GEOMETRY OF GENERAL CURVES VIA DEGENERATIONS AND DEFORMATIONS

DISSERTATION

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ABSTRACT

This thesis studies the geometric and deformational behavior of linear series under degenerations with the aim of attacking the maximal rank conjecture. There are three parts. The first part gives an explicit construction of the classical tangent-obstruction theory for deformations of the pair (X, L) to the case when X is local complete intersection scheme and L a line bundle on X. In the second part, we propose a new method, using deformation theory, to study the maximal rank conjecture. We prove that the maximal rank conjecture holds for the first unknown case: line bundles of extremal degree. Problems related to the maximal rank conjecture have become potentially accessible to this new method. In the third part, a canonical semi-stable degeneration of the d-th symmetric product $C^{(d)}$ as the curve C becomes singular is constructed.

To Ji and Eric

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CHAPTER 1

INTRODUCTION

A central problem in algebraic curve theory is to describe algebraic curves with fixed genus and degree and how they vary in families in a given projective space $\mathbb{P}^r$.

There are two approaches to this problem. The first approach is to deal with curves in projective spaces directly. We would like to analyze the extrinsic geometry of $C \subset \mathbb{P}^r$. For instance, one wants to describe the ideal of a curve $C \subset \mathbb{P}^r$, and in particular, to know how many independent hypersurfaces of each degree C lies on, what are the relations between the generators of the ideal, etc.

The second approach is via abstract curves. One may think of a curve $C \subset \mathbb{P}^r$ as a triple object: an abstract curve C, a line bundle $L = \mathcal{O}_C(1)$ and an $(r+1)$-tuple of global sections of $\mathcal{O}_C(1)$. The good thing about this approach is that the collections of objects we want to describe are themselves algebraic varieties (stacks), called moduli spaces (stacks). Thus we can apply the general machinery in algebraic geometry to study these spaces. The moduli space of smooth curves C of genus g is called $\mathcal{M}_g$, and the moduli space of linear series of degree d dimension r on C is called $G_d^r(C)$.

One of the major open problem in algebraic curve theory, the maximal rank conjecture (Eisenbud-Harris [15]), lies in the area between these two approaches. It has

to do with the extrinsic geometry of the pair (C, L), for C a general point of $\mathcal{M}_g$, L a general point of $G_d^r(C)$.

Conjecture 1.1. *(Maximal rank conjecture) For fixed d, g, $r \geq 3$, let C be a general curve of genus g and $|L|$ be a general g_d^r on C, then the multiplication map*

$$\operatorname{Sym}^k H^0(C, L) \xrightarrow{\mu^k} H^0(C, L^k) \tag{1.1}$$

is of maximal rank (either injective or surjective) for any $k \geq 1$.

In the case $|L|$ gives an embedding of C into $\mathbb{P}^r$, $\operatorname{Sym}^k H^0(C, L)$ is the space of homogeneous polynomials of degree k in $\mathbb{P}^r$ and $\operatorname{Ker}(\mu^k)$ is just the subspace consisting of those vanishing on C. Since the dimension of the domain and target of μ^k are constants only depending on k, d, r and g, the maximal rank conjecture (MRC) simply says that the number of independent hypersurfaces containing C is as small as it could be.

Since conjecture 1.1 concerns conditions that are open, it suffices to verify the assertion for one point on each component of the $\mathcal{G}_d^r$ which dominates $\mathcal{M}_g$, where

$$\mathcal{G}_d^r = \{(C, L, V) \mid L \text{ line bundle on } C, \ deg \ L = d, \ V \subset H^0(L), \ dimV = r+1\}$$

is the parameter space in question.

However, this does not seem to help. Since $\mathcal{M}_g$ is of general type for $g > 23$, it is very difficult to write down general curves satisfying this conjecture. The curves we can write down for large g, such as hyperelliptic and trigonal curves, complete intersections are all special with respect to the properties that this conjecture asserts to be general. The classic strategy to deal with this problem is to degenerate

smooth curves to some sufficiently special singular ones on which we can carry out the necessary analysis explicitly but remain general in the sense of the assertions. This approach achieved great success for the more classical theorems of Brill- Noether ([24]) and Gieseker-Petri ([22], [14]), but has not been fully successful for the maximal rank conjecture.

The aim of this thesis is to develop more tools to understand the geometric and deformational behavior of linear series under degenerations and hopefully make some progress on the maximal rank conjecture.

In chapter 2, we study the deformation theory of the pair (X, L) for X a local complete intersection scheme and L a line bundle on X. We generalize the tangent-obstruction theory the pair (X, L) from the classical case X is a smooth variety to the case X is a local complete intersection scheme (l.c.i). We prove that even though X could be singular, the functor of Artin rings

$$Def_{(X,L)}(A) = \{\text{Flat deformations of } (X, L) \text{ over } A\}/\text{isomorphisms}$$

still behaves well in the sense that there is a tangent-obstruction theory for this deformation functor, with tangent space $\mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{P}^1_X(L), L)$ and obstruction space $\mathrm{Ext}^2_{\mathcal{O}_X}(\mathcal{P}^1_X(L), L)$, where $\mathcal{P}^1_X(L)$ is the sheaf of one jets or sheaf of principle parts of the line bundle L on X. Moreover, a criterion for sections of L to extend is given. When X is a l.c.i curve, L is a complete g^r_d on X, this result is directly related to the local behavior of the parameter space $\mathcal{G}^r_d$ near the boundary. For instance, the tangent space of $\mathcal{G}^r_d$ at the point $(X, L, H^0(L))$ consists of all vectors $\xi \in \mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{P}^1_X(L), L)$ such that all global sections of L extend along ξ.

In chapter 3, we take a first step to attack the maximal rank conjecture via deformations. We prove that the maximal rank conjecture holds for the first unknown case: line bundles of extremal degree, i.e. line bundles such that

$$deg \ L = 2g - 2h^1(L) - Cliff(C),$$

or equivalently,

$$Cliff(L) = Cliff(C),$$

where $Cliff(C)$ is the Clifford index of C.

The method in the proof is different from classical degeneration methods. The general idea is as follows: instead of constructing some (C, L) such that the multiplication map μ^k in (1.1) is of maximal rank there, we relax the requirement by considering a one parameter family of pairs $(C_t, L_t) \in \mathcal{G}_d^r$, degenerating to some (C_0, L_0) (C_0 is singular) for which $\mu^k(0)$ is not of maximal rank, then use deformation theory to show, however, only a subspace of "correct" dimension in $\text{Ker}(\mu^k(0))$ can extend to $\text{Ker}(\mu^k(t))$ and therefore nearby $\mu^k(t)$ is of maximal rank.

Chapter 4 takes another more geometric point of view to study the degenerations of linear series. We view a g_d^r on C as a geometric object $\mathbb{P}^r$ sitting inside the d-th symmetric product $C^{(d)}$ of C and the hope is to study the geometric properties of the linear series by studying the geometry of the subvariety $\mathbb{P}^r \subset C^{(d)}$. For instance, L satisfies the Gieseker-Petri theorem, i.e the natural map

$$H^0(L) \otimes H^0(K_C \otimes L^{-1}) \longrightarrow H^0(K_C)$$

is injective, if and only if $\mathbb{P}^r$ is unobstructed in $C^{(d)}$ ([12]). Taking this point of view, to understand the degenerations of linear series, we need to first understand

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the degenerations of $C^{(d)}$ when C become singular. For a one parameter family of smooth curves C_t degenerating to a nodal curve C_0, the suitable degeneration we want is a smooth space X over the t-disk Δ such that the fiber of X over $t \neq 0$ is isomorphic to $C_t^{(d)}$, and the fiber over $t = 0$ has simple normal crossing support.

It is proved in [33] that the total space of relative Hilbert scheme $\mathcal{H}_d$ parametrizing length-d dimension-0 subschemes of the fiber is a partial resolution of singularities of the relative d-th symmetric product $\mathcal{C}_{\Delta}^{(d)}$. Based on this result, we study in this chapter the toric singularities appeared in $\mathcal{H}_d$ and give an algorithm to canonically subdivide the cones corresponding to the toric singularities in question and describe a canonical sequence of blowing-ups of $\mathcal{H}_d$ along smooth centers that leads to a canonical log resolution $\widetilde{\mathcal{H}_d}$ of $(\mathcal{H}_d, H_0^d)$.

Throughout this thesis, we will work over the complex numbers $\mathbb{C}$.

CHAPTER 2

DEFORMATION OF PAIRS (X, L) WHEN X IS SINGULAR

2.1 Background

The deformation theory of the pair (X, L) for X a smooth variety and L a line bundle on X was first used to study Petri's conjecture by Arbarello and Cornalba in [5]. It was proved there that first-order deformations of the pair (X, L) are in natural one to one correspondence with

$$\xi \in H^1(X, \mathcal{D}_1(L)),$$

where $\mathcal{D}_1(L)$ is the sheaf of holomorphic first-order differential operators, and $H^2(X, \mathcal{D}_1(L))$ is an obstruction space. Given a first-order deformation $\phi \in H^1(X, T_X)$ of X, there is a first-order deformation of L along ϕ if and only if $\phi \cup c(L) = 0 \in H^2(X, \mathcal{O}_X)$, where $c(L) \in H^1(X, \Omega_X^1)$ is the first Chern class of L in the sense of Atiyah.

Moreover, there is a natural differentiation map

$$H^1(X, \mathcal{D}_1(L)) \xrightarrow{\ M\ } Hom(H^0(X, L), H^1(X, L)) \tag{2.1}$$

such that a section $s \in H^0(X, L)$ extends to first order along ξ if and only if the element

$$M(\xi)(s) \in H^1(X, L)$$

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is zero.

The map M together with the tangent obstruction spaces have numerous deformation theoretic applications. For instance, for any first-order deformation of (X, L), at least $h^0(L) - h^1(L)$ linearly independent sections of L extend; $\mathrm{Ker}(M) \subset H^1(X, \mathcal{D}_1(L))$ is the space of first-order deformations of (X, L) to which all sections of L extend. If X is a complete curve, a dual form of (2.1) is the higher μ-map μ_1 in [7]. In case L gives an embedding of X into some projective space $\mathbb{P}$, $\mathrm{Coker}(M)$ is naturally isomorphic to $H^1(X, N_{X|\mathbb{P}})$ (cf. [5]), and therefore the surjectivity of M implies that $X \subset \mathbb{P}$ is unobstructed. Another direct consequence is that the deformations of the pair (X, L) is unobstructed for a smooth curve X, since $H^2(X, \mathcal{D}_1(L)) = 0$. If X is a smooth $K3$-surface, the map $H^1(T_X) \xrightarrow{\cup c(L)} H^2(\mathcal{O}_X) \cong \mathbb{C}$ is surjective for every nontrival line bundle L. This means that L deforms along a 19-dimensional subspace of $H^1(T_X)$, because $h^1(X, T_X) = 20$.

In this chapter, we give an elementary approach to the deformation theory of the pair (X, L) for X a separated reduced local complete intersection scheme (l.c.i) of finite type over $\mathbb{C}$. We prove that even though X could be singular, the functor of Artin rings

$$Def_{(X,L)}(A) = \{\text{Flat deformations of } (X, L) \text{ over } A\}/\text{isomorphisms}$$

still behaves well in the sense that there is a tangent-obstruction theory for this deformation functor, with tangent space $\mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{P}^1_X(L), L)$ and obstruction space $\mathrm{Ext}^2_{\mathcal{O}_X}(\mathcal{P}^1_X(L), L)$, where $\mathcal{P}^1_X(L)$ is the sheaf of one jets or sheaf of principle parts

of L on X. Moreover, there is a natural map analogous to M characterizing obstructions for sections of L to extend. Therefore, all the nice consequences mentioned above generalize to reduced l.c.i schemes. If X is smooth, $\mathcal{P}_X^1(L) = \mathcal{D}_1(L)^* \otimes L$, where $\mathcal{D}_1(L)$ is the sheaf of first-order differential operators on L, and $\mathrm{Ext}^i_{\mathcal{O}_X}(\mathcal{P}_X^1(L), L) = H^i(X, \mathcal{D}_1(L))$. We go back to the classical case. The tangent and obstruction spaces for deformations of (X, L) was known to experts and was stated implicitly in [29], [30]. Our approach is new and more elementary. In particular, it does not use the more abstract machinery of cotangent complexes. The author believes that the generalization of the map M in (2.1) to singular varieties is also new.

2.2   The sheaf of one jets

In this section, we briefly review some basic facts and definitions about the sheaf of one jets.

Let $g : X \to Y$ be a morphism between two algebraic schemes (separated schemes of finite type over $\mathbb{C}$), L be a line bundle on X, and let $\Delta \subset X \times_Y X$ be the diagonal defined by ideal sheaf $\mathcal{I}_\Delta$. Consider the first order neighborhood $Spec\frac{\mathcal{O}_{X \times_Y X}}{\mathcal{I}_\Delta{}^2}$ of Δ with two projections π_1, π_2 to X. The sheaf of one jets $\mathcal{P}^1_{X/Y}(L)$ of X over Y is defined to be $\mathcal{P}^1_{X/Y}(L) := \pi_{1*}\pi_2^*(L)$. $\mathcal{P}^1_{X/Y}(L)$ has a natural left $\mathcal{O}_X$-module structure induced by π_1 and a right $\mathcal{O}_X$-module structure induced by π_2 which, in general, is not equivalent to the left one. Throughout this chapter, we will only use the left

$\mathcal{O}_X$-module structure of $\mathcal{P}^1_{X/Y}(L)$. Consider the short exact sequence

$$0 \longrightarrow \frac{\mathcal{I}_\Delta}{\mathcal{I}_\Delta^2} \longrightarrow \frac{\mathcal{O}_{X \times_Y X}}{\mathcal{I}_\Delta^2} \longrightarrow \frac{\mathcal{O}_{X \times_Y X}}{\mathcal{I}_\Delta} \longrightarrow 0$$

Tensoring the above sequence with $\pi_2^* L$ then applying the functor π_{1*}, we get a short exact sequence of left $\mathcal{O}_X$-modules on X

$$0 \longrightarrow \Omega^1_{X/Y}(L) \xrightarrow{\ i\ } \mathcal{P}^1_{X/Y}(L) \longrightarrow L \longrightarrow 0 \qquad (2.2)$$

where $\Omega^1_{X/Y}$ is the sheaf of relative Kähler differentials. The sequence is exact on the right because there is no higher derived image for π_{1*} (π_1 has relative dimension 0). When $Y = Spec(\mathbb{C})$, we will write $\mathcal{P}^1_X(L)$ for $\mathcal{P}^1_{X/Y}(L)$. The "fibre" of the sheaf $\mathcal{P}^1_{X/Y}(L)$ at a closed point $x \in X$ is the stalk of $L|_{g^{-1}(g(x))}$ at x mod the maximal ideal squared, i.e.

$$\mathcal{P}^1_{X/Y}(L)_x \otimes_{\mathcal{O}_{X,x}} \frac{\mathcal{O}_{X,x}}{m_x} \cong \frac{L_x}{(m_x{}^2 + g^{-1}(m_{g(x)}))L_x}.$$

This is the reason $\mathcal{P}^1_{X/Y}(L)$ is called the sheaf of (relative) one jets. There is a $\mathcal{O}_Y$-linear splitting $p_1 : L \to \mathcal{P}^1_{X/Y}(L)$, which sends a section s of L to its one jet $\pi_{1*}\pi_2^* s$. p_1 satisfies the property that

$$p_1(fs) = i(df \otimes s) + fp_1(s) \qquad (2.3)$$

for any $f \in \mathcal{O}_X(U)$ and $s \in L(U)$ where $U \subset X$ is any open subset. (In fact, p_1 is $\mathcal{O}_X$-linear if we use the right $\mathcal{O}_X$-module structure of $\mathcal{P}^1_{X/Y}(L)$). If X is smooth, $Y = Spec(\mathbb{C})$, $\mathcal{P}^1_X(L)$ is the vector bundle $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{D}_1(L), L)$, where $\mathcal{D}_1(L)$ is the sheaf of first-order differential operators on L.

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2.3 Computation of the tangent space

In this section, let X be a reduced algebraic scheme. Applying the functor $\mathcal{H}om_{\mathcal{O}_X}(-, L)$ to (2.2), we get a long exact sequence

$$\ldots \longrightarrow \mathrm{Ext}^1_{\mathcal{O}_X}(L, L) \longrightarrow \mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{P}^1_X(L), L) \longrightarrow \mathrm{Ext}^1_{\mathcal{O}_X}(\Omega^1_X(L), L) \longrightarrow$$

$$\longrightarrow \mathrm{Ext}^2_{\mathcal{O}_X}(L, L) \longrightarrow \mathrm{Ext}^2_{\mathcal{O}_X}(\mathcal{P}^1_X(L), L) \longrightarrow \mathrm{Ext}^2_{\mathcal{O}_X}(\Omega^1_X(L), L) \longrightarrow \ldots$$

Notice that $\mathrm{Ext}^1_{\mathcal{O}_X}(\Omega^1_X(L), L) = \mathrm{Ext}^1_{\mathcal{O}_X}(\Omega^1_X, \mathcal{O}_X)$ is the tangent space of the deformations of X, and $\mathrm{Ext}^1_{\mathcal{O}_X}(L, L) = H^1(\mathcal{O}_X)$ is the tangent space of deformations of L with the base X fixed. This suggests that $\mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{P}^1_X(L), L)$ is the tangent space of deformations of the pair (X, L) and $\mathrm{Ext}^2_{\mathcal{O}_X}(\mathcal{P}^1_X(L), L)$ is an obstruction space. If X is smooth, $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{P}^1_X(L), L)$ is the sheaf of first-order differential operators $\mathcal{D}_1(L)$, and $\mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{P}^1_X(L), L) = H^1(X, \mathcal{D}_1(L))$ is the correct tangent space. In this section and the next, we will prove this is indeed the correct generalization of the tangent-obstruction theory for deformations of the pair (X, L).

