On Shifted Convolution Sums Involving the Fourier Coefficients of Theta Functions Attached to Quadratic Forms

Dissertation

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Abstract

The study of shifted convolution sums has acquired a prominent place in current number theory research owing to its potential applications to the sub-convexity problem, while quadratic forms have fascinated mathematicians since antiquity. This thesis, which deals with both these topics, studies shifted convolution sums involving the Fourier coefficients of Theta Series associated to a positive definite integral quadratic form and a cuspidal Hecke eigenform of integral weight. Our aim is to generalize the work of W. Luo, J. Hafner, and H. Iwaniec et al. in this new setting. Three independent approaches are used in this endeavour – the spectral theory of the hyperbolic Laplacian, the δ-symbol method (a variant of the Hardy-Littlewood-Ramanujan circle method), and the theory of Poincaré series via a Poisson-Voronoï summation formula. We establish asymptotic formulae in all three aspects with the spectral theory approach providing the optimal estimate for the error term when one of the forms involved is cuspidal, while the δ-symbol method gives a sharp error term when only Theta Series are involved in the shifted convolution sum.

This dissertation is dedicated to my parents, for everything and much more besides.

Acknowledgments

I would like to begin by thanking my advisor, Professor Wenzhi Luo, for his guidance, enthusiasm, and support throughout the span of my graduate education. In teaching me the rudiments of analytic number theory and sharing his perspective borne out of years of experience with me, I owe him no mean debt. Without him, this dissertation would have remained a distant glimmer in the eye, not bound concretely in paper. A few lines here cannot do justice to my gratitude. Much of what I have learnt as a graduate student can be traced back to Professor Luo and I am profoundly grateful that he was my advisor during this undertaking.

I would also like to thank Professors Warren Sinnott, Jeff McNeal and Roman Holowinsky for taking time away from their busy schedules to pore over the preliminary thesis draft and serve on the defense committee.

To my parents, this dissertation is the culmination of a journey that began a decade ago. Without their support and understanding, in more ways than one, my enthusiasm in pursuing this line of study might have petered out and come to a heaving stop a whole lot earlier. This dissertation is dedicated to them, with deep love and gratitude.

No acknowledgement would be complete without mentioning all the friends I made here at Ohio State. To Tapo Majumdar, Pawas Ranjan, and Rohit Prabhavalkar, 'bags of fun' is seriously understating the nature of the good times we shared. In their company, boredom was an ill-defined exotic word forever without context. To José Cervantes, Donald Robertson, Samir Chowdhury, Yeonghwan Son and Yeongseong Jo, thanks for all those Friday sessions where the food was divine and the talk (all of it mine) less so. Boring you to death with fatuous (and some might say, fat-headed) pronouncements issued with all the solemnity of youth was never part of the plan, but I enjoyed it nonetheless. Thanks for reading the first chapter and chiming in with your reactions as well. There are many who deserve to be listed here, but my memory is like a leaky sieve and many names seem to be hiding in the dim recesses of my memory. To them, my apologies.

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Fields of Study

Major Field: Mathematics

Specialization: Analytic Number Theory

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Chapter 1: Introduction

"I don't know where to begin."
"At the beginning, don't you think? I often feel that that is best. Then work through the middle and from there, taking your time, carry on to the end."

P.G. Wodehouse, *Service with a Smile*

1.1 Prologue

To begin this dissertation, which contributes to the field of Analytic Number Theory, at its beginning, we introduce its main object of study – the sum:

$$\sum_{n\geq 1} a(n+b) r_Q(n) \phi(n), \tag{1.1}$$

where b is a positive integer, $r_Q(n)$ is the number of ways of representing an integer n by a positive definite quadratic form $Q(x)$ of rank l and weight $k = l/2$, $a(n)$ is either the normalized Fourier coefficient of a holomorphic cusp form $f(z)$ (to be denoted by $a_f(n)$ in the pages that follow) in $\Gamma_0(N)$ or $r_Q(n)$ itself and $\phi(x)$ is a suitably nice test function on $(0, \infty)$. The sum in (1.1) is a specific example of a Shifted Convolution Sum, which has been extensively studied by contemporary number theorists as a part of the Shifted Convolution Problem (henceforth abbreviated as SCP, borrowing Philipe Michel's notation from [26]). To put it succinctly (for the time being), the SCP refers to the problem of obtaining sufficient non-trivial upper bounds for Shifted Convolution Sums.

In this expository chapter, we hope to shed some light on a few of the key players involved in the study of the sum in (1.1). Along the way, we will also define the SCP from a mathematically rigorous point of view and on a related note, furnish a few details regarding its relevance to modern day analytic number theorists.

1.2 Statement of thesis results

Before we go on to explore the various facets to the study of the sum in (1.1), we would like to provide a brief summary of the results that arise out of this research. The results themselves were obtained using a variety of different approaches. In light of this multi pronged approach, the chapters of this dissertation have been organized on a methodological basis, as listed below:

(i) Chapter II – Poincaré Series Approach

(ii) Chapter III – Spectral Method

(iii) Chapter IV – δ-symbol Method

Each chapter devotes itself entirely to the exposition of a single approach to the problem at hand. This way, we hope that the thesis will also serve to illustrate the potential benefits and disadvantages of some of the more common techniques used by analytic number theorists in action as it were.

To begin with, Chapters II and IV are linked thematically even if the objects under scrutiny in both the chapters differ. Here, we study the sum in (1.1) using relatively 'elementary' (in the sense that a more mathematically sophisticated and powerful approach is eschewed in favor of obtaining exploratory estimates) techniques, such as the δ-symbol method and variations thereof. We can trace the origins of these approaches back to the circle method of Hardy, Littlewood, and Ramanujan used to study the asymtpotics of the partition function in 1916-17. We also use ideas inspired by the work of Siegel (and even further beyond to Dirichlet, as a matter of fact) in [36] wherein automorphic forms such as Θ functions associated to quadratic forms $Q(\mathbf{x})$ were extensively used to find, among other things, formulae relating to $r_Q(n)$. Upon applying the aforementioned techniques to the sum

$$D_f(\Theta, b) = \sum_{n \leq X} a_f(n+b) r_Q(n), \tag{1.2}$$

we manage to exert a certain degree of control over the order of magnitude of the sum in (1.2), by getting an upper bound that is marginally worse than the bound predicted by the 'square-root' cancellation heuristic. This heuristic is the generally reliable rule of thumb that in certain families of finite oscillating sums, the order of magnitude is roughly given by the square root of the number of terms (the table towards the end of this section shows this in a very concrete way). To wit, we obtain the following result:

$$\sum_{n \geq 1} a_f(n+b) r_Q(n) \phi(n) \ll X^{l/2-(l-1)/4+\epsilon} P^g, \tag{1.3}$$

where X, P are parameters whose values reflect the nature of the applications that employ (1.3) by determining the support of the test function $\phi(x)$, while g is the smallest integer such that $g \geq l/2+1$ with l representing the rank of the quadratic form $Q(x)$.

In Chapter IV, variations on the techniques used in Chapter II yield a similar result for sums of the form

$$D(\Theta, \phi) = \sum_{n \geq 1} r_Q(n) r_Q(n+1) \phi(n),$$

giving us the following asymptotic expansion

$$D(\Theta, \phi) = \int_0^\infty g(x, x+1)\, dx + O\left(P^{l/2+1/4}(X+Y)^{l/2-3/4}(XY)^{1/4+\epsilon}\right). \tag{1.4}$$

Here, X, Y, P are, as mentioned in the previous paragraph, yet-to-be determined parameters that control the support of the test function $\phi(x)$, while $g(x, y)$ is defined as follows:

$$g(x, y) = \Gamma(l/2)^{-2}\,(2\pi)^l \sum_{q=1}^{\infty} q^{-l} \mu(q) f(x, y)(xy)^{l/2-1} \tag{1.5}$$

with $\mu(x)$ representing the Möbius function and $f(x, y)$ being a smooth test-function defined in terms of $\phi(x)$. The error term in (1.4) supersedes the trivial bound

$$D(\Theta, \phi) \ll (XY)^{(l-1)/2}$$

whenever

$$P^{l/2+1/4} \ll (X+Y)^{-(l/2-3/4)}(XY)^{l/2-3/4-\epsilon}.$$

Finally, in Chapter III, we shall resort to the far more powerful spectral theory approach in order to study (1.1). However, sophistication notwithstanding, this method has a drawback in that the bounds obtained are restricted to Θ functions of integral weight. Owing to the non-obvious extension of one of the key ideas used, the process of extending the result to include Θ functions of half-integral weight becomes a murky issue and one which we have not analysed to the fullest extent possible. We obtain the following bound for the unsmoothed sum:

$$\sum_{n \le X} r_Q(n) a_f(n+b) \ll X^{l/2-1/2+\delta},$$

where δ is a positive number whose value depends on the locations of eigenvalues of the non-Euclidean Laplacian operator on quotients of $SL(2,\mathbb{R})$ by congruence groups.

The following table summarizes all the results mentioned in this section for the special case where f and Θ live in the full modular group, while Θ is assumed to be of integral weight. We use $\widetilde{r_Q}(n)$ to represent a normalized $r_Q(n)$, i.e., since $r_Q(n) \ll n^{k-1}$, we set $\widetilde{r_Q}(n) = r_Q(n)/n^{k-1}$, thus removing the main contributor towards the size of $r_Q(n)$. We also assume that f is a normalized eigencuspform for the full modular group. In so doing, all the shifted convolution sums considered in this dissertation are placed on an equal footing. Since the size of each normalized sum is trivially bounded by X (the 'length' of the sum), we get to see the extent of the power saving affected by each method. Unsurprisingly, the spectral method performs the best when one of the summands is cuspidal, while the δ-symbol method provides a rather sharp estimate for the error term when the weight k is large, implying that the main term in the asymptotic expansion approximates the shifted convolution sum quite closely.

Table 1.1: A table displaying the asymptotics obtained in terms of the weight k for the unweighted shifted convolution sums considered in this dissertation across three independent methods.

Sum under consideration	Approach used	Asymptotics
$\sum_{n\le X} \widetilde{r_Q}(n)a_f(n+1)$	Poincaré Series	$X^{1-\frac{2k-1}{4k+8}+\epsilon}$
$\sum_{n\le X} \widetilde{r_Q}(n)\widetilde{r_Q}(n+1)$	δ-symbol	$c(Q)X + X^{1-\frac{4k-3}{4k+5}+\epsilon}$
$\sum_{n\le X} \widetilde{r_Q}(n)a_f(n+1)$	Spectral Theory	$X^{1-1/2+\epsilon}$

On a final note, in the vein of the familiar adage, a picture is worth a thousand words, we present a diagram from J.L. Hafner's expository article [14] on the following page which highlights the areas pertinent to our research amidst the rich interplay that exists between the various branches of modern number theory.

1.3 Automorphic forms in a (tiny) nutshell

Seeing as how automorphic cusp forms – mathematical objects that we have not defined in any capacity so far – play an integral role in the course of this thesis, we would like to take the opportunity to review their definitions at this juncture. A few cursory paragraphs can hardly do justice to automorphic forms (along with their equally well-known relatives, modular forms) and their role in modern mathematics when entire tomes have been written about these tantalizing entities. We refer the interested reader to H. Iwaniec's [21] or Diamond and Shurman's [3], both of which devote considerable space and energy towards developing and motivating the

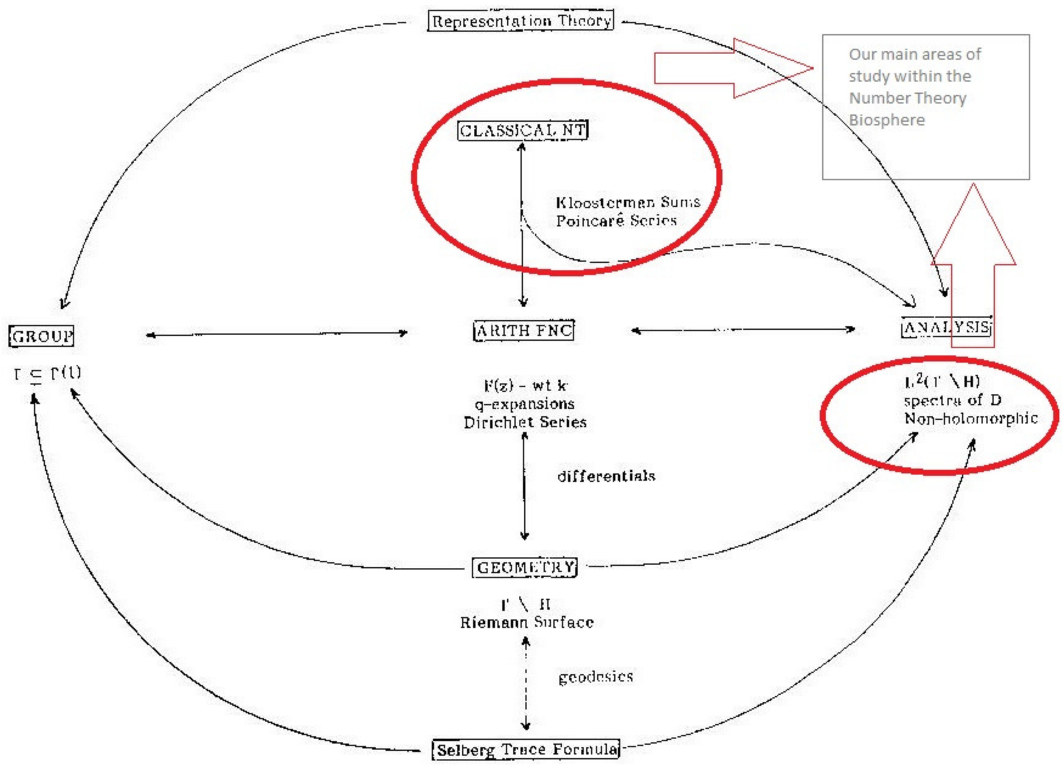

Figure 1.1: The interrelated world of number theory and our areas of interest

relevant theory. So without further ado, we shall jump headlong into the pertinent definitions.

Let $\mathbb{H}$ denote the complex upper half-plane, i.e., those complex numbers $x+iy$ with $y>0$, and let $\overline{\mathbb{H}} = \mathbb{H} \cup \mathbb{Q} \cup \{\infty\}$. Let $\Gamma = SL(2,\mathbb{Z})$, the group of 2×2 matrices with integer coefficients and determinant 1 under multiplication. It can be shown that Γ acts on $\overline{\mathbb{H}}$ by fractional linear transformations:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}(z) = \frac{az+b}{cz+d}.$$

Moreover, $\overline{\mathbb{H}}$ also has an action by any subgroup of Γ. A central object in the theory of modular forms is the set of cusps of Γ which is nothing more than the set of points in the Γ-orbits of $\Gamma\backslash(\mathbb{Q}\cup\{\infty\})$. In this thesis, we pay particular attention to the action of the subgroup $\Gamma_0(N)$ consisting of those matrices in $SL(2,\mathbb{Z})$ for which the lower left-hand entry is divisible by N.

Definition 1.3.1. *Let $f : \mathbb{H} \to \mathbb{C}$ be a holomorphic function and $k \in \mathbb{N}\cup\{0\}$. The function f is a (holomorphic) modular form of weight k and level N if*

$$f(\gamma z) = (cz+d)^k f(z) \text{ for } z \in \mathbb{H}, \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N), \tag{1.6}$$

and f is "holomorphic at each cusp of" $\Gamma_0(N)$. We denote the space of modular forms of weight k and level N by $M_k(N)$.

As far as the notion of "being holomorphic at a cusp" is concerned, it suffices (for our purposes) to merely say that the condition requires the function $f(z)$ satisfy a polynomial growth condition as z approaches a cusp of $\Gamma_0(N)\backslash\mathbb{H}$.

Let $f \in M_k(N)$. Since $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in \Gamma_0(N)$ for all N, then (1.6) with $\gamma = T$ says that $f(z+1) = f(z)$, i.e., the function f is periodic with period 1. Hence, it has a Fourier expansion

$$f(z) = \sum_{n \in \mathbb{Z}} a_n e^{2\pi i n z}.$$

Setting $q = e^{2\pi i z}$ and expressing $f(z)$ as the function $F(q)$, we can re-write $f(z)$ as

$$f(z) = F(q) = \sum a_n q^n.$$

Thus, the Fourier expansion of f may be viewed as a power series expansion of $F(q)$ at $q = 0$, where we have used the fact that as $z \to i\infty$, $q \to 0$. The condition that f be holomorphic at the cusps forces $a_n = 0$ for $n < 0$ thereby removing the possibility of negative powers of q from showing up in the $F(q)$ series expansion. For larger values of N, $\Gamma_0(N)$ has multiple cusps, and we can find similar q-expansions for $f(z)$ about each cusp.

Definition 1.3.2. *Let* $f \in M_k(N)$. *We say that* f *is a cusp form if* f *vanishes at each cusp, i.e., if* $a_0 = 0$ *in the* q*-expansion*

$$f(z) = \sum_{n \geq 0} a_n q^n$$

about any cusp of $\Gamma_0(N)$. *The space of cusp forms in* $M_k(N)$ *is denoted* $S_k(N)$.

We proceed to define a multiplier system of weight k for $\Gamma \subset SL(2, \mathbb{R})$ as a precursor to the definition of an automorphic form from [21, p. 42]. Let $A, B \in SL(2, \mathbb{R})$. The function $w(A, B)$ is defined as follows:

$$2\pi w(A, B) = -\arg j_{AB}(z) + \arg j_A(Bz) + \arg j_B(z),$$

where $j_g(z) = cz + d$ for any $g = \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in SL(2, \mathbb{R})$.

Definition 1.3.3. *A multiplier system of weight* k *for* Γ *is a function* $\vartheta : \Gamma \to \mathbb{C}$ *such that*

$$|\vartheta(\gamma)| = 1,$$

$$\vartheta(\gamma_1\gamma_2) = w(\gamma_1, \gamma_2)\, \vartheta(\gamma_1)\, \vartheta(\gamma_2).$$

We shall also require that

$$\vartheta(-1) = e(-k/2), \text{ if } -1 \in \Gamma,$$

which is called the *consistency condition.* Here, and for the rest of the dissertation, $e(z) = e^{2\pi i z}$.

For any $A \in \mathrm{SL}(2, \mathbb{R})$, we define the "slash" operator $|A$ acting on functions $f : \mathbb{H} \to \mathbb{C}$ by

$$f_{|A}(z) = j_A(z)^{-k} f(Az).$$

Definition 1.3.4. *A holomorphic function satisfying the transformation rule*

$$f_{|\gamma} = \vartheta(\gamma) f, \text{ for any } \gamma \in \Gamma$$

is called an automorphic form for Γ *of weight* k *with respect to the multiplier system* ϑ*.*

We shall also require that an automorphic form be holomorphic not only in $\mathbb{H}$ but also at every cusp. See [21, p. 43] for a more concrete explanation of this condition.

We now focus on automorphic cusp forms (i.e., automorphic forms which vanish at each cusp). We begin by noting that we can make the space of automorphic cusp forms a finite-dimensional Hilbert space with the Petersson inner product defined as follows

$$\langle f, g\rangle_k = \iint\limits_{\Gamma\backslash\mathbb{H}} y^{k-2} f(z)\bar{g}(z)\,dx\,dy.$$

On this note, we introduce the classical Poincaré series at ∞ for $\Gamma = \Gamma_0(N)$ given by

$$P(m,k,N;z) := \sum_{\Gamma_\infty\backslash\Gamma_0(N)} j_\gamma(z)^{-k} e(m\gamma z),$$

where $\Gamma_\infty = \left\{ \begin{pmatrix} 1 & t \\ & 1 \end{pmatrix} : t \in \mathbb{Z} \right\}$ is the stabilizer of the cusp ∞. The collection $\{P(m,k,N;z)\}_{m\geq 1}$ spans the finite-dimensional space of cusp forms on $\Gamma_0(N)$. This is a fact that will be of some importance to us in Chapter II. See §3.3 of [21] for a more comprehensive exposition on this topic.

1.4 To sub-convexity and beyond via shifted convolutions

In order to understand the relevance of the Shifted Convolution Problem (a.k.a *SCP*) in modern analytic number theory, we will first need to phrase and understand the statement of the Sub-convexity Problem (a.k.a *ScP*). Much of what follows has been inspired by the writings of P.Michel in [26] and [27].

We shall begin by defining L-functions in a rather broad and abstract context, following H.Iwaniec and P. Sarnak in [20]. Succinctly put, L-functions are a type of generating function associated to an arithmetic-geometric object or an automorphic form. As a classical object whose origins can be traced back to Dirichlet (or even Riemann, if one is willing to take a few liberties with the definition), L-functions have been a fount of research in contemporary analytic number theory owing in no small part to the various sweeping conjectures that dot its mathematical landscape such as the Generalized Riemann Hypothesis (GRH), Generalized Ramanujan Conjecture (GRC), Birch Swinnerton-Dyer Conjecture (BSC), etc.

