MASS EQUIDISTRIBUTION OF HECKE EIGENFORMS ON THE HILBERT MODULAR VARIETIES

DISSERTATION

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By

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ABSTRACT

In this thesis we study the analogue of Arithmetic Quantum Unique Ergodicity conjecture on the Hilbert modular variety. Let F be a totally real number field with ring of integers $\mathcal{O}$, and let $\Gamma = SL(2, \mathcal{O})$ be the Hilbert modular group. Given the orthonormal basis of Hecke eigenforms in $S_{2k}(\Gamma)$, the space of cusp forms of weight $(2k, 2k, \cdots, 2k)$, one can associate a probability measure $d\mu_k$ on the Hilbert modular variety $\Gamma\backslash\mathbb{H}^n$. We prove that $d\mu_k$ tends to the invariant measure on $\Gamma\backslash\mathbb{H}^n$ weakly as $k \to \infty$. This shows that the analogue of Arithmetic Quantum Unique Ergodicity conjecture is true on the average on Hilbert modular variety. Our result generalizes Luo's result [Lu] for the case $F = \mathbb{Q}$.

Our approach is using Selberg trace formula, Bergman kernel, and Shimizu's dimension formula.

Dedicated to my family

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INTRODUCTION

1.1 Arithmetic Quantum Unique Egrodicity

Suppose Y is a compact Riemannian manifold with the normalized Riemannian measure $d\nu$ and the associated Laplace operator Δ. Let $\{\phi_j\}_{j\geq 0}$ be an orthonormal basis of $L^2(Y)$ consisting of Laplace eigenfunctions with increasing eigenvalues, i.e.

$$\Delta\phi_j + \lambda_j\phi_j = 0 \text{ and } 0 = \lambda_0 \leq \lambda_1 \leq \lambda_2 \leq \cdots \to \infty.$$

From quantum physics Δ is the quantization of the Hamiltonian generating the geodesic flow and the weak* limits of the sequence of probability measures

$$d\nu_j = |\phi_j(x)|^2 d\nu$$

are called *quantum limits.* One can find more details from the physics viewpoint in the conference volume *"Chaos and Quantum Physics"* [GVZ]. An important result in studying quantum limits is due to Shnirelman, Zelditch and Colin de Verdière ([Shn], [Ze], [CV]) as follows:

Theorem 1.1.1. *(Quantum Ergodicity).*

If the geodesic flow is ergodic, then there exists a full density subsequence $\{\lambda_{j_k}\}$ *of* $\{\lambda_j\}$, *i.e.*

$$\lim_{T\to\infty} \frac{\#\{j_k|\lambda_{j_k} \le T\}}{\#\{j|\lambda_j \le T\}} = 1$$

such that

$$d\nu_{j_k} \longrightarrow d\nu \quad \text{as } \lambda_{j_k} \to \infty.$$

This phenomenon is called *Quantum Ergodicity* and it is known that if Y has negative sectional curvature, then the geodesic flow is ergodic. However quantum ergodicity does not give an explicit subsequence having quantum limit $d\nu$. It does not eliminate the possibility of existing an exceptional subsequence having a quantum limit different from $d\nu$ either. Such exceptional weak limits are called *strong scars*. In the special case of an arithmetic hyperbolic surface, Rudnick and Sarnak conjectured the non-existence of strong scars which is the main topic we will discuss here.

Let $X = \Gamma\backslash\mathbb{H}$ be an *arithmetic* hyperbolic surface where $\Gamma \subset SL(2,\mathbb{R})$ is a discrete *arithmetic* subgroup with finite covolume. We denote by Δ the Laplace-Beltrami operator associated to X. In this case the Laplacian is given by

$$\Delta = y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right).$$

We also denote $\{\phi_j\}_{j\ge 0}$ an orthonormal basis of eigenfunctions with increasing eigenvalues $\{\lambda_j\}_{j\ge 0}$, (*i.e.* $0 = \lambda_0 \le \lambda_1 \le \lambda_2 \le \cdots \to \infty$). According to Weyl's law,

$$\#\{j : \lambda_j \le T\} \sim \frac{Area(\Gamma\backslash\mathbb{H})}{4\pi} T, \quad \text{as } T \to \infty.$$

One can define a commutative family $\mathcal{H}$ of Hecke operators on $L^2(X)$ (see [IR] or [Iw1] for details). Moreover $\mathcal{H}$ commutes with Δ. Hence we may consider that ϕ_j

are also eigenfunctions of $\mathcal{H}$. This is the case we will concern from now on. We may note that the spectrum of the Laplace operator is expected to be simple, so that any eigenfunction of Δ would be automatically an eigenfunction of all Hecke operators. However this has not been proved.

An important problem of arithmetic quantum chaos is understanding the asymptotic behavior of such ϕ_j as the eigenvalue $\lambda_j \to \infty$. The equidistribution problem of ϕ_j asks whether $|\phi_j(z)|$ is approximately constant as $\lambda_j \to \infty$. One approach is to bound the L^∞-norm or L^p-norm of ϕ_j in terms of λ_j. Iwaniec and Sarnak ([IR]) gave a nontrivial bound of L^∞-norm: $\|\phi_j\|_\infty \ll_\varepsilon \lambda_j^{\frac{5}{24}+\varepsilon}$. Sarnak and Watson had some unpublished results concerning the bound of L^4-norm. The conjecture in this direction is $|\phi_j(z)| \ll_{\varepsilon,z} \lambda_j^\varepsilon$ as $\lambda_j \to \infty$ and it is still out of reach at the present.

Another approach is to study the probability measures $d\mu_j = |\phi_j(z)|^2 d\mu$ on X as $\lambda_j \to \infty$. Here $d\mu = \dfrac{1}{\mathrm{vol}(X)} \dfrac{dxdy}{y^2}$ is the normalized Γ-invarant measure on X. In 1994, Rudnick and Sarnak [RS] formulated the following conjecture predicting the behavior of Maass-Hecke eigenforms on arithmetic surfaces as their corresponding Laplace eigenvalues tend to infinity. The conjecture is known as the *Arithmetic Quantum Unique Ergodicity* (AQUE) conjecture for modular surfaces.

Conjecture 1.1.2 (Arithmetic Quantum Unique Ergodicity)**.** *Let $X = \Gamma\backslash\mathbb{H}$ be an arithmetic hyperbolic surface of negative curvature. Then the measure $d\mu$ is the unique arithmetic quantum limit. This means the measures $d\mu_j$ is weak* convergent to $d\mu$ as $\lambda_j \to \infty$. i.e. for any $h(z)$ in $C_0(X)$, we have*

$$\lim_{\lambda_j \to \infty} \left| \int_X h(z)|\phi_j(z)|^2 d\mu - \int_X h(z) d\mu \right| = 0.$$

The AQUE conjecture for compact arithmetic hyperbolic surfaces (associated with quaternion algebras) was proved by Lindenstrauss [Li] by using ergodic theory.

Theorem 1.1.3 (Lindenstrauss)**.** *Let* $X = \Gamma\backslash\mathbb{H}$ *be an arithmetic hyperbolic surface. If* X *is compact, then the only arithmetic quantum limit is the normalized invariant measure* $d\mu$*. If* X *is not compact, then the arithmetic quantum is of the form* $c \cdot d\mu$ *for some* $c \in [0, 1]$*.*

The most interesting arithmetic group in number theory is the full modular group $\Gamma = SL(2, \mathbb{Z})$. In this case $X = SL(2, \mathbb{Z})\backslash\mathbb{H}$ is noncompact and the conjecture was just proved by Soundararajan [So1]. He based on Lindenstrauss' result to prove $c = 1$ in above theorem. However Soundararajan's result does not give the rate of convergence. Luo and Sarnak formulated the following quantitative form of the conjecture predicting the rate of convergence: For $\varepsilon > 0$,

$$\int_X |\phi_i(z)|^2 h(z) d\mu = \int_X h(z) d\mu + O_{\varepsilon,f}(\lambda_i^{-1/4+\varepsilon}).$$

They showed in [LS] that the conjecture holds on average. More precisely, for any $\varepsilon > 0$, they showed that

$$\sum_{\lambda_i \leq \lambda} \left| \int_X h(z) d\mu_j - \int_X h(z) \right|^2 \ll_{\varepsilon,h} \lambda^{1/2+\varepsilon}.$$

1.2 Holomorphic Analogue of AQUE in Weight Aspect

It is of great interest to consider a natural analogue of the AQUE conjecture for holomorphic cusp forms with the weight going to infinity.