Let's first recall that for any reduced algebraic scheme over $\mathbb{C}$, we have a one-to-one correspondence between isomorphism classes of extensions of X by a coherent locally free $\mathcal{O}_X$-module $\mathcal{I}$ and $\mathrm{Ext}^1_{\mathcal{O}_X}(\Omega^1_X, \mathcal{I})$ in the following way:

Given an isomorphism class of extensions of $\mathcal{O}_X$ by $\mathcal{I}$,

$$0 \longrightarrow \mathcal{I} \longrightarrow \mathcal{O}_{\mathcal{X}} \longrightarrow \mathcal{O}_X \longrightarrow 0,$$

i.e. a closed immersion $X \subset \mathcal{X}$ defined by ideal sheaf $\mathcal{I}$, and $\mathcal{I}^2 = 0$ in $\mathcal{O}_{\mathcal{X}}$, we associate to it (the isomorphism class of) the conormal sequence

$$\mathcal{E} : 0 \longrightarrow \mathcal{I} \longrightarrow \Omega^1_{\mathcal{X}}|_X \longrightarrow \Omega^1_X \longrightarrow 0$$

(which is also exact on the left since $\mathcal{I}$ is locally free.)

This conormal sequence corresponds to an element $c_{\mathcal{E}}$ in $\operatorname{Ext}^1_{\mathcal{O}_X}(\Omega^1_X, \mathcal{I})$.

Conversely, for any $\mathcal{O}_X$-module extension

$$0 \longrightarrow \mathcal{I} \longrightarrow \mathcal{E} \overset{h}{\longrightarrow} \Omega^1_X \longrightarrow 0 , \qquad (2.4)$$

let $d : \mathcal{O}_X \to \Omega^1_X$ be the canonical derivation. Let $\mathcal{O} = \mathcal{O}_X \times_{\Omega^1_X} \mathcal{E}$ be the fibre product sheaf: over an open subset $U \subset X$ we have $\mathcal{O}(U) = \{(f,a) : h(a) = df\}$, with ring structure given by

$$(f,a)(f',a') = (ff', fa' + f'a)$$

We get a commutative diagram:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{I} & \overset{j}{\longrightarrow} & \mathcal{O} & \longrightarrow & \mathcal{O}_X & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle d'} & & \downarrow{\scriptstyle d} & & \\
0 & \longrightarrow & \mathcal{I} & \longrightarrow & \mathcal{E} & \longrightarrow & \Omega^1_X & \longrightarrow & 0
\end{array}
$$

It is easy to check that $d' : \mathcal{O} \to \mathcal{E}$ is a $\mathbb{C}$-derivation, thus factors through $\Omega^1_{\mathcal{O}} \otimes_{\mathcal{O}} \mathcal{O}_X$.

Therefore $\Omega^1_{\mathcal{O}} \otimes_{\mathcal{O}} \mathcal{O}_X \cong \mathcal{E}$ by 5-lemma and we recover $\mathcal{X}$ from (2.4).

In case $\mathcal{I} = \mathcal{O}_X$, we can give $\mathcal{O}_{\mathcal{X}}$ a $\mathbb{C}[\epsilon]$-module structure by sending ϵ to $j(1) \in \mathcal{O}_{\mathcal{X}}$. The fact that $\epsilon \mathcal{O}_{\mathcal{X}} \cong \mathcal{O}_X$ means that $\mathcal{X}$ is flat over $Spec(\mathbb{C}[\epsilon])$. Therefore $\mathcal{X}$ is a first-order infinitesimal deformation of X.

For the deformations of the pair (X, L), we have the following result:

Theorem 2.1. *Let X be a reduced scheme of finite type over $\mathbb{C}$, L be a line bundle on X.*

(1) *The tangent space of the functor of Artin rings* $Def_{(X,L)}$ *is canonically identified with* $\mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{P}^1_X(L), L)$.

(2) *There exists a natural pairing*

$$\mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{P}^1_X(L), L) \otimes H^0(X, L) \xrightarrow{\ p\ } H^1(X, L) \ . \tag{2.5}$$

such that for any first-order deformation of the pair (X, L) *corresponding to* $\xi \in \mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{P}^1_X(L), L)$, *a section* $s \in H^0(L)$ *extends to first order along* ξ *if and only if* ξ *and* s *pair to zero under* p.

Proof. (1) Given a first-order deformation of the pair (X, L), i.e. the following fibered diagram with $\mathcal{O}_{\mathcal{X}}$ flat over $Spec(\mathbb{C}[\epsilon])$ and $\mathcal{L}$ line bundle on $\mathcal{X}$:

$$
\begin{array}{ccc}
L & \hookrightarrow & \mathcal{L} \\
\downarrow & & \downarrow \\
X & \hookrightarrow & \mathcal{X} \\
\downarrow & & \downarrow \\
Spec(\mathbb{C}) & \hookrightarrow & Spec(\mathbb{C}[\epsilon])
\end{array}
$$

We have a diagram of (left) $\mathcal{O}_X$-modules:

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$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & L & \longrightarrow & \Omega^1_{\mathcal{X}}(\mathcal{L})|_X & \longrightarrow & \Omega^1_X(L) & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow{\scriptstyle i} & & \\
0 & \longrightarrow & L & \longrightarrow & \mathcal{P}^1_{\mathcal{X}}(\mathcal{L})|_X & \overset{r}{\longrightarrow} & \mathcal{P}^1_X(L) & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \mathcal{L}|_X & = & L & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}
$$

The two right columns are exact by (2.2), and the fact that restriction to X is exact. (left exactness of restriction follows from the fact that $\mathcal{T}or^1_{\mathcal{O}_{\mathcal{X}}}(\mathcal{L}, \mathcal{O}_X) = 0$, since $\mathcal{L}$ is a locally free $\mathcal{O}_{\mathcal{X}}$-module!) The first row is the conormal sequence of $X \subset \mathcal{X}$ twisted by L, which is exact. Thus by Snake Lemma, $ker(r) = L$ and the second row is exact . Therefore we can associate any first-order deformation of the pair (X, L) the second row exact sequence, which corresponds to an element of $\mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{P}^1_X(L), L)$.

Now consider the commutative diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & L & \longrightarrow & \mathcal{L} & \longrightarrow & L & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle p'_1} & & \downarrow{\scriptstyle p_1} & & \\
0 & \longrightarrow & L & \longrightarrow & \mathcal{P}^1_{\mathcal{X}}(\mathcal{L})|_X & \overset{r}{\longrightarrow} & \mathcal{P}^1_X(L) & \longrightarrow & 0
\end{array}
$$

where p'_1 is the composition of $\tilde{p}_1 : \mathcal{L} \to \mathcal{P}^1_{\mathcal{X}}(\mathcal{L})$ and the restriction map to X. Thus p'_1 factors through $L \times_{\mathcal{P}^1_X(L)} \mathcal{P}^1_{\mathcal{X}}(\mathcal{L})|_X$ and therefore $\mathcal{L} \cong L \times_{\mathcal{P}^1_X(L)} \mathcal{P}^1_{\mathcal{X}}(\mathcal{L})|_X$ as a $\mathcal{O}_X$-module. We can give $L \times_{\mathcal{P}^1_X(L)} \mathcal{P}^1_{\mathcal{X}}(\mathcal{L})|_X$ a $\mathcal{O}_{\mathcal{X}}$-module structure via

this isomorphism (This $\mathcal{O}_{\mathcal{X}}$-module structure turns out to be given by formula (2.6)). This fact suggests that we can recover $\mathcal{L}$ from $\mathcal{P}^1_{\mathcal{X}}(\mathcal{L})|_X$ and L.

Conversely, for any element $\xi \in \mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{P}^1_X(L), L)$ corresponding to an $\mathcal{O}_X$-module extension:

$$0 \longrightarrow L \longrightarrow \mathcal{E} \xrightarrow{\ r\ } \mathcal{P}^1_X(L) \longrightarrow 0.$$

The pull back extension $\mathcal{E}' = \mathcal{E} \times_{\mathcal{P}^1_X(L)} \Omega^1_X(L)$ by the natural inclusion

$$i : \Omega^1_X(L) \longrightarrow \mathcal{P}^1_X(L),$$

sits naturally in the diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & L & \longrightarrow & \mathcal{E}' = \mathcal{E} \times_{\mathcal{P}^1_X(L)} \Omega^1_X(L) & \longrightarrow & \Omega^1_X(L) & \longrightarrow & 0 \\
& & \big\| & & \downarrow{\scriptstyle i'} & & \downarrow{\scriptstyle i} & & \\
0 & \longrightarrow & L & \longrightarrow & \mathcal{E} & \xrightarrow{\ r\ } & \mathcal{P}^1_X(L) & \longrightarrow & 0
\end{array}
$$

The first row exact sequence corresponds to an element in $\mathrm{Ext}^1_{\mathcal{O}_X}(\Omega^1_X(L), L) = \mathrm{Ext}^1_{\mathcal{O}_X}(\Omega^1_X, \mathcal{O}_X)$, which corresponds to a first-order infinitesimal deformation $\mathcal{X}$ of X as described in the beginning of this section.

To recover the deformation of L, let $\mathcal{E}'' = \mathcal{E}' \otimes L^{-1}$ and let

$$\mathcal{L} = L \times_{\mathcal{P}^1_X(L)} \mathcal{E} = \{(s, e) \in L \oplus \mathcal{E} \,|\, p_1(s) = r(e)\}.$$

$\mathcal{L}$ sits naturally in the diagram

$$0 \longrightarrow L \longrightarrow \mathcal{L} = L \times_{\mathcal{P}_X^1(L)} \mathcal{E} \longrightarrow L \longrightarrow 0$$

(diagram with vertical maps: identity on left, a map in the middle, p_1 on the right, down to)

$$0 \longrightarrow L \longrightarrow \mathcal{E} \xrightarrow{\ r\ } \mathcal{P}_X^1(L) \longrightarrow 0$$

and has a natural $\mathcal{O}_{\mathcal{X}}$-module structure (recall that $\mathcal{O}_{\mathcal{X}} = \mathcal{O}_X \times_{\Omega_X^1} \mathcal{E}''$ by the construction in the beginning of this section) as below

$$(f, a)(s, e) = (fs, fe + i'(a \cdot s)) \tag{2.6}$$

where $(f, a) \in \mathcal{O}_X \times_{\Omega_X^1} \mathcal{E}'' = \mathcal{O}_{\mathcal{X}}$, $(s, e) \in \mathcal{L} = L \times_{\mathcal{P}_X^1(L)} \mathcal{E}$ and $a \cdot s \in \mathcal{E}'$. This is a well defined $\mathcal{O}_{\mathcal{X}}$-module structure because

$$p_1(fs) = i(df \otimes s) + fp_1(s) = r(i'(a \cdot s)) + fr(e).$$

In order to see $\mathcal{L}$ is a locally free $\mathcal{O}_{\mathcal{X}}$-module of rank one, it suffices to prove the case L is the trivial bundle since the question is local. In this case, (2.2) splits (as left $\mathcal{O}_X$-module) and $\mathcal{P}_X^1(\mathcal{O}_X) \cong \mathcal{O}_X \oplus \Omega_X^1$. The statement follows immediately from this.

(2) For any $\xi \in \mathrm{Ext}_{\mathcal{O}_X}^1(\mathcal{P}_X^1(L), L)$ corresponding to the extension

$$0 \longrightarrow L \longrightarrow \mathcal{P}_{\mathcal{X}}^1(\mathcal{L})|_X \longrightarrow \mathcal{P}_X^1(L) \longrightarrow 0.$$

Define the natural pairing $p(\xi \otimes s) := \delta(p_1(s)) \in H^1(L)$. Where $\delta : H^0(\mathcal{P}_X^1(L)) \to H^1(L)$ is the connecting homomorphism of the long exact cohomology sequence corresponding to ξ:

$$\cdots \longrightarrow H^0(\mathcal{P}^1_{\mathcal{X}}(\mathcal{L})|_X) \xrightarrow{\;r\;} H^0(\mathcal{P}^1_X(L)) \xrightarrow{\;\delta\;} H^1(L) \longrightarrow \cdots$$

$\delta(p_1(s)) = 0$ means there exists some $e \in H^0(\mathcal{P}^1_{\mathcal{X}}(\mathcal{L})|_X)$ such that $r(e) = p_1(s)$, thus (s, e) determines a global section of $\mathcal{L} = L \times_{\mathcal{P}^1_X(L)} \mathcal{P}^1_{\mathcal{X}}(\mathcal{L})|_X$.

$\square$

2.4 Obstructions

In this section, let X be as in section 2.3 and we assume furthermore that X is a local complete intersection scheme. We will show that $\mathrm{Ext}^2_{\mathcal{O}_X}(\mathcal{P}^1_X(L), L)$ is an obstruction space for deformations of the pair (X, L).

The general idea is to apply Vistoli's construction of obstruction spaces for deformations of l.c.i schemes (cf. sections 3, 4 of [35]) to the total space of $L^\vee$ and keep track of the bundle structure using a $\mathbb{C}^*$-action.

For any $z \in \mathbb{C}^*$, denote $\phi_z : L^\vee \to L^\vee$ be the multiplication map by z in the fiber direction. Define a $\mathbb{C}^*$-action on $\mathcal{O}_{L^\vee}$ and $\Omega^1_{L^\vee}$ by

$$z \cdot f = z^{-1}\phi_z^* f \tag{2.7}$$

$$z \cdot \omega = z^{-1}\phi_z^* \omega \tag{2.8}$$

for local sections $f \in \mathcal{O}_{L^\vee}$, $\omega \in \Omega^1_{L^\vee}$.

Let $\mathcal{O}^{\mathbb{C}^*}_{L^\vee}$ and $\Omega^{\mathbb{C}^*}_{L^\vee}$ be the sheaf of sections which are invariant under the $\mathbb{C}^*$-action. Under some trivialization of $L^\vee$ over $U \subset X$: $L^\vee_U \cong U \times \mathbb{A}^1_t$, $\mathcal{O}^{\mathbb{C}^*}_{L^\vee}$ consists of functions on $L^\vee$ of the form $f(x)t$, and $\Omega^{\mathbb{C}^*}_{L^\vee}$ consists of 1-forms $f(x)d_{L^\vee}t + \omega(x)t$ where f is the pull back of a function on U and $\omega \in \Omega^1_U$.

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Both $\mathcal{O}_{L^\vee}^{\mathbb{C}^*}$ and $\Omega_{L^\vee}^{\mathbb{C}^*}$ have natural $\mathcal{O}_X$-module structures and we have natural isomorphisms of $\mathcal{O}_X$-modules $\mathcal{O}_{L^\vee}^{\mathbb{C}^*} \cong L$ and $\mathcal{P}_X^1(L) \cong \Omega_{L^\vee}^{\mathbb{C}^*}$. The isomorphisms can be described as follows: for any section $s \in L$, we can naturally view it as a function on the total space of $L^\vee$ which restricts to a linear function on the fiber. Such functions are invariant under the $\mathbb{C}^*$-action and vice versa. This gives the first isomorphism. The second isomorphism is the natural one which identifies $p_1(s)$ with $d_{L^\vee}(f_s)$, where s is any section of L, f_s is the function on $L^\vee$ corresponding to s and $d_{L^\vee}$ is the exterior derivative on $L^\vee$. Under some local trivialization of $L^\vee$ over U, it sends $(f, \omega) \in \mathcal{P}_X^1(L)|_U \cong \mathcal{O}_X(U) \oplus \Omega_X(U)$ to $f(x)d_{L^\vee}t + \omega(x)t \in \Omega_{L^\vee}^{\mathbb{C}^*}$.

Let

$$e : \quad 0 \longrightarrow J \longrightarrow \widetilde{A} \longrightarrow A \longrightarrow 0$$

be a small extension of local artinian $\mathbb{C}$-algebras with $m_{\widetilde{A}} \cdot J = 0$. Suppose we have a flat deformation $(\mathcal{X}, \mathcal{L})$ of the pair (X, L) over $Spec(A)$:

$$
\begin{array}{ccc}
L & \longrightarrow & \mathcal{L} \\
\downarrow & & \downarrow \\
X & \longrightarrow & \mathcal{X} \\
\downarrow & & \downarrow{\scriptstyle f} \\
Spec(\mathbb{C}) & \longrightarrow & Spec(A)
\end{array}
$$

Let $(\widetilde{\mathcal{X}}_\alpha, \widetilde{\mathcal{L}}_\alpha)$ and $(\widetilde{\mathcal{X}}_\beta, \widetilde{\mathcal{L}}_\beta)$ be two liftings of $(\mathcal{X}, \mathcal{L})$ to $Spec(\widetilde{A})$. We would like to measure the difference of two such liftings.

Let's restrict ourselves to the local situation first. Suppose that $\mathcal{X}$ is affine, embedded in $\mathcal{S} = Spec(A[x_1, ..., x_n])$ and the total space of $\widetilde{\mathcal{L}}_i^\vee$ are both embedded into $Spec(\widetilde{A}[x_1, ..., x_n]) \times \mathbb{A}^1 = \widetilde{\mathcal{S}} \times \mathbb{A}^1$ with image $\widetilde{\mathcal{X}}_i \times \mathbb{A}^1$.

