DISCRETE GROUPS AND CAT(0) ASYMPTOTIC CONES

DISSERTATION

Presented in Partial Fulfillment of the Requirements for
the Degree Doctor of Philosophy in the Graduate
School of the Ohio State University

By

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The Ohio State University
2008

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ABSTRACT

In this dissertation, we introduce the class of **asymptotically CAT(0) groups**, the principal objective being to study their algebraic properties and provide examples. The initial focus is on **δ-CAT(0)** groups, which form a special class of asymptotically CAT(0) groups. These have many desirable algebraic properties, in particular, they are semihyperbolic and satisfy Novikov's Conjecture on Higher Signatures. We observe that there are examples of metric spaces which are asymptotically CAT(0) but not δ-CAT(0).

We proceed to study the general theory of asymptotically CAT(0) groups, explaining why such a group has finitely many conjugacy classes of finite subgroups, is F_∞ and has solvable word problem. We provide techniques to combine asymptotically CAT(0) groups via direct products, amalgams and HNN extensions.

The universal cover of the Lie group $PSL(2,\mathbb{R})$ is shown to be an asymptotically CAT(0) metric space. Therefore, cocompact lattices in $\widetilde{PSL(2,\mathbb{R})}$ provide the first examples of asymptotically CAT(0) groups which are neither CAT(0) nor hyperbolic. Another potential rich source of examples is the class of relatively hyperbolic groups.

We conclude with a selection of interesting questions which arise out of this dissertation.

ACKNOWLEDGMENTS

I am greatly indebted to my doctoral superviser Indira Chatterji. She has played a decisive role in shaping my mathematical career, first in bringing me to The Ohio State University and then in supervising my research.

I am grateful to Ian Leary for the many hours he spent answering my questions and for carefully reading the first ever draft of this work.

I want to thank Jean-Francois Lafont for patiently discussing aspects of Riemannian geometry with me and for agreeing to serve on my doctoral committee.

I also want to thank Michael Davis for agreeing to be a member of the doctoral committee.

The Mathematics department of The Ohio State University has been immensely supportive throughout my graduate career. I am specially grateful to Professor Fangyang Zheng and the Graduate Studies Committee for their help and advice. I also thank Denise Witcher and Marilyn Radcliff for their help.

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CHAPTER 1

INTRODUCTION

Metric spaces of non-positive curvature have been the central objects of study among geometric group theorists for more than two decades. A fundamental theorem of Riemannian Geometry states that the universal cover of a complete Riemannian n-manifold with constant sectional curvature is isometric to $\mathbb{H}^n$, $\mathbb{R}^n$ or $\mathbb{S}^n$. In geometric group theory, $CAT(\kappa)$ spaces are 'modelled' on these three spaces. They encapsulate in a metric fashion, the traditional notion of sectional curvature that is bounded above.

The study of groups of isometries of non-positively curved spaces has proved to be very fruitful in enhancing our understanding of finitely presented groups. A CAT(0) space (see [4]) has many fascinating properties. Indeed, it is always contractible and it exhibits a rather desirable local-to-global phenomenon.

A group G is said to act geometrically on a metric space (X, d) if it acts properly and co-compactly by isometries on X. Groups acting geometrically on CAT(0) spaces or CAT(0) groups have soluble word and conjugacy problems. All free abelian subgroups of CAT(0) groups are finitely generated and a CAT(0) group can contain only finitely many conjugacy classes of finite subgroups. These are a few highlights of the subject but they portray how well the geometry of the space complements the algebra of the group acting geometrically on it.

The other profound notion that greatly enhanced our knowledge of infinite groups was δ-hyperbolicity (found in [12]). The isoperimetric function of a hyperbolic group is linear and so there is an efficient solution to the word problem. Hyperbolic groups satisfy all the desirable properties we mentioned earlier with regard to CAT(0) groups. Further, many notoriously difficult conjectures like the Novikov and the Baum-Connes are known to be true for this class of groups.

There seems to be great merit in generalizing the notion of non-positive curvature for the purpose of studying infinite discrete groups. It has been done in the past. Gromov suggested the idea of relatively hyperbolic groups which turned out to be invaluable in the study of fundamental groups of complex hyperbolic manifolds with cusps. Juan Alonso and Martin Bridson defined and studied semihyperbolic groups in [2]. Many of the results regarding $CAT(0)$ groups can be extended to semihyperbolic groups. However it is still unknown whether a semihyperbolic group can have infinitely many conjugacy classes of finite subgroups or if a torsion free semihyperbolic group has a finite Eilenberg-MacLane space or for that matter, if an abelian subgroup of a semihyperbolic group can be of infinite rank.

More recently, systolic complexes were introduced by Tadeusz Januszkiewicz and Jacek Swiatkowski in [15] and independently by Frederic Haglund in [13]. These are simply connected simplicial complexes satisfying a local combinatorial condition which is reminiscent of nonpositive curvature. Systolic groups too share many properties with $CAT(0)$ groups. As explained in [15] an aspherical manifold M of dimemsion at least 3 can never be systolic. It is worth mentioning here, that we are still in the dark as to whether every δ-hyperbolic group acts geometrically on a metric space of nonpositive curvature. The same question may be asked about the class of systolic groups.

In this thesis, I propose the theory of asymptotically CAT(0) groups: these are groups acting geometrically on a geodesic space all of whose asymptotic cones are CAT(0). Heuristically speaking, asymptotic cones provide the perspective of a metric space from infinitely far away. Hence, an asymptotically CAT(0) space appears to have non-positive curvature when viewed from increasingly distant observation points.

The objective of my thesis is to provide examples and investigate the algebraic properties of groups acting geometrically on asymptotically CAT(0) geodesic spaces.

Before I proceed to outline the principal results appearing in the different chapters, I want to explain the relationship between the CAT(0) property and quasi-isometries (see Appendix A for definition). The class of CAT(0) metric spaces is not invariant under quasi-isometries. For example, the metric spaces $(\mathbb{R}^n, ||.||_1)$ and $(\mathbb{R}^n, ||.||_2)$ are quasi-isometric (in fact, bi-Lipschitz equivalent). The latter is the quintessential example of a CAT(0) space; on the other hand, $\mathbb{R}^n$ endowed with the l^1 norm is very far from being CAT(0). Indeed, a normed real vector space is CAT(0) if and only if the norm arises from an inner product. Moreover, the class of CAT(0) groups is not invariant under quasi-isometry. Co-compact lattices in $\widetilde{PSL(2,\mathbb{R})}$ are quasi-isometric to co-compact lattices in $\mathbb{H}^2 \times \mathbb{R}$; however, they cannot act properly by semisimple isometries on any CAT(0) space (Section 3.3).

Outline of Chapter 2

Chapter 2 concentrates on *δ-CAT(0) groups*, which form a special class of asymptotically CAT(0) groups. The main definitions are presented and some basic properties of δ-CAT(0) spaces are described. The first examples of δ-CAT(0) spaces are CAT(0) spaces. I verify that hyperbolic spaces are δ-CAT(0).

A *graph* is a 1-complex endowed with the path metric in which every edge has length 1. For graphs the notion of hyperbolicity coincides with that of geodesic triangles being δ-CAT(0).

Proposition 4 *A graph is δ-CAT(0) if and only if it is hyperbolic.*

I investigate how one may combine δ-CAT(0) groups to obtain new ones. Two results in this vein are the following.

Theorem 5 and 6 *The class of δ-CAT(0) groups for $\delta \geq 0$ is closed under taking finite direct products and free products.*

I show that *all δ-CAT(0) groups are semihyperbolic* and this provides some vital corollaries. Indeed, it follows that a δ-CAT(0) group G is always finitely presented and satisfies a quadratic isoperimetric inequality. Further, the word and conjugacy problems for G are solvable, G is of type FP_∞ and every abelian subgroup is finited generated and quasi-isometrically embedded in G with respect to any choice of word metrics. Finally, G has no Baumslag Solitar subgroups, $BS(p,q)$, for $|p| \neq |q|$.

A deep and intriguing conjecture in group theory is Novikov's Conjecture about the homotopy invariance of the 'higher signatures'. Kasparov and Skandalis proved that the Conjecture is true for a large class of groups which can act on metric spaces with some distinctive properties. These are the bolic spaces and I demonstrate that δ-CAT(0) spaces are bolic, a result that also follows from the work of Bucher and Karlsson in [5]. This yields

Corollary 14 *A δ-CAT(0) group satisfies Novikov's Conjecture on higher signatures.*

In the final sections of the second chapter, I introduce asymptotic cones of δ-CAT(0) spaces. The main content of this part of the dissertation is an example, 'The plane with the Wrinkled Quadrant', which is used to prove the proposition below.

Proposition 18 *There exists an asymptotically CAT(0) space which is not δ-CAT(0), for any $\delta \geq 0$.*

Outline of Chapter 3

The third chapter is devoted exclusively to the theory of asymptotically CAT(0) spaces and groups. A metric characterization (**Theorem 19**) of asymptotically CAT(0) geodesic spaces is obtained. This says that a geodesic space is asymptotically CAT(0) if and only if balls of radius r satisfy the $f(r)$-CAT(0) inequality for triangles, where $f : \mathbb{R}_+ \to \mathbb{R}$ is a monotonically non-decreasing function with special decay properties.

Using the metric characterization, I investigate finite subgroups of groups acting on asymptotically CAT(0) geodesic spaces. It is possible to prove the existence of a fixed point for any finite subgroup in a CAT(0) group. Unfortunately, the same may not be true for an asymptotically CAT(0) group. Rather I show that if H is a finite subgroup of an asymptotically CAT(0) group then there exists an H-invariant subspace of uniformly bounded diameter. The consequence of this is Theorem 20.

Theorem 20 *An asymptotically CAT(0) group has finitely many conjugacy classes of finite subgroups.*

Asymptotically CAT(0) groups have nice finiteness properties: indeed, every asymptotically CAT(0) group is of type F_∞, which means that there is a CW-complex $K(\pi, 1)$ with finitely many cells in each dimension. Moreover, the word problem for an asymptotically CAT(0) group is solvable. These are explained in Sections 3.5 and 3.6.

In Section 3.4, I provide techniques for combining asymptotically CAT(0) groups. The class of asymptotically CAT(0) groups is shown to be closed under taking finite direct products. Moreover one can form amalgams and HNN extensions of asymptotically CAT(0) groups, provided the hypotheses of **Theorems 25, 27** are true. In particular I consider amalgamating along finite subgroups and prove the theorems below.

Theorem 28 and 29 *The class of asymptotically CAT(0) groups is closed under free products with amalgamation and HNN extensions along finite subgroups.*

The remaining part of the third chapter is used to outline examples of asymptotically CAT(0) groups. The bolic spaces of Kasparov and Skandalis are asymptotically CAT(0) (Lemma 21). I explain why co-compact lattices in $\widetilde{PSL(2,\mathbb{R})}$ are examples of groups which are neither hyperbolic nor CAT(0); the proof that they are asymptotically CAT(0) is presented in Chapter 5.

Outline of Chapter 4

A potentially rich source of examples is the class of relatively hyperbolic groups and these are the focus of Chapter 4. Given that the group G is relatively hyperbolic with respect to a subgroup H and that H acts geometrically on a space X, one wonders if there is a natural choice for a space Y that supports a geometric G-action. I provide a construction for such a space Y; using the space Y, I prove the result below.

Theorem 35 *Let the group G be relatively hyperbolic with respect to a subgroup H. Then, if H is asymptotically $CAT(0)$, so is G.*

As a corollary, we obtain that *systolic groups with isolated flats are asymptotically CAT(0).*

Outline of Chapter 5

In many ways, $\widetilde{PSL(2,\mathbb{R})}$ is the most intriguing of Thurston's eight geometries. It is neither hyperbolic, nor does it support a $CAT(0)$ metric (See Section 3.3). I exploit the Riemannian geometry of $\widetilde{PSL(2,\mathbb{R})}$ to show that $\widetilde{PSL(2,\mathbb{R})}$, endowed with the Sasaki metric is $(1,\pi)$-quasiisometric to $\mathbb{H}^2\times\mathbb{R}$, endowed with its Riemannian metric. The induced maps at the level of asymptotic cones are therefore, isometries. Moreover, ‘taking asymptotic cones’ commutes with direct products. Hence,

Theorem 44 *The Lie group $\widetilde{PSL(2,\mathbb{R})}$, endowed with the Sasaki metric is asymptotically CAT(0); in particular, co-compact lattices in $\widetilde{PSL(2,\mathbb{R})}$ are asymptotically CAT(0).*

Ouline of Chapter 6

In this chapter I gather some of the interesting questions that arise out of this dissertation. These involve asymptotically CAT(0) graphs, Artin groups, Novikov's Conjecture for asymptotically CAT(0) groups and some special Lie groups.

CHAPTER 2

GROUPS ACTING ON δ-CAT(0) SPACES

2.1 Main Definitions, Properties and Examples

A geodesic segment, denoted $[xy]$, joining two points x and y of a metric space (X, d) is the isometric image of a path of length $d(x, y)$ joining x and y. A geodesic triangle in X consists of its three vertices, call them x, y, z and a choice of geodesic segments $[xy]$, $[yz]$ and $[zx]$ joining these vertices. We will denote such a geodesic triangle by $\triangle(x, y, z)$ (Caution: X may not be uniquely geodesic).

A triangle $\bar{\triangle}(\bar{x}, \bar{y}, \bar{z})$ in E^2 is called a comparison triangle for $\triangle(x, y, z)$ if $d(x, y) = d(\bar{x}, \bar{y})$, $d(y, z) = d(\bar{y}, \bar{z})$, and $d(z, x) = d(\bar{z}, \bar{x})$. It is a consequence of the triangle inequality that given a triangle in X, there is always a comparison triangle in $\mathbb{E}^2$.

Definition 1. *(δ-CAT(0) metric spaces) Let $\triangle$ be a geodesic triangle in X with comparison triangle $\bar{\triangle}$ in $\mathbb{E}^2$. Let $\delta > 0$. Then, $\triangle$ is said to satisfy the δ-CAT(0) inequality if for all p, $q \in \triangle$ and comparison points $\bar{p}$, $\bar{q}$, we have*

$$d(p, q) \leq d(\bar{p}, \bar{q}) + \delta.$$

X is called a δ-CAT(0) space if X is a geodesic metric space and there is a $\delta \geq 0$ such that all geodesic triangles in X satisfy the δ-CAT(0) inequality.

Some properties of δ-CAT(0) spaces

It is useful to investigate the convexity properties of δ-CAT(0) spaces in some detail. We say that a function $\alpha : X \to \mathbb{R}$ is k-convex if, there exists a $k > 0$ such that for any geodesic path $c : I \to X$, parametrized proportional to arc-length, the function $t \mapsto \alpha(c(t))$ defined on the interval I satisfies $\alpha(c(s)) \leq (s-1)\alpha(c(0)) + s\alpha(c(1)) + k$, for all $s \in [0,1]$.

Proposition 1. *If X is a δ-CAT(0) space, then the distance function $d : X \times X \to \mathbb{R}$ is 2δ-convex, that is, given any pair of geodesics f,$g : [0,1] \to X$, parametrized proportional to arc-length, the following inequality holds for all $t \in [0,1]$: $d(f(t), g(t)) \leq (1-t)d(f(0), g(0)) + td(f(1), g(1)) + 2\delta$.*

Proof. We first assume that $f(0) = g(0)$ and consider a comparison triangle $\bar{\triangle} \subseteq \mathbb{R}^2$ for $\triangle(f(0), f(1), g(1))$. Given $t \in [0,1]$, we know that $d(\bar{f(t)}, \bar{g(t)}) = td(\bar{f(1)}, \bar{g(1)}) = td(f(1), g(1))$. The δ-CAT(0)inequality implies that

$$d(f(t), g(t)) \leq d(\bar{f(t)}, \bar{g(t)}) + \delta.$$

Hence, we obtain, $d(f(t), g(t)) \leq td(f(1), g(1)) + \delta$.

In the general case, consider a linearly reparametrized geodesic $h : [0,1] \to X$ with $h(0) = f(0)$ and $h(1) = g(1)$. By applying the preceeding case to f and h and then to h and g with reversed orientation, we get: $d(f(t), h(t)) \leq td(f(1), h(1)) + \delta$ and $d(h(t), g(t)) \leq (1-t)d(h(0), g(0)) + \delta$.

This implies that $d(f(t), g(t)) \leq d(f(t), h(t)) + d(h(t), g(t))$

$\leq td(f(1), g(1)) + (1-t)d(f(0), g(0)) + 2\delta$.

Hence, the metric on X is 2δ-convex. $\square$

Remarks

1. A δ-CAT(0) space X is not necessarily uniquely geodesic. However, the above proposition shows that *geodesics with common endpoints stay uniformly close to each other*. Suppose $f,\ g : [0, d(p, q)] \to X$ are geodesics issuing from p and ending in q. Then, $d(f(t), g(t)) \leq 2\delta$ for all $t \in [0, d(p, q)]$.

2. Unlike CAT(0) spaces, balls in a general δ-CAT(0) space X may not be convex or contractible. Moreover, X may not be simply connected. However, it is easy to see that *balls in X are δ-quasiconvex* i.e. a geodesic joining two points of a ball B in X stays within a δ-neighbourhood of B.

3. Geodesics vary *almost* continuously with their endpoints, and the extent of discontinuity is always uniformly bounded in terms of δ. Let p_n and q_n be sequences of points converging to p and q, respectively. Let c, c_n and c'_n be linear parametrizations of geodesic segments $[p, q]$, $[p_n, q_n]$ and $[p, q_n]$, respectively. Then, $d(c(t), c_n(t)) \leq d(q, q_n) + d(p, p_n) + 2\delta$. Hence, for n large enough, $d(c(t), c_n(t)) \leq 2\delta$, for all $t \in [0, 1]$.

Triangles to Quadrilaterals

Lemma 2. *Let X be a δ-CAT(0) space and let $(x_1, \ldots, x_4)$ be a geodesic quadrilateral in X. Then there exists a convex quadrilateral in the Euclidean plane with vertices $(\bar{x}_1, \ldots, \bar{x}_4)$, such that $d(x_i, x_{i+1}) = d(\bar{x}_i, \bar{x}_{i+1})$ for all i, modulo 4, and $d(x_i, x_j) \leq d(\bar{x}_i, \bar{x}_j) + \delta$, for all $i \neq j = 1, ..., 4$.*

Proof. Let X be a δ-CAT(0) space and $(x_1, \ldots, x_4)$ be a geodesic quadrilateral in X. Form a quadrilateral in $\mathbb{E}^2$ as follows: fix a line segment $[\bar{x}_1, \bar{x}_3]$ with $d(x_1, x_3) = d(\bar{x}_1, \bar{x}_3)$ and form the comparison triangles $[\bar{x}_1, \bar{x}_3, \bar{x}_2]$ and $[\bar{x}_1, \bar{x}_3, \bar{x}_4]$ so that $\bar{x}_2$ and $\bar{x}_4$ are on the opposite sides of this line segment.

If the resulting quadrilateral is convex to start with, then the diagonals $[\bar{x}_1, \bar{x}_3]$ and $[\bar{x}_2, \bar{x}_4]$ intersect at a point say, $\bar{a}$ in the interior of the quadrilateral. Choose a point a on a geodesic joining x_1 and x_3 so that $d(a, x_1) = d(\bar{a}, \bar{x}_1)$. Note that $d(x_2, x_4) \leq d(x_2, a) + d(a, x_4) \leq d(\bar{x}_2, \bar{a}) + d(\bar{a}, \bar{x}_4) + 2\delta = d(\bar{x}_2, \bar{x}_4) + 2\delta$. By construction, $d(x_1, x_3) = d(\bar{x}_1, \bar{x}_3)$. This shows that $(\bar{x}_1, \ldots, \bar{x}_4)$ is a quadrilateral in $\mathbb{E}^2$ of the required description.

