}{dx}\bigg|_{x=-b} = \frac{dg}{dx}\bigg|_{x=b}, \tag{5.2.9}$$

and hence

$$\frac{\partial}{\partial x}\left(y_2(\nu;x)\right)\bigg|_{x=-b} = \frac{\partial}{\partial x}\left(y_2(\nu;x)\right)\bigg|_{x=b}. \tag{5.2.10}$$

Again by the $c = -1$ case of Lemma 2.5.5, part 1,

$$\frac{\partial}{\partial x}\left(D_\nu(-x)\right)\bigg|_{x=-b} = -\frac{dD_\nu}{dx}\bigg|_{x=b}. \tag{5.2.11}$$

Therefore, (5.2.6) becomes

$$\left(-\gamma\frac{\partial}{\partial x}\left(y_1(\nu;x)\right)\bigg|_{x=b} + \delta\frac{\partial}{\partial x}\left(y_2(\nu;x)\bigg|_{x=b}\right)\right) + \beta\frac{dD_\nu}{dx}\bigg|_{x=b} = -z\beta_0 D_\nu(-[-b]). \tag{5.2.12}$$

Combining, and recollecting the terms in (5.2.2),(5.2.1), (5.2.12), and (5.2.5), we see by Proposition 3.3.1 that $\nu \notin \mathbb{N}_0$ is an eigenvalue of $L_{\mathrm{PC}}(z,b)$ if and only if

$$\beta D_\nu(b) - \gamma y_1(\nu;b) + \delta y_2(\nu;b) = 0, \tag{5.2.13a}$$

$$\gamma D_\nu(b) + \delta D_\nu(-b) - \alpha D_\nu(b) = 0, \tag{5.2.13b}$$

$$\beta\left(zD_\nu(b) + \frac{dD_\nu}{dx}\bigg|_{x=b}\right) - \gamma\frac{\partial}{\partial x}\left(y_1(\nu;x)\right)\bigg|_{x=b} + \delta\frac{\partial}{\partial x}\left(y_2(\nu;x)\right|_{x=b} = 0, \tag{5.2.13c}$$

$$\gamma\frac{\partial}{\partial x}\left(y_1(\nu;x)\right)\bigg|_{x=b} + \delta\frac{\partial}{\partial x}\left(y_2(\nu;x)\right)\bigg|_{x=b} + \alpha\left(zD_\nu(b) - \frac{dD_\nu}{dx}\bigg|_{x=b}\right) = 0. \tag{5.2.13d}$$

Equivalently, creating the shorthands

$$\mathcal{P} := D_\nu(b), \tag{5.2.14a}$$

$$\mathcal{Q} := \left.\frac{dD_\nu}{dx}\right|_{x=b}, \tag{5.2.14b}$$

$$\mathcal{R} := y_1(\nu; b), \tag{5.2.14c}$$

$$\mathcal{T} := \left.\frac{\partial}{\partial x}\left(y_1(\nu; x)\right)\right|_{x=b} \tag{5.2.14d}$$

$$\mathcal{U} := y_2(\nu; b), \tag{5.2.14e}$$

$$\mathcal{W} := \left.\frac{\partial}{\partial x}\left(y_2(\nu; x)\right)\right|_{x=b} \tag{5.2.14f}$$

we have that

$$\begin{pmatrix} \mathcal{P} & -\mathcal{R} & \mathcal{U} & 0 \\ 0 & \mathcal{R} & \mathcal{U} & -\mathcal{P} \\ z\mathcal{P}+\mathcal{Q} & -\mathcal{T} & \mathcal{W} & 0 \\ 0 & \mathcal{T} & \mathcal{W} & z\mathcal{P}-\mathcal{Q} \end{pmatrix} \cdot \begin{pmatrix} \beta \\ \gamma \\ \delta \\ \alpha \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{5.2.15}$$

if and only if $\nu \in \operatorname{Sp} L_{\mathrm{PC}}(z, b)$. More precisely, we have the following.

Proposition 5.2.1. *Fix $b > 0$, $z \in \mathbb{C}$, and $\nu \in \mathbb{C}$, and let $\mathcal{L}_\nu$ denote the ν-eigenspace of $\operatorname{Sp} L_{PC}(z, b)$ (which can be $\{0\}$). Then defining $\mathcal{P}, \ldots, \mathcal{W}$ as in (5.2.14), and defining*

$$A(\nu; z; b) := \begin{pmatrix} \mathcal{P} & -\mathcal{R} & \mathcal{U} & 0 \\ 0 & \mathcal{R} & \mathcal{U} & -\mathcal{P} \\ z\mathcal{P}+\mathcal{Q} & -\mathcal{T} & \mathcal{W} & 0 \\ 0 & \mathcal{T} & \mathcal{W} & z\mathcal{P}-\mathcal{Q} \end{pmatrix} \tag{5.2.16}$$

there is a linear bijection from $\ker A(\nu; z; b)$ (regarded as a subspace of $\mathbb{R}^4$) to $\mathcal{L}_\nu$

defined by

$$\begin{pmatrix} \beta \\ \gamma \\ \delta \\ \alpha \end{pmatrix} \mapsto y_{\beta,\gamma,\delta,\alpha} := \begin{cases} \beta D_\nu(-x), & x \leq -b \\ \gamma y_1(\nu;x) + \delta y_2(\nu;x), & -b \leq x \leq b \\ \alpha D_\nu(x), & x \geq b \end{cases} . \tag{5.2.17}$$

Proof. Fix $b > 0$, $z \in \mathbb{C}$, and $\nu \in \mathbb{C}$. Then by Corollary 5.1.2, any element of $\mathcal{L}_\nu$ must be for the form $y_{\beta,\gamma,\delta,\alpha}$ for some $\beta, \gamma, \delta, \alpha \in \mathbb{C}$. Moreover, we have shown that $y_{\beta,\gamma,\delta,\alpha} \in \mathcal{L}_\nu$ if and only if $(\beta, \gamma, \delta, \alpha)^T \in \ker A(\nu; z; b)$, so the map in (5.2.17) exists and is a surjection. We now argue that the map is an injection. Suppose that $(\beta, \gamma, \delta, \alpha)^T \neq (\mathfrak{b}, \mathfrak{c}, \mathfrak{d}, \mathfrak{a})^T$ are elements of $\ker A(\nu; z; b)$. Then the vectors disagree in some index. If $\beta \neq \mathfrak{b}$, then $y_{\beta,\gamma,\delta,\alpha} \neq y_{\mathfrak{b},\mathfrak{c},\mathfrak{d},\mathfrak{a}}$ on $(-\infty, -b)$. Similarly, if $\alpha \neq \mathfrak{a}$, the functions disagree on (b, ∞). If $\gamma \neq \mathfrak{c}$, or $\delta \neq \mathfrak{d}$, we know by (3.1.21) and Proposition 2.5.4 that $y_1(\nu, x)$ and $y_2(\nu; x)$ are linearly independent functions of x, so

$$\gamma y_1(\nu; x) + \delta y_2(\nu; x) \neq \mathfrak{c} y_1(\nu; x) + \mathfrak{d} y_2(\nu; x). \tag{5.2.18}$$

Finally, since summation of functions is linear, linearity is assured. $\square$

5.3 The Algebraic Criterion

For any $b > 0$ and $z \in \mathbb{C}$, by Proposition 5.2.1 it is clear that $\nu \in \mathbb{C}$ is an eigenvalue of $L_{\mathrm{PC}}(z, b)$ if and only if $\ker A(\nu; z; b) \neq \{0\}$. Yet a nontrivial kernel implies that $A(\nu; z; b)$ is noninvertible, and hence that $\det A(\nu; z; b) = 0$. Calculating this determinant, one arrives at

$$\det A(\nu; z; b) = 2 \left[(\mathcal{R}\mathcal{Q} - \mathcal{T}\mathcal{P})(\mathcal{U}\mathcal{Q} - \mathcal{W}\mathcal{P}) - z^2 \mathcal{P}^2 \mathcal{R}\mathcal{U} \right]. \tag{5.3.1}$$

We now rewrite the terms in parentheses: recalling what $\mathcal{P}$, etc. mean,

$$\begin{aligned}\mathcal{R}\mathcal{Q} - \mathcal{T}\mathcal{P} &= y_1(\nu; b)\, \frac{dD_\nu}{dx}\bigg|_{x=b} - D_\nu(b)\, \frac{\partial}{\partial x}\left(y_1(\nu, x)\right)\bigg|_{x=b} \\ &= \mathrm{Wr}\left[y_1(\nu;\cdot), D_\nu\right](b)\end{aligned} \tag{5.3.2}$$

and similarly,

$$\mathcal{U}\mathcal{Q} - \mathcal{W}\mathcal{P} = \mathrm{Wr}\left[y_2(\nu;\cdot), D_\nu\right](b) \tag{5.3.3}$$

Altogether, then, we have that

$$\det A(\nu; z; b) = 2\left[\mathrm{Wr}\left[y_1(\nu;\cdot), D_\nu\right](b) \cdot \mathrm{Wr}\left[y_2(\nu;\cdot), D_\nu\right](b) - z^2 D_\nu^2(b) y_1(\nu; b) y_2(\nu; b)\right]. \tag{5.3.4}$$

We now simplify those Wronskians. First, by Lemma 2.5.3, the Wronskian is bilinear, so given (3.1.22),

$$\begin{aligned}&\mathrm{Wr}\left[y_1(\nu;\cdot), D_\nu\right](b) \\ &= \frac{2^{\nu/2}\sqrt{\pi}}{\Gamma\left(-\frac{\nu}{2}+\frac{1}{2}\right)}\,\mathrm{Wr}\left[y_1(\nu;\cdot), y_2(\nu;\cdot)\right](b) - \frac{2^{(\nu+1)/2}\sqrt{\pi}}{\Gamma\left(-\frac{\nu}{2}\right)}\,\mathrm{Wr}\left[y_1(\nu;\cdot), y_2(\nu;\cdot)\right](b) \\ &= -\frac{2^{(\nu+1)/2}\sqrt{\pi}}{\Gamma\left(-\frac{\nu}{2}\right)}\,\mathrm{Wr}\left[y_1(\nu;\cdot), y_2(\nu;\cdot)\right](b).\end{aligned} \tag{5.3.5}$$

Similarly, since we know the Wronskian is alternating (see Lemma 2.5.3),

$$\begin{aligned}&\mathrm{Wr}\left[y_2(\nu;\cdot), D_\nu\right](b) \\ &= \frac{2^{\nu/2}\sqrt{\pi}}{\Gamma\left(-\frac{\nu}{2}+\frac{1}{2}\right)}\,\mathrm{Wr}\left[y_2(\nu;\cdot), y_1(\nu;\cdot)\right](b) - \frac{2^{(\nu+1)/2}\sqrt{\pi}}{\Gamma\left(-\frac{\nu}{2}\right)}\,\mathrm{Wr}\left[y_2(\nu;\cdot), y_2(\nu;\cdot)\right](b) \\ &= -\frac{2^{\nu/2}\sqrt{\pi}}{\Gamma\left(-\frac{\nu}{2}+\frac{1}{2}\right)}\,\mathrm{Wr}\left[y_1(\nu;\cdot), y_2(\nu;\cdot)\right](b).\end{aligned} \tag{5.3.6}$$

Then the product of the Wronskians becomes

$$\begin{aligned}
&\mathrm{Wr}\left[y_1(\nu;\cdot), D_\nu\right](b)\,\mathrm{Wr}\left[y_2(\nu;\cdot), D_\nu\right](b) \\
&= \frac{2^{\nu+\frac{1}{2}}\pi}{\Gamma\left(-\frac{\nu}{2}\right)\Gamma\left(-\frac{\nu}{2}+\frac{1}{2}\right)}\,\mathrm{Wr}\left[y_1(\nu;\cdot), y_2(\nu;\cdot)\right](b)\,.
\end{aligned} \tag{5.3.7}$$

This simplifies, however, thanks to the formula

$$\Gamma(\zeta)\Gamma\left(\zeta+\frac{1}{2}\right) = 2^{(1/2)-2\zeta}\sqrt{2\pi}\,\Gamma(2\zeta), \quad \zeta \in \mathbb{C}\setminus -\mathbb{N}_0; \tag{5.3.8}$$

see [Sze75, (1.7.3), p. 14] or [Con78, Exercise VII.7.3, p. 183]. Using the case $\zeta = -\dfrac{\nu}{2}$ of (5.3.8), then, (5.3.7) becomes

$$\begin{aligned}
&\mathrm{Wr}\left[y_1(\nu;\cdot), D_\nu\right](b)\,\mathrm{Wr}\left[y_2(\nu;\cdot), D_\nu\right](b) \\
&= \frac{\sqrt{\pi}}{\sqrt{2}\Gamma(-\nu)}\,\mathrm{Wr}\left[y_1(\nu;\cdot), y_2(\nu;\cdot)\right](b)
\end{aligned} \tag{5.3.9}$$

Since $y_1(\nu;x)$ and $y_2(\nu;x)$ are solutions to (3.1.1a), a second order linear, homogeneous differential equation with no first-derivative term, the Wronskian function is constant; see [Olv74, Chap. V, (1.10), p. 142]. Therefore,

$$\mathrm{Wr}\left[y_1(\nu;\cdot), y_2(\nu;\cdot)\right](b) = \mathrm{Wr}\left[y_1(\nu;\cdot), y_2(\nu;\cdot)\right](0) \tag{5.3.10}$$

Recall from (3.1.21), it follows that

$$\mathrm{Wr}\left[y_1(\nu;\cdot), y_2(\nu;\cdot)\right](0) = 1. \tag{5.3.11}$$

Thus, finally,

$$\det A(\nu;z;b) = 2\left[\frac{\sqrt{\pi}}{\sqrt{2}\Gamma(-\nu)} - z^2 D_\nu^2(b)y_1(\nu;b)y_2(\nu;b)\right]. \tag{5.3.12}$$

Since the presence of eigenvalues is asking whether this determinant is 0, and $2 \neq 0$, we know that $\nu \in \operatorname{Sp} L_{\mathrm{PC}}(z, b)$ if and only if

$$\frac{\sqrt{\pi}}{\sqrt{2}\Gamma(-\nu)} - z^2 D_\nu^2(b) y_1(\nu; b) y_2(\nu; b) = 0 \tag{5.3.13}$$

i.e., we have proven Theorem 1.

5.4 Reconciliation with Demiralp's work, and another algebraic criterion

Our objective is to reconcile the above formula with the work of [Dem05] mentioned above. In his work, in the case $z = ir$, he uses the basis of solutions $\{D_\nu(x), D_\nu(-x)\}$, which restricts him to the case $\nu \notin \mathbb{N}_0$, and his results are essentially as follows.

Lemma 5.4.1 ([Dem05, p. 3])**.** *Fix $b > 0$, $r \in \mathbb{R}$, and $\nu \in \mathbb{C} \backslash \mathbb{N}_0$. Then $\nu \in L_{PC}(ir, b)$ if and only if*

$$1 + \frac{|r|^2}{W^2} D_\nu^2(b) \left(D_\nu^2(-b) - D_\nu^2(b) \right) = 0, \tag{5.4.1}$$

where $W = \operatorname{Wr}[D_\nu, D_\nu(-\cdot)](b) = \operatorname{Wr}[D_\nu, D_\nu(-\cdot)](-b)$.

Our objective is to reconcile these values. First, by (3.1.17), $W = \dfrac{\sqrt{2\pi}}{\Gamma(-\nu)}$. Since $|r|^2 = r^2 = -z^2$ under the identification $z = ir$, we have

$$1 - z^2 \frac{\Gamma^2(-\nu)}{2\pi} D_\nu^2(b) \left(D_\nu^2(-b) - D_\nu^2(b) \right) = 0 \tag{5.4.2}$$

Moreover, we may simplify the $D_\nu^2(-b) - D_\nu^2(b)$ term. First, we note that by the difference–of–squares formula,

$$D_\nu^2(-b) - D_\nu^2(b) = (D_\nu(-b) + D_\nu(b))(D_\nu(-b) - D_\nu(b)). \tag{5.4.3}$$

Since the unperturbed equation (3.1.1a) is linear and homogeneous, it is clear that this is the evaluation at b of an odd and an even solution for (3.1.1a). However, we already know from (3.1.22) the precise factorization of $D_\nu(b)$ into an odd and an even solution; from (3.1.22) we get that

$$D_\nu(-b) + D_\nu(b) = 2 \cdot \frac{2^{\nu/2}\sqrt{\pi}}{\Gamma\left(-\frac{\nu}{2} + \frac{1}{2}\right)} \cdot y_1(\nu; b) \tag{5.4.4a}$$

$$D_\nu(-b) - D_\nu(b) = 2 \cdot \frac{2^{(\nu+1)/2}\sqrt{\pi}}{\Gamma\left(-\frac{\nu}{2}\right)} y_2(\nu; b) \tag{5.4.4b}$$

Plugging (5.4.4) into (5.4.3), and again using (5.3.8) to simplify, we have that

$$D_\nu^2(-b) - D_\nu^2(b) = \frac{2^{3/2}\sqrt{\pi}}{\Gamma(-\nu)} y_1(\nu; b) y_2(\nu; b). \tag{5.4.5}$$

Therefore, with this improvement, we may rewrite (5.4.2) as

$$1 - z^2 \frac{\sqrt{2}\Gamma(-\nu)}{\sqrt{\pi}} D_\nu^2(b) y_1(\nu; b) y_2(\nu; b) = 0 \tag{5.4.6}$$

We see that the only difference between (5.3.13) and (5.4.6) is a factor of $\dfrac{2\Gamma(-\nu)}{\pi}$; certainly, for $\nu \notin \mathbb{N}_0$, $\Gamma(-\nu) \neq 0$ and hence both forms are indeed valid. In short, we have proven Theorem 2; i.e., for $b > 0$, $z \in \mathbb{C}$, and $\nu \in \mathbb{C} \setminus \mathbb{N}_0$, $\nu \in \operatorname{Sp} L_{\mathrm{PC}}(z, b)$ if and only if

$$1 - z^2 M(\nu; b) = 0, \tag{5.4.7}$$

where

$$M(\nu; b) := \frac{\Gamma(-\nu)\sqrt{2}}{\sqrt{\pi}} D_\nu^2(b) y_1(\nu; b) y_2(\nu; b). \tag{5.4.8}$$

Again, for $z \neq 0$, (5.4.7) can be rewritten as

$$M(\nu; b) = \frac{1}{z^2}. \tag{5.4.9}$$

This is useful for many reason, in particular, the following restrictions on eigenvalues of $L_{\mathrm{PC}}(\nu, b)$.

Lemma 5.4.2. *Fix $b > 0$, $z \in \mathbb{C}$, and $\nu \in \mathbb{C} \setminus \mathbb{N}_0$. If $M(\nu; b) = 0$, then $\nu \notin \operatorname{Sp} L_{PC}(z, b)$.*

Proof. Fix $b > 0$, $z \in \mathbb{C}$, and $\nu \notin \mathbb{N}_0$, so that $\Gamma(-\nu)$ is defined. If $M(\nu; b) = 0$, then (5.4.7) to become $1 = 0$, which is never true. □

Before finishing the section, we note that the product-factorization of $M(\nu; b)$ in (5.4.8) has the following consequence.

Lemma 5.4.3. *Fix $b > 0$ and $\nu \in \mathbb{C} \setminus \mathbb{N}_0$. Then $M(\nu; b) = 0$ if and only if $y_1(\nu; b) = 0$, $y_2(\nu; b) = 0$, or $D_\nu(b) = 0$.*

Proof. We are working in $\mathbb{C}$, so a product is zero if and only if one of the factors is 0. Certainly $\frac{\sqrt{2}}{\sqrt{\pi}}$ is nonzero. Since $\frac{1}{\Gamma(\zeta)}$ is entire, it has no poles, so $\Gamma(\zeta)$, hence $\nu \mapsto \Gamma(-\nu)$, has no zeroes. The statement follows. □

5.5 Integer eigenvalues, and related observations

Before continuing, we use the general equation (5.3.13) to the question of whether or not for $z \neq 0$, $\nu = n \in \mathbb{N}_0$ can be an eigenvalue of $L_{\mathrm{PC}}(z, b)$. Of course, for $\nu = n \in \mathbb{N}_0$, $\frac{1}{\Gamma(-\nu)} = 0$ and hence (5.3.13) reduces to the case of

$$-z^2 D_n(b) y_1(n, b) y_2(n, b) = 0. \tag{5.5.1}$$

Hence, a nonnegative integer n is an eigenvalue of $L_{\mathrm{PC}}(z, b)$ for $z \neq 0$ if and only if one of $D_n(b)$, $y_1(n; b)$, and $y_2(n; b)$ is 0. Hence, as a corollary to Theorem 1, we have:

Corollary 5.5.1. *Fix $b > 0$, and $n \in \mathbb{N}_0$. Then if at least one of*

$$D_n(b) = 0, \tag{5.5.2a}$$

$$y_1(n; b) = 0, \tag{5.5.2b}$$

$$y_2(n; b) = 0 \tag{5.5.2c}$$

holds, then $n \in L_{PC}(z, b)$ for all $z \in \mathbb{C}$. If none of the statements in (5.5.2) *hold, then for all $z \neq 0$, $n \notin L_{PC}(z, b)$.*

Example 5.5.2. We note that the equations in (5.5.2) can be satisfied. For an explicit example, we note that the second Hermite polynomial is $H_2(x) = 4x^2 - 2$, and by (3.2.5), $D_2(x)$ has as a factor

$$H_2\left(\frac{x}{\sqrt{2}}\right) = 2x^2 - 2 = 2(x^2 - 1) = 2(x + 1)(x - 1),$$

so $D_2(1) = 0$; similarly, one may see from (3.1.19a) that $y_1(2; 1) = 0$.

We must note that in a sense, (5.5.2) contains redundancies. This is because, recalling (3.2.6),

$$D_n(-x) = (-1)^n D_n(x), \quad n \in \mathbb{N}_0,$$

it follows that $D_n(x)$ is an even function for n even and an odd function for n odd; hence it must be a multiple of $y_1(n; x)$ or $y_2(n; x)$, respectively. More specifically, by

(3.1.22),

$$D_{2k}(x) = \frac{2^k\sqrt{\pi}}{\Gamma\left(-k-\frac{1}{2}\right)} y_1(2k,x), \quad k \in \mathbb{N}_0, \tag{5.5.3a}$$

$$D_{2k+1}(x) = \frac{2^{k+1}\sqrt{\pi}}{\Gamma\left(-k-\frac{1}{2}\right)} y_2(2k+1,x), \quad k \in \mathbb{N}_0. \tag{5.5.3b}$$

Therefore, (5.5.2) is indeed redundant. Yet emphasizing the possibility for $\nu \mapsto M(\nu;b)$ has possible zeroes at nonnegative integers provides the following technical value. Suppose $n \in \mathbb{N}_0$ is a zero of $\nu \mapsto D(\nu;b)$ of order m. Then defining

$$\widetilde{M}(\nu;b) := \left[D_\nu(b)\right]^2 y_1(\nu;b)y_2(\nu;b), \tag{5.5.4}$$

$\nu \mapsto \widetilde{M}(\nu;b)$ has a zero of order at least $2m+1$ at n, since $\nu \mapsto [D_\nu(b)]^2$ has a zero of order $2m$, and either $y_1(n,b)$ or $y_2(n,b)$ must also be zero, contributing at least one additional zero. The expression in (5.5.4), of course, is up to constant factors $\dfrac{M(\nu;b)}{\Gamma(-\nu)}$, and the $\Gamma(-\nu)$ term only has a pole of order 1 at $\nu = n$. Therefore, by the theory of functions of one complex variable (e.g., [Con78, Chapter V, Section 1]), $\nu \mapsto M(\nu;b)$ can be analytically extended to n and its vicinity; we write the extension as $\mathbf{M}(\nu;b)$ for clarity. Of course, this holds for any integer zero of $\nu \mapsto D(\nu;b)$. We state this as a lemma/corollary pair for future reference.

Lemma 5.5.3. *Fix $b > 0$. Suppose that for some $n \in \mathbb{N}_0$, the map $\nu \mapsto \widetilde{M}(\nu;b)$ has a zero of order $m+1$, $m \geq 0$, at $\nu = n$. Then the function $\nu \mapsto M(\nu;b)$ has a removable singularity at $\nu = n$. Denoting the analytic extension by $\mathbf{M}(\nu;b)$, $\nu \mapsto \mathbf{M}(\nu;b)$ has a zero of order $m \geq 0$ at $\nu = n$.*

Corollary 5.5.4. *Fix $b > 0$. Suppose that $\nu \mapsto D_\nu(b)$ has a nonnegative integer zero $\nu = n$, $n \in \mathbb{N}_0$, of order $m \geq 1$. Then $\nu \mapsto \widetilde{M}(\nu;b)$ has a zero of order at least $2m+1$*

at $\nu = n$, so $\nu \mapsto M(\nu; b)$ has a removable singularity at $\nu = n$, and the extension $\nu \mapsto \mathbf{M}(\nu; b)$ has a zero of order at least $2m$.

5.6 Numerical Evidence for Behavior of the Eigenvalues

We now present some numerical computations using `Mathematica` to find the eigenvalues of $L_{\mathrm{PC}}(z, 2)$. (Our code is in Appendix We used both (5.3.13) and (5.4.9) to find the eigenvalues. In each of Figures 5.1, 5.2, and 5.3, the eigenvalues of $L_{\mathrm{PC}}\big(re^{i\theta}, 2\big)$ are shown, for $r \in \left[\frac{1}{2}, 10\right]$ and $\theta \in \left\{0, \frac{\pi}{4}, \frac{\pi}{2}\right\}$.

According to the diagrams, we certainly confirm that for z real, the eigenvalues are real, and the results suggest that for $z = ir$, certainly many eigenvalues appear to become nonreal. To suggest a solution, we add to our diagrams the (numerically computed) degree-1 and degree-2 zeroes of $\nu \mapsto M(\nu; 2)$, and the zeroes of the first ν-derivative of $\nu \mapsto M(\nu; 2)$ that are not zeroes of $\nu \mapsto M(\nu; 2)$. The results are in Figures 5.4, 5.5, and 5.6.

It appears that the zeroes of $M(\nu; b)$ attract eigenvalues of $L_{\mathrm{PC}}(z, 2)$ regardless of the argument of z, and that for $z = ir$, the zeroes of the ν-derivative of $M(\nu; 2)$ that are not zeroes of $M(\nu; 2)$ have a role in determining when the eigenvalues of $L_{\mathrm{PC}}(ir, 2)$ coalesce into complex-conjugate pairs as r increases. We discuss the former in great detail in Chapter 6, and discuss some issues of the latter in Chapter 7.