Let $\mathcal{I}_0$ be the ideal sheaf of $L^\vee$ in $S \times \mathbb{A}^1$. The conormal sequence

$$0 \longrightarrow \frac{\mathcal{I}_0}{\mathcal{I}_0^2} \xrightarrow{\;d\;} \Omega_{S \times \mathbb{A}^1}|_{L^\vee} \longrightarrow \Omega_{L^\vee} \longrightarrow 0$$

is exact because $L^\vee$ is l.c.i. Taking the invariant part under the $\mathbb{C}^*$-action we get an exact sequence of $\mathcal{O}_X$-modules

$$0 \longrightarrow \left(\frac{\mathcal{I}_0}{\mathcal{I}_0^2}\right)^{\mathbb{C}^*} \xrightarrow{\;d'\;} \left(\Omega_{S \times \mathbb{A}^1}|_{L^\vee}\right)^{\mathbb{C}^*} \longrightarrow \Omega_{L^\vee}^{\mathbb{C}^*} \longrightarrow 0 \ . \tag{2.9}$$

The difference of $\widetilde{\mathcal{L}}_\alpha^\vee$ and $\widetilde{\mathcal{L}}_\beta^\vee$ as embedded deformations corresponds to an $\mathcal{O}_{L^\vee}$-module homomorphism $v_{\alpha\beta} : \frac{\mathcal{I}_0}{\mathcal{I}_0^2} \to J \otimes_{\mathbb{C}} \mathcal{O}_{L^\vee}$. The fact that $\widetilde{\mathcal{L}}_i^\vee$ is embedded as $\widetilde{\mathcal{X}}_i \times \mathbb{A}^1$ implies that $v_{\alpha\beta}$ sends the invariant part $\left(\frac{\mathcal{I}_0}{\mathcal{I}_0^2}\right)^{\mathbb{C}^*}$ to the invariant part $J \otimes_{\mathbb{C}} \mathcal{O}_{L^\vee}^{\mathbb{C}^*} = J \otimes_{\mathbb{C}} L$. Denote the restriction $v'_{\alpha\beta}$.

Now, take the push-out of (2.9) under $v'_{\alpha\beta}$, we obtain an $\mathcal{O}_X$-module extension $\mathcal{E}_{\alpha\beta}$ of $\mathcal{P}_X^1(L)$ by $J \otimes_{\mathbb{C}} L$:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \left(\frac{\mathcal{I}_0}{\mathcal{I}_0^2}\right)^{\mathbb{C}^*} & \xrightarrow{\;d'\;} & \left(\Omega_{S \times \mathbb{A}^1}|_{L^\vee}\right)^{\mathbb{C}^*} & \longrightarrow & \Omega_{L^\vee}^{\mathbb{C}^*} & \longrightarrow & 0 \\
& & \downarrow{\scriptstyle v'_{\alpha\beta}} & & \downarrow{\scriptstyle \psi_{\alpha\beta}} & & \| & & \\
0 & \longrightarrow & J \otimes_{\mathbb{C}} L & \xrightarrow{\;l_{\alpha\beta}\;} & \mathcal{E}_{\alpha\beta} & \longrightarrow & \mathcal{P}_X^1(L) & \longrightarrow & 0
\end{array} \tag{2.10}$$

Lemma 2.2. *The extension $\mathcal{E}_{\alpha\beta}$ does not depend on the choice of $\widetilde{\mathcal{S}}$.*

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Proof. Suppose there are embeddings $\widetilde{\mathcal{L}}_i^\vee \to \widetilde{\mathcal{S}}_j \times \mathbb{A}^1$, where $i = \alpha, \beta$, $j = 1, 2$; reducing to embeddings $\mathcal{L}^\vee \to \mathcal{S}_1 \times \mathbb{A}^1$ and $\mathcal{L}^\vee \to \mathcal{S}_2 \times \mathbb{A}^1$. These induce embeddings

$$\widetilde{\mathcal{L}}_i^\vee \to \widetilde{\mathcal{S}}_1 \times_{Spec(\widetilde{A})} \widetilde{\mathcal{S}}_2 \times \mathbb{A}^1$$

reducing to

$$\mathcal{L}^\vee \to \mathcal{S}_1 \times_{Spec(A)} \mathcal{S}_2 \times \mathbb{A}^1.$$

Let C_1, C_2, C_{12} be the conormal bundles of $L^\vee$ in $S_1 \times \mathbb{A}^1$, $S_2 \times \mathbb{A}^1$, $S_1 \times S_2 \times \mathbb{A}^1$ respectively. Denote by $v_j' : C_j^{\mathbb{C}^*} \to J \otimes_\mathbb{C} L$ the invariant part of the corresponding sections of the normal bundles, $\mathcal{E}_j = v_{j*}'(\Omega_{S_j \times \mathbb{A}^1}|_{L^\vee})^{\mathbb{C}^*}$, $\mathcal{E}_{12} = v_{12*}'(\Omega_{S_1 \times S_2 \times \mathbb{A}^1}|_{L^\vee})^{\mathbb{C}^*}$ and $p_j : C_j^{\mathbb{C}^*} \to C_{12}^{\mathbb{C}^*}$ be the natural map between conormal bundles. Then

$$v_{12}' \circ p_j = v_j' : C_j^{\mathbb{C}^*} \to J \otimes_\mathbb{C} L.$$

We have the following diagram

By the universal property of push out, this diagram induces isomorphism of extensions $\psi_j : \mathcal{E}_j \cong \mathcal{E}_{12}$. We define the canonical isomorphism between $\mathcal{E}_2$ and $\mathcal{E}_1$ to be $\psi_1 \circ \psi_2^{-1}$. $\square$

Proposition 2.3. *For any two liftings of line bundles $\widetilde{\mathcal{L}}_\alpha^\vee$, $\widetilde{\mathcal{L}}_\beta^\vee$ inside $\widetilde{\mathcal{S}} \times \mathbb{A}^1$ as above, there is an $\mathcal{O}_X$-module extension $\mathcal{E}_{\alpha\beta}$ of $\mathcal{P}_X^1(L)$ by $J \otimes_\mathbb{C} L$, well defined up to canonical isomorphism, with the following properties.*

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(a) For any three liftings $\widetilde{\mathcal{L}}_\alpha^\vee$, $\widetilde{\mathcal{L}}_\beta^\vee$, and $\widetilde{\mathcal{L}}_\gamma^\vee$, there is a canonical isomorphism of extensions

$$F_{\alpha\beta\gamma} : \mathcal{E}_{\alpha\beta} + \mathcal{E}_{\beta\gamma} \cong \mathcal{E}_{\alpha\gamma}{}^1$$

such that for any four liftings,

$$F_{\alpha\gamma\delta} \circ (F_{\alpha\beta\gamma} + id_{\mathcal{E}_{\beta\gamma}}) = F_{\alpha\beta\delta} \circ (id_{\mathcal{E}_{\alpha\beta}} + F_{\beta\gamma\delta}) \tag{2.11}$$

as homomorphism of extensions from $\mathcal{E}_{\alpha\beta} + \mathcal{E}_{\beta\gamma} + \mathcal{E}_{\gamma\delta}$ to $\mathcal{E}_{\alpha\delta}$.

(b) Given an $\mathcal{O}_X$-module extension $\mathcal{E}$ of $\mathcal{P}_X^1(L)$ by $J \otimes_{\mathbb{C}} L$, and a lifting $\widetilde{\mathcal{L}}_\alpha^\vee$ of $\mathcal{L}^\vee$, there is an abstract lifting $\widetilde{\mathcal{L}}_\beta^\vee$ such that $\mathcal{E}_{\alpha\beta}$ is isomorphic to $\mathcal{E}$.

(c) There is a natural bijection between bundle isomorphisms $\Phi : \widetilde{\mathcal{L}}_\alpha^\vee \cong \widetilde{\mathcal{L}}_\beta^\vee$ with splittings of $\mathcal{E}_{\alpha\beta}$.

Proof. (a) As embedded deformations we certainly have $v'_{\alpha\beta} + v'_{\beta\gamma} = v'_{\alpha\gamma}$ as homomorphisms from $\left(\frac{\mathcal{I}_0}{\mathcal{I}_0^2}\right)^{\mathbb{C}^*}$ to $J \otimes_{\mathbb{C}} L$ Then $\mathcal{E}_{\alpha\beta} + \mathcal{E}_{\beta\gamma}$ fits into the diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \left(\frac{\mathcal{I}_0}{\mathcal{I}_0^2}\right)^{\mathbb{C}^*} & \xrightarrow{d'} & (\Omega_{S\times\mathbb{A}^1}|_{L^\vee})^{\mathbb{C}^*} & \longrightarrow & \Omega_{L^\vee}^{\mathbb{C}^*} & \longrightarrow & 0 \\
& & \Big\downarrow{\scriptstyle v'_{\alpha\beta}+v'_{\beta\gamma}} & & \Big\downarrow{\scriptstyle (\psi_{\alpha\beta},\psi_{\beta\gamma})} & & \Big\| & & \\
0 & \longrightarrow & J \otimes_{\mathbb{C}} L & \longrightarrow & \mathcal{E}_{\alpha\beta} + \mathcal{E}_{\beta\gamma} & \longrightarrow & \mathcal{P}_X^1(L) & \longrightarrow & 0
\end{array}
$$

By the universal property of push-out, there is a unique isomorphism $F_{\alpha\beta\gamma}^{-1} : \mathcal{E}_{\alpha\gamma} \to \mathcal{E}_{\alpha\beta} + \mathcal{E}_{\beta\gamma}$ such that $(\psi_{\alpha\beta},\psi_{\beta\gamma})$ factors through $F_{\alpha\beta\gamma}^{-1}$. The compatibility condition (2.11) follows from the universal property of push-out as well.

---

[1]The sum of two extensions of $\mathcal{O}_X$-module $0 \longrightarrow \mathcal{G} \xrightarrow{l_i} \mathcal{E}_i \xrightarrow{k_i} \mathcal{F} \longrightarrow 0$ is defined to be the quotient of the submodule $\mathcal{B} = \{(e_1, e_2) \in \mathcal{E}_1 \oplus \mathcal{E}_2 : k_1(e_1) = k_2(e_2)\}$ by sections of the form $(l_1(y), -l_2(y))$, $y \in \mathcal{G}$.

The oposite extension $-\mathcal{E}$ is defined to be $0 \longrightarrow \mathcal{G} \xrightarrow{-l} \mathcal{E} \xrightarrow{k} \mathcal{F} \longrightarrow 0$.

(b) Applying the derived functor $\mathrm{Hom}_{\mathcal{O}_X}(-, J \otimes_{\mathbb{C}} L)$ to (2.9), we obtain exact sequence

$$\mathrm{Hom}_{\mathcal{O}_X}((\tfrac{\mathcal{I}_0}{\mathcal{I}_0^2})^{\mathbb{C}^*}, J \otimes_{\mathbb{C}} L) \longrightarrow \mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{P}^1_X(L), J \otimes_{\mathbb{C}} L) \longrightarrow \mathrm{Ext}^1_{\mathcal{O}_X}((\Omega_{S \times \mathbb{A}^1}|_{L^\vee})^{\mathbb{C}^*}, J \otimes_{\mathbb{C}} L)$$

where the last term is zero because X is affine and $(\Omega_{S \times \mathbb{A}^1}|_{L^\vee})^{\mathbb{C}^*}$ is locally free. Thus for any

$$\mathcal{E} \in \mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{P}^1_X(L), J \otimes_{\mathbb{C}} L),$$

there is

$$v' \in \mathrm{Hom}_{\mathcal{O}_X}((\frac{\mathcal{I}_0}{\mathcal{I}_0^2})^{\mathbb{C}^*}, J \otimes_{\mathbb{C}} L)$$

such that $v'_*(\Omega_{S \times \mathbb{A}^1}|_{L^\vee})^{\mathbb{C}^*} \cong \mathcal{E}$. v' can be uniquely extended to a $\mathcal{O}_{L^\vee}$-module homomorphism $v : \frac{\mathcal{I}_0}{\mathcal{I}_0^2} \to J \otimes_{\mathbb{C}} \mathcal{O}_{L^\vee}$. Now choose $\widetilde{\mathcal{L}}^\vee_\beta \subset \widetilde{\mathcal{S}} \times \mathbb{A}^1$ such that the difference of $\widetilde{\mathcal{L}}^\vee_\beta$ and $\widetilde{\mathcal{L}}^\vee_\alpha$ as embedded deformations corresponds to v, then by construction $\mathcal{E}_{\alpha\beta} \cong \mathcal{E}$.

(c) First notice that by the construction of push-out, to give a splitting $s : \mathcal{E}_{\alpha\beta} \to J \otimes_{\mathbb{C}} L$ is equivalent to give a $\mathcal{O}_X$-module homomorphism $D : (\Omega_{S \times \mathbb{A}^1}|_{L^\vee})^{\mathbb{C}^*} \to J \otimes_{\mathbb{C}} L$ such that $D \circ d' = v'_{\alpha\beta}$.

Now let $\phi : \mathcal{O}_{\widetilde{\mathcal{L}}^\vee_\alpha} \cong \mathcal{O}_{\widetilde{\mathcal{L}}^\vee_\beta}$ be a bundle isomorphism inducing identity on $\mathcal{O}_{\mathcal{L}^\vee}$. Consider the two projections $\pi_i : \mathcal{O}_{\widetilde{\mathcal{S}} \times \mathbb{A}^1} \to \mathcal{O}_{\widetilde{\mathcal{L}}^\vee_i}$. The difference

$$D = \pi_\beta - \phi \circ \pi_\alpha : \mathcal{O}_{\widetilde{\mathcal{S}} \times \mathbb{A}^1} \to \mathcal{O}_{\widetilde{\mathcal{L}}^\vee_\beta}$$

will have its image inside $J\mathcal{O}_{\widetilde{\mathcal{L}}^\vee_\beta} = J \otimes_{\mathbb{C}} \mathcal{O}_{L^\vee}$. It is easy to check that

$$D \in Der_{\widetilde{A}}(\mathcal{O}_{\widetilde{\mathcal{S}} \times \mathbb{A}^1}, J \otimes_{\mathbb{C}} \mathcal{O}_{L^\vee}) = Der_{\mathbb{C}}(\mathcal{O}_{S \times \mathbb{A}^1}, J \otimes_{\mathbb{C}} \mathcal{O}_{L^\vee}) \cong \mathrm{Hom}_{\mathcal{O}_{L^\vee}}(\Omega_{S \times \mathbb{A}^1}|_{L^\vee}, J \otimes_{\mathbb{C}} \mathcal{O}_{L^\vee})$$

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and $D \circ d = v_{\alpha\beta}$. The fact that ϕ is a bundle isomorphism implies that D sends

$(\Omega_{S \times \mathbb{A}^1}|_{L^\vee})^{\mathbb{C}^*}$ to $J \otimes_{\mathbb{C}} \mathcal{O}_{L^\vee}^{\mathbb{C}^*} = J \otimes_{\mathbb{C}} L$. This gives a spliting of $\mathcal{E}_{\alpha\beta}$.

Conversely, any $\mathcal{O}_X$-module homomorphism

$$D : (\Omega_{S \times \mathbb{A}^1}|_{L^\vee})^{\mathbb{C}^*} \to J \otimes_{\mathbb{C}} \mathcal{O}_{L^\vee}^{\mathbb{C}^*}$$

with $D \circ d' = v'_{\alpha\beta}$ can be extended uniquely to a $\mathcal{O}_{L^\vee}$-module homomorphism $D : \Omega_{S \times \mathbb{A}^1}|_{L^\vee} \to J \otimes_{\mathbb{C}} \mathcal{O}_{L^\vee}$ with $D \circ d = v_{\alpha\beta}$. Now consider

$$\pi_\beta - D \circ d : \mathcal{O}_{\widetilde{\mathcal{S}} \times \mathbb{A}^1} \longrightarrow \mathcal{O}_{\widetilde{\mathcal{L}}_\beta^\vee}$$

If $\tilde{f}$ is a local function on $\widetilde{\mathcal{S}} \times \mathbb{A}^1$ which vanishes on $\widetilde{\mathcal{L}}_\alpha^\vee$ and f be its restriction to $S \times \mathbb{A}^1$, then

$$\pi_\beta(\tilde{f}) - (D \circ d)f = \pi_\beta(\tilde{f}) - v_{\alpha\beta}(f)$$

is zero in $\mathcal{O}_{\widetilde{\mathcal{L}}_\beta^\vee}$ by the construction of $v_{\alpha\beta}$.

Thus $\pi_\beta - (D \circ d)$ factors through π_α, and therefore we recover the bundle isomorphism ϕ from such D.

$\square$

Remark. Proposition 2.3 still holds in the global case. Since the local extension does not depending on the choice of embeddings, one can construct a global extension for any two abstract liftings $\widetilde{\mathcal{L}}_2^\vee$ and $\widetilde{\mathcal{L}}_1^\vee$ by glueing together the local extensions using the canonical isomorphisms in lemma 2.2 on the overlap of two open affine subsets.

One checks easily that the glued extension satisfies the properties in the proposition. We will not need the global case in the construction of the obstruction space.

$\square$

The rest of the proof is entirely based on the construction in [35]. The idea is to use extension cocycles to measure the obstructions to patching together local liftings (which always exist since X is l.c.i) coherently.

Here we collect some useful results about extension cocycles and refer to [35] for details.

Definition 2.4. *Let $\mathcal{F}$, $\mathcal{G}$ be sheaves of $\mathcal{O}_X$-modules, $\{U_\alpha\}$ be an open covering of X. An extension cocycle*

$$(\{\mathcal{E}_{\alpha\beta}\}, \{F_{\alpha\beta\gamma}\})$$

of $\mathcal{F}$ by $\mathcal{G}$ on $\{U_\alpha\}$ is a collection of extensions $\{\mathcal{E}_{\alpha\beta}\}$ of $\mathcal{F}|_{U_{\alpha\beta}}$ by $\mathcal{G}|_{U_{\alpha\beta}}$, and isomorphisms

$$F_{\alpha\beta\gamma} : \mathcal{E}_{\alpha\beta} + \mathcal{E}_{\beta\gamma} \cong \mathcal{E}_{\alpha\gamma}$$

on $U_{\alpha\beta\gamma}$ satisfying the compatibility condition as in (2.11).

Two extension cocycles $(\{\mathcal{E}_{\alpha\beta}\}, \{F_{\alpha\beta\gamma}\})$, $(\{\mathcal{E}'_{\alpha\beta}\}, \{F'_{\alpha\beta\gamma}\})$ are isomorphic if there exist isomorphism of extensions

$$\phi_{\alpha\beta} : \mathcal{E}_{\alpha\beta} \cong \mathcal{E}'_{\alpha\beta}$$

such that

$$\phi_{\alpha\gamma} \circ F_{\alpha\beta\gamma} = F'_{\alpha\beta\gamma} \circ (\phi_{\alpha\beta} + \phi_{\beta\gamma}).$$

Definition 2.5. *We say an extension cocycle is a boundary if it is isomorphic to*

$$\partial\{\mathcal{E}_\alpha\} = (\{\mathcal{E}_\alpha - \mathcal{E}_\beta\}, F_{\alpha\beta\gamma})$$

for a collection of extensions $\{\mathcal{E}_\alpha\}$ *of* $\mathcal{F}|_{U_\alpha}$ *by* $\mathcal{G}|_{U_\alpha}$*, where*

$$F_{\alpha\beta\gamma} : \mathcal{E}_\alpha - \mathcal{E}_\beta + \mathcal{E}_\beta - \mathcal{E}_\gamma \longrightarrow \mathcal{E}_\alpha - \mathcal{E}_\gamma$$

is the obvious isomorphism.