Now consider the case when the quadrilateral $(\bar{x}_1, \ldots, \bar{x}_4)$ is not convex. We may assume that $\bar{x}_3$ lies in the interior of the convex hull of $\bar{x}_1$, $\bar{x}_2$, and $\bar{x}_4$. This implies that the interior angle at $\bar{x}_3$ is more than π. Extend the line joining $\bar{x}_2$ and $\bar{x}_3$ up to the point x_4' such that $d(\bar{x}_3, x_4') = d(\bar{x}_3, \bar{x}_4)$. Note that $d(\bar{x}_2, x_4') \geq d(\bar{x}_2, \bar{x}_4)$. Clearly, there is a Euclidean triangle with sides $d(\bar{x}_1, \bar{x}_2)$, $d(\bar{x}_1, \bar{x}_4)$ and $d(\bar{x}_2, \bar{x}_3) + d(\bar{x}_3, \bar{x}_4)$. This triangle satisfies all properties of the quadrilateral we were looking for. □

Examples

1. *CAT(0) spaces.* Trivially, a CAT(0) space is δ-CAT(0), for $\delta = 0$.

2. *Hyperbolic spaces.* A short proof of this is provided in Section 2.2. In fact, by Proposition 4, a graph is δ-CAT(0) if and only if it is hyperbolic, for some

$\delta \geq 0$: a graph is a one-dimensional complex endowed with the path metric in which every edge has unit length.

3. *Direct products of finitely many δ-CAT(0) metric spaces.* This is proved in Section 2.3.

4. *Banach spaces:* A finite dimensional real vector space endowed with the l^p norm is δ-CAT(0) if and only if $p = 2$. See Section 2.7.

5. *Tree of Spaces:* A tree of spaces in which every vertex space is δ-CAT(0) and every edge space is trivial is also δ-CAT(0). This is the content of Lemma 7.

2.2 Hyperbolic Spaces

There are several equivalent definitions of hyperbolicity and here, we use the 'Thin Triangles' condition. Given any three positive numbers a, b and c, the tripod $T(a, b, c)$ is a simplicial metric tree with at most three edges, of length a, b and c, and at most one vertex of valency greater than one.

Now let $\mathcal{M}$ be a metric space. Suppose that $\triangle(A, B, C)$ is a geodesic triangle in $\mathcal{M}$. By the triangle inequality there exist unique, non-negative numbers a, b and c such that $d(A, B) = a + b$, $d(A, C) = a + c$ and $d(B, C) = b + c$. The collection of vertices $\{A, B, C\}$ of Δ map isometrically to the vertices of the tripod $T(a, b, c)$. Moreover, this isometry extends uniquely to a map $\pi : \Delta \to T(a, b, c)$ whose restriction to each side of Δ is an isometry.

Let $\delta \geq 0$. The triangle Δ is said to be δ-thin if $P, Q \in \pi^{-1}(t) \Rightarrow d(P,Q) \leq \delta$, for all $t \in T(a,b,c)$.

Definition 2. *A geodesic metric space $\mathcal{M}$ is said to be δ-hyperbolic if all geodesic triangles in M are δ-thin for some $\delta \geq 0$.*

Proposition 3. *A δ-hyperbolic space is δ-CAT(0).*

Proof. Let $\mathcal{M}$ be a geodesic space, equipped with a δ-hyperbolic metric. We will show that every geodesic triangle in $\mathcal{M}$ satisfies the δ-CAT(0) inequality.

Suppose that $\triangle(A,B,C)$ is a geodesic triangle in $\mathcal{M}$ and let a, b and c be as above. Let X be the point on the geodesic joining A and B such that $d(A,X) = a$. Similarly, choose Y on BC and Z on AC such that $d(B,Y) = b$ and $d(C,Z) = c$. By the δ-thin condition, we know that the three quantities $d(X,Y)$, $d(Y,Z)$ and $d(X,Z)$ are no larger than δ.

Now let P and Q be two points on $\triangle$. Take a comparison triangle $\bar{\triangle}(\bar{A},\bar{B},\bar{C})$ in the Euclidean plane and mark off the comparison points $\bar{X}$, $\bar{Y}$, $\bar{Z}$, $\bar{P}$ and $\bar{Q}$. We may assume that P lies on the segment AX, whence it suffices to consider the following four possibilities for the point Q.

1) The point Q lies on the segment AZ. Let P' be the point on AZ such that $d(A,P') = d(A,P)$. Then, by the Thin Triangles condition, $d(P,P') \leq \delta$. Moreover, $d(P',Q) = |d(A,Q) - d(A,P)|$. Therefore $d(P,Q) \leq |d(A,Q) - d(A,P)| + \delta = |d(\bar{A},\bar{Q}) - d(\bar{A},\bar{P})| + \delta \leq d(\bar{P},\bar{Q}) + \delta$.

2) The point Q lies on the segment BY. In this case, $d(P,Q) \leq d(P,X) + d(Q,Y) + \delta$.

On the other hand, $d(\bar{P},\bar{Q}) \geq |d(\bar{B},\bar{P}) - d(\bar{B},\bar{Q})| = |d(\bar{B},\bar{X}) + d(\bar{X},\bar{P}) - (d(\bar{B},\bar{Y}) - d(\bar{Y},\bar{Q}))| = d(\bar{P},\bar{X}) + d(\bar{Q},\bar{Y}) = d(P,X) + d(Q,Y)$.

3) The point Q lies on the segment CY. Once again, $d(P,Q) \leq d(P,X) + d(Q,Y) + \delta$. Note that $d(\bar{P},\bar{Q}) \geq |d(\bar{A},\bar{C}) - d(\bar{A},\bar{P}) - d(\bar{C},\bar{Q})|$. But, this last quantity is equal to $d(P,X) + d(Q,Y)$.

4) The point Q lies on the segment CZ. This is similar to case 2.

Hence, in all possible cases, $d(P,Q) \leq d(\bar{P},\bar{Q}) + \delta$. □

Proposition 4. *A graph is hyperbolic if and only if it is δ-CAT(0), for some $\delta \geq 0$.*

Proof. By the preceeding proposition, every δ-hyperbolic space is δ-CAT(0). The converse is a consequence of a simple geometric fact, proved in [19]; that geodesic triangles in a graph are *thin* if *bigons* in the graph are thin.

A bigon in a graph Γ is a pair of geodesics γ, γ' with $\gamma(0) = \gamma'(0)$ and $\gamma(l) = \gamma'(l)$ where $l = length(\gamma)$. We say that bigons in Γ are ϵ-thin if for any bigon, (γ, γ'), we have $d(\gamma(t), \gamma'(t)) < \epsilon$, for every t, $0 < t < l$. On the other hand, a bigon (γ, γ') is M-thick if $d(\gamma(t), \gamma'(t)) > M$, for some $t \in (0, l)$.

Now, suppose that Γ is a graph. If Γ is not hyperbolic, then by [19], it contains r-thick bigons, for all $r > 0$. But this is impossible in a δ-CAT(0) space, as geodesics joining a pair of points in a δ-CAT(0) space stay uniformly close together. □

2.3 Direct Products

Theorem 5. *The class of δ-CAT(0) metric spaces for all $\delta \geq 0$, is closed under taking finite direct products. More precisely, if (X,d) and (Y,d') are δ-CAT(0) and δ'-CAT(0) respectively, then $(X \times Y, \sqrt{d^2 + (d')^2})$ is $\sqrt{2}max\{\delta, \delta'\}$-CAT(0).*

Proof. Let (X,d) and (Y,d') be δ-CAT(0) metric spaces. Consider $X \times Y$, denoted Z, with the product metric, that is, $d'' = \sqrt{d^2 + d'^2}$. Let $p = (p_1, p_2)$, $q = (q_1, q_2)$ and $r = (r_1, r_2)$ be three points in Z and $\triangle$ denote a geodesic triangle in Z with vertices p, q and r. Fix $x = (x_1, x_2)$ and $y = (y_1, y_2) \in \triangle$. We may assume that x lies on the geodesic side $[p,q]$ joining p and q while y lies on the geodesic $[q,r]$ joining q and r.

Since $d'' = \sqrt{d^2 + d'^2}$, the geodesics $z(t)$ in Z are obtained as a product $z(t) = (x(t), y(t))$ of geodesics $x(t), y(t)$ in X and Y respectively. This gives that

$$\frac{d(p_1, x_1)}{d(x_1, q_1)} = \frac{d'(p_2, x_2)}{d'(x_2, q_2)} \quad \text{and} \quad \frac{d(q_1, y_1)}{d(y_1, r_1)} = \frac{d'(q_2, y_2)}{d'(y_2, r_2)}. \tag{2.1}$$

Consider the geodesic triangle $\triangle_X = \triangle(p_1, q_1, r_1)$ in X. Note that x_1 lies on the geodesic side $[p_1, q_1]$ of $\triangle_X$ and $y_1 \in [p_1, r_1]$. Let $\bar{\triangle}_X \subset \mathbb{E}^2$ be its comparison triangle having vertices $\bar{p}_1, \bar{q}_1, \bar{r}_1$. Similarly, let $\triangle_Y = \triangle(p_2, q_2, r_2) \subset \mathbb{E}^2$ be a geodesic triangle in Y whose comparison triangle $\bar{\triangle}_Y$ has vertices $\bar{p}_2, \bar{q}_2, \bar{r}_2$.

Let $\bar{x}_1 \in [\bar{p}_1, \bar{q}_1]$ be the point on the segment $[\bar{p}_1, \bar{q}_1]$ of $\bar{\triangle}_X$ which is of distance $d(p_1, x_1)$ from $\bar{p}_1$, i.e. so that $d(\bar{p}_1, \bar{x}_1) = d(p_1, x_1)$. Let $\bar{y}_1 \in [\bar{q}_1, \bar{r}_1]$ be chosen so that $d(\bar{q}_1, \bar{y}_1) = d(q_1, y_1)$. Thus the points $\bar{x}_1$ and $\bar{y}_1$ on the sides of $\bar{\triangle}_X$ are comparison points for x_1 and $y_1 \in \triangle_X$.

Similarly define points $\bar{x}_2, \bar{y}_2 \in \bar{\triangle}_Y$ which are comparison points for x_2 and y_2 in $\triangle_Y$. This means $d(\bar{p}_2, \bar{x}_2) = d'(p_2, x_2)$ and $d(\bar{q}_2, \bar{y}_2) = d'(q_2, y_2)$.

Using the δ-CAT(0) inequality , we have

$$d(x_1, y_1) \leq d(\bar{x}_1, \bar{y}_1) + \delta, \quad d'(x_2, y_2) \leq d(\bar{x}_2, \bar{y}_2) + \delta. \tag{2.2}$$

We now build a comparison triangle $\bar{\triangle} = \triangle(\bar{p}, \bar{q}, \bar{r})$ in $\mathbb{E}^4$ for $\triangle$ with vertices $\bar{p}, \bar{q}, \bar{r}$ defined in the following way:

$$\bar{p} := (\bar{p}_1, \bar{p}_2), \quad \bar{q} := (\bar{q}_1, \bar{q}_2), \quad \bar{r} := (\bar{r}_1, \bar{r}_2).$$

Since $d''^2 = d^2 + d'^2$, it is immediate that the sides of $\bar{\triangle}$ have the same length as the sides of $\triangle$. Moreover, $\bar{\triangle}$ lies in a 2-dimensional subspace inside $\mathbb{E}^4$ and therefore can be considered as a comparison triangle for $\triangle$ in $\mathbb{E}^2$.

Let $\bar{x} := (\bar{x}_1, \bar{x}_2)$. Since $\bar{x}_1 \in [\bar{p}_1, \bar{q}_1]$ and $\bar{x}_2 \in [\bar{p}_2, \bar{q}_2]$ and the first equality in (2.1) holds we see that $\bar{x}$ is on the side $[\bar{p}, \bar{q}]$ of $\bar{\triangle}$ and

$$d''(p, x) = \sqrt{d(p_1, x_1)^2 + d'(p_2, x_2)^2} = \sqrt{d(\bar{p}_1, \bar{x}_1)^2 + d(\bar{p}_2, \bar{x}_2)^2} = d(\bar{p}, \bar{x}).$$

Thus, $\bar{x}$ is a comparison point for x in $\bar{\triangle}$. Similarly define $\bar{y} := (\bar{y}_1, \bar{y}_2)$ which by (2.1) lies on the side $[\bar{q}, \bar{r}]$ of $\bar{\triangle}$ and in the same way see that $\bar{y}$ is a comparison points for y. Now, by (2.2) we have

$d''(x, y)^2 = d(x_1, y_1)^2 + d'(x_2, y_2)^2$
$\leq (d(\bar{x}_1, \bar{y}_1) + \delta)^2 + (d(\bar{x}_2, \bar{y}_2) + \delta')^2$
$\leq d(\bar{x}_1, \bar{y}_1)^2 + d(\bar{x}_2, \bar{y}_2)^2 + 2\Delta(d(\bar{x}_1, \bar{y}_1) + d(\bar{x}_2, \bar{y}_2)) + 2\Delta^2$, where $\Delta = max\{\delta, \delta'\}$
$\leq (\sqrt{d(\bar{x}_1, \bar{y}_1)^2 + d(\bar{x}_2, \bar{y}_2)^2} + \sqrt{2}\Delta)^2 = (d(\bar{x}, \bar{y}) + \sqrt{2}\Delta)^2.$

In the above argument, we have used the fact for real numbers a and b, $(a + b) \leq \sqrt{2(a^2 + b^2)}$. $\square$

2.4 Free Products

Theorem 6. *The free product of two δ-CAT(0) groups is δ-CAT(0).*

Proof. Let A and B be two groups acting geometrically on the δ-CAT(0) spaces X and Y respectively. We build a space Z on which the free product $A * B$ acts geometrically and show that Z supports a δ-CAT(0) metric. Set $G = A * B$.

Recall that the free product G acts without inversions on a tree T, with fundamental domain an edge (for more on Bass Serre Theory, see [14]). The tree T is unique up to graph isomorphism. Select representatives γ_α for the left cosets α of A in G and similarly, choose θ_β, representatives for the left cosets of B in G. The edges of T are in one-one correspondence with the elements of G; the vertices of T are in one-one correspondence with the set $G/A \coprod G/B$, that is the disjoint union of the left cosets of A in G and of B in G. Each edge is identified with two vertices of the tree using the natural maps $\phi : G \to G/A$ and $\psi : G \to G/B$.

We build the space Z, with the tree T as guide. Choose a base point $x_0 \in X$ and similarly, a base point $y_0 \in Y$. Set

$$Z = \frac{(G/A \times X) \coprod (G \times [0,1]) \coprod (G/B \times Y)}{\sim}$$

where the equivalence relation $\sim$ is given by $(g, 0) \sim (\phi(g), x_0)$, and $(g, 1) \sim (\psi(g), y_0)$, for all $g \in G$. One makes Z into a metric space in a natural way using the quotient metric.

We now describe the action of the free product G on Z. For each $\alpha \in G/A$ and $g \in G$, let $a_{(\alpha,g)}$ be the element of A such that $g\gamma_\alpha = \gamma_{g\alpha}\ a_{(\alpha,g)}$. Similarly, define $b_{(\beta,g)} \in B$, such that $g\theta_\beta = \theta_{g\beta}\ b_{(\beta,g)}$.

Define the action of G on $G/A \times X$ by

$$g.(\gamma_\alpha, x) = (\gamma_{g\alpha}, a_{(\alpha,g)}x), \quad \text{for all} \ \ g \in G,\ x \in X,\ \alpha \in G/A.$$

Similarly define the action of G on $G/B \times Y$ by

$$g.(\theta_\beta, y) = (\theta_{g\beta}, b_{(\theta,g)}y), \quad \text{for all} \ \ g \in G,\ y \in Y,\ \beta \in G/B.$$

Finally, G acts on $G \times [0,1]$ by left multiplication on the first component.

It is clear that this is an action on each component of the space Z prior to gluing. Using Bass Serre theory, we deduce that the gluing respects this action and so we have an induced action of G on Z. Note that

$$Z/G \cong \frac{X/A \coprod [0,1] \coprod Y/B}{\sim}$$

where the relation $\sim$ simply identifies the two ends of the interval $[0,1]$ with the equivalence class of x_0 on one side and with the class of y_0 on the other. This implies that the action of G on Z is co-compact. On the other hand, G acts properly on the tree T. Hence, it follows from the construction of Z that the action of G on Z is proper if the actions of A and B on X and Y are proper. Therefore, if A and B act geometrically on X and Y then G acts geometrically on Z.

We now observe that Z is a geodesic metric space. The tree T maps on to the space Z via the canonical projection π of Z onto T which sends any point of the form (α, x) (and (β, y)) to α (and β, respectively). Let $p \neq q$ be two points of Z. If $\pi(p)$ and $\pi(q)$ coincide then they lie in the same copy of X or Y, hence they can be joined by a geodesic in X or Y. On the other hand, if $\pi(p) \neq \pi(q)$, it suffices to consider the situation when p comes from a point (α, x) and q comes from a point (β, y). In this case, p and q may be joined by a geodesic that is the union of three paths: a geodesic in X joining (α, x) to (α, x_0), followed by a geodesic in the (image of the) tree joining (α, x_0) to (β, y_0), followed by a geodesic in Y joining (β, y_0) to (β, y).

Lemma 7. *The metric space Z is δ-CAT(0).*

Proof of Lemma 7. Let P, Q and R be three distinct points in Z. If the three points project to the same point of T, then they belong to the same copy of X or Y in Z. In contrast, if they project to three distinct points in T, then any triangle with vertices at these points is a tripod. In these two cases, the claim is clearly true. It thus suffices to consider points, two of which project to some α and the third projects to some β.

Let $P = (\beta, y)$, $Q = (\alpha, x_1)$ and $R = (\alpha, x_2)$ and let Δ denote a choice of triangle in Z with vertices at these three points. Now, $\Delta - [Q, R]$ is a tripod with a branching point at (α, x_0). Set $P' = (\alpha, x_0)$. Then $\Delta = [P, P'] \cup \Delta'$, where Δ' is a geodesic triangle in (α, X) with vertices P', Q and R. We claim that Δ satisfies the δ-CAT(0) inequality.

First, we prove a simple Lemma from Euclidean geometry.

Lemma 8. *Let LMN and $L'M'N'$ be two triangles in the Euclidean Plane and $r > 0$. Suppose that $d(M,N) = d(M',N')$, $d(L',M') = d(L,M) + r$ and $d(L',N') = d(L,N) + r$. Then, $\angle LMN \leq \angle L'M'N'$.*

Proof of Lemma 8. Let X and Y denote the points on $L'M'$ and $L'N'$ respectively such that $d(L',X) = r = d(L',Y)$. Then to prove that $\angle LMN \leq \angle L'M'N'$, the cosine law says we need to show that $d(L,N) \leq d(X,N')$.

Now $\Delta L'XY$ is an isosceles triangle. Hence, $\angle L'YX$ is an acute angle and consequently $\angle XYN'$ is an obtuse angle. This implies that XN' is the largest side of the triangle XYN' and so the side of XN' is at least as large as the side YN'. Therefore, $d(L,N) \leq d(X,N')$ and we conclude that $\angle LMN \leq \angle L'M'N'$. □

Let S and T be two points on the triangle Δ. We assume first that S lies on the geodesic QR and that T lies on one of the geodesic segments QP', $P'R$ or $P'P$. The cases when T lies on QP' and RP' are similar and so we will only consider the first.

We denote the comparison triangle $\bar{P}\bar{Q}\bar{R}$ for Δ by $\bar{\Delta}$ and the comparison triangle $\bar{P}'\bar{Q}'\bar{R}'$ for Δ' by $\bar{\Delta}'$. Let $\bar{S}$ and $\bar{S}'$ be the comparison points for S on $\bar{\Delta}$ and $\bar{\Delta}'$, respectively.

Case 1. The point T lies on $P'Q$

Choose comparison points $\bar{T}$ and $\bar{T}'$ for T on $\bar{\Delta}$ and $\bar{\Delta}'$, respectively. As X is δ-CAT(0), we have $d(T,S) \leq d(\bar{S}',\bar{T}') + \delta$. But by Lemma 8, $\angle \bar{P}'\bar{Q}'\bar{R}' \leq \angle \bar{P}\bar{Q}\bar{R}$. By the Cosine Law, $d(\bar{S}',\bar{T}') \leq d(\bar{S},\bar{T})$. Therefore, $d(T,S) \leq d(\bar{S},\bar{T}) + \delta$.

Case 2a. The point T lies on PP' and T coincides with P.