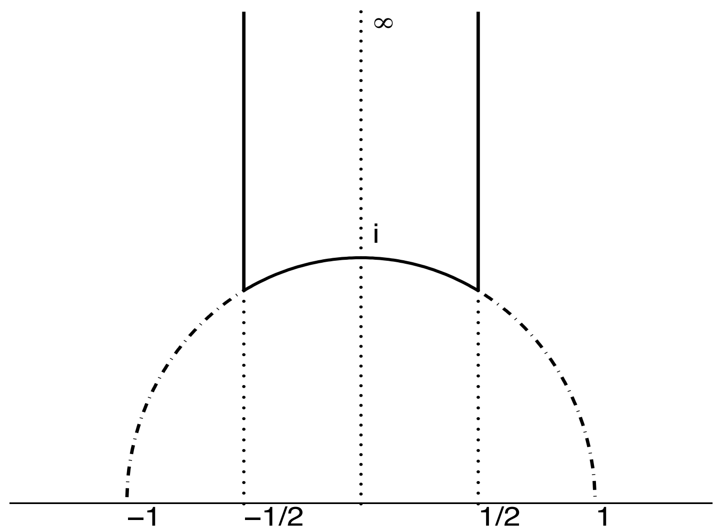


Figure 1.1: A fundamental domain for $SL(2,\mathbb{Z})$

Let $\Gamma = SL(2,\mathbb{Z})$ be the full modular group and let $X = \Gamma\backslash\mathbb{H}$ (see Figure 1.1). Let $\{f_{j,k}\}_{1\leq j\leq J_k}$ be the orthonormal basis of Hecke eigenforms in $S_{2k}(\Gamma)$, the space of holomorphic cusp forms of weight $2k$ with respect to the full modular group $\Gamma = SL(2,\mathbb{Z})$. Thus by the Riemann-Roch theorem

$$J_k = \dim_{\mathbb{C}} S_{2k}(\Gamma) = \begin{cases} \lfloor \frac{k}{6} \rfloor - 1, & \text{if } k \equiv 1 (mod\ 6) \\ \lfloor \frac{k}{6} \rfloor, & \text{if } k \not\equiv 1 (mod\ 6). \end{cases}$$

Let

$$d\mu = \frac{1}{\mathrm{vol}(X)} \frac{dxdy}{y^2} = \frac{3}{\pi} \frac{dxdy}{y^2}$$

be the normalized invariant measure on X and let

$$d\mu_{j,k} = |f_{j,k}|^2 y^{2k} d\mu.$$

As an analogue of unique ergodicity, Shiffman-Zelditch [SZ] proved the following theorem.

Theorem 1.2.1 (Shiffman-Zeldithch)**.** *There exists a full density subsequence of* $\{f_{j,k}\}_{1\le j\le J_k}$, *i.e. there exist a subset* $\Lambda_k \subseteq \{1, 2, \cdots, J_k\}$ *satisfying*

$$\lim_{k\to\infty} \frac{\#\Lambda_k}{J_k} = 1,$$

such that for any compact region $A \subset SL(2,\mathbb{Z})\backslash\mathbb{H}$, *we have*

$$\lim_{k\to\infty, j\in\Lambda_k} \int_A d\mu_{j,k} = \int_A d\mu.$$

Moreover using the potential theory, they showed that the zeros of the sequence $f_{j,k}$ $(j \in \Lambda_k)$ *are also equidistributed, i.e.*

$$\lim_{k\to\infty, j\in\Lambda_k} \frac{\#\{z \in A : f_{j,k}(z) = 0\}}{J_k} = \int_A d\mu.$$

One expects the following mass equidistribution conjecture on the modular surface $X = SL(2,\mathbb{Z})\backslash\mathbb{H}$ should be true (i.e. no exceptional subsequence).

Conjecture 1.2.2 (Mass Equidistribution Conjecture)**.** *For any* $h(z)$ *in* $C_0(X)$, *we have*

$$\lim_{k\to\infty} \max_{1\le j\le J_k} \left| \int_X h(z)|f_{j,k}(z)|^2 y^{2k} d\mu - \int_X h(z) d\mu \right| = 0.$$

This conjecture was proved just recently by Holowinsky and Soundararajan ([HS], [Ho][So]). Their approaches use in an essential way that the Hecke eigenvalues of a holomorphic cusp eigenform satisfy the Ramanujan conjecture (Deligne's theorem). Assuming the Ramanujan conjecture for Maass forms, their methods would obtain the AQUE conjecture. In Holowinsky and Soundararajan's proof, it is necessary to consider the Fourier expansion of a holomorphic cusp form at a cusp. Hence it does not apply to the holomorphic forms on compact modular surface and in this case the mass equidistribution conjecture is still open.

Remarks.

1) Note that by Riemann-Roch theorem, we have

$$\#\{z \in X : f_{j,k}(z) = 0\} \sim J_k, \quad \text{as } k \to \infty.$$

Rudnick [Ru] proved that the mass equidistribution conjecture implies that the zeros of a Hecke eigen-cusp form become equidistributed in $SL(2,\mathbb{Z})\backslash\mathbb{H}$ with respect to $d\mu$ as $k \to \infty$. Now this is a theorem. This feature is not clear in Maass form case.

2) In the Maass form case, the Laplacian eigenform is expected to be a Hecke eigenform. However in the holomorphic case, it is necessary to restrict to consider Hecke eigenforms in this conjecture. For example, the measure associated to the cusp form $\triangle(z)^{\frac{k}{12}}$ in $S_k(SL_2(\mathbb{Z}))$ does not tend to equidistribution as $k \to \infty$. Here $12|k$ and $\triangle(z)$ is the Ramanujan cusp form (see [Iw]).

3) From Watson's explicit triple product L-function formula [Wa], it follows that the generalized Riemann hypothesis for certain triple product L-functions implies the AQUE conjecture for Maass forms and the mass equidistribution conjecture for

holomorphic Hecke eigen-cusp forms and provides sharp convergent rate. We will give more precise discussion in Chapter 5.

4) The conjecture may be formulated for holomorphic newforms on $X_N = \Gamma_0(N)\backslash\mathbb{H}$. Holowinsky and Soundararajan's methods would also apply in this case. For the special case of dihedral forms in the holomorphic (respectively non-holomorphic) case, the conjecture has been established by Sarnak [Sa] (respectively Liu-Ye [LY]).

5)It is not clear how to extend Lindenstrauss' ergodic method [Li] to the holomorphic setting.

One may consider quantum unique ergodicity in higher dimensional symmetric spaces. See Silberman-Venkatesh [SV] for a locally symmetric space and Cogdell-Luo [CL] for the Siegel modular variety. In this thesis we will consider an analogue equidistribution problem for holomorphic Hecke eigenforms on the Hilbert modular varieties $SL(2,\mathcal{O})\backslash\mathbb{H}^n$, where F is a totally real number field and $\mathcal{O}$ is the ring of integers in F (See chapter 3 for details). In this case, $SL(2,\mathcal{O})\backslash\mathbb{H}^n$ is noncompact. We expect the following mass equidistribution conjecture on the Hilbert modular variety $\Gamma\backslash\mathbb{H}^n$ should be true (where $\Gamma = SL(2,\mathcal{O})$).

Conjecture. (Mass Equidistribution)

$$\lim_{k\to\infty} \max_{1\le i\le J_k} \left| \int_A (Ny)^{2k} |f_{i,k}(z)|^2 d\mu - \int_A d\mu \right| = 0$$

where $A \subset \Gamma\backslash\mathbb{H}^n$ *is compact and* $\{f_{i,k}\}_{i=1}^{J_k}$ *is the orthonormal Hecke basis of* $S_{2k}(\Gamma)$.

Our main result is to prove the mass equidistribution conjecture on the average and give a sharp bound of the rate of convergence as follows:

Theorem. *Let* $\{f_{i,k}\}_{i=1}^{J_k}$ *be an orthonormal basis of* $S_{2k}(\Gamma)$. *Set*

$$d\mu_k = \frac{1}{J_k}\left(\sum_{i=1}^{J_k} |f_{i,k}(z)|^2\right)(Ny)^{2k} d\mu.$$

Then for any compact subset $A \subset \Gamma\backslash\mathbb{H}^n$ *and any* $0 < \epsilon < 1$, *we have*

$$\int_A d\mu_k = \int_A d\mu + O_{\epsilon,A}\left((k^{-1+\epsilon})^n\right)$$

as $k \longrightarrow \infty$.

Our approach is using Selberg trace formula, Bergman kernel, and Shimizu's dimension formula.

CHAPTER 2

PRELIMINARIES

2.1 The Hilbert Modular Group

Let F be a totally real number field of degree n over $\mathbb{Q}$ with ring of integers $\mathcal{O}$ and $\sigma_1, \sigma_2, \cdots, \sigma_n$ be all the real embeddings of F. The group $\Gamma = SL(2, \mathcal{O})$ is called the (full) *Hilbert modular group*. It can be shown that Γ acts discontinuously on the product of n upper half planes $\mathbb{H}^n$ in the following way:

For $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$, and $z = (z_1, \cdots, z_n) \in \mathbb{H}^n$, we define

$$\gamma z = (\gamma_1 z_1, \cdots, \gamma_n z_n)$$

where

$$\gamma_i = \begin{pmatrix} \sigma_i(a) & \sigma_i(b) \\ \sigma_i(c) & \sigma_i(d) \end{pmatrix}, \quad \gamma_i z_i = \frac{\sigma_i(a) z_i + \sigma_i(b)}{\sigma_i(c) z_i + \sigma_i(d)} \quad (1 \leq i \leq n).$$

Remark. We may also identify Γ *with its image in* $SL(2, \mathbb{R})^n$ *via*

$$\gamma \in \Gamma, \ \gamma = (\gamma_1, \cdots, \gamma_n) \in SL(2, \mathbb{R})^n.$$

It is well known that Γ has finite co-volume (see [Fr]), i.e.

$$vol(\Gamma\backslash\mathbb{H}^n) = \int_{\Gamma\backslash\mathbb{H}^n} \frac{dxdy}{(Ny)^2} < \infty,$$

where $z = (x_1 + iy_1, \cdots, x_n + iy_n) \in \mathbb{H}^n$, $dx = dx_1 \cdots dx_n$, $dy = dy_1 \cdots dy_n$, and $Ny = y_1 \cdots y_n$.