The set of isomorphism classes of extension cocycles form an abelian group, and the boundaries form a subgroup. The quotient group is called the group of extension classes, and is denoted by $\Xi_{\mathcal{O}_X}(U_\alpha; \mathcal{F}, \mathcal{G})$. We refer to section 3 in [35] for the proofs of the above facts.

The following theorem is taken from theorem (3.13) of [35]. For the convenience of the reader, we sketch the proof here.

Theorem 2.6. *For* $\{U_\alpha\}$ *a good cover, there is canonical group isomorphism of* $\Xi_{\mathcal{O}_X}(U_\alpha; \mathcal{F}, \mathcal{G})$ *with the kernel of the localization map* $\text{Ext}^2_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) \to H^0(X, \mathcal{E}xt^2(\mathcal{F}, \mathcal{G}))$.

Proof. Let $\mathcal{J}$ be an injective sheaf of $\mathcal{O}_X$-modules containing $\mathcal{G}$ and $\mathcal{Q}$ be the quotient.

$$0 \longrightarrow \mathcal{G} \xrightarrow{j} \mathcal{J} \xrightarrow{\pi} \mathcal{Q} \longrightarrow 0.$$

Then the boundary map

$$\text{Ext}^1_{\mathcal{O}_X}(\mathcal{F}, \mathcal{Q}) \xrightarrow{\partial} \text{Ext}^2_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$$

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is an isomorphism and we have a commutative diagram

$$\begin{array}{ccc} \text{Ext}^1_{\mathcal{O}_X}(\mathcal{F}, \mathcal{Q}) & \xrightarrow{\ \cong\ } & \text{Ext}^2_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) \\ \downarrow & & \downarrow \\ H^0(X, \mathcal{E}xt^1_{\mathcal{O}_X}(\mathcal{F}, \mathcal{Q})) & \xrightarrow{\ \cong\ } & H^0(X, \mathcal{E}xt^2_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})) \end{array}$$

where the vertical arrows are localization maps. Hence the kernel of the left column is isomorphic to the kernel of the right column. But from the local-to-global spectral sequence, we get an exact sequence

$$0 \longrightarrow H^1(X, \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{Q})) \longrightarrow \text{Ext}^1_{\mathcal{O}_X}(\mathcal{F}, \mathcal{Q}) \longrightarrow H^0(X, \mathcal{E}xt^1_{\mathcal{O}_X}(\mathcal{F}, \mathcal{Q}))$$

Thus theorem follows from lemma 2.7. $\square$

Lemma 2.7. *There is a canonical isomorphism*

$$\Xi_{\mathcal{O}_X}(U_\alpha; \mathcal{F}, \mathcal{G}) \cong \check{H}^1(U_\alpha, \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{Q})).$$

Proof. Let $(\{\mathcal{E}_{\alpha\beta}\}, \{F_{\alpha\beta\gamma}\})$ be an isomorphism class of extension cocycles of $\mathcal{F}$ by $\mathcal{G}$. Since $\mathcal{J}$ is injective, we can we can find a homomorphism $\sigma_{\alpha\beta} : \mathcal{E}_{\alpha\beta} \to \mathcal{J}$ such that the diagram on $U_{\alpha\beta}$

$$
\begin{array}{ccccccccc}
& & 0 & & & & & & \\
& & \downarrow & & & & & & \\
0 & \longrightarrow & \mathcal{G} & \xrightarrow{\ j\ } & \mathcal{J} & \xrightarrow{\ \pi\ } & \mathcal{Q} & \longrightarrow & 0 \\
& & \;\downarrow{\scriptstyle l_{\alpha\beta}} & \;\nearrow{\scriptstyle \sigma_{\alpha\beta}} & & & & & \\
& & \mathcal{E}_{\alpha\beta} & & & & & & \\
& & \;\downarrow{\scriptstyle k_{\alpha\beta}} & & & & & & \\
& & \mathcal{F} & & & & & & \\
& & \downarrow & & & & & & \\
& & 0 & & & & & &
\end{array}
$$

is commutative. We claim that we can do this coherently in the sense that

$$\sigma_{\alpha\beta} + \sigma_{\beta\gamma} = \sigma_{\alpha\gamma} \circ F_{\alpha\beta\gamma} : \mathcal{E}_{\alpha\beta} + \mathcal{E}_{\beta\gamma} \longrightarrow \mathcal{J} \tag{2.12}$$

where $\sigma_{\alpha\beta} + \sigma_{\beta\gamma}$ sends $(e_1, e_2) \in \mathcal{E}_{\alpha\beta} + \mathcal{E}_{\beta\gamma}$ to $\sigma_{\alpha\beta}(e_1) + \sigma_{\alpha\beta}(e_2)$.

Choose arbitrary homomorphisms $\sigma_{\alpha\beta}$ such that $\sigma_{\alpha\beta} \circ l_{\alpha\beta} = j$ and consider

$$\tau_{\alpha\beta\gamma} = (\sigma_{\alpha\beta} + \sigma_{\beta\gamma}) - \sigma_{\alpha\gamma} \circ F_{\alpha\beta\gamma} : \mathcal{F} \longrightarrow \mathcal{J}.$$

It is easy to check that $\{\tau_{\alpha\beta\gamma}\}$ is a Čech 2-cocycle in $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{J})$. Since $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{J})$ is flasque and thus has no higher cohomology. Hence we can find 1-cochain $\{\tau_{\alpha\beta}\}$ such that $\tau_{\alpha\beta\gamma} = \tau_{\alpha\beta} - \tau_{\alpha\gamma} + \tau_{\beta\gamma}$. If we set

$$\widetilde{\sigma}_{\alpha\beta} = \sigma_{\alpha\beta} + \tau_{\alpha\beta}$$

we see easily that $\widetilde{\sigma}_{\alpha\beta}$ satisfies the coherence condition (2.12). Claim is proved.

Now we describe a homomorphism from $\Xi_{\mathcal{O}_X}(U_\alpha; \mathcal{F}, \mathcal{G})$ to $\check{H}^1(U_\alpha, \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{Q}))$ as below. Choose $\sigma_{\alpha\beta}$ satisfying the coherence condition (2.12). Then

$$\pi \circ \sigma_{\alpha\beta} : \mathcal{E}_{\alpha\beta} \to \mathcal{Q}$$

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sends $\mathcal{G}$ to zero and therefore induces $\eta_{\alpha\beta} : \mathcal{F} \to \mathcal{Q}$ satisfying $\eta_{\alpha\beta} + \eta_{\beta\gamma} = \eta_{\alpha\gamma}$. So we have associated to the extension cocycle $(\{\mathcal{E}_{\alpha\beta}\}, \{F_{\alpha\beta\gamma}\})$ an element $\{\eta_{\alpha\beta}\} \in \check{H}^1(U_\alpha, \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{Q}))$. This does not depend on the choice of $\sigma_{\alpha\beta}$. If $\{\eta_{\alpha\beta}\}$ represent zero class, let $\eta_{\alpha\beta} = \eta_\beta - \eta_\alpha$ and set $\mathcal{E}_\alpha = \eta_\alpha^* \mathcal{J}$. Here we view $\mathcal{J}$ as an extension of $\mathcal{Q}$ by $\mathcal{G}$. Since $\mathcal{E}_{\alpha\beta}$ is isomorphic to $\eta_{\alpha\beta}^* \mathcal{J}$, we conclude that $\mathcal{E}_{\alpha\beta}$ is isomorphic to the boundary extension cocyle $\{\mathcal{E}_\beta - \mathcal{E}_\alpha\}$. This gives the injectivity. For the surjectivity, let $\eta_{\alpha\beta}$ represent a class in $\check{H}^1(U_\alpha, \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{Q}))$. Set

$$\mathcal{E}_{\alpha\beta} = \eta_{\alpha\beta}^* \mathcal{J}$$

and

$$F_{\alpha\beta\gamma} : \mathcal{E}_{\alpha\beta} + \mathcal{E}_{\beta\gamma} = \eta_{\alpha\beta}^* \mathcal{J} + \eta_{\beta\gamma}^* \mathcal{J} \cong (\eta_{\alpha\beta} + \eta_{\beta\gamma})^* \mathcal{J} = \eta_{\alpha\gamma}^* \mathcal{J} = \mathcal{E}_{\alpha\gamma}$$

is the natural homomorphism.

$\square$

To finish the construction of the obstruction class, we cover $\mathcal{X}$ by open affine subscheme $\{\mathcal{U}_\alpha\}$ such that $\mathcal{L}_\alpha^\vee = \mathcal{L}^\vee|_{\mathcal{U}_\alpha}$ has a lifting $\widetilde{\mathcal{L}}_\alpha^\vee$ over $\widetilde{\mathcal{U}}_\alpha$. The difference of $\widetilde{\mathcal{L}}_\alpha^\vee$ and $\widetilde{\mathcal{L}}_\beta^\vee$ on the overlap corresponds to an extension $\mathcal{E}_{\alpha\beta}$ of $\mathcal{P}^1_{U_{\alpha\beta}}(L_{\alpha\beta})$ by $J \otimes_{\mathbb{C}} L_{\alpha\beta}$. For each triple α, β, γ, consider the isomorphism

$$F_{\alpha\beta\gamma} : \mathcal{E}_{\alpha\beta} + \mathcal{E}_{\beta\gamma} \cong \mathcal{E}_{\alpha\gamma}$$

in proposition 2.3 (a).

Then $(\mathcal{E}_{\alpha\beta}, F_{\alpha\beta\gamma})$ is an extension cocycle, which we will denote simply by $(\mathcal{E}_{\alpha\beta})$. If

$\overline{\mathcal{L}}_\alpha^\vee$ is another collections of liftings, coresponding to another extension cocycle $(\mathcal{E}'_{\alpha\beta})$, we get isomorphisms

$$\mathcal{E}_{\alpha\beta} \cong \mathcal{E}'_{\alpha\beta} + \mathcal{E}(\widetilde{\mathcal{L}}_\alpha^\vee, \overline{\mathcal{L}}_\alpha^\vee) - \mathcal{E}(\widetilde{\mathcal{L}}_\beta^\vee, \overline{\mathcal{L}}_\beta^\vee).$$

by proposition 2.3 (a). One checks that this is an isomorphism of extension cocycles. Thus the class of

$$[\mathcal{E}_{\alpha\beta}] \in \Xi_{\mathcal{O}_X}(U_\alpha; \mathcal{P}_X^1(L), J \otimes_{\mathbb{C}} L)$$

is independent of the choice of local liftings.

A global lifting exists if and only if we can choose local liftings $\widetilde{\mathcal{L}}_\alpha^\vee$ and isomorphisms of line bundles $\phi_{\alpha\beta} : \widetilde{\mathcal{L}}_\alpha^\vee \to \widetilde{\mathcal{L}}_\beta^\vee$ satisfying the cocycle condition

$$\phi_{\alpha\beta} \circ \phi_{\beta\gamma} = \phi_{\alpha\gamma}.$$

By proposition 2.3 (c), to give $\phi_{\alpha\beta}$ is equivalent to assigning splittings for $\mathcal{E}_{\alpha\beta}$. It is easy to check that $\phi_{\alpha\beta}$ satisfies cocycle condition if and only if $(\mathcal{E}_{\alpha\beta})$ is isomorphic to the trivial extension cocycle.

Conversely, if the class

$$[\mathcal{E}_{\alpha\beta}] \in \Xi_{\mathcal{O}_X}(U_\alpha; \mathcal{P}_X^1(L), J \otimes_{\mathbb{C}} L)$$

is zero, $(\mathcal{E}_{\alpha\beta})$ is isomorphic to a boundary $(\mathcal{E}_\alpha - \mathcal{E}_\beta)$. By proposition 2.3 (b), we can choose local lifting $\overline{\mathcal{L}}_\alpha^\vee$ such that $\mathcal{E}(\widetilde{\mathcal{L}}_\alpha^\vee, \overline{\mathcal{L}}_\alpha^\vee) \cong \mathcal{E}_\alpha$. Then $\overline{\mathcal{L}}_\alpha^\vee$ will patch together to give a global lifting.

Combine the above discussion with theorem 2.6 and the fact that $\mathcal{E}xt_{\mathcal{O}_X}^2(\mathcal{P}_X^1(L), L) = 0$ (since (2.9) is a locally free resolution of $\mathcal{P}_X^1(L)$), we get

Theorem 2.8. *Let X be a l.c.i scheme, L a line bundle on X. For any small extension*

$$e: \quad 0 \longrightarrow J \longrightarrow \widetilde{A} \longrightarrow A \longrightarrow 0$$

and any deformation $(\mathcal{X}, \mathcal{L})$ of (X, L) over A,

(a) *There is an element*

$$\circ(e) \in J \otimes_{\mathbb{C}} \mathrm{Ext}^2_{\mathcal{O}_X}(\mathcal{P}^1_X(L), L),$$

such that $\circ(e) = 0$ if and only if a lifting $(\widetilde{\mathcal{X}}, \widetilde{\mathcal{L}})$ of $(\mathcal{X}, \mathcal{L})$ to $\widetilde{A}$ exists.

(b) *If a lifting exists, the set of isomorphism classes of liftings is a principal homogeneous space for the group*

$$J \otimes_{\mathbb{C}} \mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{P}^1_X(L), L).$$

CHAPTER 3

THE MAXIMAL RANK CONJECTURE FOR LINE

BUNDLES OF EXTREMAL DEGREE

3.1 Introduction

In this chapter, we take a first step to attack the maximal rank conjecture. We start
with a result by Green and Lazarsfeld [25] asserting that any very ample line bundle
L on C with

$$deg\ L \geq 2g + 1 - 2h^1(L) - Cliff(C)$$

or equivalently

$$Cliff(L) < Cliff(C), \text{(this implies that } h^1(L) \leq 1)$$

is projectively normal, where $Cliff(C)$ is the clifford index of C:

$$Cliff(C) := \min\{Cliff(A) \mid A \text{ line bundle on } C,\ h^0(A) \geq 2,\ h^1(A) \geq 2\}$$

and

$$Cliff(A) = deg\ A - 2r(A).$$

and for a general curve C, $Cliff(C) = \lfloor \frac{g-1}{2} \rfloor$.

It is also showed by Green and Lazarsfeld that the bound $2g+1-2h^1(L)-Cliff(C)$ is the best possible. There are line bundles of degree one less than this bound which are not normally generated. We say a line bundle L on C has extremal degree if

$$deg\ L = 2g - 2h^1(L) - Cliff(C),$$

that is,

$$Cliff(L) = Cliff(C).$$

On the other hand, if the maximal rank conjecture were true, we should still expect projective normality for general line bundles of extremal degree on general curves. Thus the extremal degree range may be thought of as the first unknown case to test the maximal rank conjecture. There are four cases according to the value of $h^1(L)$:

(1) $h^1(L) = 0$. L is non special and the MRC follows from [11].

(2) $h^1(L) = 1$. If $g = 2l$ even, L is a g^l_{3l-1} and $\rho = l - 1$; if $g = 2l + 1$ odd, L is a g^l_{3l} and $\rho = l$.

(3) $h^1(L) = 2$. If $g = 2l$ even, L is a g^{l-1}_{3l-3} and $\rho = 0$; if $g = 2l + 1$ odd, L is a g^{l-1}_{3l-2} and $\rho = 1$.

(4) $h^1(L) \geq 3$. The Brill-Noether number is negative. There are no such g^r_d's $(r \geq 3)$ on a general curve.

In this chapter, we prove the MRC for the remaining open cases (2) and (3).

Theorem 3.1. *Let C be a general curve of genus g ($g \geq 10$ if g even, $g \geq 13$ if g odd), L be a general line bundle of extremal degree on C, then (C, L) satisfies the MRC, or equivalently, is projectively normal.*

By the H^0 lemma of Green (theorem (4.e.1) of [23]), in our degree range, $H^0(L) \otimes H^0(L^k) \to H^0(L^{k+1})$ is surjective for any $k \geq 2$. Thus it suffices to prove μ^2 in (1.1) is surjective. We apply a new method, using deformation theory, to prove this fact. The general idea of this method is as follows. Instead of looking for some (C_0, L_0) such that μ^2 is of maximal rank there, consider a one parameter family of pairs $(C_t, L_t) \in \mathcal{W}_d^r$, specializing to some (C_0, L_0) (C_0 could be singular) with $\mu^2(0)$ not necessarily of maximal rank. Suppose moreover that all global sections of L_0 extends to L_t. Then one can construct obstruction maps

$$\delta_1 : \mathrm{Ker}(\mu^2(0)) \longrightarrow \mathrm{Coker}(\mu^2(0))$$

and inductively

$$\delta_{n+1} : \mathrm{Ker}(\delta_n) \longrightarrow \mathrm{Coker}(\delta_n)$$

such that an element $s \in \mathrm{Ker}(\mu^2(0))$ extends to $\mathrm{Ker}(\mu^2(t))$ modulo t^{n+1} if and only if $\delta_i(s) = 0$ for $i = 0, ..., n$.

For the decreasing sequence

$$\mathrm{Ker}(\mu^2(0)) \supset \mathrm{Ker}(\delta_1) \supset ... \supset \mathrm{Ker}(\delta_n) \supset ...$$

if we can show that the vector space $V = \bigcap_i \mathrm{Ker}(\delta_i)$ consisting of elements which deform to $\mathrm{Ker}(\mu^2(t))$ to any order is of "correct dimension", then $\mu^2(t)$ is of maximal rank. Said differently, it suffices to prove that δ_n is of maximal rank for some $n \in \mathbb{Z}_+$.