To assist us in the definition, we shall denote L-functions by $L(f,s)$ where the symbol f merely imply that L-functions are usually attached to some interesting arithmetic object and s refers to a point in the complex plane at which the L-function is evaluated. Following §5.1 of [10], we shall say that $L(f,s)$ is an L-function if we have the following data and conditions:

1. A Dirichlet series with Euler product of degree $d \geq 1$,

$$L(f,s) = \sum_{n\geq 1} \lambda_f(n) n^{-s} = \prod_p (1-\alpha_1(p)p^{-s})^{-1} \dots (1-\alpha_d(p)p^{-s})^{-1}$$

   with $\lambda_f(1) = 1$, $\lambda_f(n) \in \mathbb{C}$, $\alpha_i(p) \in \mathbb{C}$. The series and Euler products must be absolutely convergent for $\mathrm{Re}(s) > 1$. The $\alpha_i(p), 1 \leq i \leq d$, are called the local roots or local parameters of $L(f,s)$ at p, and they satisfy

$$|\alpha_i(p)| < p \text{ for all } p.$$

2. A gamma factor

$$\gamma(f,s) = \pi^{-ds/2} \prod_{j=1}^{d} \Gamma\left(\frac{s+\kappa_j}{2}\right)$$

   where the numbers $\kappa_j \in \mathbb{C}$ are called the local parameters of $L(f,s)$ at infinity. We assume these numbers are either real or come in conjugate pairs. Moreover, Re $(\kappa_j) > -1$. This last condition tells us that $\gamma(f,s)$ has no zero in $\mathbb{C}$ and no pole for $\mathrm{Re}(s) \geq 1$.

3. An integer $q(f) \geq 1$, called the conductor of $L(f,s)$ such that $\alpha_i(p) \neq 0$ for $p \nmid q(f)$ and $1 \leq i \leq d$. A prime $p \nmid q(f)$ is said to be unramified.

Using these quantities, we can define the 'complete' L-function

$$\Lambda(f,s) = q(f)^{s/2}\gamma(f,s)L(f,s).$$

$\Lambda(f,s)$ is a holomorphic function in the half-plane $\mathrm{Re}(s) > 1$, and admits an analytic continuation to a meromorphic function for $s \in \mathbb{C}$ of order one with at most poles at $s = 0$ and $s = 1$. Moreover, it must satisfy the functional equation

$$\Lambda(f,s) = \epsilon(f)\Lambda(\bar{f}, 1-s), \tag{1.7}$$

where $\bar{f}$ is an object associated with f for which $\lambda_{\bar{f}}(n) = \bar{\lambda}_f(n)$, $\gamma(\bar{f},s) = \gamma(f,s)$, $q(\bar{f}) = q(f)$ and $\epsilon(f)$, also known as the 'root number' of $L(f,s)$, is a complex number of absolute value 1.

Analytic number theorists are primarily interested in uniform estimates for various analytic quantities related to $L(f,s)$ such as the conductor, degree, local parameters, etc. To this end, as in [20], one can define the analytic conductor, $\mathfrak{Q}(f,s)$, which encapsulates all the relevant information about $L(f,s)$ and in so doing, allows us to measure the 'size' of $L(f,s)$. We start by defining

$$\mathfrak{Q}_\infty(s) = \prod_{j=1}^{d}(|s+\kappa_j|+3).$$

Multiplying this $q(f)$, we get the analytic conductor

$$\mathfrak{Q}(f,s) = q(f)\mathfrak{Q}_\infty(s) = q(f)\prod_{j=1}^{d}(|s+\kappa_j|+3).$$

We also denote

$$\mathfrak{Q}(f) = \mathfrak{Q}(f,0) = q(f)\prod_{j=1}^{d}(|\kappa_j|+3).$$

Following [10], we note that $\mathfrak{Q}(f,s) \geq 3^d q(f)$, so $d < \log \mathfrak{Q}(f)$, and that

$$\mathfrak{Q}(f,s) \leq \mathfrak{Q}(f)(|s|+3)^d,$$

which allows us to perform estimates in terms of $\mathfrak{Q}(f)$ without sacrificing a lot in terms of powers of the quantities s and d.

The *ScP* addresses the question of the size of $L(f, s)$ when s is on the critical line. As a consequence of the functional equation (1.7), one has the convexity bound, $\forall \epsilon > 0$, for Re(s) =1/2,

$$L(f, s) \ll \mathfrak{Q}(f, s)^{1/4+\epsilon}.$$

The convexity bound can typically be obtained by applying the Phrágmen Lindelöf interpolation method together with bounds on $L(f, s)$ in $\mathfrak{R}(s) > 1$ and $\mathfrak{R}(s) < 0$. The *ScP* then, simply put, asks for any non-trivial improvement over the convexity bound. Thus, the *ScP* then can be stated as:

Definition 1.4.1. [Sub-convexity Problem] *Find an absolute $\delta > 0$ such that*

$$L(f, s) \ll \mathfrak{Q}(f, s)^{1/4-\delta}.$$

This interest in the *ScP* arises out of its applications to problems that, at first glance, have nothing at all to do with L-functions. Any progress made towards the *ScP*, in turn, can be used in the resolution of equidistribution problems arising from 'Quantum Chaos' and Hilbert's 11^{th} Problem. In the interests of brevity, we shall refrain from expounding on these topics, instead referring the reader to §6 in [20] for a more in-depth explanation pertaining to these topics.

We shall now proceed to define the Shifted Convolution Problem, i.e., the *SCP*. Given $g(z)$, a primitive modular form of some level D, with Hecke eigenvalues $\lambda_g(n)$, the *SCP* involves the non-trivial estimation of the following kinds of sums:

$$\sum(g, l_1, l_2, h) = \sum_{l_1 m - l_2 n = h} \overline{\lambda_g}(m)\lambda_g(n)V(m, n) \tag{1.8}$$

where $h \neq 0$ and V is a smooth compactly supported function in $[M, 2M] \times [N, 2N]$. The trivial bound for this sum is given by

$$\sum (g, l_1, l_2, h) \ll (MN)^\epsilon \max(M, N).$$

The *SCP* can then be defined as follows:

Definition 1.4.2. [Shifted Convolution Problem] *Find* $\delta > 0$ *such that*

$$\sum (g, l_1, l_2, h) = \text{ Main Term}(h) + O\left(M^{1-\delta}\right)$$

A.E. Ingham first studied a sum as in (1.8) in [19] where $g = \frac{\partial}{\partial s} E(z, s)|_{s=1/2}$ with $E(z, s)$ referring to the Eisenstein series for $SL(2, \mathbb{Z})$ (see Chapter IV for a definition). The divisor function of n, $\tau(n)$ under standard notation, played the role of $\lambda_g(n)$. We shall be using this historical fact again in later chapters, as an analogy that will serve to motivate the various problem(s) we will be studying.

Now, in order to finally tie the knot that binds the *SCP* and *ScP* together, we note that many instances of the *SCP* have been solved for the L-functions of $GL(1, \mathbb{Q})$ and $GL(2, \mathbb{Q})$ automorphic forms (via work done by Good in [11], Meurman in [25], Duke-Friedlander-Iwaniec in a series of papers [5], [8], [7], [6], [9], etc.) and – this is the key – all can be reduced to an instance of the *SCP*. Thus, any improvement in the error term of a shifted convolution sum could have potential ramifications in breaking the convexity barrier of a related L-function. This marks the end of our brief exposé on the relevance of the role played by the *SCP* in modern analytic number theory. However, a caveat is in order. Ours is a sum quite unlike the ones traditionally considered in a *SCP*. The cause of this unlikeness is the subject of the next section.

1.5 On $r_Q(n)$ and related historical perspectives

A distinguishing feature of the Shifted Convolution Sum under consideration in (1.1) is the presence of the $r_Q(n)$ term. If we were to leaf through the pages of relatively recent mathematical history as it pertains to analytic number theory, we would notice the repeated attempts of mathematicians to solve and improve upon existing solutions to *SCP*s. In these attempts, one is drawn to the fact that questions concerning the *SCP* were, to get to the heart of the matter, mainly focused on summands involving the Hecke eigenvalues of Hecke-Maass cusp forms (see Chapter IV for a defintion). Remarks outlining the reasons behind this emphasis were made in the previous section. Seen in the light of prior research then, the sum in (1.1) does not, strictly speaking, fall under the purview of the *SCP*. This is because $\Theta(z, Q)$, i.e., the theta function associated to the quadratic form $Q(x)$ whose Fourier coefficient is given by $r_Q(n)$, is not a Hecke-Maass cusp form. However, in spite of this apparent dissimilarity, we can use the *SCP* and other techniques that crop up in its study as a motivating analogy for our own research.

Moreover, sums such as (1.1) and variations thereof have also been studied by earlier mathematicians and deemed worthy of interest aside from any considerations involving the *SCP*. So, this thesis could be seen as an attempt to translate the established *SCP* techniques to this new setting and study their efficacy. An example of antecedent work, albeit one using rather different methods of investigation, similar to this dissertation can be seen in [18] ,where the following result was established:

$$\sum_{1 \leq n \leq X} r_2(n)\tau(n+1) = \sum_{1 \leq n \leq X} \tau(n^2+1) = XP(\log X) + O\left(X^{8/9+\epsilon}\right). \qquad (1.9)$$

Here, $\tau(n)$ is the divisor function , while $P(x)$ is a polynomial of degree one. In fact, [18] was itself an attempt by C. Hooley to place R. Bellman's flawed paper [1] of 1950 on a mathematically rigorous footing. Further improvements to the error term in [18] were made by Deshouillers/Iwaniec, Sarnak, and Bikovski. See the expository survey [2] by Deshouillers for more details regarding related research.

In fact, the question of studying $r_Q(n)$ as a mathematical entity in its own right sports an illustrious history, judging solely by the various mathematical luminaries who have pondered this question and made serious contributions towards its long and interesting history. As a specific example, when $Q(x) = x_1^2 + \ldots + x_m^2$, then $r_Q(n)$ boils down to the question of studying the representation of an integer n by a sum of m squares. In this instance, we shall henceforth denote $r_Q(n)$ by $r_m(n)$. Recounting the history of the various attempts made at obtaining formulae for $r_m(n)$ pertaining to specific values of m and n takes up more than a hundred pages in volume II of L.E. Dickson's encyclopedic history of number theory [4]. Various approaches to this problem have been developed from two of the best-known results of elementary number theory: Fermat's theorem that any prime of the form $p = 4m + 1$ is representable in a unique way as a sum of two squares and Lagrange's theorem that every positive integer is a sum of at most four squares of integers. Moreover, to quote directly from [31]:

> the crowning achievement of the modern theory of modular forms pertaining to the problem of representing integers as a sum of squares is the clear understanding of the role played by the theta series
>
> $$\theta(z) = \sum_{n\in\mathbb{Z}} q^{n^2}, \text{ where } q = e^{2\pi i z}$$
>
> in the representation theory of the group of $SL(2)$ of 2×2 matrices.

A key, yet deceptively simple, observation that underlies the importance of $\theta(z)$ towards an understanding of $r_m(n)$ is given by

$$\theta^m(z) = 1 + \sum_{n=1}^{\infty} r_m(n) q^n.$$

Combinatorial arguments involving this Fourier series expansion of $\theta^m(z)$ leads us to expressions of the following type for specific instances of $r_m(n)$, many of which were first derived by Jacobi (as noted in [16]):

$$r_2(n) = 4 \sum_{2l+1|n} (-1)^l$$

$$r_4(n) = 8 \cdot 3^\delta \sum_{2l+1|n} (2l+1), \text{ where } \delta = \begin{cases} 1, & \text{if } n \text{ is even.} \\ 0, & \text{otherwise.} \end{cases}$$

$$r_6(n) = 4 \sum_{2l+1|n} (-1)^l \left\{ \left(\frac{2n}{2l+1} \right)^2 - (2l+1)^2 \right\}.$$

Jacobi's successes in this matter can be attributed to the many identities he had established earlier in the course of his investigation into the intricacies of $\theta(z)$, some of which are reproduced below:

$$\theta(z)^2 = 1 + 4 \left(\frac{q}{1-q} - \frac{q^3}{1-q^3} + \frac{q^5}{1-q^5} + \cdots \right)$$

$$\theta(z)^4 = 1 + 8 \left(\frac{q}{1-q} + \frac{2q^2}{1+q^2} + \frac{3q^3}{1-q^3} + \cdots \right)$$

$$\theta(z)^8 = 1 + 16 \left(\frac{q}{1+q} - \frac{2^3 q^2}{1-q^2} + \frac{3^3 q^3}{1+q^3} + \cdots \right)$$

Exact expressions for $r_m(n)$ can be derived from these alternative expansions of powers of $\theta(z)$ by equating coefficients between the Fourier series expansions of powers of $\theta(z)$ and the generating-function identities quoted above.

An analytic tool that we have failed to mention so far is the Hardy-Littlewood circle method. Important as it is, we will not be touching upon this topic in this thesis. Besides having been applied towards the study of $r_Q(n)$, this method has also been used with marked success in dealing with the intricacies of the partition function and some success towards a solution of Waring's Problem. Lest it be considered an antiquated tool, H. Helfgott also uses it in his solution of the ternary Goldbach Problem [17] announced last year.

The work in this dissertation is based on an historical idea that exploits the relationship between a particular instance of $r_Q(n)$ and the divisor function $\tau(n)$. In order to describe this idea in further detail, we consider two classical examples: *Dirichlet's divisor problem* and *Gauss' circle problem*. Both these problems have geometric interpretations, with Dirichlet's divisor problem counting lattice points in the region $\{x, y > 0, xy \leq X\}$, and Gauss' circle problem counting lattice points in the disc $\{x^2 + y^2 \leq X\}$. To formulate the problems in a more precise manner, Gauss' circle problem involves finding the 'correct' error term $E(X)$ for the following asymptotic expression

$$\sum_{n=1}^{X} r_2(n) = \pi X + E(X),$$

while Dirichlet's divisor problem concerns itself with estimating the error term $\Delta(X)$ of

$$\sum_{n=1}^{X} \tau(n) = X \log X + (2\gamma - 1)X + \Delta(X),$$

where $\gamma \approx 0.5772$ is Euler's constant. A lot of work has gone into improving the estimates for $E(X)$ and $\Delta(X)$. See [32, p. 106 - 114].

From the geometric perspective, the analysis involved in optimizing the error term for $\sum r_2(n)$ differs rather crucially from that of $\sum \tau(n)$ in that the former counts the number of lattice points enclosed by a closed curve (i.e., a circle in this case), while the latter counts the number of lattice points enclosed by an unbounded curve (i.e., a hyperbola).While this difference might seem rather trivial, we expect – and this is perhaps borne out by the chronological order of the discovery of the results in question and a perusal of the reasoning required to establish these classical results – that the nature of the analysis involved becomes a bit easier when confronting a closed curve as opposed to an unbounded one. However, the counterintuitive nature of what we are professing here is not to be understated – one the one hand, it would not be surprising to expect the analysis involved in sums comprising $r_Q(n)$ to be more complicated than the analysis of sums containing $\tau(n)$ owing to the more complicated structure (and definition) of the former as compared to the latter; on the other hand, what we actually predict is an easier analysis of the sums containing $r_Q(n)$. As history goes on to show, one could conceivably develop similar techniques to study the two aforementioned kinds of sums concurrently. In a series of papers starting with [39] in 1904, G. Voronoï developed similar geometric and analytic methods to improve both Dirichlet's and Gauss' initial bounds simultaneously. This set the stage for the development of the Voronoï Summation Formula, a technique of great import which will be discussed in greater detail in the next section.

Motivated by the work of Voronoï, in this dissertation, we base our techniques for studying the sum (1.1) on comparable work dealing with the case involving the divisor function $\tau(n)$. However, the path of discovery is seldom smooth and our efforts at clearing the hurdles that bar our way will be the subject of the pages to come.

1.6 On Voronoï Summation

In this final section of the introduction, we would like to describe the Voronoï Summation Formula, an oft-used tool belonging to a broader class of summation formulae. It is this Voronoï formula, first seen in Chapter II, which serves as the foundation for much of the work done in later chapters. Summation formulae in general have played an important role in the analysis and number theory, with some accounts (as in [28]) dating their origins back to the Poisson Summation Formula, a variant of which when applied to functions $f(x)$ of bounded variation supported on a finite interval assumes the following shape:

$$\sum_{a<n\leq b}{}' f(n) = \int_a^b f(x)\,dx + 2\sum_{n=1}^{\infty}\int_a^b f(x)\cos(2\pi nx)\,dx. \tag{1.10}$$

Another prominent example of a summation formula is the Euler-Maclaurin formula which is often used by analytic number theorists as a way of passing from the analysis of sums of certain types of smooth functions to its corresponding integrals accompanied by a relatively simple error term. The Euler-Maclaurin formula manifests itself in rather diverse versions, with its most useful incarnation as follows:

$$\sum_{n=a}^{b} f(n) \sim \int_a^b f(x)\,dx + \frac{f(a)+f(b)}{2} + \sum_{k=1}^{\infty}\frac{B_{2k}}{(2k)!}\left(f^{(2k-1)}(b) - f^{(2k-1)}(a)\right).$$

See Chapter IV of [10] for examples of summation formulae in action.

In what follows, amidst the various notable summation formulae clamoring for attention, we will be confining our attention solely on the Voronoï Summation Formula and its journey through history. The following account has been inspired by the excellent expository article [28] of W. Schmid and S. Miller.

Picking up the narrative thread of the previous section once again, we note that in its manifestation as (1.10), the Poisson Summation Formula appealed to mathematicians like Voronoï and his contemporaries because of the promise it held in evaluating certain finite sums of arithmetic quantities. Such sums featured quite prominently in several questions of analytic number theory and hence, were (and continue to be) much studied by number theorists.

G. Voronoï, in his seminal work of 1904-05, managed to generalize the Poisson Summation Formula as stated in (1.10) by considering weighted sums involving piecewise continuous and piecewise monotone functions $f(x)$. His formula for the divisor function runs as follows :

$$\sum_{a<n\le b}{}' \tau(n)f(n) = \int_a^b f(x)(\log x + 2\gamma)\,dx + \sum_{n=1}^{\infty} \tau(n) \int_a^b f(x)(4K_0(4\pi\sqrt{nx}) - 2\pi Y_0(4\pi\sqrt{nx}))\,dx, \quad (1.11)$$

where Y_0, K_0 denote Bessel functions.

Under the same hypotheses for $f(x)$, he also asserted a formula for $r_2(n)$ as displayed below (using notation established earlier, $r_2(n)$ refers to the number of ways of representing an integer as the sum of two squares), later rigorously proved by Hardy and Sierpenski.

$$\sum_{a<n\le b}{}' r_2(n)f(n) = \sum_{n=0}^{\infty} r_2(n) \int_a^b f(x)\pi J_0(2\pi\sqrt{nx})\,dx, \quad (1.12)$$

with J_0 referring to the J-Bessel function.

From (1.10), (1.11), and (1.12), Voronoï conjectured the existence of analogous formulae for sums $\sum_{a\le n\le b} a_n f(n)$ corresponding to any 'arithmetic' sequence of coefficients a_n. In modern parlance, such formulae are collectively referred to as Voronoï Summation Formulae. While the phrase 'arithmetic sequence of coefficients' in the statement of Voronoï's conjecture may come across as a bit vague in scope and meaning, modern number theorists prefer to think of the phrase as alluding to the coefficients of L-functions, around the understanding of which much, if not all, of analytic number theory revolves. These coefficients are regarded as the natural class of coefficients $(a_n)_{n\geq 1}$ for which Voronoï summation formulae can be proved. In fact, and this point is to be emphasized as far as the utility of Voronoï formulae is concerned, the properties of an L-function (as encountered in §2 of this chapter) serve as a wellspring of summation formulae. These formulae can then be used, via related analytic techniques, towards an attempt at understanding the nature of the coefficients $(a_n)_{n\geq 1}$ of the L-function itself. Thus, the Voronoï Summation Formula is a worthwhile tool that gives us an additional insight (albeit in a roundabout manner) into the behavior of the L-function itself.

For an exemplar of the point made in the previous line, we refer the interested reader towards §2 of [28]. There, the authors exhibit an 'equivalence' between

1. the functional equation of the Riemann ζ-function, and
2. the Poisson Summation Formula for Schwartz functions.