In fact, this volume is calculated precisely by Siegel [Sie]:

$$vol(\Gamma\backslash\mathbb{H}^n) = 2\pi^{-n}\zeta_F(2)D_F^{3/2}$$

where $\zeta_F(s)$ is the Dedekind zeta function of F and D_F is the discriminant of F.

Although we know the fundamental domain has finite volume, there is no easy way to determine its shape. One way to describe a fundamental domain for $GL^+(2, \mathcal{O})$ is given by Herrmann [He] as follows.

A fundamental domain for $GL^+(2, \mathcal{O})$ is the set of $z \in \mathbb{H}^n$ satisfying the inequalities:

1) $|j(\gamma, z)|^2 \geq 1$ for all $\gamma \in GL^+(2, \mathcal{O})$;

2) $Tr(\log \varepsilon(\log \varepsilon + 2 \log \mathrm{Im}(z))) \geq 0$ for all $\varepsilon \in \mathcal{O}_+^\times$;

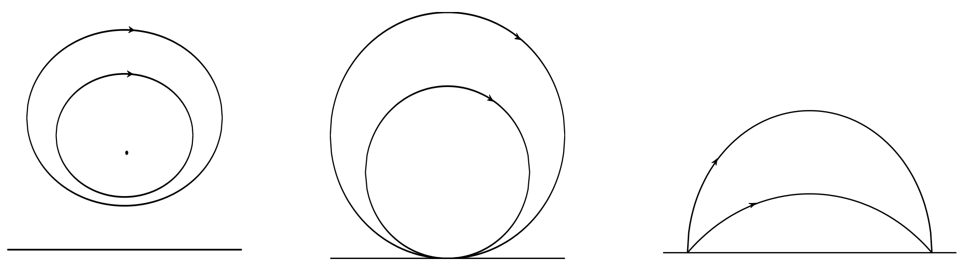
3) $Tr(\nu(\nu + \mathrm{Re}(z)) \geq 0$ for all $\nu \in \mathcal{O}$,

where $\mathcal{O}_+^\times$ is the group of totally positive units, Tr and $j(\gamma, z)$ are defined in next two sections. He also showed that the domain can be already described by a finite number of above inequalities.

Some properties of Γ.

An element $\gamma \neq \pm 1$ in $SL(2, \mathbb{R})$ is called elliptic (respectively parabolic and hyperbolic) if $tr(\gamma) < 2$ (respectively $tr(\gamma) = 2$ and $tr(\gamma) > 2$).

An elliptic element has one fixed point in $\mathbb{H}$ and moves points along hyperbolic circles centered at its fixed point. The fixed point is the hyperbolic center. A parabolic element has one fixed point on $\overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ and moves points along horocycles (circles in $\mathbb{H}$ tangent to $\overline{\mathbb{R}}$). A hyperbolic element has two distinct fixed points in $\overline{\mathbb{R}}$ and moves points along hypercycles (the segments in $\mathbb{H}$ of circles in $\mathbb{C}$ through the fixed points on $\overline{\mathbb{R}}$). See Figure 2.1.



(a) elliptic (b) parabolic (c) hyperbolic

Figure 2.1: motions in $\mathbb{H}$

Recall that for $\gamma \in \Gamma$ we identify $\gamma = (\gamma_1, \cdots, \gamma_n) \in SL(2, \mathbb{R})^n$. We say that an element $\gamma(\neq \textit{identity})$ of Γ is *elliptic* (respectively *parabolic* and *hyperbolic*) if all the γ_i are elliptic (respectively parabolic and hyperbolic). If $\gamma(\neq \textit{identity})$ is not of above types, we say that γ is *mixed.* A point z in $\mathbb{H}^n$ is called an *elliptic point* if it is fixed by an elliptic element in Γ. A point κ in $\overline{\mathbb{R}}^n$ (where $\overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$) is called a *cusp* if it is fixed by a parabolic element in Γ.

Definition 2.1.1. *Two points z_1 and z_2 in $\mathbb{H}^n \cup \overline{\mathbb{R}}^n$ are called Γ-equivalent if there exists a γ in Γ such that $\gamma z_1 = z_2$.*

Proposition 2.1.2. ([Sh] Theorem 6) *The number of the Γ-inequivalent elliptic points of Γ is finite.*

Proposition 2.1.3. ([Sh] Lemma 15) *Let $e_1, \cdots, e_s \in \mathbb{H}^n$ be complete representatives of Γ-inequivalent elliptic points of Γ. Then the union of $\Gamma_{e_i} \setminus \{1\}$ $(1 \le i \le s)$ forms a complete representatives of non-conjugate elliptic elements in Γ, where $\Gamma_{e_i} = \{\gamma \in \Gamma | \gamma e_i = e_i\}$ $(1 \le i \le s)$.*

Since Γ_{e_i} is a discrete subgroup of a compact subgroup, Γ_{e_i} is a finite subgroup. Hence we have the following lemma.

Lemma 2.1.4. *There are only finitely many elliptic conjugacy classes in Γ.*

Proposition 2.1.5. ([Fr] 3.5 Corollary) *There are only finitely many Γ-inequivalent cusps. Moreover the number of Γ-inequivalent cusps is equal to the class number of F.*

We may note that the Hilbert modular variety $X = \Gamma \backslash \mathbb{H}^n$ is *noncompact*.

2.2 Hilbert Modular Forms

Let $\Gamma = SL(2, \mathcal{O})$.

Definition 2.2.1. Let $\mathbf{k} = (k_1, \cdots, k_n) \in \mathbb{Z}^n$. A *Hilbert modular form* of weight $\mathbf{k}$ with respect to Γ is a holomorphic function $f(z)$ on $\mathbb{H}^n$ that satisfies

$$f(\gamma z) = j(\gamma, z)^{\mathbf{k}} f(z) = N(cz + d)^{\mathbf{k}} f(z)$$

for all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$.

Here for $z = (z_1, \cdots, z_n) \in \mathbb{H}^n$,

$$N(cz+d) = \prod_{i=1}^{n} (\sigma_i(c) z_i + \sigma_i(d)),$$

and we use a standard multi-index notation:

$$j(\gamma, z)^{\mathbf{k}} = N(cz+d)^{\mathbf{k}} = \prod_{i=1}^{n} (\sigma_i(c) z_i + \sigma_i(d))^{k_i}.$$

For $F = \mathbb{Q}$, one has to add the condition that f is holomorphic at the cusps. For $n = [F, \mathbb{Q}] > 1, f$ is automatically holomorphic at cusps by the *Koecher's principle* which we will explain in a moment.

We may consider the cusp at infinity. A Hilbert modular form f is invariant under the subgroup

$$\left\{ \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} : u \in \mathcal{O} \right\}.$$

Thus f has a Fourier expansion

$$f(z) = \sum_{\nu \in \mathcal{O}^*} a_\nu \exp(2\pi i Tr(\nu z)),$$

where Tr is the $\mathbb{C}$-linear extension to $\mathbb{C}^m \to \mathbb{C}$ of the Galois trace $F \to \mathbb{Q}$ and where

$$\mathcal{O}^* = \{\nu \in F : Tr(\nu \mathcal{O}) \subset \mathcal{O}\}$$

is the inverse of the different.

Definition 2.2.2. A Hilbert modular form f of weight $\mathbf{k} = (k_1, \cdots, k_n) \in \mathbb{Z}^n$ is a *cusp form* if the constant term $a_0 = 0$ in the Fourier expansion of

$$f|\gamma(z) := N(cz+d)^{-\mathbf{k}} f(\gamma z)$$

for all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, F)$.

Theorem 2.2.3 (Koecher's Principle)**.** *Suppose $[F : \mathbb{Q}] = n > 1$ and let f be a Hilbert modular form of weight $\mathbf{k} = (k_1, \cdots, k_n) \in \mathbb{Z}^n$. Then in the Fourier expansion*

$$f(z) = \sum_{\nu \in \mathcal{O}^*} a_\nu \exp(2\pi i Tr(\nu z))$$

of f, we have $a_\nu = 0$ unless $\nu = 0$ or ν is totally positive. Moreover, unless $k_1 = k_2 = \cdots = k_n$, we also have $a_0 = 0$.

Corollary 2.2.4. *For $[F : \mathbb{Q}] > 1$, every holomorphic Hilbert modular form of weight $\mathbf{k} = (k_1, \cdots, k_n) \in \mathbb{Z}^n$ is a cusp form unless $k_1 = k_2 = \cdots = k_n$.*

Denote by $S_{2\mathbf{k}}(\Gamma)$ the space of Hilbert modular cusp forms of weight $2\mathbf{k} = (2k_1, \cdots, 2k_n)$. Let

$$d\mu = \frac{1}{vol(\Gamma \backslash \mathbb{H}^n)} \frac{dxdy}{(Ny)^2}.$$

For f and g in $S_{2\mathbf{k}}(\Gamma)$, we define the (normalized) Petersson inner product by

$$\langle f, g \rangle = \int_{\Gamma \backslash \mathbb{H}^n} f(z)\overline{g(z)}(Ny)^{2\mathbf{k}} d\mu$$

where we use the multi-index notation $(Ny)^{2\mathbf{k}} = y_1^{2k_1} \cdots y_n^{2k_n}$.