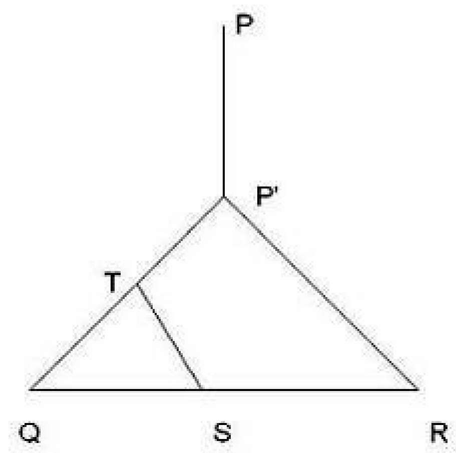

Figure 2.1: A generic triangle in a tree of δ-CAT(0) spaces.

As X is δ-CAT(0), we have $d(P', S) \leq d(\bar{P}', \bar{S}') + \delta$. Moreover, $d(P, S) = d(P, P') + d(P', S)$. So we need to show that $d(P, P') + d(\bar{P}', \bar{S}') \leq d(\bar{P}, \bar{S})$. Identifying $[\bar{Q}, \bar{R}]$ with $[\bar{Q}', \bar{R}']$, we see that the points $\bar{P}$, $\bar{P}'$, $\bar{Q}$, $\bar{R}$ and $\bar{S}$ in the Euclidean plane satisfy $d(\bar{Q}, \bar{P}) - d(\bar{Q}, \bar{P}') = d(P, P') = d(\bar{R}, \bar{P}) - d(\bar{R}, \bar{P}')$.

Therefore, these points may be placed in a configuration such that $\bar{Q}$ and $\bar{R}$ lie on the same branch of a hyperbola with foci at $\bar{P}$ and $\bar{P}'$. As $\bar{S}$ lies on the line joining two points which belong to the same branch of a hyperbola and the latter is convex, we have $d(\bar{S}, \bar{P}) - d(\bar{S}, \bar{P}') \geq d(P, P')$. This is the required inequality.

Case 2b. The point T lies on PP'

We argue that this follows from cases 1 and 2a. Suppose that $\tilde{T}\tilde{Q}\tilde{R}$ is a comparison triangle for TQR. Then we deduce from the analysis of case 2a that $d(S, T) \leq d(\tilde{S}, \tilde{T}) + \delta$, where $\tilde{S}$ is the comparison point for S on the segment $\tilde{Q}\tilde{R}$. We are now in a position to appeal to case 1, which tells us that $d(\tilde{S},\tilde{T}) \leq d(\bar{S},\bar{T})$. Therefore, $d(S,T) \leq d(\bar{S},\bar{T}) + \delta$.

Finally we consider the case when S lies on $P'Q$ and T lies on $P'R$. As before, choose comparison points $\bar{S}$ and $\bar{T}$ on $\bar{\Delta}$ for S and T. Similarly choose $\bar{S}'$ and $\bar{T}'$ on $\bar{\Delta}'$. Now, $d(S,T) \leq d(\bar{S}',\bar{T}')+\delta$ and so it suffices to establish the inequality $d(\bar{S}',\bar{T}') \leq d(\bar{S},\bar{T})$. This is done by direct computation with the Cosine Law.

Set $d(P,P') = r$, $d(P',S) = s$, $d(P',T) = t$, $d(P',Q) = a$, $d(P',R) = b$, $d(\bar{S}',\bar{T}') = h$, $d(\bar{S},\bar{T}) = \bar{h}$ and $d(Q,R) = c$. Using the Cosine Law, we have

$$h^2 = s^2 + t^2 - 2st\left[\frac{a^2+b^2-c^2}{2ab}\right] = (s-t)^2 + st\left[\frac{c^2-(a-b)^2}{ab}\right]$$

$$\begin{aligned}\bar{h}^2 &= (s+r)^2 + (t+r)^2 - 2(s+r)(t+r)\left[\frac{(a+r)^2+(b+r)^2-c^2}{2(a+r)(b+r)}\right]\\ &= (s-t)^2 + (s+r)(t+r)\left[\frac{c^2-(a-b)^2}{(a+r)(b+r)}\right]\end{aligned}$$

$$\begin{aligned}\text{Therefore, } \bar{h}^2 - h^2 &= (c^2-(a-b)^2)\left[\frac{(s+r)(t+r)}{(a+r)(b+r)} - \frac{st}{ab}\right]\\ &= (c^2-(a-b)^2)\left[\frac{asr(b-t)+brt(a-s)+r^2(ab-st)}{ab(a+r)(b+r)}\right].\end{aligned}$$

Now, $c^2-(a-b)^2 \geq 0$, by the triangle inequality. Clearly, $(b-t)$, $(a-s)$ and $(ab-st)$ are all non-negative. Therefore, $\bar{h}^2 - h^2 \geq 0$ and so $d(\bar{S}',\bar{T}') \leq d(\bar{S},\bar{T})$.

This shows that Z is δ-CAT(0). $\square$

The proof of the above lemma completes the proof of Theorem 6. $\square$

2.5 Semihyperbolicity and δ-CAT(0) Groups

In this section, we prove that groups acting geometrically on δ-CAT(0) spaces are semihyperbolic and list some consequences. We use the characterisation of semihyperbolic groups given below.

Let G be a finitely generated group with generating set A. The free monoid $(A^{\pm 1})^*$ consists of all words in the alphabet $A^{\pm 1}$. Let $\mathcal{P}(G)$ denote the set of all subsets of G. There is a natural map from $(A^{\pm 1})^*$ to $\mathcal{P}(G)$ that takes a word w in G to the discrete path $t \mapsto w(t)$, where $w(t)$ is the image in G of the prefix of length t in w.

Proposition 9 (Proposition III.Γ.4.5 [4]). *Let G be a finitely generated group with generating set A. Then, G is semihyperbolic if and only if there exist positive constants λ, ϵ, l and a choice of words $\{w_g | \ g \in G\} \subseteq (A^{\pm 1})^*$, such that $w_g = g$ in G and the discrete paths $t \mapsto w_g(t)$ are (λ, ϵ)-quasigeodesics satisfying the property: $d(w_g(t), a.w_{a^{-1}ga'}(t)) \leq l$, for all a, $a' \in A^{\pm 1} \cup \{1\}$ and for all $t \in \mathbb{N}$.*

Theorem 10. *A δ-CAT(0) group is semihyperbolic.*

Proof. We first choose a convenient set of generators for G. Fix $x_0 \in X$. As G acts geometrically on a δ-CAT(0) space X, there is a $D > 0$ such that $G.B(x_0, D/3) = X$. In this case, the collection $A = \{a \in G \mid B(x_0, D) \cap aB(x_0, D) \neq \emptyset\}$ is a generating set of G. Further, the map $g \mapsto g.x_0$ defines a (say, (λ, κ)-) quasi-isometry from G to X.

To each $g \in G$, we associate a word w_g in the generators A as follows: let c_g be a geodesic joining $x_0 = c_g(0)$ to $g.x_0$ in X. For each $n \in \mathbb{N}$, let $w_g(n)$ be such that $d(c_g(n), w_g(n)x_0) \leq D/3$, with the convention that $w_g(0) = 1$ and $w_g(n) = g$ for

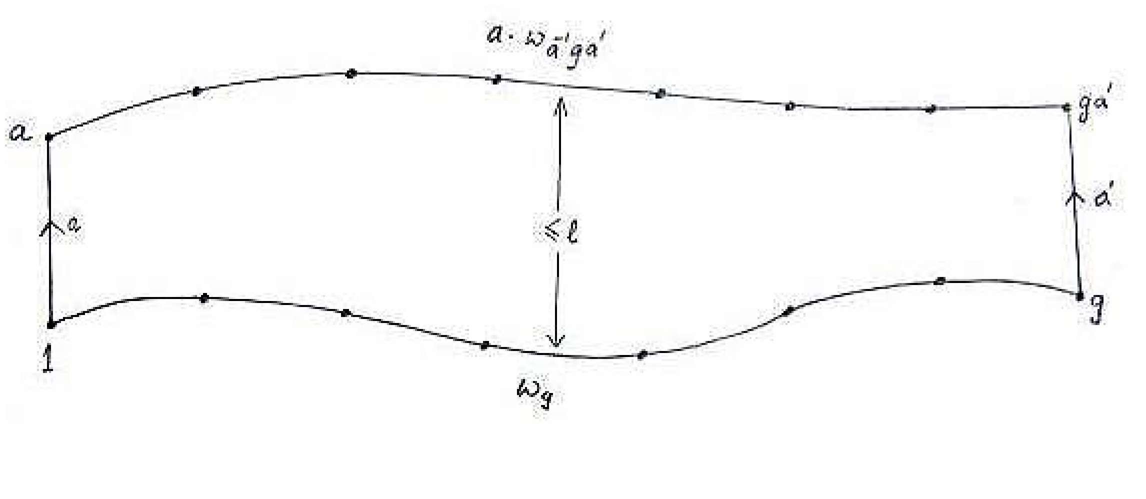

Figure 2.2: Semihyperbolicity

$n \geq d(x_0, g.x_0)$. Note that $a_n := w_g(n-1)^{-1}w_g(n) \in A \cup \{1\}$. Define w_g to be the empty word if $1 = g \in G$ or else, define w_g to be $a_1a_2...a_k$, where k is the smallest integer exceeding the quantity $d(x_0, gx_0)$. Note that $g = w_g$ in G.

Now, by construction, for $t \in \mathbb{N}$, $w_g(t) = a_1a_2...a_t$ with the understanding that if $t \geq k$, then $w_g(t) = a_1a_2...a_k$. Hence, the discrete paths $t \mapsto w_g(t)$ are quasigeodesics, being images of geodesics from x_0 to $g.x_0$ under the natural quasi-isometry between G and X.

For any a, $a' \in A^{\pm 1} \cup \{1\}$ and $t \in \mathbb{N}$ we have

$d_G(w_g(t), a.w_{a^{-1}ga'}(t)) \leq \lambda(d(w_g(t).x_0, a.w_{a^{-1}ga'}(t).x_0) + \kappa)$

$\leq \lambda\,(d(w_g(t).x_0, c_g(t)) + d(c_g(t), a.c_{a^{-1}ga'}(t)) + d(a.c_{a^{-1}ga'}(t), a.w_{a^{-1}ga'}(t).x_0) + \kappa)$

$= \lambda(d(w_g(t).x_0, c_g(t)) + d(c_g(t), a.c_{a^{-1}ga'}(t)) + d(c_{a^{-1}ga'}(t), w_{a^{-1}ga'}(t).x_0) + \kappa)$

$\leq \lambda \left(\frac{D}{3} + (2\delta + 1) + \frac{D}{3} + \kappa\right) \leq \lambda \left(D + 2\delta + \kappa + 1\right).$

In the last line we have used the inequality $d(c_g(t), a.c_{a^{-1}ga'}(t)) \leq 2\delta + 1$ which is a consequence of Proposition 1.

Therefore, we can take the constant l in Proposition 9 to be $\lambda(D + 2\delta + \kappa + 1)$ and it follows that G is semihyperbolic. □

Corollary 11. *Suppose that the group G acts on a δ-CAT(0) space. Then,*

1. *The group G is finitely presented and satisfies a quadratic isoperimetric inequality. Further, the word and conjugacy problems for G are solvable.*
2. *The group G is of type FP_∞.*
3. *If H is a finitely generated abelian group, then every monomorphism $\phi : H \hookrightarrow G$ is a quasi-isometric embedding with respect to any choice of word metrics.*
4. *If $S \subset G$ is a finite subset, then the centraliser of S in G is finitely generated and so the centre $Z(G)$ of G is also a finitely generated abelian group.*
5. *A polycyclic group P is a subgroup of G if and only if P is virtually abelian.*
6. *If $|p| \neq |q|$, then $\langle x, t \mid t^{-1}x^p t = x^q\rangle$ cannot be a subgroup of G.*

Proof. Indeed, all these follow from the fact that G is semihyperbolic. For 1 and 3-6, see [BH] and for 2, see [A]. □

2.6 Novikov Conjecture and δ-CAT(0) Groups

In [16], Gennadi Kasparov and Georges Skandalis introduce a class of metric spaces which they call bolic and go on to prove that Novikov's Conjecture on Higher Signatures (see Appendix B) is true for any discrete group acting properly by isometries on a weakly bolic, weakly geodesic metric space of bounded coarse geometry. In what follows, we show that δ-CAT(0) metric spaces are 4δ-bolic and use their criteria to deduce the Novikov Conjecture for δ-CAT(0) groups. We first present a few definitions as they appear in the aforesaid paper.

Let ϵ be a nonnegative real number. A function (not necessarily continuous) $f : X \to X'$ between metric spaces (X, d) and (X', d') is said to be a δ-isometry if for every pair (x, y) of elements of X we have $|d'(f(x), f(y)) - d(x, y)| \leq \epsilon$.

Further, a metric space (X, d) is said to be ϵ-geodesic if for every pair $(x, y) \in X$, there exists an ϵ-isometry f between $[0, d(x, y)]$ and X such that $f(0) = x$ and $f(d(x, y)) = y$.

Definition 3. *The space (X, d) is said to be weakly ϵ-geodesic if for every pair (x, y) of points of X, and every $t \in [0, d(x, y)]$ there exists a point $a \in X$ such that $d(a, x) \leq t + \epsilon$ and $d(a, y) \leq d(x, y) - t + \epsilon$. The point $a \in X$ is called a ϵ-middle point of x and y if $|2d(x, a) - d(x, y)| \leq 2\epsilon$ and $|2d(y, a) - d(x, y)| \leq 2\epsilon$.*

We will say that the space (X, d) admits ϵ-middle points if there exists a map $m : X \times X \to X$ such that for any x, $y \in X$, the point $m(x, y)$ is a ϵ-middle point of x and y. Note that every ϵ-geodesic space is weakly ϵ-geodesic.

Definition 4. *A metric space (X, d) is δ-bolic if:*

1. *For all $r > 0$, there exists $R > 0$ such that for every quadruple x, y, z, t of points of X satisfying $d(x,y) + d(z,t) \leq r$ and $d(x,z) + d(y,t) \geq R$, then we have $d(x,t) + d(y,z) \leq d(x,z) + d(y,t) + 2\delta$.*

2. *There exists a map $m : X \times X \to X$ such that for all x, y, $z \in X$, we have $2d(m(x,y),z) \leq \sqrt{2d(x,z)^2 + 2d(y,z)^2 - d(x,y)^2} + 4\delta$.*

Theorem 12 (Main Theorem of [16])**.** *If a group G acts geometrically by isometries on a metric space X which is ϵ-geodesic, δ-bolic and of bounded coarse geometry, then it satisfies Novikov's Conjecture on Higher Signatures.*

In [5], the authors prove that the second condition in the definition of δ-bolicity implies the first. It is a technical result, an immediate consequence of which is the following:

Theorem 13. *(Bucher, Karlsson) A δ-CAT(0) space X is 4δ-bolic.*

We give a simple alternative proof that bypasses the result in [5].

Proof of Theorem 13. Let X be a δ-CAT(0) metric space and r, a positive real number. Suppose that x, y, z and t are four points in X such that $d(x,y) + d(z,t) \leq r$. By the δ-CAT(0) inequality for quadrilaterals, there exist four points say $\bar{x}$, $\bar{y}$, $\bar{z}$ and $\bar{t}$ in the Euclidean plane satisfying $d(x,y) = d(\bar{x},\bar{y})$, $d(y,t) = d(\bar{y},\bar{t})$, $d(z,t) = d(\bar{z},\bar{t})$, $d(x,z) = d(\bar{x},\bar{z})$, $d(x,t) \leq d(\bar{x},\bar{t}) + 2\delta$ and $d(y,z) \leq d(\bar{y},\bar{z}) + 2\delta$.
Now, we know that the Euclidean plane is δ-bolic for arbitrarily small δ. In fact, there is a choice of $R > 0$ for which If $d(\bar{x},\bar{y}) + d(\bar{z},\bar{t}) \leq r$ and $d(\bar{x},\bar{z}) + d(\bar{y},\bar{t}) \geq R$, then $d(\bar{x},\bar{t}) + d(\bar{y},\bar{z}) \leq d(\bar{x},\bar{z}) + d(\bar{y},\bar{t})$.

Hence, applying the above inequalities, we get:

if $d(x, y) + d(z, t) \leq r$ and $d(x, z) + d(y, t) \geq R$, then $d(x, t) + d(y, z) \leq d(\bar{x}, \bar{t}) + d(\bar{y}, \bar{z}) + 4\delta \leq d(\bar{x}, \bar{z}) + d(\bar{y}, \bar{t}) + 4\delta = d(x, z) + d(y, t) + 4\delta$.

We now need to consider middle point maps. By definition, our spaces are geodesic and hence, midpoints exist. Moreover, all geodesics joining a pair of points stray at most δ apart and therefore any two choice of middle points vary by the same amount. Moreover, in the Euclidean plane there are unique midpoints. So, suppose that m is a midpoint between two points, x and y in our δ-CAT(0)space X and z is some other point in X. Take a geodesic triangle with vertices x, y and z. Let m' be the midpoint of the geodesic joining x and y. Then $d(m, m') \leq \delta$.

Now draw a comparison triangle $\bar{\triangle}(\bar{x}, \bar{y}, \bar{z})$ in the Euclidean plane, taking $\bar{m}'$ to be the comparison point for m' in the segment $[\bar{x}, \bar{y}]$. By the cosine law, $2d(\bar{z}, \bar{m}') = \sqrt{2d(\bar{x}, \bar{z})^2 + 2d(\bar{z}, \bar{y})^2 - d(\bar{x}, \bar{y})^2}$. Therefore, $2d(z, m) \leq 2d(z, m') + 2\delta \leq 2d(\bar{z}, \bar{m}') + 4\delta$. Hence, $2d(z, m) \leq \sqrt{2d(x, z)^2 + 2d(y, z)^2 - d(x, y)^2} + 4\delta$. $\square$

We can now appeal to Theorem 12 to deduce Novikov's Conjecture for δ-CAT(0) groups. We consider an orbit Y of our group G in the δ-CAT(0) space X on which it acts. Then Y with the subspace metric is weakly ϵ-geodesic for a suitable ϵ and δ-bolic. Since the action is assumed to be geometric, it is of bounded coarse geometry. This implies the following theorem:

Corollary 14. *If a group G acts geometrically on a δ-CAT(0) metric space, then it satisfies Novikov's Conjecture on higher signatures.*

Remark. In the light of Theorem 13, one may view the δ-CAT(0) property as a combinatorial (though potentially weaker) version of δ-bolicity.

2.7 Asymptotic Cones of δ-CAT(0) Spaces

In this section we discuss asymptotic cones of δ-CAT(0) spaces. For a definition of asymptotic cones, see Appendix A. The following statement is a consequence of Lemma 21.

Observation 15. *All asymptotic cones of a δ-CAT(0) space are CAT(0).*

Using the above observation, one can identify the finite dimensional Banach spaces which are δ-CAT(0). By Lemma 52, if V is a finite dimensional Banach space, then $Cone_\omega(V)$ is canonically isomorphic to V. Moreover the space $\mathbb{R}^n$ with the l_p metric is CAT(0) if and only if $p = 2$. (See proposition II.1.14 in [4]).

Remark 16. *The Banach space, $\mathbb{R}^n$, endowed with the l_p norm is δ-CAT(0) for some δ if and only if $p = 2$.*

It is tempting to ask if the converse to Observation 15 is true. More precisely, if all asymptotic cones of a geodesic space X are CAT(0), then is it true that X is δ-CAT(0) for some $\delta \geq 0$? The answer turns out to be negative and we present an example that demonstrates this in the following section. The lemma below proves to be very useful for the example and elsewhere.

A Preliminary Lemma

If $F : X \to Y$ is a map of metric spaces which distorts distances in a controlled fashion then there is an induced map at the level of asymptotic cones taking the class of (x_n) to the class of $(F(x_n))$. We make this precise in the lemma below and prove that under suitable hypotheses, the induced map is an isometry.