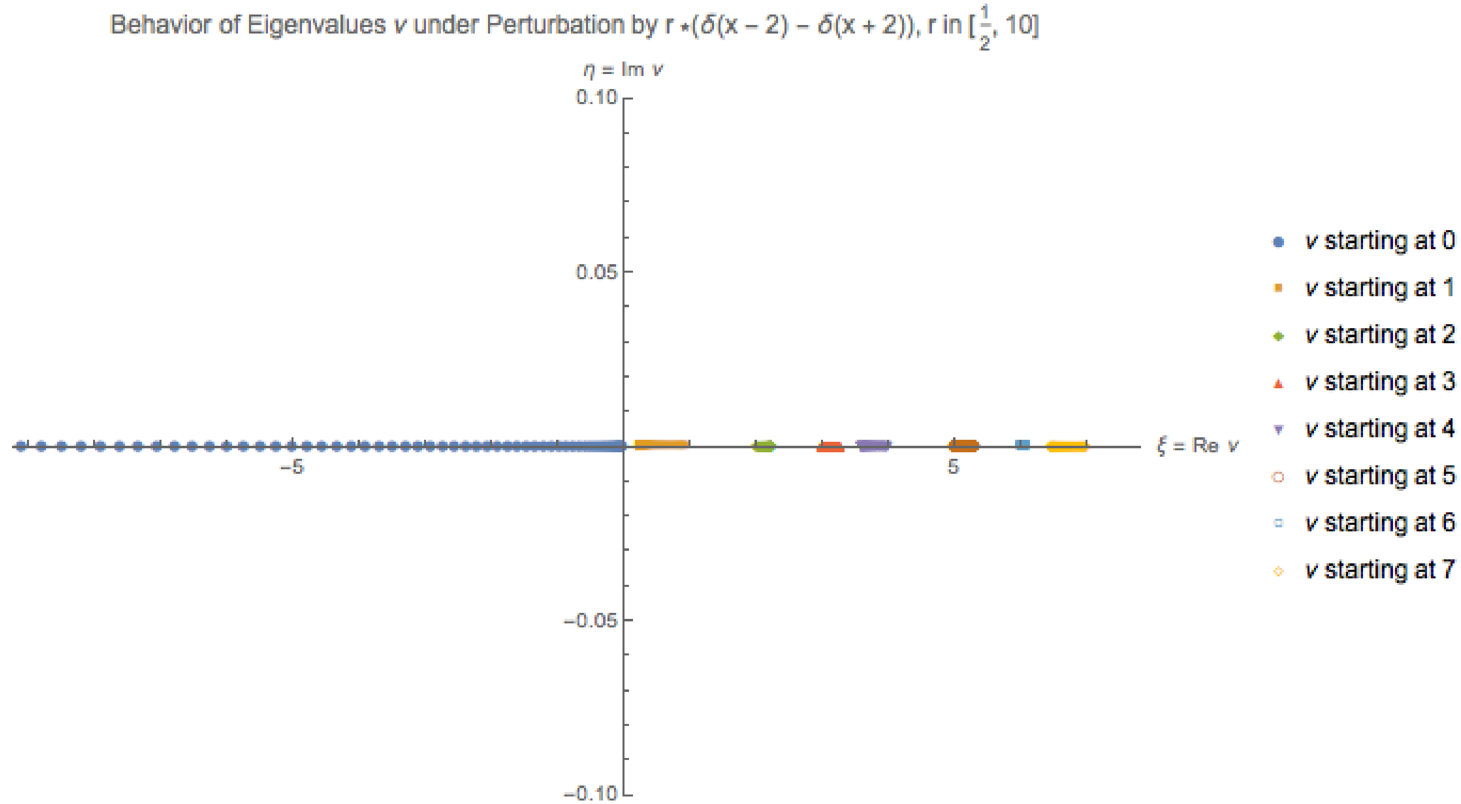

Figure 5.1: Evolution of the small eigenvalues of $L_{\mathrm{PC}}(r, 2)$ as r increases

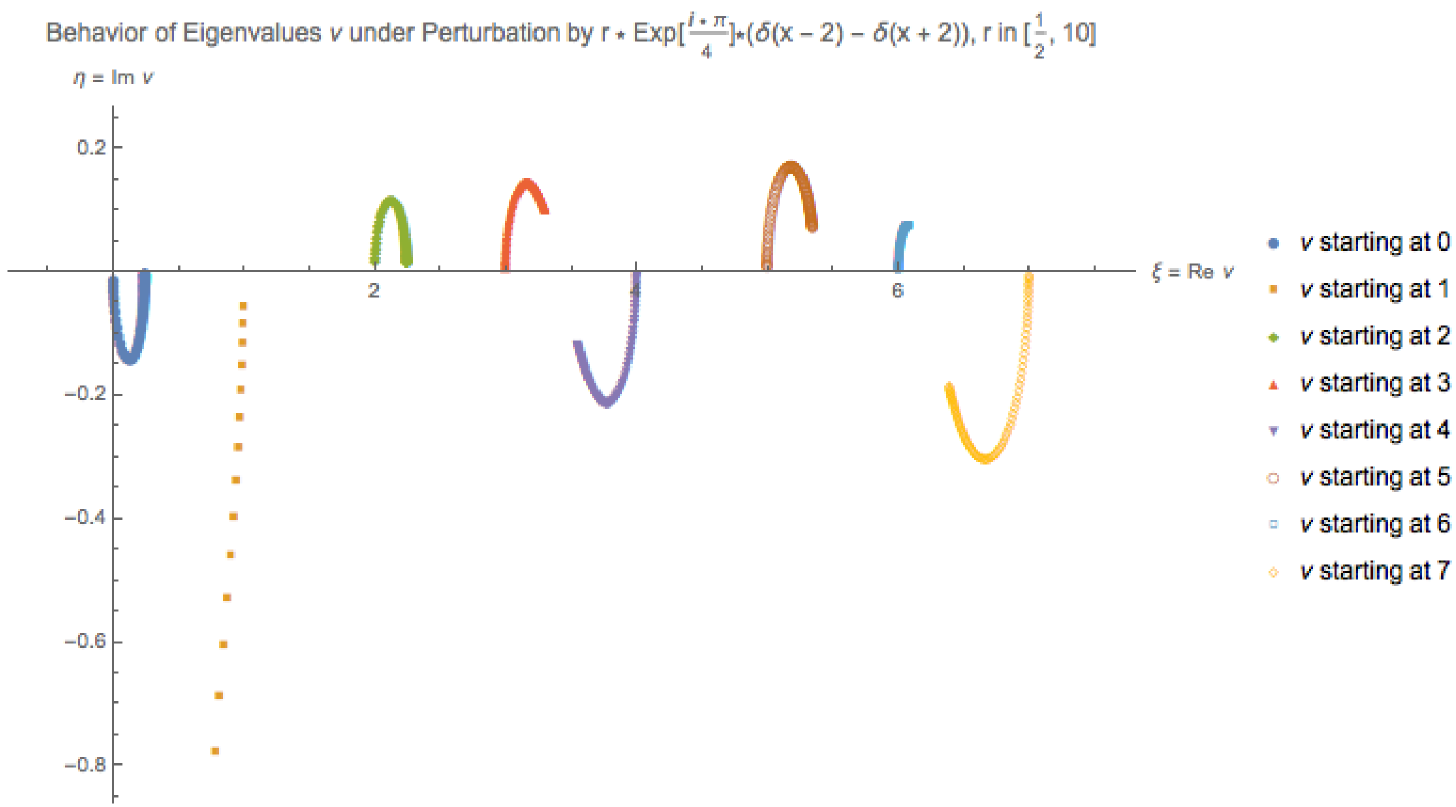

Figure 5.2: Evolution of the small eigenvalues of $L_{\mathrm{PC}}\left(re^{i\pi/4}, 2\right)$ as r increases

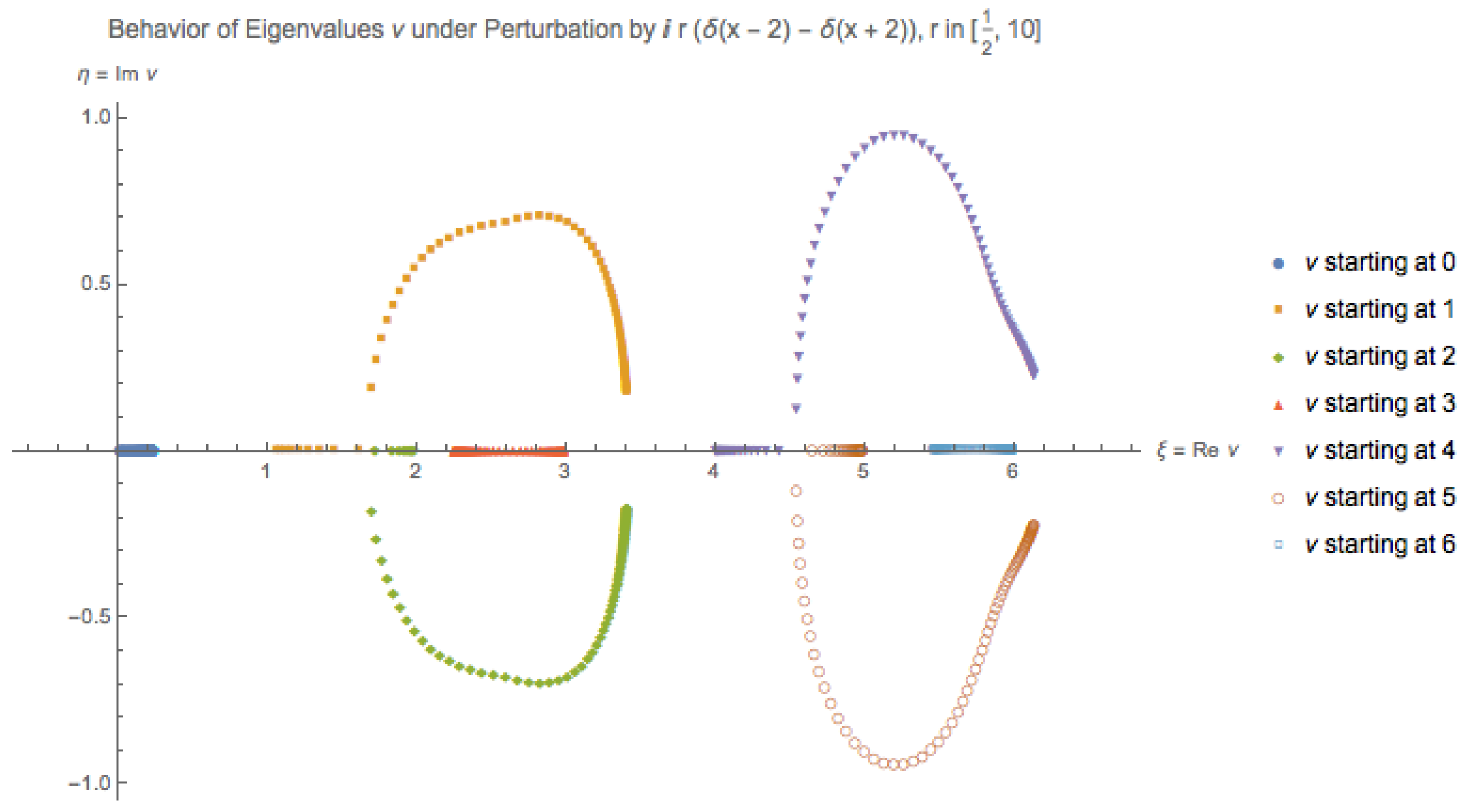

Figure 5.3: Evolution of the small eigenvalues of $L_{\mathrm{PC}}(ir, 2)$ as r increases

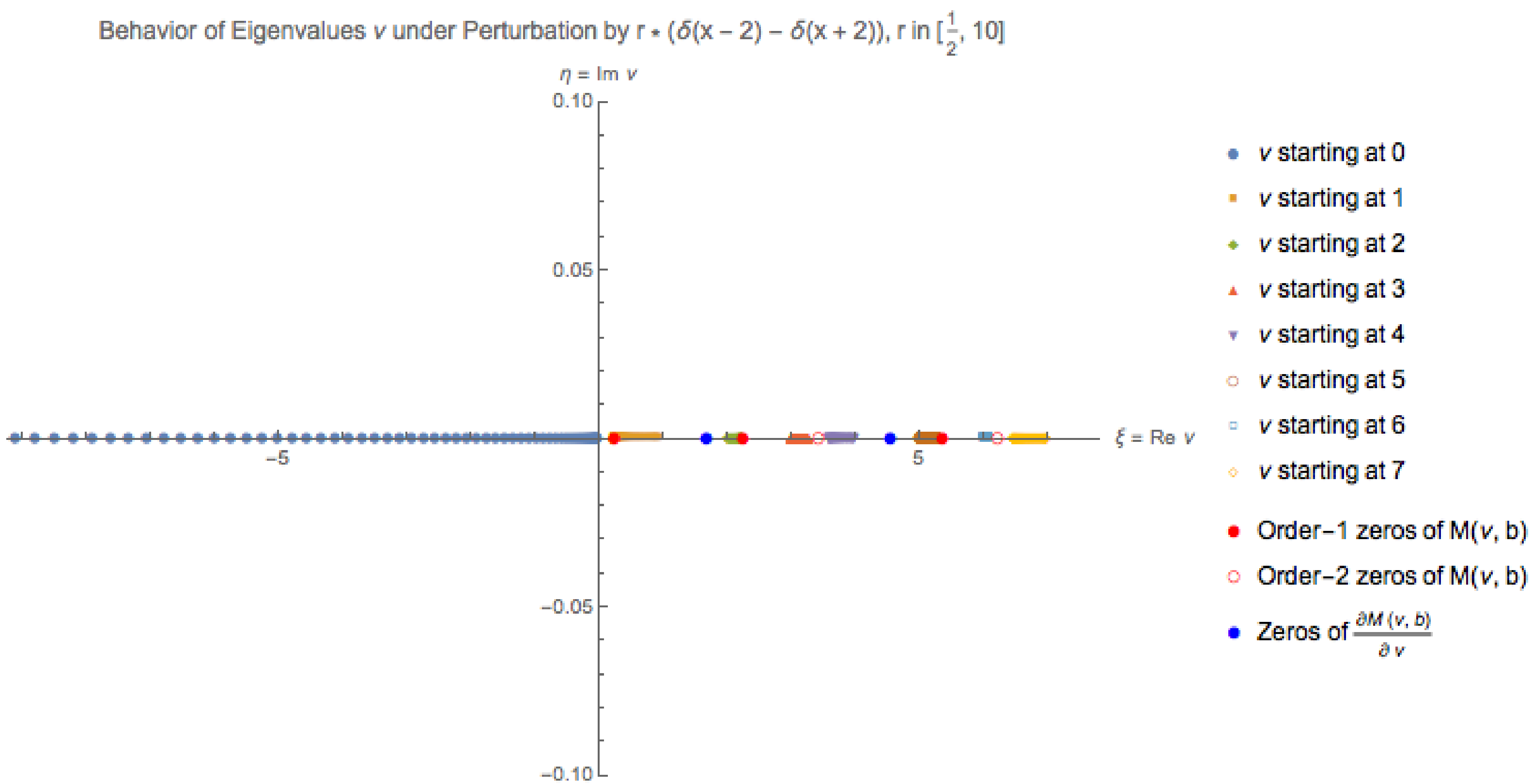

Figure 5.4: Evolution of the small eigenvalues of $L_{\text{PC}}(r, 2)$ as r increases. Zeroes of $M(\nu; 2)$ and its first ν-derivative are marked.

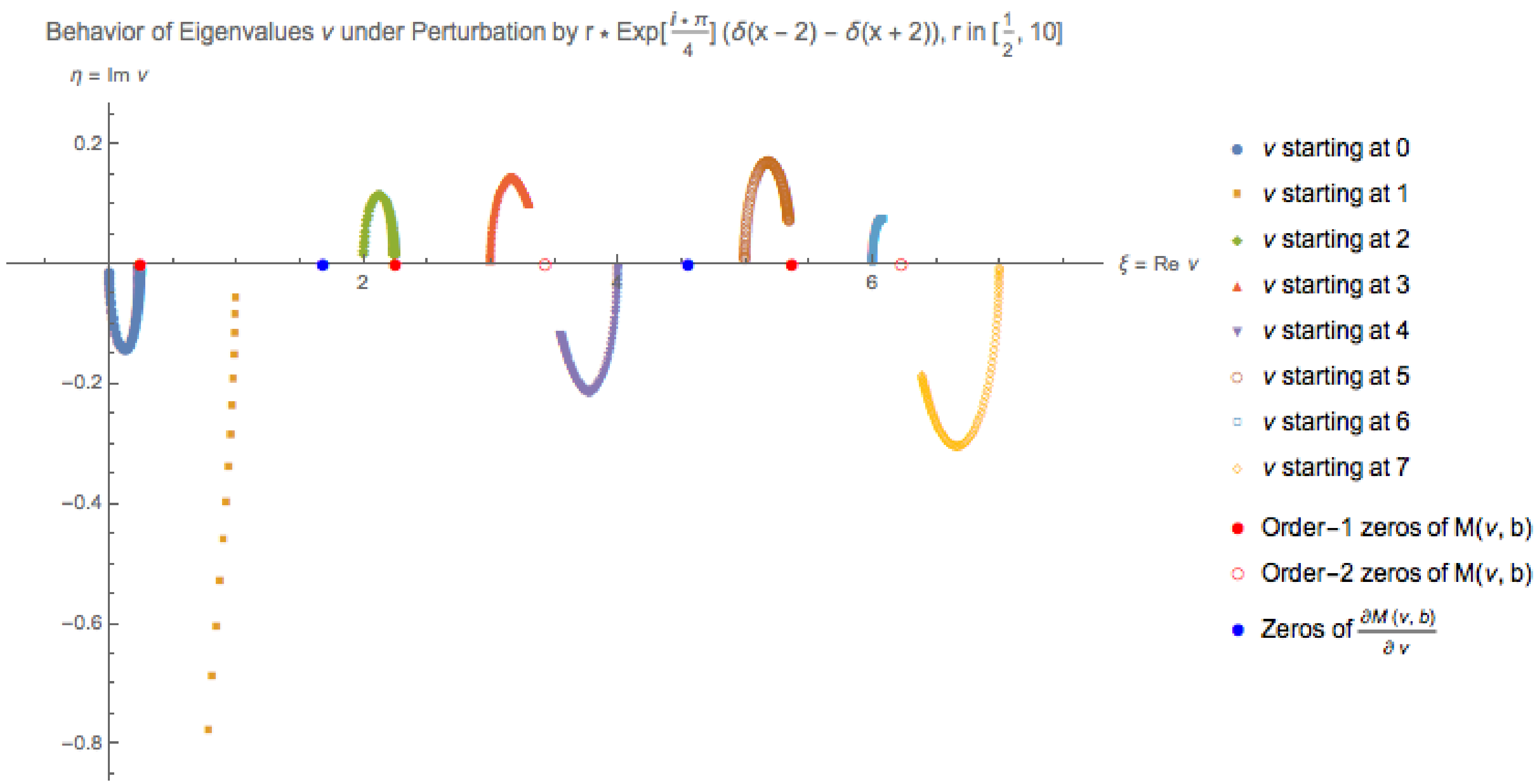

Figure 5.5: Evolution of the small eigenvalues of of $L_{\mathrm{PC}}(re^{i\pi/4}, 2)$ as r increases. Zeroes of $M(\nu; 2)$ and its first ν-derivative are marked.

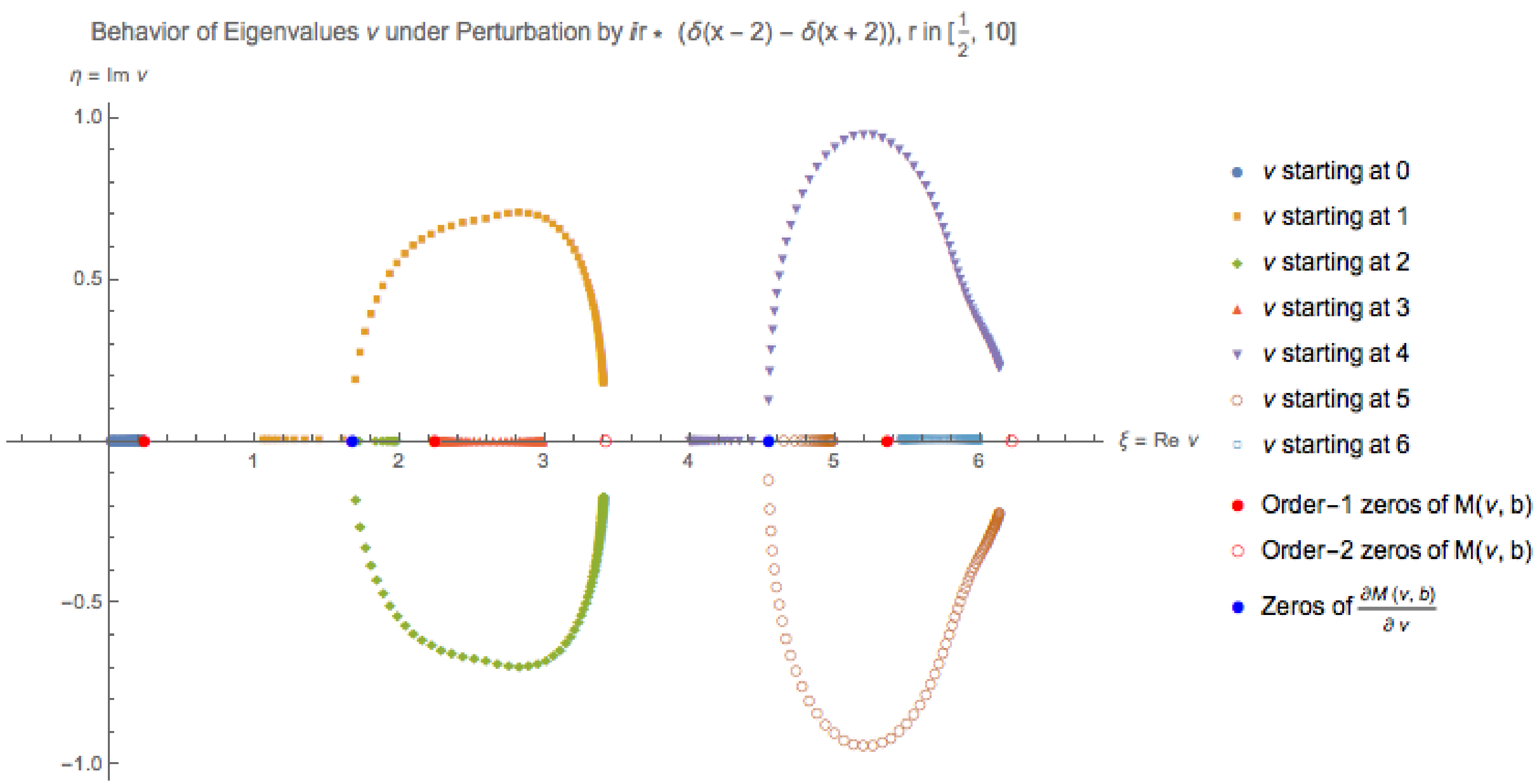

Figure 5.6: Evolution of the small eigenvalues of $L_{\text{PC}}(ir, 2)$ as r increases. Zeroes of $M(\nu; 2)$ and its first ν-derivative are marked.

CHAPTER 6

EXISTENCE OF NONREAL EIGENVALUES

In this section, we prove Theorem 3. To do so, we pay particular attention to the criterion for eigenvalues from Theorem 2,

$$M(\nu; b) = \frac{1}{z^2}, \tag{6.0.1}$$

where

$$M(\nu; b) = \frac{\Gamma(-\nu)\sqrt{2}}{\sqrt{\pi}} D_\nu^2(b) y_1(\nu; b) y_2(\nu; b). \tag{6.0.2}$$

We first show some basic properties of $M(\nu; b)$ and its component functions.

6.1 Preliminary Properties of $M(\nu; b)$

We start with the proof that we are dealing with a nonconstant function.

Lemma 6.1.1. *For all $b > 0$, the function $\nu \mapsto M(\nu; b)$ is holomorphic on the domain $\mathbb{C} \setminus \mathbb{N}_0$, and meromorphic on $\mathbb{C}$. Moreover, $\nu \mapsto M(\nu; b)$ is nonconstant on any open subset of its domain.*

Proof. Fix $b > 0$. By (6.0.2), and the fact that $\nu \to D_\nu(b)$, $\nu \to y_1(\nu; b)$, and $\nu \to y_2(\nu; b)$ are holomorphic by Proposition 2.5.2,

$$\nu \mapsto \widetilde{M}(\nu) b = [D_\nu(b)]^2 y_1(\nu; b) y_2(\nu; b)$$

is holomorphic on $\mathbb{C}$. Since

$$M(\nu; b) = \frac{\Gamma(-\nu)\sqrt{2}}{\sqrt{\pi}}\widetilde{M}(\nu)b,$$

and $\Gamma(-\nu)$ is meromorphic on $\mathbb{C}$ with poles at every $n \in \mathbb{N}_0$, it follows that $M(\nu; b)$ is holomorphic on $\mathbb{C} \setminus \mathbb{N}_0$ and meromorphic on $\mathbb{C}$, with poles at a subset of $\mathbb{N}_0$.

To demonstrate that $M(\nu; b)$ is nonconstant, we will show that $y_1(0, b)$, $y_2(0, b)$ and $D_0(b)$ are all nonzero, and so by (6.0.2), there is no cancellation of the pole of $\Gamma(-\nu)$ at $\nu = 0$, so $\nu \mapsto M(\nu; b)$ will have a pole at $\nu = 0$. By the rules of poles, then (e.g., [Con78, Chapter V, Defn. 1.3, p. 105], it follows that $\lim_{\nu \to 0} |M(\nu; b)| = \infty$, so $M(\nu; b)$ cannot possibly be a constant function in a punctured neighborhood of 0.

$y_1(0, b) \neq 0$. By (3.1.19a),

$$y_1(0, b) = 1 + (0)\frac{b^2}{2!} + (0)(2)\frac{b^4}{4!} + \cdots = 1 \neq 0. \tag{6.1.1}$$

$y_2(0, b) \neq 0$. Similarly, by (3.1.19b),

$$y_2(0, b) = b + (1)\frac{b^3}{3!} + (1)(3)\frac{b^5}{5!} + \cdots > b > 0. \tag{6.1.2}$$

$D_0(b) \neq 0$. By (3.2.5),it follows that

$$D_0(x) = \pi^{1/4} e^{-x^2/4}$$

which is clearly nonzero for any $x \in \mathbb{C}$, hence for $x = b$.

Hence, $\nu \mapsto M(\nu; b)$ is nonconstant in a punctured neighborhood centered at $\nu = 0$. Yet the domain of $M(\nu; b)$, $\mathbb{C} \setminus \mathbb{N}_0$ and hence is connected and path-connected; hence, by the Identity Principle (e.g., [Con78, Chapter IV, Thm. 3.7 and Cor. 3.8, p. 79]) $M(\nu; b)$ is not constant on any open set in $\mathbb{C} \setminus \mathbb{N}_0$. □

Corollary 6.1.2. *Suppose $\nu \mapsto M(\nu; b)$ has a removable singularity at some $n \in \mathbb{N}$. Then the analytic extension $\mathbf{M}(\nu; b)$ is nonconstant.*

We also make the following observations.

Lemma 6.1.3. *Fix $b > 0$ and $\nu \in \mathbb{R}$ (as opposed to $\mathbb{C}$). Then $y_1(\nu; b)$, $y_2(\nu; b)$, and $D_\nu(b)$ are real. If in addition, $\nu \notin \mathbb{N}_0$, $\Gamma(-\nu)$ and $M(\nu; b)$ are real. Also, if $M(\nu; b)$ analytically continues to any $n \in \mathbb{N}$, the analytic extension $\mathbf{M}(\nu; b)$ satisfies $\mathbf{M}(n; b) \in \mathbb{R}$.*

Proof. Fix $b \in \mathbb{R}$ and $\nu \in \mathbb{R}$. Note that by (3.1.19a) and (3.1.19b), $y_1(\nu; b)$ and $y_2(\nu; b)$ are real-valued by ν and b being real. By the product definition of $\dfrac{1}{\Gamma(\zeta)}$, as in, e.g., B. Ya. Levin's text [Lev96, Lecture 5, p. 32],

$$\frac{1}{\Gamma(\zeta)} = \zeta e^{\gamma\zeta} \prod_{j=1}^{\infty} \left(1 + \frac{\zeta}{j}\right) e^{-\zeta/j}, \tag{6.1.3}$$

where γ is the Euler-Mascheroni constant, a real constant. Thus, for ζ real, $\dfrac{1}{\Gamma(\zeta)}$ is a product of real numbers and hence is real. Since the transformations $\nu \mapsto -\dfrac{\nu}{2}$ and $\nu \mapsto -\dfrac{\nu}{2} + \dfrac{1}{2}$ preserve the real line, it is clear from (3.1.22) that $D_\nu(b)$ is real-valued by ν, b real.

If $\nu \notin \mathbb{N}_0$, by (6.1.3), $\Gamma(-\nu)$ is therefore real-valued by $\nu \in \mathbb{R} \setminus \mathbb{N}_0$. Thus, by (6.0.2), $M(\nu; b)$ is real-valued.

Finally, if $\nu \mapsto M(\nu; b)$ has a removable singularity at $\nu = n$, the extended function

$\mathbf{M}(\nu; b)$ is of course real-valued for $\nu \in \mathbb{R} \setminus \mathbb{N}_0$, and by continuity of the extension,

$$\begin{aligned} \mathbf{M}(n; b) &= \lim_{\substack{\epsilon \to 0^+ \\ \epsilon > 0}} \mathbf{M}(n + \epsilon; b) \\ &= \lim_{\substack{\epsilon \to 0^+ \\ \epsilon > 0}} M(n + \epsilon; b) \end{aligned} \tag{6.1.4}$$

and thus $\mathbf{M}(n; b)$ is the limit of real values and hence is real. □

Corollary 6.1.4. *Fix $b > 0$. Any power-series expansion for $M(\nu; b)$ centered at any $\nu \in \mathbb{R} \setminus \mathbb{N}_0$ must have real coefficients. The same is true for any analytic extension $\mathbf{M}(\nu; b)$ at any real number in its domain of analyticity.*

6.2 Zeroes outside $\mathbb{N}_0$ of $\nu \mapsto M(\nu; b)$

We now show that the existence of zeroes of $\nu \mapsto M(\nu; b)$, more specifically of $\nu \mapsto D_\nu(b)$, solves the problem.

Proposition 6.2.1. *Fix $b > 0$. If there exists $\mu \in \mathbb{C}$ such that μ is a zero of $\nu \mapsto D_\nu(b)$, then $\mu \in \mathbb{R}^+ = (0, \infty)$ and the zero is simple. Moreover, for sufficiently large $r > 0$, $L_{PC}(ir, b)$ has 2 nonreal eigenvalues in a neighborhood of ν.*

For technical reasons, however, we choose to split the proof into the case of zeroes of $M(\nu; b)$ (including $D_\nu(b)$), and zeroes of the extension at any nonnegative integers. We handle the first case here. First, for any fixed $\mu \in \mathbb{C} \setminus \mathbb{N}_0$ with $M(\mu; b) = 0$, we may apply Proposition 2.6.3 to the function $\nu \mapsto M(\nu; b)$ at the point μ, and get:

Proposition 6.2.2. *Let $b > 0$, and fix $z \in \mathbb{C} \setminus \{0\}$; let $z = re^{i\theta}$, $r \in \mathbb{R}^+$, $\theta \in [-\pi, \pi)$. Suppose $\mu \in \mathbb{C} \setminus \mathbb{N}_0$ is a zero of $M(\nu; b)$ of order m, $m \in \mathbb{N}$. Suppose the power-series expansion of $M(\nu; b)$ in ν centered at $\nu = \mu$ is given as*

$$M(\nu; b) = \sum_{j=m}^{\infty} c_j (\nu - \mu)^j, \tag{6.2.1}$$

where $c_m \neq 0$; hence, let $c_m = \rho e^{i\psi}$, $\rho \in \mathbb{R}^+$, $\psi \in [-\pi, \pi)$.