Riemann, in his seminal work [33], used 1) to prove 2). This is a classical result and a proof can be found, for example, in §2.4 of [38]. The opposite implication, wherein the functional equation of the ζ-function implies the Poisson Summation Formula follows from an application of the following techniques in order: Mellin inversion, a contour shift assisted by employing Cauchy's Residue Theorem, an application of the functional equation of the ζ-function, and an application of the classical integral identity

$$\int_{\mathbb{R}} \cos(2\pi x)|x|^{s-1}\, dx = \frac{\pi^{-s/2}\,\Gamma\left(\frac{s}{2}\right)}{\pi^{(s-1)/2}\,\Gamma\left(\frac{1-s}{2}\right)}$$

which is valid for $s \in \mathbb{C}$ by meromorphic continuation. This example also illustrates to great effect the basic model followed by contemporary analytic number theorists in deriving Voronoï formulae from the functional equations of the L-functions. As the brief sketch provided above may prove to be rather cryptic in its brevity to the point of incomprehensibility, we urge the reader to consult §2 of [28] to see this rather elegant argument written in unarguably more lucid detail.

A similar argument also allows us to derive Voronoï summation formulae (1.11) and (1.12) from the functional equations of the Dirichlet series

$$\sum_{n=1}^{\infty} d(n)n^{-s}, \text{ and } \frac{1}{4}\sum_{n=1}^{\infty} r_2(n)n^{-s}. \tag{1.13}$$

More pertinently, modular forms and automorphic forms on $]GL(2,\mathbb{Z})$ are a rich source of L-functions, and each such L-function satisfies a functional equation from which a Voronoï formula can be deduced. In analytic number theory, the Voronoï Summation Formula for $GL(2,\mathbb{Z})$ has become an essential tool in dealing with problems such as the sub-convexity of L-functions or the analysis of Petersson and Kuznetsov trace formulae. The Voronoï Summation Formula for $GL(2,\mathbb{Z})$ mainly deals with sums of the following form:

$$\sum_{n\neq 0} a_n e\left(\frac{-na}{c}\right) f(n). \tag{1.14}$$

To quote [29] directly, 'The presence of the additive twists in (1.14), as signified by the factors of $e(-na/c)$, are absolutely crucial. These additive twists lead to estimates for sums of modular form coefficients over arithmetic progressions.' Another feature of Voronoï summation that plays an important role in applications is the phenomenon of 'dualizing' whereby the Voronoï formula lengthens the sum on one side of the equation while simultaneously shortening the sum on the other side. This fundamental technique in analytic number theory allows one to detect and exploit cancellation in sums. Attempts at generalizing the Voronoï formula to $\mathrm{GL}(3,\mathbb{Z})$ proved futile because additively twisted L-functions for $\mathrm{GL}(3,\mathbb{Z})$ do not satisfy functional equations, a necessary precursor of the Voronoï formula as explained earlier in this section. This obstacle was surmounted by S. Miller and W. Schmid in [29] by relying on the notion of an automorphic distribution coupled with the use of representation theoretic methods.

With this, we draw to a close this section (and thence, this chapter) on Voronoï summation formulae and their role in modern analytic number theory. We have yet to introduce the other methods pertinent to our study – such as the δ-symbol method and the spectral method – in this chapter. In order to prevent inundating the reader with details divorced from their applications in this introductory chapter, we will expound on the relevant techniques later in this dissertation.

Chapter 2: An Approach using Poincaré Series

2.1 Introduction and motivation

In this chapter, we focus our attention on getting estimates for sums of the form

$$D_f(\Theta, b) = \sum_{n \geq 1} a_f(n+b) r_Q(n) \phi(n), \tag{2.1}$$

where b is an integer, $r_Q(n)$ is the number of ways of representing an integer n by a positive definite quadratic form $Q(x)$, $a_f(n)$ is the normalized Fourier coefficient of a holomorphic cusp form f and ϕ is a suitable nice weight function on $(0, \infty)$. Variations of such shifted convolution sums have been studied extensively by analytic number theorists of yore such as in works such as [12] and [25] (see §2 of the Introduction for further elucidation regarding this matter) and continue to be a source of further research conducted by modern analytic number theorists. As we noted in the Introduction, a primary reason for the relative importance of such sums in the field of analytic number theory lies in its application towards the study of the Sub-convexity Problem (*ScP*) which, as described in complete generality by H. Iwaniec and P. Sarnak in [20], addresses the size of certain families of L-functions on the real half line. One of the earliest examples of a result involving such shifted convolution sums was obtained by A.E. Ingham in 1927 wherein he proved

$$\sum_{n \leq X} \tau(n)\tau(n+1) = \frac{6}{\pi^2} X \log^2 X + O\left(X \log X\right), \text{ as } X \to \infty.$$

He was the first to obtain an asymptotic formula for the shifted convolution sum in question, in turn spurring further research attempting to improve upon his estimate of the error term.This quest for a better estimate is nowadays referred to as the Shifted Divisor Problem and concerns, in particular, the study of the following sum:

$$D(\tau, b) = \sum_{n \leq N} \tau(n+b)\tau(n).$$

While $D_f(\Theta, b)$ and $D(\tau, b)$ look like unrelated number theoretic entities, the tie that binds them together is a strong and deep one. In order to realize the analogy between the two sums under consideration, one must look towards the field of modular forms and their resulting Fourier expansions. Seen in this light, $\tau(n)$ is the n-th Fourier expansion of the modular form $\frac{\partial}{\partial s}E(z,s)|_{s=1/2}$ where $E(z,s)$ is the Eisenstein series for $SL_2(\mathbb{Z})$, while $r_Q(n)$ is the n-th Fourier coefficient of the Θ-function associated to the quadratic form $Q(x)$. Under this analogy then, the only difference between $D_f(\Theta, b)$ and $D(\tau, b)$ lies in the fact that $D(\tau, b)$ involves the Fourier coefficients of the same modular form, while $D_f(\Theta, b)$ involves two different modular forms.

This paper can be seen as a straightforward generalization of [24] which uses the framework introduced in [5] for studying the shifted convolution sums as in (2.1). This involves combining an 'elementary' approach using Poincaré series, with a Voronoï-type summation formula followed by an application of Weil's estimate for the individual Kloosterman sums that arise.

This chapter (together with [24] and [23]) make no use of the δ-symbol method. Rather, we resort to the Petersson trace formula in order to capture the shift. The δ-symbol method has the advantage of employing characters of much smaller moduli relative to the shift thereby. This makes it useful in establishing sub-convexity estimates (see §20.5 of [10] for an explanation), while the sort of estimates we're interested in are much more exploratory in nature. Consequently, we're not that concerned about controlling the moduli of the characters involved and moreover, using the Petersson trace formula allows us the freedom of not having to worry about controlling the variation of any additional auxiliary functions introduced in the course of applying the δ-symbol method. See Chapter IV for further clarification in this matter.

2.2 Background on the theta function $\Theta(z, Q)$ associated to a quadratic form $Q(\mathbf{x})$

Let $Q(\mathbf{x})$ be a positive definite quadratic form in $l \geq 2$ variables. In Siegel's notation $Q(\mathbf{x}) = \frac{1}{2}A[x]$, where $A = (a_{ij})$ is a symmetric, positive definite matrix of rank l. We assume that A has integral entries which are even on the diagonal. Then, $Q(\mathbf{x})$ has integral coefficients. Now, we define the theta function $\Theta(z, Q)$ associated to the quadratic form $Q(\mathbf{x})$ as follows

$$\Theta(z, Q) = \sum_{m \in \mathbb{Z}^l} e(Q(m)z) = \sum_{n=0}^{\infty} r(n, Q)e(nz)$$

where the representation numbers $r(n, Q)$ are the Fourier coefficients of $\Theta(z, Q)$.

For a positive integer N satisfying $NA^{-1} \in M_l(\mathbb{Z})$, we note (as in [21, p. 185]) that $\Theta(z, Q)$ is an automorphic form for $\Gamma_0(4N)$ of weight $k = l/2$ and multiplier

$$\theta(\tau) = \left(\frac{|A|}{d}\right)\left(\bar{\epsilon}_d\left(\frac{c}{d}\right)\right)^l \text{ where } \tau = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(4N). \tag{2.2}$$

Here, $\left(\frac{\cdot}{\cdot}\right)$ denotes the Kronecker symbol, while N is a positive integer such that NA^{-1} is an integral matrix and might not be the minimal level. ϵ_d denotes the sign of the Gauss sum

$$\epsilon_d = \begin{cases} 1 & \text{if } d \equiv 1 \pmod 4 \\ i & \text{if } d \equiv 3 \pmod 4 \end{cases}$$

as usual. For our purposes, we let $N = |A|$. Consequently, if A is a unimodular matrix, then $N = 1$. We also define

$$L_Q\left(s, \frac{a}{c}\right) = \sum_{n=1}^{\infty} r(n, Q) e\left(an/c\right) n^{-s},$$

which is absolutely convergent for $\Re(s) > k$.

2.3 Statement of results

Let $f(z) = \sum_{n \geq 1} a_f(n) n^{(k-1)/2} e(nz) \in S_k(\Gamma_0(N))$ and $\phi(x)$ be a smooth function with support in $[X/2, 5X/2]$, satisfying $\phi^{(p)}(x) \ll (X/P)^{-p}$ for integers $p \geq 0$, where $1 \leq P \leq X^{\beta}$, $\beta < 1$. Let $Q(x)$ be a positive definite quadratic form in $l \geq 2$ variables. Our object of interest is the shifted convolution sum

$$\sum_{n \geq 1} a_f(n+b) r_Q(n) \phi(n), \tag{2.3}$$

where $b > 0$ is a fixed integer, and $l \geq 2, k \geq l/2 + 3$. Here, X and P are parameters whose values can be determined depending upon the type of application. We wish to prove the following theorem.

Theorem 2.3.1. *With notation and assumptions as in the preceding paragraph and defining g as the smallest integer such that $g > l/2 + 1$, we have that*

$$\sum_{n \geq 1} a_f(n+b) r_Q(n) \phi(n) \ll X^{l/2-(l-1)/4+\epsilon} P^g. \tag{2.4}$$

The proof of Theorem (in all its detailed glory) will be given in §6 of this chapter. We now apply the bound obtained from Theorem 2.3.1 to get the following result for an unsmoothed sum.

Corollary 2.3.2. *When* $l \geq 2, k \geq l/2 + 3$, *we have the following bound on the sum*

$$\sum_{n \leq X} a_f(n+b) r_Q(n) \ll X^{\frac{l}{2} - \frac{l-1}{4(g+1)} + \epsilon}, \tag{2.5}$$

where g *is defined as in Theorem 2.3.1.*

Proof of the Corollary. We shall prove (2.5) for the restricted range $X \leq n \leq 2X$ first. The following analysis can then be extended to the required range $1 \leq n \leq X$ by means of a standard 'dyadic partition' argument. Please see §5 of [15] for further discussion regarding this technique. Let $\phi(x)$ be a smooth function with $\mathrm{Supp}(\phi) \subset [X - X/P, 2X + X/P]$, satisfying

$$\phi(x) = 1, \text{ if } x \in [X, 2X], \text{ and } \phi^{(p)}(x) \ll (X/P)^{-p}, \ \forall p \in \mathbb{Z}. \tag{2.6}$$

Using the definition of $\phi(x)$ (to be specific, we are using the fact that its support lies in $[X - X/P, 2X + X/P]$), we obtain the following breakdown of the sum

$$\begin{aligned} \sum_{X \leq n \leq 2X} a_f(n+b) r_Q(n) = \sum_{n \geq 1} a_f(n+b) r_Q(n) \phi(n) &- \sum_{X - X/P \leq n \leq X} a_f(n+b) r_Q(n) \phi(n) \\ &- \sum_{2X \leq n \leq 2X + X/P} a_f(n+b) r_Q(n) \phi(n). \end{aligned} \tag{2.7}$$

Upon applying the standard bound $r_Q(n) \ll n^{l/2-1}$, the inequality $\phi(x) \leq 1$ from (2.6), and the fact that the normalized Fourier coefficient $a_f(n)$ satisfies $a_f(n) \ll n^{\epsilon}$ directly to the second term on the right-hand side of (2.7), we get the following inequality

$$\sum_{2X \leq n \leq 2X + X/P} a_f(n+b) r_Q(n) \phi(n) + \sum_{X - X/P \leq n \leq X} a_f(n+b) r_Q(n) \phi(n) \ll (X/P)^{l/2+\epsilon}. \tag{2.8}$$

Using Theorem 2.3.1, we also know that

$$\sum_{n \geq 1} a_f(n+b) r_Q(n) \phi(n) \ll X^{l/2 - (l-1)/4 + \epsilon} P^g, \tag{2.9}$$

where g is as defined as in Theorem 2.3.1. Putting all of the these inequalities into (2.7), we get

$$\sum_{X \le n \le 2X} a_f(n+b) r_Q(n) = O\left(X^{l/2-(l-1)/4+\epsilon} P^g\right) + O\left((X/P)^{l/2+\epsilon}\right).$$

We now proceed to optimize the error term by setting the two Big-O terms equal to each other and solving for P, giving us $P = X^{(l-1)/(2l+4g)}$.

$\square$

2.4 Some Lemmas pertaining to Voronoï summation

We now proceed to state and prove two important results that will help us set the stage for things to come.

Lemma 2.4.1. [A functional equation for $L_Q(s, a/c)$] *The completed L-function $\left(\frac{c}{2\pi}\right)^s \Gamma(s) L_Q\left(s, \frac{a}{c}\right)$ admits a meromorphic continuation to $\mathbb{C}$, with only simple poles at $s = 0, k$. Moreover,*

$$\operatorname*{Res}_{s=0} \left(\frac{c}{2\pi}\right)^s \Gamma(s) L_Q\left(s, \frac{a}{c}\right) = -1, \tag{2.10}$$

and

$$\operatorname*{Res}_{s=k} \left(\frac{c}{2\pi}\right)^s \Gamma(s) L_Q\left(s, \frac{a}{c}\right) = i^k \left(\frac{|A|}{a}\right)^{-1} \left(\bar{\epsilon}_a \left(\frac{-2c}{a}\right)\right)^{-2k}. \tag{2.11}$$

It also satisfies the functional equation

$$\begin{aligned}\left(\frac{c}{2\pi}\right)^s \Gamma(s) L_Q\left(s, \frac{a}{c}\right) = i^k \left(\frac{|A|}{a}\right)^{-1} \left(\bar{\epsilon}_a \left(\frac{-2c}{a}\right)\right)^{-2k} \left(\frac{c}{2\pi}\right)^{k-s} \\ \times \Gamma(k-s) L_Q\left(k-s, \frac{-d}{c}\right).\end{aligned} \tag{2.12}$$

Let $\phi \in C_0^\infty(\mathbb{R}^+)$ be a smooth function with compact support in $(0, \infty)$,and define its Mellin transform

$$G(s) = \int_0^\infty \phi(x) x^{s-1}\, dx, \tag{2.13}$$

which is entire on $\mathbb{C}$. Moreover, for any positive A and B, we have

$$G(s) \ll_{A,B} (1+|s|)^{-A}, \tag{2.14}$$

uniformly for $|\mathcal{R}(s)| \leq B$.

We proceed to sketch a proof of (2.14). The main ingredient in the proof turns out to be integration by parts. Integrating the right-hand side of (2.13) by parts once, we get

$$\int_0^\infty \phi(x)x^{s-1}\,dx = \phi(x)\frac{x^s}{s}\bigg|_0^\infty - \frac{1}{s}\int_0^\infty \phi'(x)x^s\,dx.$$

Since $\phi(x)$ has compact support on the open interval $(0,\infty)$, the first term on the right evaluates to zero. Continuing to integrate by parts, we get the following relation:

$$\int_0^\infty \phi(x)x^{s-1}\,dx = \frac{(-1)^k}{(s)(s+1)\cdots(s+k-1)}\int_0^\infty \phi^k(x)x^{s+k-1}\,dx.$$

Note that the higher derivatives of $\phi(x)$ also have compact support in $(0,\infty)$. Within the vertical strip $|\mathcal{R}(s)| \leq B$, we note that for $|z| > B$, we can use the expression above to get the following bound for $G(s)$:

$$G(s) = \frac{(-1)^k}{(s)(s+1)\cdots(s+k-1)}\int_0^\infty \phi^k(x)x^{s+k-1}\,dx \ll_B (1+|s|)^{-k}.$$

The rest of the values within the strip (i.e., those with $|\mathcal{R}(s)| \leq B$ and $|z| \leq B$) form a compact set H. Since $G(s)$ is holomorphic in this region, and $1/(1+|s|)^k$ is a continuous function over a compact set H, we have

$$G(s) \ll_B (1+|s|)^{-k}.$$

Finally, putting all the bounds together, and by letting $k = \lceil A \rceil$, we get (2.14).

We now proceed to state our Voronoï summation formula for $L_Q(s, a/c)$ which we obtain from Lemma 2.4.1.

Lemma 2.4.2. [A Voronoï summation formula for $L_Q(s, a/c)$] *Let $\phi(x)$ be a smooth function of compact support in $(0, \infty)$. Then, we have the following Voronoï-type summation formula:*

$$\begin{aligned}&\sum_{n=1}^{\infty} r_Q(n) e(an/c)\phi(n)\\ =&\frac{(2\pi)^k}{c^k}\Gamma(k)^{-1}G(k)i^k\left(\frac{|A|}{a}\right)^{-1}\left(\bar{\epsilon}_a\left(\frac{-2c}{a}\right)\right)^{-2k}+\frac{2\pi}{c}i^k\left(\frac{|A|}{a}\right)^{-1}\left(\bar{\epsilon}_a\left(\frac{-2c}{a}\right)\right)^{-2k}\\ &\times\sum_{n=1}^{\infty}\left(r_Q(n)e(-dn/c)n^{(1-k)/2}\int_0^{\infty}\phi(x)x^{(k-1)/2}J_{k-1}\left(\frac{4\pi\sqrt{nx}}{c}\right)dx\right)\end{aligned}$$

2.5 Proof of the Lemmas

Proof of Lemma 1. Consider the integral

$$I_Q(s, a/c) = \int_0^{\infty} \left(\Theta(a/c + iy) - 1\right) y^{s-1}\, dy. \tag{2.15}$$

Note that via (2.2), $\Theta(z)$ satisfies the transformation law given by

$$\Theta(z) = \left(\frac{|A|}{a}\right)\left(\bar{\epsilon}_a\left(\frac{-c}{a}\right)\right)^{-2k}(-cz+a)^{-k}\,\Theta\left(\frac{dz-b}{-cz+a}\right), \text{ where } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(4N). \tag{2.16}$$

We first need to make sure that the integral in (2.15) converges for large and small values of y. As $y \to \infty$, we note that $(\Theta(a/c + iy) - 1) \ll e^{-2\pi y}$. Hence, the integral given by (2.15) converges for all values of s as y tends to infinity. Moreover, (2.16) tells us that $(\Theta(a/c + iy) - 1)\, y^{s-1} \ll y^{\Re(s)-k-1}$ as $y \to 0$. Therefore the integral in (2.15) is convergent and analytic for $\Re(s) > k$ in a small neighborhood of the origin. In the region $\Re(s) > k$, we also have the relation

$$I_Q(s, a/c) = \left(\frac{1}{2\pi}\right)^s \Gamma(s) L_Q(s, a/c). \tag{2.17}$$

Thus, we see that

$$
\begin{aligned}
I_Q(s, a/c) =& \int_{1/c}^{\infty} (\Theta(a/c + iy) - 1)\, y^{s-1}\, dy + \int_{0}^{1/c} (\Theta(a/c + iy) - 1)\, y^{s-1}\, dy \\
=& -\frac{c^{-s}}{s} - i^k \left(\frac{|A|}{a}\right)^{-1} \left(\bar{\epsilon}_a \left(\frac{-2c}{a}\right)\right)^{-2k} \frac{c^{-s}}{k-s} + \int_{1/c}^{\infty} (\Theta(a/c + iy) - 1)\, y^{s-1}\, dy \\
&+ i^k \left(\frac{|A|}{a}\right)^{-1} \left(\bar{\epsilon}_a \left(\frac{-2c}{a}\right)\right)^{-2k} c^{k-2s} \int_{1/c}^{\infty} (\Theta(-d/c + iy) - 1)\, y^{k-s-1}\, dy.
\end{aligned}
\tag{2.18}
$$

This gives the meromorphic continuation of $\left(\frac{c}{2\pi}\right)^s \Gamma(s) L_Q(s, a/c)$. Moreover, we can also establish the functional equation

$$
c^s I_Q(s, a/c) = i^k \left(\frac{|A|}{a}\right)^{-1} \left(\bar{\epsilon}_a \left(\frac{-2c}{a}\right)\right)^{-2k} c^{k-s} I_Q(k-s, -d/c) \tag{2.19}
$$

in a straightforward manner by replacing $I_Q(k-s, -d/c)$ on the right-hand side of the equation with its equivalent expression from (2.18). Hence, using the definition of $I_Q(s, a/c)$ from (2.15), (2.12) follows and this proves Lemma 2.4.1. □