Lemma 17. *Let $c \geq 0$ and let $F : X \to Y$ be a map of metric spaces such that $d(y, F(X)) \leq c$, for all $y \in Y$. Suppose that there exists a monotonically non-decreasing function $f : \mathbb{R} \to \mathbb{R}$ satisfying $\lim_{r\to\infty} \frac{f(r)}{r} = 0$. Moreover, $|d(x, x') - d(F(x), F(x'))| \leq f(d(x, x'))$, for all x, $x' \in X$. Then $F_\omega : Cone_\omega(X, (a_n), (p_n)) \to Cone_\omega(Y, (a_n), (F(p_n)))$, given by $(x_n) \mapsto (F(x_n))$, is an isometry.*

Proof. Let $F : X \to Y$ be a map of metric spaces satisfying the hypotheses of the lemma. Let (x_n) be an element of X_ω. Then the sequence $(d(F(x_n), F(p_n))/a_n)$ is bounded because $d(F(x_n), F(p_n)) \leq d(x_n, p_n) + f(d(x_n, p_n))$ for all $n \in \mathbb{N}$ and $(d(x_n, p_n)/a_n)$ is a bounded sequence. This means that F_ω is well-defined.

By hypothesis, there exists a $c \geq 0$ such that $d(y, F(X)) \leq c$ for all $y \in Y$. Let (y_n) be an element of Y_ω. For each $n \in \mathbb{N}$, there exists an $x_n \in X$ such that $d(y_n, F(x_n)) \leq c$. But then the class of (y_n) is the same as the class of $(F(x_n))$ in Y_ω. This shows that the map F_ω is surjective.

Finally, let (x_n), $(x'_n) \in X_\omega$. Then, for each $n \in \mathbb{N}$, we have the inequality

$$\left|\frac{d(x_n, x'_n)}{a_n} - \frac{d(F(x_n), F(x'_n))}{a_n}\right| \leq \frac{f(d(x_n, x'_n))}{d(x_n, x'_n)} \frac{d(x_n, x'_n)}{a_n}$$

If $\lim_\omega d(x_n, x'_n)$ is finite then $\lim_\omega \frac{d(x_n,x'_n)}{a_n} = 0$. In this case, the points (x_n) and (x'_n) coincide and consequently $d_\omega((x_n),(x'_n)) = d_\omega((F(x_n)),(F(x'_n)))$.

On the other hand, if $\lim_\omega d(x_n, x'_n)$ is not finite, then $\lim_\omega \frac{f(d(x_n,x'_n))}{d(x_n,x'_n)} = 0$. It follows that the right hand side of the above inequality is equal to zero and so $d_\omega((x_n),(x'_n)) = d_\omega((F(x_n)),(F(x'_n)))$. This proves that F_ω is an isometry. □

2.8 The Plane with the Wrinkled Quadrant

Proposition 18 (The plane with the wrinkled quadrant)**.** *There is a metric space Y which is not δ-CAT(0), for any $\delta \geq 0$ but all its asymptotic cones are isometric to a CAT(0) space.*

Proof. For each integer $n \geq 2$, take S_n to be the trapezium in the first quadrant of the Euclidean plane, bounded by the x-axis, the y-axis, and the lines, $x+y = n(n+1)$ and $x + y = n(n-1)$. Now let P_n be a solid with five faces: the base of P_n is the trapezium S_n; two isosceles triangles, each of base length $2n$ and side length $\sqrt{n^2+1}$ form two of the faces. The remaining two faces are trapezia, one with sides $\sqrt{n^2+1}$, $\sqrt{2}n(n+1)$, $\sqrt{n^2+1}$ and $\sqrt{2}n^2$ and the other with sides, $\sqrt{n^2+1}$, $\sqrt{2}n(n-1)$, $\sqrt{n^2+1}$ and $\sqrt{2}n^2$. Note that each prism is of height 1. Attach P_n isometrically along its base to the trapezium S_n. Finally, remove the interior of each P_n, along with the interior of the base, S_n. Give the resulting space Y, the induced path metric; it can be loosely described as 'the plane with the wrinkled quadrant'.

We claim that the space Y is not δ-CAT(0) for any $\delta \geq 0$ but all of its asymptotic cones are CAT(0).

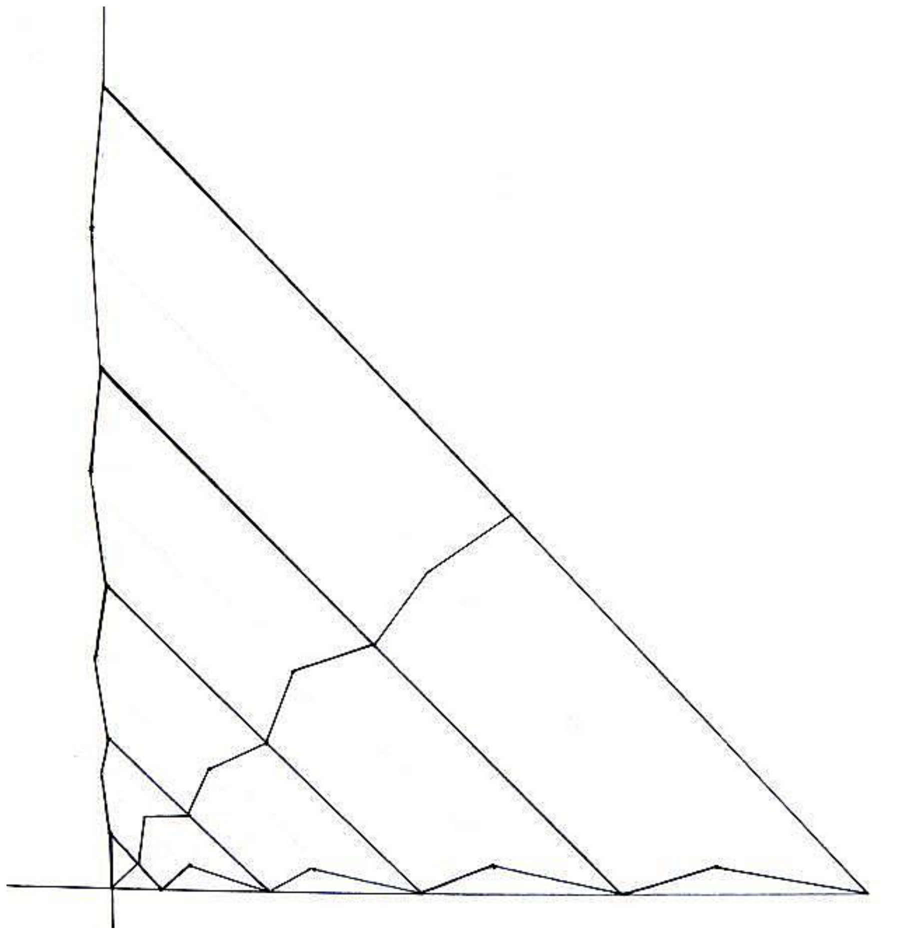

Figure 2.3: The Plane with The Wrinkled Quadrant, showing T_5 and γ_5.

The geodesic γ_n in Y joining the origin to the point $\left(\frac{n(n+1)}{2}, \frac{n(n+1)}{2}\right)$ is by direct computation, of length $d_n = \sqrt{2} + \sum_{k=2}^{n} 2\sqrt{1+\frac{k^2}{2}}$.

Now consider triangles T_n, one for each integer $n \geq 1$, whose vertices are at the origin, and the points $(0, n(n+1))$ and $(n(n+1), 0)$. Then, $(\frac{n(n+1)}{2}, \frac{n(n+1)}{2})$ is the mid-point m_n of the side $[(0, n(n+1)), (n(n+1), 0)]$ and the Euclidean distance $\bar{d}_n := d_{\mathbb{E}^2}((0,0), m_n)$ is exactly $\sum_{k=1}^{n} k\sqrt{2}$.

The triangle T_n, for each $n \geq 1$ therefore coincides along its boundary with its Euclidean comparison triangle. The difference between d_n and $\bar{d}_n$ is given by

$$\sum_{k=2}^{n} \left(2\sqrt{\frac{k^2}{2}+1} - \sqrt{2}k \right).$$

The summand is equal to $\frac{4}{\sqrt{2k^2+4}+k\sqrt{2}}$. As this is no smaller than $\frac{4}{k(\sqrt{2}+\sqrt{3})}$ and the harmonic series diverges, Y is not δ-CAT(0), for any $\delta \geq 0$.

We claim that every asymptotic cone of Y is isometric to the Euclidean plane. Imagine a juxtaposition of Y and $\mathbb{E}^2$ in which Y lies above $\mathbb{E}^2$ and the x and y axes in $\mathbb{E}^2$ coincide with the copy of the axes in Y. There is a projection π of Y onto $\mathbb{E}^2$ that maps every point in Y to the point in $\mathbb{E}^2$ directly below it.

We want to estimate the quantity $f(p,q) := |d(p,q) - d(\pi(p), \pi(q))|$. If a path joining two points p and q in Y crosses n wrinkles, then by the triangle inequality, the difference $f(p,q)$ is at most $2n$. On the other hand, any path crossing n wrinkles must travel a distance of at least $k + (k+1) \cdots + (n+k)$. Now, $k + (k+1) \cdots + (n+k)$ is equal to $(n+k)(n+k+1)/2 - k(k+1)/2$, which is no smaller than $n^2/2$. Therefore we see that the quantity $f(p,q)$ is bounded above by a linear function of $\sqrt{d(p,q)}$.

The function π satisfies the hypotheses of Lemma 17. Hence, every asymptotic cone of Y is isometric to the Euclidean plane. $\square$

Proposition 18 motivates an exploration of what we call asymptotically CAT(0) spaces.

Definition 5. *A metric space X is said to be asymptotically CAT(0) if all asymptotic cones of X are CAT(0).*

Unlike negatively curved or finite dimensional Banach spaces, not all metric spaces have unique asymptotic cones. The isometry type of an asymptotic cone depends on the choice of the ultrafilter and the base point. However, if the metric space supports a cocompact group action, then one can remove this dependence on the base point.

The dependence on the choice of the ultrafilter is a far more delicate matter. In [27], Simon Thomas and Boban Velickov present an example of a finitely generated group whose Cayley Graph gives non-isometric asymptotic cones for different choices of non-principal ultrafilters.

A weaker notion to 'asymptotically CAT(0)' is that of a metric space being lacunary CAT(0), i.e. it has at least one CAT(0) asymptotic cone. In [18], the authors provide an example of a finitely generated group G such that at least one asymptotic cone of G is the infinitely branching homogeneous $\mathbb{R}$-tree while some other asymptotic cones of G are not even simply connected. This shows that there exist metric spaces which are lacunary CAT(0) but not asymptotically CAT(0). We study asymptotically CAT(0) spaces in detail in the following chapter.

CHAPTER 3

ASYMPTOTICALLY CAT(0) GROUPS

Recall that a metric space X is said to be asymptotically CAT(0) if all its asymptotic cones are CAT(0).

Definition 6. *A group G is asymptotically CAT(0) if it acts geometrically on an asymptotically CAT(0) geodesic space.*

Convention As before we will refer to a proper and co-compact action of a group by isometries as a 'geometric' action.

3.1 Asymptotically CAT(0) metric spaces

The purpose of this section is to obtain a characterisation of asymptotically CAT(0) spaces in terms of their metric properties.

Theorem 19. *A geodesic metric space is asymptotically CAT(0) if and only if there exists a function $f : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\lim_{r\to\infty} \frac{f(r)}{r} = 0$ and every ball of radius r in X is $f(r)$-CAT(0).*

Caveat In the statement above we do not assume that the balls in X are convex. We simply mean that any geodesic triangle in X with vertices in a ball of radius r satisfies the $f(r)$-CAT(0) inequality.

Proof of Theorem 19. We first consider the sufficiency statement. Take a 4-tuple of points $(x_1, x_2, x_3, x_4) \in Cone_\omega(X)$, where $x_i = (x_{i,n})$ for $i = 1, 2, 3, 4$. Note that for each n, $(x_{1,n}, x_{2,n}, x_{3,n}, x_{4,n})$ is a 4-tuple of points in (X, d_n) of some diameter r_n. Since any ball of radius r_n satisfies a $f(r_n)$-CAT(0) inequality for triangles, we know that it also satisfies a $f(r_n)$-CAT(0) inequality on quadrilaterals. This implies that there is a 4-tuple of points $(y_{1,n}, y_{2,n}, y_{3,n}, y_{4,n})$ in the Euclidean plane $\mathbb{E}^2$ such that $d(x_{i,n}, x_{i+1,n}) = d(y_{i,n}, y_{i+1,n})$ for $i = 1, 2, 3, 4$, modulo 4 and $d(x_{i,n}, x_{j,n}) \leq d(y_{i,n}, y_{j,n}) + 2f(r_n)$ for $1 \leq i < j \leq 4$. For each n, we may choose $y_{1,n}$ to be the origin.

Then $y_i = (y_{i,n})$, $i = 1, 2, 3, 4$ is a 4-tuple of points in $Cone_\omega(\mathbb{E}^2)$. By Lemma 52 in Appendix A, the Euclidean plane is isometric to any of its asymptotic cones. The above construction therefore provides us with a 4-tuple of points in $\mathbb{E}^2$ which satisfies $d(x_i, x_{i+1}) = d(y_i, y_{i+1})$, for $i = 1, 2, 3, 4$, modulo 4, and for $1 \leq i < j \leq 4$, $d(x_i, x_j) = \lim_\omega d_n(x_{i,n}, x_{j,n}) \leq \lim_\omega (d_n(y_{i,n}, y_{j,n}) + 2\frac{f(r_n)}{a_n})$. But, $\lim_\omega \frac{f(r_n)}{a_n} = \lim_\omega \frac{f(r_n)}{r_n} \frac{r_n}{a_n}$. By hypothesis, $\lim_{r \to \infty} \frac{f(r)}{r} = 0$ and further, $\lim_\omega \frac{r_n}{a_n}$ is the diameter of the four tuple of points in the asymptotic cone. We therefore conclude that $d(x_i, x_j) \leq d(y_i, y_j)$ for $1 \leq i < j \leq 4$.

Conversely, suppose that all asymptotic cones of a geodesic space X are CAT(0). Define $f(r)$ to be the supremum of the difference between $d(p, q)$ and $d(\bar{p}, \bar{q})$, where p and q are points on a geodesic triangle in X, whose vertices lie in a ball of radius r. We claim that $\lim_{r \to \infty} \frac{f(r)}{r} = 0$.

Suppose not. Then there exists a non-principal ultrafilter ω and a sequence (a_n) of positive real numbers such that $\lim_{n\to\infty} a_n = \infty$ and

$$\text{for some } \epsilon > 0,\ \omega(\{n : \frac{f(a_n)}{a_n} > 2\epsilon\}) = 1.$$

For each $n \in \mathbb{N}$, there exists a geodesic triangle Δ_n, with vertices in a ball of radius a_n in X and a comparison triangle $\bar{\Delta}_n$ for Δ_n in the Euclidean plane such that a pair p_n, q_n of points in Δ_n satisfies the condition

$$f(a_n) \geq d(p_n, q_n) - d(\bar{p}_n, \bar{q}_n) \geq f(a_n) - 1.$$

Here, $\bar{p}_n$ and $\bar{q}_n$ are as usual the comparison points for p_n and q_n in $\bar{\Delta}_n$.

Now consider the asymptotic cone X_ω of X with respect to the scaling sequence (a_n), ultrafilter, ω and sequence of base points (p_n). The ω-limit of the triangles Δ_n is a geodesic triangle Δ in a ball of radius 1 in X_ω. Observe that if $\bar{\Delta}$ denotes the ω-limit of the triangles $\bar{\Delta}_n$, then $\bar{\Delta}$ is a comparison triangle for Δ in the Euclidean plane. The comparison points for $\lim_\omega p_n$ and $\lim_\omega q_n$ in $\bar{\Delta}$ are precisely the ω-limits of the sequences $(\bar{p}_n)$ and $(\bar{q}_n)$, respectively.

As $\omega\{n \in \mathbb{N} \mid \frac{1}{a_n} < \epsilon\} = 1$, we deduce that

$$\omega\left\{n \in \mathbb{N} \mid \frac{d(p_n, q_n)}{a_n} - \frac{d(\bar{p}_n, \bar{q}_n)}{a_n} \geq \epsilon\right\} = 1.$$

Therefore, $d_\omega(\lim_\omega p_n, \lim_\omega q_n) \geq d_\omega(\lim_\omega \bar{p}_n, \lim_\omega \bar{q}_n)) + \epsilon$. This contradicts the assumption that X_ω is CAT(0). We conclude that $\lim_{r\to\infty} \frac{f(r)}{r} = 0$. □

3.2 Finite Subgroups

Theorem 20. *An asymptotically CAT(0) group G has finitely many conjugacy classes of finite subgroups.*

Proof. Let Y be a non-empty bounded subset of a proper asymptotically CAT(0) metric space. Define

$r_Y = \inf\{r > 0 | Y \subset B(x, r) \text{ for some } x \in X\}$, and

$C(Y) = \{x \in X | Y \subset B(x, r_Y)\}$.

With this notation, r_Y is the (circum)radius of Y and $C(Y)$ is the set of barycentres of Y. By a standard argument, $C(Y)$ is not empty. We wish to estimate the diameter of $C(Y)$. Now, by Theorem 19, we know there exists a function f and $a > 0$ such that if $r > a$, then $f(r) < \frac{r}{32}$ and every ball of radius r is $f(r)$-CAT(0).

Choose x_1, $x_2 \in C(Y)$ and let $\epsilon > 0$ be given. For each $y \in Y$, consider a geodesic triangle $[y, x_1, x_2]$ along with $[O_y, \bar{x}_1, \bar{x}_2]$, its comparison triangle in the Euclidean plane. Suppose m is the midpoint of the geodesic joining x_1 and x_2. Denote its comparison point in $[O_y, \bar{x}_1, \bar{x}_2]$ by m_y. Now, if $d(m_y, O_y) \leq r_Y - f(r_Y) - \epsilon$ for all $y \in Y$, then $d(m, y) \leq r_Y - \epsilon$ for all $y \in Y$ and this violates the definition of r_Y. Therefore there must exist some $z \in Y$ for which the distance between the points O_z and m_z exceeds $r_Y - (f(r_Y) + \epsilon)$. Thus,

$d(\bar{x}_1, \bar{x}_2) \leq 2\sqrt{d(\bar{x}_1, O_z)^2 - d(O_z, m_z)^2}$

$\leq 2\sqrt{r_Y^2 - (r_Y - (f(r_Y) + \epsilon))^2}$.

$= 2\sqrt{2r_Y(f(r_Y) + \epsilon) - (f(r_Y) + \epsilon)^2}$.

$\Rightarrow 2r_{C(Y)} \leq 2\sqrt{2r_Y f(r_Y) - f(r_Y)^2}$.

$\Rightarrow$ If $r_Y > a$, then $r_{C(Y)} \leq \frac{r_Y}{4}$.

Let H be a finite subgroup of G. Fix $x \in X$. Set $Y = Hx$, the orbit of x under the action of H. Then Y is a bounded subset of X. If $r_Y > a$, then we inductively define a sequence (Y_n) of subsets of X, by setting Y_0 to be Y and Y_n, to be $C(Y_{n-1})$. We deduce from the previous paragraphs that

$$\text{for } n > \log_4 \frac{r_Y}{a}\ , \text{ we have } r_{Y_n} < a.$$

Choose m to be the least such n. Note that, by construction, the sets Y_n are invariant under the action of H. Let $\bar{x}$ be an element of Y_m and $D > 0$ be such that $G.B(x, D) = X$. There exists some $g \in G$ with $d(gx, \bar{x}) \leq D$. Hence, for any $z \in Y_m$, we have $d(g^{-1}z, x) \leq d(g^{-1}z, g^{-1}\bar{x}) + d(g^{-1}\bar{x}, x) \leq 2a + D$.

Let $h \in H$. Then,

$d(g^{-1}hgx, x) \leq d(g^{-1}hgx, g^{-1}hg.g^{-1}z) + d(g^{-1}hg.g^{-1}z, x)$

$= d(x, g^{-1}z) + d(g^{-1}(hz), x)$.

The set $g^{-1}Y_m$ is invariant under the action of $g^{-1}Hg$ and $g^{-1}(hz) \in g^{-1}Y_m$. Hence, $d(g^{-1}hgx, x) \leq 2(2a + D)$ and $g^{-1}Hg.x \subset B(x, 2(2a + D))$.