It is well known that $S_{2\mathbf{k}}(\Gamma)$ is a finite dimensional Hilbert space. Furthermore, if we let $J_{\mathbf{k}} = \dim_{\mathbb{C}} S_{2\mathbf{k}}(\Gamma)$, then it was shown by Shimizu [Sh] (using the Selberg trace formula) that

$$J_{\mathbf{k}} = \frac{vol(\Gamma\backslash\mathbb{H}^n)}{(4\pi)^n}\prod_{i=1}^{n}(2k_i - 1) + O(1). \tag{2.2.1}$$

2.3 Poincare Series

Recall that F is a totally real number field of degree n over $\mathbb{Q}$ with ring of integers $\mathcal{O}$ and $\sigma_1, \sigma_2, \cdots, \sigma_n$ are all the real embeddings of F.

Definition 2.3.1. *An element η of F is called totally positive, denoted $\eta \gg 0$, if $\sigma_i(a) > 0$ for $i = 1, \cdots, n$.*

Definition 2.3.2. *Let ν be a totally positive element of $\mathcal{O}^*$ and let $2\mathbf{k} = (2k_1, 2k_2, \cdots, 2k_n)$. We define the ν-th Poincare series of weight $2\mathbf{k}$ with respect to Γ by*

$$P(z; \mathbf{k}, \nu) = \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} j(\gamma, z)^{-2\mathbf{k}} \exp(2\pi i Tr(\nu(\gamma z)))$$

where $\Gamma_\infty = \left\{ \gamma = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \in \Gamma \right\}$.

Proposition 2.3.3. *Suppose $k_j \geq 2$ for $j = 1, 2, \cdots, n$. Then the Poincare series $P(z; \mathbf{k}, \nu)$ is absolutely convergent and uniformly for z in compact subsets of $\mathbb{H}^n$. Moreover $P(z; \mathbf{k}, \nu)$ is in $S_{2\mathbf{k}}(\Gamma)$.*

Proof. See [Ga 1.13]. ∎

Proposition 2.3.4. *Let* $2\mathbf{k} = (2k_1, 2k_2, \cdots, 2k_n)$ *with each* $k_j \geq 2$. *Let*

$$f(z) = \sum_{\mu} a_\mu \exp(2\pi i Tr(\nu z))$$

be in $S_{2\mathbf{k}}(\Gamma)$ *and let* $P(z; \mathbf{k}, \nu)$ *be the* ν*-th Poincare series of weight* $2\mathbf{k}$ *for* Γ. *Then*

$$\langle f, P(\cdot\,; \mathbf{k}, \nu)\rangle = a_\nu N(\nu)^{1-2\mathbf{k}} \times \frac{vol(\Lambda\backslash\mathbb{R}^n)}{vol(\Gamma\backslash\mathbb{H}^n)} \times \prod_{j=1}^{n} (4\pi)^{1-2k_j}\Gamma(2k_j - 1)$$

where

$$\Lambda = \left\{ x = (\sigma_1(b), \sigma_2(b), \cdots, \sigma_n(b)) \in \mathbb{R}^n : \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \in \Gamma \right\}.$$

Proof. The proof is using the unfolding method.

$$\begin{aligned}
&\langle f, P(\cdot\,; \mathbf{k}, \nu)\rangle \\
&= \int_{\Gamma\backslash\mathbb{H}^n} f(z)\overline{P(z; \mathbf{k}, \nu)}(Ny)^{2\mathbf{k}} \frac{1}{vol(\Gamma\backslash\mathbb{H}^n)} \frac{dxdy}{(Ny)^2} \\
&= \int_{\Gamma_\infty\backslash\mathbb{H}^n} \sum_{\mu} a_\mu \exp(2\pi i Tr(\nu z)) \exp(2\pi i Tr(\nu z))(Ny)^{2\mathbf{k}} \frac{1}{vol(\Gamma\backslash\mathbb{H}^n)} \frac{dxdy}{(Ny)^2}
\end{aligned}$$

We integrate over x first and use the othogonality of exponential functions.

$$\begin{aligned}
&\int_{\Gamma_\infty\backslash\mathbb{H}^n} \sum_{\mu} a_\mu \exp(2\pi i Tr(\nu z)) \exp(2\pi i Tr(\nu z))(Ny)^{2\mathbf{k}} \frac{1}{vol(\Gamma\backslash\mathbb{H}^n)} \frac{dxdy}{(Ny)^2} \\
&= a_\nu \times \frac{vol(\Lambda\backslash\mathbb{R}^n)}{vol(\Gamma\backslash\mathbb{H}^n)} \times \prod_{j=1}^{n} \int_0^\infty y_j^{2k_j-1} e^{-4\pi\sigma_j(\nu)y_j} \frac{dy_j}{y_j} \\
&= a_\nu \times \frac{vol(\Lambda\backslash\mathbb{R}^n)}{vol(\Gamma\backslash\mathbb{H}^n)} \times \prod_{j=1}^{n} (4\pi\sigma_j(\nu))^{1-2k_j}\Gamma(2k_j - 1).
\end{aligned}$$

∎

Corollary 2.3.5. *Let* $2\mathbf{k} = (2k_1, 2k_2, \cdots, 2k_n)$ *with each* $k_j \geq 2$. *The Poincere series* $\{P(z; \mathbf{k}, \nu) : \nu \gg 0\}$ *span* $S_{2\mathbf{k}}(\Gamma)$.

Proof. Suppose f in $S_{2\mathbf{k}}(\Gamma)$ with $\langle f, P(\cdot\ ; \mathbf{k}, \nu)\rangle = 0$ for all $\nu \gg 0$. Then all its Fourier coefficients $a_\nu = 0$, hence $f \equiv 0$. ∎

2.4 Hecke Operators

One may define a commutative family of Hecke operators on $S_{2\mathbf{k}}(\Gamma)$. However it is not easy to define them in a classical way in general. In the special case that F has narrow class number one, we may define them classically. In general, one needs to lift the Hilbert modular forms to the automorphic forms on $GL(2, \mathbb{A}_F)$ and consider the spherical Hecke algebra. In this section we restrict to consider F has narrow class number one and give a classical definition of Hecke operators. For general F and adelic setting one can find the definition in Garrett's book [Ga]. An important property of the Hecke operators in all cases is that they are commutative and normal. Hence $S_{2\mathbf{k}}(\Gamma)$ has an orthonormal basis consisting of the eigenforms of the Hecke operators.

In this section, we assume that F has *narrow class number one* which means that every ideal of F has a generator that is totally positive. In this case, its ring of integers $\mathcal{O}$ is a principal ideal domain and the totally positive units are squares of units.

Let $\mathcal{O}^\times$ denote the group of units and let $\mathcal{O}^\times_+$ denote the group of totally positive units. Recall that $\mathcal{O}^*$ is the inverse different.

Definition 2.4.1. *Let $\mathfrak{n} = (\eta)$ be an ideal of $\mathcal{O}$, with η totally positive. Let*

$$\Delta(\mathfrak{n}) = \left\{ \delta = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2, F) : a, b, c, d \in \mathcal{O}, \eta^{-1} \det \delta \in \mathcal{O}_+^\times \right\}.$$

Let $Z(\mathcal{O})$ be the center of $GL^+(2, \mathcal{O})$, the set of elements of $GL(2, \mathcal{O})$ with totally positive determinant. We define the $\mathfrak{n}$-th Hecke operator $T_\mathfrak{n}$ on $S_{2\mathbf{k}}(\Gamma)$ by

$$(T_\mathfrak{n} f)(z) = \sum_{\delta \in \Gamma Z(\mathcal{O}) \backslash \Delta(\mathfrak{n})} (f|\delta)(z)$$

where $(f|\delta)(z) = N(\det \delta)^{\mathbf{k}} j(\delta, z)^{-2\mathbf{k}} f(\delta z)$.

We can use the assumption on narrow class number one to give an explicit set of representatives

$$X(\mathfrak{n}) = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} : ad = r\eta,\ d \gg 0,\ r \in \mathcal{O}_+^\times,\ b \in \mathcal{O}/(d) \right\}$$

for $\Gamma Z(\mathcal{O}) \backslash \Delta(\mathfrak{n})$. One can use the explicit representatives to prove the following proposition.

Proposition 2.4.2. *Let $\mathfrak{m}, \mathfrak{n}$ be non-zero ideals of $\mathcal{O}$. Then*

$$T_\mathfrak{m} T_\mathfrak{n} = \sum_{\mathfrak{d} \supset \mathfrak{m} + \mathfrak{n}} N(\mathfrak{d}) T_{\frac{\mathfrak{m}\mathfrak{n}}{\mathfrak{d}^2}}.$$

In particular, the Hecke operators commute.

The Hecke operators act on Poincare series can be computed precisely as follows.

Proposition 2.4.3. *Let $\mathfrak{n} = (\eta)$ be an ideal of $\mathcal{O}$ with η totally positive.*

$$T_\mathfrak{n} P(\cdot\ ; \mathbf{k}, \nu) = (N\eta)^{\mathbf{k}} \sum_d (Nd)^{1-2\mathbf{k}} P(z; \mathbf{k}, \nu\eta/d^2)$$

where the sum is over totally positive divisors d of η modulo $\mathcal{O}^\times$ such that $\nu/d \in \mathcal{O}^*$.