We take particular roots of z and c_m as follows. Let $\mathbf{z} = r^{1/m} e^{i\theta/m}$ denote a particular nth root of z; we note that an mth root of $\frac{1}{z^2}$ is $\frac{1}{\mathbf{z}^2} = r^{-2/m} e^{-2i\theta/m}$, and the other mth roots are the above figure mutliplied by $e^{2\pi ik/m}$, $k = 1, \ldots, n-1$. Also, let $\mathbf{c} = \rho^{1/m} e^{i\psi/m}$ denote a particular mth root of c_m.

Then for r large enough, there exists m solutions $\{\nu_k\}_{k=0}^{m-1} = \{\nu_k(z)\}_{k=0}^{m-1}$ to (6.0.1), i.e.,

$$M(\nu; b) = \frac{1}{z^2},$$

for ν in some neighborhood of μ, with leading-order expansion given by

$$\begin{aligned} \nu_k(re^{i\theta}) &= \mu + \frac{1}{\mathbf{z}^2 \mathbf{c}} \exp\left(\frac{2\pi ik}{m}\right) + O\left(\mathbf{z}^{-4/m}\right) \\ &= \mu + \frac{1}{r^{2/m}\rho^{1/m}} \exp\left(i\left[\frac{-2\theta - \psi + 2\pi ik}{m}\right]\right) + \Theta\left(r^{-4/m}\right), \\ & \qquad k = 0, 1, \ldots m-1. \end{aligned} \tag{6.2.2}$$

In particular, as $r \to \infty$, $\nu_k \to 0$ for all k; indeed,

$$|\nu_k - \mu| = \Theta(\frac{1}{r^{2/m}}), \quad k = 0, 1, \ldots m-1. \tag{6.2.3}$$

As $r \to \infty$, for fixed θ, we have that

$$\lim_{r\to\infty} \frac{\nu_k - \mu}{|\nu_k - \mu|} = \exp\left(i\left[\frac{-2\theta - \psi + 2\pi k}{m}\right]\right), \quad k = 0, 1, \ldots m-1. \tag{6.2.4}$$

6.2.1 Restrictions From Real z

Applying Proposition 6.2.2 to the z real case, we discover the following statements about the zeroes of $M(\nu; b)$.

Lemma 6.2.3. *Fix $b > 0$. If $\mu \in \mathbb{C} \setminus \mathbb{N}_0$ is a zero of $\nu \mapsto M(\nu; b)$ then $\mu \in \mathbb{R} \setminus \mathbb{N}_0$.*

Proof. Fix $b > 0$, and let $\mu \in \mathbb{C} \setminus \mathbb{N}_0$ be a zero of $\nu \mapsto M(\nu; b)$, of order m. As a positive real parameter r tends to ∞ along the positive real axis, by Proposition 6.2.2, there exists an solution $\nu_0 = \nu_0(r)$ to (6.0.1) satisfying (6.2.3). Fix $R > 0$ such that we can find positive constants c, C so that by (6.2.3), for all $r > R$,

$$\frac{c}{r^{2/n}} < |\nu_0(z) - \mu| < \frac{C}{r^{2/n}}. \tag{6.2.5}$$

Let $\{r_j\}_{j=1}^{\infty}$ be defined by

$$r_j = 2^j R, \quad j \in \mathbb{N}. \tag{6.2.6}$$

Then define

$$\nu_0^{(j)} := \nu_0(r_j). \tag{6.2.7}$$

By (6.2.5), we have that for $j \in \mathbb{N}$,

$$|\nu_0^{(j)} - \mu| < \frac{C}{R^{2/n} 2^{j/n}}$$

so by the Squeeze Theorem,

$$\lim_{j \to \infty} |\nu_0^{(j)} - \mu| = 0,$$

i.e.,

$$\lim_{j \to \infty} \nu_0^{(j)} = \mu. \tag{6.2.8}$$

Yet $\nu_0^{(j)}$ are solutions to the $z = r_j$ case of (6.0.1), so by Theorem 2, they must be eigenvalues of $L_{\mathrm{PC}}(r_j, b)$. Yet by the r_j real, $\nu_0^{(j)}$ must be real by Proposition 3.3.1. Thus, μ, being the limit of real numbers, must be real. □

Thus, the simple fact that the eigenvalues of $L_{\mathrm{PC}}(z, b)$ approach the zeroes of $\nu \mapsto M(\nu; b)$ as $|z| \to \infty$, when applied to z real, forces the zeroes real. We now demonstrate that the zeros are positive real.

Lemma 6.2.4. *Fix $b > 0$. The zeroes of $\nu \mapsto D_\nu(b)$, $\nu \mapsto y_1(\nu; b)$, and $\nu \mapsto y_2(\nu; b)$ are in $\mathbb{R}^+$. Also, the zeroes of $\nu \mapsto M(\nu; b)$ are in $\mathbb{R}^+$.*

Proof. Fix $b > 0$. We know by Lemma 5.4.3 that any zero of $\nu \mapsto y_1(\nu; b)$, $\nu \mapsto y_2(\nu; b)$, or $\nu \mapsto D_\nu(b)$ that is not in $\mathbb{N}_0$ must be a zero of $\nu \mapsto M(\nu; b)$. By Lemma 6.2.3, such zeroes must be real. Of course, any zero of $\nu \mapsto y_1(\nu; b)$, $\nu \mapsto y_2(\nu; b)$, or $\nu \mapsto D_\nu(b)$ which is in $\mathbb{N}_0$ is real; hence, the zeroes of $\nu \mapsto D_\nu(b)$, $\nu \mapsto y_1(\nu; b)$, and $\nu \mapsto y_2(\nu; b)$ are in $\mathbb{R}$.

We now show that these functions are positive on the nonpositive real ν-axis, for $b > 0$.

$\nu \mapsto y_1(\nu; b)$. Since for $\nu \leq 0$, $0 \leq (-\nu) \leq (-\nu + 2) \leq (-\nu + 4) \leq \cdots$, by (3.1.19a) it is clear that for any $\nu \leq 0$ and any real b, $y_1(\nu; b) \geq 1$.

$\nu \mapsto y_2(\nu; b)$. Similarly, for $\nu \leq 1$, $0 \leq (-\nu + 1) \leq (-\nu + 1)(-\nu + 3) \leq \cdots$, and by $b > 0$, $\dfrac{b^{2j+1}}{(2j+1)!} > 0$, so by (3.1.19b), $y_2(\nu; b) \geq b > 0$.

$\nu \mapsto D_\nu(b)$. If $\nu = 0$, then again, by (3.2.5), $D_0(x)$ is a nonzero constant multiple of $e^{-x^2/4}$, so for any $b > 0$, $D_0(b)$ is positive real.

For $\nu < 0$, by (3.1.2) and [Tem14, Section 12.5(i), (12.5.1)], we have that

$$D_\nu(b) = \frac{e^{-b^2/4}}{\Gamma(-\nu)} \int_0^\infty t^{-\nu} e^{-\frac{1}{2}t^2 - bt}\, dt \tag{6.2.9}$$

and hence it follows that $D_\nu(b)$ is the integral of a nonnegative integrand, strictly positive for $t > 0$, multiplied by the positive $\exp\left[-\dfrac{b^2}{4}\right]$ and $\dfrac{1}{\Gamma(-\nu)}$, where $\nu < 0$ implies $-\nu > 0$ so that $\Gamma(-\nu)$ is positive and finite. Thus, for $\nu < 0$, any real b, $D_\nu(b)$ is positive.

By Lemma 5.4.3, $\nu \mapsto M(\nu; b)$ is therefore nonzero on the negative real axis, and by the proof of Lemma 6.1.1, $M(\nu; b)$ has a pole at 0. Hence, any real zero of $\nu \mapsto M(\nu; b)$

is positive real. By Lemma 6.2.3, every zero of $\nu \mapsto M(\nu; b)$ is real, so by the above, every zero of $\nu \mapsto M(\nu; b)$ is positive real. $\square$

Now, we use the angles of approach, as guaranteed in (6.2.4), to limit the orders of the zeros.

Lemma 6.2.5. *Fix $b > 0$, and suppose that $\mu \in \mathbb{R}^+ \setminus \mathbb{N}_0$ is a zero of $\nu \mapsto M(\nu; b)$ of order m. Then $m \leq 2$. Moreover, if $m = 2$, then in the power-series expansion at $\nu = \mu$, i.e. (6.2.1), $c_2 > 0$.*

Proof. Fix $b > 0$, and let $\mu \in \mathbb{R}^+ \setminus \mathbb{N}_0$ be a zero of $\nu \mapsto M(\nu; b)$, of order m. By Proposition 6.2.2, for $z = r > 0$, r large enough, we have solutions $\{\nu_k\}_{k=0}^{m-1} = \{\nu_k(r)\}_{k=0}^{m-1}$ to (6.0.1) satisfying (6.2.4).

We first claim that for r large enough, the ν_k are real. For ν_k satisfies the $z = r$ case of (6.0.1), equivalently (1.0.18), so by Theorem 2, the $\nu_k(r)$ are eigenvalues of $L_{\mathrm{PC}}(r, b)$. Yet by Proposition 3.3.1, since r is real, all eigenvalues of $L_{\mathrm{PC}}(r, b)$ are real.

Hence, in (6.2.4),, i.e.,

$$\lim_{r \to \infty} \frac{\nu_k - \mu}{|\nu_k - \mu|} = \exp\left(i \left[\frac{-2\theta - \psi + 2\pi k}{n} \right] \right), \quad k = 0, 1, \ldots m-1, \tag{6.2.10}$$

for each $k \in \{0, 1, \ldots, m-1\}$, $\dfrac{\nu_k(r) - \mu}{|\nu_k(r) - \mu|}$ is a real expression by both $\nu_k(r)$ and μ real, so by the limits of real functions being real,

$$\exp\left(i \left[\frac{-2\theta - \psi + 2\pi k}{n} \right] \right), \quad k = 0, 1, \ldots, m-1. \tag{6.2.11}$$

In other words,

$$\frac{-2\theta - \psi + 2\pi k}{n} \in \pi\mathbb{Z}, \quad k = 0, 1, \ldots, m-1. \tag{6.2.12}$$

Yet we may simplify (6.2.12). For one thing, since $z = r$ is positive real, $\theta = 0$. Moreover, since we now know $\mu \in \mathbb{R}$, we may use Corollary 6.1.4 to note that in the power-series expansion for $M(\nu; b)$ at $\nu = \mu$, i.e.,

$$M(\nu; b) = \sum_{j=m}^{\infty} c_j (\nu - \mu)^j, \tag{6.2.13}$$

all c_j are real. In particular, c_m is real, so $\psi \in \{-\pi, 0\}$. We have two cases.

Case 1: $\psi = 0$. Then we have that

$$\frac{2\pi k}{m} \in \pi\mathbb{Z}, \quad k = 0, 1, \ldots, m-1.$$

If $m = 1$, we have a zero of order 1. Else, $m \geq 2$, and the $k = 1$ case is available, so

$$\frac{2\pi}{m} \in \pi\mathbb{Z}, \quad \text{or} \quad \frac{2}{m} \in \mathbb{Z}.$$

Since m is a positive integer, the only possibility is $m = 2$.

Case 2: $\psi = -\pi$. Then we have that

$$\frac{2\pi k - \pi}{m} \in \pi\mathbb{Z}, \quad k = 0, 1, \ldots, m-1.$$

From the $k = 0$ case we have that

$$\frac{-\pi}{m} \in \pi\mathbb{Z}, \quad \text{or} \quad \frac{-1}{m} \in \mathbb{Z},$$

and by m a nonnegative integer, the only possibility is $m = 1$.

In all cases, $m \leq 2$, and in the case $m = 2$, $c_2 < 0$ cannot occur. □

6.2.2 Consequences for Imaginary z

Proposition 6.2.6. *Fix $b > 0$, and suppose that $\mu \in \mathbb{R}^+ \setminus \mathbb{N}_0$ is a zero of $\nu \mapsto M(\nu; b)$ of order 2. Then as $r \to \infty$, $r > 0$,*

$$M(\nu; b) = \frac{1}{z^2} = -\frac{1}{r^2}$$

has 2 nonreal zeroes in a neighborhood of μ.

Proof. Fix $b > 0$, and suppose that $\mu \in \mathbb{R}^+ \setminus \mathbb{N}_0$ is a zero of $\nu \mapsto M(\nu; b)$ of order 2. Then for $z = ir = re^{i\pi/2}$, we have that in the language of Proposition 6.2.2, $\theta = \frac{\pi}{2}$. Moreover, in the power-series expansion at μ,

$$M(\nu; b) = \sum_{j=2}^{\infty} c_j (\nu - \mu)^j, \tag{6.2.14}$$

$c_2 > 0$ by Lemma 6.2.5, so $\psi = 0$. Then by Proposition 6.2.2, we have that the two guaranteed solutions of $M(\nu; b) = \frac{1}{z^2} = -\frac{1}{r^2}$ satisfy (in this case $\theta = \frac{\pi}{2}$)

$$\lim_{r \to \infty} \frac{\nu_k(ir) - \mu}{|\nu_k(ir) - \mu|} = \exp(i \left[-\frac{\pi}{2} + \pi k \right]), \quad k = 0, 1. \tag{6.2.15}$$

The right-hand side is simply $(-1)^k(-i)$, $k - 0, 1$. Thus, since the ratio between $\nu_0 - \mu$ and its absolute value is approximately $\pm i$, eventually $\arg(\nu_0 - \mu)$ is at least $\frac{\pi}{4}$ away from any integer multiple of π, so $\nu_0 = \nu_0 - \mu + \mu$ is the sum of a nonreal number $\nu_0 - \mu$ and a real number μ, hence is nonreal. Similarly for ν_1. □

Adding in Theorem 2, we have:

Corollary 6.2.7. *Fix $b > 0$, and suppose that $\mu \in \mathbb{R}^+ \setminus \mathbb{N}_0$ is a zero of $\nu \mapsto M(\nu; b)$ of order 2. Then for $r > 0$, r sufficiently large, $L_{PC}(ir, b)$ has 2 nonreal eigenvalues in a neighborhood of μ.*

To restrict the discussion from $\nu \mapsto M(\nu; b)$ to the case $\nu \mapsto D_\nu(b)$, we require a lemma.

Lemma 6.2.8. *Fix $b > 0$, and suppose that for some $\mu \in \mathbb{C} \setminus \mathbb{N}_0$, $D_\mu(b) = 0$. Then $y_1(\mu; b) \neq 0$ and $y_2(\mu; b) \neq 0$.*

Proof. First, we recall from Lemma 3.1.2 that $y_1(\nu; b)$ and $y_2(\nu; b)$ cannot be simultaneously zero. Now suppose, by way of contradiction, that $D_\mu(b) = 0$ and $y_1(\mu; b) = 0$ for $\mu \notin \mathbb{N}_0$. Then by (3.1.22),

$$0 = -\frac{2^{(\mu+1)/2}\sqrt{\pi}}{\Gamma\left(-\frac{\mu}{2}\right)} y_2(\mu; b)$$

Yet certainly $-2^{(\mu+1)/2}\sqrt{\pi} \neq 0$, and since $\mu \notin \mathbb{N}_0$, $-\dfrac{\mu}{2} \notin \mathbb{N}_0$ and so $\dfrac{1}{\Gamma\left(-\dfrac{\mu}{2}\right)} \neq 0$. Therefore, $y_2(\mu; b) = 0$. Thus, $y_1(\mu; b) = 0$ and $y_2(\mu, b) = 0$. Contradiction.

Thus, for $\mu \notin \mathbb{N}_0$, $D_\mu(b)$ and $y_1(\mu; b)$ cannot be simultaneously 0. Similarly, if $\mu \notin \mathbb{N}_0$, $D_\mu(b)$ and $y_2(\mu; b)$ cannot be simultaneously 0. □

We now prove that a simple case of finding zeroes of $\nu \mapsto M(\nu; b)$ is finding noninteger zeroes of $\nu \mapsto D_\nu(b)$.

Corollary 6.2.9. *Fix $b > 0$. Then if $\mu \in \mathbb{R}^+ \setminus \mathbb{N}_0$ is a zero of $\nu \mapsto D_\nu(b)$, then the zero is simple. In addition, if r is large enough, then there are two nonreal eigenvalues of $L_{PC}(ir, b)$ in a neighborhood of μ.*

Proof. Fix $b > 0$, and suppose that $\mu \in \mathbb{R}^+ \setminus \mathbb{N}_0$ is a zero of $\nu \mapsto D_\nu(b)$ of order m, $m \geq 1$. Then we know that $\nu \mapsto [D_\nu(b)]^2$ is a factor of $M(\nu; b)$ as in (6.0.2), and $M(\nu; b)$ is analytic except for the (possible) poles at $\mathbb{N}_0$. Thus, $\nu \mapsto M(\nu; b)$ has a zero at μ of order $2m$. Yet by Lemma 6.2.5, we know that any zero of $\nu \mapsto M(\nu; b)$ is of order at most 2. Hence, $m = 1$ and the zero is simple. Moreover, by Lemma 6.2.8,

$y_1(\mu; b) \neq 0$ and $y_2(\mu; b) \neq 0$, and of course the Gamma function times a constant has no zeroes. Hence, $\nu \mapsto M(\nu; b)$ has a zero of order exactly 2 at μ. We may invoke Corollary 6.2.7 to show the existence of the nonreal eigenvalues of $L_{\mathrm{PC}}(ir, b)$ in a neighborhood of μ. □

We therefore have almost proven Proposition 6.2.1, except that we still have the case of positive integer zeroes.

6.3 Zeroes in $\mathbb{N}$ of $\nu \mapsto M(\nu; b)$

We now consider the case when $\nu \mapsto M(\nu; b)$ has a removable singularity at $\nu = n$ in $\mathbb{N}_0$: by Lemma 6.1.1, of course we cannot make the extension at $n = 0$, so we may restrict to $n \in \mathbb{N}$. By our work in Section 5.5, we know that the singularity at $\nu = n$ is removable whenever

$$\widetilde{M}(\nu; b) = [D_\nu(b)]^2 \, y_1(\nu; b) y_2(\nu; b) \tag{6.3.1}$$

has a zero at n, but by Lemma 5.5.3 we need n to be a zero of $\nu \mapsto \widetilde{M}(\nu; b)$ of order $m + 1$ to force the analytic extension $\nu \mapsto \mathbf{M}(\nu; b)$ of $\nu \mapsto M(\nu; b)$ to have a zero of order m. In such a case, by Corollary 6.1.2, the analytic extension is nonconstant in a neighborhood of $n \in \mathbb{N}$, and by Corollary 6.1.4, in the power-series expansion of $\mathbf{M}(\nu; b)$ at $\nu = n$,

$$\mathbf{M}(\nu; b) = \sum_{j=m}^{\infty} c_j (\nu - n)^j, \tag{6.3.2}$$

all c_k are real-valued. We thus have the analogue of Proposition 6.2.2, which follows purely from the properties of zeroes of analytic functions.

Proposition 6.3.1. *Let $b > 0$, and fix $z \in \mathbb{C} \setminus \{0\}$; let $z = re^{i\theta}$, $r \in \mathbb{R}^+$, $\theta \in [-\pi, \pi)$. Suppose that $n \in \mathbb{N}_0$ is such that $\nu \mapsto \widetilde{M}(\nu; b)$ has a zero of order $m + 1$, $m \geq 1$, at*

$\nu = n$; hence $\mathbf{M}(\nu; b)$ is defined and analytic in a neighborhood of $\nu = n$, with a zero of order m. Suppose the power-series expansion of $M(\nu; b)$ in ν centered at $\nu = n$ is given as in (6.3.2), so that $c_m \neq 0$; hence, let $c_m = \rho e^{i\psi}$, $\rho \in \mathbb{R}^+$, $\psi \in \{-\pi, 0\}$. Moreover, let $\mathbf{c} = \rho^{1/m} e^{i\psi/m}$ denote a particular mth root of c_m.

Then for r large enough, there exists m distinct solutions $\{\nu_k\}_{j=0}^{m-1} = \{\nu_k(z)\}_{j=0}^{m-1}$ to

$$\mathbf{M}(\nu; b) = \frac{1}{z^2} \tag{6.3.3}$$

for ν in some punctured neighborhood of n, with leading-order expansion given by

$$\begin{aligned} \nu_k &= n + \frac{1}{\mathbf{z}^2 \mathbf{c}} \exp\left(\frac{2\pi i k}{m}\right) + O\left(\mathbf{z}^{-4/m}\right) \\ &= n + \frac{1}{r^{2/m} \rho^{1/m}} \exp\left(i \left[\frac{-2\theta - \psi + 2\pi i k}{m} \right] \right) + \Theta\left(r^{-4/m}\right), \quad k = 0, 1, \ldots n-1. \end{aligned} \tag{6.3.4}$$

In particular, as $r \to \infty$, $\nu_k \to n$ for all k; indeed,

$$|\nu_k - n| = \Theta(\frac{1}{r^{2/m}}), \quad k = 0, 1, \ldots m-1. \tag{6.3.5}$$

As $r \to \infty$, for fixed θ, we have that

$$\lim_{r \to \infty} \frac{\nu_k - n}{|\nu_k - n|} = \exp\left(i \left[\frac{-2\theta - \psi + 2\pi k}{m} \right] \right), \quad k = 0, 1, \ldots m-1. \tag{6.3.6}$$

The constraint (6.3.5) is quite important, for it reminds us that the ν_k, $k = 0, 1, \ldots m-1$ are not equal to n, yet close enough to n not to be another integer. Yet for ν in a punctured disk of radius, say, $\frac{1}{2}$ around n, $\mathbf{M}(\nu; b) = M(\nu; b)$, so we have that the ν_k, being non-$\mathbb{N}_0$ solutions to (6.3.3), are also non-$\mathbb{N}_0$ solutions to (5.4.9), i.e.

$$M(\nu; b) = \frac{1}{z^2}.$$

Hence, Theorem 2 informs us that the ν_k are indeed eigenvalues of $L_{\mathrm{PC}}(z, b)$.

Corollary 6.3.2. *Let $b > 0$, and suppose $n \in \mathbb{N}_0$ is such that $\nu \mapsto \widetilde{M}(\nu; b)$ has a zero of order $m + 1$, $m \geq 1$, at $\nu = n$; hence $\mathbf{M}(\nu; b)$ is defined and analytic in a neighborhood of $\nu = n$, with a zero of order m. Then for $z = re^{i\theta}$, $\theta \in [-\pi, \pi)$, $r > 0$ and large enough, the solutions $\{\nu_k(z)\}_{j=0}^{n-1}$ are solutions to* (6.3.3)*, guaranteed by Proposition 6.3.1, are not in $\mathbb{N}_0$. Hence, they are also solutions to* (6.0.1)*, equivalently* (1.0.20)*, and hence by Theorem 2 they are eigenvalues of $L_{PC}(z, b)$.*

6.3.1 Restrictions from Real z

Corollary 6.3.2 allows us to get the analogue of Lemma 6.2.5.

Lemma 6.3.3. *Fix $b > 0$, and suppose that there exists $n \in \mathbb{N}_0$ such that $\nu \mapsto \widetilde{M}(\nu; b)$ has a zero of order $m+1$, $m \geq 1$, at $\nu = n$, so that $\mathbf{M}(\nu; b)$ is defined and analytic in a neighborhood of $\nu = n$, and $\mathbf{M}(\nu; b)$ has a zero of order m at $\nu = n$. Then $m \leq 2$; if in fact $m = 2$, and the power-series expansion at $\nu = n$ is*

$$\mathbf{M}(\nu; b) = \sum_{k=2}^{\infty} c_j (\nu - n)^j, \tag{6.3.7}$$

then $c_2 > 0$.

Proof. Fix $b > 0$, and suppose $n \in \mathbb{N}_0$ is such that $\nu \mapsto \widetilde{M}(\nu; b)$ has a zero of order $m + 1$, $m \geq 1$, at $\nu = n$; hence $\mathbf{M}(\nu; b)$ is defined and analytic in a neighborhood of $\nu = n$, with a zero of order m. By Proposition 6.3.1, for $z = r > 0$, r large enough, we have solutions $\{\nu_k\}_{k=0}^{m-1} = \{\nu_k(r)\}_{k=0}^{m-1}$ to (6.0.1) satisfying (6.3.6).

We note for r large enough, the $\nu_k(r)$ are real. By Corollary 6.3.2, for r large enough the $\nu_k(r)$ are eigenvalues of $L_{\mathrm{PC}}(r,b)$. Yet by Proposition 3.3.1, since r is real, all eigenvalues of $L_{\mathrm{PC}}(r,b)$ are real.

Hence, in (6.3.6), i.e.,

$$\lim_{r\to\infty} \frac{\nu_k - n}{|\nu_k - n|} = \exp\left(i \left[\frac{-2\theta - \psi + 2\pi k}{m} \right] \right), \quad k = 0, 1, \dots m-1, \tag{6.3.8}$$

for each $k \in \{0, 1, \dots, m-1\}$, $\dfrac{\nu_k(r) - n}{|\nu_k(r) - n|}$ is a real expression by both $\nu_k(r)$ and n real, so by the limits of real functions being real,

$$\exp\left(i \left[\frac{-2\theta - \psi + 2\pi k}{m} \right] \right), \quad k = 0, 1, \dots, m-1. \tag{6.3.9}$$

In other words, (6.2.12) holds. The remainder of the proof proceeds as in the proof of Lemma 6.2.5. □

6.3.2 Consequences for Imaginary z

We now have the analogues of Section 6.2.2.

Proposition 6.3.4. *Fix $b > 0$, and suppose $n \in \mathbb{N}_0$ is such that $\nu \mapsto \widetilde{M}(\nu; b)$ has a zero of order 3 at $\nu = n$; hence $\mathbf{M}(\nu; b)$ is defined and analytic in a neighborhood of $\nu = n$, with a zero of order 2. Then as $r \to \infty$, $r > 0$,*

$$M(\nu; b) = \frac{1}{z^2} = -\frac{1}{r^2}$$

has 2 nonreal zeroes in a neighborhood of μ.

Proof. Fix $b > 0$, and suppose that $n \in \mathbb{N}_0$ is such that $\nu \mapsto \widetilde{M}(\nu; b)$ has a zero of order 3 at $\nu = n$, hence $\mathbf{M}(\nu; b)$ is defined and analytic in a neighborhood of $\nu = n$, with a zero of order 2. Let $z = ir = re^{i\pi/2}$ for $r > 0$; we have that in the language of Proposition 6.3.1, $\theta = \frac{\pi}{2}$. Moreover, in the power-series expansion at n,

$$M(\nu; b) = \sum_{j=2}^{\infty} c_j(\nu - n)^j, \tag{6.3.10}$$

$c_2 > 0$ by Lemma 6.2.5, so $\psi = 0$. Then by Proposition 6.3.1, we have that the two guaranteed solutions of $M(\nu; b) = \frac{1}{z^2} = -\frac{1}{r^2}$ satisfy

$$\lim_{r\to\infty} \frac{\nu_k(ir) - n}{|\nu_k(ir) - n|} = \exp(i\left[-\frac{\pi}{2} + \pi k\right]), \quad k = 0, 1. \tag{6.3.11}$$

The right-hand side is simply $(-1)^k(-i)$, $k-0, 1$. Thus, since the ratio between $\nu_0 - \mu$ and its absolute value is approximately $\pm i$, eventually $\arg(\nu_0 - \mu)$ is at least $\frac{\pi}{4}$ away from any integer multiple of π, so $\nu_0 = \nu_0 - n + n$ is the sum of a nonreal number $\nu_0 - n$ and the integer n, hence nonreal. Similarly for ν_1. □

Adding in Corollary 6.3.2, we have:

Corollary 6.3.5. *Fix $b > 0$, and suppose $n \in \mathbb{N}_0$ is such that $\nu \mapsto \widetilde{M}(\nu; b)$ has a zero of order 3 at $\nu = n$; hence $\mathbf{M}(\nu; b)$ is defined and analytic in a neighborhood of $\nu = n$, with a zero of order 2. Then for $r > 0$, r sufficiently large, $L_{PC}(ir, b)$ has 2 nonreal eigenvalues in a neighborhood of μ.*

We again show that it suffices to restrict attention to (integer) zeroes of $\nu \mapsto D_\nu(b)$.

Corollary 6.3.6. *Fix $b > 0$. Then if $n \in \mathbb{N}$ is a zero of $\nu \mapsto D_\nu(b)$, then the zero is simple. Moreover, for $r > 0$ large enough, $L_{PC}(ir, b)$ has two nonreal eigenvalues in a neighborhood of n, in addition to the eigenvalue at n guaranteed by Corollary 5.5.1.*

Proof. Fix $b > 0$, and suppose that $n \in \mathbb{N}$ is a zero of $\nu \mapsto D_\nu(b)$ of order m, $m \geq 1$. Then by Corollary 5.5.4, the analytic extension in a neighborhood of n, $\mathbf{M}(\nu; b)$, has a zero of order at least $2m$ at $m = n$. Yet by Lemma 6.3.3, the zero of $\nu \mapsto \mathbf{M}(\nu; b)$ can be of order at most 2, so $m = 1$; in other words, the zero of $\nu \mapsto D_\nu(b)$ is simple.

Moreover, by Lemma 3.1.2, $y_1(n, b)$ and $y_2(n, b)$ cannot both be 0, but by $D_n(b) = 0$ and (5.5.3), at least one of them is 0, so exactly one of them is 0. Suppose n is even so that $y_1(n, b) = 0$. The order of the zero of $\nu \mapsto y_1(\nu, b)$ at $\nu = n$ must be exactly 1, otherwise the order of the zero of $\nu \mapsto \widetilde{M}(\nu; b)$ would be at least $2 + 2 = 4$, and the order of $\nu \mapsto \widetilde{M}(\nu; b)$ at $\nu = n$ would be at least 3, violating Lemma 6.3.3. Similarly if n is odd.