Proof of Lemma 2. By Mellin inversion,

$$
\phi(x) = \frac{1}{2\pi i} \int_{(2)} G(s) x^{-s}\, ds.
$$

Consequently, following an interchange of sums and integrals justified by an application of the Lebesgue Dominated Convergence Theorem, we have that

$$
\frac{1}{2\pi i} \int_{(k+1)} L_Q(s, a/c) G(s)\, ds = \sum_{n=1}^{\infty} r_Q(n) e(an/c) \phi(n).
$$

We then shift contour lines from $\Re(s) = (k+1)$ to $\Re(s) = (-k-1)$, following an application of Cauchy's Theorem along a (gradually enlarging) rectangle. The bound given in (2.14) guarantees that the integral vanishes as we move the horizontal line of integration upwards, as a result of which only the vertical sides of the rectangle contribute towards the contour integral. In the process we pick up the residue at the pole $s = k$, this giving giving us the following equation

$$\begin{aligned}\frac{1}{2\pi i}\int_{(k+1)} L_Q(s, a/c)G(s)\,ds = \frac{1}{2\pi i}\int_{(-k-1)} L_Q(s, a/c)G(s)\,ds \\ + \left(\frac{c}{2\pi}\right)^{-k}\Gamma(k)^{-1}G(k)i^k\left(\frac{|A|}{a}\right)^{-1}\left(\bar{\epsilon}_a\left(\frac{-2c}{a}\right)\right)^{-2k}\end{aligned} \tag{2.20}$$

We now proceed to replace the first term on the right-hand side of (2.20) with its equivalent expression from the functional equation (2.12). In addition, we also shift contour lines from $\Re(s) = (-k-1)$ to $\Re(s) = -1$ in order to make use of an integral identity towards the end.

$$\begin{aligned}&\frac{1}{2\pi i}\int_{(k+1)} L_Q(s, a/c)G(s)\,ds \\ &= i^k\left(\frac{|A|}{a}\right)^{-1}\left(\bar{\epsilon}_a\left(\frac{-2c}{a}\right)\right)^{-2k}\frac{1}{2\pi i}\int_{(-1)}\left(\frac{c}{2\pi}\right)^{k-2s}\frac{\Gamma(k-s)}{\Gamma(s)}L_Q\left(k-s, \frac{-d}{c}\right)G(s)\,ds \\ &\quad + \left(\frac{c}{2\pi}\right)^{-k}\Gamma(k)^{-1}G(k)i^k\left(\frac{|A|}{a}\right)^{-1}\left(\bar{\epsilon}_a\left(\frac{-2c}{a}\right)\right)^{-2k} \\ &= i^k\left(\frac{|A|}{a}\right)^{-1}\left(\bar{\epsilon}_a\left(\frac{-2c}{a}\right)\right)^{-2k}\sum_{n=1}^{\infty} r_Q(n)e(-dn/c)\psi(n, c) \\ &\quad + \left(\frac{c}{2\pi}\right)^{-k}\Gamma(k)^{-1}G(k)i^k\left(\frac{|A|}{a}\right)^{-1}\left(\bar{\epsilon}_a\left(\frac{-2c}{a}\right)\right)^{-2k},\end{aligned}$$

where

$$\psi(n, c) = \frac{1}{2\pi i}\int_{(-1)}\left(\frac{c}{2\pi}\right)^{k-2s}\frac{\Gamma(k-s)}{\Gamma(s)}n^{s-k}G(s)\,ds.$$

Upon regrouping terms within the integral above and using Mellin-Barnes integral representation of the Bessel function given below,

$$J_{k-1}(x) = \frac{1}{2\pi i}\int_{(\sigma)}\frac{\Gamma(k-s)}{\Gamma(s)}\left(\frac{x}{2}\right)^{2s-k-1}ds,$$

we have

$$\psi(n, c) = \frac{2\pi}{c}n^{(1-k)/2}\int_0^{\infty}\phi(x)x^{(k-1)/2}J_{k-1}\left(\frac{4\pi\sqrt{nx}}{c}\right)dx.$$

This proves Lemma 2.4.2. $\square$

2.6 Proof of Theorem 2.3.1

As in [24], we use the idea of Poincaré series reduction. We first note that $S_k(\Gamma_0(N)) \subset S_k(\Gamma_0(4N))$ as a result of which, we can consider $S_k(\Gamma_0(N))$ as a linear space spanned by the following Poincaré series (see §3.3 of [21]) for $m \geq 1$,

$$P_m(z) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma_0(4N)} \frac{e(m\gamma z)}{(cz+d)^k} = \sum_{n \geq 1} a_{P_m}(n) n^{(k-1)/2} e(nz), \tag{2.21}$$

where $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, and

$$m^{(k-1)/2} a_{P_m}(n) = \delta_{mn} + \sum_{c \geq 1, 4N|c} c^{-1} S(m, n; c) J_{k-1}\left(\frac{4\pi\sqrt{mn}}{c}\right). \tag{2.22}$$

For any normalized cuspform $f(z) = \sum_{n \geq 1} a_f(n) n^{(k-1)/2} e(nz) \in S_k(\Gamma_0(N))$, we can use the above embedding to express $a_f(n)$ as a linear combination of $a_{P_m}(n)$. thus, instead of estimating the shifted convolution sum $\sum_{n \geq 1} a_f(n+b) r_Q(n) \phi(n)$, we replace $a_f(n)$ with the right hand side of (2.22). Moreover, in view of the bound $J_{k-1}(z) \ll \min(|z|^{k-1}, |z|^{-1/2})$ and the Weil-Salié bound for Kloosterman sums, viz.

$$|S_k(m, n; c)| \leq (m, n, c)^{1/2} d(c) c^{1/2},$$

we may assume that $c \ll X^A$ for sufficiently large $A > 0$.

By replacing $a_{P_m}(n+b)$ with the right-hand side of (2.22), we have the following upper bound

$$\sum_{n \geq 1} a_{P_m}(n+b) r_Q(n) \phi(n) \ll \sum_{c \geq 1, 4N|c} c^{-1} \sum_{n \geq 1} r_Q(n) \phi(n) S(m, n+b; c) J_{k-1}\left(\frac{4\pi\sqrt{m(n+b)}}{c}\right).$$

We now proceed to open up the Kloosterman sum via its definition, resulting in the following expression for the right-hand side of the above inequality

$$\sum_{c \geq 1, 4N|c} c^{-1} \sideset{}{^*}\sum_{a \mod c} e\left(\frac{m\bar{a}}{c}\right) \sum_{n \geq 1} r_Q(n) \phi(n) e\left(\frac{(n+b)a}{c}\right) J_{k-1}\left(\frac{4\pi\sqrt{m(n+b)}}{c}\right)$$

We now apply Lemma 2.4.2 to the innermost sum in the expression above, noting that the role of the smooth function with compact support in Lemma 2.4.2 is played by $\phi(x)J_{k-1}(4\pi\sqrt{m(x+b)}/c)$. We also note that in the following lines, after applying Lemma 2.4.2, we leave the Mellin transform of $\phi(x)J_{k-1}(4\pi\sqrt{m(x+b)}/c)$ in its integral form, as opposed to condensing the expression and writing it out as $G\left(l/2\right)$. This will allow us to better understand the role played by the individual terms in the estimates for the resulting expression. Thus, we can rewrite (2.23) in the following manner (which renders it more suitable for any analysis that we might do)

$$\begin{aligned}
&\sum_{c\geq 1, 4N|c} c^{-1} \sum_{a \,(\mathrm{mod}\ c)}^{*} e\left(\frac{m\bar{a}+ba}{c}\right)\left(\frac{c}{2\pi}\right)^{-l/2}\Gamma\left(\frac{l}{2}\right)^{-1} i^{l/2}\left(\frac{|A|}{a}\right)\left(\bar{\epsilon}_a\left(\frac{-c}{a}\right)\right)^{-l} \\
&\quad\times \int_0^\infty \phi(x)J_{k-1}\left(\frac{4\pi\sqrt{m(x+b)}}{c}\right)x^{l/2-1}\,dx \\
&+\sum_{c\geq 1, 4N|c} c^{-1} \sum_{a \,(\mathrm{mod}\ c)}^{*} e\left(\frac{m\bar{a}+ba}{c}\right)\frac{2\pi}{c} i^{l/2}\left(\frac{|A|}{a}\right)\left(\bar{\epsilon}_a\left(\frac{-c}{a}\right)\right)^{-l}\sum_{n=1}^{\infty} r_Q(n)e(-dn/c)n^{\frac{1-l/2}{2}} \\
&\quad\times \int_0^\infty \phi(x)x^{(l/2-1)/2}J_{l/2-1}\left(\frac{4\pi\sqrt{nx}}{c}\right)J_{k-1}\left(\frac{4\pi\sqrt{m(x+b)}}{c}\right)dx \\
&=\sum_1+\sum_2.
\end{aligned}$$

We now focus on getting an estimate for $\sum_1$. We begin by showing that

$$\int_0^\infty \phi(x)J_{k-1}\left(\frac{4\pi\sqrt{m(x+b)}}{c}\right)x^{l/2-1}\,dx \ll \left(\frac{cP}{\sqrt{X}}\right)^p X^{l/2}, \tag{2.23}$$

for any integer $p\geq 0$. Then, for a fixed $\epsilon>0$, the contribution of the above integral is negligible unless $cP \gg X^{1/2-\epsilon}$. Hence, we can ignore the first few terms of the sum $\sum_1$ satisfying $cP \ll X^{1/2-\epsilon}$ since by choosing a value of p large enough in (2.23), we can essentially nullify the contribution of these terms towards the value of sum.

Consequently, (2.23) allows us to focus our attention on the tail-end of the series where $cP \gg X^{1/2-\epsilon}$ in order to estimate $\sum_1$.

In order to prove (2.23), we will use an iterative procedure, where p in (2.23) represents the number of iterations, which makes use of repeated integration by parts and a 'recurrence' relation satisfied by the Bessel function which goes as

$$(z^k J_k(z))' = z^k J_{k-1}(z). \tag{2.24}$$

We proceed to show the first few steps of this iterative method in all its gory detail. By rearranging the terms occurring in (2.24), we can rewrite the 'recurrence' relation in a slightly more suggestive form

$$J_{k-1}\left(\frac{4\pi\sqrt{nx}}{c}\right)\, dx = \frac{d\left[\left(\frac{4\pi\sqrt{nx}}{c}\right)^k J_k\left(\frac{4\pi\sqrt{nx}}{c}\right)\right]}{\left(\frac{4\pi\sqrt{nx}}{c}\right)^k \quad \left(\frac{4\pi\sqrt{nx}}{c}\right)'}. \tag{2.25}$$

Consider the integral under consideration in (2.23). By first replacing the Bessel function $J_{k-1}\left(4\pi\sqrt{m(x+b)}/c\right)$ occurring in the integral with the equivalent expression given by the right hand side of (2.25), we can then integrate by parts to get the following relation

$$\int_0^\infty \phi(x) x^{l/2-1} J_{k-1}\left(\frac{4\pi\sqrt{m(x+b)}}{c}\right) dx = \int_0^\infty \phi'(x) x^{l/2-1} \frac{J_k\left(\frac{4\pi\sqrt{m(x+b)}}{c}\right)}{\left(\frac{4\pi\sqrt{m(x+b)}}{c}\right)'} dx +$$

$$\int_0^\infty \phi(x) x^{l/2-2} \frac{J_k\left(\frac{4\pi\sqrt{m(x+b)}}{c}\right)}{\left(\frac{4\pi\sqrt{m(x+b)}}{c}\right)'} \left(\frac{l-2}{2}\right) dx + \int_0^\infty \phi(x) x^{l/2-1} \frac{J_k\left(\frac{4\pi\sqrt{m(x+b)}}{c}\right)}{\left(\frac{4\pi\sqrt{m(x+b)}}{c}\right)'}$$

$$\times \left(\frac{1-k}{2x}\right) dx. \tag{2.26}$$

We shall now see why we expect the bound in (2.23) to be true. Let I_1 refer to the first term on the right-hand side of (2.26). Then, I_1 can be rewritten in the following manner:

$$I_1 = \int_0^\infty \left(\frac{x+b}{2\pi P} \phi'(x) \left(\frac{x+b}{X} \right)^{-1/2} \right) x^{l/2-1} J_k \left(\frac{4\pi\sqrt{m(x+b)}}{c} \right) \frac{cP}{(mX)^{1/2}} \, dx.$$

The function $\phi(x)$ was assumed to be a smooth function with support in $[X/2, 5X/2]$ satisfying the bound $\phi^{(p)}(x) \ll (P/X)^p$. In fact, the properties of $\phi(x)$ carry over to that of $g(x) = ((x+b)/2\pi P)\, \phi'(x)\, ((x+b)/X)^{-1/2}$, in that $g(x)$ is also a smooth function with support in $[X/2, 5X/2]$ satisfying the bound $g^{(p)}(x) \ll (P/X)^p$. Utilizing a standard bound satisfied by the J-Bessel functions, $J_k(x) \leq 1$ where $k \geq 1$, we bound the integral in question by

$$I_1 \ll \int_{X/2}^{5X/2} 1 \cdot x^{l/2-1} \cdot 1 \cdot \frac{cP}{(mX)^{1/2}} \, dx \ll \left(\frac{cP}{\sqrt{X}} \right) X^{l/2},$$

which agrees with (2.23) when $p = 1$.

Similarly, let I_2 and I_3 refer to the second and third integrals, respectively, on the right-hand side of (2.26). Getting upper bounds for these integrals mirrors the process that we have outlined for I_1 above, giving us

$$I_1, I_2 \ll \left(\frac{c}{\sqrt{X}} \right) X^{l/2}.$$

Putting all of these bounds together, we get

$$\begin{aligned} \int_0^\infty \frac{cP}{(mX)^{1/2}} \, g(x) x^{l/2-1} J_k \left(\frac{4\pi\sqrt{m(x+b)}}{c} \right) dx &\ll \left(\frac{cP}{\sqrt{X}} \right) X^{l/2} + 2 \left(\frac{c}{\sqrt{X}} \right) X^{l/2} \\ &\ll \left(\frac{cP}{\sqrt{X}} \right) X^{l/2} \end{aligned}$$

We note that the last line of the series of inequalities above agrees with (2.23) when $p = 1$.

Proceeding further into the iterative steps laid out by integrating each of the integrals I_1, I_2, and I_3 by parts, we see that all subsequent steps follow the same basic paradigm laid out previously. Upon performing the integration by parts and estimating each term with the standard estimates as shown above, we notice that each additional bout of integration picks up a factor of $(cP/\sqrt{X})$. Therefore, upon performing integration by parts p times, we will get the desired bound as in (2.23).

When $cP \gg X^{1/2-\epsilon}$, we can bound the integral in (2.23) directly by appealing to a standard bound for the Bessel functions given by $J_{k-1}(z) \ll \min\left(|z|^{k-1}, |z|^{-1/2}\right)$. This bound is then given by

$$\int_0^\infty \phi(x) J_{k-1}\left(\frac{4\pi\sqrt{m(x+b)}}{c}\right) x^{l/2-1}\, dx \ll \min\left(\left(\frac{\sqrt{X}}{c}\right)^{k-1}, \left(\frac{\sqrt{X}}{c}\right)^{-1/2}\right) X^{l/2}. \tag{2.27}$$

We now proceed to establish a bound for the 'finite sum' component of $\sum_1$, viz.,

$$\sideset{}{^*}\sum_{a \,(\mathrm{mod}\ c)} e\left(\frac{m\bar{a}+ba}{c}\right)\left(\frac{|A|}{a}\right)\left(\bar{\epsilon}_a\left(\frac{-c}{a}\right)\right)^{-l} \ll (m,b,c)^{1/2}\, c^{1/2}\, \tau(c), \tag{2.28}$$

where $\tau(\cdot)$ is the divisor function.

When l is odd, (2.28) can be rewritten as

$$\begin{aligned}
&\sideset{}{^*}\sum_{a \,(\mathrm{mod}\ c)} e\left(\frac{m\bar{a}+ba}{c}\right)\left(\frac{|A|}{a}\right)\left(\frac{-c}{a}\right)(\bar{\epsilon}_a)^{-l}\\
&= \sideset{}{^*}\sum_{\substack{a \,(\mathrm{mod}\ c)\\ a\equiv 1 \,(\mathrm{mod}\ 4)}} e\left(\frac{m\bar{a}+ba}{c}\right)\left(\frac{|A|}{a}\right)\left(\frac{-c}{a}\right) + i^l \sideset{}{^*}\sum_{\substack{a \,(\mathrm{mod}\ c)\\ a\equiv 3 \,(\mathrm{mod}\ 4)}} e\left(\frac{m\bar{a}+ba}{c}\right)\left(\frac{|A|}{a}\right)\left(\frac{-c}{a}\right).
\end{aligned}$$

We complete the sum in the first term above as follows

$$\begin{aligned}
&\sideset{}{^*}\sum_{\substack{a \,(\mathrm{mod}\ c)\\ a\equiv 1 \,(\mathrm{mod}\ 4)}} e\left(\frac{m\bar{a}+ba}{c}\right)\left(\frac{|A|}{a}\right)\left(\frac{-c}{a}\right)\\
&= \sum_{\substack{a \,(\mathrm{mod}\ c)\\ a\equiv 1 \,(\mathrm{mod}\ 4)}} e\left(\frac{m\bar{a}+ba}{c}\right)\left(\frac{|A|}{a}\right)\left(\frac{-c}{a}\right) \sum_{\chi \,(\mathrm{mod}\ 4)} \frac{1}{\phi(4)}\chi(a)\\
&= \sum_{\chi \,(\mathrm{mod}\ 4)} \frac{1}{\phi(4)} \sum_{a \,(\mathrm{mod}\ c)} \left(\frac{|A|}{a}\right)\left(\frac{-c}{a}\right)\chi(a) e\left(\frac{m\bar{a}+ba}{c}\right)
\end{aligned}$$

We now note that since $|A|$ is the determinant of a symmetric, positive definite matrix, $|A| \equiv 0, 1 \pmod 4$. Hence, $\left(\frac{|A|}{\cdot}\right)$ is a character mod $|A|$. Moreover, since we are working with Poincaré series in $S_k(\Gamma_0(4N))$, we also have the following condition $c \equiv 0 \pmod{4N}$. As noted earlier, N can be taken to be $|A|$, and by virtue of the fact that $c \equiv 0 \pmod 4$, we know that $\left(\frac{-c}{\cdot}\right)$ is a character mod c. And since $c \equiv 0 \pmod{|A|}$, the product of the two characters $\left(\frac{-c}{\cdot}\right)$, $\left(\frac{|A|}{\cdot}\right)$ also gives rise to a character mod c, say $\widetilde{\chi}_c$. In light of this analysis, we have

$$\begin{aligned}\left|\sideset{}{^*}\sum_{\substack{a \pmod c\\ a\equiv 1 \pmod 4}} e\left(\frac{m\bar{a}+ba}{c}\right)\left(\frac{|A|}{a}\right)\left(\frac{-c}{a}\right)\right| &\ll \left|\sum_{a \pmod c} \widetilde{\chi}_c(a) e\left(\frac{m\bar{a}+ba}{c}\right)\right| \\ &\ll (m,b,c)^{1/2}\, c^{1/2}\, \tau(c),\end{aligned}$$

where we have used a bound for the Salié sum (cf. [21], Chapter 4) in order to get the last inequality.

Similarly, we can deal with the second term in the following manner

$$\begin{aligned}&\sideset{}{^*}\sum_{\substack{a \pmod c\\ a\equiv 3 \pmod 4}} e\left(\frac{m\bar{a}+ba}{c}\right)\left(\frac{|A|}{a}\right)\left(\frac{-c}{a}\right)\\ &= \sum_{\chi \pmod 4} \frac{1}{\phi(4)} \sum_{a \pmod c} (-1)^a \left(\frac{|A|}{a}\right)\left(\frac{-c}{a}\right)\chi(a) e\left(\frac{m\bar{a}+ba}{c}\right)\\ &\ll (m,b,c)^{1/2}\, c^{1/2}\, \tau(c).\end{aligned}$$

When l is even and $4 \nmid l$, the bound obtained for (2.28) is exactly the same as the case when l is odd, with the only difference being that after completing the sum, the character by which we are twisting the exponential term is slightly different than in the preceding case. However, that does not affect our bound for (2.28) in any way.

When $4|l$, the 'finite sum' component of $\sum_1$ reduces to

$$\sum_{a \mod c}^{*} e\left(\frac{m\bar{a}+ba}{c}\right)\left(\frac{|A|}{a}\right) \ll (m,b,c)^{1/2}\ c^{1/2}\ \tau(c),$$

where we are using the bound for the Salié sum, since as noted in the case when l is odd, $\left(\frac{|A|}{\cdot}\right)$ can be thought of as a character mod c.