We find a nice singular curve C_0 on which the computation of obstructions is surprisingly simple. Enough information in the obstruction maps δ_n is captured by the natural multiplication map

$$\kappa_n : H^0(C_0, L_n) \otimes H^0(C_0, L_{-n}) \longrightarrow H^0(C_0, L_0^2)$$

in the following theorem:

Theorem 3.2. *Let $(C_t, L_t) \in \mathcal{W}_d^r$ be a one parameter family of pairs (with smooth total space $\mathcal{L} \to \mathcal{C}$) degenerating to (C_0, L_0) with $C_0 = X \cup Y$ a nodal curve consisting of two smooth curves of genus g_X, g_Y meeting at a point p. Write $L_n = \mathcal{L}(nY)|_{C_0}$. Suppose all (global) sections of L_n extend to L_t for $|n| \leq a$ and the natural map*

$$\bigoplus_{n=0}^{a} H^0(C_0, L_n) \otimes H^0(C_0, L_{-n}) \xrightarrow{\kappa = \oplus_n \kappa_n} H^0(C_0, L_0^2) \tag{3.1}$$

is surjective (resp. of rank $= \dim_{\mathbb{C}} \operatorname{Sym}^2 H^0(L_0)$) for some $a \in \mathbb{Z}_+$, then the multi-plication map $\mu^2(t)$ is surjective (resp. injective) for small $t \neq 0$.

Notice that κ only depends on (C_0, L_0), not on the actual family specializing to it. This simple way to describe higher order obstructions is new and it is reasonable to expect that it will have other applications.

The significance of theorem 3.2 is that we are now reduced to finding a smoothable (C_0, L_0) such that all sections of L_n extend to the nearby fiber and $\oplus_{n=0}^{a} \kappa_n$ (instead of $\kappa_0 = \mu^2(0)$) is of required rank. By making a good choice of (C_0, L_0), we manage to prove theorem 3.1 by showing that κ in theorem 3.2 is surjective.

This chapter is organized as follows:

In section 3.2, we set up some machinery which measures the obstructions for elements of $\text{Ker}(\mu^k(0))$ to extend to $\text{Ker}(\mu^k(t))$.

In section 3.3, we compute the obstruction maps δ_n for the special degeneration described in theorem 3.2 and give a proof of this theorem.

Section 3.4 contains a proof of the main theorem 3.1.

Finally, in section 3.5, we include some technical facts about canonical bundles on general curves which are needed in the proof of the main theorem.

3.2   Infinitesimal study of the degeneracy loci

Let C_0 be a reduced l.c.i curve over $\mathbb{C}$ and L_0 be a degree d line bundle on C_0 with $h^0(L_0) = r + 1$. By theorem 2.8 of chapter 2, the deformations of the pair (C_0, L_0) are unobstructed. Let S be the versal deformation space of (C_0, L_0), then S is smooth near (C_0, L_0). Let $\mathcal{W}_d^r$ be the subvariety of S consisting of (C, L) such that $h^0(L) \geq r + 1$. Consider the multiplication map

$$\text{Sym}^k H^0(C, L) \xrightarrow{\mu^k} H^0(C, L^k) . \tag{3.2}$$

We may think of this map as a morphism between two vector bundles (at least near the point (C_0, L_0)) over $\mathcal{W}_d^r$ as (C, L) varies in $\mathcal{W}_d^r$. We are interested in the infinitesimal properties of the locus D consisting of (C, L) such that the multiplication map is not of maximal rank, i.e it is neither injective nor surjective. Our goal is to show that D is a proper subvariety of $\mathcal{W}_d^r$ (assuming $\mathcal{W}_d^r$ irreducible near (C_0, L_0)).

Suppose now that there is a (flat) one parameter family (C_t, L_t) of pairs specializing to (C_0, L_0) such that all sections of L_0 extend to L_t. If $\mu^k(0)$ is not of maximal rank at

(C_0, L_0), then the dimension of $\text{Ker}(\mu^k(0))$ is bigger than expected. We would like to knock down this dimension by showing that only an expected number of independent sections of $\text{Ker}(\mu^k(0))$ can extend to $\text{Ker}(\mu^k(t))$. Thus $\mu^k(t)$ is of maximal rank for $t \neq 0$.

Our goal in this section is to set up some machinery which measures the obstructions for elements of $\text{Ker}(\mu^k(0))$ to extend to $\text{Ker}(\mu^k(t))$.

To this end, let $(\mathcal{C}, \mathcal{L})$ be the total space of the one parameter family and $(\mathcal{C}_n, \mathcal{L}_n)$ be the restriction of $(\mathcal{C}, \mathcal{L})$ to $Spec\frac{\mathbb{C}[t]}{(t^{n+1})} =: SpecR_n$. Let $M_n = \text{Sym}^k H^0(\mathcal{C}_n, \mathcal{L}_n)$, $N_n = H^0(\mathcal{C}_n, \mathcal{L}_n^k)$ and $\mu_n : M_n \to N_n$ be the multiplication map.

We have $M_{i+1} \otimes_{R_{i+1}} R_i = M_i$, $N_{i+1} \otimes_{R_{i+1}} R_i = N_i$ compatibly with μ_i for any $i \geq 0$.

Lemma 3.3. *Under the above notations and assumptions, there exist obstruction maps*

$$\delta_{n+1} : \text{Ker}(\delta_n) \to \text{Coker}(\delta_n)$$

for $n \geq 0$ such that $\delta_0 = \mu_0$ and $s \in \text{Ker}(\mu_0)$ can be lifted to $\text{Ker}(\mu_n)$ if and only if $\delta_i(s) = 0$ for $i = 0, ..., n$.

Proof.   For each $n \geq 0$ consider

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & M_0 & \xrightarrow{\cdot t^{n+1}} & M_{n+1} & \xrightarrow{p_{n+1}} & M_n & \longrightarrow & 0 \\
& & \downarrow{\mu_0} & & \downarrow{\mu_{n+1}} & & \downarrow{\mu_n} & & \\
0 & \longrightarrow & N_0 & \xrightarrow{\cdot t^{n+1}} & N_{n+1} & \xrightarrow{q_{n+1}} & N_n & \longrightarrow & 0
\end{array}
\qquad (3.3)
$$

Let $\delta'_{n+1} : \text{Ker}(\mu_n) \to \text{Coker}(\mu_0))$ be the connecting homomorphism of (3.3) from

snake lemma. Fix $s_0 \in \mathrm{Ker}(\mu_0)$ and $n \geq 1$. Suppose s_0 has a lifting $s_n \in \mathrm{Ker}(\mu_n)$.

Let $s'_{n+1} \in M_{n+1}$ be any lifting of s_n. Suppose that $\mu_{n+1}(s'_{n+1}) = t^{n+1}v$ with

$$v = \mu_0(s'_0) + \delta'_1(p_1(s'_1)) + \ldots + \delta'_n(p_n(s'_n)), \tag{3.4}$$

then

$$s'_{n+1} - (t^{n+1}s'_0 + t^n s'_1 + \ldots + ts'_n) \in \mathrm{Ker}(\mu_{n+1}).$$

On the other hand, if (3.4) has no solution for any collection $s'_j \in M_j$ with $p_j(s'_j) \in \mathrm{Ker}(\mu_{j-1})$, then s_0 has no lifting to $\mathrm{Ker}(\mu_{n+1})$.

Now, simply define $\delta_{n+1}(s_0) = \delta'_{n+1}(s_n) = v$ as an element of

$$\frac{\mathrm{Coker}(\mu_0)}{\sum_{i=1}^{n} \mathrm{Im}(\delta'_i)} = \mathrm{Coker}(\delta_n).$$

Then $\delta_{n+1}(s_0)$ does not depend on the choice of s_n and is equal to zero in $\mathrm{Coker}(\delta_n)$ if and only if s_0 can be lifted to $\mathrm{Ker}(\mu_{n+1})$.

Finally, we check

$$\frac{\mathrm{Coker}(\mu_0)}{\sum_{i=1}^{n+1} \mathrm{Im}(\delta'_i)} = \mathrm{Coker}(\delta_{n+1}).$$

$\square$

By theorem 2.1 of chapter 2, $(\mathcal{C}_1, \mathcal{L}_1)$ determines a tangent vector $\xi \in T_{(C_0, L_0)}S = \mathrm{Ext}^1_{\mathcal{O}_{C_0}}(\mathcal{P}^1_{C_0}(L_0), L_0)$ which annihilates $H^0(C_0, L_0)$. This means exactly that ξ is tangent to $\mathcal{W}^r_d \subset S$ at (C_0, L_0). ξ is a tangent direction such that the rank of μ_0^k does not increase if and only if

$$\delta_1 : \mathrm{Ker}(\mu_0^k) \longrightarrow \mathrm{Coker}(\mu_0^k)$$

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is zero. If this is the case, then every element in $\mathrm{Ker}(\mu_0^k)$ extends to $\mathrm{Ker}(\mu_1^k)$, thus to first order, the rank of the map μ^k does not increase (It does not decrease either, by lower semicontinuity of the rank).

On the other hand, if δ_1 is of maximal rank, there are two cases:

1) δ_1 is injective. No elements of $\mathrm{Ker}(\mu_0^k)$ will extend to $\mathrm{Ker}(\mu_1^k)$, thus for a general $t \neq 0$, the multiplication map (3.2) is injective at (C_t, L_t).

2) δ_1 is surjective. Only a subspace of $\mathrm{Ker}(\mu_0^k)$ of dimension equal to

$$dim_{\mathbb{C}} \mathrm{Ker}(\mu_0^k) - dim_{\mathbb{C}} \mathrm{Coker}(\mu_0^k) = dim_{\mathbb{C}} \mathrm{Sym}^k H^0(L_0) - h^0(L_0^k)$$

will extend to first order, therefore, for the nearby (C_t, L_t), the multiplication map (3.2) is surjective.

Suppose now that δ_1 is not of maximal rank. It is not possible to test if the nearby multiplication map is of maximal rank to first order. We have to look at the higher order obstruction maps δ_n.

By lemma 3.3, any $s \in \mathrm{Ker}(\mu_0^k)$ can be extended to $\mathrm{Ker}(\mu_n^k)$ if and only if $\delta_i(s) = 0$ for $i = 0, ..., n$. Let n be the smallest integer such that δ_n is of maximal rank (if it exists). Since the index of δ_i $Ind\ \delta_i := dim_{\mathbb{C}} \mathrm{Ker}(\delta_i) - dim_{\mathbb{C}} \mathrm{Coker}(\delta_i)$ is always constant for any i, we see that only a subspace of $\mathrm{Ker}(\mu_0^k)$ of expected dimension (0 if δ_n injective, $Ind\ \delta_n$ if δ_n surjective) can be extended to $\mathrm{Ker}(\mu_n^k)$. Therefore, the multiplication maps for nearby fibers are of maximal rank.

We have proved the following proposition:

Proposition 3.4. *If δ_n is of maximal rank for some $n \in \mathbb{Z}_+$, the multiplication map μ^k is of maximal rank for nearby fibers.*

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3.3 A nice degeneration

To use proposition 3.4, we need to compute the obstruction maps δ_n, which in general is difficult. However, for the $k = 2$ case, there is a nice degeneration on which the computation is surprisingly simple.

Let $\mathcal{L} \to \mathcal{C}$ be the total space of a one parameter family $(C_t, L_t) \in \mathcal{W}_d^r$ degenerating to (C_0, L_0) with $C_0 = X \cup Y$ a nodal curve consisting of two smooth curves of genus g_X, g_Y meeting at a point p. Write $L_n = \mathcal{L}(nY)|_{C_0}$. Suppose all sections of L_n extend to L_t for $|n| \le a$. Notice that $L_n|_X = L_0|_X(np)$ and $L_n|_Y = L_0|_Y(-np)$, thus L_n only depends on (C_0, L_0), not on the family specializing to it.

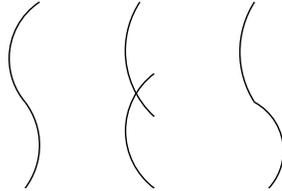

Figure 3.1: Smooth curves degenerating to a nodal curve

The multiplication map

$$\mathrm{Sym}^2 H^0(C_0, L_0) \xrightarrow{\mu(0)} H^0(C_0, L_0^2) \tag{3.5}$$

is usually not of maximal rank here, which means dimension of $\mathrm{Ker}(\mu(0))$ is bigger than it should be.

There are some obvious elements in $\mathrm{Ker}(\mu(0))$. Let W be the subspace of $\mathrm{Sym}^2 H^0(L_0)$ spanned by

$$\{\sigma \cdot \tau \ : \ \sigma, \tau \in H^0(L_0), \ \sigma|_X \equiv 0, \ \tau|_Y \equiv 0, \}.$$

Clearly W is a subspace of $\mathrm{Ker}(\mu(0))$. Let's compute the image of W under

$$\delta_1 : \mathrm{Ker}(\mu(0)) \longrightarrow \mathrm{Coker}(\mu(0)).$$

Let $\tilde{\sigma}$, $\tilde{\tau}$ be sections of $\mathcal{L}$ which extend σ, τ respectively (They always exist, since by assumption, all sections of L_0 extend to L_t). Then $\tilde{\sigma} = \tilde{\sigma}' s_X$, $\tilde{\tau} = \tilde{\tau}' s_Y$, where s_X (resp. s_Y) is a section of $\mathcal{O}_{\mathcal{C}}(X)$ (resp. $\mathcal{O}_{\mathcal{C}}(Y)$) which vanishes exactly on X (resp Y) and $\tilde{\sigma}'$ (resp. $\tilde{\tau}'$) is a section of $\mathcal{L}(-X)$ (resp. $\mathcal{L}(-Y)$). By the construction of δ_1,

$$\delta_1(\sigma \cdot \tau) = \frac{\tilde{\sigma}\tilde{\tau}}{t}|_{C_0} = \frac{\tilde{\sigma}' s_X \tilde{\tau}' s_Y}{t}|_{C_0} = \tilde{\sigma}'\tilde{\tau}'|_{C_0} \mod \mathrm{Im}(\mu(0))$$

Therefore, the image of W under δ_1 is equal to the image of the composition

$$H^0(L_1) \otimes H^0(L_{-1}) \xrightarrow{\kappa_1} H^0(L_0^2) \longrightarrow \mathrm{Coker}(\mu(0)).$$

δ_1 in general is not injective. Let α be any section of $\mathcal{L}(-2X)$, β be any section of $\mathcal{L}(-2Y)$. Then $(\alpha s_X^2)|_{C_0} \cdot (\beta s_Y^2)|_{C_0} \in W \subset \mathrm{Ker}(\mu(0))$, and $\delta_1((\alpha s_X^2)|_{C_0} \cdot (\beta s_Y^2)|_{C_0}) = \frac{\alpha s_X^2 \cdot \beta s_Y^2}{t}|_{C_0} \equiv 0$. Clearly $(\alpha s_X^2) \cdot (\beta s_Y^2) \in \mathrm{Ker}(\mu_1)$ as in diagram (3.3), and therefore

$$\delta_2((\alpha s_X^2)|_{C_0} \cdot (\beta s_Y^2)|_{C_0}) = \frac{\alpha s_X^2 \cdot \beta s_Y^2}{t^2}|_{C_0} = \alpha\beta|_{C_0}.$$

Thus the image of δ_2 contains the image of the composition

$$H^0(L_2) \otimes H^0(L_{-2}) \xrightarrow{\kappa_2} H^0(L_0^2) \longrightarrow \mathrm{Coker}(\delta_1).$$

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Similarly, the image of δ_n contains the image of the composition

$$H^0(L_n) \otimes H^0(L_{-n}) \xrightarrow{\kappa_n} H^0(L_0^2) \longrightarrow \mathrm{Coker}(\delta_{n-1}).$$

Therefore, we have a surjection

$$\frac{H^0(L_0^2)}{\sum_{n=0}^{a} \mathrm{Im}(\kappa_n)} \longrightarrow \mathrm{Coker}(\delta_a).$$

for any positive integer a.

The above analysis immediately gives a proof of theorem 3.2 because if

$$\bigoplus_{n=0}^{a} H^0(C_0, L_n) \otimes H^0(C_0, L_{-n}) \xrightarrow{\kappa = \oplus_n \kappa_n} H^0(C_0, L_0^2)$$

is surjective (resp. of rank $= dim_{\mathbb{C}} \mathrm{Sym}^2 H^0(L_0)$) for some $a \in \mathbb{Z}_+$, then δ_a is of maximal rank and therefore by proposition 3.4, the multiplication map $\mu^2(t)$ is surjective (resp. injective) for small $t \neq 0$.

3.4   Proof of the main theorem

We will prove theorem 3.1 in this section.

Notice that in our degree and genus range, the MRC for L is equivalent to the statement that L is projectively normal. By theorem $(4.e.1)$ of [23], in our degree range, $H^0(L) \otimes H^0(L^k) \to H^0(L^{k+1})$ is surjective for any $k \geq 2$. Thus to show such L is projectively normal, it suffices to show the multiplication map μ^2 in (3.2) is surjective.

The idea is to make a good choice of (C_0, L_0) such that the hypothesis of theorem 3.2 is satisfied. We need the following lemma:

Lemma 3.5. *Let (C_0, L_0) be the same as theorem 3.2. Write $L_X = L_0|_X$ and $L_Y = L_0|_Y$. Suppose the restriction maps $H^0(C_0, L_n) \to H^0(X, L_X(np))$ and $H^0(C_0, L_n) \to H^0(Y, L_Y(-np))$ are surjective for $-a \le n \le a$. If*

$$\bigoplus_{n=0}^a H^0(L_X(np)) \otimes H^0(L_X(-np)) \longrightarrow H^0(L_X^2) \qquad (3.6)$$

and

$$\bigoplus_{n=0}^a H^0(L_Y(np)) \otimes H^0(L_Y((-n-1)p)) \longrightarrow H^0(L_Y^2(-p)) \qquad (3.7)$$

are both surjective, then the natural map κ in (3.1) is surjective.