Hence, the order of the zero of $\nu \mapsto \widetilde{M}(\nu; b)$ at $\nu = n$ is exactly 3, and the zero of $\nu \mapsto \mathbf{M}(\nu; b)$ at $\nu = n$ is of order exactly 2. We may invoke Corollary 6.3.5 to get the two nonreal eigenvalues in a neighborhood of n. □

Combining Lemma 6.2.4, Corollary 6.2.9, and Corollary 6.3.6, we have Proposition 6.2.1.

Example 6.3.7. To demonstrate how this works in practice, we recall the example that $D_2(1) = y_1(2, 1) = 0$. Hence, in the $b = 1$ case, there is an eigenvalue at $\nu = 2$ by Corollary 5.5.1, and there are also 2 eigenvalues approaching $\nu = 2$ as $|z| \to \infty$. Numerically, it appears that for the case $z = ir$, the perturbations of the $\nu = 0$ and $\nu = 1$ eigenvalues approach $\nu = 2$, as shown in Figure 6.1

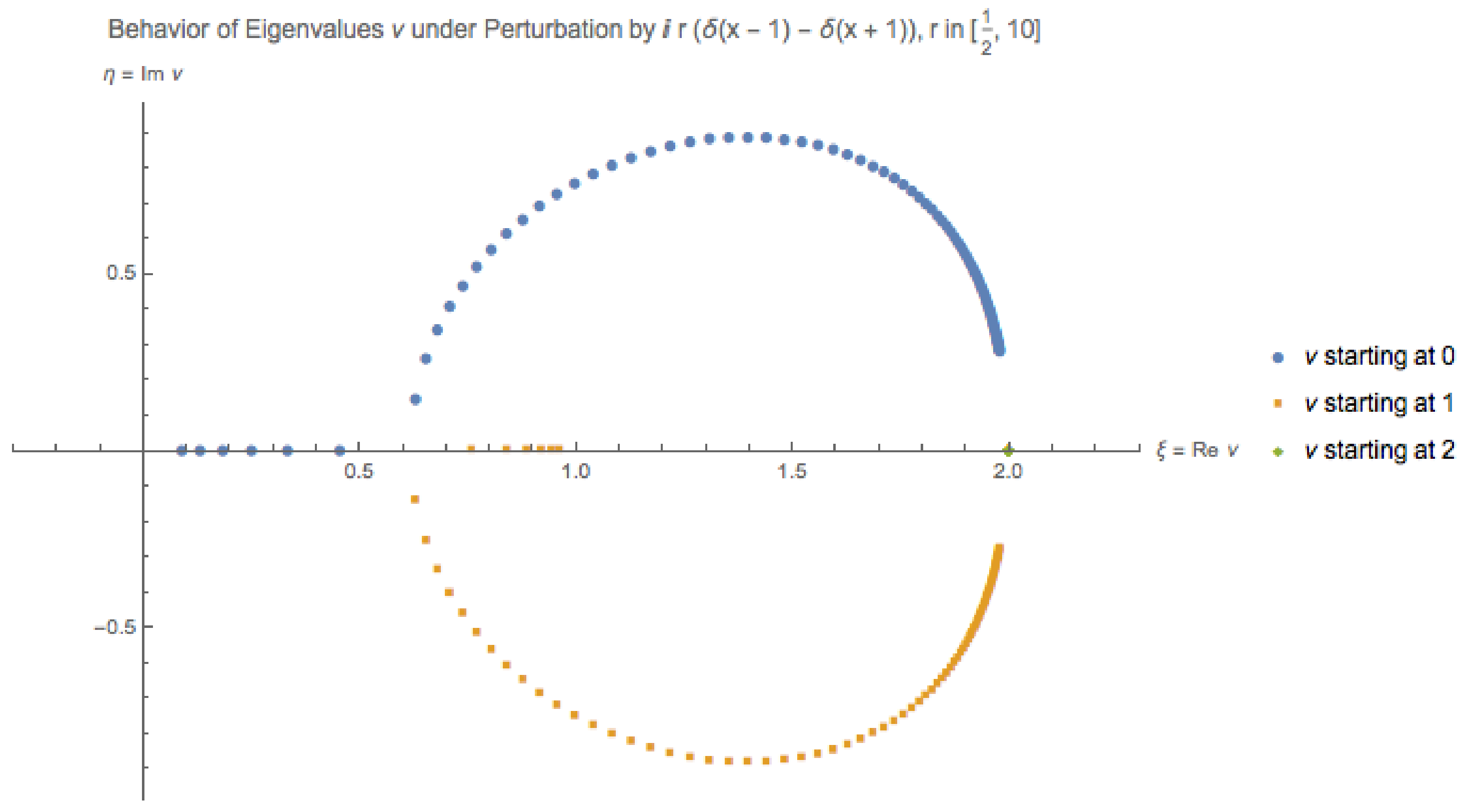

Figure 6.1: Evolution of the small eigenvalues of $L_{\mathrm{PC}}(r, 2)$ as r increases

6.4 Existence of zeroes of $\nu \mapsto D_\nu(b)$

To unconditionally show the existence of nonreal eigenvalues for $L_{\mathrm{PC}}(ir, b)$ with $r > 0$ large enough, by Proposition 6.2.1, we must show that for all $b > 0$, there exists ν with $D_\nu(b) = 0$. To show the existence of zeroes of $\nu \mapsto D_\nu(b)$, we note a paper of Dean, [Dea66], which studies the zeroes of the parabolic cylinder functions and states:

Proposition 6.4.1 ([Dea66, pp. 281–2])**.** *There exist countably many functions* $\{g_k(x)\}_{k=0}^{\infty}$*,* $g_k : [0, \infty) \to \mathbb{R}^+$*, such that for all* $x \in \mathbb{R}$ *and* $k \in \mathbb{N}_0$*,* $D_{g_k(x)}(x) = 0$ *and* $g_k(0) = 2k + 1$*.*

Dean's paper [Dea66] is quite terse, however, so we give a complete proof in Appendix C.

To prove the full extent of Theorem 3, we also need the following:

Lemma 6.4.2. *For all* $b > 0$*, for all* $j, k \in \mathbb{N}_0$*,* $g_j(b) = g_k(b)$ *if and only if* $j = k$*; hence, for each* $b > 0$ *there are countably many zeroes of* $\nu \mapsto M(\nu; b)$*.*

Proof. Suppose that for some $b > 0$, and for some $j, k \in \mathbb{N}_0$, $g_j(b) = g_k(b)$. Then by the Implicit Function Theorem applied in the vicinity of $(b, g_j(b))$, there is a *unique* function $g(x)$ such that $D_{g(x)}(x) = 0$ for x near b, and $g(b) = g_j(b) = g_k(b)$. In other words, $g_j(x) = g_k(x)$ in a neighborhood of b, call it (a, c). By Lemma C.1.4, $g(x)$ extends uniquely to a function on $(-\epsilon, c)$ for some $\epsilon > 0$. Yet by uniqueness of the extension, $g(x) = g_j(x) = g_k(x)$ on $[0, c)$, so

$$\begin{aligned} g_j(0) &= g_k(0) \\ 2j + 1 &= 2k + 1 \\ j &= k. \end{aligned} \tag{6.4.1}$$

□

Proof of Theorem 3. Fix $b > 0$. By Proposition 6.4.1, there are countably many numbers $\{g_k(b)\}_{k=0}^{\infty}$ with $g_k(b) = 0$. Fix $N \in \mathbb{N}$, and consider the subcollection $\{g_k(b)\}_{k=0}^{N}$ for some $N \in \mathbb{N}$. Then for each $g_k(b)$, there exists $R_k = R_k(b)$ such that for $r > R_k$, $L_{\mathrm{PC}}(ir, b)$ has two nonreal eigenvalues in a neighborhood of $\pm g_k(b)$. Then for $R^*(N) := \max_{0 \leq k \leq n} R_k(b)$, $r > R^*$ implies that there are $2(N+1)$ nonreal eigenvalues of $L_{\mathrm{PC}}(ir, b)$. We denoted the counting-function of the non-real eigenvalues of $L_{\mathrm{PC}}(ir, b)$ by $\mathcal{N}(r)$, so

$$r > R^*(N) \text{ implies } \mathcal{N}(r) \geq 2(N+1). \tag{6.4.2}$$

This works for all $N \in \mathbb{N}_0$, so $\mathcal{N}(r)$ is unbounded above as $r \to \infty$. □

CHAPTER 7

CREATION OF NONREAL EIGENVALUES: PARTIAL RESULTS

We now know that for $z = ir$, $r \to \infty$, the number of nonreal eigenvalues becomes unbounded. What is less clear from the above results is *how* the eigenvalues become nonreal.

7.1 An observation characterizing real eigenvalues

We first make the following observation, a sort of converse to Lemma 5.4.2.

Lemma 7.1.1. *Fix $b > 0$, and $\nu \in \mathbb{C} \setminus \mathbb{N}_0$. Suppose in addition that $M(\nu; b) \neq 0$. Then $\nu \in \mathrm{Sp}(L_{PC}(z, b))$ for exactly two $z \in \mathbb{C}$; namely, if $\mathbf{m}$ is a square root of $M(\nu; b)$, then*

$$z = \pm \frac{1}{\mathbf{m}}. \tag{7.1.1}$$

Proof. $M(\nu; b)$ is defined for $b > 0$ and $\nu \notin \mathbb{N}_0$. Hence, if $M(\nu; b) \neq 0$, from (6.0.1),

$$M(\nu; b) = \frac{1}{z^2}, \tag{7.1.2}$$

or

$$z^2 = \frac{1}{M(\nu; b)} \tag{7.1.3}$$

it is thus necessary and sufficient that $z = \pm\frac{1}{\mathbf{m}}$, for $\mathbf{m}$ a square root of $M(\nu; b)$. □

Corollary 7.1.2. *Suppose $b > 0$, $\nu \in \mathbb{R}\backslash\mathbb{N}_0$. Then if $M(\nu; b) > 0$, then $\nu \in L_{PC}(r, b)$ for some $r \in \mathbb{R}^+$, and if $M(\nu; b) < 0$, then $\nu \in L_{PC}(ir, b)$ for some $r \in \mathbb{R}^+$.*

Proof. By Lemma 6.1.3, for $b > 0$ and $\nu \in \mathbb{R} \setminus \mathbb{N}_0$, $M(\nu; b)$ is real. If $M(\nu; b) > 0$, it has a positive square root, and if $M(\nu; b) < 0$, it has a purely imaginary square root, so the rest follows from Lemma 7.1.1. □

7.2 Demonstration of the Necessity of the Zero-Derivative Condition

We first need a technical lemma.

Lemma 7.2.1. *Fix $r_0 > 0$. Then for r close enough to r_0, in particular $|r - r_0| < \frac{1}{2}r_0$,*

$$\left|\frac{1}{r^2} - \frac{1}{r_0^2}\right| = \Theta\left(|r - r_0|\right), \tag{7.2.1}$$

with the implicit constant depending on r_0.

Proof. Fix $r_0 > 0$. We calculate:

$$\begin{aligned} \left|\frac{1}{r_0^2} - \frac{1}{r^2}\right| &= \left|\frac{r^2 - r_0^2}{r^2 r_0^2}\right| \\ &= |r - r_0| \cdot \frac{r + r_0}{r^2 r_0^2}. \end{aligned} \tag{7.2.2}$$

We wish to show that the coefficient of $|r - r_0|$ is small. If we suppose that the relative error is small, i.e., that $|r - r_0| < \dfrac{1}{2} r_0$, then we have that

$$
\begin{aligned}
\frac{r_0}{r_0^2 r_0^2} \leq \frac{r + r_0}{r^2 r_0^2} &\leq \cdot \frac{\frac{3}{2} r_0}{r_0^2 \left(\frac{1}{2} r_0\right)^2} \\
\frac{1}{r_0^3} \leq \cdot \frac{r + r_0}{r^2 r_0^2} &\leq 6 \frac{1}{r_0^3}
\end{aligned}
\tag{7.2.3}
$$

Since we let the constants depend on r_0, we let $c = c(r_0) = r_0^{-3}$ and $C = C(r_0) = 6r_0^{-3}$, and we have that if $|r - r_0| < \dfrac{1}{2} r_0$, plugging back (7.2.3) into (7.2.2),

$$
c|r - r_0| \leq \left| \frac{1}{r_0^2} - \frac{1}{r^2} \right| \leq C|r - r_0|. \tag{7.2.4}
$$

□

We now argue that for z real or pure imaginary and nonzero, under a certain hypothesis, real eigenvalues remain real in a vicinity of the eigenvalue.

Proposition 7.2.2. *Fix $b > 0$, fix $z = r_0$ (respectively, $z = ir_0$) for $r_0 \in \mathbb{R} \setminus \{0\}$, and fix $\mu \in \mathbb{R} \setminus \mathbb{N}_0$ with $\mu \in \mathrm{Sp}(L_{PC}(z, b))$; in particular, by Lemma 5.4.2, $M(\nu; b) \neq 0$. Suppose in addition that $\left.\dfrac{\partial}{\partial \nu}(D_\nu(b))\right|_{\nu = \mu} \neq 0$. Then for r sufficiently close to r_0, there exists a unique eigenvalue $\nu_0 = \nu_0(r)$ of $L_{PC}(r, b)$ (respectively, $L_{PC}(ir, b)$) in a neighborhood of μ, and $\nu_0(r)$ is real.*

Proof. Fix $b > 0$, and for the first case, fix $z = r_0$ for $r_0 \in \mathbb{R} \setminus \{0\}$. Suppose that $\mu \in \mathrm{Sp}(L_{\mathrm{PC}}(r_0, b))$, and $\left.\dfrac{\partial}{\partial \nu}(D_\nu(b))\right|_{\nu = \mu} \neq 0$. By Theorem 2,

$$
M(\mu; b) = \frac{1}{r_0^2}.
$$

Then consider the function

$$f(\nu) := M(\nu; b) - M(\mu; b) = M(\nu; b) - \frac{1}{r_0^2}. \tag{7.2.5}$$

; by the hypothesis, $f(\nu)$ has a zero of order 1 at $\nu = \mu$. Then for r sufficiently close to r_0; by Lemma 7.2.1, $\frac{1}{r^2}$ is close enough to $\frac{1}{r_0^2}$ to invoke Proposition 2.6.3, with ν in the role of ζ and

$$\frac{1}{r^2} - \frac{1}{r_0^2} \tag{7.2.6}$$

in the role of γ. Therefore, there exists a unique $\nu_0 = \nu_0(r)$ in a vicinity of μ, (in particlar, such that $\nu_0 \notin \mathbb{N}_0$) satisfying

$$f(\nu_0) = \frac{1}{r^2} - \frac{1}{r_0^2}, \tag{7.2.7}$$

i.e.,

$$M(\nu_0; b) = \frac{1}{r^2}. \tag{7.2.8}$$

By Theorem 2, $\nu_0(r)$ is the unique eigenvalue of $L_{\mathrm{PC}}(r, b)$ in a neighborhood of μ.

Moreover, μ is real, so by Corollary 6.1.4, the power-series expansion of $M(\nu; b)$ at μ has real coefficients; hence, so does $f(\nu)$. By Lemma 2.6.4, $\nu_0(r)$ is real, for r close to r_0.

In the case $z = ir_0$, we have $\frac{1}{z^2} = -\frac{1}{r_0^2}$, which is still real. The proof works similarly, replacing $\frac{1}{r_0^2}$ by $-\frac{1}{r_0^2}$, and similarly for r^{-2}. □

Thus, for $\mu \in \mathbb{R} \setminus \mathbb{N}_0$, by Corollary 7.1.2, $M(\mu; b) \neq 0$ is enough to ensure that $\mu \in \mathrm{Sp}(L_{\mathrm{PC}}(r, b))$ or $\mathrm{Sp}(L_{\mathrm{PC}}(ir, b))$ for some $r \in \mathbb{R}^+$. Yet if the ν-derivative of $M(\nu; b)$ at $\nu = \mu$ is nonzero, by Proposition 7.2.2 nearby values of r still have a real eigenvalue. It suggests to us that to have a coalesence of real eigenvalues into complex eigenvalues, we need $M(\nu; b) \neq 0$, but the first ν-derivative equal to 0.

Hence, suppose that $\nu_{\text{crit}} \in \mathbb{R} \setminus \mathbb{N}_0$ satisfies $M(\nu; b) \neq 0$ and $\left.\frac{\partial}{\partial \nu}(M(\nu; b))\right|_{\nu=\nu_{\text{crit}}} = 0$. We know by Lemma 6.1.1 that $M(\nu; b)$ is nonconstant, so there exists a minimal $m \geq 2$ with $\left.\frac{\partial^m M(\nu; x)}{\partial \nu^m}\right|_{\nu=\nu_{\text{crit}}, x=b} \neq 0$. Then the power-series expansion in ν at $\nu = \nu_{\text{crit}}$ can be written

$$M(\nu; b) = c_0 + \sum_{k=m}^{\infty} c_k (\nu - \nu_{\text{crit}})^k; \quad c_0 = M(\nu_{\text{crit}}, b \neq 0) \neq 0,\ c_n \neq 0. \tag{7.2.9}$$

7.3 Restrictions from real z

We now show that the self-adjointness of $L_{\text{PC}}(r, b)$ gives us a restriction on the relevant power-series expansions.

Proposition 7.3.1. *Suppose that there exists $\nu_{crit} \in \mathbb{R} \setminus \mathbb{N}_0$ with $M(\nu_{crit}; b) \neq 0$ and $\left.\frac{\partial}{\partial \nu}(M(\nu; b))\right|_{\nu=\nu_{crit}} = 0$. Then $M(\nu_{crit}; b) < 0$.*

Proof. Suppose, by way of contradiction, that there existed $\nu_{\text{crit}} \in \mathbb{R} \setminus \mathbb{N}_0$ with $M(\nu_{\text{crit}}; b) > 0$ and $\left.\frac{\partial}{\partial \nu}(M(\nu; b))\right|_{\nu=\nu_{\text{crit}}} = 0$. Then the critical equation for the $z = r$ (r real) case of (5.4.9) can be rewritten with the help of (7.2.9) as

$$\begin{aligned} c_0 + \sum_{k=n}^{\infty} c_k (\nu - \nu_{\text{crit}})^k &= \frac{1}{r^2} \\ \sum_{k=n}^{\infty} c_k (\nu - \nu_{\text{crit}})^k &= \frac{1}{r^2} - c_0 \\ \sum_{k=n}^{\infty} c_k (\nu - \nu_{\text{crit}})^k &= \frac{1}{r^2} - |c_0|, \end{aligned} \tag{7.3.1}$$

with $c_0 > 0$, $n \geq 2$, $c_n \neq 0$. Therefore, defining

$$r_{\text{crit}} := \sqrt{\frac{1}{|c_0|}} \in \mathbb{R}^+ \tag{7.3.2}$$

it is clear that when $r = r_{\text{crit}}$, the right-hand side of the last line in (7.3.1) is 0, and for r near to r_{crit}, $\frac{1}{r^2} - |c_0| \approx 0$. Therefore, for r sufficiently close to r_{crit}, we may apply Proposition 2.6.3 to

$$f(\nu) := M(\nu; b) - M(\nu_{\text{crit}}; b), \tag{7.3.3}$$

, and we have that there are m solutions $\{\nu_k\}_{k=0}^{n-1}$ to (7.3.1). We let $\mathbf{z} := \frac{1}{r^2} - c_0 = \mathbf{r}e^{i\theta}$, $\mathbf{r} \in \mathbb{R}^+$, $\theta \in [-\pi, \pi)$; since $\frac{1}{r^2} - c_0$ is real, $\theta \in \{-\pi, 0\}$. We also fix $c_n = \rho e^{i\psi}$, $-\pi \leq \psi < \pi$; by Lemma 6.1.3, $M(\nu; b)$ is real-valued on the real line, so all c_k are real and so $\psi \in \{-\pi, 0\}$. Thus, (2.6.5) gives

$$\lim_{r \to r_{\text{crit}}} \frac{\nu_k(r) - \nu_{\text{crit}}}{|\nu_k(r) - \nu_{\text{crit}}|} = \exp\left(\frac{i(\theta - \psi + 2\pi k)}{m}\right), \quad 0 \leq k \leq m - 1. \tag{7.3.4}$$

Yet since $\nu_{\text{crit}} \in \mathbb{R}$, by Lemma 6.1.3 we know that the ν_k are real for all $k \in \{0, 1, \ldots, n-1\}$; hence, so are the $\nu_k - \nu_{\text{crit}}$, since ν_{crit} is real; thus, moreover, $\nu_k \neq \nu_{\text{crit}}$, even by the weaker Proposition 2.6.1. Hence,

$$\frac{\nu_k(r) - \nu_{\text{crit}}}{|\nu_k(r) - \nu_{\text{crit}}|}$$

is real and nonzero for all real r close enough to r_{crit}. Hence, the limit of such expressions is real-valued, so

$$\exp\left(\frac{i(\theta - \psi + 2\pi k)}{m}\right) \in \mathbb{R} \text{ for all } k, 0 \leq k \leq m - 1. \tag{7.3.5}$$

Thus,

$$\frac{(\theta - \psi + 2\pi k)}{m} \in \pi\mathbb{Z} \text{ for all } k, 0 \leq k \leq n-1. \tag{7.3.6}$$

Now, $\pi\mathbb{Z}$ is a group under addition, so the difference of any two such expressions is in the set as well. Hence, taking the difference of the $k = 0$ and $k = 1$ case,

$$\frac{2\pi}{m} \in \pi\mathbb{Z},$$

so $\frac{2}{m} \in \mathbb{Z}$. Since $m \geq 2$, the only consistent possibility is $m = 2$.

Case 1: $c_m = c_2 > 0$; i.e., $\psi = 0$. Then for r slightly greater than r_{crit}, $\frac{1}{r^2} - c_0 < 0$ so $\theta = -\pi$. Hence, from the $k = 2$ case of (7.3.6),

$$-\frac{\pi}{2} + \pi k \in \pi\mathbb{Z}, \quad k \in \{0, 1\},$$

but $-\frac{\pi}{2}$ is not an integer multiple of π. Contradiction.

Case 2: $c_m = c_2 < 0$; i.e., $\psi = -\pi$. Then for r slightly smaller than r_{crit}, $\frac{1}{r^2} - c_0 > 0$ so $\theta = 0$. Hence, from the $k = 2$ case of (7.3.6),

$$+\frac{\pi}{2} + \pi k \in \pi\mathbb{Z}, \quad k \in \{0, 1\},$$

but again, $\frac{\pi}{2}$ is not an integer multiple of π. Contradiction.

Since c_2 is real and nonzero by hypothesis, all cases lead to contradiction. Thus, $c_0 = M(\nu_{\text{crit}}; b)$ is not greater than 0. □

7.4 Consequences for Imaginary z

Hence, if $\nu_{\text{crit}} \in \mathbb{R}\backslash\mathbb{N}_0$, $M(\nu_{\text{crit}}; b) \neq 0$, and $\frac{\partial}{\partial \nu}\left(M(\nu; b)\right)\Big|_{\nu=\nu_{\text{crit}}} = 0$, then $M(\nu_{\text{crit}}; b) < 0$. We now wish to apply this to the case of $z = ir$, and we wish to consider, in a neighborhood of ν_{crit}, the equation

$$M(\nu; b) = \frac{1}{(ir)^2} \tag{7.4.1}$$

We reduce it to a question of a zero by rewriting as

$$f(\nu) := M(\nu; b) - M(\nu_{\text{crit}}; b) = -\frac{1}{r^2} - M(\nu_{\text{crit}}; b). \tag{7.4.2}$$

Using the power-series expansion (7.2.9), we have

$$\begin{aligned} c_0 + \sum_{k=n}^{\infty} c_k(\nu - \nu_{\text{crit}})^k &= -\frac{1}{r^2} \\ \sum_{k=n}^{\infty} c_k(\nu - \nu_{\text{crit}})^k &= -c_0 - \frac{1}{r^2} \\ \sum_{k=n}^{\infty} c_k(\nu - \nu_{\text{crit}})^k &= |c_0| - \frac{1}{r^2} \end{aligned} \tag{7.4.3}$$

We now (re)define

$$r_{\text{crit}} := \sqrt{\frac{1}{|c_0|}} \tag{7.4.4}$$

and note that $|c_0| - \frac{1}{r_{\text{crit}}^2} = 0$. As in the previous case, for $r \neq r_{\text{crit}}$ we let $\mathbf{z} = \mathbf{r}e^{i\theta}$, $\mathbf{r} \in \mathbb{R}^+$, $\theta \in [-\pi, \pi)$; since $|c_0| - \frac{1}{r^2} \in \mathbb{R}$, we have that $\theta \in \{-\pi, 0\}$. Similarly, we again set $c_m = \rho e^{i\psi}$; again, the c_j are real by the assumption that ν_{crit} is real, and Lemma 6.1.3, so $\psi \in \{-\pi, 0\}$. Applying Proposition 2.6.3 to the map $\nu \mapsto M(\nu; b) - M(\nu_{\text{crit}}; b)$, we get the following.

Proposition 7.4.1. *Suppose that there exists $\nu_{crit} \in \mathbb{R} \setminus \mathbb{N}_0$ with $M(\nu_{crit}; b) < 0$ and $\frac{\partial}{\partial \nu}(M(\nu; b))\Big|_{\nu=\nu_{crit}} = 0$, and let m be the smallest index greater than or equal to 2 such that $\frac{\partial^m}{\partial \nu^m}(M(\nu; b))\Big|_{\nu=\nu_{crit}} \neq 0$. Then for r sufficiently close to $r_{crit} \in$ (7.4.4), but* *not equal to it, then there exist exactly* m *solutions* $\{\nu_k\}_{k=0}^{m-1}$ *to* $M(\nu;b) - M(\nu_{crit};b) = -\frac{1}{r^2} - M(\nu_{crit};b)$. *If* $\mathcal{Z} = \mathbf{r}^{1/m}e^{i\theta/m}$ *and* $\mathbf{c} = \rho^{1/m}e^{i\psi/m}$ *be specific* m*th roots of* $\mathbf{z} = -\frac{1}{r^2} - M(\nu_{crit};b) = |c_0| - \frac{1}{r^2}$ *and* c_m *respectively, then the leading-order expansions are given by*

$$\nu_k = \nu_{crit} + \frac{\mathcal{Z}}{\mathbf{c}}\exp\left(\frac{2\pi ik}{m}\right) + \Theta(|r - r_{crit}|^{2/m}), \quad 0 \le k \le m-1, \tag{7.4.5}$$

where the constants in the $\Theta|r - r_{crit}|^{2/m}$ *term depend on* $|M(\nu_{crit};b)|$. *Moreover, as* $r \to r_{crit}$, $\nu_k - \nu_{crit} \to 0$ *for all* k, $0 \le k \le n-1$*; indeed,*

$$|\nu_k - \nu| = \Theta(|r - r_{crit}|^{1/m}), \quad 0 \le k \le m-1 \tag{7.4.6}$$

Also,

$$\lim_{r\to r_{crit}} \frac{\nu_k(r) - \nu_{crit}}{|\nu_k(r) - \nu_{crit}|} = \exp\left(\frac{i(\theta - \psi + 2\pi k)}{m}\right), \quad 0 \le k \le m-1. \tag{7.4.7}$$

Proof. The only issues left to show are the asymptotic size of the error terms in (7.4.5) and (7.4.6). Fortunately, since $M(\nu_{\text{crit}};b) = c_0 = \frac{1}{r_{\text{crit}}^2}$, this follows from Lemma 7.2.1; using $r_0 = r_{\text{crit}}$, we get that Indeed, we know from (2.6.2) that the error is of order $\mathcal{Z}^2$, and $\mathcal{Z}$ is an nth root of $|c_0| - \frac{1}{r^2}$; yet by definition of r_{crit}, $|c_0| = \frac{1}{r_{\text{crit}}^2}$. Hence,

$$|c_0| - \frac{1}{r^2} = \frac{1}{r_{\text{crit}}^2} - \frac{1}{r^2}, \tag{7.4.8}$$

so by Lemma 7.2.1, we have that for $r > 0$, $|r - r_{\text{crit}}| < \frac{1}{2}r_{\text{crit}}$,

$$\frac{1}{r_{\text{crit}}^3}|r - r_{\text{crit}}| \le \left|\frac{1}{r_{\text{crit}}^2} - \frac{1}{r^2}\right| \le 6|\frac{1}{r_{\text{crit}}^3}|r - r_{\text{crit}}|, \tag{7.4.9}$$

, or, by definition of r_{crit},

$$|c_0|^{3/2}|r - r_{\text{crit}}| \leq \left|\frac{1}{r_{\text{crit}}^2} - \frac{1}{r^2}\right| \leq 6|c_0|^{3/2}|r - r_{\text{crit}}| \tag{7.4.10}$$

Hence, the discrepancy $\left|\frac{1}{r_{\text{crit}}^2} - \frac{1}{r^2}\right|$ is $\Theta(|r - r_{\text{crit}}|)$, and hence

$$\begin{aligned} |\mathcal{Z}| &= \left||c_0| - \frac{1}{r^2}\right|^{1/n} \\ &= \left|\frac{1}{r_{\text{crit}}^2} - \frac{1}{r^2}\right|^{1/n} \\ &= \Theta(|r - r_{\text{crit}}|)^{1/n} \end{aligned} \tag{7.4.11}$$

and so, the error in the leading-order expansion (7.4.5), of order $|\mathcal{Z}|^2$ by Proposition 2.6.3, is really $\Theta(|r - r_{\text{crit}}|)^{2/n}$. We are done. □

7.4.1 Application to the case $m = 2$

To explain the behaviour shown in Figure 5.6, we take the case $m = 2$, or alternatively, the condition $\left.\frac{\partial^2}{\partial \nu^2}(M(\nu; b))\right|_{\nu=\nu_{\text{crit}}} \neq 0$, and note that $\frac{2\pi k}{m} = \pi k$ in this case. Moreover, in the power-series expansion (7.2.9), c_2 is real by $M(\nu; b)$ real-valued for ν, b, real, so $\psi \in \{-\pi, 0\}$. We split into cases depending on the sign of c_2.