When P is 'small', i.e., when $\sqrt{X}/P \geq 1$, we can use the bounds given by (2.27) and (2.28) and establish a series of inequalities for $\sum_1$, culminating in our bound for the sum.

$$\begin{aligned}
\sum_1 &\ll\ X^{l/2} \sum_{c \geq \sqrt{X}/P} c^{-1-l/2+1/2} d(c)\ \min\left(\left(\frac{\sqrt{X}}{c}\right)^{k-1}, \left(\frac{\sqrt{X}}{c}\right)^{-1/2}\right) \\
&\ll X^{l/2} \sum_{\sqrt{X}/P \leq c \leq \sqrt{X}} c^{-(1+l)/2} d(c) \left(\frac{\sqrt{X}}{c}\right)^{-1/2} +\ X^{l/2} \sum_{c \geq \sqrt{X}} c^{-(1+l)/2} d(c) \left(\frac{\sqrt{X}}{c}\right)^{k-1} \\
&\ll X^{l/2-1/4} \sum_{c \geq \sqrt{X}/P} c^{-l/2} d(c)\ +\ X^{l/2+(k-1)/2} \sum_{c \geq \sqrt{X}} c^{(-l+1)/2-k} d(c) \\
&\ll X^{l/2-1/4} \left(\frac{\sqrt{X}}{P}\right)^{-l/2+1+\epsilon} +\ X^{l/2+(k-1)/2} \left(\sqrt{X}\right)^{(-l+3)/2-k+\epsilon} \\
&\ll X^{l/2-(l-1)/4+\epsilon} P^{l/2-1}.
\end{aligned}$$

When P is 'large', i.e., when $\sqrt{X}/P \leq 1$, then (2.23) is not quite helpful. This is because the inequality in (2.23) doesn't help us isolate the tail-end of the sum. In fact, when $c \geq 1$, then we automatically have $c \geq \sqrt{X}/P$. So, for no value of c within the given range of the sum does the integral actually manage to dominate the rest of the terms within the sum. This forces us to deal with the sum as a whole. However, there is a workaround this issue as shown in the following steps.

Let $P_0 = \sqrt{X}$. Note that $P \geq P_0$. Then, we see that

$$\begin{aligned}
\sum_1 &\ll \sum_{c\geq 1} c^{-1-l/2+1/2} d(c)\ \min\left(\left(\frac{\sqrt{X}}{c}\right)^{k-1}, \left(\frac{\sqrt{X}}{c}\right)^{-1/2}\right) X^{l/2} \\
&= \sum_{c\geq \sqrt{X}/P_0} c^{-1-l/2+1/2} d(c)\ \min\left(\left(\frac{\sqrt{X}}{c}\right)^{k-1}, \left(\frac{\sqrt{X}}{c}\right)^{-1/2}\right) X^{l/2} \\
&\ll X^{l/2-(l-1)/4+\epsilon} P_0^{l/2-1} \\
&\leq X^{l/2-(l-1)/4+\epsilon} P^{l/2-1}, \text{ as desired.}
\end{aligned}$$

We now wish to get an estimate for $\sum\limits_2$. To this end, we have the inequality

$$\begin{aligned}
&\int_0^\infty \phi(x) x^{(l/2-1)/2} J_{k-1}\left(\frac{4\pi\sqrt{m(x+b)}}{c}\right) J_{l/2-1}\left(\frac{4\pi\sqrt{nx}}{c}\right)\, dx \\
&\ll \left(\left[Pc(Xn)^{-1/2}\right]^p + n^{-p/2}\right) X^{(l/2+1)/2} \min\left(\left(\frac{\sqrt{X}}{c}\right)^{k-1}, \left(\frac{\sqrt{X}}{c}\right)^{-1/2}\right) \\
&\qquad \times \min\left(1, \left(\frac{\sqrt{nX}}{c}\right)^{-1/2}\right).
\end{aligned} \tag{2.29}$$

This inequality plays the same role as (2.23) in that it allows us to control $\sum_2$ in terms of the parameters X and P.

The method of proof is quite similar to the one used to prove (2.23). We shall once again use an iterative method relying on integration by parts, the relation (2.24), together with an additional relation involving Bessel functions in the form of

$$J_k'(z) = kz^{-1} J_k(z) - J_{k+1}(z). \tag{2.30}$$

For the sake of brevity, we won't go into the minutiae of the proof, as we did for (2.23). However, we shall provide an outline of the details starting with the integration by parts of the integral that we are interested in.

$$\int_0^\infty \phi(x) J_{k-1}\left(\frac{4\pi\sqrt{m(x+b)}}{c}\right) J_{l/2-1}\left(\frac{4\pi\sqrt{nx}}{c}\right)\, dx$$

$$= \int_0^\infty \phi'(x) J_{k-1}\left(\frac{4\pi\sqrt{m(x+b)}}{c}\right) \frac{J_{l/2}\left(\frac{4\pi\sqrt{nx}}{c}\right)}{\left(\frac{4\pi\sqrt{nx}}{c}\right)'}\, dx$$

$$+ \int_0^\infty \phi(x) J'_{k-1}\left(\frac{4\pi\sqrt{m(x+b)}}{c}\right) \frac{J_{l/2}\left(\frac{4\pi\sqrt{nx}}{c}\right)}{\left(\frac{4\pi\sqrt{nx}}{c}\right)'}\, dx$$

$$+ \int_0^\infty \phi(x) J_{k-1}\left(\frac{4\pi\sqrt{m(x+b)}}{c}\right) \frac{J_{l/2}\left(\frac{4\pi\sqrt{nx}}{c}\right)}{\left(\frac{4\pi\sqrt{nx}}{c}\right)'} \left(\frac{1-l/2}{2x}\right)\, dx\,. \qquad (2.31)$$

We shall now proceed to rewrite (2.30) in a more functional form given by

$$J'_{k-1}\left(\frac{4\pi\sqrt{m(x+b)}}{c}\right) = (k-1)\,\frac{J'_{k-1}\left(\frac{4\pi\sqrt{m(x+b)}}{c}\right)}{\left(\frac{4\pi\sqrt{m(x+b)}}{c}\right)} - J_k\left(\frac{4\pi\sqrt{m(x+b)}}{c}\right).$$

Using the relation above to replace the $J'_{k-1}\left(4\pi\sqrt{m(x+b)}/c\right)$ occurring in the second term of the right-hand side of (2.31), together with a slight rearrangement of terms within each integral gives us the following inequality

$$\int_0^\infty \phi(x) J_{k-1}\left(\frac{4\pi\sqrt{m(x+b)}}{c}\right) J_{l/2-1}\left(\frac{4\pi\sqrt{nx}}{c}\right)\, dx \qquad (2.32)$$

$$\ll \left|\int_0^\infty \frac{1}{n^{1/2}}\left(\frac{x}{x+b}\,\phi(x)\right) J_k\left(\frac{4\pi\sqrt{m(x+b)}}{c}\right) J_{l/2}\left(\frac{4\pi\sqrt{nx}}{c}\right)\, dx\right|$$

$$+ \left|\int_0^\infty \frac{cP}{(nX)^{1/2}}\left(\frac{x}{P}\,\phi'(x)\left(\frac{x}{X}\right)^{-1/2}\right) J_{k-1}\left(\frac{4\pi\sqrt{m(x+b)}}{c}\right) J_{l/2}\left(\frac{4\pi\sqrt{nx}}{c}\right)\, dx\right|$$

$$+ \left|\int_0^\infty \frac{c}{(nX)^{1/2}}\left(\frac{x}{x+b}\,\phi(x)\left(\frac{x}{X}\right)^{-1/2}\right) J_{k-1}\left(\frac{4\pi\sqrt{m(x+b)}}{c}\right) J_{l/2}\left(\frac{4\pi\sqrt{nx}}{c}\right)\, dx\right|$$

$$+ \left|\int_0^\infty \frac{c}{(nX)^{1/2}}\left(\phi(x)\left(\frac{x}{X}\right)^{-1/2}\right) J_{k-1}\left(\frac{4\pi\sqrt{m(x+b)}}{c}\right) J_{l/2}\left(\frac{4\pi\sqrt{nx}}{c}\right)\, dx\right|$$

We now note that the functions $f(x) = x/P\, \phi'(x)\, (x/X)^{-1/2}$, $x/(x+b)\, \phi(x)\, (x/X)^{-1/2}$, $x/(x+b)\, \phi(x)$, and $\phi(x)\, (x/X)^{-1/2}$ all share the same properties as $\phi(x)$, viz. , $f(x)$ is a smooth function with support in $[X/2, 5X/2]$ satisfying the bound $f^{(p)}(x) \ll (P/X)^p$. Upon applying this bound directly to each of the right-hand side integrals of (2.32) together with the standard bound for the Bessel functions given by $J_{k-1}(z) \ll \min\left(|z|^{k-1}, |z|^{-1/2}\right)$– keeping in mind all the while that we do not wish to lose our dependency on n, c, and P – we obtain the following upper bound for the integral constituting the left-hand side of (2.32):

$$\int_0^\infty \phi(x) x^{(l/2-1)/2} J_{k-1}\left(\frac{4\pi\sqrt{m(x+b)}}{c}\right) J_{l/2-1}\left(\frac{4\pi\sqrt{nx}}{c}\right)\, dx$$
$$\ll \left(\frac{cP}{(nX)^{1/2}} + \frac{1}{n^{1/2}}\right) X^{(l/2+1)/2} \min\left(\left(\frac{\sqrt{X}}{c}\right)^{k-1}, \left(\frac{\sqrt{X}}{c}\right)^{-1/2}\right) \min\left(1, \left(\frac{\sqrt{nX}}{c}\right)^{-1/2}\right),$$

which agrees with (2.29) when $p = 1$. Integrating by parts each of the integrals occurring on the right-hand side of (2.32) $p - 1$ more times and repeating the analysis seen above will yield (2.29).

The bound (2.29) tells us that for a fixed $\epsilon > 0$, the contribution of the integral towards the sum in $\sum_2$ is negligible unless $cP \gg (Xn)^{1/2} X^{-\epsilon}$. Consequently, when $n \gg X^{4\epsilon}$, the contribution of the integral in $\sum_2$ is negligible unless $cP \gg X^{1/2+\epsilon}$. Using (2.28) in conjunction with (2.29), we now proceed to estimate the contribution of those terms in $\sum_2$ with $n \ll X^{4\epsilon}$, i.e.,

$$\sum_{c \geq 1, 4N|c} c^{-1} \sum_{a \mod c}^{*} e\left(\frac{m\bar{a} + ba}{c}\right) \frac{2\pi}{c} i^{l/2} \left(\frac{|A|}{a}\right) \left(\bar{\epsilon}_a \left(\frac{-c}{a}\right)\right)^{-l} \sum_{n \ll X^{4\epsilon}} r_Q(n) e(-dn/c) n^{\frac{1-l/2}{2}}$$
$$\times \int_0^\infty \phi(x) x^{(l/2-1)/2} J_{l/2-1}\left(\frac{4\pi\sqrt{nx}}{c}\right) J_{k-1}\left(\frac{4\pi\sqrt{m(x+b)}}{c}\right)\, dx \tag{2.33}$$

$$\ll \sum_{c\geq 1} c^{-3/2} d(c) \sum_{n \ll X^{4\epsilon}} r_Q(n) n^{\frac{1-l/2}{2}} X^{\frac{l/2+1}{2}} \min\left(\left(\frac{\sqrt{X}}{c}\right)^{k-1}, \left(\frac{\sqrt{nX}}{c}\right)^{-1/2}\right)$$
$$\times \min\left(1, \left(\frac{\sqrt{nX}}{c}\right)^{-1/2}\right)$$
$$\ll X^{\frac{l/2+1}{2}+\epsilon} \sum_{c\geq 1} c^{-3/2} d(c) \min\left(\left(\frac{\sqrt{X}}{c}\right)^{k-1}, \left(\frac{\sqrt{X}}{c}\right)^{-1/2}\right)$$
$$\ll X^{\frac{l/2+1}{2}+\epsilon} \sum_{c\leq \sqrt{X}} c^{-3/2} d(c) \left(\frac{\sqrt{X}}{c}\right)^{-1/2} + X^{\frac{l/2+1}{2}+\epsilon} \sum_{c\geq \sqrt{X}} c^{-3/2} d(c) \left(\frac{\sqrt{X}}{c}\right)^{k-1}$$
$$\ll X^{\frac{l/2+1}{2}-\frac{1}{4}+\epsilon} \sum_{c\leq \sqrt{X}} c^{-1} d(c) + X^{\frac{l/2+1}{2}+\frac{k-1}{2}+\epsilon} \sum_{c\geq \sqrt{X}} c^{-1/2-k} d(c)$$
$$\ll X^{\frac{l/2+1}{2}-\frac{1}{4}+\epsilon}$$

Dealing with the tail-end of the double sum, i.e., when $cP \gg X^{1/2+\epsilon}$ and $n \gg X^{4\epsilon}$ proceeds in more or less the same manner. Here, the difference is that, in order to force convergence in n (which constitutes the outer sum of the double sum), we will need to utilize the full expression from (2.29). Thus, the tail-end of the sum is going to be no bigger than

$$X^{\frac{l/2+1}{2}} \sum_{n \gg X^{4\epsilon}} r_Q(n) n^{\frac{1-l/2}{2}} \sum_{c\geq 1} c^{-3/2} d(c) \left(\left[Pc(Xn)^{-1/2}\right]^p + n^{-p/2}\right)$$
$$\times \min\left(\left(\frac{\sqrt{X}}{c}\right)^{k-1}, \left(\frac{\sqrt{nX}}{c}\right)^{-1/2}\right)$$
$$\ll X^{\frac{l/2+1}{2}+\epsilon} \sum_{n \gg X^{4\epsilon}} n^{\frac{l/2-1-p}{2}} \sum_{c\geq 1} c^{-3/2} d(c) \min\left(\left(\frac{\sqrt{X}}{c}\right)^{k-1}, \left(\frac{\sqrt{X}}{c}\right)^{-1/2}\right)$$
$$\ll X^{\frac{l/2+1}{2}-\frac{1}{4}+\epsilon} P^p \sum_{n \gg X^{4\epsilon}} n^{\frac{l/2-1-p}{2}}.$$

In order to guarantee the convergence of the sum in n, we choose p (where p represents the number of times integration by parts has been performed to the integrals in (2.32)) such that p is the smallest integer satisfying $p > 1 + l/2$. Upon combining all the bounds obtained in estimating Σ_2, we see that it is $O\left(X^{(l/2+1)/2-1/4+\epsilon}\right) P^g$, where g is as in Theorem 2.3.1. Merging together the bounds for Σ_1 and Σ_2 gives the estimate we stated in Theorem 2.3.1. This marks the end of its proof and thence, that of the chapter as well.

Chapter 3: An Approach using Spectral Theory

3.1 Introduction and motivation

In this chapter, our object of interest will be the sum

$$\sum_{m \leq X} r_Q(m)\bar{\tilde{a}}_g(m+n), \tag{3.1}$$

where n is a positive integer, $r_Q(n)$ is the number of ways of representing an integer n by a positive definite quadratic form $Q(x)$ of weight k, and $\tilde{a}_g(n)$ is the normalized Fourier coefficient of a holomorphic cusp form $G(z)$ also of weight k, i.e., $\tilde{a}_g(n) = a_g(n)/n^{(k-1)/2}$.

Over the last half-century or so, inspired by the work of A. Selberg in [35], a lot of work has been done in Number Theory building on the connections linking together the seemingly disparate fields of spectral theory and number theory. It is a very natural question to ask what in the world L-functions attached to automorphic forms have to do with the spectral theory of hyperbolic manifolds. Rather than write an expository account affirming this connection (which, we would like to point out, has been done already by J.L. Hafner in [14] and that too with aplomb), we hope to convince the reader of the relevance of spectral theory by illustrating its use in a specific example. The methods utilized in this chapter are based upon the works of J.L. Hafner in [13] and A. Good in [11]. Much as complex variables empowered mathematicians in the nineteenth century to make great inroads towards a theory of primes, spectral theory has allowed modern mathematicians to make significant headway in problems such as the Artin conjecture for primitive roots, the Brun-Titchmarsh theorem, etc.

In what follows, let $Q(x)$ be a positive definite quadratic form in $l \geq 2$ variables. For much of this chapter, we shall mainly concern ourselves with the case where $\Theta(z, Q)$ is an automorphic form for the full modular group $SL(2, \mathbb{Z})$ and is of integral weight $k = l/2$ (i.e., we are assuming that l is an even integer). Such a theta function $\Theta(z, Q)$ is guaranteed to exist if the symmetric matrix A giving rise to the quadratic form Q satisfies the following conditions:

(i) $|A| = 1$.

(ii) All the diagonal components of A and A^{-1} are even.

This follows from corollaries 4.9.5 and 4.9.6 from [30]. An example of such a matrix A, attributed to Minkowski, is given on the following page. We also assume, for now, that $G(z)$ is a holomorphic cusp form on $SL(2, \mathbb{Z})$ as well.

While these restrictions may seem rather binding, they allow us to focus on the crux of the argument used without getting sidetracked in the numerous little details that would have plagued us had we inadvisably jumped in at the deep end as far as the generality of our results are concerned. Towards the end of this chapter, we shall attempt at relaxing these restrictive conditions, taking care to point out any deviations from the methods of the full modular group case.

$$\begin{pmatrix} 2 & 1 & & & & & & \\ 1 & 2 & 1 & & & & & \\ & 1 & 4 & 3 & & & & \\ & & 3 & 4 & 5 & & & \\ & & & 5 & 20 & 3 & & \\ & & & & 3 & 12 & 1 & \\ & & & & & 1 & 4 & 1 \\ & & & & & & 1 & 2 \end{pmatrix}$$

Figure 3.1: An example of a symmetric unimodular matrix A generating a quadratic form $Q(\mathbf{x})$ whose Θ function is an automorphic form for the full modular group.

We shall now introduce some of the salient details regarding the application of spectral theory to number theory. Our exposition in this section will be a mere echo of related material from texts such as [10], [37] and [22] among others.

Let $\Gamma = SL(2, \mathbb{Z})$ and $X = \Gamma\backslash\mathbb{H}$. The hyperbolic metric $ds^2 = y^{-2}\left(dx^2 + dy^2\right)$ induces the volume element $dz = y^{-2}\, dx\, dy$. With these, X has finite volume with a corresponding fundamental domain of

$$\mathcal{F} := \{z \in \mathbb{H} |\, |z| \geq 1, |x| \leq 1/2\}.$$

X is non-compact, having a cusp which corresponds to the point $i\infty$ in the fundamental domain. The hyperbolic Laplacian

$$\Delta = -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$$

is a positive, self-adjoint, unbounded operator on $L^2(X)$. Under its action, $L^2(X)$ is a direct sum of closed, infinite-dimensional subspaces

$$L^2(X) = L^2_{\text{disc}} \oplus L^2_{\text{cont}},$$

such that, as the name suggests, Δ has its discrete spectrum on $L^2_{\text{disc}}(X)$ and purely continuous spectrum on $L^2_{\text{cont}}(X)$. We shall now proceed to briefly describe the spectra.

3.1.1 The discrete spectrum: Hecke-Maass forms

Before we proceed to define the discrete spectrum, we shall begin by explicitly defining the Hecke operators T_n, for $n \geq 1$, as follows:

$$T_n(f(z) = \frac{1}{\sqrt{n}} \sum_{\substack{n=ad \\ a,d\geq 1}} \sum_{b \pmod{d}} f\left(\frac{az+b}{d}\right).$$

To these T_n's, we add the following "symmetry" operator:

$$T_{-1}f(z) = \overline{f(-\bar{z})},$$

and finally, let $T_\infty = \Delta$. Then, $\{T_n\}_{-1\leq n\leq\infty}$ is a commuting family of self-adjoint operators, each of which preserves the subspaces L^2_{disc} and L^2_{cont}.

Returning to the spectral decomposition of Δ on $L^2(X)$, the discrete part of the spectrum is characterized by an orthonormal basis $\{e_j(z)\}$ which consists of the constant function $e_0(z) = \sqrt{\pi/3}$ and the L^2 - normalized, joint eigenfunctions of all T_n:

$$\left.\begin{aligned} \Delta e_j(z) &= \lambda_j e_j(z),\ 0 = \lambda_0 < \lambda_1 \leq \cdots, \quad \lambda_j \to \infty \\ T_n e_j(z) &= \lambda_j(m) e_j(z), \quad n \geq 1 \\ T_{-1} e_j(z) &= \epsilon_j e_j(z), \quad \epsilon_j = \pm 1. \\ \int_0^1 e_j(x+iy)\,dx &= 0, \quad y > 0.\ \text{(Cuspidality condition)} \end{aligned}\right\} \tag{3.2}$$

Following the usual notation, we write

$$\lambda_j = s_j(1-s_j) = \frac{1}{4} + r_j^2, \text{ whereupon we have that } s_j = \frac{1}{2} + ir_j.$$

The eigenfunctions $\{e_j(z)\}_{0\leq n\leq\infty}$ are called the Hecke-Maass cusp forms. The ones with $\epsilon_j = 1$ are called *even*, while those with $\epsilon_j = -1$ are called *odd*. A theorem of Hecke identifies the Hecke eigenvalues with the coefficients of the Fourier expansion, giving us the following Fourier series expansion for the Hecke-Maass cusp forms

$$e_j(z) = \sum_{n\neq 0} \rho_j(n)\sqrt{y}\, K_{ir_j}(2\pi|n|y)e(nx), \tag{3.3}$$

where K_{ir_j} is the modified Bessel function of the second kind, $\rho_j(1)$ is a normalizing factor ensuring $\|e_j\| = 1$ and $\rho_j(-n) = \epsilon_j\rho_j(n)$.