The properness of the action of G ensures that there are only finitely many subgroups with the property that the orbit of a point x lies in the $2(2a+D)$-ball around x. This proves the theorem. $\square$

3.3 Examples

In this section, we concentrate on providing examples of asymptotically CAT(0) metric spaces and groups. The following are known classes of examples:

1. **Hyperbolic Groups**: Indeed every asymptotic cone of a hyperbolic group is isometric to an $\mathbb{R}$-tree. See [6].

2. **CAT(0) metric spaces**: Every asymptotic cone of a CAT(0) metric space is also CAT(0). A proof of this may be found in [4]. Alternatively, see Lemma 21 below.

Bolic Spaces

We have already encountered the concept of δ-bolicity in Section 2.6. Recall that a metric space (X, d) is said to be δ-bolic if there exists a map $m : X \times X \to X$ such that for all x,y, and $z \in X$,

$$2d(m(x,y),z) \leq \sqrt{2d(x,z)^2 + 2d(y,z)^2 - d(x,y)^2} + \delta.$$

This class not only includes both hyperbolic and CAT(0) groups, but also groups acting geometrically on a δ-CAT(0) space. In the following paragraph we present a general proof of the fact that any asymptotic cone of a δ-bolic metric space is CAT(0).

Lemma 21. *Let (X, d) be a δ-bolic metric space. Then for any non-principal ultra filter ω, scaling sequence (a_n) and base point (x_n), the asymptotic cone $X_\omega := Cone_\omega(X, (x_n), (a_n))$ is CAT(0).*

Proof. Recall that a geodesic metric space Y is CAT(0) if for all p,q, $r \in Y$, and all $m \in Y$ with $d(q, m) = d(r, m) = d(q, r)/2$, one has

$$d(p,q)^2 + d(p,r)^2 \geq 2d(m,p)^2 + \frac{1}{2}d(q,r)^2.$$

(See [4], for this characterisation of the CAT(0) property).

We first argue that X_ω is a geodesic space. The asymptotic cone X_ω is a complete metric space since the ultralimits of all sequences of metric spaces are complete ([4], Lemma I.5.53). By definition, between any two points in X, is an 'approximate' midpoint and so mid-points exist in the asymptotic cone. But, a complete metric space in which midpoints exist is a geodesic space.

Now take any choice of points (p_n), (q_n) and (r_n) in X_ω and let M be the midpoint of the geodesic joining (q_n) and (r_n). Then for each n, $m(q_n, r_n)$ satisfies the inequality

$$2d(m(q_n, r_n), p_n) \leq \sqrt{2d(q_n, p_n)^2 + 2d(r_n, p_n)^2 - d(q_n, r_n)^2} + \delta.$$

Note that the equivalence class of M is the same as that of $(m(q_n, r_n))$. Taking the ω-limit of the last inequality, we have that

$2 \lim_\omega d(m(q_n, r_n), p_n)$

$\leq \sqrt{2 \lim_\omega d(q_n, p_n)^2 + 2 \lim_\omega d(r_n, p_n)^2 - \lim_\omega d(q_n, r_n)^2}$

$\Rightarrow d((p_n), (q_n))^2 + d((p_n), (r_n))^2 \geq 2d(M, (p_n))^2 + \frac{1}{2} d((q_n), (r_n))^2.$

This proves that X_ω is CAT(0). $\square$

Remark 22. *It would be interesting to define an asymptotic notion of bolicity and extend existing techniques to prove Novikov's Conjecture for asymptotically CAT(0) groups.*

Co-compact Lattices of $\widetilde{PSL(2, \mathbb{R})}$

In Chapter 5, we will show that every asymptotic cone of $\widetilde{PSL(2, \mathbb{R})}$ is isometric to a direct product of the real line with the infinitely branching homogeneous $\mathbb{R}$-tree. This will establish that co-compact lattices in $\widetilde{PSL(2, \mathbb{R})}$ are examples of asymptotically CAT(0) groups.

Co-compact lattices in $\widetilde{PSL(2,\mathbb{R})}$ are central extensions of cocompact lattices in $PSL(2,\mathbb{R})$ by $\mathbb{Z}$. Typical examples of these are the fundamental groups of $T^1(S)$, where $T^1(S)$ denotes the unit tangent bundle of a closed surface S of genus at least 2. These groups are neither hyperbolic nor can they act properly by semisimple isometries on any CAT(0) space. This is a consequence of the geometry of $\widetilde{PSL(2,\mathbb{R})}$ and the following theorem about CAT(0) groups.

Theorem 23. *(Theorem II.6.12 in [4]) Let X be a CAT(0) metric space and let Γ be a finitely generated group acting by isometries on X. If Γ contains a central subgroup $A \cong \mathbb{Z}^n$ that acts faithfully by hyperbolic isometries, then there exists a subgroup of finite index $H \subset \Gamma$ which contains A as a direct factor.*

The fundamental groups of S and of $T^1(S)$ are linked by the short exact sequence given below.

$$1 \to \mathbb{Z} \to \pi_1(T^1(S)) \to \pi_1(S) \to 1$$

Observe first $\pi_1(T^1(S))$ contains free abelian subgroups of rank 2 and therefore it is not hyperbolic.

As described in Theorem 4.15 of [20], $\widetilde{PSL(2,\mathbb{R})}$ does not contain the fundamental group of any closed surface of genus ≥ 2. Every finite index subgroup of $\pi_1(S)$ is the fundamental group of such a surface and so, the short exact sequence above cannot split, even after passing to a subgroup of finite index in $\pi_1(S)$. Now, we appeal to Theorem 23 and conclude that co-compact lattices in $\widetilde{PSL(2,\mathbb{R})}$ cannot act geometrically on a CAT(0) space.

Hence, the fundamental groups of $T^1(S)$ are examples of asymptotically CAT(0) groups which are neither CAT(0) nor hyperbolic.

Relatively Hyperbolic Groups

Relatively hyperbolic groups provide further examples of asymptotically CAT(0) groups. We show in Chapter 4 that if a group G is hyperbolic relative to an asymptotically CAT(0) subgroup H, then G is also asymptotically CAT(0).

3.4 Direct Products, Amalgams and HNN Extensions

In this section, we provide methods for combining asymptotically CAT(0) groups using direct products, amalgams and HNN extensions.

Direct Products

Proposition 24. *The category of asymptotically CAT(0) groups is closed under finite direct products.*

Proof. This follows from a simple observation: if two groups G and H act geometrically on (X, d_X) and (Y, d_Y) respectively, then the direct product $G \times H$ acts geometrically on $(X \times Y, \sqrt{d_X^2 + d_Y^2})$, the action being defined component-wise. Moreover, $Cone_\omega(X \times Y, (x_n, y_n))$ is isometric to $Cone_\omega(X, (x_n)) \times Cone_\omega(Y, (y_n))$, where $x_n \in X$, $y_n \in Y$ and ω is a non-principal ultrafilter. This implies that if all asymptotic cones of X and Y are CAT(0), then so are all asymptotic cones of $X \times Y$. Hence, if G and

H are asymptotically CAT(0) then the direct product $G \times H$ is also asymptotically CAT(0). $\square$

Amalgams and HNN Extensions with Isometric Gluing

We now describe techniques to form amalgams and HNN extensions from asymptotically CAT(0) groups. The hypotheses on the amalgamated subgroup in theorem 25 and 27 appear restrictive at first. However the conditions are met among others, by finite subgroups with fixed points, virtually cyclic subgroups in CAT(0) groups or by the central infinite cyclic subgroups of lattices in $\widetilde{PSL(2,\mathbb{R})}$.

Theorem 25. *Let G_1, G_2 and H be groups acting geometrically on asymptotically CAT(0) geodesic spaces X_1, X_2 and A respectively. Suppose that for $i = 1, 2$, there exist monomorphisms, $\phi_i : H \to G_i$ and a ϕ_i-equivariant isometric embedding $f_i : A \to X_i$. Then, the amalgam $G = G_1 *_H G_2$ associated to the maps ϕ_i acts geometrically on an asymptotically CAT(0) geodesic space.*

Proof. The amalgam $G = G_1 *_H G_2$ acts simplicially on a tree T which is unique up to graph isomorphism. The vertices of T are in bijection with the cosets of G_1 and G_2 in G, while the unoriented edges of T may be identified with the cosets of H in G. Given spaces on which the groups G_1, G_2 and H act geometrically, one asks if there exists a space Z, which supports a geometric action of G by isometries. Indeed, there is a well-known construction which serves this purpose, provided the maps f_i and ϕ_i in the statement of the theorem exist.

We present the construction in some detail here, following the treatment in Theorem II.11.18 of [4]. Start with an equivalence relation $\approx$ on the disjoint union of $G \times X_1$, $G \times [0,1] \times A$ and $G \times X_2$. The equivalence relation $\approx$ is generated by: $(gg_1, x_1) \approx (g, g_1x_1)$, $(gg_2, x_2) \approx (g, g_2x_2)$, $(gh, t, a) \approx (g, t, ha)$, $(g, f_1(a)) \approx (g, 0, a)$ and $(g, f_2(a)) \approx (g, 1, a)$ for all $g \in G$, $g_1 \in G_1$, $g_2 \in G_2$, $h \in H$, $x_1 \in X_1$, $x_2 \in X_2$, $a \in A$ and $t \in [0,1]$.

For $i = 1, 2$, let $\bar{X}_i$ be the quotient of $G \times X_i$ by the above relation and similarly, let $\bar{A}$ be the quotient of $G \times [0,1] \times A$ by the above relation. Then $\bar{X}_i$ is isometric to $G/G_i \times X_i$; this is because each $g \times X_i$ contains exactly one element from each equivalence class of $G \times X_i$. Similarly, $\bar{A}$ is isometric to $G/H \times [0,1] \times A$. We are now in a position to describe Z. Recall that T denotes the Bass Serre tree of G.

The space Z is a tree of spaces with underlying tree T such that the vertex spaces are isomorphic to the X_i and the edge spaces are isomorphic to A. More precisely,

$$Z := \frac{(G/G_1 \times X_1) \coprod (G/H \times [0,1] \times A) \coprod (G/G_2 \times X_2)}{\sim}$$

The relation $\sim$ is given via the canonical surjections $G/H \to G/G_1$ and $G/H \to G/G_2$: $(gH, 0, a) \sim (gG_1, f_1(a))$ and $(gH, 1, a) \sim (gG_2, f_2(a))$, for all $gH \in G/H$ and $a \in A$.

The group G acts by left multiplication on the first component of each of $G \times X_1$, $G \times [0,1] \times A$ and $G \times X_2$. This action is compatible with the gluing and so there is an induced action of G on Z. The quotient of Z via this action of G is a compact space obtained via a gluing of X_1/G_1, X_2/G_2 along $A/H \times [0,1]$. Therefore the action is cocompact.

If G_1, G_2 and H act properly on X_1, X_2 and A, respectively, then G acts properly on Z. The subgroup of G leaving a copy of M, for $M \in \{X_1, X_2, A\}$, fixed is a conjugate of G_i or of H. On the other hand, if an element g of G does not leave a copy of M invariant then g maps this copy of M to a different one. Consequently, every point is moved by a distance of at least 2 and hence, the action of G on Z is proper.

Endowed with the quotient metric, Z is a geodesic space. There is a natural projection π from Z to the Bass Serre tree T of G, which takes the equivalence classes of (g, x_1), (g, x_2) and (g, t, a) to $(gG_1, 0)$, $(gG_2, 1)$ and (gH, t), respectively. Moreover π is G-equivariant. We describe a geodesic γ joining the equivalence class (gG_1, x_1) to the equivalence class of $(g'G_2, x_2)$. Recall that A is a proper metric space and the maps f_i are isometries. Therefore there is a point $\bar{x}_1$ in $(gG_1, f_1(A))$ which is closest to the point (gG_1, x_1) in Z. Similarly there is a point $\bar{x}_2$ in $(g'G_2, f_2(A))$ which is closest to the point $(g'G_2, x_2)$ in Z. The geodesic γ then is a concatenation of three geodesics, the first joining (gG_1, x_1) to $\bar{x}_1$ in (gG_1, X_1), the second joining $\bar{x}_1$ to $\bar{x}_2$ in $A \times T$ and the third, joining $\bar{x}_2$ to $(g'G_2, x_2)$ in $(g'G_2, X_2)$. We deduce from this and Bass Serre theory that the action of G on Z is by isometries.

Our main task now is to show that Z is asymptotically CAT(0). The space Z supports a proper cocompact G-action and so, the choice of base point is not crucial. Let ω be a non-principal ultrafilter and choose a sequence (a_n) of positive real numbers such that $\lim_{n\to\infty} a_n = \infty$. We will show that the canonical asymptotic cone (see Appendix A) $Z_\omega := Cone_\omega(Z, (a_n))$ is a CAT(0) space.

Denote the canonical asymptotic cones of X_1, X_2, T and A with respect to the non-principal ultrafilter ω and scaling sequence (a_n) by $(X_1)_\omega$, $(X_2)_\omega$, T_ω and A_ω, respectively. Observe that the isometric embeddings f_i induce isometries F_i at the level of asymptotic cones. We construct a new space $\mathcal{Z}$ as a tree of spaces with underlying tree, T_ω.

$$\mathcal{Z} := \frac{(T_\omega \times (X_1)_\omega) \coprod (T_\omega \times A_\omega) \coprod (T_\omega \times (X_2)_\omega)}{\sim},$$

where, $(\underline{t}, F_1(\underline{a}) \sim (\underline{t}, \underline{a}) \sim (\underline{t}, F_2(\underline{a}))$, for all $\underline{a} \in A_\omega$ and $\underline{t} \in T_\omega$.

The proof of the Theorem will therefore follow from the next proposition. $\square$

Proposition 26. *The spaces Z_ω and $\mathcal{Z}$ are isometric. Moreover, $\mathcal{Z}$ is CAT(0).*

Proof. We first show that $\mathcal{Z}$ is CAT(0). We have assumed that the spaces X_i and A are asymptotically CAT(0). It follows that $T_\omega \times A_\omega$, is CAT(0). By Theorem II.11.3 of [4], a tree of spaces in which every vertex and edge space is CAT(0) is also CAT(0). Hence, $\mathcal{Z}$ is CAT(0).

We now define a map η from Z_ω to $\mathcal{Z}$, which furnishes us with the required isometry. Let $(z_n) \in Z_\omega$. Define $\mathcal{X}_1 = \{n \in \mathbb{N} \mid z_n \in G/G_1 \times (X_1 - f_1(A))\}$, $\mathcal{X}_2 = \{n \in \mathbb{N} \mid z_n \in G/G_2 \times (X_2 - f_2(A))\}$ and $\mathcal{A} = \{n \in \mathbb{N} \mid z_n \in G/H \times [0,1] \times A\}$. Then, $\mathcal{X}_1 \coprod \mathcal{A} \coprod \mathcal{X}_2 = \mathbb{N}$. This implies that exactly one of these three sets has ω-measure 1, and so z_n belongs to exactly one of $T \times A$, $G/G_1 \times X_1$ and $G/G_2 \times X_2$ with ω-measure 1.

Observe that the copy of $(T \times A)_\omega \cong T_\omega \times A_\omega$ in Z_ω is isometric to the copy of $T_\omega \times A_\omega$ in $\mathcal{Z}$. Therefore, the restriction of η to $T_\omega \times A_\omega \subset Z_\omega$ can be taken to be the identity map.

Now suppose that $\omega(\mathcal{X}_1) = 1$. For each $n \in \mathbb{N}$, let t_n be the projection of z_n on to the copy of T in Z. Since the projection map decreases distances, (t_n) defines a point in the tree T_ω in Z_ω. For $n \in \mathcal{X}_1$, define w_n to be the projection of z_n onto X_1, otherwise take w_n to be any point in X_1. Let $\eta((z_n)) = ((t_n), (w_n))$. Similarly, define η for the case when $\omega(\mathcal{X}_2) = 1$.

Observe that the copies of $(T \times A)_\omega$, $(X_1)_\omega$ and $(X_2)_\omega$ in Z_ω and $\mathcal{Z}$ are isometric and moreover η is a bijection. It follows that η defines an isometry from Z_ω on to $\mathcal{Z}$. This proves that Z_ω is CAT(0). $\square$

One can construct HNN extensions of asymptotically CAT(0) groups in the same fashion.

Theorem 27. *Let G and H be groups acting properly by isometries on asymptotically CAT(0) spaces X and A. Suppose that for $i = 1, 2$, there exist monomorphisms $\phi_i : H \to G$ and ϕ_i-equivariant embedding $f_i : Y \to X$. Then the HNN extension $G*_H$ acts properly by isometries on an asymptotically CAT(0) space.*

Proof. The proof is similar to that of the previous theorem. The only difference lies in the definition of the space Z. For an HNN extension Γ of G over the subgroup H, define Z to be as follows.

$$Z := \frac{\left(\Gamma/G_1 \times \tilde{X}\right) \coprod (\Gamma/H \times [0,1] \times A)}{\sim}$$

where $(\gamma H, 0, a) \sim (\gamma G, f_1(a))$ and $(\gamma H, 1, a) \sim (\gamma G, f_2(a))$, for all $a \in A$ and for all $\gamma \in \Gamma$. Conclude as before that Z and hence Γ is asymptotically CAT(0). $\square$

Amalgams and HNN Extensions along Finite Subgroups

Theorem 28. *Let G_1 and G_2 be asymptotically CAT(0) groups and let C be a finite group, endowed with monomorphisms $\phi : C \to G_1$ and $\psi : C \to G_2$. Then the amalgam $G := G_1 *_C G_2$ associated to ϕ and ψ is also asymptotically CAT(0).*

Proof. Let G_1, G_2 and C be as above; let X_1 and X_2 be the asymptotically CAT(0) spaces associated to G_1 and G_2 respectively. To prove the theorem, we need to construct an asymptotically CAT(0) space which supports a geometric G-action.

Fix $x_1 \in X_1$ and $x_2 \in X_2$. Let H_1 be the stabilizer of x_1 in G_1. Likewise, denote the stabilizer of x_2 in G_2 by H_2. For fixed $x_1 \in X_1$, the set map $G_1 \to G_1/H_1$ gives a canonical map $\pi_1 : G_1 \to G_1.x_1$ from the group G_1 to the orbit $G_1.x_1$. Similarly, there exists a natural map $\pi_2 : G_2 \to G_2.x_2$. Define

$$Z := \frac{(G/G_1 \times X_1) \coprod (G/C \times [0,1] \times C) \coprod (G/G_2 \times X_2)}{\sim}$$

where $(\gamma C, 0, c) \sim (\gamma G_1, \pi_1 \circ \phi(c))$ and $(\gamma C, 1, c) \sim (\gamma G_2, \pi_2 \circ \psi(c))$, for all $c \in C$ and for all $\gamma \in G$.

By a similar argument as before, the amalgam G acts properly and co-compactly on Z by isometries. We claim that Z is asymptotically CAT(0). There is a natural projection of Z to the Bass Serre tree T of G. Consider the tree of spaces $\tilde{Z}$ with underlying tree T and vertex spaces X_1 and X_2; that is, a vertex of the form gG_1 of T corresponds to a copy of X_1 and a vertex of the form gG_2 of T corresponds to a copy of X_2. Observe that Z is $(1, \epsilon)$-quasi-isometric to $\tilde{Z}$, where ϵ depends solely on the diameter of $C.x_1$ in X_1 and the diameter of $C.x_2$ in X_2.

It follows from the argument in Theorem 25 that $\tilde{Z}$ is asymptotically CAT(0). Moreover, by Lemma 17, a $(1, \epsilon)$ quasi-isometry induces an isometry at the level of asymptotic cones. Hence, the space Z is asymptotically CAT(0). $\square$

Theorem 29. *The class of asymptotically CAT(0) groups is closed under HNN extensions along finite subgroups.*

Proof. Here again, it suffices to describe the space Z. Let G_1 be an asymptotically CAT(0) group with associated space X_1; let C be a finite subgroup of G_1 along with monomorphisms ϕ, $\psi : C \to G_1$. Let G be the HNN extension of G_1 over C, corresponding to ϕ and ψ.