Proof. See [Ga 1.15] ∎

Corollary 2.4.4. *Assume that the weight* $\mathbf{2k} = (2k_1, \cdots, 2k_n)$ *with each* $k_j \geq 2$. *Then the Hecke operators on* $S_{\mathbf{2k}}(\Gamma)$ *are self-adjoint with respect to the Petersson inner product.*

Proof. Since $S_{\mathbf{2k}}(\Gamma)$ is spanned by Poincare series, it suffices to examine this property for Poincare series. For detail we refer to [Ga 1.15]. ∎

Corollary 2.4.5. *Assume that the weight* $\mathbf{2k} = (2k_1, \cdots, 2k_n)$ *with each* $k_j \geq 2$. *Then there is an orthogonal basis for* $S_{\mathbf{2k}}(\Gamma)$ *consisting of simultaneous eigenfunctions for all the Hecke operators.*

CHAPTER 3

EQUIDISTRIBUTION OF HECKE EIGENFORMS

3.1 Mass Equidistribution Conjecture and Main Results

Let $\Gamma = SL(2, \mathcal{O})$. Denote by $S_{2k}(\Gamma)$ the space of holomorphic Hilbert modular cusp forms of weight $2\mathbf{k} = (2k, 2k, \cdots, 2k)$. Shimizu's dimension formula (2.2.1) tells us

$$\dim_{\mathbb{C}} S_{2k}(\Gamma) = J_k = \frac{vol(\Gamma \backslash \mathbb{H}^n)}{(4\pi)^n}(2k-1)^n + O(1) \tag{3.1.1}$$

as $k \to \infty$.

One expects the following mass equidistribution conjecture on the Hilbert modular variety $\Gamma \backslash \mathbb{H}^n$ should be true:

Conjecture 3.1.1.

$$\lim_{k\to\infty} \max_{1\leq i\leq J_k} \left| \int_A (Ny)^{2k} |f_{i,k}(z)|^2 d\mu - \int_A d\mu \right| = 0 \tag{3.1.2}$$

where $A \subset \Gamma \backslash \mathbb{H}^n$ is compact and $\{f_{i,k}\}_{i=1}^{J_k}$ is the orthonormal Hecke basis of $S_{2k}(\Gamma)$.

This is an analogue of arithmetic quantum unique ergodicity conjecture, formulated by Rudnick and Sarnak [RS].

Luo [Lu] established this conjecture on average for $n = 1$ and Lau [La] generalized Luo's result to the arithmetic surface $\Gamma_0(N) \backslash \mathbb{H}$. The purpose of this paper is to generalize Luo's and Lau's results to the Hilbert modular varieties (Theorem 3.1.2 and Theorem 4.1.1).

Let $\{f_{i,k}\}_{i=1}^{J_k}$ be an orthonormal basis of $S_{2k}(\Gamma)$. Set

$$d\mu_k = \frac{1}{J_k}\left(\sum_{i=1}^{J_k} |f_{i,k}(z)|^2\right)(Ny)^{2k} d\mu.$$

Theorem 3.1.2. *For any compact subset $A \subset \Gamma\backslash\mathbb{H}^n$ and any $0 < \epsilon < 1$, we have*

$$\int_A d\mu_k = \int_A d\mu + O_{\epsilon,A}\left((k^{-1+\epsilon})^n\right)$$

as $k \longrightarrow \infty$.

Remark 1. The key ingredients in [Lu] and [La] are the Bergman kernel for the Hecke operator and the Petersson trace formula respectively. Our approach is using the Bergman kernel on $\Gamma\backslash\mathbb{H}^n$.

Remark 2. [Lu] proved a uniform result for all measurable subsets A. In our Theorem 1, the result depends on the compact subset A. But our decay rate is sharper than in [Lu].

Corollary 3.1.3. *For any $z \in A$, we have*

$$|Ny|^k |f_{i,k}(z)| \ll_A k^{n/2}.$$

Remark 3. This corollary generalizes [Ru, prop. A.1] for the case $n = 1$.

3.2 Bergman kernel

For $k \in \mathbb{N}$, $k \geq 2$ and $z = (z_1, , \cdots z_n)$, $\quad w = (w_1, \cdots, w_n) \in \mathbb{H}^n$, we define the Bergman kernel by

$$B_k(z,w) = \sum_{\gamma \in \Gamma} N(\gamma z - \overline{w})^{-2k} j(\gamma, z)^{-2k}$$

where $N(\gamma z - \overline{w}) = \prod_{i=1}^{n} (\sigma_i(\gamma) z_i - \overline{w_i})$ and $j(\gamma, z) = N(cz+d)$, $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

Proposition 3.2.1. *(1) $B_k(z,w)$ converges absolutely and uniformly for (z,w) in compact subsets of $\mathbb{H}^n \times \mathbb{H}^n$.*

(2) For each fixed $w \in \mathbb{H}^n, B_k(z,w) \in S_{2k}(\Gamma)$ (as a function of z).

Proof. The proof can be found in [Ga, 1.14]. However there are some minor mistakes in [Ga, 1.14]. So we follow the idea of [Ga] and give a proof here.

We need a general result in complex analysis:

Lemma 3.2.2. *If a sequence of holomorphic functions $\{f_n\}$ is L^1-convergent in an open subset U of $\mathbb{C}^n$, then it is uniformly convergent to a holomorphic function on any compact subset $C \subset U$.*

We note that

$$(Ny)^k \sum_{\gamma \in \Gamma} |N(\gamma z - \overline{w})^{-2k} j(\gamma, z)^{-2k}|$$

is Γ-invariant. It suffices to show that

$$\int_{\Gamma \backslash \mathbb{H}^n} (Ny)^k \sum_{\gamma \in \Gamma} \left| N(\gamma z - \overline{w})^{-2k} j(\gamma, z)^{-2k} \right| \frac{dxdy}{(Ny)^2} < \infty.$$

By unfolding method, the above integral is equal to

$$\int_{\mathbb{H}^n} |N(z-\overline{w})^{-2k}|(Ny)^k \frac{dxdy}{(Ny)^2} = \prod_{j=1}^{n} \int_{\mathbb{H}} |(z_j+\overline{w_j})^{-2k}| y^k \frac{dx_j dy_j}{y_j^2}$$

Let $w_j = u_j + iv_j \in \mathbb{H}$. We make changes of variables by replacing x_j by $x_j - u_j$ and then replacing x_j by $x_j(y_j - v_j)$. So the integral is equal to

$$\prod_{j=1}^{n} \int_{\mathbb{R}} |x_j + i|^{-2k} dx_j \times \int_0^\infty |y_j - v_j|^{-2k+1} y_j^{k-2} dy_j.$$

Hence the integral is convergent if $k \geq 2$.

To prove $B_k(z, w)$ is a cusp form in z, one shows that it can be expressed as a linear sum of Poincare series. We refer [Ga 1.14] for detail. ∎

Proposition 3.2.3. *If $f \in S_{2k}(\Gamma)$, then*

$$\begin{aligned} f(w) &= \left(\frac{2k-1}{4\pi}\right)^n \frac{(2i)^{2kn}}{2} \int_{\Gamma\backslash\mathbb{H}^n} f(z)\overline{B_k(z,w)}(Ny)^{2k} \frac{dxdy}{(Ny)^2} \\ &= \left(\frac{2k-1}{4\pi}\right)^n \frac{(2i)^{2kn}}{2} vol(\Gamma\backslash\mathbb{H}^n)\langle f, B_k(\cdot, w)\rangle \end{aligned}$$

where $z = (x_1 + iy_1, \cdots, x_n + iy_n) \in \mathbb{H}^n, \quad w \in \mathbb{H}^n$.

Proof. By unfolding, we have

$$\langle f, B_k(\cdot, w)\rangle = \frac{2}{vol(\Gamma\backslash\mathbb{H}^n)} \int_{\mathbb{H}^n} f(z)(Ny)^{2k} N(\overline{z} - w)^{-2k} \frac{dxdy}{(Ny)^2}.$$

We insert the Fourier expansion of $f(z)$:

$$f(z) = \sum_{\nu \gg 0} a_\nu \exp(2\pi i Tr(\nu z))$$

and then factor it into n integrals

$$\frac{2}{vol(\Gamma\backslash\mathbb{H}^n)}\times\sum_{\nu\gg 0}a_\nu\prod_{j=1}^{n}\int_{\mathbb{H}}\exp(2\pi i\sigma_j(\nu)z_j)y_j^{2k}(x_j-iy_j-w_j)^{-2k}\frac{dx_jdy_j}{y_j^2}.$$

Integrating over x_j first (using Cauchy residue theorem), the inner integral becomes

$$2\pi i\int_0^\infty\frac{(2\pi i\sigma_j(\nu))^{2k-1}}{(2k-1)!}\exp(2\pi i\sigma_j(\nu)(2iy_j+w_j))y_j^{2k-1}\frac{dy_j}{y_j}$$

$$= (2\pi i)\frac{(2\pi i\sigma_j(\nu))^{2k-1}}{(2k-1)!}\exp(2\pi i\sigma_j(\nu)w_j)\int_0^\infty\exp(-4\pi\sigma_j(\nu)y_j)y_j^{2k-1}\frac{dy_j}{y_j}$$

$$= (2\pi i)\frac{(2\pi i\sigma_j(\nu))^{2k-1}}{(2k-1)!}\exp(2\pi i\sigma_j(\nu)w_j)(4\pi\sigma_j(\nu))^{1-2k}\Gamma(2k-1)$$

$$= \frac{4\pi}{2k-1}(2i)^{-2k}\exp(2\pi i\sigma_j(\nu)w_j).$$

This proves the proposition. ∎

For convenience, denote by

$$C_k^{-1}=\left(\frac{2k-1}{4\pi}\right)^n\frac{(2i)^{2kn}}{2}vol(\Gamma\backslash\mathbb{H}^n) \tag{3.2.1}$$

and note that $C_k=\overline{C_k}$ when $k\geq 2$.