Before we move on to the continuous spectrum, we would like to point out that the normalizing factor $\rho_j(1)$ is related to the symmetric square L-function:

$$\alpha_j := \frac{|\rho_j(1)|^2}{\cosh(\pi r_j)} = \frac{2}{L(1, \mathrm{sym}\ ^2 e_j)}.$$

This relation will be used implicitly in getting an estimate for $\rho_j(n)$ later in this chapter.

3.1.2 The continuous spectrum: Eisenstein series

The continuous part of the spectrum is characterized by the set of Eisenstein series

$$E_\eta(z, s) = \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} (\Im(\gamma z))^s, \quad \mathbb{R}s \geq 0,$$

where η is an index that runs through the set of cusps in X. Since there is only one cusp in X, we will refer to the sole member of the set of Eisenstein series as $E_\infty(z, s)$. This series is absolutely convergent in the right half-plane where it defines an automorphic function in $z \in X$ such that $\Delta E(z, s) = s(1 - s)E(z, s)$. Put succinctly (and rather vaguely), the Eisenstein series $E(z, 1/2 + it)$ spans the space L^2_{cont} as a direct integral. We will see what this means in a more concrete way shortly. The key to the spectral decomposition in this subspace lies in the meromorphic continuation of the Eisenstein series (in the s-variable) to the entire complex plane, which was first proved by Selberg.

Over the full modular group, the Eisenstein series has the following Fourier series expansion

$$E_\infty(z,s) = y^s + \phi(s)y^{1-s} + \frac{2}{\xi(2s)}\sum_{n\neq 0}\tau_{s-1/2}(|n|)\sqrt{y}\,K_{s-1/2}(2\pi|n|y)e(nx), \qquad (3.4)$$

where $\tau_\alpha(n) = \sum_{n=d_1d_2} d_1^\alpha d_2^{-\alpha}$ is the generalized divisor sum, $\xi(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)$ is the completed zeta-function and $\phi(s) = \xi(2s-1)/\xi(2s)$.

3.1.3 Non-holomorphic Poincaré Series

Before we proceed to reap the fruits of the automorphic Laplacian's spectral decomposition on X, we first define a collection of non-holomorphic Poincaré series first introduced by A. Selberg. For $n \geq 1$, and η representing one of the inequivalent cusps of X, we define

$$\widetilde{P}_\eta(z,s,n) := \frac{(\pi n)^{s-1/2}}{\Gamma(s+1/2)}\sum_{\gamma\in\Gamma_\eta\backslash\Gamma}(\Im(\gamma z))^s\, e(n\gamma z)$$

which satisfies

$$P_\eta(\gamma z,s,n) = \widetilde{P}_\eta(z,s,n), \quad \text{where } \gamma\in\Gamma.$$

Since there is only inequivalent cusp in X (i.e., the one at $i\infty$), we drop the index η and henceforth, refer to the pertinent member(s) of this collection of Poincaré series as $\widetilde{P}(z,s,n)$ for $n \geq 1$. Even though $\widetilde{P}(z,s,n)$ fails to be an eigenfunction of Δ, by virtue of its automorphic nature, we can deduce its spectral decomposition in $L^2(X)$:

$$\widetilde{P}(z,s,n) = \sum_{j>0}\left(\frac{\overline{\rho_j(n)}\Gamma(s-1/2-ir_j)\Gamma(s-1/2+ir_j)}{\Gamma(2s)}\right)e_j(z)$$
$$+2\sqrt{\pi}\int_{-\infty}^{\infty}\left(\frac{\tau_{-ir}(|n|)\Gamma(s-1/2-ir)\Gamma(s-1/2+ir)}{\xi(1-2ir)\Gamma(2s)}\right)E_\infty(z,1/2+ir)\,dr \quad (3.5)$$

As the reader will see shortly, the spectral decomposition of $\widetilde{P}(z,s,n)$ as displayed above will play a central role in the proof of Theorem 3.2.1.

3.2 Statement of results

We begin by introducing the series which plays

$$D_{\Theta,G}(s,n) = \sum_{m=1}^{\infty}\frac{r_Q(m)\bar{a}_g(n+m)}{(m+n)^{s+k-1}}, \quad \sigma > \frac{k+1}{2}, n \geq 1.$$

Recall that $r_Q(n) \ll n^{k-1}$, while $a_g(n) \ll n^{(k-1)/2+\epsilon}$ (Deligne's bound).

Furthermore, we will also be using the notation

$$\langle X, Y\rangle_k = \iint\limits_{\mathcal{F}} y^{k-2}X(z)\bar{Y}(z)\,dx\,dy$$

whenever this integral converges absolutely.

We shall prove the following theorem about $D_{\Theta,G}(s,n)$.

Theorem 3.2.1. *Put*

$$Q(s,n) = \frac{(4\pi)^{s+k-1}\Gamma(s+1/2)}{\Gamma(s+k-1)(\pi n)^{s-1/2}}.$$

Then $D_{\Theta,G}$ has a meromorphic continuation to the region $\sigma \geq k/2$ in the form

$$D_{\Theta,G}(s,n) = \frac{Q(s,n)}{\Gamma(2s)}\Bigg(\sum_{j>0}\rho_j(n)\Gamma(s-1/2+ir_j)\Gamma(s-1/2-ir_j)\left\langle\Theta, G\bar{e}_j\right\rangle_k$$
$$+2\sqrt{\pi}\int\limits_{-\infty}^{\infty}\frac{\tau_{-ir}(|n|)}{\xi(1-2ir)}\Gamma(s-1/2-ir)\Gamma(s-1/2+ir)$$

$$\times \left\langle \Theta, G\overline{E}_\infty(\cdot, 1/2 + ir)\right\rangle_k \, dr \Bigg), \tag{3.6}$$

where all the series and integrals are absolutely convergent in the given region of the s*-plane.*

Extending the region over which the sum $D_{\Theta,G}$ is defined via meromorphic continuation allows us to obtain the following upper bound for the sum in (3.1) by an inverse Mellin transform-type argument. Theorem 3.2.1 also explicitly exhibits the location of the singularities of $D_{\Theta,G}$ via a Petersson inner product by identifying them with the location of the poles of the Poincaré series.

Theorem 3.2.2. *Let* δ *is any positive number and* n *be as defined above in the definition of* $D_{\Theta,G}(s,n)$*. Let* $\tilde{a}_g(n)$ *represent the normalized* n*-th Fourier coefficient of the holomorphic cusp form* $G(z)$*. Then we have*

$$\sum_{m \le X} r_Q(m)\bar{\tilde{a}}_g(m+n) \ll X^{k-1/2+\delta}.$$

3.3 Application of the spectral method

The spectral approach can be traced back to the discovery of Rankin and Selberg, that for a holomorphic cusp form

$$\phi(z) = \sum_{n=1}^{\infty} a_\phi(n) e(nz)$$

of weight k, level N, and arbitrary nebentypus, there is an integral representation

$$\sum_{n=1}^{\infty} \frac{|a_\phi(n)|^2}{n^{s+k-1}} = \frac{(4\pi)^{s+k-1}}{\Gamma(s+k-1)} \int_{\Gamma\backslash\mathbb{H}} y^{k-2} \, |\phi(z)|^2 \, E(z,s) \, dx \, dy, \tag{3.7}$$

where $\Gamma_0(N)\backslash\mathbb{H}$ is a fundamental domain for the action of the Hecke congruence subgroup $\Gamma_0(N)$ on the upper half-plane $\mathbb{H}$, and $E(z,s)$ denotes the Eisenstein series introduced in the previous section. The identity above can be proved by an 'unfolding' technique, one which we ourselves shall recourse to as shown below.

To prove Theorem 3.2.1, we compute the Petersson inner product $\langle\Theta(z,Q), G\overline{\overline{P}}(z,s,n)\rangle_k$ in two different ways. On the one hand, by the same 'unfolding' technique mentioned in the previous paragraph, we obtain a similar integral representation for the series $D_{\Theta,G}$ as in (3.7) valid for $\sigma > (k+1)/2$,

$$\begin{aligned}
&\langle\Theta(z,Q), G\overline{\overline{P}}(z,s,n)\rangle_k \\
&\qquad = \frac{(\pi n)^{s-1/2}}{\Gamma(s+1/2)} \sum_{M\in\Gamma_\infty\backslash\Gamma} \iint_{\Gamma\backslash\mathbb{H}} y^{k-2}\,\Theta(z,Q)\,\overline{G}(z)\,(\Im(Mz))^s\, e(nMz)\,dx\,dy \\
&\qquad = \frac{(\pi n)^{s-1/2}}{\Gamma(s+1/2)} \iint\limits_{\Gamma_\infty\backslash\mathbb{H}} y^{s+k-2}\,\Theta(z,Q)\,\overline{G}(z) e(nz)\,dx\,dy \\
&\qquad = \frac{(\pi n)^{s-1/2}}{\Gamma(s+1/2)} \int_0^\infty y^{s+k-2} e^{-2\pi n y} \int_0^1 \left(\sum_{m=1}^\infty r_Q(m) e^{-2\pi m y} e(mx)\right) \\
&\qquad\qquad\qquad\qquad\qquad \times \left(\sum_{d=1}^\infty \bar{a}_g(d) e^{2\pi d y} e(-dx)\right) e(nx)\,dx\,dy \\
&\qquad = \frac{(\pi n)^{s-1/2}}{\Gamma(s+1/2)} \sum_{m=1}^\infty r_Q(m) \sum_{d=1}^\infty \bar{a}_g(d) \int_0^\infty y^{s+k-2} e^{-2\pi(n+m+d)y}\,dy \int_0^1 e^{2\pi i(m+n-d)x}\,dx \\
&\qquad = \frac{(\pi n)^{s-1/2}}{\Gamma(s+1/2)} \sum_{m=1}^\infty r_Q(m)\bar{a}_g(m+n) \int_0^\infty y^{s+k-2} e^{-4\pi(n+m)y}\,dy \\
&\qquad = \frac{1}{Q(s,n)}\, D_{\Theta,G}(s,n).
\end{aligned} \tag{3.8}$$

Here, we used the Γ-invariance of $y^k\Theta(z,Q)\,\overline{G}(z)$ in order to get the second line in the series of equalities above.

On the other hand, the same inner product $\langle \Theta(z,Q), G\overline{\widetilde{P}}(z,s,n)\rangle$ can be decomposed according to the spectrum of $L^2(X)$ by using (3.5), giving us

$$\begin{aligned}\langle \Theta(z,Q), G\overline{\widetilde{P}}(z,s,n)\rangle = \frac{1}{\Gamma(2s)}\Bigg(&\sum_{j>0} \rho_j(n)\Gamma(s-1/2+ir_j)\Gamma(s-1/2-ir_j)\,\langle \Theta, G\bar{e}_j\rangle_k \\ &+2\sqrt{\pi}\int_{-\infty}^{\infty} \frac{\tau_{-ir}(|n|)}{\xi(1-2ir)}\Gamma(s-1/2-ir)\Gamma(s-1/2+ir) \\ &\times \left\langle \Theta, G\overline{E}_\infty(\cdot,1/2+ir)\right\rangle_k \, dr\Bigg), \end{aligned} \tag{3.9}$$

Upon equating the right-hand sides of (3.8) and (3.9), we get the expression for $D_{\Theta,G}(s,n)$ given in Theorem 3.2.1, thus completing its proof.

In order to prove Theorem 3.2.2. we introduce the following nonnegative C^∞-function (see [11, p. 546])on the real line such that

$$\phi_P(z) = \begin{cases} 1, & \text{if } 0 \le z \le 1-1/P. \\ \le 1, & \text{if } 1-1/P \le z \le 1+1/P. \\ 0, & \text{if } z \ge 1+1/P. \end{cases} \tag{3.10}$$

.

Then the Mellin transform of $\phi_P(z)$, which is defined as follows

$$K_P(s) = \int_0^\infty \phi_P(z) z^{s-1}\, dz,$$

is analytic in $\sigma > 0$. On a related (and pertinent) note, $K_P(s)$ satisfies the following bound which can be proved by integrating by parts

$$K_P(s) \ll \frac{P^C}{(1+|t|)^{C+1}}, \forall C>0, \text{in } 1 \le \sigma \le k+1 \text{ and } P \ge 2. \tag{3.11}$$

We then normalize the Fourier coefficient of the cusp form $G(z)$ within the sum $D_{\Theta,G}(s,n)$ by setting $\bar{a}_g(n) = a_g(n)/n^{(k-1)/2}$ so that $\bar{a}_g(n) \ll n^\epsilon$. Finally, via the inverse Mellin transform, we have for $c > k$,

$$\sum_{m=1}^{\infty} r_Q(m)\bar{a}_g(m+n)\phi_P\left(\frac{n}{X}\right) = \frac{1}{2\pi i}\int_{(c)} D_{\Theta,G}(s-\left(\frac{k-1}{2}\right), n)K_P(s)X^s\, ds.$$

We shift the path of integration to $c = k - 1/2 + \delta$, where δ is any positive number, so that the argument of $D_{\Theta,G}$ within the integral becomes $k - 1/2 + \delta - (k-1)/2 = k/2 + \delta$. So, what we are doing, in essence, is that we are pushing the path of integration all the way to left up to the line $\Re(s) = k/2 + \delta$, which acts as a natural boundary owing to the presence of poles on the line $\Re(s) = k/2$ from Theorem 3.2.1. Then, upon pulling out the factor of $X^{k-1/2+\delta}$ from the integral, absorbing the rest of the convergent integral into the implied constant, and realizing that there are functions satisfying (3.10) such that the first sum in the inverse Mellin transform is majorized by

$$\sum_{m \leq X} r_Q(m) \bar{\tilde{a}}_g(m+n)$$

by manipulating the value of the parameter P, we obtain the bound in Theorem 3.2.2, barring one small detail.

In order to shift the path of integration of the inverse Mellin Transform, we will be integrating along a rectangle of some arbitrary height T and width $1/2 - \delta$. Performing the calculation above would entail verifying that integral of $D_{\Theta,G}(s - (k-1)/2, n) K_P(s) X^s$ along the horizontal components of the rectangle goes to zero as the height T of the rectangle tends to infinity. Since, from the previous paragraph, we have that $K_P(s) \ll (1 + |t|)^{-(C+1)}$, for any $C > 0$, we need to make sure that $D_{\Theta,G}(s, n)$ satisfies a polynomial estimate in $|t|$ in order to complete the proof of Theorem 3.2.2. This is what motivates the following calculations that make up the rest of the section.

Since, as we stated at the beginning of the chapter, we have assumed that $\Theta(z, Q)$ is a weight k automorphic form for the full modular group $SL(2, \mathbb{Z})$, we can obtain a general decomposition of the Θ-function into

$$\Theta(z, Q) = E_k(z) + F(z, Q) \tag{3.12}$$

where $E_k(z)$ is the unique Eisenstein series whose Fourier expansion is

$$E_k(z) = 1 + \frac{(2\pi)^k}{\zeta(k)\Gamma(k)} \sum_{1}^{\infty} \sigma_{k-1}(n) e(nz),$$

and

$$F(z, Q) = \sum_{1}^{\infty} a(n, Q) e(nz)$$

is a cusp form. Then, by the additivity of the inner product, each of the $\langle \Theta, \cdot \rangle$ terms in (3.9) splits into $\langle F(z, Q), \cdot \rangle + \langle E_k(z), \cdot \rangle$. So, after decomposing the inner products of the Θ-function, we rearrange the terms on the right-hand side of (3.9) to give us

$$\begin{aligned}
\sum_{l=1}^{\infty} \frac{r_Q(l) a_g(n+l)}{(l+n)^{s+k-1}} = {} & \frac{Q(s,n)}{\Gamma(2s)} \Bigg(\sum_{j>0} f(r_j, n, \rho, \lambda, \Gamma) \left\langle F(z, Q), G\bar{e}_j \right\rangle_k \\
& \qquad + 2\sqrt{\pi} \int_{-\infty}^{\infty} g(r, n, \tau, \xi, \Gamma) \left\langle F(z, Q), G\overline{E}_{\infty}(z, 1/2 + ir) \right\rangle_k \, dr \Bigg) \\
& + \frac{Q(s,n)}{\Gamma(2s)} \Bigg(\sum_{j>0} f(r_j, n, \rho, \lambda, \Gamma) \left\langle E_k(z), G\bar{e}_j \right\rangle_k \\
& \qquad + 2\sqrt{\pi} \int_{-\infty}^{\infty} g(r, n, \tau, \xi, \Gamma) \left\langle E_k(z), G\overline{E}_{\infty}(z, 1/2 + ir) \right\rangle_k \, dr \Bigg) \\
= {} & I_F(s, n) + II_{E_k}(s, n),
\end{aligned} \tag{3.13}$$

where $f(\rho, \lambda, \Gamma)$ and $g(\tau, \xi, \Gamma)$ represent the expressions in terms of Γ factors and/or any other constants appearing in front of the inner products in (3.9). Conveniently, estimates for the terms represented by I_F, i.e., the cuspidal part of the Θ-function decomposition appearing in the first two lines of (3.13), were already obtained by J.L.Hafner in [13] telling us that, for $s = \sigma + it$,

$$I_F(s, n) \ll n^{1/2+\epsilon} |t|^{1+\epsilon}. \tag{3.14}$$

Hafner's bounds are uniform in a variety of related parameters, but for our purposes, we are only interested in dependencies on the parameters n and s absorbing the rest into the implied constant.

This just leaves the Eisenstein series part, given by $II_{E_k}(s,n)$, of the Θ-function inner product decomposition in (3.13). We expect the estimates for $II_{E_k}(s,n)$ to be marginally worse than the estimates for the cuspidal part, since the Eisenstein series lack exponential decay at the cusps, a distinguishing feature of cusp forms. The main advantage of having the holomorphic Eisenstein series in the Petersson inner products $\langle E_k(z), G\overline{E}_\infty(z, 1/2+ir)\rangle_k$, and $\langle E_k(z), G\bar{e}_j\rangle_k$ is that its presence lends itself to the same 'unfolding' technique last seen in the proof of Theorem 3.2.1. In what follows, the analysis of the continuous spectrum can be carried out in the same way as the discrete part, so we will confine our calculations to the discrete part only. For a fixed value of j, upon 'unfolding' the inner product $\langle E_k(z), G\bar{e}_j\rangle_k$, we have

$$\begin{aligned}
&\langle E_k(z), G\bar{e}_j\rangle_k \\
&\qquad = \iint\limits_{\Gamma_\infty\backslash\mathbb{H}} y^{k-2}\,\overline{G}(z)e_j(z)\,dx\,dy \\
&\qquad = \int_0^\infty y^{k-2}\int_0^1 \left(\sum_{d=1}^\infty \bar{a}_g(d)e^{2\pi dy}e(-dx)\right) \\
&\qquad\qquad\qquad \times\left(\sum_{m=1}^\infty \rho_j(m)\sqrt{y}\,K_{ir_j}(2\pi my)e(mx)\right)\,dx\,dy \\
&\qquad = \sum_{m=1}^\infty \bar{a}_g(m)\rho_j(m)\int_0^\infty y^{k-2+1/2}e^{-2\pi my}\,K_{ir_j}(2\pi my)\,dy \\
&\qquad = \frac{1}{(2\pi)^{k-1/2}}\sum_{m=1}^\infty \frac{\bar{\bar{a}}_g(m)\rho_j(m)}{m^{k/2}}\int_0^\infty v^{k-3/2}e^{-v}\,K_{ir_j}(v)\,dv
\end{aligned}$$

$$= \left(\sum_{m=1}^{\infty} \frac{\bar{\tilde{a}}_g(m)\rho_j(m)}{m^{k/2}} \right) \frac{2^{1/2-k}\sqrt{\pi}\,\Gamma(k-1/2+ir_j)\Gamma(k-1/2-ir_j)}{(2\pi)^{k-1/2}\,\Gamma(k)} \tag{3.15}$$

In the sum

$$S(\rho_j, \tilde{a}_g) = \sum_{m=1}^{\infty} \frac{\bar{\tilde{a}}_g(m)\rho_j(m)}{m^{k/2}},$$

$\tilde{a}_g(m)$ refers to the normalized m-th Fourier coefficient of $G(z)$. From [34], we know that we can choose an orthonormal basis e_j of Hecke eigencuspforms coming from new forms and old forms such that

$$\rho_j(m) \ll_\epsilon (m)^\epsilon \cosh\left(\frac{\pi r_j}{2}\right) m^\theta \tag{3.16}$$

Here, θ refers to the Ramanujan bounds for $GL(2)$, which are the best possible bounds obtained in the pursuit of the generalized Ramanujan conjecture for number fields. We can choose $\theta = 7/64$, utilizing the state of the art bound obtained by Kim and Sarnak in 2003. For a large enough value of k, say $k > 2$, the sum $S(\rho_j, \tilde{a}_g)$ converges and consequently, will not play a role in the estimate for the inner product given below

$$\langle E_k(z), G\bar{e}_j \rangle_k \ll e^{-\pi r_j/2} |r_j|^{2k-2}. \tag{3.17}$$

Here, we have used (3.16) in conjunction with Stirling's bound for the Γ function

$$\Gamma(s) \asymp e^{-\pi|t|/2} |t|^{\sigma-1/2}.$$

Moreover, since we are uninterested at the moment in obtaining uniform estimates in terms of all the remaining parameters that make up the right-hand side of (3.15), we have absorbed all such extraneous factors (such $\Gamma(k)$, etc.) into the implied constant.