$c_2 > 0$. Hence, $\psi = 0$.

Then for $0 < r < r_{\text{crit}}$, $\mathbf{z} = |c_0| - \frac{1}{r^2} > 0$, so $\theta = 0$. For r close enough to r_{crit}, (7.4.7) gives that

$$\lim_{r \to r_{\text{crit}}} \frac{\nu_k(r) - \nu_{\text{crit}}}{|\nu_k(r) - \nu_{\text{crit}}|} = \exp\left(\frac{2\pi i k}{2}\right) = (-1)^k, \quad k = 0, 1 \tag{7.4.12}$$

Hence, for $r < r_{\text{crit}}$, by ν_{crit} real, the eigenvalues are real to leading order. Indeed,.

For $r > r_{\text{crit}}$, $\mathbf{z} = |c_0| - \dfrac{1}{r^2} < 0$, so $\theta = -\pi$. For r close enough to r_{crit}, (7.4.7) gives that

$$\lim_{r\to r_{\text{crit}}} \frac{\nu_k(r) - \nu_{\text{crit}}}{|\nu_k(r) - \nu_{\text{crit}}|} = \exp\left(\frac{-\pi + 2\pi ik}{2}\right) = (-i)(-1)^k, \quad k = 0, 1 \qquad (7.4.13)$$

Hence, for $r > r_{\text{crit}}$ and close enough, by ν_{crit} real, the eigenvalues are ν_{crit} plus an adjustment that is pure-imaginary to leading order, hence certainly nonreal. Hence, for $r > r_{\text{crit}}$, if the eigenvalues were real before, they have become nonreal.

$c_2 < 0$**.** Hence, $\psi = -\pi$.

Then for $0 < r < r_{\text{crit}}$, $\mathbf{z} = |c_0| - \dfrac{1}{r^2} > 0$, so $\theta = 0$. For r close enough to r_{crit}, (7.4.7) gives that

$$\lim_{r\to r_{\text{crit}}} \frac{\nu_k(r) - \nu_{\text{crit}}}{|\nu_k(r) - \nu_{\text{crit}}|} = \exp\left(\frac{\pi + 2\pi ik}{2}\right) = i(-1)^k, \quad k = 0, 1 \qquad (7.4.14)$$

so $\nu_k(r) = \nu_{\text{crit}} + (\nu_k(r) - \nu_{\text{crit}})$ is the sum of ν_{crit} and a leading-order-imaginary term.

For $r > r_{\text{crit}}$, $\mathbf{z} = |c_0| - \dfrac{1}{r^2} < 0$, so $\theta = -\pi$. For r close enough to r_{crit}, (7.4.7) gives that

$$\lim_{r\to r_{\text{crit}}} \frac{\nu_k(r) - \nu_{\text{crit}}}{|\nu_k(r) - \nu_{\text{crit}}|} = \exp\left(\frac{2\pi ik}{2}\right) = (-1)^k, \quad k = 0, 1 \qquad (7.4.15)$$

Hence, for $r > r_{\text{crit}}$, $\nu_k(r)$ is real, at least asymptotically. Thus, as r increases through r_{crit}, the eigenvalues transition from surely nonreal to approximately real.

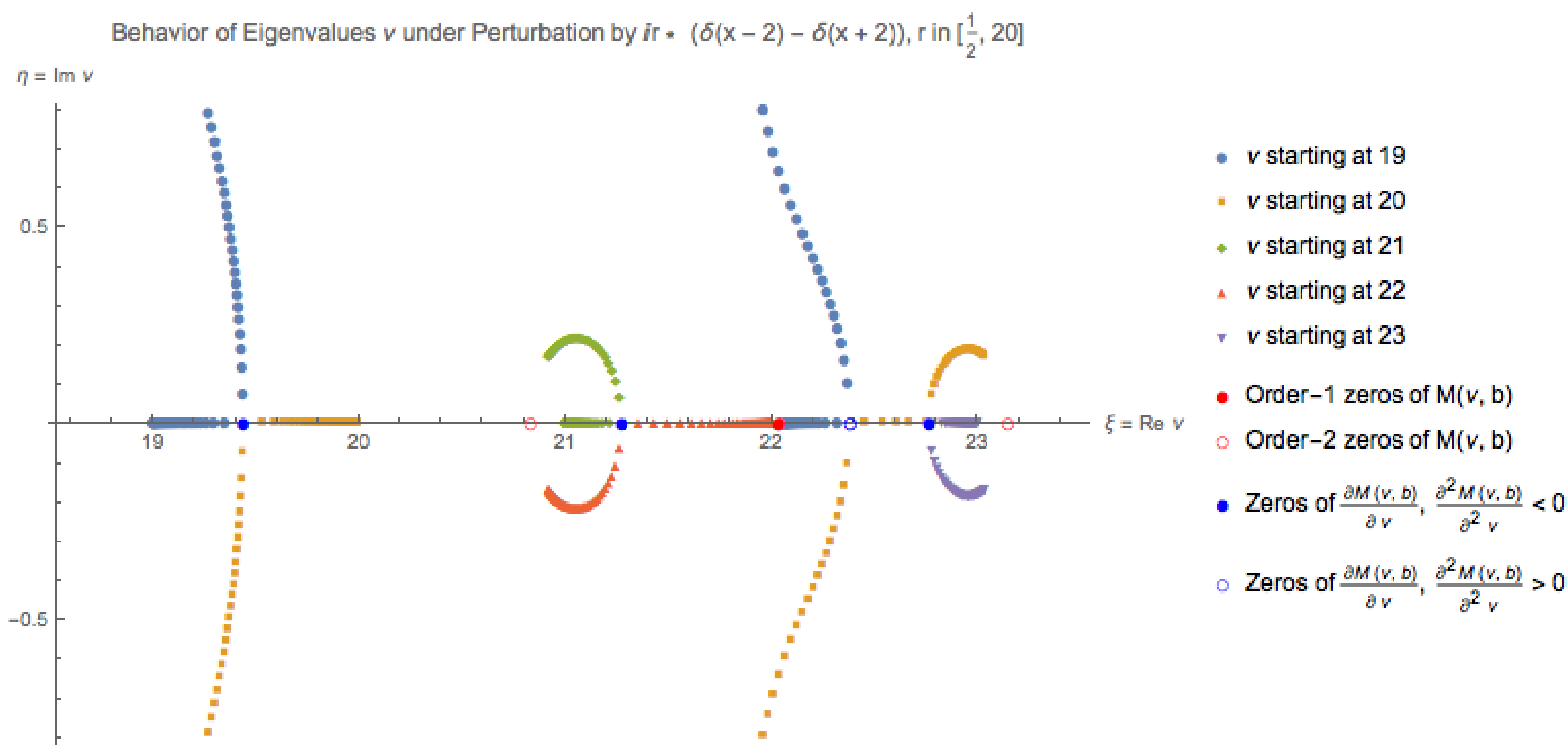

Figure 7.1: Evolution of the eigenvalues of $L_{\mathrm{PC}}(ir, 2)$ starting at $\nu = 19, 20, \ldots, 23$. Zeroes of $M(\nu; 2)$ and its first ν-derivative are marked. Note that the perturbations of the $\nu = 19$ and $\nu = 20$ eigenvalues reconverge on the real axis in finite time; one descendant of that collision coalesces with the $\nu = 23$ eigenvalue and leaves the real line again.

To see that both cases occur (at least if the calcuations are accurate), we refer to Figures 7.1 and 7.2, which show the calculated eigenvalues starting from the unperturbed eigenvalues of $19, 20, 21, 22, 23$. In particular, the eigenvalues from 19 and 20 appear to pass through a zero of the derivative and become purely complex in the $(19, 20)$ interval, and re-coalesce to the real line inbetween 22 and 23. Then one of the eigenvalues returned to the real line becomes complex again after colliding with the descendant of the $\nu = 23$ unperturbed eigenvalue.

Therefore, the zeros of the derivative seem to dictate the switch from real to non-real eigenvalues.

We do not have, however, any proof that if the first ν-derivative of $M(\nu; b)$ is zero, the second ν-derivative is nonzero. In theory, if $M(\nu_{\text{crit}}, b) < 0$, many succeeding ν-derivatives could be zero at $\nu = \nu_{\text{crit}}$ – though not all of them, by $M(\nu, b)$ nonconstant in ν. Also, Lemma 2.6.4 is not quite strong enough to prove that the inverses above are real. More work is needed here.

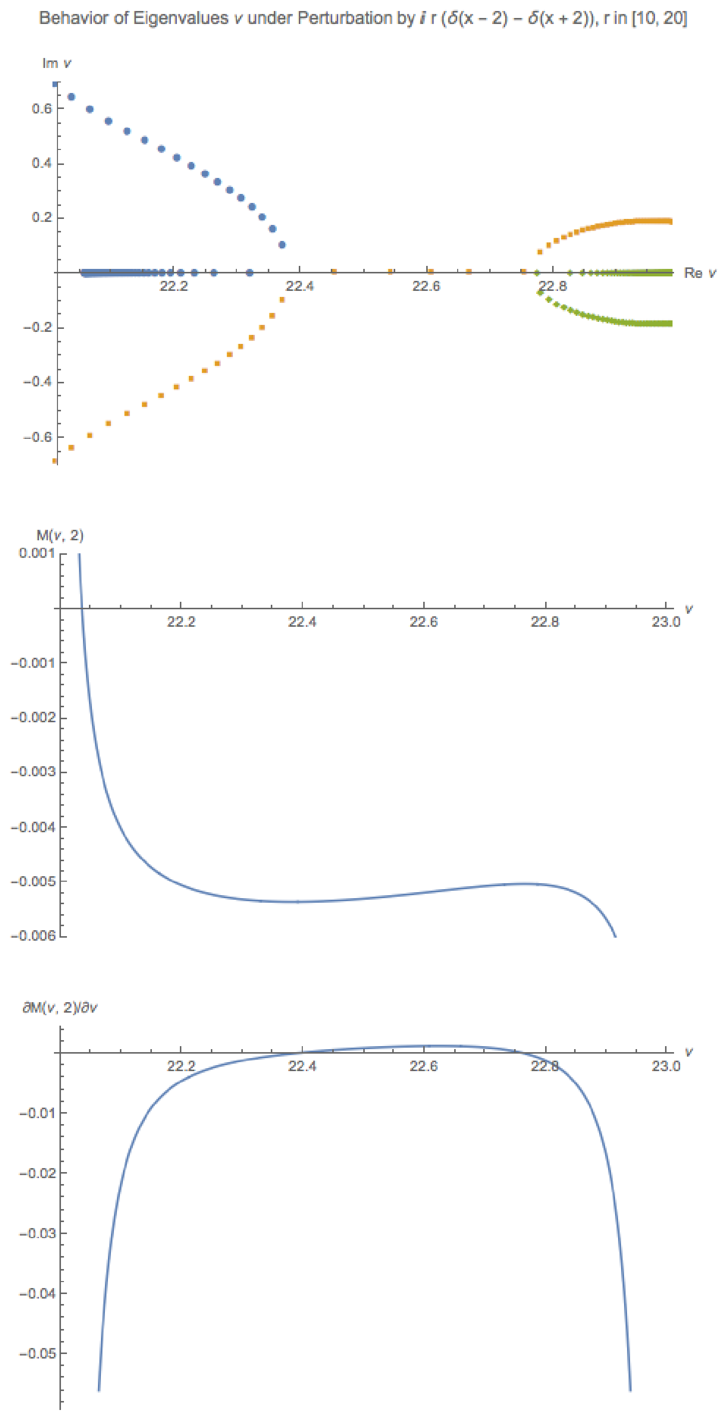

Figure 7.2: A close-up of the right-hand side of the previous figure, plotting the evolution of the $\nu = 19$, $\nu = 20$, and $\nu = 23$ eigenvalues of $L_{\mathrm{PC}}(ir, 2)$ as r increases. The second and third diagrams explicitly graph $M(\nu; 2)$ and its first ν-derivative.

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Appendix A

CONSTRUCTION OF THE PERTURBED OPERATOR

A.1 Background: Theory of Sectorial Quadratic Forms

Only certain quadratic forms will be useful for creating operators – the *densely defined*, *sectorial*, and *closed* quadratic forms. "Densely defined" is self-evident – $\mathfrak{D}(\mathfrak{t})$ should be densely defined. We now discuss sectoriality.

Definition A.1.1 ([Kat95, Chapter VI, Section 1.2, pp. 310–311])**.** Fix $\mathcal{H}$ a Hilbert space, with norm $\|\cdot\|_{\mathcal{H}}$, and let $\mathfrak{t} : \mathcal{L} \times \mathcal{L} \to \mathbb{C}$ be a quadratic form on a linear submanifold $\mathcal{L}$ of $\mathcal{H}$. The *numerical range* of $\mathfrak{t}$ is the set

$$\mathfrak{N}(\mathfrak{t}) := \{\mathfrak{t}\,(u, u) : u \in \mathcal{L}, \|u\|_{\mathcal{H}} = 1\}. \tag{A.1.1}$$

The form $\mathfrak{t}$ is called *sectorial* if $\mathfrak{N}(\mathfrak{t})$ is a subset of a sector of the form

$$|\arg(\zeta - \gamma)| \leq \theta, \quad 0 \leq \theta < \frac{\pi}{2}, \quad \gamma \text{ real}. \tag{A.1.2}$$

Sectoriality is also a sensible concept for operators.

Definition A.1.2 ([Kat95, Chapter V, Sections 2.2 and 3.10, pp. 267–268, 278–280])**.**

Fix $\mathcal{H}$ a Hilbert space, with inner product $(\cdot,\cdot)_{\mathcal{H}}$ and norm $\|\cdot\|_{\mathcal{H}}$, and T a (possibly unbounded) operator on a domain $\mathfrak{D}(T) \subseteq \mathcal{H}$. The *numerical range* of T is the set

$$\mathfrak{N}(T) := \{(Tu,u)_{\mathcal{H}} : u \in \mathfrak{D}(T) : \|u\|_{\mathcal{H}} = 1\}. \tag{A.1.3}$$

T is called *sectorial* if $\mathfrak{N}(T)$ is a subset of a sector of the form in (A.1.2). T is called *m-sectorial* if it is sectorial, and there is no proper extension $\widetilde{T}$ of T such that $\widetilde{T}$ is sectorial, or even obeys the weaker criterion

$$\mathfrak{N}\left(\widetilde{T}\right) \subseteq \{\zeta \in \mathbb{C} : \operatorname{Re}\zeta \geq c\}, \quad c \in \mathbb{R}. \tag{A.1.4}$$

Again, the form T is called *bounded below* if $\mathfrak{N}(T) \subseteq [\gamma,\infty)$ for some $\gamma \in \mathbb{R}$, or equivalently,

$$(Tu,u)_{\mathcal{H}} \geq \gamma\|u\|_{\mathcal{H}}^2, \quad \gamma \in \mathbb{R}. \tag{A.1.5}$$

Before continuing, we note certain special types of sectorial forms and operators.

Definition A.1.3 ([Kat95, Chapter 6, Sections 1.1–2, pp. 309-310])**.** Fix $\mathcal{H}$ a Hilbert space, with inner product $(\cdot,\cdot)_{\mathcal{H}}$ and norm $\|\cdot\|_{\mathcal{H}}$, and $\mathfrak{t} : \mathfrak{L} \times \mathfrak{L} \to \mathcal{H}$ a sesquilinear quadratic form on a linear submanifold $\mathfrak{L}$ of $\mathcal{H}$. $\mathfrak{t}$ is called *symmetric* if for all $u, v \in \mathfrak{D}(\mathfrak{t})$,

$$\mathfrak{t}(v,u) = \overline{\mathfrak{t}(u,v)}. \tag{A.1.6}$$

We call a symmetric form *bounded below* if $\mathfrak{N}(\mathfrak{t}) \subseteq [\gamma,\infty)$ for some $\gamma \in \mathbb{R}$; or equivalently

$$\mathfrak{t}(u,u) \geq \gamma\|u\|^2, \quad \gamma \in \mathbb{R}. \tag{A.1.7}$$

If (A.1.7) holds, we say that $\mathfrak{t} \geq \gamma$.

Definition A.1.4 ([Kat95, Chapter V, Sections 3.3 and 3.10, pp. 269, 278])**.** Fix $\mathcal{H}$ a

Hilbert space, with inner product $(\cdot,\cdot)_{\mathcal{H}}$ and norm $\|\cdot\|_{\mathcal{H}}$, and T a (possibly unbounded) linear operator on a linear submanifold $\mathfrak{D}(T)$ of $\mathcal{H}$. T is called *symmetric* if for all $u, v \in \mathfrak{D}(T)$,

$$(Tu, v)_{\mathcal{H}} = u, Tv_{\mathcal{H}}. \tag{A.1.8}$$

We call a symmetric operator *bounded below* if $\mathfrak{N}(T) \subseteq [\gamma, \infty)$ for some $\gamma \in \mathbb{R}$; or equivalently

$$(Tu, u)_{\mathcal{H}} \geq \gamma \|u\|^2, \quad \gamma \in \mathbb{R}. \tag{A.1.9}$$

Clearly, symmetric forms (resp. operators) that are bounded below are sectorial form (resp. operators).

We now define closedness of a quadratic form.

Definition A.1.5 ([Kat95, Chapter VI, Section 1.3, p. 313]). Fix $\mathcal{H}$ a Hilbert space, $\mathcal{L}$ a linear manifold in $\mathcal{H}$, and $\mathfrak{t}$ a sesquilinear quadratic form on $\mathcal{L}$. A sequence $\{u_n\}_{n=0}^{\infty}$ in $\mathcal{L}$ is called $\mathfrak{t}$-convergent to $u \in \mathcal{H}$ if and only if

$$\lim_{n\to\infty} u_n = u, \tag{A.1.10a}$$

$$\mathfrak{t}(u_n - u_m, u_n - u_m) \to 0 \text{ as } m, n \to \infty, \tag{A.1.10b}$$

In symbols, we write $u_n \xrightarrow{\mathfrak{t}} u$.

The form $\mathfrak{t}$ is *closed* if, whenever $u_n \xrightarrow{\mathfrak{t}} u$, we have that

$$u \in \mathcal{L} = \mathfrak{D}(\mathfrak{t}), \tag{A.1.11a}$$

$$\lim_{n\to\infty} \mathfrak{t}(u_n - u, u_n - u) = 0. \tag{A.1.11b}$$

Since it will be used later, we also define the *closure* of a quadratic form and the concept of a *core*.

Definition A.1.6 ([Kat95, Chapter 6, Section 1.4, pp. 315–317])**.** A sectorial form $\mathfrak{t}$ is said to be *closable* if it has a closed extension. If closed extensions exists, the *closure* $\widetilde{\mathfrak{t}}$ is the smallest closed extension.

Suppose that $\mathfrak{t}$ is a closed sectorial form on a linear manifold $\mathcal{L}$, and let $\mathcal{L}'$ be a linear sub-manifold. $\mathcal{L}'$ is called a *core* of $\mathfrak{t}$ if the restriction $\mathfrak{t}'$ of $\mathfrak{t}$ to $\mathcal{L}' \times \mathcal{L}'$ has closure $\mathfrak{t}$.

Once we have the closed, densely defined, sectorial sesquilinear form, the existence of a corresponding operator is guaranteed with the first representation theorem.

Proposition A.1.7 ([Kat95, Thm. 2.1, Cor. 2.4, and Thm. 2.6, pp. 322–323])**.** *Let $\mathfrak{t}(u, v)$ be a densely defined, closed, sectorial sesquilinear form in a Hilbert space $\mathcal{H}$, with inner product denoted $(\cdot, \cdot)_{\mathcal{H}}$ and induced norm $\|\cdot\|_{\mathcal{H}}$. There exists an m-sectorial operator T such that:*

i) $\mathfrak{D}(T) \subseteq \mathfrak{D}(\mathfrak{t})$ *and*

$$\mathfrak{t}(u, v) = (Tu, v)_{\mathcal{H}} \tag{A.1.12}$$

for all $u \in \mathfrak{D}(T)$ and $v \in \mathfrak{D}(\mathfrak{t})$.

ii) $\mathfrak{D}(T)$ *is a core of* $\mathfrak{t}$*, and*

iii) if $u \in \mathfrak{D}(\mathfrak{t})$, $w \in \mathcal{H}$ and

$$\mathfrak{t}(u, v)_{\mathcal{H}} = (w, v)_{\mathcal{H}} \tag{A.1.13}$$

for all v in a core of $\mathfrak{t}$, then $u \in \mathfrak{D}(T)$ and $Tu = w$.

The m-sectorial operator T is uniquely determined by the condition i). Moreover, we have the following.

a) If S is an operator with $\mathfrak{D}(S) \subseteq \mathfrak{D}(\mathfrak{t})$, and

$$\mathfrak{t}(u, v) = (Su, v)_{\mathcal{H}} \tag{A.1.14}$$

for every $u \in \mathfrak{D}(S)$ and every v belonging to a core of $\mathfrak{D}(\mathfrak{t})$, then $S \subseteq T$.

b) If $\mathfrak{t}$ is symmetric and bounded from below, then T is selfadjoint and bounded from below.

We also have the *second representation theorem*, for self-adjoint operators. We need some definitions, however.

Proposition A.1.8 ([Kat95, Chapter VI, Thm. 2.23, pp. 331-332])**.** *Let $\mathfrak{h}$ be a densely defined, closed symmetric form, $\mathfrak{h} \geq 0$, and let $H = H_{\mathfrak{t}}$ be the associated selfadjoint operator. Then we have $\mathfrak{D}\left(H^{1/2}\right) = \mathfrak{D}(\mathfrak{t})$ and*

$$\mathfrak{t}(u,v) = (H^{1/2}u, H^{1/2}v). \tag{A.1.15}$$

Of course, a classic example of the use of Proposition A.1.8 is an application to the harmonic oscillator itself. We now prove Lemma 2.3.4; in fact we prove something stronger.

Lemma A.1.9. *There exists a positive, self-adjoint square root of L^0_{HO}; we write it as $(L^0_{HO})^{1/2}$. Moreover, $\mathfrak{D}\left((L^0_{HO})^{1/2}\right) = \mathfrak{D}\left(\mathfrak{t}^0_{HO}\right) = \mathfrak{D}_1$, and for all $u, v \in \mathfrak{D}_1$,*

$$\mathfrak{t}^0_{HO}(u(x), v(x)) = \left((L^0_{HO})^{1/2}u(x), (L^0_{HO})^{1/2}v(x)\right)_{L^2(\mathbb{R})} \tag{A.1.16}$$

In particular, we note that

$$(L^0_{HO})^{1/2}h_k(x) = \sqrt{2k+1}\,h_k(x). \tag{A.1.17}$$

Proof. We note that for all $u(x), v(x) \in \mathfrak{D}\left(L^0_{\text{HO}}\right)$,

$$\begin{aligned}\left(L^0_{\text{HO}}u(x), v(x)\right)_{L^2(\mathbb{R})} &= \left(-u''(x) + x^2u(x), v(x)\right)_{L^2(\mathbb{R})} \\ &= \left(-u''(x), v(x)\right)_{L^2(\mathbb{R})} + \left(x^2u(x), v(x)\right)_{L^2(\mathbb{R})}\end{aligned} \tag{A.1.18}$$

Each inner product in the last line of (A.1.18) simplifies. For the second one, by x real, $\overline{x} = x$, so we have that

$$\begin{aligned}\left(x^2u(x), v(x)\right)_{L^2(\mathbb{R})} &= \int\limits_{\mathbb{R}} x^2u(x)\overline{v(x)}\,dx \\ &= \int\limits_{\mathbb{R}} xu(x)\overline{xv(x)}\,dx \\ &= \left(xu(x), xv(x)\right)_{L^2(\mathbb{R})}\end{aligned} \tag{A.1.19}$$

Next, it is known (see, e.g., [Fol99, Exercises 8.8–9, p. 246, and Exercise 9.31, p. 308]) that $u \in \mathcal{H}^k(\mathbb{R})$ for some $k \in \mathbb{N}$ implies that $u(x)$, $u'(x)$, $v(x)$, and $v'(x)$ are absolutely continuous; therefore, integration-by-parts works, and hence by (2.1.4), we note that

$$\left(-u''(x), v(x)\right)_{L^2(\mathbb{R})} = \left(u'(x), v'(x)\right)_{L^2(\mathbb{R})}\,dx \tag{A.1.20}$$

(This can also be justified with the Fourier Transform through (2.1.4) and (A.1.19)). Altogether, then,

$$\begin{aligned}\left(L^0_{\text{HO}}u(x), v(x)\right)_{L^2(\mathbb{R})} &= \left(-u''(x), v(x)\right)_{L^2(\mathbb{R})} + \left(x^2u(x), v(x)\right)_{L^2(\mathbb{R})} \\ &= \left(u'(x), v'(x)\right)_{L^2(\mathbb{R})} + \left(xu(x), xv(x)\right)_{L^2(\mathbb{R})}.\end{aligned} \tag{A.1.21}$$

Yet the final expression in (A.1.21) is clearly defined for u, v in the set

$$\mathfrak{D}_1 = \{f \in \mathcal{H}^1(\mathbb{R}) : xf(x) \in L^2(\mathbb{R})\}. \tag{A.1.22}$$

We therefore define the quadratic form $\mathfrak{t}^0_{\mathrm{HO}}$ on $\mathfrak{D}_1 \times \mathfrak{D}_1$ by setting

$$\mathfrak{t}^0_{\mathrm{HO}}(u,v) := (u'(x), v'(x))_{L^2(\mathbb{R})} + (xu(x), xv(x))_{L^2(\mathbb{R})} \tag{A.1.23}$$

Manifestly, $\mathfrak{t}^0_{\mathrm{HO}}$ is densely defined, sesquilinear, and symmetric, and

$$\mathfrak{t}^0_{\mathrm{HO}}(u,u) = \|u'(x)\|^2_{L^2(\mathbb{R})} + \|xu(x)\|^2_{L^2(\mathbb{R})} \geq 0,$$

so $\mathfrak{t}^0_{\mathrm{HO}} \geq 0$; hence, $\mathfrak{t}^0_{\mathrm{HO}}$ is certainly sectorial. The closure of $\mathfrak{t}^0_{\mathrm{HO}}$ follows along the same lines as the closure of the form $\mathfrak{t}_{z,b}$ discussed below, so we omit the proof. By the first representation theorem (Proposition A.1.7), there exists an operator H coming from the quadratic form $\mathfrak{t}^0_{\mathrm{HO}}$, self-adjoint by part b), and by part a), it is clear that $L^0_{\mathrm{HO}} \subseteq H$. Yet since L^0_{HO} and H are both self-adjoint, with domains contained in $\mathfrak{D}\left(\mathfrak{t}^0_{\mathrm{HO}}\right) = \mathfrak{D}_1$, by standard uniqueness theorems (e.g., Reed/Simon volume 2, specifically [RS75, Thm. X.23, p.177]) it follows that $L^0_{\mathrm{HO}} = H$.

Then by the second representation theorem (Proposition A.1.8), we have the statement that $\mathfrak{D}\left((L^0_{\mathrm{HO}})^{1/2}\right) = \mathfrak{D}_1$.