Fix $x \in X_1$. Let H be the stabilizer of x in G_1 and let $\pi : G_1 \to G_1/H \cong G_1.x$ be the canonical map from G_1 onto the orbit of x. Define the space Z to be as follows.

$$Z := \frac{(G/G_1 \times X_1) \coprod (G/C \times [0,1] \times C)}{\sim}$$

where $(\gamma C, 0, c) \sim (\gamma G_1, \pi \circ \phi(c))$ and $(\gamma C, 1, c) \sim (\gamma G_2, \pi \circ \psi(c))$, for all $c \in C$ and for all $\gamma \in G$. Conclude as before that Z and hence G is asymptotically CAT(0). $\square$

3.5 Finiteness Properties

Let G be an asymptotically CAT(0) group. There exists a space X on which G acts geometrically and such that all asymptotic cones of X are CAT(0). Any CAT(0) space is contractible. The fact that all asymptotic cones of X are contractible has implications for the finiteness properties of G. In fact, it implies that G is of type F_∞ (and hence FP_∞), as we shall now explain.

Given a group G, a $K(G,1)$ is a path-connected space whose fundamental group is isomorphic to G and which has a contractible universal covering space. The existence of a 'nice' $K(G,1)$ has a central place in algebraic topology.

Definition 7. *A group G is said to be of type F_n if there exists a CW-complex $K(G,1)$, whose n-skeleton is finite.*

Definition 8. *A group G is said to be of type FP_n if there exists a resolution*

$$P_n \to P_{n-1} \to \cdots \to P_0 \to \mathbb{Z} \to 0$$

of the trivial G-module $\mathbb{Z}$ by finitely generated projective G-modules.

Remarks If G is of type F_n, then G is of type FP_n. We say that G is of type F_∞ if there is a CW-complex $K(G,1)$ with finitely many cells in each dimension; G is of type FP_∞ if there is projective resolution of $\mathbb{Z}$ by finitely generated G-modules.

Theorem 30. *(Theorem 2.6.D of [26]) If G is a finitely generated group with a word metric such that all asymptotic cones of G are n-connected, then G is of type F_{n+1}.*

Therefore, if all asymptotic cones of G are contractible, then there exists a CW-complex with finitely many cells in each dimension and whose fundamental group is isomorphic to G.

Now, suppose that G is asymptotically CAT(0) and X is as above. The group G is finitely generated and by the well-known *Svărc-Milnor Lemma* (Proposition I.8.19 in [4]), G with any word metric, is quasi-isometric to the space X. But, by Observation 53, a quasi-isometry induces a bi-Lipschitz homeomorphism at the level of asymptotic cones. Since X is asymptotically CAT(0), all asymptotic cones of X are contractible. This implies that all asymptotic cones of G are also contractible and hence, G is of type F_∞. We have therefore the following proposition.

Proposition 31. *An asymptotically CAT(0) group is of type F_∞.*

3.6 The Word problem

Theorem 32. *(Theorem 4.6 in [6]) Let X be a geodesic space. If the isoperimetric function for every asymptotic cone of X is quadratic then the following is true: for every $\epsilon > 0$, there exists l_ϵ such that the 'area' of a minimal diagram of boundary length l is at most $l^{2+\epsilon}$, for all $l \geq l_\epsilon$.*

As the isoperimetric function for any CAT(0) space is quadratic, this theorem applies to asymptotically CAT(0) spaces. One wonders if the above estimate can be improved to a quadratic bound. Nevertheless the isoperimetric function for any asymptotically CAT(0) group is sub-cubic.

Proposition 33 (See [23])**.** *If G is a finitely presented group, then the word problem for G is solvable if and only if the isoperimetric function for G is recursive.*

We conclude from the above discussion that

Observation 34. *The word problem for asymptotically CAT(0) groups is solvable.*

CHAPTER 4

RELATIVE HYPERBOLICITY

The aim of this chapter is to prove the following theorem:

Theorem 35. *If a group G is relatively hyperbolic with respect to a subgroup H and H is asymptotically CAT(0), then so is G.*

Relatively hyperbolic groups were first introduced by Gromov in [12] and later studied by Farb ([9]), and Bowditch ([3]), among others. The motivating examples were the fundamental groups of complex hyperbolic manifolds with cusps. The presence of the cusp subgroups ensure that these groups are not negatively curved. The class of relatively hyperbolic groups also includes (1) groups acting geometrically by isometries on 'CAT(0) spaces with isolated flats'; these include limit groups; (2) fundamental groups of hyperbolic manifolds of finite volume (that is, non-uniform lattices in rank one semisimple groups with trivial center); these are hyperbolic relative to their cusp subgroups; (3) hyperbolic groups; these are hyperbolic relative to the trivial subgroup (4) free products of groups; these are hyperbolic relative to their factor subgroups; and (5) fundamental groups of non-geometric Haken manifolds with at least one hyperbolic component; these are hyperbolic relative to the fundamental groups of the maximal graph-manifold components and to the fundamental groups of the tori and Klein bottles not contained in graph-manifold components.

We will define relatively hyperbolic groups in terms of the 'coned-off Cayley graph' and bounded coset penetration.

4.1 Definitions

The coned-off Cayley Graph

Let G be a finitely generated group with generating set S and let H be a subgroup of G. We may assume for simplicity that S contains a generating set for H. Consider the Cayley graph C of G with respect to the given generating set. Let $\mathcal{X}$ be an enumeration of the cosets of H in G. Build a new graph $\mathcal{C}$, whose vertex set contains all the vertices of C, along with new vertices, v_X, one for each $X \in \mathcal{X}$. The edge set of $\mathcal{C}$ contains all the edges of C, along with edges $e(X, g)$, where $X \in \mathcal{X}$ and g is an element of the coset X. Assign length one to each edge. The resulting graph $\mathcal{C}$ is called the 'coned-off' Cayley graph of G with respect to H.

Bounded Coset Penetration Property

Given a path $w \in C$, locate all maximal subwords in w formed by the generators of H, reading w from left to right. Suppose that such a subword z goes from g to $g.\bar{z}$ in C. Replace the path labelled z by a concatenation of two edges, the first joining g to the cone point v_{gH} and the second running from v_{gH} to the vertex gz in C. Repeat for all maximal subwords. This procedure produces from w, a path $\tilde{w}$ in $\mathcal{C}$.

Terminology

If w is a geodesic (or λ-quasigeodesic), then $\tilde{w}$ is called a *relative geodesic* (or λ-quasigeodesic, respectively).

A path $w \in C$ is said to *penetrate a coset* if $\tilde{w}$ passes through a cone point.

A path w is *without backtracking* if w does not penetrate a given coset twice.

Definition 9 (Bounded Coset Penetration)**.** *The pair (G, H) is said to have bounded coset penetration if for every $\lambda > 0$ there exists $\alpha = \alpha(\lambda) > 0$ with the following property. Suppose u and v are two relative λ-quasigeodesics without backtracking such that $d_C(u, v) \leq 1$. Then,*

1. *If u penetrates a coset X and v does not, then u travels a C-distance of at most α in X, and*
2. *If both u and v penetrate a coset X, then the vertices of C at which they first enter X are at most distance α apart in C. Similarly the vertices at which u and v exit the coset are at most distance α apart in C.*

Definition 10 (Relative hyperbolicity)**.** *A finitely generated group G is said to be relatively hyperbolic with respect to a subgroup H if*

1. *the coned-off Cayley graph of G with respect to H is δ-hyperbolic for some $\delta \geq 0$*
2. *the pair (G, H) has the bounded coset penetration property.*

The subgroup H is called a parabolic subgroup.

Remark. The definition of relative hyperbolicity has been proved to be independent of the generating set. See [9], for instance.

4.2 Proof of Theorem 35

Let a group G be hyperbolic relative to an asymptotically CAT(0) subgroup H. There exists a geodesic space X such that all asymptotic cones of X are CAT(0) and H acts geometrically on X. We construct a geodesic metric space Y which supports a geometric G-action and go on to prove that all asymptotic cones of Y are CAT(0). The space Y has been described in [8], in the special case where H is a finitely generated free abelian group.

The space Y

Assume that the group G is hyperbolic with respect to the subgroup H. Let B be a generating set for H such that there exists a point x_0 in X which is not fixed by any element of B. Since H acts geometrically on X such a set exists. Now choose a generating set A for G which intersects H in B. The word metric induced on G by the generating set A is written d_A and similarly, d_B denotes the word metric on H coming from B. Note that the metric d_B gives a natural choice of metric for each coset gH of H in G. Start with the Cayley graph C for G with respect to the generating set A. The elements of a coset gH along with all edges between them that are labelled by B comprise a copy of the Cayley graph for H in C.

Define $c = \min\{d(b.x_0, x_0) \mid b \in B\}$. By choice of x_0, c is strictly positive. Now, for each of the generators b in B, assign the distance $d(x_0, b.x_0)$ to the edge in C that connects the identity to b. Extend this metric to all copies of the Cayley graph of H in C, using the action of G on itself by left multiplication. Continue to call the Cayley graph of G as C.

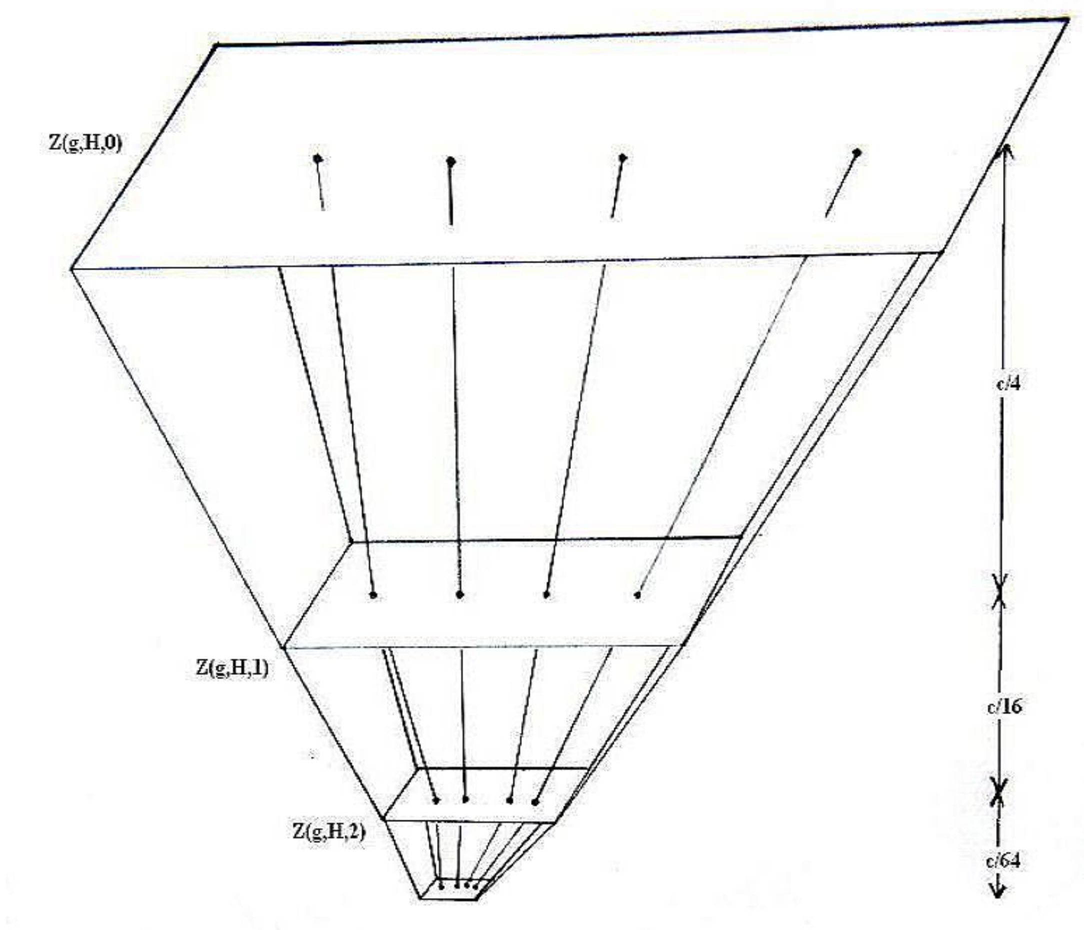

Figure 4.1: 'The Cusped Cayley Graph'.

The subgraph of C with vertices from the coset gH and with edges labelled by B is denoted $Z(g,H,0)$. Form $Z(g,H,1)$ by taking $Z(g,H,0)$ and making each edge in the latter a fourth of its original length. Join corresponding edges of $Z(g,H,0)$ and $Z(g,H,1)$ with edges of length $\frac{c}{4}$. Perform this construction on each coset of H in G. The space obtained is denoted Y^1.

Define Y^n inductively from Y^{n-1} by forming for every coset gH, a graph $Z(g,H,n)$ with edges of length 2^{-2n}-th of the original and then gluing it to $Z(g,H,n-1)$ via edges of length $c/2^{2n}$. Endow Y^n with the natural path metric and form the metric completion Y^∞ of $\bigcup Y^n$. In Y^∞, each coset has a cone point which is at a distance of $\sum 2^{-2n} < \frac{1}{2}$. The graph Y^∞ is quasi-isometric to the coned off Cayley graph of G and therefore, it is δ-hyperbolic, for some δ.

Lemma 36. *Let Y_g be the space obtained by performing the above construction to all cosets of H other than gH. There exists $\rho > 0$ such that for all x and y in gH, $d_{gH}(x,y) \le \rho d_{Y_g}(x,y)$.*

The lemma is largely a consequence of the property of Bounded Coset Penetration.

Lemma 37. *There exists a $k \ge 0$ such that each of the graphs $Z(g,H,k)$ is isometrically embedded in Y^k.*

Proof of Lemma 37. Let $C(g,H)$ denote the union of the $Z(g,H,j)$'s for $j = 1,\ldots,k$, along with all the edges of length $c/2^{2j}$ that join a $Z(g,H,j-1)$ to $Z(g,H,j)$. Suppose there exists a pair u, $v \in Z(g,H,k)$ such that the geodesic γ joining them does not lie completely in $Z(g,H,k)$. Then, this geodesic must leave the set $C(g,H)$.

Let x be the point at which γ leaves $C(g, H)$ and y denote the point at which it re-enters $C(g, H)$. Let x_1 and y_1 be the points at which γ exits $Z(g, H, k)$ and re-enters $Z(g, H, k)$. By construction, the part of γ between u and x is made of the edges of length $c/2^{2j}$ that join a $Z(g, H, j-1)$ to $Z(g, H, j)$ and the same applies to the part of γ between y and v. Therefore, $d_{gH}(x, y) = 2^{2k} d_{Z(g,H,k)}(x_1, y_1)$. Moreover, by the previous Lemma,

$$d_{Y_g}(x, y) + \frac{c}{2} \leq d_{Z(g,H,k)}(x_1, y_1) = \frac{d_{gH}(x, y)}{2^{2k}} \leq \rho \frac{d_{Y_g}(x, y)}{2^{2k}}.$$

This implies that $\frac{\rho}{2^{2k}} - 1 > 0$. Therefore, $k = [\frac{\log_2 \rho}{2}] + 1$ is the required constant. This proves the Lemma. $\square$

Choose k large enough such that each of the graphs $Z(g, H, k)$ is isometrically embedded in Y^k. Such a k exists by Lemma 37. Let C_H denote a copy of the Cayley graph of H in C. With C perturbed as described before, there is a natural embedding μ of C_H into X such that for all $g \in G$ and $b \in B$, the vertex g of C_H maps to the point gx_0 and the edge (g, gb) of C_H maps isometrically to a geodesic joining gx_0 to gbx_0. Glue a copy of X to every copy of C_H in Y^k via the embedding μ, rescaling metrics involved by 2^{2k}. Denote the resulting space by Y. Let Ξ denote the collection of X's that are adjoined to C. Since each of the graphs $Z(g, H, k)$ is isometrically embedded in Y, the same is true for every element of Ξ. Moreover, the left action of G on itself induces a natural action of G on Y by isometries.

The action of H on X is cocompact and so we deduce that the action of G on Y is co-compact. Indeed, the quotient of Y by the action of G comprises the standard

1-complex of G corresponding to the generating set A and a concatenation of k edges emanating from the unique vertex of the 1-complex and terminating in the compact space X/H. At each vertex other than the initial and terminal ones in the aforesaid path of k edges, is a suitably rescaled copy of the standard 1-complex of H with respect to the generating set B. An element of G either fixes a $C(g, H)$ or else maps it to a different one. Therefore, the properness of the action of H on X implies that the action of G on Y is proper.

The Asymptotic Cones of Y

In order to describe the asymptotic cones of Y, we have to introduce 'tree graded' spaces.

Definition 11. *Let M be a complete geodesic space and let $\mathcal{P}$ be a collection of closed geodesic subsets (called pieces) of M. Then, M is said to be tree graded with respect to $\mathcal{P}$ if the following two properties are satisfied:*

1. *Two different pieces have at most one common point.*
2. *Every simple geodesic triangle (a simple loop composed of three geodesics) in M is contained in one piece.*

One characterisation of relative hyperbolicity is via tree graded asymptotic cones.

Theorem 38 (Theorem 1.11 in [7])**.** *A finitely generated group G is relatively hyperbolic with respect to H if and only if every asymptotic cone $Cone_\omega(G)$ is tree-graded with respect to ω-limits of sequences of cosets of the subgroup H.*

Let ω be a non-principal ultrafilter and (a_n), a sequence of positive real numbers such that $\lim_{n\to\infty} a_n = 0$. Choose for each $n \in \mathbb{N}$, a point $p_n \in Y$.

Recall that Ξ is the collection of the different copies of X that are attached to C. Set $\mathcal{X} = \{Cone_\omega(Z) \mid Z \in \Xi\}$, where $Cone_\omega(Z)$ refers to the ω-limit of Z in $Cone_\omega(Y, (a_n), (p_n))$.

Claim. The asymptotic cone $Y_\omega := Cone_\omega(Y, (a_n), (p_n))$ is tree graded with respect to $\mathcal{X}$.

The fact that Y_ω is a complete geodesic space follows from Lemmas 50 and 51.

As the action of G on Y is proper and co-compact, G is quasi-isometric to Y. But by the above theorem, G is asymptotically tree graded with respect to the cosets of H. Theorem 5.1 in [7] states that the property of being asymptotically tree-graded is preserved under quasi-isometries. It follows, that Y is asymptotically tree-graded with respect to $\mathcal{X}$.

Lemma 39. *The space Y_ω is CAT(0).*

Proof. The asymptotic cone Y_ω is tree-graded with respect to $\mathcal{X}$ and by hypothesis, each piece is CAT(0). We know that every simple triangle in Y_ω is contained in a piece. Hence, we may assume that our triangle ABC in Y_ω has the form $A'B'C' \cup AA' \cup BB' \cup CC'$, where $A'B'C'$ is a geodesic triangle that lies in some piece of the asymptotic cone while AA', BB' and CC' are simply geodesics.

We will use the 'Bruhat-Tits' inequality for CAT(0) spaces. This says that a geodesic

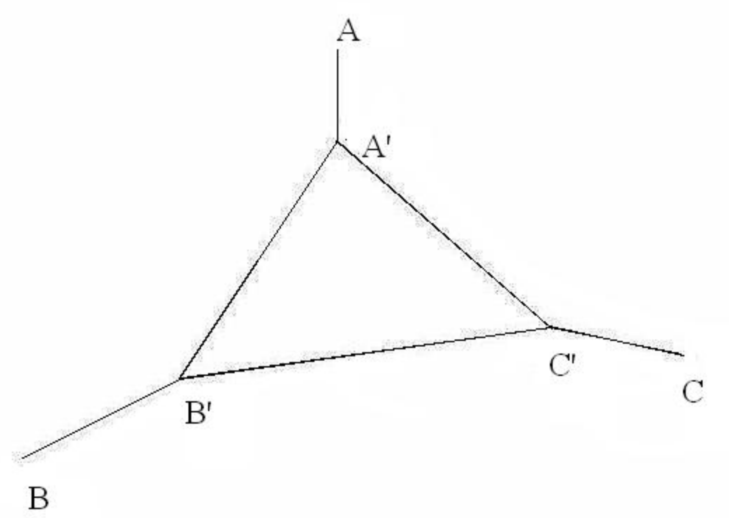

Figure 4.2: A generic triangle in Y_ω.

space $\mathcal{M}$ is CAT(0) if and only if for all triples $(p,q,r) \in \mathcal{M}^3$ and all $m \in \mathcal{M}$ with $d(q,m) = d(m,p) = d(p,q)/2$, we have

$$d(p,q)^2 + d(q,r)^2 \geq 2d(m,p)^2 + d(q,r)^2/2.$$

To show that the triangle ABC satisfies the CAT(0) property, take M to be the midpoint of the side BC. The case when M lies on the geodesic BB' or the geodesic CC' is trivial. So assume that $M \in B'C'$. There is a comparison triangle $A_1'B_1'C_1'$ with comparison point M_1 on $B_1'C_1'$ for M. Since each piece of the asymptotic cone is CAT(0) we have $d(A',M) \leq d(A_1',M_1)$.