Proof. Let s be any element of $H^0(C_0, L_0^2)$. Since (3.6) is surjective, and any section of $L_X(np)$ extends to a section of L_n, we can modify s by some element in the image of κ such that $s|_X \equiv 0$. Thus we can assume $s|_X \equiv 0$, then $s|_Y \in H^0(L_Y^2(-p))$. Since (3.7) is surjective, $s|_Y = \sum_{n=0}^a (x_n y_n)$ with $x_n \in H^0(L_Y(np))$, $y_n \in H^0(L_Y((-n-1)p))$. We can view y_n as a section of $H^0(L_Y(-np))$ which vanishes at p, thus can be extended constantly 0 to X as a section of L_n. Still call it y_n. Extend x_n arbitrarily to X as a section of L_{-n}. Then $\sum_{n=0}^a \kappa(x_n \otimes y_n) = s$. $\qquad \square$

We now take $C_0 = X \cup Y$, where X and Y are general curves of genus g_X, g_Y meeting transversely at a general point p. In particular, p is not a Weierstrass point of either X or Y. We will divide the proof of the main theorem 3.1 into two parts, according to the value of $h^1(L)$.

1. $h^1(L) = 2$ **case**.

Since the residual series N of L is either a g^1_{l+1} or g^1_{l+2} depending on $g = 2l$ even or $g = 2l + 1$ odd. We will work backwards by starting with a line bundle N with $h^0(N) = 1$ and take its residual line bundle. Here we take a simple N whose restriction to X and Y are just suitable multiples of $\mathcal{O}_X(p)$ and $\mathcal{O}_Y(p)$, then take L_0 as the residual series of N.

There are two subcases:

(1) $g = 2l$ even. Here we are dealing with g^{l-1}_{3l-3}'s. ($l \geq 6$. $l = 5$ needs a special argument and is proved in the appendix.) Let $g_X = g_Y = l$, $L_X = K_X(-\lceil \frac{l-1}{2} \rceil p)$ and $L_Y = K_Y(-\lfloor \frac{l-1}{2} \rfloor p)$.

(2) $g = 2l+1$ odd. L is a g^{l-1}_{3l-2} ($l \geq 6$). Take $g_X = l+1$, $g_Y = l$, $L_X = K_X(-\lceil \frac{l}{2} \rceil p)$, and $L_Y = K_Y(-\lfloor \frac{l}{2} \rfloor p)$

In both cases, it is easy to prove using the theory of limit linear series (see [16]) that (C_0, L_0) are smoothable in such a way all sections of L_0 extend to nearby. More precisely, the corresponding limit linear series on C_0 has aspects $V_X = (l-1)p + |K_X|$, $V_Y = (l-1)p + |K_Y|$ in case $g = 2l$, and $V_X = (l-1)p + |K(-p)|$, $V_Y = lp + |K_Y|$ in case $g = 2l+1$. They are both smoothable because the variety of limit liner series with the same ramification sequence as (V_X, V_Y) at p has expected dimensions.

Proposition 3.6. *For (C_0, L_0) as described above, the natural map (3.1) is surjective.*

Proof. Case (1). Here $L_n|_X = K_X((-\lceil \frac{l-1}{2} \rceil + n)p)$, $L^2_X = K^2_X(-2\lceil \frac{l-1}{2} \rceil p)$. Apply lemma 3.8 or 3.9 for $a = -\lfloor \frac{l-1}{2} \rfloor$, L_X satisfies (3.6). Meantime, $L^2_Y(-p) =$

$K_Y^2(-(2\lfloor \frac{l-1}{2} \rfloor + 1)p)$ and $2\lfloor \frac{l-1}{2} \rfloor + 1$ is either $l - 1$ or l depending on l even or odd. Again by lemma 3.8 or 3.9, L_Y satisfies (3.7).

Case (2). Take $a = \lceil \frac{l}{2} \rceil - 1$. $L_n|_X = K_X((-\lceil \frac{l}{2} \rceil + n)p)$, $L_X^2 = K_X^2(-2\lceil \frac{l}{2} \rceil p)$, $L_Y^2 = K_Y^2(-2\lfloor \frac{l}{2} \rfloor p)$.

If l even, by lemma 3.9, we see that L_Y satisfies (3.6) and L_X satisfies (3.7) (notice $g_X = l + 1$).

If l odd, again by lemma 3.9, L_X satisfies (3.6) and L_Y satisfies (3.7).

Thus in either case, the hypotheses of lemma 3.5 are satisfied, and therefore κ in (3.1) is surjective. $\qquad\square$

2. $h^1(L) = 1$ case

We are dealing with the residual series of g_{l-1}^0's if $g = 2l$ and g_l^0's if $g = 2l + 1$, so smoothability is not a problem. Again there are two subcases:

(1) $g = 2l$ even. L is a g_{3l-1}^l. Let $g_X = g_Y = l$, D_X (resp. D_Y) be a divisor consisting of $\lceil \frac{l-1}{2} \rceil$ (resp. $\lfloor \frac{l-1}{2} \rfloor$) general points on X (resp. Y). Take $L_X = K_X(p - D_X)$, $L_Y = K_Y(p - D_Y)$.

(2) $g = 2l + 1$ odd. L is a g_{3l}^l. Let $g_X = l + 1$, $g_Y = l$ D_X a general divisor of degree $\lceil \frac{l}{2} \rceil$ on X and D_Y a general divisor of degree $\lfloor \frac{l}{2} \rfloor$ on Y. Take $L_X = K_X(p - D_X)$, $L_Y = K_Y(p - D_Y)$.

Proposition 3.7. *For (C_0, L_0) as described above, the natural map (3.1) is surjective.*

Proof. Case (1). Let $M_X = L_X((\lceil \frac{l-1}{2} \rceil + 1)p) = K_X((\lceil \frac{l-1}{2} \rceil + 2)p - D_X)$, $M_Y = L_Y((\lfloor \frac{l-1}{2} \rfloor + 1)p) = K_Y((\lfloor \frac{l-1}{2} \rfloor + 2)p - D_Y)$.

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If l even, $L_X^2 = K_X^2(2p - 2D_X) = M_X^2(-(l+2)p)$. By lemma 3.10, L_X satisfies (3.6). $L_Y^2(-p) = K_Y^2(p - 2D_Y) = M_Y^2(-(l+1)p)$, L_Y satisfies (3.7).

If l odd, $L_X^2 = K_X^2(2p - 2D_X) = M_X^2(-(l+1)p)$, L_X satisfies (3.6). $L_Y^2(-p) = K_Y^2(p - D_Y) = M_Y^2(-(l+2)p)$, thus L_Y satisfies (3.7) by lemma 3.10.

Case (2). Let $M_X = L_X((\lceil \frac{l}{2} \rceil + 1)p) = K_X((\lceil \frac{l}{2} \rceil + 2)p - D_X)$, $M_Y = L_Y((\lfloor \frac{l}{2} \rfloor + 1)p) = K_Y((\lfloor \frac{l}{2} \rfloor + 2)p - D_Y)$.

If l even, $L_Y^2 = K_Y^2(2p - 2D_Y) = M_Y^2(-(l+2)p)$, thus L_Y satisfies (3.6). $L_X^2(-p) = K_X^2(p - 2D_X) = M_X^2(-(l+3)p) = M_X^2(-(g_X + 2)p)$, L_X satisfies (3.7).

If l odd, $L_X^2 = K_X^2(2p - 2D_X) = M_X^2(-(l+3)p) = M_X^2(-(g_X + 2)p)$, L_X satisfies (3.6). $L_Y^2(-p) = K_Y^2(p - 2D_Y) = M_Y^2(-(l+2)p))$, L_Y satisfies (3.7).

Thus in either case, the hypotheses of lemma 3.5 are satisfied, and therefore κ in (3.1) is surjective.

$\square$

Combining propositions 3.6, 3.7 and theorem 3.2,

$$\mathrm{Sym}^2 H^0(C_t, L_t) \xrightarrow{\mu^2(t)} H^0(C_t, L_t^2)$$

is surjective for small $t \neq 0$. Since there is a unique component P of $\mathcal{W}_d^r$ which dominate $\mathcal{M}_g$ ($\rho > 0$ case follows from the Gieseker-Petri theorem and the connectedness of $W_d^r(C)$; $\rho = 0$ case follows from [17]), to prove theorem 3.1, it suffices to arrange so that $(C_t, L_t) \in P$. But this is immediate because C_0 is a general point of the boundary of $\overline{\mathcal{M}_g}$.

3.5 Some facts about canonical bundles on general curves

We present some technical facts about canonical bundles for a general curve in this section. They are needed in the proof of the main theorem. Lemma 3.8 to 3.10 are used in the previous section to show that maps of the type of (3.6) and (3.7) are surjective for the specific L_0 we choose in section 3.4.

Lemma 3.8. *For a general smooth curve X of genus $l \geq 4$, and $p \in X$ a general point (in particular, not a Weierstrass point), the natural map*

$$\sum_{i+j=l-1,i,j\geq 0} H^0(K_X(-ip)) \otimes H^0(K_X(-jp)) \xrightarrow{m_{l-1}} H^0(K_X^2(-(l-1)p))$$

is surjective and

$$\sum_{i+j=l,i,j\geq 1} H^0(K_X(-ip)) \otimes H^0(K_X(-jp)) \xrightarrow{m_l} H^0(K_X^2(-lp))$$

is of corank at most 1.

Proof. Choose $\{\omega_0, ..., \omega_{l-1}\}$ a basis of $H^0(K_X)$ adapted to the flag

$$H^0(K_X) \supsetneq H^0(K_X(-p)) \supsetneq ... \supsetneq H^0(K_X(-(l-1)p),$$

i.e. $H^0(K_X(-ip)) = span\{\omega_i, ..., \omega_{l-1}\}$ for any i. By generality of X and p, we can assume $K_X(-(l-2)p)$ is a base point free pencil. By base point free pencil trick, the kernel of the map

$$H^0(K_X) \otimes H^0(K_X(-(l-2)p) \xrightarrow{m'} H^0(K_X^2(-(l-2)p)) \tag{3.8}$$

is $H^0(K_X \otimes K_X^{-1}((l-2)p) = H^0(\mathcal{O}_X(l-2)p)$. By Riemann-Roch, $h^0(\mathcal{O}_X(l-2)p) = h^0(K_X(-(l-2)p))+l-2-l+1 = 1$. Thus, by dimension count, m' is surjective with

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a one dimension kernel generated by $\omega_{l-1} \otimes \omega_{l-2} - \omega_{l-2} \otimes \omega_{l-1}$. We obtain a basis of $H^0(K_X^2(-(l-2)p)$:

$$\omega_{l-1}^2, \ \omega_{l-1}\omega_{l-2}, \ ..., \ \omega_{l-1}\omega_1, \ \omega_{l-1}\omega_0, \tag{3.9}$$

$$\omega_{l-2}^2, \ \omega_{l-2}\omega_{l-3}, \ ..., \ \omega_{l-2}\omega_1, \ \omega_{l-2}\omega_0.$$

Except $\omega_{l-2}\omega_0$, every other element of the above basis lies in the image of m_{l-1}, thus m_{l-1} is surjective. Similarly, except $\omega_{l-2}\omega_0$, $\omega_{l-1}\omega_0$, $\omega_{l-2}\omega_1$, every other element of the above basis lies in the image of m_l. Therefore, m_l is of corank at most 1.

$\square$

Lemma 3.9. *For a general curve X of genus $l \geq 6$, and $p \in X$ a general point, m_l in lemma 3.8 is surjective.*

Proof. It suffices to find some (X, p) for which m_l is surjective. Take a special (X, p) such that $K_X(-(l-2)p)$ is a pencil with a base point $q \neq p$ and that q is not a base point of $K_X(-(l-3)p)$. This is equivalent to find a X and $p, q \in X$ such that $h^0(\mathcal{O}_X((l-2)p+q)) = 2$ and $h^0(\mathcal{O}_X((l-3)p+q) = h^0(\mathcal{O}_X((l-2)p)) = 1$. One can actually choose X to be a general point of $\mathcal{M}_l$ (but (X, p) is not general in $\mathcal{M}_{l,1}$). This is because by theorem 3.13, there exists a g_{l-1}^1 on a general curves X with vanishing sequence $(0, l-2)$ at some point $p \in X$, and by theorem 3.12, such a g_{l-1}^1 is base point free, complete and its residual series has a unique base point q. For such (X, p), any element in

$$V = \mathrm{Im}(\textstyle\sum_{i=1}^2 H^0(K_X(-ip)) \otimes H^0(K_X(-(l-i)p)) \xrightarrow{m_l} H^0(K_X^2(-lp)))$$

vanishes at q. Now, take $\omega_{l-3} \in H^0(K_X(-(l-3)p))$ which does not vanish at q, then ω_{l-3}^2 is in the image of m_l but does not lie in V. It remains to show for this special (X, p), V is still of codimension 1 in $H^0(K_X^2(-lp))$. Since $K_X(-(l-2)p)$ has a unique base point q, by base point free pencil trick, the kernel of m' in (3.8) is isomorphic to $H^0(\mathcal{O}_X((l-2)p+q)))$, which is 2 dimensional. By dimension count, m' is corank 1, and there is exactly 1 linear relation among the generators in (3.9). Let $\tau \in H^0(\mathcal{O}_X((l-2)p+q))$ be a section viewed as a rational function having a pole of order exactly $l-2$ at p and a pole of order 1 at q, then the kernel of m' is spanned by

$$\tau\omega_{l-1} \otimes \omega_{l-2} - \tau\omega_{l-2} \otimes \omega_{l-1}$$

and

$$\omega_{l-1} \otimes \omega_{l-2} - \omega_{l-2} \otimes \omega_{l-1}.$$

Where $\{\omega_{l-2}, \omega_{l-1}\}$ span $H^0(K_X(-(l-2)p))$. Since a general curve only has normal Weierstrass points, $h^0(K_X(-(l-1)p)) = 1$. We can assume ω_{l-2} vanishes to order exactly $l-2$ at p. Thus $\tau\omega_{l-2} \in H^0(K_X)$ does not vanish at p, and the linear relation between the generators in (3.9) will have non-zero coefficient in $\omega_{l-1}\omega_0$. Thus

$$\omega_{l-1}^2, \ \omega_{l-1}\omega_{l-2}, \ ..., \ \omega_{l-1}\omega_1,$$

$$\omega_{l-2}^2, \ \omega_{l-2}\omega_{l-3} \ , ..., \ \omega_{l-2}\omega_2.$$

are still linearly independent, and therefore V is still of codimension 1 in $H^0(K_X^2(-lp))$.

$\square$

Lemma 3.10. *Let X be a general curve of genus $l \geq 6$, D is a divisor of degree $0 < d < l - 2$ consisting of d distinct general points, $p \in X$ is a general point. Let $M = K_X(-D + (d+2)p)$. Then the multiplication maps*

$$\sum_{i+j=l+1, i,j \geq 0} H^0(M(-ip)) \otimes H^0(M(-jp)) \longrightarrow H^0(M^2(-(l+1)p)) \qquad (3.10)$$

and

$$\sum_{i+j=l+2, i,j \geq 1} H^0(M(-ip)) \otimes H^0(M(-jp)) \longrightarrow H^0(M^2(-(l+2)p)) \qquad (3.11)$$

are both surjective.

Proof. The idea is similar to the previous two lemmata. We have $\deg M = 2l$, $h^0(M) = l+1$ and $M(-lp) = K_X(-D + (d+2-l)p)$ is a base point free pencil since p is general. Consider

$$H^0(M(-lp)) \otimes H^0(M) \xrightarrow{\quad m' \quad} H^0(M^2(-lp)) \qquad (3.12)$$

By base point free pencil trick, $\text{Ker}(m')$ is isomorphic to $H^0(\mathcal{O}_X(lp))$ and is 1 dimensional since p is not a Weierstrass point of X. Since $h^0(M^2(-lp)) = 2l + 1$, by dimension count, m' is surjective. Extend a basis $\{\omega_l, \ \omega_{l+1}\}$ of $H^0(M(-lp))$ to a basis of $H^0(M)$:

$$\{ \ \omega_0, \ \omega_1, ..., \omega_d, \ \widehat{\omega_{d+1}}, \ \omega_{d+2}, ..., \omega_l, \ \omega_{l+1} \ \} \qquad (3.13)$$

with each ω_i vanish to order exactly i at p. The gap $\widehat{\omega}_{d+1}$ occurs because

$$h^0(M(-(d+1)p) = h^0(K_X(-D+p)) = h^0(K_X(-D)) = h^0(M(-(d+2)p).$$

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By the same argument as lemma 3.8, (3.10) is surjective and (3.11) is at most corank 1.

To prove that (3.11) is actually surjective for general (X, D, p), we specialize to some (X, D, p) such that $M(-lp)$ is a pencil with a base point q but q is not a base point of $M(-(l-1)p)$. This is equivalent to $h^0(M(-lp-q)) = h^0(M(-lp)) = 2$ and $h^0(M(-(l-1)p-q)) = 2$ or equivalently $h^0(\mathcal{O}_X(D+q+(l-d-2)p)) = 2$ and $h^0(\mathcal{O}_X(D+(l-d-2)p)) = 1$. We can even choose (X, p) be a general point of $\mathcal{M}_{l,1}$. This is because by the existence half of theorem 3.11 (or by [32]), there exists a g^1_{l-1} with ramification sequence $(0, l-d-2)$ at p, and by the second half of theorem 3.11, a general such g^1_{l-1} is base point free, complete, and its residual series has a unique base point q. Then, as in lemma 3.9, ω^2_{l-1} is in the image of (3.11) but not in

$$V = \mathrm{Im}(\textstyle\sum_{i=1}^2 H^0(M(-ip)) \otimes H^0(M(-(l+2-i)p)) \longrightarrow H^0(M^2(-(l+2)p)))$$

It remains to prove for this special (X, D, p), V is still of codimension 1 in $H^0(M^2(-(l+2)p))$. The problem here is that due to the base point q of $M(-lp)$, $\mathrm{Ker}(m')$ is isomorphic to $H^0(\mathcal{O}_X((lp+q)))$, which is 2 dimensional (because p is not a Weierstrass point). Thus m' is not surjective but corank 1. However, if $H^0(\mathcal{O}_X(ip+q))$ is span by $\{1, \tau\}$ with τ a rational function having pole of order exactly l at p, then $\mathrm{Ker}(m')$ is spanned by

$$\tau\omega_l \otimes \omega_{l+1} - \tau\omega_{l+1} \otimes \omega_l$$

and

$$\omega_l \otimes \omega_{l+1} - \omega_{l+1} \otimes \omega_l.$$

Again by theorem 3.11, we can even assume $h^0(M(-(l+1)p)) = 1$, and therefore

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can assume m_l vanishes to order exactly l at p. Thus $\tau\omega_l$ does not vanish at p and by the same argument as lemma 3.9, V is still of codimension 1 in $H^0(M^2(-(l+2)p))$.