The only thing left to show is that $(L^0_{\mathrm{HO}})^{1/2}$ has the expected effect on Hermite functions, i.e,

$$(L^0_{\mathrm{HO}})^{1/2} h_k(x) = \sqrt{2k+1}\, h_k(x). \tag{A.1.24}$$

Yet of course, by the $(h_k(x))_{k=0}^{\infty}$ an orthonormal-basis, and the properties of the Harmonic Oscillator Operator, and the Second Representation Theorem, we have for all $j, k \in \mathbb{N}_0$,

$$\begin{aligned}(2j+1)\delta_{jk} &= (2j+1)(h_j(x), h_k(x))_{L^2(\mathbb{R})} \\ &= \big(L^0_{\mathrm{HO}} h_j(x), h_k(x)\big)_{L^2(\mathbb{R})} \\ &= \big((L^0_{\mathrm{HO}})^{1/2} h_j(x), (L^0_{\mathrm{HO}})^{1/2} h_k(x)\big)_{L^2(\mathbb{R})},\end{aligned} \tag{A.1.25}$$

so in particular,

$$\big\|(L^0_{\mathrm{HO}})^{1/2} h_k(x)\big\|_{L^2(\mathbb{R})} = 2k+1, \quad k \in \mathbb{N}_0. \tag{A.1.26}$$

Moreover, we have that for all $j, k \in \mathbb{N}_0$, by $(L^0_{\mathrm{HO}})^{1/2}$ self-adjoint, $[(L^0_{\mathrm{HO}})^{1/2}]^2 = L^0_{\mathrm{HO}}$, and properties of the Hermite Functions,

$$\begin{aligned}\big((L^0_{\mathrm{HO}})^{1/2} h_j(x), h_k(x)\big)_{L^2(\mathbb{R})} &= \frac{1}{2k+1}\big((L^0_{\mathrm{HO}})^{1/2} h_j(x), (2k+1) h_k(x)\big)_{L^2(\mathbb{R})} \\ &= \frac{1}{2k+1}\big((L^0_{\mathrm{HO}})^{1/2} h_j(x), L^0_{\mathrm{HO}} h_k(x)\big)_{L^2(\mathbb{R})} \\ &= \frac{1}{2k+1}\big((L^0_{\mathrm{HO}})^{1/2} h_j(x), (L^0_{\mathrm{HO}})^{1/2}(L^0_{\mathrm{HO}})^{1/2} h_k(x)\big)_{L^2(\mathbb{R})} \\ &= \frac{1}{2k+1}\big((L^0_{\mathrm{HO}})^{1/2}(L^0_{\mathrm{HO}})^{1/2} h_j(x), (L^0_{\mathrm{HO}})^{1/2} h_k(x)\big)_{L^2(\mathbb{R})} \\ &= \frac{1}{2k+1}\big(L^0_{\mathrm{HO}} h_j(x), (L^0_{\mathrm{HO}})^{1/2} h_k(x)\big)_{L^2(\mathbb{R})} \\ &= \frac{2j+1}{2k+1}\big(h_j(x), (L^0_{\mathrm{HO}})^{1/2} h_k(x)\big)_{L^2(\mathbb{R})} \\ &= \frac{2j+1}{2k+1}\big((L^0_{\mathrm{HO}})^{1/2} h_j(x), h_k(x)\big)_{L^2(\mathbb{R})},\end{aligned} \tag{A.1.27}$$

so if $\big((L^0_{\mathrm{HO}})^{1/2} h_j(x), h_k(x)\big)_{L^2(\mathbb{R})} \neq 0$, $2j+1 = 2k+1$, so $j = k$. Hence, $(L^0_{\mathrm{HO}})^{1/2} h_j(x)$ must be a multiple of $h_j(x)$. Combined with nonnegativity of $(L^0_{\mathrm{HO}})^{1/2}$ by definition, and (A.1.26), (A.1.24) follows. □

A.2 Application to the perturbed Parabolic Cylinder Operator

To apply the theory of the previous section, we use the quadratic form defined by

$$\begin{aligned}
\mathfrak{D}\left(\mathfrak{t}_{z,b}\right) &:= \mathfrak{D}_1 \\
\mathfrak{t}_{z,b}\left(u,v\right) &:= \left(u'(x),v'(x)\right)_{L^2(\mathbb{R})} + \frac{1}{4}\left(xu(x),xv(x)\right)_{L^2(\mathbb{R})} - \frac{1}{2}\left(u(x),v(x)\right)_{L^2(\mathbb{R})} \\
&\qquad + zu(b)v(b) - zu(-b)v(-b), \quad u,v \in \mathfrak{D}_1
\end{aligned} \tag{A.2.1}$$

By Lemma 2.4.3, $\mathcal{S} \subset \mathfrak{D}_1 = \mathfrak{D}\left(\mathfrak{t}_{z,b}\right)$, so we immediately have that $\mathfrak{t}_{z,b}$ is densely defined. To use the first representation theorem, Proposition A.1.7, we must show that $\mathfrak{t}_{z,b}$ is sectorial and closed. We start with the proof of sectoriality, which uses a definition and a lemma.

Definition A.2.1. For f a bounded continous function on $\mathbb{R}$, let $\|f\|_{\text{unif}}$ denote the uniform norm or sup-norm over $\mathbb{R}$:

$$\|f\|_{\text{unif}} := \sup_{x\in\mathbb{R}}|f(x)|. \tag{A.2.2}$$

Lemma A.2.2. *For all $\epsilon > 0$, there exists $M = M(\epsilon)$ such that for all $f \in \mathcal{H}^1(\mathbb{R})$,*

$$\|f\|_{unif} \leq \epsilon\|u'(x)\|_{L^2(\mathbb{R})} + M\|u(x)\|_{L^2(\mathbb{R})}. \tag{A.2.3}$$

Proof. We first note that by $u \in \mathcal{H}^1(\mathbb{R})$, u is absolutely continuous on any finite interval (for the second part, see [Fol99, Exercise 8.9, p.246, and Exercise 9.31, p. 308]). We know that the product of absolutely continuous functions on an interval is absolutely continuous on the same interval, with the expected integration-by-parts formula (see, e.g., [Fol99, Exercise 3.35, p. 108]). Thus, we have that for a, c, $-\infty <$

$a < c < \infty$,

$$u^2(c) - u^2(a) = \int_a^c 2u(x)u'(x)\,dx \tag{A.2.4}$$

Yet by $u \in \mathcal{H}^1(\mathbb{R})$, $u(x)$ and $u'(x)$ are in $L^2(\mathbb{R})$, so $2u(x)u'(x) \in L^1(\mathbb{R})$. Hence, we know by the Dominated Convergence Theorem that

$$\begin{aligned} \lim_{a\to-\infty}(u^2(c) - u^2(a)) &= \lim_{a\to-\infty}\int_a^c 2u(x)u'(x)\,dx \\ &= \int_{-\infty}^c 2u(x)u'(x)\,dx. \end{aligned} \tag{A.2.5}$$

In particular then, we know that $\lim_{a\to-\infty} u^2(a)$ exists, so $\lim_{a\to-\infty} |u^2(a)|$ exists. Yet this latter limit must be zero: if $\lim_{a\to-\infty} |u^2(a)| = C \neq 0$, then there exists $M > 0$ such that for $a < -M$,

$$\begin{aligned} \big||u^2(a)| - C\big| &< \frac{C}{2} \\ |u^2(a)| &> \frac{C}{2} \end{aligned} \tag{A.2.6}$$

and so the set

$$\{x \in \mathbb{R} : |u^2(x)| > \frac{C}{2}\} \tag{A.2.7}$$

has infinite measure, containing the interval $(-\infty, -M)$. Yet this would force

$$\int_{-\infty}^{\infty} |u(x)|^2\,dx \geq \frac{C^2}{4}\int_{-\infty}^{-M} 1\,dx = \infty,$$

contradicting $u \in L^2(\mathbb{R})$.

Hence, $\lim_{a\to-\infty} |u^2(a)| = 0$, so by (A.2.5),

$$u^2(c) = \int_{-\infty}^c 2u(x)u'(x)\,dx$$

and thus

$$\begin{aligned} |u(c)|^2 &\le \left| \int_{-\infty}^{c} 2u(x)u'(x)\,dx \right| \\ &\le 2\int_{-\infty}^{c} |u(x)u'(x)|\,dx \\ &\le 2\int_{-\infty}^{\infty} |u(x)u'(x)|\,dx = 2\|u(x)u'(x)\|_{L^1(\mathbb{R})}. \end{aligned} \tag{A.2.8}$$

By Cauchy-Schwartz, this is bounded by $2\|u(x)\|_{L^2(\mathbb{R})}\|u'(x)\|_{L^2(\mathbb{R})}$. We may rewrite this, for any $\epsilon > 0$, as

$$2\left(\epsilon\|u'(x)\|_{L^2(\mathbb{R})}\right)\cdot\left(\frac{1}{\epsilon}\|u(x)\|_{L^2(\mathbb{R})}\right)$$

By the standard estimate $2\alpha\beta \le \alpha^2+\beta^2$ for real α, β, we have that

$$2\left(\epsilon\|u'(x)\|_{L^2(\mathbb{R})}\right)\cdot\left(\frac{1}{\epsilon}\|u(x)\|_{L^2(\mathbb{R})}\right) \le \left(\epsilon^2\|u'(x)\|^2_{L^2(\mathbb{R})} + \frac{1}{\epsilon^2}\|u(x)\|^2_{L^2(\mathbb{R})}\right).$$

Finally, we know for positive α, β that $\alpha^2+\beta^2 \le (\alpha+\beta)^2$, so we have

$$\left(\epsilon^2\|u'(x)\|^2_{L^2(\mathbb{R})} + \frac{1}{\epsilon^2}\|u(x)\|^2_{L^2(\mathbb{R})}\right) \le \left(\epsilon\|u'(x)\|_{L^2(\mathbb{R})} + \frac{1}{\epsilon}\|u(x)\|_{L^2(\mathbb{R})}\right)^2. \tag{A.2.9}$$

Altogether, then,

$$\begin{aligned} |u(c)|^2 &\le \left(\epsilon\|u'(x)\|_{L^2(\mathbb{R})} + \frac{1}{\epsilon}\|u(x)\|_{L^2(\mathbb{R})}\right)^2 \\ |u(c)| &\le \epsilon\|u'(x)\|_{L^2(\mathbb{R})} + \frac{1}{\epsilon}\|u(x)\|_{L^2(\mathbb{R})}. \end{aligned} \tag{A.2.10}$$

This works for all $c \in \mathbb{R}$, so letting $M(\epsilon) = \dfrac{1}{\epsilon}$, we are done. □

With this estimate, one straightforwardly adapts the $p(x) = 1$, $q(x) = \dfrac{x^2}{4} - \dfrac{1}{2}$,

$r(x) = s(x) = 0$, $a = -b$ case of Example 1.7 of Chapter VI on pp. 312–313 of [Kat95] to show:

Lemma A.2.3. *$\mathfrak{t}_{z,b}$ is sectorial; indeed, the aperture of the sector in which $\mathfrak{N}(\mathfrak{t}_{z,b})$ is contained can be made arbitrarily small if the offset γ is taken large enough.*

We now must show that $\mathfrak{t}_{z,b}$ is closed. We use a lemma.

Lemma A.2.4. *Suppose that $(u_n)_{n=1}^{\infty}$ is a sequence in $L^2(\mathbb{R})$ such that for some continuous function $\varphi : \mathbb{R} \to \mathbb{C}$, $\varphi(x)u_n(x) \in L^2(\mathbb{R})$ for all $n \in \mathbb{N}$. Further suppose that there exist $u, \Phi \in L^2(\mathbb{R})$ with $u_n(x) \xrightarrow{L^2(\mathbb{R})} u(x)$ and $\varphi(x)u_n(x) \xrightarrow{L^2(\mathbb{R})} \Phi(x)$. Then $\varphi(x)u(x) \in L^2(\mathbb{R})$ and $\Phi(x) = \varphi(x)u(x)$ almost everywhere (with respect to Lebesgue measure).*

Proof. Since $u_n(x) \xrightarrow{L^2(\mathbb{R})} u(x)$, we have by the standard theory of integration, e.g., [Fol99, Cor. 2.32, p. 62], that there exists a subsequence $(n_k)_{k=1}^{\infty}$ with $u_{n_k} \to u$ pointwise almost everywhere (with respect to Lebesgue measure). (Hereinafter, "almost everywhere," is assumed to refer to Lebesgue measure and is abbreviated a.e..) Then, of course, multiplying by φ,

$$\varphi(x)u_{n_k}(x) \to \varphi(x)u(x) \quad \text{for a.e. } x \in \mathbb{R} \tag{A.2.11}$$

Yet we know that $\varphi(x)u_n(x) \xrightarrow{L^2(\mathbb{R})} \Phi(x)$, so since subsequences have the same limits as the full sequences,

$$\lim_{k\to\infty} \varphi(x)u_{n_k}(x) = \Phi(x) \quad \text{in } L^2(\mathbb{R}),$$

and hence by the standard theory again, there exists a further subsequence $(n_{k_\ell})_{\ell=1}^{\infty}$ with

$$\lim_{\ell\to\infty} \varphi(x)u_{n_{k_\ell}}(x) = \Phi(x) \quad \text{for a.e. } x \in \mathbb{R} \tag{A.2.12}$$

Yet again, by subsequences having the same limits as the full sequences, we have that

$$\lim_{\ell\to\infty} \varphi(x)u_{n_{k_\ell}}(x) = \varphi(x)u(x) \quad \text{for a.e. } x \in \mathbb{R} \tag{A.2.13}$$

Therefore, $\varphi(x)u(x)$ and $\Phi(x)$ are the same functions pointwise a.e., so $\varphi(x)u(x) \in L^2(\mathbb{R})$ and

$$\varphi(x)u(x) = \Phi(x) \quad \text{for a.e. } x \in \mathbb{R}, \tag{A.2.14}$$

and hence, by $\Phi \in L^2(\mathbb{R})$, $\varphi(x)u(x) \in L^2(\mathbb{R})$. □

Corollary A.2.5. *If $(u_n)_{n=1}^{\infty}$ is a sequence in $\mathcal{H}^1(\mathbb{R})$, and there exist functions $u, \Psi \in L^2(\mathbb{R})$ such that $u_n \xrightarrow{L^2(\mathbb{R})} u$ and $\frac{du_n}{dx} \xrightarrow{L^2(\mathbb{R})} \Psi$, then $u \in \mathcal{H}^1(\mathbb{R})$ and $\Psi = \frac{du}{dx}$.*

Proof. Let $(u_n)_{n=1}^{\infty}$ is a sequence in $\mathcal{H}^1(\mathbb{R})$, such that the functions u, Ψ in $L^2(\mathbb{R})$ exist such that $u_n \xrightarrow{L^2(\mathbb{R})} u$ and $\frac{du_n}{dx} \xrightarrow{L^2(\mathbb{R})} \Psi$. Then taking the Fourier Transform, and applying the norm-preservation of the Fourier transform and the inequality (IX.1) of [RS75, p.2], we know that

$$\widehat{u_n}(\xi) \xrightarrow{L^2(\mathbb{R})} \widehat{u}(\xi) \tag{A.2.15a}$$

and

$$i\xi\widehat{u_n}(\xi) \xrightarrow{L^2(\mathbb{R})} \widehat{\Psi}(\xi) \tag{A.2.15b}$$

Applying Lemma A.2.4 with $x = \xi$, $\varphi(\xi) = i\xi$ and $\Phi(\xi) = \widehat{\Psi}(\xi)$, it follows that $i\xi\widehat{u}(\xi) \in L^2(\mathbb{R})$ and $i\xi\widehat{u}(\xi) = \widehat{\Psi}$ a.e. In particular, $u \in \mathcal{H}^1(\mathbb{R})$. Taking the inverse Fourier Transform, it follows that

$$\frac{du_n}{dx} \xrightarrow{L^2(\mathbb{R})} \frac{du}{dx}. \tag{A.2.16}$$

$\square$

Lemma A.2.6. $\mathfrak{t}_{z,b}$ *is closed.*

Proof. Suppose that there exists $(u_n)_{n=1}^{\infty} \in \mathfrak{D}(\mathfrak{t}_{z,b})$ and $u \in L^2(\mathbb{R})$ such that the conditions

$$u_n(x) \xrightarrow{L^2(\mathbb{R})} u(x), \tag{A.2.17a}$$

$$\mathfrak{t}_{z,b}(u_n - u_m, u_n - u_m) \to 0 \text{ as } m, n \to \infty, \tag{A.2.17b}$$

hold. First, note that by the definition of $\mathfrak{t}_{z,b}$,

$$\begin{aligned} \mathfrak{t}_{z,b}(u_n - u_m, u_n - u_m) = &\left\| \frac{du_n}{dx} - \frac{du_m}{dx} \right\|_{L^2(\mathbb{R})}^2 \\ &+ \frac{1}{4}\|xu_n - xu_m\|_{L^2(\mathbb{R})}^2 \\ &- \frac{1}{2}\|u_n - u_m\|_{L^2(\mathbb{R})}^2 \\ &+ z|u_n(b) - u_m(b)|^2 - z|u_n(-b) - u_m(-b)|^2 \end{aligned} \tag{A.2.18}$$

Also, note by u_n and u_m in $\mathfrak{D}(\mathfrak{t}_{z,b})$, by the $\epsilon = \dfrac{1}{4\sqrt{|z|}}$ case of Lemma A.2.2, we have that

$$\begin{aligned} &\left| z|u_n(b) - u_m(b)|^2 - z|u_n(-b) - u_m(-b)|^2 \right| \\ &\leq |z| \cdot |u_n(b) - u_m(b)|^2 - |z| \cdot |u_n(-b) - u_m(-b)|^2 \\ &\leq 2|z| \left(\frac{1}{4\sqrt{|z|}} \left\| \frac{du_n}{dx} - \frac{du_m}{dx} \right\|_{L^2(\mathbb{R})} + M\|u_n(x) - u_m(x)\|_{L^2(\mathbb{R})} \right)^2 \\ &\leq 4|z| \left(\frac{1}{16|z|} \left\| \frac{du_n}{dx} - \frac{du_m}{dx} \right\|_{L^2(\mathbb{R})}^2 + M^2\|u_n(x) - u_m(x)\|_{L^2(\mathbb{R})}^2 \right) \\ &= \frac{1}{4} \left\| \frac{du_n}{dx} - \frac{du_m}{dx} \right\|_{L^2(\mathbb{R})}^2 + 4|z|M^2\|u_n(x) - u_m(x)\|_{L^2(\mathbb{R})}^2, \end{aligned} \tag{A.2.19}$$

where in the next-to-last line we used the inequality for nonnegative constants α, β,

$$(\alpha+\beta)^2 \leq 2(\alpha^2+\beta^2).$$

Therefore, we have by (A.2.18) that

$$\begin{aligned}
&\left|\mathfrak{t}_{z,b}\left(u_n-u_m, u_n-u_m\right)\right| \geq \\
&\geq \left\|\frac{du_n}{dx}-\frac{du_m}{dx}\right\|^2_{L^2(\mathbb{R})}+\frac{1}{4}\|xu_n-xu_m\|^2_{L^2(\mathbb{R})} \\
&\quad -\frac{1}{2}\|u_n-u_m\|^2_{L^2(\mathbb{R})} \\
&\quad -\left|z|u_n(-b)-u_m(-b)|^2-z|u_n(-b)-u_m(-b)|^2\right| \qquad \text{(A.2.20)} \\
&\geq \frac{3}{4}\left\|\frac{du_n}{dx}-\frac{du_m}{dx}\right\|^2_{L^2(\mathbb{R})} \\
&\quad +\frac{1}{4}\|xu_n-xu_m\|^2_{L^2(\mathbb{R})} \\
&\quad -\left(\frac{1}{2}+4|z|M^2\right)\|u_n-u_m\|^2_{L^2(\mathbb{R})}.
\end{aligned}$$

Bounding $\frac{1}{4}\|xu_n-xu_m\|^2_{L^2(\mathbb{R})}$ below by 0, and rearranging, we have

$$\begin{aligned}
\left\|\frac{du_n}{dx}-\frac{du_m}{dx}\right\|^2_{L^2(\mathbb{R})} \leq \frac{4}{3}\Big(&\left|\mathfrak{t}_{z,b}\left(u_n-u_m, u_n-u_m\right)\right|+ \\
&+\left[\frac{1}{2}+4|z|M^2\right]\|u_n-u_m\|^2_{L^2(\mathbb{R})}\Big)
\end{aligned} \qquad \text{(A.2.21)}$$

By the hypothesis (A.2.17b), we know that as $n, m \to \infty$, $\left|\mathfrak{t}_{z,b}\left(u_n-u_m, u_n-u_m\right)\right| \to 0$. Yet we know by (A.2.17a) that u_n converges to u in $L^2(\mathbb{R})$, and hence $(u_n)_{n=1}^{\infty}$ is Cauchy in $\mathcal{H}$, so as $n, m \to \infty$,

$$\|u_n-u_m\|_{L^2(\mathbb{R})} \to 0.$$

Hence, by the Squeeze Theorem,

$$\lim_{n,m\to\infty}\left\|\frac{du_n}{dx}-\frac{du_m}{dx}\right\|^2_{L^2(\mathbb{R})}=0; \tag{A.2.22}$$

i.e., $\left(\dfrac{du_n}{dx}\right)_{n=1}^{\infty}$ is Cauchy in the Hilbert space $L^2(\mathbb{R})$, hence convergent: there exists some $\Psi(x)\in L^2(\mathbb{R})$ such that $\dfrac{du_n}{dx}\to\Psi(x)$. By Corollary A.2.5, it follows that $u\in\mathcal{H}^1(\mathbb{R})$ and

$$\frac{du_n}{dx}\xrightarrow{L^2(\mathbb{R})}\frac{du}{dx}. \tag{A.2.23}$$

On the other hand, if we start with (A.2.20) and bound the derivative norm below by 0, we have that , we have that

$$\begin{aligned}&|\mathfrak{t}_{z,b}\left(u_n-u_m,u_n-u_m\right)|\geq\\&\geq\frac{1}{4}\|xu_n-xu_m\|^2_{L^2(\mathbb{R})}\\&\quad-\left(\frac{1}{2}+4|z|M^2\right)\|u_n-u_m\|^2_{L^2(\mathbb{R})}.\end{aligned} \tag{A.2.24}$$

Rearranging,

$$\begin{aligned}\|xu_n-xu_m\|^2_{L^2(\mathbb{R})}\leq 4|\mathfrak{t}_{z,b}\left(u_n-u_m,u_n-u_m\right)|\\+(2+16|z|M)\|u_n-u_m\|^2_{L^2(\mathbb{R})}\end{aligned} \tag{A.2.25}$$

Yet again, both terms on the right-hand side become arbitrarily small for m, n large enough, and hence we see that $(xu_n(x))_{n=1}^{\infty}$ is Cauchy in $L^2(\mathbb{R})$, hence convergent. By Lemma A.2.4, with $\varphi(x)=x$, it follows that $xu(x)\in L^2(\mathbb{R})$ and

$$xu_n(x)\xrightarrow{L^2(\mathbb{R})}xu(x). \tag{A.2.26}$$

Thus, $u \in \mathfrak{D}(\mathfrak{t}_{z,b})$; moreover, we see that by definition of $\mathfrak{t}_{z,b}$,

$$\mathfrak{t}_{z,b}(u_n - u, u_n - u) \tag{A.2.27a}$$
$$= \left\| \frac{du_n}{dx} - \frac{du}{dx} \right\|^2_{L^2(\mathbb{R})} \tag{A.2.27b}$$
$$+ \frac{1}{4} \|xu_n(x) - xu(x)\|^2_{L^2(\mathbb{R})} \tag{A.2.27c}$$
$$- \frac{1}{2} \|u_n(x) - u(x)\|^2_{L^2(\mathbb{R})} \tag{A.2.27d}$$
$$+ z|u_n(b) - u(b)|^2 - z|u_n(-b) - u(-b)|^2 \tag{A.2.27e}$$

By (A.2.23), the norm in (A.2.27b) tends to 0 as $n \to \infty$. By (A.2.26), the term in (A.2.27c) tends to 0 as $n \to \infty$. By (A.2.17a), the term (A.2.27d) converges to 0 as $n \to \infty$. Replacing u_m by u in (A.2.19), and using both (A.2.23) and (A.2.17a), the terms in (A.2.27e) converge to 0 as $n \to \infty$. All in all, we see that

$$\lim_{n \to \infty} \mathfrak{t}_{z,b}(u_n - u, u_n - u) = 0, \tag{A.2.28}$$

as required. $\mathfrak{t}_{z,b}$ is indeed closed. □

Remark A.2.7. *By similar arguments, one may indeed show that* $\mathfrak{t}^0_{HO}$ *is closed, as we indicated earlier.*

Before continuing, we need a technical lemma, stating essentially that stating the effect on the Schwartz class $\mathcal{S}$ is enough to describe the quadratic form above.

Lemma A.2.8. $\mathcal{S}$ *is a core for* $\mathfrak{t}_{z,b}$.

Proof. In fact, we will show that finite linear combinations of Hermite functions are a core for $\mathfrak{t}_{z,b}$; the Hermite functions are in $\mathcal{S}$ by Proposition 2.2.2, hence finite linear combinations of them are.

The trick is to use Lemma 2.4.4 and its proof. For any $u(x)$, $v(x)$ in $\mathfrak{D}_1$, then by Lemma 2.4.4, we have that $u(x)$, $v(x)$ are in $\iota(\mathfrak{L}_1)$, i.e., we have that

$$u(x) = \sum_{k=0}^{\infty} c_k h_k(x), \qquad (c_k)_{k=0}^{\infty}, (c_k\sqrt{k+1})_{k=0}^{\infty} \in \ell^2, \tag{A.2.29a}$$

$$v(x) = \sum_{k=0}^{\infty} a_k h_k(x), \qquad (a_k)_{k=0}^{\infty}, (a_k\sqrt{k+1})_{k=0}^{\infty} \in \ell^2. \tag{A.2.29b}$$

Moreover, by the proof of $\iota(\mathfrak{L}_1) \subseteq \mathfrak{D}_1$, we know that if

$$u_J(x) := \sum_{k=0}^{J} c_k h_k(x), \quad J \in \mathbb{N}_0, \tag{A.2.30a}$$

$$v_J(x) := \sum_{k=0}^{J} a_k h_k(x), \quad J \in \mathbb{N}_0, \tag{A.2.30b}$$

then $xu_J(x) \xrightarrow{L^2(\mathbb{R})} xu(x)$, and similarly for $xv_J(x)$; moreover, for all $J \in \mathbb{N}_0$, $u_J(x)$ and $v_J(x)$ are in $\mathcal{S}$. Therefore, by a consequence of the Schwarz Inequality (see, e.g, [Fol99, Prop. 5.21, p. 173])

$$\lim_{J\to\infty} (xu_J(x), xv_J(x))_{L^2(\mathbb{R})} = (xu(x), xv(x))_{L^2(\mathbb{R})}. \tag{A.2.31}$$

Similarly, we have that $\xi\widehat{u_J}(\xi) \xrightarrow{L^2(\mathbb{R})} \xi\widehat{u}(\xi)$ and $\xi\widehat{v_J}(\xi) \xrightarrow{L^2(\mathbb{R})} \xi\widehat{v}(\xi)$, which implies by (2.1.4) that $u_J'(x) \xrightarrow{L^2(\mathbb{R})} u'(x)$ and $v_J'(x) \xrightarrow{L^2(\mathbb{R})} v'(x)$. Hence,

$$\lim_{J\to\infty} (u_J'(x), v_J'(x))_{L^2(\mathbb{R})} = (xu(x), xv(x))_{L^2(\mathbb{R})}. \tag{A.2.32}$$

Finally, by Lemma A.2.2 and the more immediate fact that $u_J(x) \xrightarrow{L^2(\mathbb{R})} u(x)$ and $v_J(x) \xrightarrow{L^2(\mathbb{R})} v(x)$, we know that $\lim_{J\to\infty} u_J(b) = u(b)$ and $\lim_{J\to\infty} v_J(b) = v(b)$, similarly for

$-b$. Hence,

$$\lim_{J\to\infty} zu_J(b)v_J(b) - zu_J(-b)v_J(-b) = zu(b)v(b) - zu(-b)v(-b). \qquad \text{(A.2.33)}$$

Combining (A.2.31), (A.2.32), and (A.2.33), we have that

$$\lim_{J\to\infty} \mathfrak{t}_{z,b}\left(u_J(x), v_J(x)\right) = \mathfrak{t}_{z,b}\left(u(x), v(x)\right), \qquad \text{(A.2.34)}$$

for all $u(x), v(x) \in \mathfrak{D}_1$. The u_J and v_J are finite linear combinations of Hermite functions and hence are in $\mathcal{S}$. Thus, the closure of $\mathfrak{t}_{z,b}|_{\mathcal{S}}$ contains $\mathfrak{t}_{z,b}$. Since $\mathfrak{t}_{z,b}$ is closed, the reverse inclusion holds. Hence, $\mathcal{S}$ is a core of $\mathfrak{t}_{z,b}$. □

Now we may invoke the first representation theorem, and construct a closed, densely-defined operator – call it $L^1_{\mathrm{PC}}(z,b)$ – from the application of Proposition A.1.7. The question is why $L^1_{\mathrm{PC}}(z,b)$ has anything to do with the operator $L_{\mathrm{PC}}(z,b)$ defined in (3.3.1).

Proposition A.2.9. *For all $b > 0$ and $z \in \mathbb{C}$, $L^1_{PC}(z,b) \subseteq L_{PC}(z,b)$.*

Proof. For any $y \in \mathfrak{D}\left(L^1_{\mathrm{PC}}(z,b)\right)$, $L^1_{\mathrm{PC}}(z,b)y \in L^2(\mathbb{R})$ by definition. Recall that any L^2 function f forms a *tempered* distribution under the rule

$$\langle f, \varphi\rangle_{\mathcal{S}',\mathcal{S}} = \int f(x)\varphi(x)\,dx, \quad \varphi(x) \in \mathcal{S}. \qquad \text{(A.2.35)}$$

Note that we have that for $f \in L^2(\mathbb{R})$,

$$(f,\varphi)_{L^2(\mathbb{R})} = \int_{\mathbb{R}} f(x)\overline{\varphi(x)}\,dx = \langle f, \overline{\varphi}\rangle_{\mathcal{S}',\mathcal{S}}, \qquad \text{(A.2.36)}$$

but since the Schwartz class is closed under conjugation, this causes no essential difficulty.