Let $\bar{A}\bar{B}\bar{C}$ be a comparison triangle for ABC with comparison point $\bar{M}$ for M. Let $a = d(A',B')$, $b = d(A',C')$, $c = d(B',C')$, $x = d(M,B')$, $r = d(A,A')$, $p = d(B,B')$,

$q = d(C, C')$, $h = d(\bar{A}, \bar{M})$ and $h' = d(A'_1, M_1)$. We know that $d(A, M) = d(A, A') + d(A', M) \leq h' + r$. Hence it suffices to prove that $h' + r \leq h$.

Case 1. The value of r is 0.

Note that $x = (c + q - p)/2$, so using the Cosine Law,

$$
\begin{aligned}
h'^2 &= a^2 + \left(\frac{c+q-p}{2}\right)^2 - \left(\frac{a^2+c^2-b^2}{c}\right)\left(\frac{c+q-p}{2}\right) \\
&= \frac{2a^2+2b^2-c^2}{4} + \frac{(q-p)^2}{4} + \left(\frac{q-p}{2}\right)\left(\frac{b^2-a^2}{c}\right).
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
h^2 &= \frac{2(a+p)^2 + 2(b+q)^2 - (c+q+p)^2}{4} \\
&= \frac{2a^2+2b^2-c^2}{4} + \frac{(q-p)^2}{4} + \frac{4ap+4bq-2pc-2cq}{4}.
\end{aligned}
$$

$$
\begin{aligned}
h^2 - h'^2 &= \frac{4ap+4bq-2pc-2cq}{4} - \left(\frac{q-p}{2}\right)\left(\frac{b^2-a^2}{c}\right) \\
&= \frac{p(b^2-(a-c)^2) + q(a^2-(b-c)^2)}{2c}.
\end{aligned}
$$

That the final expression is non-negative is a consequence of the triangle inequality for $A'B'C'$.

Case 2. The value of r is not zero.

By case 1 the result holds for the triangle $A'BC$. Now, let $\alpha = d(A', B)$, $\beta = d(A', C)$ and $\gamma = d(B, C)$. Further, set $\alpha' = \alpha + r$ and $\beta' = \beta + r$.

Then,

$$
h' + r = \sqrt{\frac{2\alpha^2 + 2\beta^2 - \gamma^2}{4}} + r, \text{ and } h = \sqrt{\frac{2\alpha'^2 + 2\beta'^2 - \gamma^2}{4}}.
$$

Manipulating the above two expressions, one reduces the inequality $h' + r \leq h$ to $\sqrt{2\alpha^2 + 2\beta^2 - \gamma^2} \leq \alpha + \beta$ or equivalently to $(\alpha + \beta)^2 - \gamma^2 \leq 0$. This again is a consequence of the triangle inequality. This proves that Y_ω is CAT(0). $\square$

The proof of the lemma completes the proof of Theorem 35.

Systolic groups with Isolated Flats

As a corollary to Theorem 35, we obtain that systolic groups with isolated flats are asymptotically CAT(0).

Definition 12. *A simplicial complex is said to be flag if every finite set of vertices pairwise connected by edges spans a simplex. A simplicial complex X is said to be 6-large if it is flag and every cycle of length 4 or 5 has a diagonal. A simplicial complex X is said to be systolic if it is connected, simply connected and the link of every non-empty simplex in X is 6-large.*

A two dimensional flat in a systolic complex X is a subcomplex F which is isomorphic to the triangulation of $\mathbb{R}^2$ by congruent equilateral triangles. There is no systolic triangulation of $\mathbb{R}^n$ for $n \geq 3$ (see [15]) and so one does not consider flats of dimension more than 2. Two flats in a systolic complex are considered to be equivalent if they are at finite Hausdorff distance to one another.

A cocompact systolic complex X has isolated flats property if there exists a function $\psi : \mathbb{N} \to \mathbb{N}$ such that the c-neighbourhood of any flat F intersects the c-neighbourhood of a non-equivalent flat F' over a diameter of at most $\psi(c)$.

The main theorem about systolic groups with isolated flats is the following.

Theorem 40. *(Theorem B in [25]) Let X be a systolic complex with the Isolated Flats Property and G a group acting geometrically on X. Then the group G is relatively hyperbolic with respect to a family of maximal virtually abelian subgroups of rank 2.*

The above theorem, together with Theorem 35 implies the corollary below.

Corollary 41. *Let X be a systolic complex with the Isolated Flats Property and G, a group acting cocompactly and properly discontinuously on X. Then the group G is asymptotically CAT(0).*

Question 42. *Are all systolic groups, asymptotically CAT(0)?*

CHAPTER 5

THE UNIVERSAL COVER OF $PSL(2,\mathbb{R})$

5.1 Geometry of Unit Tangent Bundles

In this section we describe the geometry of $\widetilde{PSL(2,\mathbb{R})}$ which is a Riemannian manifold with the additional structure of a Lie group. It is well known that $PSL(2,\mathbb{R})$ acts on the hyperbolic plane by Mobius transformations. This action can be used to identify the group with the unit tangent bundle $T^1(\mathbb{H}^2)$ of the hyperbolic plane. The universal cover of the latter is then $\widetilde{PSL(2,\mathbb{R})}$.

It is possible to give the tangent bundle TM of a Riemannian manifold M a Riemannian metric. There is therefore an induced Riemmanian metric on the unit tangent bundle $T^1(M)$ of M. Unit tangent bundles of Riemannian manifolds have been studied in some detail by Sasaki in [21] and [22].

Convention In this section alone, the word 'geodesic' will refer to curves in a Riemannian manifold with constant speed parametrization. More precisely, if M is a Riemannian manifold and ∇ is its Riemannian connection, then a geodesic in M is a curve γ such that $\nabla_{\frac{d\gamma}{dt}}\frac{d\gamma}{dt} = 0$.

The Sasaki metric on $T(M)$

Let M be an n-dimensional Riemannian manifold and let TM be its tangent bundle. Consider the projection $\pi : TM \to M$ that sends a point $\theta = (x, v)$ in TM to the point $x \in M$. The map π is a Riemannian submersion. The kernel $V(\theta)$ of its differential is made of vectors in $T_\theta TM$ that are tangent to the fibre $T_x(M)$ at x. These vectors are said to be 'vertical'. On the other hand, the vectors which are orthogonal to the fibre at x are the 'horizontal' vectors. These are denoted $H(\theta)$ and more formally, they form the kernel of the covariant map.

A curve σ in TM is given by the pair $(\alpha(t), v(t))$, where α is a path in M and $v(t)$ is a vector field along α. If V is an element of $T_\theta TM$, then V comes from an infinitesimal path $\sigma : (-\epsilon, \epsilon) \to TM$, which satisfies $\sigma'(0) = V$. One may now compute the covariant derivative of the vector field $v(t)$ along α'. This measures the rate at which $v(t)$ varies from the tangent vector to the curve α.

The covariant map K_θ at the point V is defined to be $(\nabla_{\alpha'} v)(0)$, where ∇ denotes the Riemannian connection of M. One can show that $H(\theta)$ is precisely the kernel of the covariant map, that the linear map $d_\theta \pi$ gives an isomorphism of $T_x M$ with $H(\theta)$ while K_θ gives a linear isomorphism of $V(\theta)$ with $T_x M$. Moreover, the vector space $T_\theta TM$ is a direct sum of $H(\theta)$ and $V(\theta)$.

One defines the Sasaki metric $\langle\langle ., . \rangle\rangle_\theta$ on $T_\theta TM$ so that these two components are orthogonal. For V and W in $T_\theta TM$,

$$\langle\langle V, W \rangle\rangle_\theta := \langle d_\theta \pi(V), d_\theta \pi(V) \rangle_x + \langle K_\theta(V), K_\theta(V) \rangle_x \,.$$

Note that a curve $\sigma(t) = (\alpha(t), v(t))$ in TM is *horizontal* if its tangent vector is horizontal, which is the same as saying that the vector field $v(t)$ is the parallel transport of its initial vector along $\alpha(t)$.

The collection of unit tangent vectors $T^1(M)$ is a Riemannian subspace of TM with the induced Riemannian metric.

Geodesics in $T^1(\mathbb{H}^2)$

We will now describe the geodesics of the unit tangent bundle of the hyperbolic plane, endowed with the induced Sasaki metric. The account in this paragraph follows Sasaki's work from [22].

A curve Γ on $T^1(\mathbb{H}^2)$ is a unit vector field $y(\sigma)$ along a curve $x(\sigma) = \pi(\Gamma)$ in $\mathbb{H}^2$, where σ is the arc length of Γ. Let x' denote $\frac{dx}{d\sigma}$ and ∇ denote the Riemannian connection of $\mathbb{H}^2$. (We work throughout with the upper half plane model of the hyperbolic plane). Then $\langle\langle \Gamma', \Gamma' \rangle\rangle = 1$, which is equivalent to

$$\langle x', x' \rangle + \langle \nabla_{x'} y, \nabla_{x'} y \rangle = 1.$$

Putting $c^2 = \langle \nabla_{x'} y, \nabla_{x'} y \rangle$ we have $\langle x', x' \rangle = 1 - c^2$ and $0 \leq c \leq 1$. The conditions for Γ to be a geodesic in $T^1(\mathbb{H}^2)$ are that c is a constant and $x(\sigma)$ and $y(\sigma)$ satisfy the differential equations

$$x'' = by - a\nabla_{x'} y, \;\; \nabla_{x'} \nabla_{x'} y = -c^2 y.$$

where $a = \langle x', y \rangle$ and $b = \langle x', \nabla_{x'} y \rangle$.

Using c, one may characterize the geodesics in $T^1(\mathbb{H}^2)$ into the following types:

1. Horizontal type or $c = 0$: In this case, $\nabla_{x'} y = 0$ and so Γ is a horizontal geodesic in the unit tangent bundle. Its corresponding π image in the hyperbolic plane is also a geodesic.

2. Vertical type or $c = 1$: In this case the image of Γ under π is a point and the geodesic Γ is a great circle that lives entirely in the fibre above that point.

3. Oblique type or $0 < c < 1$: See below.

Geodesics of Oblique type

Definition 13. *An equidistant curve is the locus of points which lie at a constant distance from a given geodesic.*

Let T, N denote the unit tangent vector and the principal normal vector of x in $\mathbb{H}^2$; let κ be the curvature of x and s, its arclength. Then, $\frac{ds}{d\sigma} = \sqrt{1-c^2}$ and $x' = \sqrt{1-c^2}T$ while $x'' = (1-c^2)\kappa N$. One can show that $(1-c^2)^2\kappa^2 = c^2$ and so κ is always constant. Hence, x is an equidistant curve, a horocycle or a circle in the hyperbolic plane, depending on whether κ^2 is less than 1, equal to 1 or greater than 1.
In [22], Sasaki shows that the vector field component y has the form

$$y(\sigma) = \cos 2c\sigma T(\sigma) - \sin 2c\sigma N(\sigma).$$

Observe that the vector field y has period $\frac{\pi}{c}$.

Length in $\widetilde{PSL(2,\mathbb{R})}$

Assume that $c \in (\frac{1}{\sqrt{2}}, 1)$. Then the projection of a $T^1(\mathbb{H}^2)$-geodesic is a proper circle. Consider the following configuration of points in the upper half plane: $P = (0, 1)$,

$Q = (0, e^D)$ and M is the midpoint of the geodesic joining P and Q. Note that $d(P,Q) = D$ while $M = (0, e^{\frac{D}{2}})$.

Consider the geodesic in the upper half plane which passes through M and is perpendicular to the y-axis at M. Every point on this semi-circle corresponds to the center of a circle that passes through the points P and Q. Fix such a point C.

Let θ denote the angle PCM and $L = d(C,P)$. The hyperbolic sine law implies that $\sinh L = \frac{\sinh D/2}{\sin\theta}$. Moreover the circumference of a circle in the upper half plane of radius L is given by $2\pi \sinh L$. Hence the length of the arc of the circle at C subtended by the angle PCQ is given by $\frac{2\theta \sinh \frac{D}{2}}{\sin\theta}$. Therefore, for $\theta \in (0,\pi)$, a geodesic in $T^1(\mathbb{H}^2)$ covering the arc from P to Q has length $\frac{2\theta \sinh \frac{D}{2}}{\sqrt{1-c^2}\sin\theta}$.

Using all previous relations for c, we deduce that this length is the same as

$$\frac{2}{\sqrt{2c^2-1}} Arcsin\left(\sqrt{\frac{2c^2-1}{1-c^2}} \sinh\frac{D}{2}\right),$$

where $Arcsin$ is the continuous version of arcsin, taking all values between $-\infty$ to $+\infty$. More precisely, for any real number x, we have

$$Arcsin(x) = \frac{(2k-1)\pi}{2} + \arcsin\left(x - (2k-1)\right).$$

where $k = [\frac{x-1}{2}]$. Denote this length by l.

Geodesics in $\widetilde{PSL(2,\mathbb{R})}$

Fix a section of the hyperbolic plane in $\widetilde{PSL(2,\mathbb{R})}$ so that parallel transport of tangent vectors along vertical geodesics in the upper half plane preserve the section. With respect to this section, any point in $\widetilde{PSL(2,\mathbb{R})}$ maybe given in the form (P,r), where

P is a point in the hyperbolic plane and r is a real number. Suppose that a geodesic between $\widetilde{P} = (P, r)$ and $\widetilde{Q} = (Q, s)$ projects to a geodesic of length l, as above in $T^1(\mathbb{H}^2)$. Since the corresponding vector field has period $\frac{\pi}{c}$, we deduce that $|s - r| = \frac{l}{\pi/c}$. Hence, any geodesic in $T^1(\mathbb{H}^2)$ that lifts to a geodesic between $\widetilde{P}$ and $\widetilde{Q}$ must satisfy

$$|s - r| = \frac{2c}{\pi\sqrt{2c^2 - 1}} Arcsin\left(\sqrt{\frac{2c^2 - 1}{1 - c^2}} \sinh \frac{D}{2}\right).$$

Call this function $\phi_D(c)$.

To understand geodesics in $\widetilde{PSL(2, \mathbb{R})}$, one has to study the function ϕ_D. From the definition of $Arcsin$, it follows that $\lim_{c \to 1^-} \phi_D(c) = \infty$. On the other hand, since for sufficiently small values of x, $Arcsin$ may be approximated by x, we have

$$\lim_{c \to \frac{1}{\sqrt{2}}^+} \phi_D(c) = \frac{2}{\pi} \sinh \frac{D}{2}.$$

The function ϕ_D for fixed D is monotonic for values of c bounded away from $\frac{1}{\sqrt{2}}$. Close to $\frac{1}{\sqrt{2}}$, the function is oscillatory, with the number of oscillations depending on the value of D. However, for every value of D, there are only finitely many pre-images corresponding to a given function value.

Observation 43. *The length of the distance-minimising geodesic joining $\widetilde{P}$ and $\widetilde{Q}$ is given by $\frac{\pi}{x}|s - r|$, where x is the largest value of c that satisfies $\phi_D(c) = |s - r|$.*

5.2 Asymptotic Cones of $\widetilde{PSL(2, \mathbb{R})}$

Theorem 44. *There exists a $(1, \pi)$-quasi-isometry from $\widetilde{PSL(2, \mathbb{R})}$ to $\mathbb{H} \times \mathbb{R}$.*

Proof. The proof exploits the structure of $\widetilde{PSL(2,\mathbb{R})}$ as a Riemannian manifold. We saw that the action of $PSL(2,\mathbb{R})$ on the hyperbolic plane by Mobius transformations gives an identification of $PSL(2,\mathbb{R})$ with the unit tangent bundle $T^1(\mathbb{H}^2)$ of $\mathbb{H}^2$. Thus, $\widetilde{PSL(2,\mathbb{R})}$ is the universal cover of $T^1(\mathbb{H}^2)$. One uses the Sasaki metric to make the tangent bundle of $\mathbb{H}^2$ into a Riemannian manifold. The Riemannian metric on $T^1(\mathbb{H}^2)$ is the induced Sasaki metric. Let $\pi : \widetilde{PSL(2,\mathbb{R})} \to \mathbb{H}^2$ denote the canonical projection of $\widetilde{PSL(2,\mathbb{R})}$ onto the hyperbolic plane.

We now describe an identification of $\widetilde{PSL(2,\mathbb{R})}$ with $\mathbb{H}^2 \times \mathbb{R}$. Fix a base point $*$ on $\mathbb{H}^2$ and a reference unit vector $v \in T^1_*(\mathbb{H}^2)$. For any curve $\alpha \in \mathbb{H}^2$, denote by $P_\alpha(w)$, the parallel transport of a vector w along α. Using $(*, v)$, form a section s of $T^1(\mathbb{H}^2) \to \mathbb{H}^2$ as follows: given $x \in \mathbb{H}^2$, let γ be the unique geodesic joining $*$ to x in the hyperbolic plane. Define $s(x) = (x, P_\gamma(v))$. Since $\mathbb{H}^2$ is simply connected, the section s lifts to a section $\tilde{s}$ of $\mathbb{H}^2$ to $\widetilde{PSL(2,\mathbb{R})}$. With this global section, one can describe a point P in $\widetilde{PSL(2,\mathbb{R})}$ with an ordered pair $(\pi(P), \theta(P)) \in \mathbb{H}^2 \times \mathbb{R}$, where $\pi(P)$ is the projection of P to $\mathbb{H}^2$ and $\theta(P)$ is the distance of P from $\tilde{s}(\pi(P))$ in $\widetilde{PSL(2,\mathbb{R})}$.

We know that a geodesic in $\widetilde{PSL(2,\mathbb{R})}$ projects, via the map π, to one of the following: a point, an $\mathbb{H}^2$-geodesic or an arc of a proper circle, a horocycle or an equidistant curve. Since a $\widetilde{PSL(2,\mathbb{R})}$-geodesic η is simply a vector field along the projection $\bar{\eta} := \pi(\eta)$, the general form of a geodesic joining two points, P and Q (which are identified with $(\pi(P), \theta(P))$ and $(\pi(Q), \theta(Q))$ respectively) is given by $\eta(t) = (\bar{\eta}(t), P_{\bar{\eta}}(w) + tc_\eta)$, where w is the initial vector of η and $|c_\eta| \leq 1$ is the rate of rotation of w along $\bar{\eta}$.

Let η be a distance-minimising geodesic in $\widetilde{PSL(2,\mathbb{R})}$ from P to Q. We may assume, without loss of generality that $\theta(Q) \geq \theta(P)$. The length $l(\eta)$ in the Sasaki metric is given by

$l(\eta) = \sqrt{l(\bar{\eta})^2 + (\theta(Q) - \theta(P) - P_{\bar{\eta}}(w))^2}$.

Observe that if the points P and Q are joined by the minimal length curve α in $\widetilde{PSL(2,\mathbb{R})}$ whose projection in $\mathbb{H}^2$ is the geodesic joining $\pi(P)$ and $\pi(Q)$, then $l(\eta) \leq l(\alpha)$, and

$$l(\alpha) = \sqrt{d(\pi(P), \pi(Q))^2 + (\theta(Q) - \theta(P))^2}.$$

On the other hand, $|P_{\bar{\eta}}(w)| \leq \pi$, which implies that if $\theta(Q) - \theta(P) \geq \pi$ then the length of η is at least $\sqrt{l(\bar{\eta})^2 + (\theta(Q) - \theta(P) - \pi)^2}$.

Let $d = d(\pi(P), \pi(Q))$, $r = \theta(Q) - \theta(P)$, $L = l(\eta)$ and $D = \sqrt{d^2 + r^2}$. Note that D is the distance between the images of P and Q in $\mathbb{H}^2 \times \mathbb{R}$, while L is the distance between them in $\widetilde{PSL(2,\mathbb{R})}$. As the length $L = l(\eta)$ of η is no larger than $\sqrt{d(\pi(P), \pi(Q))^2 + (\theta(Q) - \theta(P))^2}$, we deduce that $L \leq D$.