$\square$

For the convenience of the reader, we state here some existence and non-existence results in Brill-Noether theory which we used in the proof of Lemma 3.9 and 3.10.

Theorem 3.11. *(Brill-Noether theorem, fixed ramification point) Let X be a general curve of genus g, $p \in X$ a fixed general point. For any sequence $0 \le m_0 < ... < m_r \le d$, let ρ be the adjusted Brill-Noether number*

$$\rho = g - \sum_{i=0}^{r}(m_i - i + g - d + r).$$

and let ρ_+ be the existence number

$$\rho_+ = g - \sum_{m_i - i + g - d + r \ge 0}(m_i - i + g - d + r)$$

If ρ_+ is nonnegative, then X possesses g_d^r's with vanishing sequence $(m_0, .., m_r)$ at p. Moreover, the variety $\mathcal{G}_d^r(m_0, ..., m_r)$ parametrizing such g_d^r's is empty if $\rho < 0$ and has pure dimension ρ if $\rho \ge 0$.

Proof. For the existence half, see [31] theorem 3.2-1. For the second half, see [28] theorem 5.37. $\square$

Theorem 3.12. *(Non-existence for low ρ_{mov})*

Let X be a general curve of genus g, ρ_{mov} be the moving-point Brill-Noether number

$$\rho_{mov} = 1 + g - (r+1)(g - d + r) - \sum_{i=0}^{r}(m_i - i).$$

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If $\rho_{mov} < 1 - r$, then for any $p \in X$, there is no g_d^r on X with vanishing sequence $(m_0, ..., m_r)$ at p.

Proof. See [31] theorem 4.3-6. □

Theorem 3.13. *(Existence of g_d^1's with movable ramification point)*
Let X be a general curve of genus g, fix $0 < m_1 \leq d$, then there exists a g_d^1 on X with vanishing sequence $(0, m_1)$ at some point $p \in X$ if $g - d + 1 \geq 0$ and the moving-point Brill-Noether number $\rho_{mov} = 1 + g - 2(g - d + 1) - (m_1 - 1) \geq 0$.

Proof. See [31] theorem 3.3-4 and example 3.3-8. □

3.6 Appendix

We give a special treatment for the case of genus 10 curves C and a complete g_{12}^4 $|L|$ on C. We include this case here because it does not follow from the general discuss in section 3.4 (we need $l \geq 6$ in lemma 3.9) and there is very interesting geometry behind this example.

Since $\rho = g - (r + 1)(g - d + r) = 0$ in this case, by Brill-Noether Theorem, a general genus 10 curve C has only finitely many g_{12}^4s, and each g_{12}^4 on C is very ample. It is also known that for general such C, there exists some g_{12}^4 $|L|$ such that μ^2 is not injective, i.e. C is contained in some quadric hypersurface in $\mathbb{P}^4$ under the embedding of $|L|$, if and only if C is contained in some $K3$ surface (see [20]). By proposition 2.2 of [13], the locus $\mathcal{K}$ in $\mathcal{M}_{10}$ consisting of curves contained in some $K3$ surface is a divisor. (Interestingly, Farkas and Popa [20] proved that $\mathcal{K}$ is a counter example for

the slope conjecture.) Thus, for a general genus 10 curve C, and any g_{12}^4 $|L|$ on C, the multiplication map μ^2 is injective (therefore an isomorphism, since the domain and range of μ^2 are of the same dimension).

Here we will use the results in the previous section to give a proof of this fact without using the geometry of curves contained in $K3$ surfaces.

Proof. Notation the same as theorem 3.2. Take X and Y both general curves of genus 5 meeting at a general point p. Take $L_X = K_X(-2p) = g_6^2$, $L_Y = K_Y(-2p)$ (which is smoothable). We will check κ in (3.1) is surjective. Consider the following exact sequence of sheaves:

$$0 \longrightarrow \mathcal{O}_{C_0'}(2) \longrightarrow \pi_* \mathcal{O}_{C_0}(2) \longrightarrow \bigoplus_{k=1}^{5}(\mathbb{C}_{p_k} \oplus \mathbb{C}_{q_k}) \longrightarrow 0 .$$

Where $\pi : C_0 \longrightarrow C_0' \subset \mathbb{P}^4$ is the map given by the linear series $|L_0|$. $C_0' = X' \cup Y'$ is the image of π consisting of two degree 6 plane curves with 5 nodes, and $\bigoplus_{k=1}^{5}(\mathbb{C}_{p_k} \oplus \mathbb{C}_{q_k})$ is the skyscraper sheaf of rank 10 supported on the five nodes of X' and five nodes on Y'.

Taking the long exact cohomology sequence, we obtain

$$0 \longrightarrow H^0(\mathcal{O}_{C_0'}(2)) \longrightarrow H^0(\mathcal{O}_{C_0}(2)) \xrightarrow{\phi} \bigoplus_{k=1}^{5}(\mathbb{C}_{p_k} \oplus \mathbb{C}_{q_k}) \longrightarrow H^1(\mathcal{O}_{C_0'}(2)) \longrightarrow 0$$

From the above exact sequence we see that a section $s \in H^0(L_0^2)$ is not coming from pull back of $H^0(\mathcal{O}_{\mathbb{P}^4}(2))$ if and only if $\phi(s)$ is not zero in $\mathbb{C}^{10}$, i.e. s separate at least one node on C_0'.

Now, choose σ_i sections of L_1, σ_j sections of L_{-1} according to their value at the inverse image under π of the ten nodes on C_0' as table 3.1 below. Where $i = 0, 1,$

$j = 2, 3$, p'_k, p''_k (resp. q'_k, q''_k) are points in X (resp. Y) which gets mapped to the node p_k (resp. q_k) on X' (resp. Y'). Here it suffices to consider two nodes on each component, say $k = 1, 2$. Cross means that σ_i does not vanish at the corresponding point, 0 means vanishing. For instance, σ_0 is a section of L_1, such that $\sigma_0|_X$ is a section of the $g_7^3 = K_X(-p)$ on X that vanish on p''_1 and p''_2 but not on p'_1 or p'_2 (Although the $g_6^2 = K_X(-2p)$ does not separate p'_1, p''_1 or p'_2, p''_2, the g_7^3 does). $\sigma_0|_Y$ is a section of the $g_5^1 = K_Y(-3p)$ on Y that vanishes on q'_1, q''_1, but not on q'_2 or q''_2. Similarly, for other σ_i. By the generality of X, Y and p, the assigned value in table 3.1 can be achieved.

Table 3.1:

	p'_1	p''_1	p'_2	p''_2	q'_1	q''_1	q'_2	q''_2
σ_0	×	0	×	0	×	×	0	0
σ_1	×	0	×	0	0	0	×	×
σ_2	×	×	0	0	×	0	×	0
σ_3	0	0	×	×	×	0	×	0

Table 3.2 describes the difference of $\sigma_i \sigma_j$ at p'_k, p''_k and q'_k, q''_k for $k = 1, 2$.

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Table 3.2:

	p_1	p_2	q_1	q_2
$\phi(\sigma_0\sigma_2)$	$\times$	0	$\times$	0
$\phi(\sigma_0\sigma_3)$	0	$\times$	$\times$	0
$\phi(\sigma_1\sigma_2)$	$\times$	0	0	$\times$
$\phi(\sigma_1\sigma_3)$	0	$\times$	0	$\times$

From table 3.2, we get a matrix of rank at least 3. Thus, $\mathrm{Im}(\delta_1)$ is mapped under ϕ to a subspace of dimension at least 3 in $\mathbb{C}^{10}$. In other words, we have shown that C_t for $t \neq 0$ is contained in at most one quadric in $\mathbb{P}^4$.

It remains to show that we get an extra dimension from

$$\mathrm{Im}(H^0(L_2) \otimes H^0(L_{-2}) \longrightarrow H^0(L_0^2)).$$

Choose $\lambda_1 \in H^0(L_2)$, $\lambda_2 \in H^0(L_{-2})$ according to table 3.3,

We get one more vector $\phi(\lambda_1\lambda_2)$ in $\mathbb{C}^{10}$. So we can add one row to the matrix in table 3.2, to get table 3.4.

To show the matrix in table 3.4 has rank 4, it suffices to show that the first row and the last row can be chosen linearly independently, or equivalently, that

$$\frac{\sigma_0|_X \cdot \sigma_2|_X}{\lambda_1|_X \cdot \lambda_2|_X}(p_1') \neq \frac{\sigma_0|_Y \cdot \sigma_2|_Y}{\lambda_1|_Y \cdot \lambda_2|_Y}(q_1').$$

This can be easily achieved, for instance, as follows. Take $X = Y$ and X, Y meeting at the same point $p \in X = Y$ and $q_1 = p_3$, $q_2 = p_2$. Choose $\sigma_0|_X = \sigma_2|_Y$, $\sigma_0|_Y = \sigma_2|_X$

Table 3.3:

	p_1'	p_1''	p_2'	p_2''	q_1'	q_1''	q_2'	q_2''
λ_1	$\times$	0	0	0	$\times$	$\times$	$\times$	$\times$
λ_2	$\times$	$\times$	$\times$	$\times$	$\times$	0	0	0

Table 3.4:

	p_1	p_2	q_1	q_2
$\phi(\sigma_0\sigma_2)$	$\times$	0	$\times$	0
$\phi(\sigma_0\sigma_3)$	0	$\times$	$\times$	0
$\phi(\sigma_1\sigma_2)$	$\times$	0	0	$\times$
$\phi(\sigma_1\sigma_3)$	0	$\times$	0	$\times$
$\phi(\lambda_1\lambda_2)$	$\times$	0	$\times$	0

$\lambda_1|_X = \lambda_2|_Y$ and $\lambda_1|_Y = \lambda_2|_X$ as the unique (up to scalar) sections satisfying the conditions in table 3.5:

Then

$$\frac{\sigma_0|_X \cdot \sigma_2|_X}{\lambda_1|_X \cdot \lambda_2|_X} = \frac{\sigma_0|_Y \cdot \sigma_2|_Y}{\lambda_1|_Y \cdot \lambda_2|_Y}$$

Table 3.5:

	p_1'	p_1''	p_2'	p_2''	p_3'	p_3''
$\sigma_0\|_X = \sigma_2\|_Y$	$\times$	0	$\times$	0	$\times$	0
$\sigma_0\|_Y = \sigma_2\|_X$	$\times$	$\times$	0	0	$\times$	$\times$
$\lambda_1\|_X = \lambda_2\|_Y$	$\times$	0	0	0	$\times$	0
$\lambda_1\|_Y = \lambda_2\|_X$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$

as rational functions, but since everything is general,

$$\frac{\sigma_0|_X \cdot \sigma_2|_X}{\lambda_1|_X \cdot \lambda_2|_X}(p_1') \neq \frac{\sigma_0|_Y \cdot \sigma_2|_Y}{\lambda_1|_Y \cdot \lambda_2|_Y}(p_3').$$

In conclusion, we can arrange so that the rank of the matrix in table 3.4 is exactly 4 and therefore the image in $\mathbb{P}^4$ of a general genus 10 curve under a general (thus every) g_{12}^4 is not contained in any quadric.

$\square$

Corollary 3.14. *For $g > 10$, a general curve in $\mathbb{P}^4$ with degree $d \geq \frac{4}{5}g + 4$ is not contained in any quadric.*

Proof. Consider the curve consists of a general curve X of genus 10 and a general curve Y of genus $g - 10$ meeting at a general point p. Consider the limit linear series with aspects $V_X = (d-12)p + |g_{12}^4|$, $V_Y = 8p + |g_{d-8}^4|$. Since everything is general, V_X has vanishing sequence $(d-12, d-11, ..., d-8)$ at p, and V_Y has vanishing sequence

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$(8, 9, 10, 11, 12)$. Since limit g_d^4 on $X \cup Y$ with the above specified vanishing sequence have dimension $\rho(g-10, 4, d-8) + \rho(10, 4, 12) = (g-10) - 5(g-10-(d-8)+4) + 0 = g - 5(g-d+4) = \rho(g, 4, d)$, by the smoothing theorem of limit linear series (see [16]), V_X, V_Y are smoothable. On the other hand, the image of X in $\mathbb{P}^4$ under ϕ_{V_X} is not contained in any quadric and $V_X|_Y$ is a $|g_{d-12}^0|$. Thus, by degenerating to such limit g_{12}^4 on $X \cup Y$, we have our conclusion.

$\square$

CHAPTER 4

DEGENERATIONS OF SYMMETRIC PRODUCTS OF CURVES

4.1 The relative Hilbert scheme of points

Suppose $\mathcal{C} \to \Delta$ is a flat one parameter family of smooth projective curves over the punctured disk $\Delta^* = \Delta - \{0\}$, degenerating to a reduced nodal curve at $t = 0 \in \Delta$. We will, in addition, assume the total space $\mathcal{C}$ is smooth over $\mathbb{C}$.

Denote $\mathcal{C}_{\Delta^*}^{(d)}$ the relative symmetric product of this family over Δ^*, parameterizing effective divisors of degree d on fibres C_t for $t \neq 0$. We would like to construct a compactification $\widetilde{\mathcal{H}}_d$ of $\mathcal{C}_{\Delta^*}^{(d)}$ over Δ, such that $\widetilde{\mathcal{H}}_d$ has smooth total space and the fibre over $t = 0$ has simple normal crossing support. By studying the fibre over $t = 0$ of $\widetilde{\mathcal{H}}_d$, we understand how the symmetric products of smooth curves degenerate as the curves degenerate to a nodal curve.

The first candidate for this compactification is the relative symmetric product $\mathcal{C}_{\Delta}^{(d)}$ over Δ. But this is singular. In fact, let $p \in C_0$ be a node of C_0, locally at p, this family is given by $\{xy = t\}$ in affine 3 space. Thus locally (analytically) near the cycle $d[p]$, $\mathcal{C}_{\Delta}^{(d)}$ is a quotient of a complete intersection by the symmetric group S_d

$$(x_1y_1 = x_2y_2 = ... = x_dy_d)/S_d$$

For a smooth curve C, the Hilbert scheme parameterizing length-d dimension-0 subschemes of C is isomorphic to the symmetric product $C^{(d)}$. So if we consider the relative Hilbert scheme $\mathcal{H}_d = Hilb_d(\mathcal{C}/\Delta)$ parameterizing length-d dimension-0 subschemes in the fibers of the family $\mathcal{C}/\Delta$, it is a natural compactification of $\mathcal{C}_{\Delta^*}^{(d)}$. Moreover, there is a natural morphism

$$\pi : \mathcal{H}_d \to \mathcal{C}_{\Delta}^{(d)}$$

defined by

$$\pi(Z_t) = \sum_{x \in C_t} length_x(Z_t)[x],$$

where Z_t is a length-d, dimension-0 subscheme of C_t.

In fact, $\mathcal{H}_d$ is a partial resolution of singularities of $\mathcal{C}_{\Delta}^{(d)}$. To explain why this is the case. Let's look at the simplest case:

Example. Suppose $d = 2$ and the special fibre C_0 has only two components X and Y meeting at a point p. $C_0^{(2)}$ has three components: $X^{(2)}, Y^{(2)}$, and $X \times Y$. We can figure out the scheme structure of $\mathcal{C}_{\Delta}^{(2)}$. Locally at p, this family is given by $\{xy = t\}$ with projection to the t-disk. The relative symmetric product is given by taking the fibre product of the family with itself,

$$\mathcal{C} \times_{\Delta} \mathcal{C} \cong Spec(\mathbb{C}[x, y, z, w, t]/(xy = zw = t)) =: SpecA,$$

modulo the $\mathbb{Z}_2$ action $x \to z, y \to w$.

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The ring of invariants $A^{\mathbb{Z}_2}$ under this action is generated by

$$\mathbb{C}[x+z, y+w, xz, yw, xw+yz, t] \subset Spec\frac{\mathbb{C}[x,y,z,w,t]}{(xy-t, \ zw-t)}.$$

It is easy to write down all relations between the above generators, and we find that $A^{\mathbb{Z}_2}$ is isomorphic to

$$\frac{\mathbb{C}[X_1, X_2, X_3, X_4, X_5, t]}{(X_5+2t-X_1X_2, \ X_1X_4-tX_2, \ X_2X_3-tX_1, \ X_3X_4-t^2)}$$

Eliminating X_5, we get,

$$\frac{\mathbb{C}[X_1, X_2, X_3, X_4, t]}{(X_1X_4-tX_2, \ X_2X_3-tX_1, \ X_3X_4-t^2)}.$$

The fibre over $t=0$ is

$$Spec\frac{\mathbb{C}[X_1, X_2, X_3, X_4]}{(X_1X_4, \ X_2X_3, \ X_3X_4)}.$$

In particular, $C_0^{(2)}$ is not normal crossing. One of these components, namely, $X \times Y$ (defined by $X_3 = X_4 = 0$ in the local description of $C_0^{(2)}$,) is distinguished in that it meets both of the others along curves, the other pair of components meet only at the bad point $2[p]$ (see figure 4.1).