Now, suppose $u \in \mathfrak{D}\left(L^1_{\mathrm{PC}}(z,b)\right)$, so that $L^1_{\mathrm{PC}}(z,b)u \in L^2(\mathbb{R})$. On the one hand, by (A.2.36), we have that for all $\varphi \in \mathcal{S}$,

$$\left(L^1_{\mathrm{PC}}(z,b)u, \varphi\right)_{L^2(\mathbb{R})} = \left\langle L^1_{\mathrm{PC}}(z,b)u, \overline{\varphi}\right\rangle_{\mathcal{S}',\mathcal{S}}. \tag{A.2.37}$$

On the other hand, by property a), we know that for all $\varphi(x) \in \mathcal{S}$, which are certainly in $\mathfrak{D}\left(\mathfrak{t}_{z,b}\right)$,

$$\left(L^1_{\mathrm{PC}}(z,b)u, \varphi\right)_{L^2(\mathbb{R})} = \mathfrak{t}_{z,b}\left(u, \varphi\right) \tag{A.2.38}$$

and writing this out, we have

$$(u'(x), \varphi'(x))_{L^2(\mathbb{R})} \tag{A.2.39a}$$
$$+ \frac{1}{4}(xu(x), x\varphi(x))_{L^2(\mathbb{R})} \tag{A.2.39b}$$
$$- \frac{1}{2}(u(x), \varphi(x))_{L^2(\mathbb{R})} \tag{A.2.39c}$$
$$+ zu(b)\overline{\varphi(b)} - zu(b)\overline{\varphi(-b)} \tag{A.2.39d}$$

We convert these to distributional form. For (A.2.39a), by (A.2.36), we have that

$$(u'(x), \varphi'(x))_{L^2(\mathbb{R})} = \left\langle u', \overline{\varphi'(x)}\right\rangle_{\mathcal{S}',\mathcal{S}} \tag{A.2.40}$$

Yet by the rules of (tempered) distributions and derivatives, this simply becomes

$$-\left\langle u''(x), \overline{\varphi(x)}\right\rangle_{\mathcal{S}',\mathcal{S}}. \tag{A.2.41}$$

For (A.2.39b), we apply (A.2.36) and then move the x over to the other side, since for $x \in \mathbb{R}$, $\overline{x} = x$:

$$\begin{aligned} \frac{1}{4}(xu(x), x\varphi(x))_{L^2(\mathbb{R})} &= \frac{1}{4}\int_{\mathbb{R}} xu(x)\overline{x\varphi(x)}\,dx \\ &= \frac{1}{4}\int_{\mathbb{R}} x^2u(x)\overline{\varphi(x)}\,dx \\ &= \frac{1}{4}\left\langle x^2u(x), \overline{\varphi(x)}\right\rangle_{\mathcal{S}',\mathcal{S}} \end{aligned} \tag{A.2.42}$$

For (A.2.39c), a straightforward application of (A.2.36) holds. Finally, for (A.2.39d), we have that

$$zu(b)\overline{\varphi(b)} = zu(b)\left\langle \delta\left(x-b\right), \overline{\varphi(x)}\right\rangle_{\mathcal{S}',\mathcal{S}} = \left\langle zu(b)\delta\left(x-b\right), \overline{\varphi(x)}\right\rangle_{\mathcal{S}',\mathcal{S}}, \tag{A.2.43}$$

and similarly for $-zu(b)\overline{\varphi(-b)}$. Putting it all together, we have

$$\begin{aligned} &\left(L^1_{\mathrm{PC}}(z,b)u, \varphi\right)_{L^2(\mathbb{R})} = \\ &= \Big\langle -u''(x) + \frac{1}{4}x^2u(x) - \frac{1}{2}u(x) + zu(b)\delta\left(x-b\right) - \\ &\qquad -zu(-b)\delta\left(x+b\right), \overline{\varphi(x)}\Big\rangle_{\mathcal{S}',\mathcal{S}} \end{aligned} \tag{A.2.44}$$

Comparing with (A.2.37), and noting that conjugation preserves the Schwartz class, we see that $L^1_{\mathrm{PC}}(z,b)u(x)$ and

$$-u''(x) + \frac{1}{4}x^2u(x) - \frac{1}{2}u(x) + zu(b)\delta\left(x-b\right) - zu(-b)\delta\left(x+b\right) \tag{A.2.45}$$

are identical as distributions. Yet $L^1_{\mathrm{PC}}(z,b)u$ is an $L^2(\mathbb{R})$ function, so it follows that (A.2.45) is an $L^2(\mathbb{R})$ function. In other words, $u \in \mathfrak{D}\left(L_{\mathrm{PC}}(z,b)\right)$ and $L^1_{\mathrm{PC}}(z,b)u = L_{\mathrm{PC}}(z,b)u$. This works for all $u \in \mathfrak{D}\left(L_{\mathrm{PC}}(z,b)\right)$, so $L^1_{\mathrm{PC}}(z,b) \subseteq L_{\mathrm{PC}}(z,b)$. □

Proposition A.2.10. *For all $b > 0$ and $z \in \mathbb{C}$, $L_{PC}(z,b) \subseteq L^1_{PC}(z,b)$.*

Proof. Fix $b > 0$, $z \in \mathbb{C}$, and $y \in \mathfrak{D}\left(L_{\mathrm{PC}}(z,b)\right)$. Then $L_{\mathrm{PC}}(z,b)y(x) \in L^2(\mathbb{R})$, so by (A.2.36) we can write, for any $\varphi \in \mathcal{S}$,

$$(L_{\mathrm{PC}}(z,b)y(x), \varphi)_{L^2(\mathbb{R})} = \Big\langle L_{\mathrm{PC}}(z,b)y(x), \overline{\varphi(x)} \Big\rangle_{\mathcal{S}',\mathcal{S}}, \tag{A.2.46}$$

and by writing out and separating the terms, this works out to

$$\begin{aligned}
&- \Big\langle y''(x), \overline{\varphi(x)} \Big\rangle_{\mathcal{S}',\mathcal{S}} \\
&+ \frac{1}{4}\Big\langle x^2 y(x), \overline{\varphi(x)} \Big\rangle_{\mathcal{S}',\mathcal{S}} \\
&- \frac{1}{2}\Big\langle y(x), \overline{\varphi(x)} \Big\rangle_{\mathcal{S}',\mathcal{S}} \\
&+ zy(b)\Big\langle \delta\left(x-b\right), \overline{\varphi(x)} \Big\rangle_{\mathcal{S}',\mathcal{S}} \\
&- zy(-b)\Big\langle \delta\left(x+b\right), \overline{\varphi(x)} \Big\rangle_{\mathcal{S}',\mathcal{S}}.
\end{aligned} \tag{A.2.47}$$

Yet reversing the work of the previous section, this simply becomes

$$\begin{aligned}
\Big\langle y'(x), \overline{\varphi'(x)} \Big\rangle_{\mathcal{S}',\mathcal{S}} &+ \frac{1}{4}\Big\langle xy(x), \overline{x\varphi(x)} \Big\rangle_{\mathcal{S}',\mathcal{S}} \\
- \frac{1}{2}\Big\langle y(x), \overline{\varphi(x)} \Big\rangle_{\mathcal{S}',\mathcal{S}} &+ zy(b)\overline{\varphi(b)} - zy(b)\overline{\varphi(-b)}
\end{aligned} \tag{A.2.48}$$

Moreover, for all $y \in \mathfrak{D}\left(L_{\mathrm{PC}}(z,b)\right)$, $y'(x) \in L^2(\mathbb{R})$ and $xy(x) \in L^2(\mathbb{R})$, so we can convert (A.2.48) to (A.2.39) by (A.2.36). In other words, for all $y \in \mathfrak{D}\left(L_{\mathrm{PC}}(z,b)\right)$, and all $\varphi \in \mathcal{S}$, we have that

$$(L_{\mathrm{PC}}(z,b)y(x), \varphi(x))_{L^2(\mathbb{R})} = \mathfrak{t}_{z,b}\left(y(x), \varphi(x)\right). \tag{A.2.49}$$

This works for all $y(x)$ in $\mathfrak{D}\left(L_{\mathrm{PC}}(z,b)\right)$ and all $\varphi(x)$ in $\mathcal{S}$, which by Lemma A.2.8 is a core of $\mathfrak{t}_{z,b}$. Hence, by part a) of Proposition A.1.7, $L_{\mathrm{PC}}(z,b) \subseteq L^1_{\mathrm{PC}}(z,b)$. □

To finish the proof of Propostion 3.3.1, we now must establish certain remaining properties.

Lemma A.2.11. *For all $r \in \mathbb{R}$ and $b > 0$, $L_{PC}(r, b)$ is self-adjoint.*

Proof. The main idea is to use part b) of the first representation theorem, Prop. A.1.7. We must show that $L_{\mathrm{PC}}(r, b)$ is symmetric or bounded below. To show symmetry, we must show that for $b > 0$ and r real,

$$\mathfrak{t}_{r,b}\left(v(x), u(x)\right) = \overline{\mathfrak{t}_{r,b}\left(u(x), v(x)\right)}. \tag{A.2.50}$$

Yet by the inner product $(u, v)_{L^2(\mathbb{R})}$ havind the desired symmetry, the individual terms $(u'(x), v'(x))_{L^2(\mathbb{R})}$, $(xu(x), xv(x))_{L^2(\mathbb{R})}$, $u(b)\overline{v(b)}$, and $u(-b)\overline{v(-b)}$ are quadratic forms with the desired symmetry, and by r real, they are multiplied by real weights, and so the whole expression has the desired symmetry.

We already showed that $L_{\mathrm{PC}}(r, b)$ is sectorial, which in particular implies that $\mathfrak{N}(\mathfrak{t}_{r,b})$ is contained in a half-plane of the form $\{\zeta \in \mathbb{C} : \operatorname{Re}\zeta \geq \gamma\}$. Since symmetric forms have numerical range on the real line, we have that $L_{\mathrm{PC}}(r, b)$ is bounded below. We may now use part b) of Prop. A.1.7. □

Lemma A.2.12. *For all $b > 0$ and $z \in \mathbb{C}$, $L_{PC}(z, b)$ has compact resolvent.*

Proof. We first show that for any fixed $b > 0$, the family $\{\mathfrak{t}_{z,b} : z \in \mathbb{C}\}$ of forms is a *holomorphic family of quadratic forms of type (a)* in the sense of Kato ([Kat95, Chapter VII, Section 4.2, p. 395]). This means that all of the forms have the same domain, and for any $u(x)$ in that domain, $\mathfrak{t}_{z,b}\left(u(x), u(x)\right)$ varies holomorphically in z. Of course, $\mathfrak{t}_{z,b}$ was defined on $\mathfrak{D}_1$ independently of z, and for fixed $b > 0$ and $u(x) \in \mathfrak{D}_1$, $\mathfrak{t}_{z,b}\left(u(x), u(x)\right)$ is *linear* in z! Thus, the requirement is satisfied.

It follows that for fixed $b > 0$, the family $\{\mathfrak{t}_{z,b} : z \in \mathbb{C}\}$ is a *holomorphic family of quadratic forms of type (a)* in the sense of Kato ([Kat95, Chapter VII, Section 4.2, p.

395]). The family of associated operators $\{L^1_{\mathrm{PC}}(z,b) : z \in \mathbb{C}\} = \{L_{\mathrm{PC}}(z,b) : z \in \mathbb{C}\}$ is called a *holomorphic family of quadratic forms of type (B)* in the sense of Kato ([Kat95, Chapter VII, Section 4.2, p. 395]). This is valuable because by [Kat95, Chapter VII, Section 4.2, Thm. 4.3, p. 396], the members of a holomorphic family of type (B) either all have compact resolvent, or none do. Yet by Proposition 3.2.1, L^0_{PC} has compact resolvent, and for any $b > 0$, it is clear from (3.3.1b) that $L_{\mathrm{PC}}(0,b) = L^0_{\mathrm{PC}}$. Therefore, we may indeed conclude that for any $b > 0$, for all $z \in \mathbb{C}$, $L_{\mathrm{PC}}(z,b)$ has compact resolvent. $\square$

This finishes the proof of Proposition 3.3.1.

Appendix B
PROOF OF CONTINUITY AND JUMP CONDITIONS ON EIGENFUNCTIONS

B.1 Reminders on Distributions

In this section, we give the promised proof on the jump conditions on eigenfunctions, namely Proposition 3.3.2. To do so, we remind ourselves of the "regular" distributions.

Definition B.1.1. Fix Ω open in $\mathbb{R}$. Let $C^\infty_{\text{cpct}}(\Omega)$ denote the set of C^∞ functions compactly supported in Ω. A *distribution on* Ω is a continuous linear functional on Ω. We denote the space of distributions on Ω by $D'(\Omega)$. We denote the pairing between a distribution F and a test-function $\varphi \in C^\infty_{\text{cpct}}(\Omega)$ by

$$\langle F, \varphi \rangle_{D'(\Omega),\, C^\infty_{\text{cpct}}(\Omega)}. \tag{B.1.1}$$

Lemma B.1.2 (Locally integrable functions and distributions; e.g., [Fol99, p.283])**.** *The analogues of* (A.2.35) *and* (A.2.36) *hold for "regular" distributions; i.e., we have that for all $f \in L^2(\mathbb{R})$ all Ω open in $\mathbb{R}$, and all $\varphi \in C^\infty_{cpct}(\Omega)$,*

$$\langle f, \varphi \rangle_{D'(\Omega),\, C^\infty_{cpct}(\Omega)} = \int f(x)\varphi(x)\, dx. \tag{B.1.2}$$

Moreover,

$$(f, \varphi)_{L^2(\mathbb{R})} = \int_{\mathbb{R}} f(x)\overline{\varphi(x)}\, dx = \langle f, \overline{\varphi} \rangle_{D'(\Omega), C^\infty_{cpct}(\Omega)}, \tag{B.1.3}$$

B.2 The Proof

Proof of Proposition 3.3.2. Fix $b > 0$ and $z \in \mathbb{C}$. Suppose that $y(x) \in \mathfrak{D}\left(L_{\text{PC}}(z, b)\right)$ is an eigenfunction of $L_{\text{PC}}(z, b)$. We first show that $y(x)$, restricted to (b, ∞), is C^∞ on (b, ∞); the infinite differentiability on the other intervals will follow similarly. First, since $y \in \mathfrak{D}\left(L_{\text{PC}}(z, b)\right) \subseteq \mathfrak{D}\left(\mathfrak{t}_{z,b}\right) = \mathfrak{D}_1 \subseteq \mathcal{H}^1(\mathbb{R})$, it is continuous, in $L^2(\mathbb{R})$, and satisfies $L_{\text{PC}}(z, b)y(x) \in L^2(\mathbb{R})$, so both $L_{\text{PC}}(z, b)y(x)$ and $\nu y(x)$ form distributions on (b, ∞) in the manner of (B.1.2). Yet of course, for any $\varphi \in C^\infty_{\text{cpct}}\left((b, \infty)\right)$, we have that $\varphi(b) = 0$ and $\varphi(-b) = 0$, so

$$\begin{aligned}
&\langle L_{\text{PC}}(z, b)y(x), \varphi(x)\rangle_{D'((b,\infty)), C^\infty_{\text{cpct}}((b,\infty))} \\
&= \left\langle -y''(x) + \left(\frac{x^2}{4} - \frac{1}{2}\right) y(x), \varphi(x) \right\rangle_{D'((b,\infty)), C^\infty_{\text{cpct}}((b,\infty))} \\
&\qquad + \left\langle zy(b)\delta\left(x - b\right) - zy(-b)\delta\left(x + b\right), \varphi(x)\right\rangle_{D'((b,\infty)), C^\infty_{\text{cpct}}((b,\infty))} \\
&= \left\langle -y''(x) + \left(\frac{x^2}{4} - \frac{1}{2}\right) y(x), \varphi(x) \right\rangle_{D'((b,\infty)), C^\infty_{\text{cpct}}((b,\infty))} + 0,
\end{aligned} \tag{B.2.1}$$

since of course the point masses at $\pm b$ do not affect functions that are 0 at them. Yet since y is a ν-eigenvector, of $L_{\text{PC}}(z, b)$, it follows that

$$\begin{aligned}
&\langle \nu y(x), \varphi(x)\rangle_{D'((b,\infty)), C^\infty_{\text{cpct}}((b,\infty))} \\
&= \langle L_{\text{PC}}(z, b)y(x), \varphi(x)\rangle_{D'((b,\infty)), C^\infty_{\text{cpct}}((b,\infty))} \qquad , \quad \varphi \in C^\infty_{\text{cpct}}\left((b, \infty)\right) \\
&= \left\langle -y''(x) + \left(\frac{x^2}{4} - \frac{1}{2}\right) y(x), \varphi(x) \right\rangle_{D'((b,\infty)), C^\infty_{\text{cpct}}((b,\infty))}
\end{aligned} \tag{B.2.2}$$

or

$$\left\langle -y''(x) + \left(\frac{x^2}{4} - \left[\nu + \frac{1}{2}\right]\right) y(x), \varphi(x) \right\rangle_{D'((b,\infty)),\, C^\infty_{\text{cpct}}((b,\infty))} = 0 \qquad \text{(B.2.3)}$$

for all $\varphi \in C^\infty_{\text{cpct}}((b, \infty))$. In other words, defining the differential operator

$$Ly := -y'' + \left(\frac{x^2}{4} - \left[\nu + \frac{1}{2}\right]\right) y(x), \qquad \text{(B.2.4)}$$

, for all eigenfunctions y of $L_{\text{PC}}(z, b)$, Ly is indistinguishable from the 0 perturbation as a distribution on (b, ∞). Moreover, L is an second-order elliptic differential operator, since its principal symbol is

$$P_2(x, \xi) = -\xi^2,$$

which clearly satisfies

$$P_2(x, \xi) \neq 0 \text{ unless } \xi = 0, x \in RR.$$

Because of the $\frac{x^2}{4}$ term in the zero-order term, all we can say is that L has C^∞ coefficients. By the Local Regularity Theorem and its corollaries (see Folland, *Introduction to Partial Differential Equations*, in particular[Fol76, Thm. 6.30 and Cor. 6.31, pp. 269 – 270]), it follows that since $Ly = 0$ in $D'((b, \infty))$, and since 0 is C^∞, then y is $C^\infty(b, \infty)$.

We now show that y satisfies (3.3.3) on (b, ∞), i.e., "weak solutions with sufficient regularity are classical solutions." First, we note that by that $-y''(x)$ on (b, ∞) can now be considered the classical, and continuous, derivative $-\frac{d^2y}{dx^2}(x)$, the left-hand element of the distribution pair (B.2.3) is now continuous, hence locally integrably, so for all $\varphi \in C^{\infty}_{\text{cpct}}\left((b,\infty)\right)$,

$$\begin{aligned} 0 &= \left\langle -y''(x) + \left(\frac{x^2}{4} - \left[\nu + \frac{1}{2}\right]\right) y(x), \varphi \right\rangle_{D'((b,\infty)),\, C^{\infty}_{\text{cpct}}((b,\infty))} \\ &= \int_b^{\infty} \left\{ -\frac{d^2y}{dx^2}(x) + \left(\frac{x^2}{4} - \left[\nu + \frac{1}{2}\right]\right) y(x) \right\} \varphi(x)\, dx. \end{aligned} \tag{B.2.5}$$

For any α, β in $\mathbb{R}$, $b < \alpha < \beta < \infty$, we can take a series $\varphi_n(x)$ of $C^{\infty}_{\text{cpct}}\left((b,\infty)\right)$ functions , uniformly supported in $(\alpha - \epsilon, \beta + \epsilon)$ for some $\epsilon > 0$, approximating the step function $\frac{1}{\beta - \alpha} \mathbb{1}_{(\alpha,\beta)}$ in $L^2((b,\infty))$ norm. Hence, by Cauchy-Schwartz, for all $b < \alpha < \beta < \infty$, replacing the continous function $-\frac{d^2y}{dx^2}(x) + \left(\frac{x^2}{4} - \left[\nu + \frac{1}{2}\right]\right) y(x)$ by its truncation to $(\alpha - \epsilon, \beta + \epsilon)$ for L^2-estimates,

$$\begin{aligned} 0 &= \lim_{n\to\infty} 0 \\ &= \lim_{n\to\infty} \int_b^{\infty} \left\{ -\frac{d^2y}{dx^2}(x) + \left(\frac{x^2}{4} - \left[\nu + \frac{1}{2}\right]\right) y(x) \right\} \varphi_n(x)\, dx \\ &= \frac{1}{\beta - \alpha} \int_{\alpha}^{\beta} -\frac{d^2y}{dx^2}(x) + \left(\frac{x^2}{4} - \left[\nu + \frac{1}{2}\right]\right) y(x)\, dx. \end{aligned} \tag{B.2.6}$$

Yet we know that for any $g(x)$ continuous on an interval (α, β), for any $c \in (\alpha, \beta)$, we have by averaging arguments that

$$\lim_{\epsilon\to 0^+} \frac{1}{2\epsilon} \int_{c-\epsilon}^{c+\epsilon} g(x)\, dx = g(c); \tag{B.2.7}$$

see, e.g., [Fol99, Proof of Thm. 3.18, p. 97]. Hence, for any $c \in (b, \infty)$, we have by (B.2.6) that

$$\begin{aligned} 0 &= \lim_{\epsilon\to 0^+} 0 \\ &= \lim_{\epsilon\to 0^+} \frac{1}{2\epsilon} \int_{c-\epsilon}^{c+\epsilon} -\frac{d^2y}{dx^2}(x) + \left(\frac{x^2}{4} - \left[\nu + \frac{1}{2}\right]\right) y(x)\, dx \\ &= -\left.\frac{d^2y}{dx^2}\right|_{x=c} + \left(\frac{c^2}{4} - \left[\nu + \frac{1}{2}\right]\right) y(c); \end{aligned} \tag{B.2.8}$$

i.e., (3.3.3) holds at $x = c$. This works for all $c \in (b, \infty)$, so (3.3.3) holds on (b, ∞). Similarly, (3.3.3) will hold on the intervals $(-\infty, -b)$ and $(-b, b)$.

Now we must show that (3.3.4) holds; we show that (3.3.4b) holds, and (3.3.4a) will hold Now, y is continuous. Also, since $y(x)$ is a solution to (3.3.3) on $(-\infty, -b)$, $(-b, b)$ and (b, ∞), but by Proposition 2.5.1, any solution to (3.3.3) will freely extend to an analytic function on $\mathbb{C}$, we certainly have that the limits $y'(b+)$ and $y'(b-)$, and $y'(-b+)$ and $y'(-b-)$ exist. Then by iteration of [Fol99, Exercise 9.5, p. 289], the distributional derivative y'' satisfies on $(0, \infty)$

$$y''(x) = \frac{d^2y}{dx^2} + (y'(b+) - y'(b-))\delta\,(x - b) + (y'(-b+) - y'(-b-))\delta\,(x + b)\,, \quad \text{(B.2.9)}$$

but by hypothesis, y is an eigenvector of $\mathfrak{D}\,(L_{\mathrm{PC}}(z, b))$, hence is in $L^2(\mathbb{R}) \cap C(\mathbb{R})$ and satisfies

$$-y''(x) + \left(\frac{x^2}{4} - \frac{1}{2}\right) y(x) + zy(b)\delta\,(x - b) - zy(-b)\delta\,(x + b) \in L^2(\mathbb{R}).$$

In particular, the point-masses must cancel, and the point-masses on the left-hand side add up to

$$[zy(b) - (y'(b+) - y'(b-))]\,\delta\,(x - b) + [-zy(b) - (y'(-b+) - y'(-b-))]\,\delta\,(x + b)$$

so to cancel the point mass at $x = b$

$$\begin{aligned} zy(b) - (y'(b+) - y'(b-)) &= 0 \\ zy(b) &= (y'(b+) - y'(b-)); \end{aligned} \quad \text{(B.2.10)}$$

i.e., (3.3.4b) holds. Similarly, (3.3.4a) holds. □

Appendix C

IMPLICIT FUNCTIONS: PROOF OF DEAN'S RESULT

C.1 Preliminary Properties of any Implicit Function

Proposition 6.4.1 declares the existence of C^1 functions $g_k(x) : \mathbb{R} \to \mathbb{R}^+$ such that $D_{g_k(x)}x = 0$; i.e., we define $\nu = g_k(x)$ as an *implicit* function of x. In this section, we briefly note some properties that *any* implicit function on *any* subinterval of $\mathbb{R}$ must satisfy. For our first result, we require a few lemmas.

Lemma C.1.1. *$D_\nu(0) = 0$ if and only if $\nu \in 2\mathbb{N}_0 + 1$.*

Proof. It is known (e.g., [Tem14, Section 2, (12.2.6)] that

$$D_\nu(0) = \frac{\sqrt{\pi}2^{\nu/2}}{\Gamma\left(-\frac{\nu}{2} + \frac{1}{2}\right)}, \tag{C.1.1}$$

and hence, exponentials being nonzero, it follows that $D_\nu(0) = 0$ if and only if $\left(-\frac{\nu}{2} + \frac{1}{2}\right)$ is one of the poles of the Gamma Function; i.e., $\left(-\frac{\nu}{2} + \frac{1}{2}\right) = -n$ for some $n \in \mathbb{N}_0$. In other words,

$$D_\nu(0) = 0 \quad \text{if and only if } \nu \in 2\mathbb{N}_0 + 1. \tag{C.1.2}$$

□

Lemma C.1.2. *Suppose that for some* $x_0 \in \mathbb{R}$ *and* $\nu_0 \in \mathbb{C}$, $D_{\nu_0}(x_0) = 0$. *Then* $\frac{\partial}{\partial x}(D_\nu(x))\Big|_{x=x_0,\nu=\nu_0} \neq 0$. *If* $x_0 \geq 0$, *then* $\frac{\partial}{\partial \nu}(D_\nu(x))\Big|_{\nu=\nu_0,x=x_0} \neq 0$.

Proof. Suppose, by way of contradiction, that for some $x_0 \in \mathbb{R}$ and $\nu_0 \in \mathbb{C}$. $D_{\nu_0}(x_0) = 0$ and $\frac{\partial}{\partial x}(D_\nu(x))\Big|_{x=x_0,\nu=\nu_0} = 0$. Then since $D_{\nu_0}(x)$ is a solution to the $\nu = \nu_0$ case of (3.1.1a), with initial conditions all 0, it follows by Proposition 2.5.1 that $D_{\nu_0}(x) \equiv 0$. Yet $D_{\nu_0}(x)$ was defined so that

$$\lim_{x\to\infty} \frac{D_{\nu_0}(x)}{x^{\nu_0}e^{-x^2/4}} = 1. \tag{C.1.3}$$

In particular, for x large enough, $D_{\nu_0}(x) \neq 0$. Contradiction. Hence, if $D_{\nu_0}(x_0) = 0$, $\frac{\partial}{\partial x}(D_\nu(x))\Big|_{x=x_0,\nu=\nu_0} \neq 0$.

In the process of proving Proposition 6.2.1, in particular the proof of Corollary 6.2.9, we showed that for $x_0 > 0$, the zeroes of $\nu \mapsto D_\nu(x_0)$, outside $\mathbb{N}_0$ were simple and on the positive real ν-axis. The exceptional case of zeroes of $\nu \mapsto D_\nu(x_0)$ in $\mathbb{N}$ was handled in the proof of Corollary 6.3.6 (and again, $D_0(x) \neq 0$ for all x, being essentially an exponential). Hence, for all $x_0 > 0$, if $D_{\nu_0}(x_0) = 0$, $\frac{\partial}{\partial \nu}(D_\nu(x))\Big|_{\nu=\nu_0,x=x_0} \neq 0$.

It remains to handle the case $x_0 = 0$. Differentiating (C.1.1) with respect to ν, and using the Product Rule and Chain Rule,

$$\begin{aligned}\frac{\partial}{\partial \nu}(D_\nu(0))\Big|_{\nu=\nu=\nu_0} &= \frac{\sqrt{\pi}2^{\nu/2}\cdot\log\sqrt{2}}{\Gamma\left(-\frac{\nu}{2}+\frac{1}{2}\right)} \\ &\quad + \sqrt{\pi}2^{\nu/2}\cdot\left(-\frac{1}{2}\right)\frac{d}{dz}\left(\frac{1}{\Gamma(z)}\right)\Big|_{z=(1-\nu)/2}\end{aligned} \tag{C.1.4}$$

Yet $D_\nu(0) = 0$ if and only if $\nu = 2k+1$, $k \in \mathbb{N}_0$, by Lemma C.1.1. In this case, the first term on the right-hand side in (C.1.4) zeroes, being a multiple of $\frac{1}{\Gamma(-k)}$. Moreover, $\frac{1}{\Gamma(z)}$ has only a simple pole at $z = -k$, so $\frac{d}{dz}\left(\frac{1}{\Gamma(z)}\right)\Big|_{z=(1-\nu)/2} \neq 0$. The other factors of the second term on the right-hand side of (C.1.4) are nonzero, hence

$$\frac{\partial}{\partial \nu}\left(D_\nu(0)\right)\Big|_{\nu=2k+1} \neq 0, \quad k \in \mathbb{N}_0. \tag{C.1.5}$$

□

We now note a result about the implicit functions, at least on $(0,\infty)$.

Corollary C.1.3. *Suppose that on any finite or infinite interval $(a,c) \subseteq (0,\infty)$, there exists a C^1 function $g : (a,c) \to \mathbb{R}^+$ with $D_{g(x)}(x) = 0$. Then $g'(x)$ is never 0 on (a,c). In fact, $g'(x)$ has a consistent sign on (a,c), so g is either increasing or decreasing on (a,c).*

Proof. Let $(a,c) \subseteq (0,\infty)$ be an open interval, and suppose that there exists g : $(a,c) \to \mathbb{R}^+$ be such that $D_{g(x)}(x) = 0$. Fix $b \in (a,c)$. Then by the rules of implicit differentiation, we have that

$$\frac{\partial}{\partial x}\left(D_\nu(x)\right)\Big|_{x=b,\nu=g(b)} + \frac{\partial}{\partial \nu}\left(D_\nu(x)\right)\Big|_{\nu=g(b),x=b} \cdot g'(b) = 0, \tag{C.1.6}$$

or

$$g'(b) = -\frac{\frac{\partial}{\partial x}\left(D_\nu(x)\right)\big|_{x=b,\nu=g(b)}}{\frac{\partial}{\partial \nu}\left(D_\nu(x)\right)\big|_{\nu=g(b),x=b}}. \tag{C.1.7}$$

By Lemma C.1.2, neither numerator nor denominator is 0, so $g'(b) \neq 0$. More importantly, since the map

$$b \mapsto \frac{\partial}{\partial x}\left(D_{g(x)}(x)\right)\Big|_{x=b,\nu=g(b)}$$

is real-valued and continuous on (a,c) (by the joint continuity of $D_\nu(x)$ in ν and x), and is never 0 by Lemma C.1.2, it follows that it must be of one sign on (a, c) by the Intermediate Value Theorem. Similarly,

$$b \mapsto \frac{\partial}{\partial \nu} (D_\nu(x)) \bigg|_{\nu = g(b), x = b}$$

is of one sign on (a, c). Hence, by (C.1.7), $g'(b)$ is of one sign on all of (a, c). By standard real analysis, e.g., □

Now, we show that g can be extended in whatever direction it is decreasing, at least to a neighborhood of 0 on the left.