For $k\in\mathbb{N}$, $\gamma\in\Gamma$ and $z=(z_1,\cdots,z_n)\in\mathbb{H}^n$, let

$$h(\gamma,z)=N(z-\overline{z})^2N(\gamma z-\overline{z})^{-2}j(\gamma,z)^{-2}$$

and

$$h_k(\gamma,z)=(h(\gamma,z))^k=N(z-\overline{z})^{2k}N(\gamma z-\overline{z})^{-2k}j(\gamma,z)^{-2k}.$$

Lemma 3.2.4. $|h_k(\gamma,z)| \le 1$ *for all* $z \in \mathbb{H}^n$, *and* $\gamma \in \Gamma$. *Moreover,* $|h_k(\gamma,z)| = 1$ *if and only if* $\gamma = \pm 1$ *or* γ *is elliptic and* z *is its fixed point.*

Proof. It suffices to prove that when $n = 1$. By definition,

$$|h_k(\gamma,z)| = \left| \frac{z-\overline{z}}{\gamma z - \overline{z}} \cdot \frac{1}{cz+d} \right|^{2k}$$

where $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$. Let $\gamma z = z' = x' + iy'$ and $z = x + iy$. Then

$$\begin{aligned} \left| \frac{z-\overline{z}}{\gamma z - \overline{z}} \cdot \frac{1}{cz+d} \right| &= \frac{y^{1/2}}{\left| \frac{(x'-x)+i(y+y')}{2i} \right|} \left(\frac{y}{|cz+d|^2} \right)^{1/2} \\ &= \frac{y^{1/2}(y')^{1/2}}{|\frac{y+y'}{2} + i\frac{x-x'}{2}|} \le \frac{y^{1/2}(y')^{1/2}}{\frac{y+y'}{2}} \le 1. \end{aligned}$$

The equality holds if and only if $x = x'$ and $y = y'$. i.e. $\gamma z = z$. Hence the equality holds if and only if $\gamma = \pm 1$ or γ is elliptic and z is its fixed point. ∎

Lemma 3.2.5. *For each fixed* $k \ge 2$, $\sum_{\gamma \in \Gamma} h_k(\gamma, z)$ *converges absolutely and uniformly on any compact subset of* $\mathbb{H}^n$.

Proof. Note that

$$\sum_{\gamma \in \Gamma} h_k(\gamma,z) = N(z-\overline{z})^{2k} B_k(z,z) \tag{3.2.2}$$

and then the result follows from Proposition 3.2.1. ∎

Lemma 3.2.6. *For any* $M \in \Gamma$, *we have*

$$h_k(M^{-1}\gamma M, z) = h_k(\gamma, Mz).$$

Proof. By a simple computation or see [Fr]. ∎

3.3 Proof of Main Theorem

Before we prove the theorem, we make the following observation.

Since $B_k(z,w)$ is a cusp form in z (by Proposition 3.2.1), we have

$$\begin{aligned} B_k(z,w) &= \sum_{i=1}^{J_k} \langle B_k(\cdot,w), f_{i,k}\rangle f_{i,k}(z) \\ &= C_k \sum_{i=1}^{J_k} \overline{f_{i,k}(w)} f_{i,k}(z) \quad (\textit{by Proposition } 3.2.3). \end{aligned}$$

Let $w = z$, then we obtain the identity

$$B_k(z,z) = C_k \sum_{i=1}^{J_k} |f_{i,k}(z)|^2, \tag{3.3.1}$$

where C_k is defined in (3.2.1).

Proof of Theorem 3.1.2. Let $\chi_A(z)$ denote the characteristic function of A on $\Gamma\backslash\mathbb{H}^n$. One can extend it (with the same notation) to $\mathbb{H}^n$ as a Γ- invariant function.

By (3.3.1)and (3.2.2),

$$\begin{aligned} \textstyle\int_A d\mu_k &= \frac{1}{J_k C_k} \int_A B_k(z,z)(Ny)^{2k} d\mu \\ &= \frac{1}{(2i)^{2kn} J_k C_k} \int_{\Gamma\backslash\mathbb{H}^n} \chi_A(z) \sum_{\gamma\in\Gamma} h_k(\gamma,z) d\mu \\ &= \frac{1}{(2i)^{2kn} J_k C_k} \Bigg[\sum_{\gamma=\pm 1} \int_{\Gamma\backslash\mathbb{H}^n} \chi_A(z) h_k(\gamma,z) d\mu + \\ & \sum_{\gamma\in\Gamma, \gamma\, is\, elliptic} \int_{\Gamma\backslash\mathbb{H}^n} \chi_A(z) h_k(\gamma,z) d\mu + \\ & \int_{\Gamma\backslash\mathbb{H}^n} \chi_A(z) \left(\sum_{\gamma\in\Gamma, \gamma\neq\pm 1, \gamma\, is\, not\, elliptic} h_k(\gamma,z) \right) d\mu \Bigg]. \end{aligned}$$

We estimate the above three summation of integrals in the following cases.

Case 1. $\gamma = \pm 1$.

$$\int_{\Gamma\backslash\mathbb{H}^n} \chi_A(z) h_k(\gamma, z) d\mu = \int_{\Gamma\backslash\mathbb{H}^n} \chi_A(z) d\mu = \mu(A).$$

Case 2. For $\gamma \in \Gamma$ elliptic, let

$$\Gamma_\gamma = \{M \in \Gamma : M\gamma = \gamma M\} \quad (\textit{the centralizer of } \gamma \textit{ in } \Gamma)$$

and

$$[\gamma] = \left\{M^{-1}\gamma M : M \in \Gamma\right\}.$$

Also let Λ be a set of complete representatives of elliptic conjugate classes in Γ.

Remark. $|\Lambda| < \infty$ *by Lemma 2.1.4.*

$$\begin{aligned}
\sum_{\gamma\in\Gamma,\gamma \text{ is elliptic}} \int_{\Gamma\backslash\mathbb{H}^n} \chi_A(z) h_k(\gamma, z) d\mu &= \sum_{\gamma\in\Lambda} \sum_{\gamma'\in[\gamma]} \int_{\Gamma\backslash\mathbb{H}^n} \chi_A(z) h_k(\gamma', z) d\mu \\
&= \sum_{\gamma\in\Lambda} \sum_{M\in\Gamma_\gamma\backslash\Gamma} \int_{\Gamma\backslash\mathbb{H}^n} \chi_A(z) h_k(M^{-1}\gamma M, z) d\mu
\end{aligned}$$

Using Lemma 3.2.6 and unfolding, we have

$$\begin{aligned}
\sum_{M\in\Gamma_\gamma\backslash\Gamma} \int_{\Gamma\backslash\mathbb{H}^n} \chi_A(z) h_k(M^{-1}\gamma M, z) d\mu &= \int_{\Gamma_\gamma\backslash\mathbb{H}^n} \chi_A(z) h_k(\gamma, z) d\mu \\
&= \frac{1}{|\Gamma_\gamma|} \int_{\mathbb{H}^n} \chi_A(z) h_k(\gamma, z) d\mu \\
&= \frac{1}{|\Gamma_\gamma|} \int_{\mathbb{H}^n} \chi_A(z) \prod_{i=1}^{n} h_{k,i}(\gamma_i, z_i) d\mu \\
&\leq \frac{1}{|\Gamma_\gamma|} \frac{1}{vol(\Gamma\backslash\mathbb{H}^n)} \prod_{i=1}^{n} \int_{\mathbb{H}} h_{k,i}(\gamma_i, z_i) \frac{dx_i dy_i}{y_i^2}
\end{aligned}$$

where

$$h_{k,i}(\gamma_i, z_i) = (z_i - \overline{z_i})^{2k}(\gamma_i z_i - \overline{z_i})^{-2k} j(\gamma_i, z_i)^{-2k}.$$

Remark. $h_{k,i}(M^{-1}\gamma_i M, z_i) = h_{k,i}(\gamma_i, Mz_i)$ *for any* $M \in SL(2,\mathbb{R})$.

Hence we may assume that each γ_i is of the form

$$\begin{pmatrix} \cos\theta_i & \sin\theta_i \\ -\sin\theta_i & \cos\theta_i \end{pmatrix} \quad \theta_i \neq 0, \pi.$$

For convenience, we drop the subscripts i in γ_i, z_i, θ_i and etc $\cdots$.