Now, let's consider the Hilbert-Chow morphism

$$\pi : Hilb_2(C_0) \longrightarrow C_0^{(2)}.$$

It is an isomorphism away from the bad point $2[p]$; over $2[p]$, there is a family of ideals parameterized by $\mathbb{P}_a^1$:

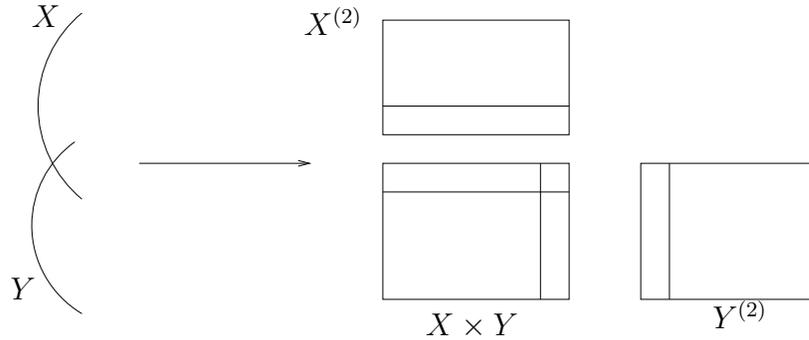$$(x+ay) \subset \frac{\mathbb{C}[x,y]}{(xy)}.$$

Figure 4.1: Components of the symmetric square of a nodal curve

Each one of these ideals corresponds to the point $(x = y = 0)$ plus a tangent direction at that point. The $\mathbb{P}^1_a$ is parameterizing the ratio of velocities of a point on X and a point on Y approaching p. Hence, we obtain $Hilb_2(C_0)$ from $C_0^{(2)}$ by blowing up the point $2[p]$ in the component $X \times Y$ and gluing back $X^{(2)}(resp.Y^{(2)})$ along the strict transform of $X \times \{p\}(resp.\{p\} \times Y)$ (see figure 4.2).

In fact, under the assumption about the special fibre C_0 as in this example, $Hilb_d(\mathcal{C}/\Delta)$ is always smooth and $Hilb_d(C_0)$ is reduced and normal crossing. So we have a nice degeneration of the symmetric product of smooth curves.

$\square$

In [33], Ran studied the local structure of the relative Hilbert scheme, i.e for $\mathcal{C} = Spec(\mathbb{C}[x,y]_{(x,y)})$, with projection map $t = xy$ to $\Delta = Spec(\mathbb{C}[t]_{(t)})$. He proved theorems 4.1 to 4.3, which describe the fibre of the Hilbert-Chow morphism, the Hilbert scheme of the special fibre, and the total space of the relative Hilbert scheme:
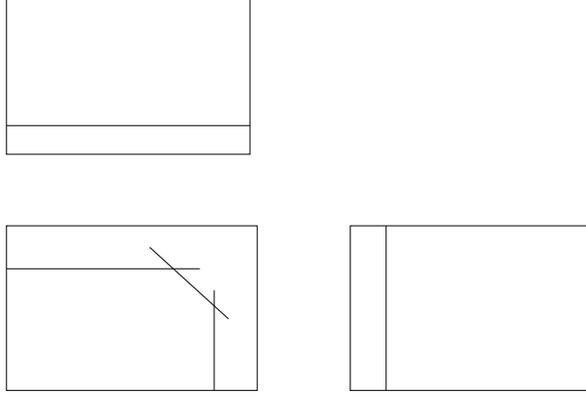
Theorem 4.1. ([*33*])

Figure 4.2: Blowing up and regluing

(1) *Every ideal I of $R = \frac{\mathbb{C}[x,y]}{(xy)}{}_{(x,y)}$ of colength d is one of the following, said to be of type $(c_i^d),(q_i^d),$ respectively:*

$$I_i^d(a) = (y^i + ax^{d-i}), \quad 0 \neq a \in \mathbb{C}, \quad i = 1, ..., d-1,$$

$$Q_i^d = (x^{d-i+1}, y^i), \quad i = 1, ..., d.$$

(2) *The closure C_i^d in the Hilbert scheme of the set of ideals of type (c_i^d) is isomorphic to $\mathbb{P}^1$ and consists of the ideals of types (c_i^d) or (q_i^d) or (q_{i+1}^d). In fact, we have*

$$\lim_{a \to 0} I_i^d(a) = Q_i^d,$$

$$\lim_{a \to \infty} I_i^d(a) = Q_{i+1}^d.$$

(3) *The punctual Hilbert scheme $Hilb_d^0(R)$, as algebraic set, is a rational chain*

$$C_1^d \cup_{Q_2^d} C_2^d \cup ... \cup_{Q_{d-1}^d} C_{d-1}^d;$$

it has ordinary nodes at $Q_2^d, ...Q_{d-1}^d$ and is smooth elsewhere.

Theorem 4.2. ([33]) *The Hilbert scheme* $Hilb_d(R)$, *where* $R = \frac{\mathbb{C}[x,y]}{(xy)}_{(x,y)}$ *is a chain*

$$W_0^d \cup W_1^d \cup ... \cup W_{d-1}^d \cup W_d^d$$

where each W_i^d *is a smooth d-dimensional germ supported on* C_i^d *for* $i = 1,...,d-1$ *or* Q_i^d *for* $i = 0, d$; *for* $i = 1,...,d-1$, W_i^d *meets its neighbors* $W_{i\pm1}^d$ *transversely in dimension* $d-1$ *and meets no other* W_i^d. *The generic point of* W_i^d *corresponds to subscheme of* $Spec(R)$ *comprised of* $d-i$ *points on the x-axis and* i *points on the y-axis.*

Theorem 4.3. ([33]) *Set* $\widetilde{R} = \mathbb{C}[x,y]_{(x,y)}$. $B = \mathbb{C}[t]_{(t)}$ *and view* $\widetilde{R}$ *as a B-module via* $xy = t$. *The relative Hilbert scheme* $Hilb^d(\widetilde{R}/B)$ *is formally smooth, formally* $(d+1)$-*dimensional over* $\mathbb{C}$.

Remark. Moreover, from the proof of theorem 4.3 in [33], $Hilb^d(\widetilde{R}/B)$ is given by equation $x_1 y_1 = t$ in $\mathbb{C}^{d+1} \times \Delta_t$. $\qquad\square$

The above theorems give the local structure of $\mathcal{H}_d = Hilb_d(\mathcal{C}/\Delta)$ completely (for any one parameter nodal degeneration). For any $Z \in \mathcal{H}_d$, suppose $\pi(Z) = \sum_i n_i[x_i]$ with x_i pairwise distinct, choose small analytic neighborhood U_i of x_i in $\mathcal{C}$, then locally at Z, $\mathcal{H}_d$ looks like the fibre product of $Hilb_{n_i}(U_i/\Delta)$ over Δ. Even if each $Hilb_{n_i}(U_i/\Delta)$ is smooth by Theorem 2.3, the fibre product could still be singular. Thus, we need a little bit more work to get a smooth compactification of $\mathcal{C}_{\Delta^*}^{(d)}$.

4.2 The augmented relative Hilbert scheme

In this section, we will describe explicitly a Log resolution $\widetilde{\mathcal{H}}_d$ of the pair $(\mathcal{H}_d, H_0^d)$ where $H_0^d = Hilb_d(C_0)$. Call it the augmented relative Hilbert scheme.

Let's first analyze the singularities of $\mathcal{H}_d$. From the remark and the last paragraph of section 4.1, we see that locally at Z, $\mathcal{H}_d$ looks like

$$Spec\frac{\mathbb{C}[x_1, y_1, ..., x_m, y_m]}{(x_1y_1 = x_2y_2 = ... = x_my_m)} = X_1 \times_\Delta X_2 \times_\Delta ... \times_\Delta X_m$$

modulo possibly crossing with an affine space. Here m is the number of distinct points of the cycle corresponding to Z, $X_i = Spec\mathbb{C}[x_i, y_i]$, and each X_i maps to Δ by $t = x_iy_i$.

To resolve the above singularity, we will do induction on m. The case $m = 2$ is simple. We just have an isolated ordinary node in a 3-fold. Blowing up the node in the 3-fold, we get a smooth space with exceptional divisor a quadric surface in $\mathbb{P}^3$. The function t vanishes to order 2 on the exceptional divisor. It is also clear that the fibre of this blowing-up at $t = 0$ has simple normal crossing support.

Let's work out the case $m = 3$ in more detail, this will help us set up the induction. Here $\mathcal{H}_d$ is locally isomorphic to

$$Spec\frac{\mathbb{C}[x_1, y_1, x_2, y_2, x_3, y_3]}{(x_1y_1 = x_2y_2 = x_3y_3)} = X_1 \times_\Delta X_2 \times_\Delta X_3$$

.

The fibre over $t = 0$ is $X_{01} \times X_{02} \times X_{03}$, where

$$X_{0i} = Spec\frac{\mathbb{C}[x_i, y_i]}{(x_iy_i)}, \quad i = 1, 2, 3.$$

The total space of $\mathcal{H}_d$ is singular at $X_{01} \times \{0\} \times \{0\} \bigcup \{0\} \times X_{02} \times \{0\} \bigcup \{0\} \times \{0\} \times X_{03}$. Each X_{0i} is a union of two lines. So the singular locus is union of six lines meeting at the point $(0, 0, 0)$. If we first blow up the point $(0, 0, 0)$ in $\mathcal{H}_d$, call

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the new space $\widetilde{\mathcal{H}}'_d$, then the proper transform of the above six lines become disjoint, and $\widetilde{\mathcal{H}}'_d$ has ordinary nodal singularity along each of the six lines (basically it is a line cross the one dimension lower case). Then, we blow up the six lines to get our smooth resolution $\widetilde{\mathcal{H}}_d$. There are seven exceptional divisors for the map from $\widetilde{\mathcal{H}}_d$ to $\mathcal{H}_d$, namely, one over the point $(0,0,0)$, and one over each of the six lines. The (reduced) fibre over $t = 0$ is still normal crossing.

For the general case, $X_1 \times_\Delta ... \times_\Delta X_m$ is singular at the points where at least two of the m coordinates are 0. Let

$$W_i = \{(x_1, ..., x_m) \in X_{01} \times ... \times X_{0m} : \text{ at least i factors are } 0\}$$

To resolve the singularity, we first blow up $X_1 \times_\Delta ... \times_\Delta X_m$ along "the most singular" locus W_m. Then the proper transform $\widetilde{W_{m-1}}$ is disjoint union of $2m$ lines. There is an open neighborhood of each line in the total space of the blow-up isomorphic to the line cross one dimension lower case. And we keep on blowing up along $\widetilde{W_{m-1}}$, etc. By induction, we will get a smooth total space $\widetilde{\mathcal{H}}_d$ after we blow up $\widetilde{W}_2$. The fibre of $\widetilde{\mathcal{H}}_d$ over $t = 0$ has simple normal crossing support.

4.3   A toric description

Since $X_1 \times_\Delta X_2 \times_\Delta X_3$ is an affine toric variety, we can describe the above resolution by subdividing the cone associated to it. Again, we will use induction on m. Denote $N = \mathbb{Z}^{m+1}$ the lattice and $M = Hom_\mathbb{Z}(N, \mathbb{Z})$ the dual lattice.

The case $m = 2$ is well known. The dual cone $\check{\sigma}$ of

$$Spec \frac{\mathbb{C}[x_1, x_2, y_1, y_2]}{(x_1 y_1 - x_2 y_2)}$$

is generated by four lattice vectors u_1, v_1, u_2, v_2 (assuming they generate $M = \mathbb{Z}^3$ as abelian group) in $M_{\mathbb{R}} = M \otimes_{\mathbb{Z}} \mathbb{R} = \mathbb{R}^3$, satisfying

$$u_1 + v_1 = u_2 + v_2$$

Without loss of generality, set $u_1 = e_1$, $u_2 = e_2$, $v_1 = e_3$. Where $\{e_i; i = 1, 2, 3\}$ is the standard basis for $M_{\mathbb{R}}$.

Under the natural pairing

$$(\bigwedge^2 \mathbb{R}^3) \bigotimes \mathbb{R}^3 \longrightarrow \bigwedge^3 \mathbb{R}^3 = \mathbb{R}(e_1 \wedge e_2 \wedge e_3),$$

we can identify $N_{\mathbb{R}} = \bigwedge^2 \mathbb{R}^3$ with $Hom_{\mathbb{R}}(M_{\mathbb{R}}, \mathbb{R})$. In this setting, the cone σ is generated (up to sign) by $u_1 \wedge u_2$, $u_1 \wedge v_2$, $v_1 \wedge u_2$, $v_1 \wedge v_2$. In fact, $u_1 \wedge u_2 + v_1 \wedge v_2 = (-u_1 \wedge v_2) + (-v_1 \wedge v_2) =: a$. Then by adding a line through a, we subdivide σ into four smaller cones. It is immediate to check each of the smaller cone is generated by a basis of the lattice N. By the general theory of toric varieties (see [21] chapter 2), this means the new variety we get by subdividing the cone is a resolution of singularities of the original variety. Geometrically, this subdivision corresponds to blowing up the singular point of the threefold node (figure 4.3).

For the general case, take $\check{\sigma}$ to be the $(m+1)$-dim cone generated in $M_{\mathbb{R}} = \mathbb{R}^{m+1} = \oplus_{i=1}^{m} \mathbb{R}e_i$ by $\{u_i, v_i; i = 1, ..., m\}$ satisfying

$$u_1 + v_1 = u_2 + v_2 = ... = u_m + v_m$$

(also assume they generate the lattice M as abelian group, say, $u_i = e_i; i = 1, ..., m$, $v_1 = e_{m+1}$). To find the generators of the dual cone σ (see [21]), we take each set of
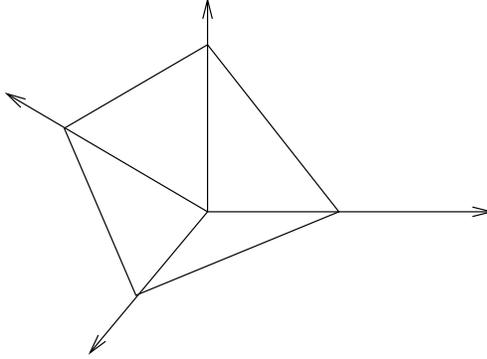
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Figure 4.3: The cone of a threefold node

m independent vectors among the generators of $\check{\sigma}$ which spans an m-plane, solve for a vector $u \in N_{\mathbb{R}}$ annihilating the m-plane. If neither u or $-u$ is nonnegative on all generators of $\check{\sigma}$, it is discarded; otherwise either u or $-u$ whichever is nonnegative on $\check{\sigma}$ is taken as a generator for σ. In the latter case, the m-plane (more precisely the cone generated by the m independent vectors) is a facet of $\check{\sigma}$. A little computation will convince the audience that there are 2^m facets of $\check{\sigma}$: each facet is generated by choosing for each subindex i either u_i or v_i, not both, among $\{u_i, v_i; i = 1, ..., m\}$.

Under the natural identification of

$$N_{\mathbb{R}} = Hom_{\mathbb{R}}(M_{\mathbb{R}}, \mathbb{R}) \cong \bigwedge^m \mathbb{R}^{m+1},$$

we get 2^m generators for σ (up to a sign) by taking wedge product of the generators for each facet. Now let $\vec{x} = (x_i; \ x_i = u_i \text{ or } v_i; \ i = 1, ..., m)$ be a set of generators for a facet of $\check{\sigma}$. Then $(-1)^{|\vec{x}|} x_1 \wedge ... \wedge x_m$ is a generator of σ. The sign $(-1)^{|\vec{x}|}$ is determined such that $(-1)^{|\vec{x}|} x_1 \wedge ... \wedge x_m \wedge b > 0$, where $b = u_1 + v_1$ (This is equivalent

to $(-1)^{|\vec{x}|}x_1 \wedge ... \wedge x_m \wedge (b - x_i) > 0;\ i = 1, ..., m)$. There is a 'complementary' facet generated by $(y_i = b - x_i; i = 1, ..., m)$. A little algebra tells us

$$(-1)^{|\vec{x}|}x_1 \wedge ... \wedge x_m + (-1)^{|\vec{y}|}y_1 \wedge ... \wedge y_m$$

does not depend on $\vec{x}$. Call this vector a (well defined up to a scalar multiple) the 'barycenter' of σ. So far, we have seen the 2^m facets of $\check{\sigma}$ which are 'dual' to the 2^m generators of σ . For the same reason, the $2m$ generators of $\check{\sigma}$ correspond to $2m$ facets of σ. The correspondence is the following: u_i corresponds to the facet in the m-plane $u_i \wedge (\bigwedge^{m-1} \mathbb{R}^{m+1}) \subset N_{\mathbb{R}} = \bigwedge^m \mathbb{R}^{m+1}$. This facet is generated by all possible $x_1 \wedge ... \wedge x_m$ with $x_i = u_i$. It is an m-dim cone with 2^{m-1} generators. Applying the argument as before, we see there is also a 'barycenter' for this facet. Therefore, we get a 'barycenter' for each facet. Then we work inductively to find a 'barycenter' for each $(m - 1)$-dim face of σ and so on. Finally we can construct a 'barycenter' for each face of σ of dimension at least three. It follows immediately that

Proposition 4.4. *Notation as last paragraph. If we subdivide σ by adding all 'barycenters' of faces of σ. The resulting toric variety is the resolution of singularity described in section 4.2.*

Finally, we observe that if we cover the relative Hilbert scheme $\mathcal{H}_d$ by analytic open subset like above, and take the resolution for each open subset. The resulting nonsingular open sets will patch together to form $\widetilde{\mathcal{H}}_d$, since on each small open set the resolution is canonical.

In conclusion, we have proved

Theorem 4.5. *For any one parameter family $\mathcal{C} \to \Delta$ with smooth total space degenerating to a reduced nodal curve C_0, there is a canonical sequence of blow-ups of $\mathcal{H}_d$ along smooth centers in $H_0^d = Hilb_d(C_0)$ that leads to a canonical log resolution $\widetilde{\mathcal{H}}_d$ of $(\mathcal{H}_d, H_0^d)$.*

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