Lemma C.1.4. *Suppose that on any finite or infinite interval $(a, c) \subseteq (0, \infty)$, there exists a C^1 function $g : (a, c) \to \mathbb{R}^+$ with $D_{g(x)}(x) = 0$. If g is increasing, then g extends to a C^1 function on $(-\epsilon, c)$ for some $\epsilon > 0$, still with the property that $D_{g(x)}(x) = 0$. If g is decreasing, then g extends to a function on (a, ∞), still with the property that $D_{g(x)}(x) = 0$.*

Proof. We prove the case that g is increasing; the case that g is decreasing is similar. Suppose, then, that on any finite or infinite interval $(a, c) \subseteq (0, \infty)$, there exists an increasing function $g : (a, c) \to \mathbb{R}^+$ with $D_{g(x)}(x) = 0$. Let

$$\begin{aligned} \mathcal{A}_0 &:= \{d \leq a : g \text{ extends to a } C^1 \text{ function on } (d, c) \text{ with } D_{g(x)}(x) = 0.\}, \\ a_0 &:= \inf \mathcal{A}_0. \end{aligned} \tag{C.1.8}$$

$a \in \mathcal{A}_0$ by hypothesis, so $a_0 \leq a$. Suppose, by way of contradiction, that $a_0 \geq 0$. Then by a_0 being the infimum, for $\epsilon > 0$ arbitrarily small, then there exists $a(\epsilon) \in \mathcal{A}_0$, $a_0 < a(\epsilon) < a_0 + \epsilon$, so in particular, g extends to a C^1 function on $(a_0 + \epsilon, c)$ with $D_{g(x)}(x) = 0$. This works for all $\epsilon > 0$, so g extends to a function on (a_0, c) with $D_{g(x)}(x) = 0$. Since g was increasing on (a, c), it follows from Corollary C.1.3 that g

is increasing on (a_0, c). Hence, $g\left(a_0 + \frac{1}{n}\right)$ is a decreasing sequence in $\mathbb{R}$. Yet $D_0(x)$ is *never* 0, so the sequence is bounded below by 0. A decreasing sequence, bounded below, must have a limit; hence,

$$L := \lim_{n\to\infty} g\left(a_0 + \frac{1}{n}\right) \text{ exists and is } \geq 0.$$

We now show that $L = \lim_{x\to a_0^+} g(x)$: on the one hand, for all $x > a_0$, there exists $n \in \mathbb{N}$ with $a_0 + \frac{1}{n} < x$, so by decreasingness of the sequence and the function g,

$$L \leq g\left(a_0 + \frac{1}{n}\right) \leq g(x); \tag{C.1.9}$$

Hence,

$$\liminf_{x\to a_0^+} g(x) \geq L. \tag{C.1.10}$$

On the other hand, for all $n \in \mathbb{N}$, eventually $x < a_0 + \frac{1}{n}$, so

$$\limsup_{x\to a_0^+} g(x) < g\left(a_0 + \frac{1}{n}\right), \tag{C.1.11}$$

and this holds for all $n \in \mathbb{N}$; hence,

$$\limsup_{x\to a_0^+} g(x) \leq \lim_{n\to\infty} g\left(a_0 + \frac{1}{n}\right) = L. \tag{C.1.12}$$

Hence,

$$L = \lim_{x\to a_0^+} g(x), \tag{C.1.13}$$

as required.

Moreover, $D_\nu(x)$ is jointly continuous in x and ν, so

$$\begin{aligned} D_L(a_0) &= \lim_{x \to a_0^+} D_{g(x)}(x) \\ &= \lim_{x \to a_0^+} 0 = 0. \end{aligned} \tag{C.1.14}$$

In particular, $L \neq 0$ since $D_0(x) \neq 0$ for all x, being a multiple of an exponential; hence, $L > 0$. Moreover, by $a_0 \geq 0$, we have by Lemma C.1.2 that

$$\frac{\partial}{\partial \nu} \left(D_\nu(x) \right) \bigg|_{\nu = L, x = a_0} \neq 0,$$

so by the Implicit Function Theorem, there exists a unique $C^1(\mathbb{R})$ function $\widetilde{g}(x)$ in a neighborhood of a_0 such that $D_{\widetilde{g}(x)} x = 0$; by uniqueness and $L = \lim_{x \to a_0^+} g(x)$, it follows that $\widetilde{g}(x)$ and $g(x)$ agree on their common domain, an interval of the form $(a_0, a_0 + \epsilon)$ for some $\epsilon > 0$. By shrinking ϵ if necessary, we also have that $\widetilde{g}(x)$ is defined on $(a_0 - \epsilon, a_0 + \epsilon)$. Hence, we may define the function

$$g_1(x) = \begin{cases} \widetilde{g(x)}, & a_0 - \epsilon < x < a_0 + \epsilon, \\ g(x), a_0 + \epsilon \leq x \leq c, \end{cases} \tag{C.1.15}$$

and so $g(x)$ extends to a $C^1(a_0 - \epsilon, c)$ function $g_1(x)$. Hence, $a_0 - \epsilon \in \mathcal{A}_\prime$. Yet $a_0 = \inf \mathcal{A}_{0,}$, so $a_0 \leq a_0 - \epsilon$ for some $\epsilon > 0$; contradiction. Therefore, $a_0 \leq 0$. $\square$

Our next argument will show that $g(x)$ *must* be increasing. To do so, we need an argument that uniformizes the rule (C.1.3).

Lemma C.1.5. *There exists $t_0 > 0$ such that for all $\nu \in [0, \infty)$,*

$$x \geq t_0 \sqrt{2\nu + 1} \quad \textit{implies} \quad D_\nu(x) \neq 0. \tag{C.1.16}$$

Proof. We adjust the arguments of [Olv74, Chapter 6, Section 6.6]. We make the change of variables

$$u = \nu + \frac{1}{2} \tag{C.1.17a}$$

$$x = (2u)^{1/2} t = t\sqrt{2\nu + 1} \tag{C.1.17b}$$

to convert (3.1.1a) into

$$\frac{d^2 w}{dt^2} = u^2 (t^2 - 1)\, dt. \tag{C.1.18}$$

Note that $\nu \geq 0$ implies $u \geq \frac{1}{2}$ and $2u \geq 1$; hence, for $t \geq 2$, $t^2 - 1 \geq 3$. Then by the standard error estimates in [Olv74, Chapter 6, Section 6.2, Thm. 2.1, p. 183], the decaying solution at $t \to \infty$ has an expansion

$$w(u,t) = (t^2 - 1)^{-1/4} \exp\{-u\xi(t)\} \left(1 + \epsilon(u,t)\right), \quad u \geq \frac{1}{2},\, t \geq 2, \tag{C.1.19}$$

where $\xi(t)$ is an antiderivative of $\sqrt{t^2 - 1}$ in the domain of validity of the expansion; in particular, we require $t \geq 2$ and $u \geq \frac{1}{2}$ so that $u^2(t^2 - 1) \geq \frac{3}{2} > 0$.

Moreover, since $D_{u-\frac{1}{2}}((2u)^{1/2} t)$ is decaying (recessive) in the same regime, we have that it is a (nonzero) multiple of this $w(u,t)$.

Our goal is to show that uniformly in $u \geq \frac{1}{2}$, there exists $t \geq t_0$ such that $\epsilon(u,t) < \frac{1}{2}$. Then $w(u,t) \in$ (C.1.19) would be nonzero, by

$$|1 + \epsilon(u,t)| \geq |1| - |\epsilon(u,t)| \geq 1 - \frac{1}{2} = \frac{1}{2},$$

$\frac{1}{\sqrt[4]{t^2 - 1}}$ positive, the exponential obviously nonzero, and the constant factor between $w(u,t)$ to $D_{u-\frac{1}{2}}((2u)^{1/2} t)$ being nonzero.

By the standard theory (e.g., [Olv74, Chapter 6, Section 6.2, Thm. 2.1, p. 183]) the error term $\epsilon(u,t)$, in terms of $f(u,t) = u^2(t^2-1)$, for $t \geq 2$, say, is bounded by

$$\int_t^\infty \left| \frac{1}{[f(u,s)]^{1/4}} \frac{d^2}{ds^2} \left(\frac{1}{[f(u,s)]^{1/4}} \right) \right| ds \tag{C.1.20}$$

With some effort, this becomes

$$\frac{1}{4u} \int_t^\infty \frac{s^2+2}{(s^2-1)^{5/2}}\, ds \tag{C.1.21}$$

The integrand is in $L^1[2,\infty)$, of course: for $s \geq 2$, $s^2+2 \leq s^2+s \leq 2s^2$, and $(s^2-1)^{5/2} \geq \left[(s-1)^2\right]^{5/2} = (s-1)^5 \geq \dfrac{s^5}{2^5}$, so the integral on $[2,\infty)$ is bounded above by $\displaystyle\int_2^\infty \frac{2^6}{s^3}\, ds$, which is of course finite. Therefore,

$$\lim_{t\to\infty} \int_t^\infty \frac{s^2+2}{(s^2-1)^{5/2}}\, ds = 0, \tag{C.1.22}$$

and hence there is t_0 such that $t \geq t_0$ implies that

$$\int_t^\infty \frac{s^2+2}{(s^2-1)^{5/2}}\, ds < \frac{1}{2},$$

and by $\dfrac{1}{4u} = \dfrac{1}{2}\cdot\dfrac{1}{2u} = \dfrac{1}{2}\cdot\dfrac{1}{2\nu+1} \leq \dfrac{1}{2}$ for $\nu \geq 0$, we see that for $t \geq t_0$, uniformly in $u \geq \dfrac{1}{2}$,

$$|\epsilon(u,t)| \leq \frac{1}{2}.$$

In particular, for $t \geq t_0$, $u \geq \dfrac{1}{2}$,

$$\begin{aligned} D_{u-\frac{1}{2}}((2u)^{1/2}t) &\neq 0, \text{or} \\ D_\nu(x) &\neq 0. \end{aligned} \tag{C.1.23}$$

In particular, this holds for $x \geq t_0\sqrt{2\nu+1}$, by (C.1.17). □

Corollary C.1.6. *Suppose that on some finite interval $(a, c) \subset (0, \infty)$, there exists a C^1 function $g : (a, c) \to \mathbb{R}^+$ such that $D_{g(x)}(x) = 0$. Then $g(x)$ is increasing.*

Proof. Suppose that on some finite interval $(a, c) \subset (0, \infty)$, there exists a C^1 function $g : (a, c) \to \mathbb{R}^+$ such that $D_{g(x)}x = 0$. Suppose by way of contradiction that g is decreasing. Then by Lemma C.1.4, g extends to a function on (a, ∞), still satisfying $D_{g(x)}(x) = 0$, which by Corollary C.1.3 must be decreasing on all of (a, ∞). Then in particular, for $x \geq c$, $g(x) \leq g(c)$. Then for t_0 as defined in Lemma C.1.5, $x \geq \max\{c, t_0\sqrt{2g(c) + 1}\}$, $g(x) \leq g(c)$ so $t_0\sqrt{2g(c) + 1} \geq t_0\sqrt{2g(x) + 1}$. Hence, for $x \geq \max\{c, t_0\sqrt{2g(c) + 1}\}$, $x \geq t_0\sqrt{2g(x) + 1}$, so by Lemma C.1.5 and (C.1.16), $D_{g(x)}x \neq 0$. Yet $D_{g(x)}(x) = 0$ by definition of g (and its extension). Contradiction. □

C.2 Induction Argument

We now use the theory of the zeroes of the Hermite Polynomials to start creating our $g_k(x)$. We remind the reader of some properties of the zeroes of the Hermite Polynomials.

Lemma C.2.1 (Properties of the Zeroes of the Hermite Polynomials)**.**

Reality, Simplicity. *The nth Hermite Polynomial's zeroes are all real and simple. This follows from the standard theory of orthogonal polynomials, e.g. [Sze75, Thm. 3.3.1, p. 43].*

Symmetry. *As the Hermite Polynomials are either odd or even, $H_n(x) = 0$ if and only if $H_n(-x) = 0$, $n \in \mathbb{N}_0$, $x \in \mathbb{R}$.*

Number of Positive Zeros. *Combining the above statements, $H_n(x)$ has exactly $\lfloor \frac{n}{2} \rfloor$ strictly positive zeroes.*

Interpolation. *If* $x_0 > x_1 > \cdots > x_{n-1}$ *are the zeroes of* $H_n(x)$*, then in each interval* $[x_{n-j}, x_{n-j-1}]$*,* $1 \leq j \leq n$*, there exists exactly one zero of* $H_{n+1}(x)$*. Moreover, there exists exactly one zero of* $H_{n+1}(x)$ *in* $[x_0, \infty)$ *and* $(-\infty, -x_0]$*. Again, this follows from the general theory of orthogonal polynomials, e.g. [Sze75, Thm. 3.3.2, p. 45].*

Behavior of largest zero. *It is known that there exists a positive constant* C *and* $N \in \mathbb{N}$ *such that for* $n \geq N$*, the largest positive zero* $x_0^{(n)}$ *of* $H_n(x)$ *satisfies*

$$\sqrt{2n+1} - \frac{C+1}{6^{1/3}(2n+1)^{1/6}} < x_0^{(n)} < \sqrt{2n+1} - \frac{C}{6^{1/3}(2n+1)^{1/6}}; \qquad \text{(C.2.1)}$$

see, e.g., [Sze75, Chapter VI, Section 6.32, Thm. 6.32, pp. 127–128]. (C is actually the first positive zero of one of the Airy functions, but this is not important). In particular,

$$\lim_{n\to\infty} x_0^(n) = \infty. \qquad \text{(C.2.2)}$$

and

$$x_0^{(n)} = o(n) \qquad \text{(C.2.3)}$$

Density. *Combining* (C.2.3) *and the reflective symmetry of the zeroes of the Hermite Polynomial, we see that the zero of greatest modulus of* $H_n(x)$ *is* $o(n)$*. Since the weight-function* $w(x) = e^{-x^2}$ *is positive and continuous everywhere on* $\mathbb{R}$*, we may use [Sze75, Chapter 6, Section 6.1, Thms. 6.1.1 and 6.1.2, pp. 107 – 108] to show that for any closed, finite interval* $[a, c] \subseteq \mathbb{R}$ *of nonzero length, there exists* $N = N([a, c])$ *such that* $n \geq N$ *implies that at least one zero of* $H_n(x)$ *lies in* $[a, c]$*.*

Of course, most of the results of Lemma C.2.1 are qualitiative enough to also occur for the zeroes of $H_n\left(\frac{x}{\sqrt{2}}\right)$, which is the zero-contributing factor to $D_n(x)$ by (3.2.5). We rewrite the results that we shall need.

Corollary C.2.2 (Properties of the zeroes of $D_n(x)$, $n \in \mathbb{N}$)**.**

Reality, Simplicity. *For $n \in \mathbb{N}$, the zeroes of $x \mapsto D_n(x)$ are real and simple.*

Number of Positive Zeros. *For $n \in \mathbb{N}$, $x \mapsto D_n(x)$ has $\lfloor \frac{n}{2} \rfloor$ strictly positive zeroes.*

Interpolation. *Fix $n \in \mathbb{N}$. If $b_0 > b_1 > \cdots > b_{n-1}$ are the zeroes of $x \mapsto D_n(x)$, then in each interval $[b_{n-j}, bn - j - 1]$, $1 \leq j \leq n$, there exists exactly one zero of $x \mapsto D_{n+1}(x)$. Moreover, there exists exactly one zero of $x \mapsto D_{n+1}(x)$ in $[b_0, \infty)$ and $(-\infty, -b_0]$.*

Behavior of largest zero. *It is known that if $b_0^{(n)}$ denotes the largest zero of $x \mapsto D_n(x)$*

$$\lim_{n\to\infty} b_0^{(n)} = \infty. \tag{C.2.4}$$

Therefore, for $n \in \mathbb{N}$, $k \in \{0, 1, \ldots, n-1\}$, let $b_k^{(n)}$ denote the zeroes of the nth rescaled Hermite polynomial $H_n\left(\frac{x}{\sqrt{2}}\right)$ (equivalently, of $D_n(x)$) in *decreasing* order:

$$b_0^{(n)} > b_1^{(n)} > \ldots > b_{n-1}^{(n)}. \tag{C.2.5}$$

By our comments on the positive zeroes,

$$b^{(n)}_{\lfloor \frac{n}{2} \rfloor - 1} > 0 \geq b^{(n)}_{\lfloor \frac{n}{2} \rfloor}, \quad n \geq 2,$$

where equality holds in the second inequality if and only if n is odd.

Of course, as noted in [Dea66], $D_{2k+1}(0) = 0$ for all $k \in \mathbb{N}_0$, and by Lemma C.1.2,

$$\frac{\partial}{\partial \nu}\left(D_\nu(x)\right)\bigg|_{\nu=2k+1, x=0} \neq 0,$$

so by the Implicit Function Theorem, for all $k \in \mathbb{N}_0$, there exists a C^1 function $g_k(x)$, defined for x in a neighborhood of 0, satisfying

$$\begin{cases} D_{g_k(x)}(x) & = 0 \\ g_k(0) & = 2k+1. \end{cases} \tag{C.2.6}$$

Similarly, for $n \geq 2$, for the positive zeroes $\{b_k^{(n)}\}_{k=0}^{\lfloor \frac{n}{2} \rfloor - 1}$ of $x \mapsto D_n(x)$, Lemma C.1.2 again assures us that

$$\frac{\partial}{\partial \nu}\left(D_\nu(x)\right)\bigg|_{\nu=2k+1, x=b_k^{(n)}} \neq 0, \quad n \geq 2, \quad 0 \leq k \leq \lfloor \frac{n}{2} \rfloor - 1.$$

so again by the Implicit Function Theorem, there exists a $C^1(\mathbb{R})$ function $g_{n,k}(x)$, defined in a neighborhood of $b_k^{(n)}$, satisfying

$$\begin{cases} D_{g_{n,k}(x)}(x) & = 0 \\ g_{n,k}(b_k^{(n)}) & = n. \end{cases} \tag{C.2.7}$$

Moreover, by Corollary C.1.6, $g_{n,k}$ is an increasing function in a neighborhood of $b_k^{(n)}$, and by Lemma C.1.4, $g_{n,k}$ extends to a function defined on an open interval containing $[0, b_k^{(n)}]$, still satisfying $D_{g_{n,k}(x)}(x) = 0$. Yet by Lemma C.1.1, by $D_{g_{n,k}(0)}(0) = 0$, $g_{n,k}(0) = 2j+1$ for some $j \in \mathbb{N}_0$. In particular, then, by the Uniqueness part of the Implicit Function Theorem, $g_{n,k}(x)$ extends $g_j(x)$. The only thing left to show is that the indexation was chosen correctly.

Proposition C.2.3. *For $n \geq 2$, and $0 \leq k \leq \lfloor \frac{n}{2} \rfloor - 1$, $g_{n,k}(x)$ extends $g_k(x)$ to an open interval containing $[0, b_k^{(n)}]$; i.e., in the notation above, $j = k$.*

Proof. We induct on n. For base cases, we consider both $n = 2$ and $n = 3$. For $n = 2$, of course $g_{2,0}$ is increasing, so $g_{2,0}(0) < g_{2,0}(b_0^{(2)}) = 2$, yet $g_{2,0}$ satisfies $g_{2,0}(0) \in 2\mathbb{N}_0 + 1$ by Lemma C.1.1, so we must have $g_{2,0}(0) = 1$ and $g_{2,0}(x)$ extends $g_0(x)$. Similarly, for $n = 3$, $g_{3,0}(0) < g_{3,0} b_0^{(3)} = 3$, but $D_{g_{3,0}(0)}(0) = 0$, so by Lemma C.1.1, $g_{3,0}(0) = 1$, so $g_{3,0}(x)$ extends $g_0(x)$. Now suppose the statement is true for $n = m \geq 3$; we wish to prove it for $n = m + 1$.

Case 1: m is odd. Then $m = 2j + 1$, $m + 1 = 2j + 2$ for some $j \in \mathbb{N}$. Then $D_{2j+1}(0) = 0$, so by the interpolation property in Corollary C.2.2, the positive zeros of $D_{2j+1}(x)$ and $D_{2j+2}(x)$ are related by

$$b_j^{(2j+1)} = 0 < b_j^{(2j+2)} < b_{j-1}^{(2j+1)} < b_{j-1}^{(2j+2)} < \cdots < b_1^{(2j+1)} < b_1^{(2j+2)} < b_0^{(2j+1)} < b_0^{(2j+2)} \tag{C.2.8}$$

The positive zero closest to $x = 0$ of $D_{2j+2}(x)$ is $b_j^{(2j+2)}$. We wish to show that $g_{2j+2,j}(x)$ does not extend $g_0(x), g_1(x), \ldots g_{j-1}(x)$. By (C.2.8), $b_j^{(2j+2)} < b_{j-1}^{(2j+1)}$, so for all $r \in \mathbb{N}_0$, $0 \leq r \leq j$, $b_j^{(2j+2)} < b_r^{(2j+1)}$ for all r, $0 \leq r \leq j - 1$. In particular, then, by the inductive hypothesis, $g_r(x)$ has been defined on $[0, b_r^{(2j+1)}]$, and hence by the g_r increasing by Corollary C.1.6,

$$g_r(b_j^{(2j+2)}) \leq g_r(b_r^{(2j+1)}) = g_{2j+1,r}(b_r^{(2j+1)}) = 2j + 1 < 2j + 2, \tag{C.2.9}$$

so the graph of $g_r(x)$ does not pass through the point $(b_j^{(2j+2)}, 2j + 2)$. Hence, $g_{2j+2}j$ does not extend $g_r(x)$ for $0 \leq r \leq j - 1$, yet it must extend some $g_p(x)$ with $2p + 1 \leq m + 1 = 2j + 2$, so it must extend $g_j(x)$. We now induct on ℓ, $0 \leq \ell \leq j$, the statement, "$g_{2j+2,j-q}$ extends g_{j-q}, $0 \leq q \leq \ell$." We have just proven the statement for

$\ell = 0$. If true for $\ell = \ell_0$, then for $\ell = \ell_0 + 1$, we see that $g_{2j+2,j-(\ell_0+1)}$ cannot extend g_{j-q} for $0 \leq q \leq \ell_0$, since the fact that all such implicit functions are increasing by Corollary C.1.6, and by the inductive hypothesis, $g_{j-q}(b_{j-q}^{(2j+2)}) = 2j+2$, $q \leq \ell_0$, so we cannot have $g_{j-q}(b_{j-(1+\ell_0)}^{(2j+1)}) = 2j+2$ as well. Yet equally, by (C.2.8), if $\ell_0 \neq j+1$.

$$b_{j-(1+\ell_0)}^{(2j+2)} < b_{j-(\ell_0+2)}^{(2j+1)} < \ldots < b_{,}^{(2j+1,0)} \tag{C.2.10}$$

so for $0 \leq r \leq j-(\ell_0+2)$, again by increasingness of the implicit functions and the inductive hypothesis,

$$g_r(b_{j-(1+\ell_0)}^{(2j+2)})) \leq g_r(b_{)}^{(2j+1,r)} = g_{2j+1,r}(b_r^{(2j+1)}) = 2j+1 < 2j+2, \tag{C.2.11}$$

so $g_r(b_{j-(1+\ell_0)}^{(2j+2)})$ is too small to touch $g_{2j+2,j-(1+\ell_0)}(b_{j-(1+\ell_0)}^{(2j+2)}) = 2j+2$. Hence, we have that $g_{2j+2,j-(1+\ell_0)}$ does not extend $g_r(x)$ for $0 \leq r \leq j-(\ell_0+2)$. Hence, $g_{2j+2,j-(1+\ell_0)}$ must interpolate some $g_p(x)$ with $2p+1 \leq 2j+2$, but $p \leq j-(\ell_0+2)$ and $p \geq j-\ell_0$ are both removed, so $p = j-(1+\ell_0)$ is the only possibility. Hence, $g_{2j+2,j-(1+\ell_0)}$ extends $g_{j-(1+\ell_0)}$. Hence, the statement is true for $\ell = \ell_0+1$. By this inner induction, we have, that $g_{2j+2,k}$ extends g_k for $0 \leq k \leq j = \left\lfloor \frac{m+1}{2} \right\rfloor - 1$.

Case 2: m is even. Then $m = 2j$, $m+1 = 2j+1$ for some $j \in \mathbb{N}$. Then $D_{2j+1}(0) = 0$, so by interpolation, the positive zeros of $D_{2j}(x)$ and $D_{2j+1}(x)$ are related by

$$b_j^{(2j+1)} = 0 < b_{j-1}^{(2j)} < b_{j-1}^{(2j+1)} < b_{j-2}^{(2j)} < b_{j-2}^{(2j+1)} < \cdots < b_1^{(2j)} < b_1^{(2j+1)} < b_0^{(2j)} < b_0^{(2j+1)} \tag{C.2.12}$$

The zero closest to $x = 0$ of $D_{2j+1}(x)$ is $b_{j-1}^{(2j+1)}$, the jth largest zero. We wish to show that $g_{2j+1,j}(x)$ does not extend $g_0(x), g_1(x), \ldots g_{j-2}(x)$. By (C.2.12), $b_{j-1}^{(2j+1)} < b_{j-2}^{(2j)}$, so for all $r \in \mathbb{N}_0$, $0 \leq r \leq j-2$, $b_{j-1}^{(2j+1)} < b_r^{(2j)}$ for all $r \leq j-2$. In particular, then, by the inductive hypothesis, $g_r(x)$ has been defined on $[0, b_r^{(2j)}]$, and hence by the g_r increasing by Corollary C.1.6,

$$g_r(b_{j-1}^{(2j+1)}) \leq g_r(b_r^{(2j)}) = g_{2j,r}(b_r^{(2j)}) = 2j < 2j+1, \tag{C.2.13}$$

so the graph of $g_r(x)$ does not pass through the point $(b_{j-1}^{(2j+1)}, 2j+1)$. Hence, $g_{2j+1,j-1}$ does not extend $g_r(x)$ for $0 \leq r \leq j-1$, yet it must extend some $g_p(x)$ with $2p+1 \leq m+1 = 2j+1$, so it must extend $g_{j-1}(x)$.

We now induct on ℓ, $0 \leq \ell \leq j-1$, the statement, "$g_{2j+1,j-1-q}$ extends g_{j-1-q}, $0 \leq q \leq \ell$." We have just proven the statement for $\ell = 0$. If true for $\ell = \ell_0 < j-1$, then for $\ell = \ell_0 + 1$, we see that $g_{2j+2,j-1-(\ell_0+1)}$ cannot extend g_{j-1-q} for $0 \leq q \leq \ell_0$, since the fact that all such implicit functions are increasing by Corollary C.1.6, and $g_{j-1-q}(b_{j-1-q}^{(2j+1)}) = g_{2j+1,j-1-q}(b_{j-1-q}^{(2j+1)}) = 2j+1$, so we cannot have $g_{j-q}(b_{j-1-(1+\ell_0)}^{(2j+1)}) = 2j+1$ as well. Yet equally, by (C.2.12), if $\ell_0 + 1 < j-1$.

$$b_{j-1-(1+\ell_0)}^{(2j+1)} < b_{<}^{(2j,j-1-(2+ell_0))} b_{<}^{(2j,j-1-(\ell_0+3))} \ldots < b_{,}^{(2j,0)} \tag{C.2.14}$$

so for $0 \leq r \leq j-1-(\ell_0+2)$, again by increasingness of the implicit functions and the inductive hypothesis,

$$g_r(b_{j-1-(\ell_0+1)}^{(2j+1)}) \leq g_r(b_r^{(2j)}) = g_{2j,r}(b_r^{(2j)}) = 2j < 2j+1 \tag{C.2.15}$$

so $g_r(b_{j-1-(\ell_0+1)}^{(2j+1)})$ is too small to touch $g_{2j+2,j-1-(\ell_0+1)}(b_{j-1-(\ell_0+1)}^{(2j+2)}) = 2j+1$. Hence, $g_{2j+2,j-1-(\ell_0+1)}$ does not extend $g_r(x)$ for $0 \leq r \leq j-1-(\ell_0+2)$. Hence, $g_{2j+2,j-1-(\ell_0+1)}$ must interpolate some $g_p(x)$ with $2p+1 \leq 2j+2$, but $p \leq j-1-(\ell_0+2)$ and $p \geq j-1-\ell_0$ are both removed, so $p = j-1-(\ell_0+1)$ is the only possibility. Hence,

$g_{2j+1,j-1-(\ell_0+1)}$ extends $g_{j-1-(\ell_0+1)}$. Hence, the statement is true for $\ell = \ell_0+1$. By this inner induction, we have, that $g_{2j+2,k}$ extends g_k for $0 \leq k \leq j-1 = \left\lfloor \frac{m+1}{2} \right\rfloor - 1$.

In all cases, we are done; the statement for $n = m$ implies the statement for $n = m + 1$. □