Now we make change of variables by using the *Cayley transform*

$$\begin{array}{ccc} \mathbb{H} & \longrightarrow & D\,(unit\ disc) \\ z & \longmapsto & w = \frac{z-i}{z+i} \end{array}$$

and then use the polar coordinates $w = \rho e^{i\varphi}$ of the unit disc. It yields

$$\begin{aligned} \int_{\mathbb{H}} |h_{k,i}(\gamma, z)| \frac{dxdy}{y^2} &= 4\int_0^{2\pi}\int_0^1 \frac{(1-\rho^2)^{2k-2}}{|1-e^{i\beta}\rho^2|^{2k}} \rho d\rho d\varphi \\ &= 4\pi \int_0^1 \frac{(1-t)^{2k-2}}{|1-e^{i\beta}t|^{2k}} dt \end{aligned}$$

$(where\ \beta = 2\theta \neq 0, 2\pi)$

- When $0 \leq t \leq k^{-1+\epsilon}$, $(0 < \epsilon < 1)$. It is easy to see that $\dfrac{1-t}{|1-e^{i\beta}t|} \leq 1$. Hence

$$\begin{aligned} \int_0^{k^{-1+\epsilon}} \frac{(1-t)^{2k-2}}{|1-e^{i\beta}t|^{2k}} dt &= \int_0^{k^{-1+\epsilon}} \left(\frac{1-t}{|1-e^{i\beta}t|}\right)^{2k-1} \frac{1}{|1-e^{i\beta}t|^2} dt \\ &\leq \int_0^{k^{-1+\epsilon}} \frac{1}{|1-e^{i\beta}t|^2} dt \ll k^{-1+\epsilon}. \end{aligned}$$

• When $k^{-1+\epsilon} \leq t \leq 1$. We have $\dfrac{2t}{(1-t)^2} \geq \dfrac{2k^{-1+\epsilon}}{4} = \dfrac{1}{2}k^{-1+\epsilon}$. So

$$\frac{1-t}{|1-e^{i\beta}t|} = \frac{1}{|1+\frac{2t}{(1-t)^2}(1-\cos\beta)|^{1/2}} \ll (1+k^{-1+\epsilon})^{-1/2}.$$

Hence

$$\int_{k^{-1+\epsilon}}^{1} \frac{(1-t)^{2k-2}}{|1-e^{i\beta}t|^{2k}} dt \ll [(1+k^{-1+\epsilon})^{-1/2}]^{2k-2} = (1+k^{-1+\epsilon})^{-k+1}$$

Combining these estimates, we get

$$\sum_{\gamma\in\Gamma,\gamma\ is\ elliptic} \int_{\Gamma\backslash\mathbb{H}^n} \chi_A(z) h_k(\gamma, z) d\mu \ll (k^{-1+\epsilon})^n.$$

Note that here the implicit constant only depends on ϵ.

Case 3. Let $\Gamma' = \Gamma \setminus (\{\pm 1\} \cup \{\gamma \in \Gamma : \gamma\ is\ elliptic\})$.

Since $\sum_{\gamma\in\Gamma'} |h_3(\gamma, z)|$ converges uniformly on A (by Lemma 3.2.5) and $|h_3(\gamma, z)| < 1$ for all $z \in A,\ \gamma \in \Gamma'$ (by Lemma 3.2.4), there exists a constant $0 < \lambda < 1$ (depends on A) such that $|h_3(\gamma, z)| < \lambda$ for all $z \in A, \gamma \in \Gamma'$. Hence

$$\begin{aligned} \int_{\Gamma\backslash\mathbb{H}^n} \chi_A(z) \left(\sum_{\gamma\in\Gamma'} h_k(\gamma, z)\right) d\mu &\leq \int_A \sum_{\gamma\in\Gamma'} |h_3(\gamma, z)||h_3(\gamma, z)|^{\frac{k-3}{3}} d\mu \\ &\leq \int_A \sum_{\gamma\in\Gamma'} |h_3(\gamma, z)|\lambda^{\frac{k-3}{3}} d\mu \ll (\lambda_1)^k \end{aligned}$$

where $\lambda_1 = (\lambda)^3 < 1$.

From case 1, 2, 3 and using Shimizu's asymptotic formula (3.1.1) for J_k, Theorem 3.1.2 follows directly. ∎

Proof of Corollary 3.1.3. From (3.2.2) and (3.3.1), we have

$$\begin{aligned}\sum_{\gamma\in\Gamma} h_k(\gamma, z) &= N(z-\overline{z})^{2k} B_k(z,z)\\ &= 2^{2nk}(Ny)^{2k} C_k \sum_{i=1}^{J_k} |f_{i,k}(z)|^2.\end{aligned}$$

Hence by Lemma 3.2.5

$$\begin{aligned}(Ny)^{2k}\sum_{i=1}^{J_k}|f_{i,k}(z)|^2 &\ll 2^{-2nk}C_k^{-1}\left|\sum_{\gamma\in\Gamma} h_k(\gamma,z)\right|\\ &\ll_A 2^{-2nk}C_k^{-1} \ll_A k^n.\end{aligned}$$

This implies

$$|Ny|^k |f_{i,k}(z)| \ll_A k^{n/2}.$$

∎

CHAPTER 4

MASS EQUIDISTRIBUTION ON HILBERT CONGRUENCE VARIETIES

4.1 Congruence Subgroups

Let Γ be a discrete subgroup of $SL(2,\mathbb{R})^n$ with finite co-volume which satisfies the irreducibility condition below and Assumption(F) on its fundamental domain.

Irreducibility condition: The restriction of each of the n projections

$$p_j : SL(2,\mathbb{R})^n \longrightarrow SL(2,\mathbb{R}) \ (1 \leq j \leq n)$$

to Γ is injective.

Assumption(F): Let $\kappa_v \, (1 \leq v \leq t)$ be a set of complete representatives of Γ-inequivalent cusp of Γ. For each v, take a $g_v \in SL(2,\mathbb{R})^n$ such that $g_v \kappa_v = \infty$ and put

$$U_v = \left\{ g_v^{-1} z : \prod_{i=1}^{n} Im(z_i) > d_v, z = (z_1, \cdots, z_n) \right\}$$

where d_v is a suitably chosen positive number. Let $\Gamma_{\kappa_v} = \{\gamma \in \Gamma : \gamma\kappa_v = \kappa_v\}$ and let V_v be a fundamental domain of Γ_{κ_v} in U_v. Then Γ has a fundamental domain F of the form

$$F = F_0 \cup V_1 \cup \cdots \cup V_t$$

where F_0 is relatively compact in $\mathbb{H}^n$.

In this case, Shimizu's dimension formula (3.1.1) also holds for Γ([Sh]). Moreover, our propositions, lemmas and theorem in previous chapter all remain true for Γ. Then one can follow previous argument to give an analogous theorem. In particular, for a non-zero ideal $\mathfrak{n}$ of $\mathcal{O}$, let

$$\Gamma_0(\mathfrak{n}) = \left\{ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathcal{O}) : c \equiv 0 \textit{ modulo } \mathfrak{n} \right\}.$$

Then $\Gamma = \Gamma_0(\mathfrak{n})$ satisfies the irreducible condition and Assumption(F). Hence we have the following theorem:

Theorem 4.1.1. *For any compact subset $A \subset \Gamma_0(\mathfrak{n}) \backslash \mathbb{H}^n$ and any $0 < \epsilon < 1$, we have*

$$\int_A d\mu_k = \int_A d\mu + O_{\epsilon, A}\left((k^{-1+\epsilon})^n\right)$$

as $k \longrightarrow \infty$.

Remark. Again the decay rate here is sharper than in [La], but the implicit constant depends on the compact subset A. In [La], the result is uniform.

4.2 Nonequal Weights

We may also consider mass equidistribution property for $S_{2\mathbf{k}}(\Gamma)$ with weight $2\mathbf{k} = (2k_1, 2k_2, \cdots, 2k_n)$ where $\Gamma = SL(2, \mathcal{O})$ or $\Gamma_0(\mathfrak{n})$. Let $J_{\mathbf{k}} = dim_{\mathbb{C}} S_{2\mathbf{k}}(\Gamma)$. Shimizu's dimension formula gives

$$J_{2\mathbf{k}} = \frac{vol(\Gamma \backslash \mathbb{H}^n)}{(4\pi)^n} \prod_{i=1}^{n} (2k_i - 1) + O(1) \tag{4.2.1}$$

as $|\mathbf{k}| = \min\{k_i\} \longrightarrow \infty$. The Bergman kernel in this case is given by

$$B_{\mathbf{k}}(z, w) = \sum_{\gamma \in \Gamma} N(\gamma z - \overline{w})^{-2\mathbf{k}} j(\gamma, z)^{-2\mathbf{k}}.$$

Following the similar argument, we have

Theorem 4.2.1. *For any compact subset $A \subset \Gamma \backslash \mathbb{H}^n$ and any $0 < \epsilon < 1$, we have*

$$\int_A d\mu_{\mathbf{k}} = \int_A d\mu + O_{\epsilon, A}\left(\prod_{i=1}^{n} k_i^{-1+\epsilon} \right)$$

as $|\mathbf{k}| \longrightarrow \infty$.