In order to estimate the contribution of the discrete part of the spectrum towards the size of $II_{E_k}(s,n)$, we split the sum over j into two parts as shown below

$$\sum_{j>0} \rho_j(n)\Gamma(s-1/2+ir_j)\Gamma(s-1/2-ir_j)\langle E_k(z), G\bar{e}_j\rangle_k = \sum_{\substack{j>0,\\ |r_j|\leq|t|}} + \sum_{\substack{j>0,\\ |r_j|>|t|}} .$$

Estimating the first sum using the bounds from (3.17), (3.16), and Stirling's bound rather liberally, we get

$$\begin{aligned}\sum_{\substack{j>0,\\ |r_j|\leq|t|}} \rho_j(n)\Gamma(s-1/2+ir_j)\Gamma(s-1/2-ir_j)\langle E_k(z), G\bar{e}_j\rangle_k &\ll \sum_{\substack{j>0,\\ |r_j|\leq|t|}} |r_j|^{2k-2}\, e^{-\pi|t|}\, |t|^{2\sigma-2} \\ &\ll e^{-\pi|t|}\, |t|^{2\sigma+2k-3}.\end{aligned}$$

In a similar way, we can bound the remaining sum to get

$$\sum_{\substack{j>0,\\ |r_j|>|t|}} \rho_j(n)\Gamma(s-1/2+ir_j)\Gamma(s-1/2-ir_j)\langle E_k(z), G\bar{e}_j\rangle_k \ll e^{-\pi|t|}\, |t|^{2\sigma+2k-4}.$$

Combining these bounds together with the following estimate for $Q(s,n)/\Gamma(2s)$,

$$\frac{Q(s,n)}{\Gamma(2s)} \ll e^{\pi|t|}|t|^{2-k-2\sigma}n^{1/2-\sigma},$$

and implicitly using the fact that the continuous part of the spectrum returns the same estimates for the error term as does the discrete part, we get the following bound for $II_{E_k}(s,n)$:

$$II_{E_k}(s,n) \ll |t|^{k-1+\epsilon}n^{1/2-\sigma}. \tag{3.18}$$

Hence, upon combining the estimates from (3.18) and (3.14), we see that $D_{\Theta,G}(s,n)$ satisfies an upper bound in terms of $|t|^{k-1+\epsilon}$, allowing us to proceed with the proof of Theorem 3.2.2.

3.4 Extension of the spectral method to $\Gamma_0(4N)$

Having dealt with the case where both $G(z)$ and Θ lie in the full modular group, we can now proceed to the 'congruence group of level $4N$' case with exactly the same method as before. We assume $\Theta(z, Q)$ is an automorphic form for $\Gamma_0(4N)$ of integral weight $k = l/2$, and that $G(z)$ is a cusp form of weight k for $\Gamma_0(N)$. Even though G and Θ may appear to live in different levels, since $S_k(\Gamma_0(N)) \subset S_k(\Gamma_0(4N))$, we can perform any pertinent inner product calculations by considering $G(z)$ as a cusp form in $\Gamma_0(4N)$. As before, we are interested in the sum

$$\sum_{m \leq X} r_Q(m)\bar{\tilde{a}}_g(m+n),$$

where b is a positive integer. To this end, we consider the sum $D_{\Theta,G}(s, n)$. The meromorphic continuation to its critical strip (via the spectral decomposition of the non-homolorphic Poincaré series) is our method of choice once again. The only deviation from the previous section is that we have to deal with additional cusps in this scenario as opposed to the one cusp in the full modular group case. However, as we will see, they make a negligible difference to the final estimate of the sum we are interested in.

In the general decomposition of the Θ-function for $\Gamma_0(4N)$, we now have

$$\Theta(z, Q) = \sum_{\mathfrak{a} \text{ singular}} \phi_{\mathfrak{a}}(Q)E_{\mathfrak{a},k}(z) + F(z, Q) \tag{3.19}$$

where $E_{\mathfrak{a},k}(z)$ is the holomorphic Eisenstein series associated to each cusp $\mathfrak{a}$ and

$$F(z, Q) = \sum_{1}^{\infty} a(n, Q)e(nz)$$

is a cusp form. As in the previous section, we use this decomposition in order to get polynomial estimates in $|t|$ for the discrete and continuous parts of the spectrum.

The constant $\phi_{\mathfrak{a}}(Q)$, which is just value of the $\Theta(z, Q)$ at the cusp $\mathfrak{a}$, will not play a role in the $|t|$-estimate and can be absorbed into the implied constant.

Before we proceed with the 'unfolding' done in the previous section, we will write down the Fourier expansion of E_η at a cusp κ

$$E_\eta(z,s) = \delta_{\eta\kappa} y^s + \phi_{\eta\kappa}(s) y^{1-s} + \sum_{m\neq 0} \alpha_{m\eta\kappa}(s)\sqrt{y}\, K_{s-1/2}(2\pi|m|y)e(mx).$$

Generally speaking, the Fourier coefficients of these Eisenstein series are mysterious quantities, but in the context of congruence groups, they have been pretty well studied with the following explicit formulae (see [13])

$$\phi_{\eta\kappa}(s) = \frac{\pi^{1/2}\,\Gamma(s-1/2)}{\Gamma(s)}\, L^0_{\eta\kappa}(2s),$$
$$\alpha_{m\eta\kappa}(s) = \frac{2\pi^s\,|m|^{s-1/2}}{\Gamma(s)}\, L^m_{\eta\kappa}(2s),$$

and

$$L^m_{\eta\kappa}(2s) = \sum_{\substack{c,d \\ 0\le d<c}}{}^{*} e(md/c)c^{-s}, \tag{3.20}$$

where $\sum^*$ ranges over those pairs not necessarily integral. From [13, p. 14], we also know that $L^m_{\eta\kappa}$ satisfies the following upper bound in terms of the parameters t and N

$$L^m_{\eta\kappa}(1+it) \ll (4mN|t|+1)^{\epsilon}. \tag{3.21}$$

This upper bound in conjunction with Stirling's bound gives us the following estimate for the size of the Fourier coefficients $\alpha_{n\kappa\infty}$ on the critical line

$$\alpha_{n\kappa\infty}(1/2-ir) \ll e^{\pi r/2}(8nN|r|+1)^{\epsilon}.$$

.

Let Λ represent the set of inequivalent cusps in $\Gamma_0(4N)$. Since we performed the t-estimate in detail for the discrete part of $II_{E_k}(s,n)$, we shall now calculate the r-estimate for the continuous part of the spectrum. This involves estimating the sum of inner products given below

$$\sum_{\mathfrak{a}\in\Lambda}\sum_{\kappa\in\Lambda}\alpha_{n\kappa\infty}(1/2-ir)\left\langle E_{\mathfrak{a},k}(z), G\overline{E}_{\kappa}(z,1/2+ir)\right\rangle_k. \tag{3.22}$$

Upon unfolding the integral in the inner product above, we get the expression

$$\sum_{\mathfrak{a}\in\Lambda}\sum_{\kappa\in\Lambda}\frac{\alpha_{n\kappa\infty}(1/2-ir)}{(2\pi)^{k-1/2}}\left(\sum_{m=1}^{\infty}\frac{\bar{\tilde{a}}_g(m)\,\alpha_{m\kappa\infty}(1/2+ir)}{m^{k/2}}\right)$$
$$\times\frac{\sqrt{\pi}\,2^{1/2-k}\,\Gamma(k-1/2+ir)\Gamma(k-1/2-ir)}{\Gamma(k)}$$

Using the bounds on $\alpha_{n\kappa\infty}(1/2-ir)$ established earlier, the expression in (3.22) is estimated to be of size $O\left(|r|^{\epsilon}(8n\,N|r|+1)^{\epsilon}|r|^{2k-2}\right)$, which is exactly the kind of polynomial-type dependency we need in terms of r. The rest of the proof follows the framework laid down in the previous section.

Let $b=\max_j|\Re\, ir_j|$, where $\lambda_j=1/4+r_j^2$ is the eigenvalue of the e_j's comprising the discrete part of the spectrum of the hyperbolic Laplacian. Selberg's eigenvalue conjecture is equivalent to the statement that $b=0$. The locations of these exceptional eigenvalues act as a natural barrier preventing us from shifting the line of integration in the inverse Mellin transform as close to $k-1/2$ as we did in the full modular group case. Consequently, to conclude this chapter, we have the following theorem.

Theorem 3.4.1. *Let* $\delta=b$ *if* $b\neq 0$*, else let* δ *be any positive number. Let* n *be as in the definition of* $D_{\Theta,G}(s,n)$*. Let* $\tilde{a}_g(n)$ *represent the normalized* n*-th*

Fourier coefficient of the holomorphic cusp form $G(z)$. *Then we have*

$$\sum_{m \leq X} r_Q(m)\bar{\tilde{a}}_g(m+n) \ll X^{k-1/2+\delta}.$$

Chapter 4: An Approach using the δ-symbol Method

4.1 Introduction and motivation

In Chapter II, we studied the shifted convolution sum

$$D_f(\Theta, b) = \sum_{n \geq 1} a_f(n+b) r_Q(n) \phi(n),$$

using a methodology that was similar in spirit to the one used to study shifted convolution sums in [5]. To put it succinctly, this involved the combination of an 'elementary' Poincaré series approach together with a Voronoï-type summation formula, finally followed by an application of Weil's estimate for Kloosterman sums.

In this chapter, we use an analogous approach to study the sum

$$\sum_{n \geq 1} r_Q(n) r_Q(n+b) f(n), \tag{4.1}$$

where $r_Q(n)$ is the number of ways of representing an integer n by a positive definite quadratic form $Q(\mathbf{x})$, and $f(x)$ is a suitable nice weight function on $(0, \infty)$. Our main source of inspiration in this case will be the following paper by H. Iwaniec et al. [8]. The main ingredients used to obtain the results of this chapter will be the δ-symbol method, together with a Voronoï-type summation formula once again.

In order to motivate the study of the sum in (4.1), we use a familiar object $D(\tau, b)$, last seen in Chapter II, defined in the following manner:

$$D(\tau, b) = \sum_{n \leq X} \tau(n)\tau(n+b). \tag{4.2}$$

The behavior of $D(\tau, b)$ was studied as the classical additive divisor problem. To reiterate the basis of the analogy used in Chapter II, $\tau(n)$ is the n-th Fourier expansion of the modular form $\frac{\partial}{\partial s}E(z,s)|_{s=1/2}$ where $E(z,s)$ is the Eisenstein series for $SL_2(\mathbb{Z})$, while $r_Q(n)$ is the n-th Fourier coefficient of the Θ-function associated to the quadratic form $Q(x)$. Thus, $D(\tau, b)$ and $D(\Theta, b)$ are variants of each other in the sense that the shifted convolution sums in (4.1) and (4.2) only differ by involving the Fourier coefficients of two rather different modular forms in their respective definitions. A. E. Ingham first gave an asymptotic formula for (4.2) in 1927 and then, in 1931, T. Estermann estalished the asymptotic expansion

$$D(\tau, b) = XP_b(\log X) + O\left(X^{1-1/12}\log^3 X\right), \tag{4.3}$$

where $P_b(T)$ is a quadratic polynomial with leading coefficient $6\pi^{-2}\sigma_{-1}(h)$. A crucial element in Estermann's proof of (4.3) is an estimate for Kloosterman sums. Further study by mathematicians such as Kuznetsov, Y. Motohashi, and Jutila, among others, made use of techniques from the more sophisticated spectral theory of automorphic forms leading to substantial improvements on the error term of Estermann. Other examples of results using such techniques (taken from [2]) which deal with the Ramanujan τ-function (not to be confused with the divisor function) include

$$\sum_{n \geq 1} e^{-n/X}\tau(n)\tau(n+1) = O\left(X^{12-1/2+\epsilon}\right) \quad \text{by D. Goldfeld, and}$$

$$\sum_{1\le n\le X} \tau(n)\tau(n+1) = O\left(X^{12-1/3+\epsilon}\right) \quad \text{by A Good.}$$

At this point, we would like to mention that the bounds that we are seeking for (4.1) are far more exploratory in nature, and as a result, we are not interested in optimizing the resulting error term to the fullest extent possible. Moreover, in order to simplify the exposition of the underlying calculations that make up the rest of this chapter, we assume that $b = 1$.

4.2 Statement of results

Before we state the main theorem that is the focus of this chapter, we shall proceed to rewrite the sum in question as

$$D(\Theta, f) = \sum_{m \mp n = 1} r_Q(m) r_Q(n) f(m, n), \tag{4.4}$$

where $f(x, y)$ is a nice smooth weight function on $\mathbb{R}^+ \times \mathbb{R}^+$. In what follows, we will only present the case $m - n = 1$, which corresponds to (4.1). The remaining case $m + n = 1$ can be obtained by merely changing signs where appropriate in our resulting calculations. For the applications we have in mind, we also place a small restriction on $f(x)$ and its partial derivatives in that the following estimate must be satisfied

$$x^i y^j f^{(i,j)}(x, y) \ll \left(1 + \frac{x}{X}\right)^{-1} \left(1 + \frac{y}{Y}\right)^{-1} P^{i+j} \tag{4.5}$$

with some $P, X, Y \ge 1$ for all $i, j \ge 0$. As noted in [8], this condition allows for $f(x, y)$ to oscillate mildly.

In addition, let us also recall, from chapter II, that $Q(\mathbf{x})$ is a positive definite quadratic form in $l \ge 2$ variables. In Siegel's notation $Q(\mathbf{x}) = \frac{1}{2}A[x]$, where $A = (a_{ij})$ is a symmetric, positive definite matrix of rank l. We assume that A has integral entries which are even on the diagonal. Then, $Q(\mathbf{x})$ has integral coefficients. Now, we define the theta function $\Theta(z, Q)$ associated to the quadratic form $Q(\mathbf{x})$ as follows

$$\Theta(z, Q) = \sum_{m \in \mathbb{Z}^l} e(Q(m)z) = \sum_{n=0}^{\infty} r(n, Q) e(nz), \tag{4.6}$$

where the representation numbers $r(n, Q)$ are the Fourier coefficients of $\Theta(z, Q)$.

For a positive integer N satisfying $NA^{-1} \in M_l(\mathbb{Z})$, we note (as in [30]) that $\Theta(z, Q)$ is an automorphic form for $\Gamma_0(4N)$ of weight $k = l/2$ and multiplier

$$\theta(\tau) = \left(\frac{|A|}{d}\right)\left(\bar{\epsilon}_d\left(\frac{c}{d}\right)\right)^l \text{ where } \tau = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(4N). \tag{4.7}$$

Here, $\left(\frac{\cdot}{\cdot}\right)$ denotes the Kronecker symbol, while N is a positive integer such that NA^{-1} is an integral matrix and might not be the minimal level. ϵ_d denotes the sign of the Gauss sum

$$\epsilon_d = \begin{cases} 1 & \text{if } d \equiv 1 \pmod 4 \\ i & \text{if } d \equiv 3 \pmod 4 \end{cases}$$

as usual. For our purposes, we let N equal $|A|$. Consequently, if A is a unimodular matrix, then $N = 1$.

Theorem 4.2.1. *Suppose f satisfies* (4.5) *and $\Theta(z, Q)$ is as in* (4.6) *of weight k. Then we have the following asymptotic expansion*

$$D(\Theta, f) = \int_0^{\infty} g(x, \mp x \pm 1)\, dx + O\left(P^{k+1/4}(X+Y)^{k-3/4}(XY)^{1/4+\epsilon}\right), \tag{4.8}$$

where $g(x, y) = f(x, y)(xy)^{k-1}\Gamma(k)^{-2}(2\pi)^{2k}\sum_{q=1}^{\infty} q^{-2k}\mu(q)$, $\mu(x)$ *is the Möbius function, and the implied constant depends on ϵ only.*

Before we state the corollary, we would like to point out that Theorem 1 of [8] corresponds to the case where $k = 1$ in the theorem above.

Corollary 4.2.2. *For,* $M \geq 1$*, we have*

$$\sum_{m \leq M} r_Q(m) r_Q(m+1) = \int_0^X g(x, x+1)\, dx + O\left(M^{2k-1-\frac{4k-3}{4k+5}+\epsilon}\right),$$

where $g(x, y)$ *is defined as in the theorem above.*

Proof. In proceeding from an estimate for a weighted sum to an unweighted one, we basically follow the same steps as in the proof of the corollary of [8]. We apply Theorem 4.2.1 for the test function $f(x, y) = f_1(x, y) f_2(x, y)$ where f_1, f_2, are single variable functions, smooth, non-negative, supported on $[X, X + XP^{-1}]$, $[0, 2Y]$, respectively, such that

$$f_1(x) = 1, \quad \text{when } 0 \leq x \leq X, \; f_1^{(j)} \ll \left(\frac{P}{X}\right)^j,$$

and

$$f_2(y) = 1, \quad \text{when } 0 \leq y \leq Y, \; f_2^{(j)} \ll Y^{-j}.$$

We set $X = M$, and $Y = M + 1$, so the sum $\sum_{m \leq X} r_Q(m) r_Q(m+1)$ will be majorized by $D(\Theta, f)$. Since f, as chosen, satisfies the condition given by (4.5), upon applying Theorem 4.2.1 to $D(\Theta, f)$, we get

$$D(\Theta, f) = \int_0^\infty g(x, x+1)\, dx + O\left(P^{k+1/4} M^{k-1/4+\epsilon}\right). \tag{4.9}$$

Moreover, we can also directly estimate the difference between the integral above and that given in Theorem 4.2.1 to obtain an upper bound of

$$\int_0^\infty g(x, x+1)\, dx - \int_0^M g(x, x+1)\, dx \ll \frac{M^{2k-1}}{P}. \tag{4.10}$$

We then set $P = M^{(4k-3)/(4k+5)}$ in order to make an optimal choice of the error term by setting the error estimates of (4.9) and (4.10) equal to one another and solving for P. Upon substituting this value of P back into the estimate for the error term obtained in either (4.9) or (4.10), we get the desired bound seen in the statement of the corollary. □

4.3 An Introduction to the δ-symbol method

We follow the δ-symbol method used in [8] without any modification whatsoever. For the sake of completeness of exposition, we proceed to restate the main lemmas without proof. A good exposition that motivates the techniques and results used forthwith is also given in §20.5 of [10].

Take a smooth, compactly supported function $w(u)$ on $\mathbb{R}$ such that $w(u) = w(-u)$ and $w(0) = 0$. We normalize $w(u)$ by requiring

$$\sum_{q=1}^{\infty} w(q) = 1. \tag{4.11}$$

Then for any $n \in \mathbb{Z}$, we have

$$\delta(n) = \sum_{q|n} \left(w(q) - w\left(\frac{n}{q}\right) \right) = \begin{cases} 1, & \text{if } n = 0. \\ 0, & \text{if } n \neq 0. \end{cases} \tag{4.12}$$

The following idea had its origins in [5]. Using additive characters to detect the divisibility $q|n$, we can get the following expression for 'capturing' the condition $n = 0$ which, in a nod to the Dirac delta function, will henceforth be expressed as $\delta(n)$.

$$\delta(n) = \sum_{q=1}^{\infty} \sideset{}{^*}\sum_{d(\bmod q)} e\left(\frac{dn}{q}\right) \Delta_q(n), \tag{4.13}$$

where

$$\Delta_q(n) = \sum_{r=1}^{\infty} (qr)^{-1} \left(w(qr) - w\left(\frac{n}{qr}\right) \right). \tag{4.14}$$

The following lemma tells us that $\Delta_q(n)$ approximates the Dirac distribution quite closely.