If $\theta(Q) - \theta(P) \leq \pi$ then by the triangle inequality, $D \leq d + r \leq L + \pi$. If however, $\theta(Q) - \theta(P) \geq \pi$, then $l(\eta) \geq \sqrt{d(\pi(P), \pi(Q))^2 + (\theta(Q) - \theta(P) - \pi)^2}$. So, $d^2 + (r - \pi)^2 \leq L^2$ and thus, $D^2 \leq L^2 + 2\pi r - \pi^2$. But $r \leq L + \pi$. Hence, $D^2 \leq L^2 + 2\pi(L + \pi) - \pi^2$ which implies that $D^2 \leq (L + \pi)^2$.

In all cases, we have $L \leq D \leq L + \pi$. $\square$

Corollary 45. *The asymptotic cones of* $\widetilde{PSL(2,\mathbb{R})}$ *and* $\mathbb{H} \times \mathbb{R}$ *are isometric and hence,* $\widetilde{PSL(2,\mathbb{R})}$ *is asymptotically CAT(0).*

Proof. By observation 54, a $(1, \pi)$-quasi-isometry induces an isometry at the level of asymptotic cones and so every asymptotic cone of $\widetilde{PSL(2, \mathbb{R})}$ is a direct product of the real line with the infinitely branching homogeneous $\mathbb{R}$-tree. $\square$

CHAPTER 6

CONCLUDING REMARKS

Asymptotically CAT(0) groups are a natural enlargement of the class of non-positively curved groups. There are many interesting questions one may ask about asymptotically CAT(0) groups. In the paragraphs below, I present a selection.

Asymptotically CAT(0) graphs

The conjecture below is commonly attributed to Erdós and Pach; unfortunately, I know no reference for it. In any case, it deserves a mention here.

Conjecture 1. *The integer points in the Euclidean Plane may be connected to form a graph which is* $(1, k)$*-quasi-isometric to the Euclidean plane.*

It was proved in Section 2.2 that a graph is δ-CAT(0) if and only if it is hyperbolic. One wonders if this is also the case with asymptotically CAT(0) graphs. I would like to propose the following conjecture.

Conjecture 2. *A graph is asymptotically CAT(0) if and only if it is* δ*-hyperbolic.*

Observe that an affirmative answer to the second conjecture implies a negative answer to the first. Indeed, if there exists a graph X with vertex set $\mathbb{Z}^2$ such that X is $(1, k)$-quasi-isometric to the Euclidean plane, then all asymptotic cones of X are isometric to the Euclidean plane and thus CAT(0). But then, X is hyperbolic, which contradicts the assumption that X is quasi-isometric to the Euclidean plane.

More generally, one may ask, under what conditions are simplicial complexes asymptotically CAT(0).

Novikov's Conjecture for Asymptotically CAT(0) Groups

There are many different approaches by which the Novikov Conjecture may be proved for asymptotically CAT(0) groups. One has been mentioned before: develop an asymptotic notion of δ-bolicity so that existing techniques from [16] may be extended to establish the conjecture.

Alternatively, one can appeal to boundary theory. Keeping in mind that asymptotically $CAT(0)$ spaces are genuinely non-positively curved when viewed from infinitely far away, one may define a (Tits) metric d_∞ on the equivalence classes of geodesic rays with the formula defined below.

Let c_1,$c_2 : [0, \infty) \to X$ be geodesic rays emanating from a point in an asymptotically CAT(0) space X. Set

$$2 \sin \frac{1}{2} d_\infty([c_1], [c_2]) = \lim_{t \to \infty} \frac{1}{t} d_X(c_1(t), c_2(t)).$$

Study properties of this boundary: does it provide an 'EZ-structure' for G (for details, see [10]), where G is a group acting geometrically on X?

A coherent notion of a Tits boundary can also help one study isometries of asymptotically CAT(0) groups.

Artin Groups

Some Artin groups are known to be CAT(0). Is it possible that all Artin groups are asymptotically $CAT(0)$? This will prove that the word problem is solvable for Artin groups.

Other Lie Groups

One wonders if the methods of Chapter 5 may be used to show that Lie groups other than $\widetilde{PSL(2,\mathbb{R})}$ are asymptotically CAT(0).

Recall that the universal cover of $SL(2,\mathbb{R})$ can be identified with the universal cover of the unit tangent bundle of the hyperbolic plane. Also the maximal compact subgroup of $SL(2,\mathbb{R})$ is the special orthogonal group $SO(2)$ and $\pi_1(SO(2)) \cong \mathbb{Z}$.

A symmetric space is a homogeneous space G/K, where G is a Lie group and H is the maximal compact subgroup of G. A special class of symmetric spaces are the Hermitian ones. These come equipped with additional structure: a Hermitian symmetric space is a Riemannian symmetric space endowed with a parallel complex structure compatible with the Riemannian metric. We understand Hermitian symmetric spaces of non-compact type via the theorem below.

Theorem 46 (Theorem VIII.6.1, [13]). *The non-compact irreducible Hermitian symmetric spaces are exactly the manifolds G/K where G is a connected noncompact simple Lie group with trivial center and K has non-discrete center and is the maximal compact subgroup of G. Further, the center of the group K is analytically isomorphic to the circle group.*

In view of this theorem, it can be decided immediately which of the spaces in the classification of irreducible Riemannian symmetric spaces are Hermitian symmetric. We list them in the table below.

Type	G/K	Rank	Dimension
AIII	$SU(p,q)/S(U_p \times U_q)$, $p \geq q \geq 1$	$min(p,q)$	$2pq$
BDI	$SO_\circ(p,2)/SO(p) \times SO(2)$, $p \geq 3$	2	$2p$
DIII	$SO^*(2n)/U(n)$, $n \geq 3$	$[n/2]$	$n(n-1)$
CI	$Sp(n,\mathbb{R})/U(n)$	n	$n(n+1)$
EIII	$(E_{6(-14)}, so(10) + \mathbb{R})$	2	32
EVII	$(E_{7(-25)}, E_6 + \mathbb{R})$	3	54

We isolate the rank 1 cases of minimal dimension here. There are three; namely, $SU(1,1)$, $SO^*(6)$ and $Sp(1,\mathbb{R})$. A special isomorphism identifies the group $SU(1,1)$ with the group $Sp(1,\mathbb{R})$. Both these groups are isomorphic to $SL(2,\mathbb{R})$, which was the object of study in Chapter 5. The group $SO^*(6)$ is isomorphic to $SU(3,1)$, the isometry group of complex hyperbolic 3-space.

In order to apply the methods of this dissertation to the Lie groups given above, one has to describe the geodesics of their unit tangent bundles. Let M be a symmetric space and let $T^1(M)$ denote the unit tangemt bundle of M. Then, under the canonical projection $T^1(M) \to M$, geodesics in the unit tangent bundle map to curves of constant geodesic curvature. This is proved in [17]. So the task at hand is to identify the curves of constant curvature in the symmetric spaces listed above.

Appendix A

ULTRAFILTERS AND ULTRALIMITS

There are different ways of defining a non-principal ultrafilter ω on a non-empty set N. The set theoretic approach exploits the structure of the power set $\mathcal{P}(N)$ as a Boolean algebra and a non-principal ultrafilter is defined to be a maximal ideal of this Boolean algebra . In this exposé however, we will take a different approach.

A.1 Ultrafilters

Definition 14. *A non-principal ultrafilter on a non-empty set N is a finitely additive measure on $\mathcal{P}(N)$ with values in $\{0,1\}$ such that every finite subset of N is null.*

The existence of non-principal ultrafilters is a non-trivial fact that involves Zorn's Lemma. Hence, one cannot produce explicit examples of non-principal ultrafilters. In the following paragraph we give an outline of the proof that non-principal ultrafilters exist.

Let N be a set. A *filter* on N is a function $\mu : \mathcal{P}(N) \to \{0,1\}$ which has the following properties:

1. $\mu(\emptyset) = 0$,
2. $\mu(N) = 1$

3. if $S \subset T$, then $\mu(S) \leq \mu(T)$

4. if $\mu(S) = 1 = \mu(T)$, then $\mu(S \cap T) = 1$.

An *ultrafilter* μ is a filter which has the property that for every subset S of N, $\mu(S) + \mu(S^c) = 1$. There are two types of ultrafilters : the principal and the non-principal. For any element s of N, one can define an ultrafilter μ_s by specifying $\mu_s(T) = 1 \iff s \in T$. Such an ultrafilter is said to be principal; an ultrafilter which is not principal is said to be non-principal.

The collection of all filters on a set N carry a natural ordering: $\mu_1 \leq \mu_2$ if for every $S \subset N$, $\mu_1(S) \leq \mu_2(S)$. A filter is an ultrafilter precisely when it is maximal under this ordering. In addition if N is an infinite set, an ultrafilter ω is non-principal if and only if $\omega \geq m$, where $m(S) = 1$ on a set S if and only if the complement of S is finite.

Let $\mathcal{F}= \{\mu \mid \mu \text{ is a filter on } N \text{ and } \mu \geq m\}$. This is a non-empty partially ordered set in which every chain has a maximal element. Hence, by Zorn's Lemma, $\mathcal{F}$ has a maximal element.

Proposition 47. *Let ω be a non-principal ultrafilter on $\mathbb{N}$. For any bounded sequence (a_n) of real numbers, there exists a unique $l \in \mathbb{R}$ such that:*
for every $\epsilon > 0$, $\omega(\{n \in \mathbb{N} \mid |a_n - l| < \epsilon\}) = 1$.

Definition 15. *The number l is called the ultralimit or more specifically, the ω-limit of the sequence (a_n).*

Remark 48. *Ultralimits exist in more general circumstances. In the above definition, the real line can be replaced with any metric space, whence any sequence of points in a compact subspace of the metric space will have a unique ultralimit.*

A.2 Ultralimits of Metric Spaces

We are now in a position to discuss ultralimits of metric spaces. Let (X_n, d_n) be a sequence of metric spaces and ω a non-principal ultrafilter on $\mathbb{N}$. For $n \in \mathbb{N}$, let $p_n \in X_n$.

Let $X_c = \{(x_n) \in \prod X_n \mid (d_n(x_n, p_n))$ is a bounded sequence of real numbers$\}$. By definition, for any two points (x_n) and $(y_n) \in X_c$, $(d_n(x_n, y_n))$ is a bounded sequence of real numbers and hence by proposition 47, has an ultralimit. Define this unique real number to be the pseudo-distance d_ω between the two points. The tuple (X_c, d_ω) is a pseudometric space.

Identifying all points of X_c which are at ω-distance zero to each other, one obtains a metric space (X_ω, d_ω) which is called the ultralimit or the ω-limit of the X_n's with respect to the base point (a_n).

Notation 49. *The ω-limit of a sequence (X_n, d_n) with respect to base point (p_n) is written as $\lim_\omega(X_n, d_n, (p_n))$ or X_ω, for short. The equivalence class of a sequence (x_n) in X_ω is denoted $\lim_\omega x_n$.*

The following important lemma appears in many places in the literature. A proof may be found, for example in [19].

Lemma 50. *Ultralimits of metric spaces are complete.*

Lemma 51. *Any ultralimit of a sequence of geodesic metric spaces is a geodesic space.*

Proof. Let X_ω be the ultralimit of metric spaces (X_n, d_n) with respect to a base point (a_n). Let (x_n), $(y_n) \in X_\omega$. Since each X_n is geodesic there is a geodesic $\gamma_n : [0,1] \to X_n$ joining the points x_n and y_n. Define $\gamma : [0,1] \to X_\omega$ be the map defined by $\gamma(t) = \lim_\omega \gamma_n(t)$. Clearly, γ is a geodesic joining the two given points. $\square$

A.3 Asymptotic Cones

An asymptotic cone of a metric space is a special case of an ultralimit. Let (X, d) be a metric space and ω be a non-principal ultrafilter on $\mathbb{N}$. Let (a_n) be a sequence of positive real numbers that tend to infinity as n tends to infinity. In the definition of ultralimit above, for each $n \in \mathbb{N}$, take X_n to be X and define d_n to be the metric d/a_n; that is, $d_n(x, y) = d(x, y)/a_n$. Let (p_n) be a sequence of points in X.

Definition 16. *The ω-limit of the system $(X_n, d_n, (p_n))$ is called the asymptotic cone of X with respect to the ultrafilter ω and base point (p_n). It is denoted as $Cone_\omega(X, (p_n))$ or as $Cone_\omega(X)$ or simply as X_ω.*

It is worth mentioning here, that the asymptotic cone of a metric space or for that matter, any ultralimit depends on the choice of the ultrafilter. If $\omega \neq \omega'$, then $Cone_\omega(X)$ may not in general, be isometric to $Cone_{\omega'}(X)$.

The canonical asymptotic cone: If the space X supports a cocompact group action then the isometry types of its asymptotic cones do not depend on the base point. Let $x \in X$. Then for all sequences (x_n) of points of X, $Cone_\omega(X, (x_n), (a_n))$ is isometric to $Cone_\omega(X, (x), (a_n))$. So we will refer to the asymptotic cone of a space X with respect to the 'constant' sequence base point, as the *canonical asymptotic cone* of X for the given choice of sequence (a_n) and ω.

Lemma 52. *Let V be a finite dimensional real Banach Space. Then every asymptotic cone of V is canonically isomorphic to V.*

Proof. We may assume that the base point is at (0). Define a map $\phi : V \to Cone_\omega(V)$ such that $\phi(v) = (a_n v)$. Observe that ϕ is a one-one map.
Now let (v_n) be an element of $Cone_\omega(V)$. There exists a constant $r > 0$ such that $||v_n||/a_n \leq r$. In other words, the vectors $\frac{v_n}{a_n}$ are all contained in the closed ball B of radius r in V. But as V is finite dimensional, B is compact. Consequently, the sequence $(\frac{v_n}{a_n})$ has an ultralimit in B. Set $v = \lim_\omega \frac{v_n}{a_n}$. Clearly, $\phi(v) = (v_n)$. $\square$

A.4 Asymptotic Cones and Quasi-isometries

Definition 17. *Let $f : X \to Y$ be a map of metric spaces. If there exist constants $\lambda \geq 1$ and $\epsilon \geq 0$ such that $\frac{1}{\lambda} d(x, x') - \epsilon \leq d(f(x), f(x')) \leq \lambda d(x, x') + \epsilon$, for all x and $x' \in X$, then f is called a (λ, ϵ)-quasi-isometric embedding of X into Y. If moreover $f(X)$ is quasi-dense in Y, that is, there exists a constant $c \geq 0$ such that $d(y, f(X)) \leq c$ for all $y \in Y$, then f is called a quasi-isometry.*

Observation 53. *A (λ, ϵ)-quasi-isometry induces a bi-Lipschitz homeomorphism at the level of asymptotic cones.*

Proof. Let $f : X \to Y$ be a quasi-isometry between metric spaces, with associated constants λ, ϵ and c, as above. Let ω be a non-principal ultrafilter and (a_n) a sequence of positive real numbers such that $\lim_{n\to\infty} a_n = \infty$. Let (p_n) be a sequence of points from X.

Set $X_\omega = Cone_\omega(X, (a_n), (p_n))$ and $Y_\omega = Cone_\omega(Y, (a_n), (f(p_n)))$. The function f induces a map $F : X_\omega \to Y_\omega$, defined by $\lim_\omega x_n \mapsto \lim_\omega f(x_n)$. Observe that if (x_n) denotes an equivalence class in X_ω, then $d(x_n, p_n)/a_n$ is a bounded sequence of real numbers. Since f is a quasi-isometry, it follows that $d(f(x_n), f(p_n))/a_n$ is also bounded and so the map F is well-defined.

Now let $\lim_\omega x_n$ and $\lim_\omega x'_n$ be elements of X_ω. Then, for every $n \in \mathbb{N}$, we have

$\frac{1}{\lambda}\frac{d(x_n, x'_n)}{a_n} - \frac{\epsilon}{a_n} \leq \frac{d(f(x_n), f(x'_n))}{a_n} \leq \lambda\frac{d(x_n, x'_n)}{a_n} + \frac{\epsilon}{a_n}$. This implies that

$\frac{1}{\lambda}d_\omega(\lim_\omega x_n, \lim_\omega x'_n) \leq d_\omega(F(\lim_\omega x_n), F(\lim_\omega x'_n)) \leq \lambda d_\omega(\lim_\omega x_n, \lim_\omega x'_n)$.

The function f has a 'quasi-inverse', $g : Y \to X$, which is a (λ', ϵ') quasi-isometric embedding. Moreover, there exists a constant $k \geq 0$ such that $d(gf(x), x) \leq k$ and $d(fg(y), y) \leq k$, for all $x \in X$ and for all $y \in Y$. The function g induces a map $G : Y_\omega \to X_\omega$ at the level of asymptotic cones. As before, every pair $(\lim_\omega y_n, \lim_\omega y'_n)$ of points from Y_ω satisfies

$$\frac{1}{\lambda'}d_\omega(\lim_\omega y_n, \lim_\omega y'_n) \leq d_\omega(G(\lim_\omega y_n), G(\lim_\omega y'_n)) \leq \lambda' d_\omega(\lim_\omega y_n, \lim_\omega y'_n).$$

Moreover, $GF(\underline{x}) = \underline{x}$ for all $\underline{x} \in X_\omega$ and $FG(\underline{y}) = \underline{y}$ for all $\underline{y} \in Y_\omega$.

We conclude from the above discussion that F is a bilipschitz homeomorphism between X_ω and Y_ω. $\square$

Observation 54. *A* $(1, \epsilon)$*-quasi-isometry satisfies the hypotheses of Lemma 17 and thus induces an isometry at the level of asymptotic cones.*

Appendix B

NOVIKOV'S CONJECTURE

Novikov's Conjecture on Higher Signatures is about homotopy invariants for closed orientable manifolds. In the nineteen fifties, Hirzebruch defined the L-class $\mathcal{L}(M)$ for a manifold M. This is a special element of the ring $\oplus_{i\geq 0} H^{4i}(M, \mathbb{Q})$ and its i-th component is written $\mathcal{L}_i$. Hirzebruch showed that if M is a $4k$-dimensional manifold and $[M]$ denotes the fundamental class of M, then evaluating $\mathcal{L}_k$ against $[M]$ produces the signature of M. This is the well-known Signature Theorem. The signature of M is known to be a homotopy invariant. Novikov's Conjecture involves the other components of the L-class.

One wants to investigate the rational numbers given by $\langle x \cup \mathcal{L}_i, [M]\rangle$, for all $x \in H^{n-4i}(M)$, where n is the dimension of M. Novikov proved in [24] that if $n = 4k+1$ and $x \in H^1(M)$, then these numerical expressions are indeed homotopy invariants and subsequently, proposed his conjecture.

Let G be a discrete group and BG its classifying space. In other words, BG is a CW-complex with contractible universal covering and with fundamental group isomorphic to G. Let $f : M \to BG$ be a continuous map from a closed oriented n-dimensional manifold M to BG, and $x \in H^{n-4i}(BG; \mathbb{Q})$.

The rational numbers $\langle f^*(x) \cup \mathcal{L}_i(M), [M]\rangle$ obtained for all possible choices of f and

x are the higher signatures of M. Novikov conjectured that the higher signatures are homotopy invariant; this means, given any f and x as above and $g : N \to M$, an orientation-preserving homotopy equivalence, we have

$$\langle f^*(x) \cup L_i(M), [M] \rangle = \langle g^*(x) \cup L_i(N), [N] \rangle .$$

The work of Mishchenko, Kasparov and Connes showed that if the Baum-Connes assembly map

$$\mu_G : K_*^G(BG) \to K_*(C_r^*(G))$$

is rationally injective then Novikov's Conjecture is true for G. Here, $K_*^G(BG)$ denotes the equivariant K-homology of G and $K_*(C_r^*(G))$ is the K-theory of the reduced C^*-algebra of G. In [16], the authors adopt this route to prove Novikov's Conjecture for groups acting geometrically on weakly geodesic, δ-bolic metric spaces of bounded coarse geometry.

We refer the reader to [11] for detailed references to Novikov's Conjecture.

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