CHAPTER 5

DIRECTION FOR FURTHER RESEARCH

5.1 AQUE and Subconvexity Bound of L-functions

Let f be a primitive weight zero Maass cusp form with eigenvalue $\lambda = \frac{1}{4} + t_f^2$ (resp. holomorphic cusp Hecke eigenform of weight $2k$) for the group $\Gamma = SL(2, \mathbb{Z})$. Let

$$d\mu_f = \frac{|f|^2}{\langle f, f\rangle} d\mu \ \ (resp.\ d\mu_f = \frac{|f|^2}{\langle f, f\rangle} y^{2k} d\mu).$$

The AQUE (resp. mass equidistribution) conjecture asserts that:

$$\int_{\Gamma\backslash\mathbb{H}} h(z) d\mu_f \longrightarrow \int_{\Gamma\backslash\mathbb{H}} h(z) d\mu$$

as $\lambda \to \infty$ (resp. $k \to \infty$), where h is a smooth bound function on $\Gamma\backslash\mathbb{H}$. By Weyl's equidistribution criterion, it is sufficient to show that: as $\lambda \to \infty$ (resp. $k \to \infty$),

$$\int_{\Gamma\backslash\mathbb{H}} \phi(z) d\mu_f \longrightarrow 0$$

and

$$\int_{\Gamma\backslash\mathbb{H}} E(z, \frac{1}{2} + it) d\mu_f \longrightarrow 0$$

for any primitive Maass form ϕ and the Eisentein series $E(z, \frac{1}{2} + it)$ with any t (fixed) $\in \mathbb{R}$. By unfolding method, we have

$$\int_{\Gamma\backslash\mathbb{H}} E(z, \frac{1}{2} + it) d\mu_f = \frac{\Lambda(f \otimes f, 1/2 + it)}{\langle f, f\rangle},$$

where

$$\Lambda(f\otimes f,s)=\Gamma_{\mathbb{R}}(s)^2\prod_{\pm}\Gamma_{\mathbb{R}}(s\pm 2it_f)L(f\otimes f,\frac{1}{2}+it)$$

if f is a Maass form, and

$$\Lambda(f\otimes f,s)=\Gamma_{\mathbb{R}}(s)\Gamma_{\mathbb{R}}(s+1)\Gamma_{\mathbb{R}}(s+2k-1)\Gamma_{\mathbb{R}}(s+2k)L(f\otimes f,\frac{1}{2}+it)$$

if f is holomorphic. Here $\Gamma_{\mathbb{R}}(s)=\pi^{-s/2}\Gamma(s/2)$.

Using Stirling formula and the bound (see [DFI])

$$(1+|t_f|)^{-\varepsilon}|\Gamma(\frac{1}{2}+it_f)|^2\ll_{\varepsilon}\langle f,f\rangle\ll(1+|t_f|)^{\varepsilon}|\Gamma(\frac{1}{2}+it_f)|^2$$

if f is a Maass form, or

$$k^{-\varepsilon}\frac{\Gamma(2k)}{(4\pi)^{2k}}\ll_{\varepsilon}\langle f,f\rangle\ll_{\varepsilon}k^{\varepsilon}\frac{\Gamma(2k)}{(4\pi)^{2k}}$$

if f is holomorphic, one has

$$\int_{\Gamma\backslash\mathbb{H}}E(z,\frac{1}{2}+it)d\mu_f\ll_{t,\varepsilon}(1+|t_f|)^{-\frac{1}{2}+\varepsilon}L(f\otimes f,1/2+it)$$

if f is a Maass form and

$$\int_{\Gamma\backslash\mathbb{H}}E(z,\frac{1}{2}+it)d\mu_f\ll_{t,\varepsilon}k^{-\frac{1}{2}+\varepsilon}L(f\otimes f,1/2+it)$$

if f is holomorphic.

On the other hand, Harris-Kudla [HK] and Watson [Wa] proved the formula

$$\frac{|\int_{\Gamma\backslash\mathbb{H}}\phi(z)d\mu_f|^2}{\langle\phi,\phi\rangle}=\frac{\Lambda(f\otimes f\otimes\phi,1/2)}{\Lambda(\mathrm{sym}^2f,1)^2\Lambda(\mathrm{sym}^2\phi,1)},$$

where

$$\Lambda(f\otimes f\otimes\phi,s)=\left(\prod_{\{\pm\}^3}\Gamma_{\mathbb{R}}(s\pm it_f\pm it_f\pm it_\phi+\delta_\phi)\right)L(f\otimes f\otimes\phi,s)$$

(here $\delta_\phi = \pm 1$ depends on ϕ is even or odd) if f is a Maass form and

$$\begin{aligned}
&\Lambda(f \otimes f \otimes \phi, s) \\
&\quad = \left(\prod_{\pm} \Gamma_{\mathbb{R}}(s + 2k - 1 \pm it_\phi) \Gamma_{\mathbb{R}}(s + 2k + it_\phi) \Gamma_{\mathbb{R}}(s + it_\phi) \Gamma_{\mathbb{R}}(s + 1 \pm it_\phi) \right) \\
&\qquad \times L(f \otimes f \otimes \phi, s)
\end{aligned}$$

if f is holomorphic.

By Stirling formula and the bound (see [HL])

$$(1 + |t_f|)^{-\varepsilon} \ll_\varepsilon L(\mathrm{sym}^2 f, 1) \ll_\varepsilon (1 + |t_f|)^{\varepsilon}$$

if f is an Maass form or

$$k^{-\varepsilon} \ll_\varepsilon L(\mathrm{sym}^2 f, 1) \ll_\varepsilon k^{\varepsilon}$$

if f is holomorphic, one has

$$\left| \int_{\Gamma \backslash \mathbb{H}} \phi(z) d\mu_f \right|^2 \ll_{\varepsilon, \phi} (1 + |t_f|)^{-1+\varepsilon} L(f \otimes f \otimes \phi, 1/2)$$

if f is a Maass form and

$$\left| \int_{\Gamma \backslash \mathbb{H}} \phi(z) d\mu_f \right|^2 \ll_{\varepsilon, \phi} k^{-1+\varepsilon} L(f \otimes f \otimes \phi, 1/2)$$

if f is holomorphic.

Now we use the following factorizations

$$L(f \otimes f, s) = \zeta(s) L(\mathrm{sym}^2 f, s),$$

$$L(f \otimes f \otimes \phi, s) = L(\phi, s) L(\mathrm{sym}^2 f \otimes \phi, s).$$

Moreover by Phragmén-Lindelöf convexity principle,

$$L(\mathrm{sym}^2 f, \frac{1}{2} + it) \ll_{t,\varepsilon} |t_f|^{\frac{1}{2}+\varepsilon}$$

$$L(\mathrm{sym}^2 f \otimes \phi, \frac{1}{2}) \ll_{\phi,\varepsilon} |t_f|^{1+\varepsilon}$$

if f is a Maass form and

$$L(\mathrm{sym}^2 f, \frac{1}{2} + it) \ll_{t,\varepsilon} k^{\frac{1}{2}+\varepsilon}$$

$$L(\mathrm{sym}^2 f \otimes \phi, \frac{1}{2}) \ll_{\phi,\varepsilon} k^{1+\varepsilon}$$

if f is holomorphic.

Hence any subconvexity bounds for $L(\mathrm{sym}^2 f, \frac{1}{2} + it)$ and $L(\mathrm{sym}^2 f \otimes \phi, \frac{1}{2})$ will imply the AQUE conjecture.

Remark.

1. Soundararajan [So] proved

$$L(\mathrm{sym}^2 f \otimes \phi, \frac{1}{2}) \ll_{\phi,\varepsilon} \frac{k}{(\log k)^{1-\varepsilon}}.$$

This weak subconvexity bound is slightly better than the convexity bound but it is not a power saving. It is of great interest to improve convexity bound for this family of L-function with power saving.

2. The Lindelöf hypothesis

$$L(\mathrm{sym}^2 f, \frac{1}{2} + it) \ll_{t,\varepsilon} |t_f|^{\varepsilon} \quad (\text{resp. } k^{\varepsilon})$$

$$L(\mathrm{sym}^2 f \otimes \phi, \frac{1}{2}) \ll_{\phi,\varepsilon} |t_f|^{\varepsilon} \quad (\text{resp. } k^{\varepsilon})$$

would give us the sharp convergent rate:

$$\int_{\Gamma\backslash\mathbb{H}} E(z, \frac{1}{2}+it)d\mu_f \ll_{t,\varepsilon} |t_f|^{-\frac{1}{2}+\varepsilon} \ \ (\text{resp. } k^{-\frac{1}{2}+\varepsilon})$$

$$\int_{\Gamma\backslash\mathbb{H}} \phi(z)d\mu_f \ll_{\phi,\varepsilon} |t_f|^{-\frac{1}{2}+\varepsilon} \ \ (\text{resp. } k^{-\frac{1}{2}+\varepsilon}).$$

5.2 L^∞-norms of Cusp Forms

Let ϕ be a L^2-normalized Hecke-Maass cusp form with Laplacian eigenvalue λ. As point out in [S], the bound of $\|\phi\|_\infty$ in terms of λ is related to many things, like Ramanujan conjecture, subconvexity bound of L-function and so on. It is also interesting to bound L^∞-norm of Maass cusp form or holomorphic Hecke cusp form in all aspect. For modular surfaces only few results are known and it is almost unknown for higher dimensional modular varieties. For example given a holomorphic Hilbert cusp form or a holomorphic Siegel cusp form, we may ask what is the bound of L^∞-norm in terms of its weight or level. Now we end up with stating some known results in modular surfaces.

In the spectral aspect, Iwaniec and Sarnak [IR] proved that $\|\phi\|_\infty \ll \lambda^{5/24+\varepsilon}$. In the level aspect, Blomer and Holowinsky proved that $\|f\|_\infty \ll N^{-1/38}$ for f an L^2-normalized weight zero Hecke-Maass cusp form of square-free level N and Laplacian eigenvalue $\lambda \geq 1/4$. In the weight aspect, Xia proved that $k^{1/4-\varepsilon} \ll \|f\|_\infty \ll k^{1/4+\varepsilon}$ for f an L^2-normalized holomorphic Hecke cusp form of weight $2k$ with respect to $SL(2,\mathbb{Z})$.

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