Lemma 4.3.1. *For any* $f \in C_0^\infty(\mathbb{R})$*, we have*

$$\int_{-\infty}^{\infty} f(u)\Delta_q(u)\,du = f(0)\int_0^\infty w(r)\,dr - q^j \int_0^\infty \psi_j\left(\frac{r}{q}\right) \int_{-\infty}^{\infty} \left(f(u)\left(\frac{w(r)}{r}\right)^{(j)} - w(u)u^j f^{(j)}(ru)\right) du dr, \quad (4.15)$$

where $j \geq 1$ *and*

$$\psi_j(z) = -\sum_{m=1}^{\infty} (2\pi i m)^{-j}(e(mz) + (-1)^j e(-mz)).$$

The proof of Lemma 4.3.1 is obtained by using the definition of $\Delta_q(u)$ in the left-hand side of (4.15) followed by a change of variable in the second integral getting

$$\int_{-\infty}^{\infty} f(u)\Delta_q(u)\,du = \int_{-\infty}^{\infty} f(u)\left(\sum_{r=1}^{\infty}(qr)^{-1}w(qr)\right) du - \int_{-\infty}^{\infty} w(v)\left(\sum_{r=1}^{\infty} f(qrv)\right) dv. \quad (4.16)$$

The proof is then completed by evaluating the individual sums in (4.16) using the Euler-Maclaurin summation formula.

We shall now impose a few conditions on $w(u)$ in order to further capture the Dirac distribution-like behavior exhibited in Lemma 2.4.1. Suppose $w(u)$ is supported in $Q \leq |u| \leq 2Q$ and it has derivatives bounded by $w^j \ll Q^{-j-1}, j \geq 0$. Using the fact that $|\psi_j(z)| \leq 1$, the terms on the right-hand side of (4.16) are bounded by $f(0)\left(1 + O\left(Q^{-j-1}\right)\right)$, $q^j Q^{-j-1}\left|\int f(u)\,du\right|$, and $q^j Q^{j-1}\int \left|f^{(j)}(u)\right|\,du$, respectively. Then we have the following corollary from Lemma 2.4.1.

Corollary 4.3.2. *Let* $j \geq 1$*. We have*

$$\int_{-\infty}^{\infty} f(u)\Delta_q(u)\,du = f(0) + O\left(q^j Q^{-1}\left(\int Q^{-j}|f(u)| + Q^j\left|f^{(j)}(u)\right|\,du\right)\right). \quad (4.17)$$

If $q < Q^{1-\epsilon}$, Corollary 4.3.2 tells us that $\Delta_q(u)$ approximates to the Dirac distribution very well on test functions satisfying $f^{(j)} \ll (qQ^{1+\epsilon})^{-j}$.

Finally, the following Lemma allows us, in the words of [10], to 'control the variations of $\Delta_q(u)$ in both variables q, u while separating q from u at a low cost'.

Lemma 4.3.3. *We have the following bound on $\Delta_q(u)$:*

$$\Delta_q(u) \ll (qQ + Q^2)^{-1} + (qQ + |u|)^{-1}. \tag{4.18}$$

4.4 Application of the δ-symbol method

We shall first investigate $D(\Theta, f)$ for smooth test functions $f(x, y)$ which are supported in a box $[X, 2X] \times [Y, 2Y]$ and has partial derivatives bounded by

$$f^{(i,j)} \ll X^{-i} Y^{-j} P^{i+j}. \tag{4.19}$$

Towards the end of the paper, we shall derive Theorem 4.2.1 by employing a partition of unity and decomposing a smooth test function $f(x, y)$ which satisfies (4.5). As $f(x, y)$ is supported in $[X, 2X] \times [Y, 2Y]$, we can assume that $X, Y \geq 1/2$ else $D(\Theta, f)$ vanishes trivially. As in [8], we shall also attach a redundant factor $\phi(x - y - h)$ to $f(x, y)$ where $\phi(t)$ is a smooth function supported on $|t| < U$ such that $\phi(0) = 1$ and $\phi^i \ll U^i$. Here, not only does the redundant factor $\phi(t)$ help smooth out the sum $D(\Theta, f)$, it also renders the sum vulnerable to certain analytic techniques that we will employ shortly. Finally, the parameter U also allows us a measure of control in choosing the optimal size of the error terms. Rather than deal with U directly, for the ease of computation, we will choose

$$U = \frac{XY}{P(X + Y)}, \tag{4.20}$$

so that, by (4.19), the new function

$$F(x, y) = f(x, y)\phi(x - y - 1)$$

has partial derivatives bounded by

$$F^{(i,j)} \ll U^{-(i+j)}. \tag{4.21}$$

Now, we can use the expression for $\delta(n)$ from (4.13) to capture the shift $m-n = 1$ in the sum $D(\Theta, f)$. Before we proceed, in order to optimize the magnitude of the moduli appearing in the expression of $\delta(n)$, we choose the compact support of the function $w(u)$, last seen in §3, to be $U^{1/2} \leq |u| \leq 2U^{1/2}$ (or equivalently, we set $Q = U^{1/2}$, where Q is the parameter that determines the compact support of $w(u)$). Consequently, we get the following expression for $D(\Theta, f)$:

$$\begin{aligned}
D(\Theta, f) &= D(\Theta, F) \\
&= \sum_{m-n=1} r_Q(m) r_Q(n) F(m, n) \\
&= \sum_{m} \sum_{n} r_Q(m) r_Q(n) F(m, n) \delta(m - n - 1) \\
&= \sum_{1 \leq q \leq 2Q} \sideset{}{^*}\sum_{d \,(\mathrm{mod}\, q)} e\left(\frac{-d}{q}\right) \sum_{m} \sum_{n} r_Q(m) r_Q(n) e\left(\frac{dm - dn}{q}\right) E(m, n),
\end{aligned} \tag{4.22}$$

where we replaced $\delta(m-n-1)$ with the right-hand side of the equation from (4.13) and switched the order of summation thereafter to get (4.22). In the process, we also combined the two auxiliary functions $\Delta_q(x - y - 1)$ and $F(x, y)$ into a single function as follows: $E(x, y) = F(x, y)\Delta_q(x - y - 1)$.

4.5 Implementation of the Voronoï summation formula

We shall execute the summation over m, n in (4.22) via the following Voronoï summation formula from Chapter II:

Proposition 4.5.1. *Let $\phi(x)$ be a smooth function of compact support in $(0,\infty)$. Recall that $\Theta(z,Q)$ is an automorphic form for $\Gamma_0(4N)$ of weight $k = l/2$. Then, we have*

$$\sum_{n=1}^{\infty} r_Q(n)e(an/c)\phi(n)$$
$$= \left(\frac{c}{2\pi}\right)^{-k}\Gamma(k)^{-1}G(k)i^k\left(\frac{|A|}{a}\right)^{-1}\left(\bar{\epsilon}_a\left(\frac{-2c}{a}\right)\right)^{-2k} + \frac{2\pi}{c}i^k\left(\frac{|A|}{a}\right)^{-1}\left(\bar{\epsilon}_a\left(\frac{-2c}{a}\right)\right)^{-2k}$$
$$\times\sum_{n=1}^{\infty}\left(r_Q(n)e(-dn/c)n^{(1-k)/2}\int_0^\infty \phi(x)x^{(k-1)/2}J_{k-1}\left(\frac{4\pi\sqrt{nx}}{c}\right)dx\right)$$

By Proposition 4.5.1 applied once to each variable in the innermost double sum of (4.22), we get the following expression

$$\begin{aligned}\sum_m\sum_n &= \frac{1}{\Gamma(k)^2}\left(\frac{2\pi}{q}\right)^{2k}\mathcal{I}\\
&+\frac{1}{\Gamma(k)}\left(\frac{2\pi}{q}\right)^{k+1}\sum_{m=1}^{\infty}r_Q(m)e\left(\frac{-\bar{d}m}{q}\right)m^{(1-k)/2}\mathcal{I}_1\\
&+\frac{1}{\Gamma(k)}\left(\frac{2\pi}{q}\right)^{k+1}\sum_{n=1}^{\infty}r_Q(n)e\left(\frac{\bar{d}n}{q}\right)n^{(1-k)/2}\mathcal{I}_2\\
&+\left(\frac{2\pi}{q}\right)^{2}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}r_Q(n)r_Q(m)e\left(\frac{\bar{d}n-\bar{d}m}{q}\right)(nm)^{(1-k)/2}\mathcal{I}_3,\end{aligned}\tag{4.23}$$

where

$$\begin{aligned}\mathcal{I} &= \int_0^\infty\int_0^\infty E(x,y)(xy)^{k-1}\,dy\,dx,\\
\mathcal{I}_1 &= \int_0^\infty\int_0^\infty x^{(k-1)/2}J_{k-1}\left(\frac{4\pi\sqrt{mx}}{q}\right)E(x,y)\,y^{k-1}\,dy\,dx,\\
\mathcal{I}_2 &= \int_0^\infty\int_0^\infty y^{(k-1)/2}J_{k-1}\left(\frac{4\pi\sqrt{ny}}{q}\right)E(x,y)\,x^{k-1}\,dx\,dy,\text{ and finally,}\end{aligned}$$

$$\mathcal{I}_3 = \int_0^\infty \int_0^\infty (xy)^{(k-1)/2} J_{k-1}\left(\frac{4\pi\sqrt{mx}}{q}\right) J_{k-1}\left(\frac{4\pi\sqrt{ny}}{q}\right) E(x,y)\, dx\, dy$$

Inserting (4.23) into (4.22), we obtain complete Kloosterman sums $S(n,m;q)$. Recall that the Kloosterman sum $S(n,m;q)$ is defined as follows:

$$S(n,m;q) = \sum e\left(\frac{na + m\bar{a}}{q}\right).$$

Upon rearranging and collecting the terms which are affected by the summation in d, we obtain the following formula:

$$\begin{aligned} D_Q(a,b;h) = \sum_{q<2Q} \Bigg(& \Gamma(k)^{-2} \left(\frac{2\pi}{q}\right)^{2k} S(1,0;q)\,\mathcal{I} \\ & + \sum_{m=1}^{\infty} \Gamma(k)^{-1} \left(\frac{2\pi}{q}\right)^{k+1} r_Q(m) m^{(1-k)/2} S(1,m;q)\,\mathcal{I}_1 \\ & + \sum_{n=1}^{\infty} \Gamma(k)^{-1} \left(\frac{2\pi}{q}\right)^{k+1} r_Q(n) n^{(1-k)/2} S(1,-n;q)\,\mathcal{I}_2 \\ & + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \left(\frac{2\pi}{q}\right)^{2} r_Q(m) r_Q(n) (mn)^{(1-k)/2} S(1,m-n;q)\,\mathcal{I}_3 \Bigg) \end{aligned} \tag{4.24}$$

To the Kloosterman sums in this formula, we apply Weil's bound

$$S(h,m-n;q) \ll q^{1/2}\tau(q).$$

When $m = n = 0$, we get the Ramanujan sum for which we have

$$c_q(h) = S(h,0;q) = \sum_{\nu|(h,q)} \nu\mu\left(\frac{q}{\nu}\right) \ll (h,q).$$

4.6 Evaluation of the main term

Upon rearranging and regrouping terms within the integral $\mathcal{I}$, we have:

$$\mathcal{I} = \iint C(x,y)\Delta_q(x-y-1)\, dx\, dy$$

$$= \iint C(x, x-1+u)\Delta_q(u)\, du\, dx,$$

where we set $C(x,y) = F(x,y)(xy)^{k-1}$ and performed a change-of-variable by setting $y = x - 1 + u$ in the last line. We note that, at a similar point, [8] defines $C(x,y)$ in terms of $\log x$ and $\log y$, while we have the k-th powers of x and y occurring in our Mellin transforms. This allows us to draw a rather loose analogy with the elementary fact from calculus that the anti-derivative of x^k is yet another power of x, except in the case where $k = 0$ (which returns $\log x$). So, we could conceivably consider the work of Iwaniec, et al. in [8] as dealing with the 'singular' case, i.e. $k = 0$, while we seem to be dealing with the easier and non-singular scenario. By (4.3.2) and (4.19), we obtain the following estimate

$$\int C(x, x-1+u)\Delta_q(u)\, du = C(x, x-1) + O\left(\left(\frac{q}{Q}\right)^j\right).$$

We can make the error term very small by first assuming that $q < Q^{1-\epsilon}$ and then adopting a large value of j. Upon doing so, we obtain

$$\mathcal{I} = \int C(x, x-1)\, dx + O\left(Q^{-A}\right).$$

We also have another upper bound for $\mathcal{I}$, namely,

$$\begin{aligned}
\mathcal{I} &\ll (XY)^{k-1} \iint |F(x,y)\Delta_q(x-y-1)|\, dx\, dy \\
&= (XY)^{k-1} \iint |F(x, x-1-u)\Delta_q(u)|\, dx\, du \\
&\ll (XY)^{k-1} \min(X,Y) \int_{-U}^{U} |\Delta_q(u)|\, du \ll \frac{(XY)^k}{X+Y} \log Q,
\end{aligned}$$

where the last inequality follows from (4.18). Moreover, the above inequality holds for all q as well. Putting together all the error terms obtained in this section, the first term on the right-hand side of (4.24) yields

$$\sum_{q=1}^{\infty} q^{-2k}\mu(q) \int C(x, x-1)\, dx + O\left(\frac{(XY)^k}{X+Y} Q^{-1} \log Q\right), \tag{4.25}$$

where we are dealing with q in the range $q < Q^{1-\epsilon}$.

4.7 Estimation of the error term

We now proceed to obtain estimates for the integrals given by $\mathcal{I}_1$, $\mathcal{I}_2$, and $\mathcal{I}_3$. We begin by first getting a bound for the function $E(x,y) = F(x,y)\Delta_q(x-y-1)$. Using (4.18) in conjunction with integration by parts gives us

$$E^{(ij)} \ll \frac{1}{qQ}\left(\frac{1}{qQ}\right)^{i+j}, \quad i+j>0. \tag{4.26}$$

The integrals $\mathcal{I}_1$, $\mathcal{I}_2$, and $\mathcal{I}_3$ are reminiscent of the integrals last seen in §6 of Chapter II. As in that chapter, we hope to obtain upper bounds on these integrals in terms of their arguments. These bounds will then help us identify those terms that contribute the most in the sums comprising (4.24). One of the most common ways to do this is by the method of 'estimating via integration by parts' which uses the recurrence relation

$$(z^k J_k(z))' = z^k J_{k-1}(z) \tag{4.27}$$

and follows exactly the same line of reasoning as that in §6 of Chapter II. Since we have already expounded on this a great deal in a previous chapter, we will not repeat the same calculation again. Moreover, as before, since we are keeping the weight k constant, we are not concerned with stationary phase arguments.

Upon integrating $\mathcal{I}_1$ by parts once, and keeping in mind that we are interested in bounds depending on q, Q, X, and m, we see that the size of the resulting expression changes by a factor of

$$\frac{q\sqrt{X}}{\sqrt{m}} \min\left(\frac{1}{X}, \frac{1}{qQ}\right)$$

where we are using (4.26) in our estimation of $dE(x, y)/dx$. Consequently, if we integrate $\mathcal{I}_1$ by parts p times and keep track of the pertinent parameters, then its size is given by

$$\mathcal{I}_1 \ll \left(\frac{q\sqrt{X}}{\sqrt{m}} \min\left(\frac{1}{X}, \frac{1}{qQ}\right)\right)^p.$$

The definitions of the parameter U and Q from §4 imply that $qQ \leq Q^2 = U \ll Y$, as a result of which the estimate above can be modified to get

$$\mathcal{I}_1 \ll \left(\frac{\sqrt{X}}{\sqrt{m}Q}\right)^p. \tag{4.28}$$

So, if $Q\sqrt{X}/\sqrt{m} < 1$, i.e., $m > X/Q^2$, then the contribution of $\mathcal{I}_1$ towards the error term will be negligible (since we can make its estimate as small as possible by simply increasing the number of times we integrate $\mathcal{I}_1$ by parts). A similar process can be repeated in terms of n with a resulting bound for $\mathcal{I}_2$ as in (4.28). This upshot of all these calculations is that we are only interested in the contribution towards the error term obtained by summing m and n in (4.24) within the range

$$m < X/Q^{2-\epsilon}, \quad n < Y/Q^{2-\epsilon} \tag{4.29}$$

.

For m, n in this range, we estimate the integrals trivially using the bound $J_k(z) \ll z^{-1/2}$, which gives

$$\mathcal{I}_1(m) \ll \left(\frac{q^2}{mX}\right)^{1/4} X^{(k-1)/2} Y^{k-1} \iint$$

$$\mathcal{I}_2(n) \ll \left(\frac{q^2}{nY}\right)^{1/4} Y^{(k-1)/2} X^{k-1} \iint$$

$$\mathcal{I}_3(m,n) \ll \left(\frac{q^4}{mnXY}\right)^{1/4} (XY)^{(k-1)/2} \iint$$

where

$$\iint = \iint E(x,y)\, dx dy \ll \frac{(XY)}{X+Y}\ Q^{\epsilon}.$$

Next, summing over m, n in the range (4.29), we obtain

$$\sum_m r_Q(m) m^{(1-k)/2}\ \mathcal{I}_1(m) \ll q^{1/2}\ \frac{X^{(k+1/2)}\, Y^k}{X+Y}\ Q^{-k-1/2+\epsilon}$$

$$\sum_n r_Q(n) n^{(1-k)/2}\ \mathcal{I}_2(n) \ll q^{1/2}\ \frac{X^k\, Y^{(k+1/2)}}{X+Y}\ Q^{-k-1/2+\epsilon}$$

$$\sum_m \sum_n r_Q(m) r_Q(n) (mn)^{(1-k)/2}\ \mathcal{I}_3(m,n) \ll q\ \frac{(XY)^{k+1/2}}{X+Y}\ Q^{-2k-1+\epsilon}.$$

Introducing these bounds, along with (4.25), into (4.24), we get Theorem 4.2.1 with the error term

$$\frac{(XY)^k}{X+Y}\ Q^{-2k+1+\epsilon} + \frac{(XY)^{k+1/2}}{X+Y} Q^{-2k-1/2+\epsilon}.$$

On using the relation $U = Q^2 = XY/(P(X+Y))$ from §4 in the expression above and picking out the term that contributes the largest power of XY, the above error term becomes that of Theorem 4.2.1. This completes the proof of the theorem in the case where $f(x,y)$ is supported in a dyadic box. In order to derive Theorem 4.2.1 in its general form for any function satisfying (4.5), we first consider an arbitrary smooth function

$$\rho : (0,\infty) \to \mathbb{R}$$

whose support lies in $[1,2]$ and which satisfies the following identity on the positive axis:

$$\sum_{k=-\infty}^{\infty} \rho(2^{-k/2} x) = 1.$$

At this juncture, we would like to note that the following procedure was borrowed from §5 of [15]. To obtain such a function, we take an arbitrary smooth $\eta : (0, \infty) \to \mathbb{R}$ which is 0 on $(0, 1)$ and 1 on $(\sqrt{2}, \infty)$ and we then define ρ as follows

$$\rho(x) = \begin{cases} \eta(x), & \text{if } 0 < x \leq \sqrt{2}. \\ 1 - \eta(x/\sqrt{2}), & \text{if } \sqrt{2} < x < \infty. \end{cases}$$

With this partition of unity, we decompose $f(x, y)$ as

$$f(x, y) = \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} f_{k,l}(x, y) \text{ ,where}$$

$$f_{k,l}(x, y) = f(x, y)\, \rho\left(\frac{x}{2^{k/2}X}\right) \rho\left(\frac{y}{2^{l/2}Y}\right).$$

We note the following relations, for $P \geq 1$,

$$\operatorname{supp} f_{k,l} \subseteq [A_k, 2A_k] \times [B_l, 2B_l], \quad A_k = 2^{k/2} X, \;\; B_l = 2^{l/2} Y, \text{ and}$$

$$(1 + 2^{k/2})(1 + 2^{l/2})\, f_{k,l}^{(i,j)} \ll A_k^{-i} B_l^{-j} P^{i+j}.$$

Consequently, the function $g_{k,l}(x, y)$ satisfies the condition give in (4.19) (with $X = A_k$, and $Y = B_l$) where $g_{k,l}$ is given by

$$g_{k,l}(x, y) = (1 + 2^{k/2})(1 + 2^{l/2})\, f_{k,l}(x, y).$$

The error term in Theorem 4.2.1 is finally acquired by noting that

$$D(\Theta, f) = \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} D(\Theta, f_{k,l}),$$

and summing up the error term arising from the summand across the double sum in k and l.

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